SHARP $C^{1,\alpha}$ REGULARITY OF THE SOLUTION TO AN OBSTACLE $p-$LAPLACIAN MINIMIZATION PROBLEM WITH LINEAR TERM

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ABSTRACT. In this paper, we consider the following obstacle minimization problem with linear term:

$$\min\left\{\int_\Omega w\,\frac{|\nabla u|^p}{p}\;+\;f\,u\;:\;u\in W^{1,p}(\Omega),\;\;u\geq 0,\;\;\;u=g\;\;\text{on}\;\;\partial\Omega\right\} \tag{0.1}$$

where $1<p<\infty$. Assume $w\in C^{0,\sigma}(\Omega)$ and $f\in L^q(\Omega)$ with $q\geq\frac{p}{p-1}$. If $q>N$, we will prove via an approximation argument that the solution u of Problem (0.1) is locally $C^{1,\alpha}$, where the exponent $\alpha\in(0,1]$ will be given explicitly in terms of N, p, q and σ.

Moreover, we will show that u enjoys higher regularity on the free boundary $\partial\{u>0\}$; it is locally of class $C^{1,\min\left\{\frac{1-\frac{N}{q}}{p-1},1\right\}}$. If $p\geq 2$, $w\in C^1(\Omega)$, $f\in L^\infty(\Omega)$ and $f\geq f_\star>0$, then we establish a nondegeneracy property, which implies that the regularity of the solution u on the free boundary is optimal. Finally, we prove that the free boundary has zero Lebesgue measure.

1. INTRODUCTION

The main goal of this paper is to study the sharp $C^{1,\alpha}$ local regularity of a solution u to the following obstacle $p-$Laplacian problem with varying coefficients:

$$\begin{cases}\nabla\cdot[w|\nabla u|^{p-2}\nabla u]=f & \text{in}\;\;\{u>0\},\\ \nabla\cdot[w|\nabla u|^{p-2}\nabla u]\leq f & \text{in}\;\;\;\Omega,\\ u=g & \text{on}\;\;\;\partial\Omega.\end{cases} \tag{1.1}$$

The regularity of solutions to $p-$Laplacian type equations has been extensively studied in the literature. In [11, 7], the author established $C^{1,\alpha}$ regularity estimates on the solutions of the $p-$harmonic equation

$$\nabla\cdot[|\nabla u|^{p-2}\nabla u]=0. \tag{1.2}$$

Let $\alpha_\star\in(0,1]$ be the largest exponent such that weak solutions of (1.2) belong locally to $C^{1,\alpha_\star}$. It is well known that $\alpha_\star$ depends only on p and N. In the planar case (i.e. when $N=2$), the optimal exponent $\alpha_\star$ was identified in [14, Theorem 1] and is given by the following (we note that when $p\in(1,2]$, solutions to (1.2) possess higher regularity):

$$\alpha_\star=\min\left\{\frac{1}{6}\left(\frac{p}{p-1}+\sqrt{1+\frac{14}{p-1}+\frac{1}{(p-1)^2}}\right),\,1\right\}.$$

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In higher dimensions $N \geq 3$, this optimal exponent $\alpha_\star$ is still unknown. In addition, the weighted $p-$Laplacian equation with nonhomogeneous term has been considered in [20] (see also [15])

$$\nabla \cdot [w|\nabla u|^{p-2}\nabla u] = f. \tag{1.3}$$

Thanks to [20], under the assumptions that $w \in C^1(\Omega)$ with $0 < \lambda \leq w \leq \Lambda < \infty$ and $f \in L^\infty(\Omega)$, any solution u to equation (1.3) is locally of class $C^{1,\alpha}$. In [8, 9], the authors have proved that u is C^1 in Ω provided that w is $\sigma-$Hölder continuous and f satisfies the following integrability:

$$\int_{B_\rho} |f| \leq C\rho^{N-1}h(\rho) \tag{1.4}$$

for every ball $B_\rho \subset\subset \Omega$ with radius ρ, where the function h in (1.4) satisfies the following condition:

$$\int_0^R [h(\rho)]^{\min\{\frac{1}{p-1},1\}} \frac{\mathrm{d}\rho}{\rho} < \infty \quad \text{for some} \quad R > 0.$$

On the other hand, the authors of [2, 8] have established sharp $C^{1,\alpha}$ regularity estimates on the solutions of (1.3). If $w \in C^{0,\sigma}(\Omega)$ and $f \in L^q(\Omega)$ with $q > N$, then every solution u to (1.3) is locally $C^{1,\alpha}$ with

$$\alpha = \min\{\alpha_\star^-, \alpha_{p,q,\sigma}\}$$

where

$$\alpha_{p,q,\sigma} = \min\left\{\sigma, 1 - \frac{N}{q}\right\} \cdot \min\left\{\frac{1}{p-1}, 1\right\} \tag{1.5}$$

and $\alpha_\star^-$ is any positive number that is strictly less than $\alpha_\star$. In this paper, we will extend this sharp $C^{1,\alpha}$ regularity result to the solution of Problem (0.1), or equivalently (1.1). In fact, studying the regularity of the solution u to the obstacle problem (1.1) has significant challenges. Under appropriate assumptions on f and w, the regularity results established in [2, 8, 9, 20] remain applicable to the solution u of (1.1), but only locally within the positivity set $\{u > 0\}$. So, one difficulty is to extend this regularity to the free boundary $\partial\{u > 0\}$. In [12, 16, 19], the authors proved the $C^{1,\alpha}$ regularity of the solution u to the obstacle problem (1.1) but in the homogeneous case (i.e. when $f = 0$). Moreover, the authors of [4] have obtained $C^{1,\alpha}$ regularity on the solution u of (1.1) provided that u is bounded, $w \in C^1(\Omega)$ and f belongs to $L^q(\Omega)$, for some $q > N$. In addition, it was proved in [18] that a solution u to (1.1) is locally bounded in Ω whenever $f \in L^\infty(\Omega)$. From [3, Proposition 1.2], the solution u of (1.1) is also bounded as soon as $g \in L^\infty(\Omega)$. Thanks also to [17, Theorem 5.14], the solution u of (1.1) belongs to $C^{1,\alpha}_{loc}(\Omega)$ provided that w is Hölder continuous and f is bounded. However, to the best of our knowledge, it seems that the $C^{1,\alpha}$ regularity of the solution u to (1.1) in the case where f is unbounded and w is not of class C^1 has not yet been addressed in the literature. Furthermore, a fundamental problem that arises with the study of the obstacle problem (1.1) concerns the optimal regularity of the solution u. Although the authors of [4, 17] established that $u \in C^{1,\alpha}_{loc}(\Omega)$, the value of α was unknown. By an approximation argument and thanks to the results of [2, 8], we obtain sharp local $C^{1,\alpha}$ regularity estimates on the solution u of (1.1) in the case where $f \in L^q(\Omega)$ and $w \in C^{0,\sigma}(\Omega)$.

In [1, Theorem 1], the authors have studied the regularity of the solution to (1.1) at free boundary points. More precisely, they proved that in the case where $f \in L^\infty(\Omega)$ and $w = 1$, the solution u of (1.1) is of class $C^{1,\min\{\frac{1}{p-1},1\}}$ on the free boundary $\partial\{u > 0\}$. Here, using different techniques, we extend the regularity result in [1] to the more general setting where $f \in L^q(\Omega)$ and w is $\sigma-$Hölder continuous. More surprisingly, we show that the solution u is more regular on the free boundary than away from it, contrary to what one might expect. Furthermore, the regularity of u on the free boundary is independent of the $\sigma-$Hölder regularity of w; in particular, even when $\sigma > 0$ is very small, u may still belong to $C^{1,1}$ on the free boundary.

On the other hand, the authors of [5] have also considered Problem (0.1) but in the presence of a strong absorption term $u^{\gamma+1}$ $(0 < \gamma < p-1)$. Their analysis allows the weight function w to be nonconstant, while assuming that the source term f is a nonnegative continuous function and that the boundary datum g is also continuous. More precisely, they show that if u is a bounded weak solution to Problem (1.1), then u is of class $C^{\frac{p}{p-1-\gamma}}$ along the free boundary. However, the results in [5] do not cover the limiting case $\gamma = 0$. We also note that if $\gamma > 0$ then the absorption term $u^{\gamma+1}$ is clearly C^1 and so, the minimizer u of Problem (0.1) turns out to be a weak solution to the equation

$$\nabla \cdot [w|\nabla u|^{p-2}\nabla u] = f\,[u]_+^{\gamma} \qquad \text{in} \quad \Omega.$$

Thus, thanks to the classical regularity literature (see [2, 7, 20]), we get immediately that $u \in C^{1,\alpha}_{loc}(\Omega)$, for some $\alpha \in (0,1)$. But, the lack of C^1-regularity of the absorption term in the limiting case $\gamma = 0$ makes the regularity analysis considerably more challenging. One of the main difficulties is to establish that the minimizer u satisfies

$$\nabla \cdot [w|\nabla u|^{p-2}\nabla u] = f \cdot \chi_{\{u>0\}} \qquad \text{in} \quad \Omega,$$

which is closely related to first proving that the free boundary $\partial\{u > 0\}$ has locally finite $\mathcal{H}^{N-1}-$measure.

Finally, we establish a non-degeneracy property of the solution u to Problem (1.1), provided that f is nonnegative and bounded away from zero. More precisely, we prove that

$$u(x) \geq C\text{dist}(x, \partial\{u > 0\})^{\frac{p}{p-1}}, \qquad \text{for all} \quad x \in \{u > 0\} \cap \Omega.$$

If f is bounded, this estimate yields that the solution u of (1.1) has the optimal growth near the free boundary. As a consequence, we obtain two important geometric properties of the free boundary: it is porous and therefore, has zero Lebesgue measure.

2. Sharp $C^{1,\alpha}$ regularity of the solution to the obstacle problem

In this section, we show via an approximation argument, sharp $C^{1,\alpha}$ regularity estimates on the unique solution u of Problem (0.1). For every $\varepsilon > 0$, we define

$$\mathcal{B}_\varepsilon(s) = \begin{cases} 0 & \text{if} \quad s \leq 0, \\ \frac{s}{\varepsilon} & \text{if} \quad 0 \leq s \leq \varepsilon, \\ 1 & \text{if} \quad s \geq \varepsilon. \end{cases}$$

Let $\Psi_\varepsilon \in C^1(\mathbb{R})$ be such that $\Psi'_\varepsilon = \mathcal{B}_\varepsilon$. Let us assume that $f \in L^q(\Omega)$ with $q \geq \frac{p}{p-1}$. For clarity and simplicity, we will only treat the case when $f \geq 0$ (but our results can be easily extended to the general case by decomposing f into its positive and negative parts). Then, we consider instead of (0.1) the following approximated problem:

$$\min\left\{ \int_\Omega w\,\frac{|\nabla u|^p}{p} \,+\, f\,\Psi_\varepsilon(u) \,:\, u \in W^{1,p}(\Omega), \;\; u = g \;\text{ on }\; \partial\Omega \right\}. \tag{2.1}$$

Here, we assume that $g \geq 0$ and that there exists $\tilde{g} \in W^{1,p}(\Omega)$ such that $\tilde{g} = g$ on $\partial\Omega$. Then, we start by the following simple result:

Proposition 2.1. *For every $\varepsilon > 0$, Problem (2.1) has a unique nonnegative minimizer u_ε. Moreover, u_ε is a weak solution to the following problem:*

$$\begin{cases} \nabla \cdot [w|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon] = f\,\mathcal{B}_\varepsilon(u_\varepsilon) & \text{in} \quad \Omega, \\ u_\varepsilon = g & \text{on} \quad \partial\Omega. \end{cases} \tag{2.2}$$

Proof. First of all, we note that if $u \in W^{1,p}(\Omega)$ with $u = g$ on $\partial\Omega$ then $u_+ := \max\{u, 0\}$ is also admissible in (2.1) and, we have

$$\int_\Omega w\,\frac{|\nabla u_+|^p}{p} + f\,\Psi_\varepsilon(u_+) = \int_\Omega w\,\frac{|\nabla u_+|^p}{p} + f\,\Psi_\varepsilon(u) \leq \int_\Omega w\,\frac{|\nabla u|^p}{p} + f\,\Psi_\varepsilon(u).$$

So, we can restrict (2.1) to nonnegative functions. Let $(u_n)_n$ be a minimizing sequence in (2.1). Then, there exists a constant $C < \infty$ such that

$$\int_\Omega w\,\frac{|\nabla u_n|^p}{p} + f\,\Psi_\varepsilon(u_n) \leq C.$$

Yet,

$$\left|\int_\Omega f\,\Psi_\varepsilon(u_n)\right| \leq \int_\Omega |f| u_n \leq C||f||_{L^q(\Omega)}||u_n||_{L^p(\Omega)}.$$

Moreover, one has $u_n = g$ on $\partial\Omega$ (so, $u_n - \tilde{g} \in W_0^{1,p}(\Omega)$). Using the Poincaré inequality, then we get that

$$||u_n||_{L^p(\Omega)} \leq C[||\nabla u_n||_{L^p(\Omega)} + ||\nabla\tilde{g}||_{L^p(\Omega)}] + ||\tilde{g}||_{L^p(\Omega)} \leq C||\nabla u_n||_{L^p(\Omega)} + C.$$

Hence,

$$||\nabla u_n||^p_{L^p(\Omega)} \leq C.$$

Consequently, $(u_n)_n$ is bounded in $W^{1,p}(\Omega)$ and so, up to a subsequence, $u_n \rightharpoonup u_\varepsilon$ weakly in $W^{1,p}(\Omega)$ with $u_\varepsilon = g$ on $\partial\Omega$. Thanks to the lower semicontinuity of the L^p-norm as well as the strong convergence $u_n \to u$ in $L^p(\Omega)$, we obtain

$$\int_\Omega w\,|\nabla u_\varepsilon|^p \leq \liminf_n \int_\Omega w\,|\nabla u_n|^p$$

and

$$\int_\Omega f|\Psi_\varepsilon(u_n) - \Psi_\varepsilon(u_\varepsilon)| \leq \int_\Omega f|u_n - u_\varepsilon| \leq C||f||_{L^q(\Omega)}||u_n - u_\varepsilon||_{L^p(\Omega)} \to 0.$$

Hence,

$$\int_\Omega w\,\frac{|\nabla u_\varepsilon|^p}{p} + f\,\Psi_\varepsilon(u_\varepsilon) \leq \liminf_n \left[\int_\Omega w\,\frac{|\nabla u_n|^p}{p} + f\,\Psi_\varepsilon(u_n)\right].$$

Consequently, u_ε minimizes (2.1). Since Ψ_ε is convex and $f \geq 0$, then the functional in (2.1) is clearly strictly convex and so, u_ε is the unique minimizer of (2.1). Fix $\varphi \in C_c^\infty(\Omega)$. For all δ small enough, we have thanks to the minimality of u_ε that

$$\int_\Omega w\,\frac{|\nabla u_\varepsilon|^p}{p} + f\,\Psi_\varepsilon(u_\varepsilon) \leq \int_\Omega w\,\frac{|\nabla u_\varepsilon + \delta\nabla\varphi|^p}{p} + f\,\Psi_\varepsilon(u_\varepsilon + \delta\varphi).$$

Thus,

$$\int_\Omega w|\nabla u_\varepsilon|^{p-2}\,\nabla u_\varepsilon \cdot \nabla\varphi + \int_\Omega f\,\mathcal{B}_\varepsilon(u_\varepsilon)\,\varphi = 0. \quad \square$$

Now, we will show uniform estimates on these solutions $\{u_\varepsilon\}_{\varepsilon>0}$ that will allow us to pass to the limit when $\varepsilon \to 0^+$.

Proposition 2.2. *There exists a uniform constant C that depends only on λ, Λ, $||f||_{L^q(\Omega)}$, diam(Ω) and $||\tilde{g}||_{W^{1,p}(\Omega)}$ such that*

$$||u_\varepsilon||_{W^{1,p}(\Omega)} \leq C.$$

In particular, up to a subsequence, we have $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$ for some function $u \in W^{1,p}(\Omega)$ with $u = g$ on $\partial\Omega$. In addition, the limit function u minimizes our problem

$$\min\left\{ \int_\Omega w\, \frac{|\nabla u|^p}{p} \;+\; f\, u \;:\; u \in W^{1,p}(\Omega),\;\; u \geq 0,\;\; u = g \;\; \text{on } \partial\Omega \right\}.$$

Finally, assume that u is continuous in Ω. Then, u solves the following $p-$Laplacian obstacle problem:

$$\begin{cases} \nabla\cdot[w|\nabla u|^{p-2}\nabla u] = f & \text{in } \;\{u>0\}, \\ \nabla\cdot[w|\nabla u|^{p-2}\nabla u] \leq f & \text{in } \;\Omega, \\ u = g & \text{on } \;\partial\Omega. \end{cases}$$

Proof. Since u_ε minimizes Problem (2.1), then we have

$$\int_\Omega w\frac{|\nabla u_\varepsilon|^p}{p} + f\,\Psi_\varepsilon(u_\varepsilon) \leq \int_\Omega w\frac{|\nabla \tilde{g}|^p}{p} + f\,\Psi_\varepsilon(\tilde{g}) \leq \frac{\Lambda}{p}||\nabla\tilde{g}||^p_{L^p(\Omega)} + ||f||_{L^{\frac{p}{p-1}}(\Omega)}||\tilde{g}||_{L^p(\Omega)}.$$

Hence,

$$\int_\Omega |\nabla u_\varepsilon|^p \leq \frac{\Lambda||\nabla\tilde{g}||^p_{L^p(\Omega)} + p\,||f||_{L^{\frac{p}{p-1}}(\Omega)}||\tilde{g}||_{L^p(\Omega)}}{\lambda}.$$

Yet, one has $u_\varepsilon = g$ on $\partial\Omega$ (so, $u_\varepsilon - \tilde{g} \in W^{1,p}_0(\Omega)$). Using the Poincaré inequality again, we get

$$||u_\varepsilon||_{L^p(\Omega)} \leq C[||\nabla u_\varepsilon||_{L^p(\Omega)} + ||\nabla\tilde{g}||_{L^p(\Omega)}] + ||\tilde{g}||_{L^p(\Omega)} \leq C.$$

Then, up to a subsequence, we have $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$. Finally, we show that u minimizes Problem (0.1). Let $v \in W^{1,p}(\Omega)$, $v \geq 0$ in Ω, and $v = g$ on $\partial\Omega$. From the minimality of u_ε in Problem (2.1), we have

$$\int_\Omega w\,\frac{|\nabla u_\varepsilon|^p}{p} \;+\; f\,\Psi_\varepsilon(u_\varepsilon) \leq \int_\Omega w\,\frac{|\nabla v|^p}{p} \;+\; f\,\Psi_\varepsilon(v).$$

Since $\Psi_\varepsilon(v) \leq v$, then

$$\int_\Omega w\,\frac{|\nabla u_\varepsilon|^p}{p} \;+\; f\,\Psi_\varepsilon(u_\varepsilon) \leq \int_\Omega w\,\frac{|\nabla v|^p}{p} \;+\; f\,v.$$

Passing to the limit when $\varepsilon \to 0$, we get

$$\liminf_\varepsilon \left[\int_\Omega w\,\frac{|\nabla u_\varepsilon|^p}{p} \;+\; f\,\Psi_\varepsilon(u_\varepsilon)\right] \leq \int_\Omega w\,\frac{|\nabla v|^p}{p} \;+\; f\,v.$$

But, we have

$$\int_\Omega w\,|\nabla u|^p \;\leq \liminf_\varepsilon \int_\Omega w\,|\nabla u_\varepsilon|^p.$$

Fix $x \in \{u > 0\}$. Since $u_\varepsilon(x) \to u(x)$ when $\varepsilon \to 0$ (this holds at a.e. x), then we may assume that $u_\varepsilon(x) \geq \frac{u(x)}{2} > 0$ for all ε sufficiently small. In particular, we have $\Psi_\varepsilon(u_\varepsilon) = u_\varepsilon - \frac{\varepsilon}{2}$ (this follows from the definition of Ψ_ε) and so,

$$\liminf_\varepsilon \ \Psi_\varepsilon(u_\varepsilon) = \liminf_\varepsilon \left[u_\varepsilon - \frac{\varepsilon}{2}\right] = u.$$

Thanks to Fatou's Lemma, one has

$$\int_\Omega f\, u = \int_{\{u>0\}} f\, u = \int_{\{u>0\}} [f\ \liminf_\varepsilon \Psi_\varepsilon(u_\varepsilon)] \leq \liminf_\varepsilon \int_{\{u>0\}} f\ \Psi_\varepsilon(u_\varepsilon) \leq \liminf_\varepsilon \int_\Omega f\ \Psi_\varepsilon(u_\varepsilon).$$

Consequently, we get

$$\int_\Omega w\, \frac{|\nabla u|^p}{p} + f\, u \leq \int_\Omega w\, \frac{|\nabla v|^p}{p} + f\, v.$$

Let φ be a smooth function with $K := \mathrm{spt}(\varphi) \subset \{u > 0\}$. Since u is continuous (by assumption; but we will show that this is the case later), then $u + \delta\varphi$ is clearly admissible in Problem (0.1) for any δ sufficiently small and so, thanks to the minimality of u, we have

$$\int_\Omega w\, \frac{|\nabla u|^p}{p} + f\, u \leq \int_\Omega w\, \frac{|\nabla u + \delta\nabla\varphi|^p}{p} + f\,[u + \delta\varphi]. \tag{2.3}$$

Hence,

$$\int_\Omega w|\nabla u|^{p-2}\, \nabla u \cdot \nabla\varphi + \int_\Omega f\, \varphi = 0.$$

Finally, let $\varphi \in C_c^\infty(\Omega)$ be such that $\varphi \geq 0$ (or more generally, $u + \varphi \geq 0$) in Ω. For any $\delta > 0$ small, $u + \delta\varphi$ is admissible in (0.1). By (2.3), we get that

$$\int_\Omega w|\nabla u|^{p-2}\, \nabla u \cdot \nabla\varphi + \int_\Omega f\, \varphi \geq 0. \quad \square$$

Now, we show that u_ε is smooth.

Proposition 2.3. *Assume $q > N$. Then, the solution u_ε of Problem (2.1) is C^1 in Ω, for every $\varepsilon > 0$.*

Proof. Thanks to [9, Theorem 4] ($p \geq 2$) or [8, Corollary 1.6] ($1 < p < 2$), the solution u_ε of Problem (2.2) is C^1 as soon as $\mu_\varepsilon := f\, \mathcal{B}_\varepsilon(u_\varepsilon)$ satisfies

$$\int_{B_\rho} |\mu_\varepsilon| \leq C\rho^{N-1}\, h(\rho),$$

for every ball $B_\rho \subset\subset \Omega$ with radius ρ, where $h : [0,\infty) \mapsto [0,\infty)$ is a function satisfying the Dini condition: there exists $R > 0$ such that

$$\begin{cases} \int_0^R [h(\rho)]^{\frac{1}{p-1}}\, \frac{\mathrm{d}\rho}{\rho} < \infty & \text{when} \quad p \geq 2, \\ \int_0^R h(\rho)\, \frac{\mathrm{d}\rho}{\rho} < \infty & \text{when} \ \ 1 < p < 2. \end{cases}$$

Yet, one has

$$\int_{B_\rho} |\mu_\varepsilon| \leq |B_\rho|^{\frac{q-1}{q}}\, ||\mu_\varepsilon||_{L^q(B_\rho)} \leq C\rho^{\frac{N(q-1)}{q}}\, ||f||_{L^q(B_\rho)} = C\rho^{N-1}\, h(\rho) \tag{2.4}$$

with

$$h(\rho) := ||f||_{L^q(B_\rho)}\, \rho^{\frac{q-N}{q}}.$$

Moreover, we have

$$\int_0^R [h(\rho)]^{\frac{1}{p-1}} \frac{\mathrm{d}\rho}{\rho} \le ||f||_{L^q(\Omega)}^{\frac{1}{p-1}} \int_0^R \frac{\mathrm{d}\rho}{\rho^{1-\frac{q-N}{q(p-1)}}} < \infty \quad \text{if} \quad p \ge 2$$

and

$$\int_0^R h(\rho) \frac{\mathrm{d}\rho}{\rho} \le ||f||_{L^q(\Omega)} \int_0^R \frac{\mathrm{d}\rho}{\rho^{\frac{N}{q}}} < \infty \quad \text{if} \quad p \in (1,2). \tag{2.5}$$

□

In the next proposition, we will show uniform $C^{1,\alpha}$ regularity estimates on the solution u_ε of (2.1).

Proposition 2.4. *Assume $p \ge 2$ and $w \in C^{0,\sigma}(\Omega)$. Then, the solution u of Problem (0.1) belongs to $C^{1,\alpha}_{loc}(\Omega)$ with*

$$\alpha := \min\{\alpha_{p,q,\sigma}, \alpha_\star^-\}.$$

Proof. From [10, Theorem 1.2], there exists a constant C that depends only on N, p, λ, Λ, σ, $[w]_{C^{0,\sigma}(\Omega)}$ and, $\mathrm{diam}(\Omega)$ such that the pointwise estimate

$$|\nabla u_\varepsilon(x)| \le C\left(\fint_{B(x,R)} |\nabla u_\varepsilon|^{\frac{p}{2}} \right)^{\frac{2}{p}} + C\left[\int_0^R \left(\frac{\mu_\varepsilon(B(x,\rho))}{\rho^{N-1}} \right)^{\frac{p}{2(p-1)}} \frac{\mathrm{d}\rho}{\rho} \right]^{\frac{2}{p}}$$

holds whenever $B(x,R) \subset \Omega$. Yet, thanks to Proposition 2.2 as well as the Holder's inequality, one has the following

$$\left(\fint_{B(x,R)} |\nabla u_\varepsilon|^{\frac{p}{2}} \right)^{\frac{2}{p}} \le ||\nabla u_\varepsilon||_{L^p(\Omega)} \, |B(x,R)|^{-\frac{1}{p}} \le C R^{-\frac{N}{p}}$$

where C is a uniform constant that does not depend on ε. Moreover, recalling estimate (2.4), we have

$$\int_0^R \left(\frac{\mu_\varepsilon(B(x,\rho))}{\rho^{N-1}} \right)^{\frac{p}{2(p-1)}} \frac{\mathrm{d}\rho}{\rho} \le C \int_0^R \left(\rho^{\frac{q-N}{q}} \right)^{\frac{p}{2(p-1)}} \frac{\mathrm{d}\rho}{\rho} < \infty.$$

Thus, we get

$$|\nabla u_\varepsilon(x)| \le C(R^{-\frac{N}{p}} + 1). \tag{2.6}$$

Let K be a compact set in Ω. Recalling (2.6), we have the following estimate (which is uniform in ε):

$$||\nabla u_\varepsilon||_{L^\infty(K)} \le C(N, p, q, \lambda, \Lambda, \sigma, [w]_{C^{0,\sigma}(\Omega)}, ||f||_{L^q(\Omega)}, \mathrm{diam}(\Omega), \mathrm{dist}(K, \partial\Omega)). \tag{2.7}$$

Thanks to the Gagliardo-Nirenberg interpolation inequality, there exists a constant C that depends only on N and p such that

$$||u_\varepsilon||_{L^\infty(K)} \le C \, ||\nabla u_\varepsilon||_{L^\infty(K)}^{\frac{N}{N+p}} \, ||u_\varepsilon||_{L^p(K)}^{\frac{p}{N+p}}. \tag{2.8}$$

By Proposition 2.2 and estimate (2.9), we infer that

$$||u_\varepsilon||_{C^1(K)} \le C(N, p, q, \lambda, \Lambda, \sigma, [w]_{C^{0,\sigma}(\Omega)}, ||f||_{L^q(\Omega)}, ||\tilde{g}||_{W^{1,p}(\Omega)}, \mathrm{diam}(\Omega), \mathrm{dist}(K, \partial\Omega)). \tag{2.9}$$

On the other hand, thanks to [2, Theorem 1.1], we also have that the solution u_ε of Problem (2.2) is locally $C^{1,\alpha}$, where (see (1.5))

$$\alpha := \min\{\alpha_{p,q,\sigma}\,,\,\alpha_\star^-\}.$$

Moreover, there holds

$$|\nabla u_\varepsilon(x) - \nabla u_\varepsilon(y)| \leq C|x-y|^\alpha, \quad \text{for all } \ x,\,y \in K, \tag{2.10}$$

where C is a constant depending only on N, p, q, σ, λ, Λ, $||u_\varepsilon||_{W^{1,p}(\Omega)}$, $[w]_{C^{0,\sigma}(\Omega)}$, $||\mu_\varepsilon||_{L^q(\Omega)}$ and $\mathrm{dist}(K,\partial\Omega)$. But, we recall that both $||u_\varepsilon||_{W^{1,p}(\Omega)}$ and $||\mu_\varepsilon||_{L^q(\Omega)}$ are uniformly bounded in ε. Consequently, we infer that

$$||u_\varepsilon||_{C^{1,\alpha}(K)} \leq C(N,p,q,\lambda,\Lambda,\sigma,[w]_{C^{0,\sigma}(\Omega)},||f||_{L^q(\Omega)},||\tilde{g}||_{W^{1,p}(\Omega)},\mathrm{diam}(\Omega),\mathrm{dist}(K,\partial\Omega)).$$

Hence, up to a subsequence, $u_\varepsilon \to u$ in $C^{1,\alpha}(K)$ when $\varepsilon \to 0$. Moreover, we have the following estimate:

$$\begin{aligned} ||u||_{C^{1,\alpha}(K)} &\leq \liminf_{\varepsilon} ||u_\varepsilon||_{C^{1,\alpha}(K)} \\ &\leq C(N,p,q,\lambda,\Lambda,\sigma,[w]_{C^{0,\sigma}(\Omega)},||f||_{L^q(\Omega)},||\tilde{g}||_{W^{1,p}(\Omega)},\mathrm{diam}(\Omega),\mathrm{dist}(K,\partial\Omega)). \quad \square \end{aligned} \tag{2.11}$$

Finally, we are concerned with the case when $p \in (1,2)$.

Proposition 2.5. *Assume $p \in (1,2)$. Then, the solution u of Problem (0.1) is locally Lipschitz in Ω. Moreover, we have the following:*

$$||u||_{C^{1,\alpha}(K)} \leq C(N,p,q,\lambda,\Lambda,\sigma,[w]_{C^{0,\sigma}(\Omega)},||f||_{L^q(\Omega)},||\tilde{g}||_{W^{1,p}(\Omega)},\mathit{diam}(\Omega),\mathit{dist}(K,\partial\Omega))$$

with

$$\alpha = \min\left\{\alpha_\star^-,\,\sigma,\,1-\frac{N}{q}\right\}.$$

Proof. Thanks to [8, Theorem 1.3], we know that there exists a constant C that depends on N, p, λ, Λ, σ and $[w]_{C^{0,\sigma}(\Omega)}$ such that

$$||\nabla u_\varepsilon||_{L^\infty(B(x,\frac{R}{2}))} \leq C\left|\left|\int_0^R \frac{\mu_\varepsilon(B(x',\rho))}{\rho^N}\,\mathrm{d}\rho\right|\right|_{L^\infty(B(x,R))}^{\frac{1}{p-1}} + CR^{-\frac{N}{2-p}}||\nabla u_\varepsilon||_{L^{2-p}(B(x,R))}$$

holds for any $R \in (0,1]$ with $B(x,2R) \subset \Omega$. Yet, by the estimates (2.4) & (2.5), we get that

$$||\nabla u_\varepsilon||_{L^\infty(B(x,\frac{R}{2}))} \leq C\left|||f||_{L^q(\Omega)}\int_0^R \frac{\mathrm{d}\rho}{\rho^{\frac{N}{q}}}\right|^{\frac{1}{p-1}} + CR^{-\frac{N}{2-p}}||\nabla u_\varepsilon||_{L^{2-p}(B(x,R))}.$$

From Proposition 2.2, we also have

$$||\nabla u_\varepsilon||_{L^{2-p}(B(x,R))} \leq C||\nabla u_\varepsilon||_{L^p(B(x,R))}\,R^{\frac{2N(p-1)}{p(2-p)}}.$$

Hence,

$$||\nabla u_\varepsilon||_{L^\infty(B(x,\frac{R}{2}))} \leq C\left|||f||_{L^q(\Omega)}\int_0^R \frac{\mathrm{d}\rho}{\rho^{\frac{N}{q}}}\right|^{\frac{1}{p-1}} + C||\nabla u_\varepsilon||_{L^p(B(x,R))}R^{-\frac{N}{p}}.$$

Let $K \subset \Omega$ be a compact set. Recalling (2.8), we infer that $\{u_\varepsilon\}_\varepsilon$ is uniformly bounded in

$C^1(K)$. From [8, Theorem 4.3 & Corollary 1.7] and thanks to (2.4), there exists a uniform constant C that does not depend on ε such that

$$|\nabla u_\varepsilon(x) - \nabla u_\varepsilon(y)| \le C|x-y|^\alpha, \quad \text{for all } \; x,\, y \in K.$$

Finally, this uniform $C^{1,\alpha}$ estimate on u_ε implies that the limit function u belongs to $C^{1,\alpha}_{loc}(\Omega)$. □

3. Regularity of the solution on the free boundary

In this section, we prove that the solution u to Problem (0.1) enjoys higher regularity along the free boundary $\partial\{u > 0\}$. The interesting observation is that the regularity of u near the free boundary does not depend on the Hölder regularity of the weight w. More precisely, we recall that u is locally of class $C^{1,\alpha}$ with $\alpha = \min\{\alpha_\star^-, \alpha_{p,q,\sigma}\}$. In particular, $0 < \alpha \le \sigma$. However, we will show that on the free boundary, u is of class $C^{1,\min\left\{\frac{1-\frac{N}{q}}{p-1},1\right\}}$ regardless of the value of σ.

Proposition 3.1. *Assume $f \in L^q(\Omega)$ and $w \in C^{0,\sigma}(\Omega)$. Let $\Omega' \subset\subset \Omega$ be an open set and K be a compact set in Ω'. There exists a uniform constant C that does not depend on ε such that for every $x \in K$ and for all $0 < r < d(K, \partial\Omega')$, we have*

$$\sup_{B(x,r)} u_\varepsilon \le C(r^\beta + u_\varepsilon(x))$$

with

$$\beta = \frac{p - \frac{N}{q}}{p-1}.$$

Proof. Assume by contradiction that this is not the case. This means that for every integer $n > 1$ there exists a point $x_n \in K$ and $0 < r_n < d(K, \partial\Omega')$ such that

$$\sup_{B(x_n,r_n)} u_{\varepsilon_n} > n\,(r_n^\beta + u_{\varepsilon_n}(x_n)).$$

Set $s_n = \sup\limits_{B(x_n,r_n)} u_{\varepsilon_n}$. So, we have

$$\frac{r_n^\beta}{s_n} + \frac{u_{\varepsilon_n}(x_n)}{s_n} < \frac{1}{n}. \tag{3.1}$$

We note that since $\{x_n\}_n \subset K$ then up to a subsequence, $x_n \to x_0$. In addition, we clearly have $r_n \to 0$ as $\frac{r_n^\beta}{s_n} \to 0$ and $s_n \le ||u_{\varepsilon_n}||_{L^\infty(B(x_n,r_n))} \le C$ (where $C < \infty$ is uniform in n thanks to (2.9)). Fix $\gamma >> 1$ sufficiently large. Then, we define

$$v_n(z) := \frac{u_{\varepsilon_n}(x_n + r_n z)}{s_n}, \quad \text{for all } \; z \in B_\gamma.$$

Thanks to Propositions 2.4 & 2.5, we see that $v_n \in C^{1,\alpha}(\overline{B_\gamma})$, for all n. Moreover, one has $||v_n||_{L^\infty(B_1)} = 1$. Now, we claim that v_n is a weak solution to the following equation:

$$\nabla\cdot[w_n|\nabla v_n|^{p-2}\,\nabla v_n] = f_n\,\mathcal{B}_{\varepsilon_n}(s_n v_n) \qquad \text{in} \quad B_\gamma \tag{3.2}$$

where

$$w_n(z) := w(x_n + r_n z) \quad \text{and} \quad f_n(z) = \frac{r_n^p}{s_n^{p-1}}\,f(x_n + r_n z).$$

Fix $\varphi \in C_c^\infty(B_\gamma)$. Define $\varphi_n(x) = \varphi(\frac{x-x_n}{r_n})$, for all $x \in B(x_n, r_n\gamma)$. Since u_{ε_n} is a weak solution to (2.2), then we have

$$-\int_{B(x_n,r_n\gamma)} w(x)|\nabla u_{\varepsilon_n}(x)|^{p-2}\,\nabla u_{\varepsilon_n}(x)\cdot\nabla\varphi_n(x)\,\mathrm{d}x = \int_{B(x_n,r_n\gamma)} f(x)\,\mathcal{B}_{\varepsilon_n}(u_{\varepsilon_n}(x))\,\varphi_n(x)\,\mathrm{d}x.$$

Take a change of variable $x = x_n + r_n z$, we get

$$-\int_{B_\gamma} w(x_n+r_nz)|\nabla u_{\varepsilon_n}(x_n+r_nz)|^{p-2}\,\nabla u_{\varepsilon_n}(x_n+r_nz)\cdot\nabla\varphi_n(x_n+r_nz)\,\mathrm{d}z$$

$$= \int_{B_\gamma} f(x_n+r_nz)\,\mathcal{B}_{\varepsilon_n}(u_{\varepsilon_n}(x_n+r_nz))\,\varphi_n(x_n+r_nz)\,\mathrm{d}z.$$

Yet, one has $\nabla v_n(z) = \frac{r_n}{s_n}\nabla u_{\varepsilon_n}(x_n+r_nz)$ and $\nabla\varphi(z) = r_n\,\nabla\varphi_n(x_n+r_nz)$, for all $z \in B_\gamma$. Hence, we infer that

$$-\int_{B_\gamma} w_n(z)|\nabla v_n(z)|^{p-2}\,\nabla v_n(z)\cdot\nabla\varphi(z)\,\mathrm{d}z = \int_{B_\gamma} f_n(z)\,\mathcal{B}_{\varepsilon_n}(s_nv_n(z))\,\varphi(z)\,\mathrm{d}z.$$

From the definition of s_n, we clearly have $0 \le v_n \le 1$ on B_1. Fix $1 \le r < r_\star < \gamma$. Then, we will show that v_n is uniformly bounded on B_r. By [17, Corollary 3.10], we have for any $0 < s \le p$ that

$$||v_n||_{L^\infty(B_r)} \le \frac{C}{(1-\frac{r}{r_\star})^{\frac{N}{s}}}\left[\left(\fint_{B_{r_\star}} v_n^s\,\mathrm{d}x\right)^{\frac{1}{s}} + ||f_n||_{L^q(B_{r_\star})}^{\frac{1}{p-1}}\right]$$

where

$$C = C(N,\,p,\,q,\,s,\,\Lambda,\,r_\star).$$

Yet, one has

$$||f_n||_{L^q(B_{r_\star})} = \frac{r_n^p}{s_n^{p-1}}\left(\int_{B_{r_\star}} f^q(x_n+r_nz)\,\mathrm{d}z\right)^{\frac{1}{q}} = \frac{r_n^{p-\frac{N}{q}}}{s_n^{p-1}}\left(\int_{B(x_n,r_nr_\star)} f^q(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \le ||f||_{L^q(\Omega)},$$

where the last inequality follows from the fact that $\frac{r_n^\beta}{s_n} \to 0$ (thanks to (3.1)). On the other hand, thanks to the weak Harnack inequality in [17, Theorem 3.13], there is a constant $C = C(N,\,p,\,q,\,s,\,\Lambda,\,r,\,r_\star)$ such that

$$\left(\fint_{B_{r_\star}} v_n^s\,\mathrm{d}x\right)^{\frac{1}{s}} \le C\left(\min_{B_1} v_n + ||f_n||_{L^q(B_{r_\star})}^{\frac{1}{p-1}}\right) \le C\left(1 + ||f||_{L^q(\Omega)}^{\frac{1}{p-1}}\right).$$

Hence,

$$||v_n||_{L^\infty(B_r)} \le C, \tag{3.3}$$

where C does not depend of n. Now, we claim that there exists also a uniform constant C such that

$$||\nabla v_n||_{L^p(B_r)} \le C. \tag{3.4}$$

Let $\eta \in C_c^\infty(B_r)$ be a cutoff function such that $\eta = 1$ on B_r and $0 \le \eta \le 1$ on $B_{r_\star}$. Let $\varphi = v_n\eta^p$. Using φ as a test function in (3.2), we obtain

$$-\int_{B_{r_\star}} w_n|\nabla v_n|^{p-2}\nabla v_n\cdot\nabla[v_n\eta^p] = \int_{B_{r_\star}} f_n\,\mathcal{B}_{\varepsilon_n}(s_nv_n)\,\eta^p\,v_n.$$

Hence,

$$\int_{B_{r_\star}} w_n|\nabla v_n|^p\eta^p = -\,p\int_{B_{r_\star}} w_n|\nabla v_n|^{p-2}[\nabla v_n\cdot\nabla\eta]\,v_n\eta^{p-1} - \int_{B_{r_\star}} f_n\,\mathcal{B}_{\varepsilon_n}(s_nv_n)\,\eta^p\,v_n. \tag{3.5}$$

By Hölder's inequality, we have

$$\left|\int_{B_{r_\star}} w_n|\nabla v_n|^{p-2}[\nabla v_n\cdot\nabla\eta]v_n\eta^{p-1}\right| \le \int_{B_{r_\star}} w_n^{\frac{p-1}{p}}|\nabla v_n|^{p-1}\eta^{p-1}w_n^{\frac{1}{p}}|\nabla\eta|v_n$$

$$\le \left(\int_{B_{r_\star}} w_n|\nabla v_n|^p\eta^p\right)^{1-\frac1p}\left(\int_{B_{r_\star}} w_n|\nabla\eta|^p v_n^p\right)^{\frac1p} \le C\Lambda^{\frac1p}\left(\int_{B_{r_\star}}|\nabla\eta|^p\right)^{\frac1p}\left(\int_{B_{r_\star}} w_n|\nabla v_n|^p\eta^p\right)^{1-\frac1p}$$

$$\le C\left(\int_{B_{r_\star}} w_n|\nabla v_n|^p\eta^p\right)^{1-\frac1p}.$$

Fix $\delta>0$. By Young's inequality, one has

$$\left|p\int_{B_{r_\star}} w_n|\nabla v_n|^{p-2}[\nabla v_n\cdot\nabla\eta]v_n\eta^{p-1}\right| \le \delta\int_{B_{r_\star}} w_n|\nabla v_n|^p\eta^p + C\,\delta^{1-p}.$$

Recalling (3.5), we get

$$\int_{B_{r_\star}} w_n|\nabla v_n|^p\eta^p \le \delta\int_{B_{r_\star}} w_n|\nabla v_n|^p\eta^p + C\,\delta^{1-p} + C||f_n||_{L^1(B_{r_\star})}.$$

Then,

$$\int_{B_{r_\star}} w_n|\nabla v_n|^p\eta^p \le \frac{C[\delta^{1-p}+||f_n||_{L^1(B_{r_\star})}]}{1-\delta}.$$

In particular, we have

$$\int_{B_r}|\nabla v_n|^p \le \frac{C[\delta^{1-p}+||f_n||_{L^1(B_{r_\star})}]}{\lambda(1-\delta)}.$$

Yet,

$$||f_n||_{L^1(B_{r_\star})} \le ||f_n||_{L^q(B_{r_\star})}|B_{r_\star}|^{1-\frac1q} \le C.$$

This proves our claim (3.4). Thanks to [10, Theorem 1.2] and [8, Theorem 1.3], we recall that we have the following estimate:

$$||\nabla v_n||_{L^\infty(B_r)} \le \begin{cases} C\left|\left|\int_0^{2r}\left(\frac{f_n(B(z,\rho))}{\rho^{N-1}}\right)^{\frac{p}{2(p-1)}}\frac{\mathrm{d}\rho}{\rho}\right|\right|_{L^\infty(B_{2r})}^{\frac2p} + Cr^{-\frac{2N}{p}}||\nabla v_n||_{L^{\frac p2}(B_{2r})}, & \text{if } \ p\ge 2,\\[2mm] C\left|\left|\int_0^{2r}\frac{f_n(B(z,\rho))}{\rho^N}\,\mathrm{d}\rho\right|\right|_{L^\infty(B_{2r})}^{\frac{1}{p-1}} + Cr^{-\frac{N}{2-p}}||\nabla v_n||_{L^{2-p}(B_{2r})}, & \text{if } \ p\in(1,2),\end{cases}$$

where C is a constant that depends only on N, p, λ, Λ, σ and $[w_n]_{C^{0,\sigma}}$. In fact, this constant C does not depend on n since

$$[w_n]_{C^{0,\alpha}(B_\gamma)} = r_n[w]_{C^{0,\alpha}(B(x_n,r_n\gamma))} \le [w]_{C^{0,\alpha}(\Omega)}. \tag{3.6}$$

Moreover, we have

$$\int_{B(z_0,\rho)} f_n = \frac{r_n^p}{s_n^{p-1}}\int_{B(z_0,\rho)} f(x_n+r_nz)\,\mathrm{d}z = \frac{r_n^{p-N}}{s_n^{p-1}}\int_{B(x_n+r_nz_0,r_n\rho)} f(x)\,\mathrm{d}x$$

$$\leq \frac{r_n^{p-N}}{s_n^{p-1}} |B_{r_n\rho}|^{\frac{q-1}{q}} \, ||f||_{L^q(\Omega)} \leq C \, \frac{r_n^{p-\frac{N}{q}}}{s_n^{p-1}} \, \rho^{N(1-\frac{1}{q})} \, ||f||_{L^q(\Omega)} \leq C \, \rho^{N(1-\frac{1}{q})} \, ||f||_{L^q(\Omega)}.$$

Since $q > N$, we infer thanks to (3.4) that

$$||\nabla v_n||_{L^\infty(B_r)} \leq C(N, \, p, \, q, \, \lambda, \, \Lambda, \, \sigma, \, [w]_{C^{0,\sigma}(\Omega)}, \, ||f||_{L^q(\Omega)}, \, r).$$

Thanks to [2, Theorem 1.1] ($p \geq 2$) and [8, Theorem 4.3 & Corollary 1.7] ($1 < p < 2$), we have the following estimate:

$$|\nabla v_n(x) - \nabla v_n(y)| \leq C|x-y|^\alpha, \quad \text{for all } \; x, \, y \in B_r, \tag{3.7}$$

where C depends on N, p, q, σ, λ, Λ, $||v_n||_{W^{1,p}(B_{r_\star})}$, $[w_n]_{C^{0,\sigma}(B_{r_\star})}$, $||f_n||_{L^q(B_{r_\star})}$. Recalling (3.3), (3.6) & (3.4), we infer that up to a subsequence, $v_n \to v$ in $C^{1,\gamma'}(B_r)$, for every $1 \leq r < \gamma$. Moreover, we clearly have

$$0 \leq v \leq 1 \; \text{ in } \; B_1.$$

From (3.1), we also have

$$v_n(0) = \frac{u_{\varepsilon_n}(x_n)}{s_n} \to 0, \;\; \text{when} \;\; n \to \infty.$$

Hence, $v(0) = 0$. In addition, we will show that this limit function v is $p-$harmonic, i.e. v solves the following $p-$Laplacian equation:

$$\nabla \cdot [|\nabla v|^{p-2} \, \nabla v] = 0 \qquad \text{in} \quad B_1.$$

Let $\varphi \in C_c^\infty(B_1)$. Since v_n solves (3.2), then we have

$$-\int_{B_1} w_n(z) |\nabla v_n(z)|^{p-2} \, \nabla v_n(z) \cdot \nabla \varphi(z) \, \mathrm{d}z = \int_{B_1} f_n(z) \, \mathcal{B}_{\varepsilon_n}(s_n v_n(z)) \, \varphi(z) \, \mathrm{d}z. \tag{3.8}$$

By (3.1), we recall that

$$||f_n \, \mathcal{B}_{\varepsilon_n}(s_n v_n)||_{L^q(B_1)} \leq \left(\frac{r_n^\beta}{s_n}\right)^{p-1} ||f||_{L^q(\Omega)} \to 0.$$

But, we also have $w_n \to w(x_0) > 0$ and $\nabla v_n \to \nabla v$ uniformly in $\overline{B_1}$. Hence, passing to the limit in (3.8) when $n \to \infty$, we get

$$-\int_{B_1} |\nabla v(z)|^{p-2} \, \nabla v(z) \cdot \nabla \varphi(z) \, \mathrm{d}z = 0.$$

Thus,

$$\Delta_p v = 0 \;\; \text{in} \;\; B_1.$$

Yet, thanks to the Harnack inequality in [13, Theorem 6.2], we know that there is a constant C such that

$$\sup_{B_1} v \leq C \inf_{B_1} v.$$

Since $v(0) = 0$, this yields that

$$v = 0 \;\; \text{on} \;\; B_1.$$

But, this is a contradiction as $||v_n||_{L^\infty(B_1)} = 1$ and then, by the uniform convergence of v_n to v, we must have $||v||_{L^\infty(B_1)} = 1$. $\square$

Corollary 3.2. *Let $\Omega' \subset\subset \Omega$ be an open set and K be a compact set in Ω'. Under the assumptions of Proposition 3.1, the solution u of Problem (0.1) is of class $C^{1,\min\left\{\frac{1-\frac{N}{q}}{p-1},1\right\}}$ on the free boundary $\mathcal{F} := \partial\{u>0\} \cap K$. More precisely, there exists a constant L such that for every $x \in \mathcal{F}$ and $0<r<d(K,\partial\Omega')$, one has*

$$||u||_{L^\infty(B(x,r))} \leq L\, r^\beta.$$

Proof. Thanks to Proposition 3.1, there exists a uniform constant C that does not depend on ε such that for every $x \in K$ and for all $0<r<d(K,\partial\Omega')$, we have

$$\sup_{B(x,r)} u_\varepsilon \leq C(r^\beta + u_\varepsilon(x)). \tag{3.9}$$

Recalling (2.9), we know that $u_\varepsilon \to u$ uniformly in Ω'. Passing to the limit when $\varepsilon \to 0$, we get that

$$\sup_{B(x,r)} u \leq C(r^\beta + u(x)). \tag{3.10}$$

Finally, assume $x \in \partial\{u>0\}$ (so, we have $u(x)=0$). From estimate (3.10), this implies that

$$\sup_{B(x,r)} u \leq C\, r^\beta. \quad \square$$

4. Non-degeneracy and some consequences on the free boundary

In this section, we will show that the solution u to Problem (1.1) leaves the free boundary with a growth rate of order $d(x,\partial\{u>0\})^{\frac{p}{p-1}}$. This growth estimate will allow us to prove some geometric properties of the free boundary.

Proposition 4.1. *Assume $w \in C^1(\Omega)$ and $f \geq f_\star > 0$. Then, there exists a uniform constant $L_\star > 0$ that depends on N, p, Λ, $||\nabla w||_\infty$, diam(Ω) and, $f_\star$ such that for every $x_0 \in \{u_\varepsilon > \varepsilon\}$ and $r>0$ with $B(x_0,r) \subset \Omega$, we have*

$$||u_\varepsilon||_{L^\infty(B(x_0,r))} \geq L_\star\, r^{\frac{p}{p-1}} + \varepsilon.$$

Proof. Fix $\varepsilon > 0$, $x_0 \in \{u_\varepsilon > \varepsilon\} \cap \Omega$ and, $0<r<d(x_0,\partial\Omega)$. Then, we define the barrier function as follows:

$$\Psi_\varepsilon(x) = L|x-x_0|^\beta + \varepsilon, \quad \text{for all } x \in B(x_0,r),$$

where both $\beta > 0$ and $L>0$ are to be chosen later so that Ψ_ε is a supersolution to (2.2) in $B(x_0,r)$. First, it is not difficult to check that

$$\nabla\cdot[w\,|\nabla\Psi_\varepsilon|^{p-2}\nabla\Psi_\varepsilon] = L^{p-1}\,\beta^{p-1}\nabla\cdot[w\,|x-x_0|^{(\beta-1)(p-1)-1}(x-x_0)]$$
$$= L^{p-1}\,\beta^{p-1}\,|x-x_0|^{(\beta-1)(p-1)-1}\Big(w[N+(\beta-1)(p-1)-1] + \nabla w\cdot[x-x_0]\Big).$$

Since $\Psi_\varepsilon \geq \varepsilon$, then we have

$$-\nabla\cdot[w\,|\nabla\Psi_\varepsilon|^{p-2}\nabla\Psi_\varepsilon] + f\,\mathcal{B}_\varepsilon(\Psi_\varepsilon)$$
$$\geq \left[f_\star - L^{p-1}\,\beta^{p-1}\,|x-x_0|^{(\beta-1)(p-1)-1}\Big(w[N+(\beta-1)(p-1)-1] + \nabla w\cdot[x-x_0]\Big)\right].$$

Yet,

$$L^{p-1}\,\beta^{p-1}\,|x-x_0|^{(\beta-1)(p-1)-1}\Big(w[N+(\beta-1)(p-1)-1]+\nabla w\cdot[x-x_0]\Big)$$
$$\leq C(N,p,\beta,\Lambda,||\nabla w||_\infty,\mathrm{diam}(\Omega))\,L^{p-1}\,r^{(\beta-1)(p-1)-1}$$

as soon as we choose $\beta \geq \frac{p}{p-1}$. If $\beta = \frac{p}{p-1}$, then there exists a small constant $L^\star > 0$ that depends on N, p, β, Λ, $||\nabla w||_\infty$, $\mathrm{diam}(\Omega)$ and, $f_\star$ such that

$$-\nabla\cdot[w\,|\nabla\Psi_\varepsilon|^{p-2}\nabla\Psi_\varepsilon]+f\,\mathcal{B}_\varepsilon(\Psi_\varepsilon)\geq f_\star - C\,L_\star^{p-1}\;\geq 0.$$

Consequently, Ψ_ε is a supersolution to (2.2) in $B(x_0,r)$. On the other side, u_ε is a solution to (2.2) in $B(x_0,r)$. Assume $u_\varepsilon \leq \Psi_\varepsilon$ on $\partial B(x_0,r)$. By the comparison principle (see [6, Theorem 4.9]), we infer that $u_\varepsilon \leq \Psi_\varepsilon$ in $B(x_0,r)$. Yet, $u_\varepsilon(x_0) > \varepsilon$ and $\Psi_\varepsilon(x_0)=\varepsilon$. This is a contradiction. Then, there exists a point $x_\varepsilon \in \partial B(x_0,r)$ such that $u(x_\varepsilon)\geq \Psi_\varepsilon(x_\varepsilon) = L_\star r^\beta+\varepsilon$. Hence, we get that

$$\sup_{B(x_0,r)} u_\varepsilon \geq L_\star\, r^{\frac{p}{p-1}}+\varepsilon. \quad \square$$

Under the assumptions of Proposition 4.1, we get the following non-degeneracy result.

Corollary 4.2. *There exists a constant $L_\star = L_\star(N,p,\Lambda,||\nabla w||_\infty,diam(\Omega),f_\star)>0$ such that for every $x_0\in\overline{\{u>0\}}\cap\Omega$ and $r>0$ with $B(x_0,r)\subset\Omega$, we have*

$$||u||_{L^\infty(B(x_0,r))}\geq L_\star r^{\frac{p}{p-1}}.$$

Proof. Thanks to the continuity of the solution u, it is sufficient to prove that such estimate is satisfied just at points within the positivity set $\{u>0\}\cap\Omega$. Fix $x_0\in\{u>0\}\cap\Omega$ and $0<r<d(x_0,\partial\Omega)$. Since $u(x_0)>0$ and $u_\varepsilon\to u$ locally uniformly in Ω, then $u_\varepsilon(x_0)>\varepsilon$ for all $\varepsilon>0$ small. From Proposition 4.1, this implies that

$$||u_\varepsilon||_{L^\infty(B(x_0,r))}\geq L_\star\, r^{\frac{p}{p-1}}+\varepsilon.$$

But, the constant $L_\star$ does not depend on ε and $u_\varepsilon\to u$ uniformly on $B(x_0,r)$, then letting $\varepsilon\to 0$ we get that

$$||u||_{L^\infty(B(x_0,r))}\geq L_\star\, r^{\frac{p}{p-1}}. \quad \square$$

Corollary 4.3. *Assume $p>2$. The solution u of Problem (0.1) is not of class $C^{1,\frac{1}{p-1}+\delta}$ on the free boundary $\partial\{u>0\}$, for every $\delta>0$.*

Remark 4.1. *The previous corollary shows that the regularity result obtained for the solution u of (1.1) is optimal whenever $f\in L^\infty(\Omega)$. However, it is not clear whether the $C^{1,\frac{1-\frac{N}{q}}{p-1}}$ regularity is optimal or not when $f\in L^q(\Omega)$ and $p>2$. While if $p\leq 2$, one may expect a higher regularity than $C^{1,1}$ along the free boundary.*

In the next result, we will show that the free boundary $\partial\{u>0\}$ is locally $\delta-$porous, that is for all $x\in\partial\{u>0\}$ and $r>0$ small, there exists a point $y\in B(x,r)$ such that $B(y,\delta r)\subset B(x,r)\backslash\partial\{u>0\}$. More precisely, we have

Proposition 4.4. *Assume $p \geq 2$, $w \in C^1(\Omega)$ and $f \in L^\infty(\Omega)$ with $f \geq f_\star > 0$. Let $\Omega' \subset \Omega$ be an open set and K be a compact set in Ω'. Then, there exists a uniform constant $\delta > 0$ such that for all $x \in \partial\{u > 0\} \cap K$ and $0 < r < d(K, \partial\Omega')$, we have*

$$\frac{\mathcal{L}^N(B(x,r) \cap \{u > 0\})}{\mathcal{L}^N(B(x,r))} \geq \delta.$$

Proof. Fix $x_0 \in \partial\{u > 0\} \cap K$ and $0 < r < d(K, \partial\Omega')$. Let $x^\star \in B(x_0, r)$ be such that $||u||_{L^\infty(B(x_0,r))} = u(x^\star)$. Thanks to Corollary 4.2, we have

$$u(x^\star) \geq L_\star r^\beta, \qquad \beta = \frac{p}{p-1}.$$

But, by Corollary 3.2, we also have

$$u(x^\star) \leq L|x^\star - x_0|^\beta.$$

Hence,

$$\left(\frac{L_\star}{L}\right)^{\frac{1}{\beta}} r \leq |x^\star - x_0| \leq r.$$

Let $1 \leq \lambda \leq \left(\frac{L_\star}{L}\right)^{-\frac{1}{\beta}}$ be such that $|x^\star - x_0| = \lambda\left(\frac{L_\star}{L}\right)^{\frac{1}{\beta}} r$. Let $z^\star$ be a point on the line segment $[x^\star, x_0]$ such that $|z^\star - x_0| = (\lambda - \frac{1}{2})\left(\frac{L_\star}{L}\right)^{\frac{1}{\beta}} r$. Now, we claim that there exists a uniform constant $\delta > 0$ (to be chosen later) such that

$$B(z^\star, \delta r) \subset B(x_0, r) \cap \{u > 0\}.$$

Assume $x \in B(z^\star, \delta r)$. So, one has

$$|x - x_0| \leq |x - z^\star| + |z^\star - x_0| < \left[\delta + (\lambda - \frac{1}{2})\left(\frac{L_\star}{L}\right)^{\frac{1}{\beta}}\right] r < r$$

as soon as

$$\lambda\left(\frac{L_\star}{L}\right)^{\frac{1}{\beta}} + \left(\delta - \frac{1}{2}\left(\frac{L_\star}{L}\right)^{\frac{1}{\beta}}\right) < 1.$$

Since $\lambda\left(\frac{L_\star}{L}\right)^{\frac{1}{\beta}} \leq 1$, then this is possible provided we choose $\delta < \frac{1}{2}\left(\frac{L_\star}{L}\right)^{\frac{1}{\beta}}$. Therefore, $B(z^\star, \delta r) \subset B(x_0, r)$. On the other hand, we also have $B(z^\star, \delta r) \subset \{u > 0\}$. Assume that this is not the case, then there exists a point $x \in B(z^\star, \delta r)$ such that $u(x) = 0$. Thanks again to Corollary 3.2, we must have

$$u(x^\star) \leq L|x^\star - x|^\beta.$$

Hence, we get

$$L_\star r^\beta \leq u(x^\star) \leq L|x^\star - x|^\beta \leq L\, 2^{\beta-1}(|x^\star - z^\star|^\beta + |z^\star - x|^\beta) \leq L\, 2^{\beta-1}(|x^\star - z^\star|^\beta + \delta^\beta r^\beta).$$

Yet,

$$|x^\star - z^\star| = |x^\star - x_0| - |z^\star - x_0| = \frac{1}{2}\left(\frac{L_\star}{L}\right)^{\frac{1}{\beta}} r.$$

Thus,

$$L_\star r^\beta \leq L\, 2^{\beta-1}\left(\frac{L_\star}{2^\beta L} + \delta^\beta\right) r^\beta.$$

Therefore,

$$\frac{L_\star}{2} \leq L\, 2^{\beta-1}\delta^{\beta}.$$

Assume $\delta < \left(\frac{L_\star}{2^\beta L}\right)^{\frac{1}{\beta}}$, then we infer that $B(z^\star, \delta r) \subset \{u > 0\}$. Thus, our claim is proved. Consequently, we get that

$$\frac{\mathcal{L}^N(B(x_0,r)\cap\{u>0\})}{\mathcal{L}^N(B(x_0,r))} \geq \frac{\mathcal{L}^N(B(z^\star,\delta r))}{\mathcal{L}^N(B(x_0,r))} = \delta^N. \quad \square$$

As an immediate consequence of the previous estimate, we obtain the following:

Corollary 4.5. *Under the assumptions of Proposition 4.4, the free boundary $\partial\{u > 0\}$ has zero Lebesgue measure.*

Proof. Thanks to the Lebesgue point theorem, we know that there exists a negligible set $\mathcal{N}$ such that for all $x \notin \mathcal{N}$, we have

$$\fint_{B(x,r)} \chi_{\{u>0\}} \to \chi_{\{u>0\}}(x) = \begin{cases} 1 & \text{if} \quad u(x) > 0, \\ 0 & \text{if} \quad u(x) = 0. \end{cases}$$

Let K be a compact set in Ω. Assume that $[\partial\{u > 0\} \cap K] \setminus \mathcal{N} \neq \emptyset$. Thus, by Proposition 4.4, there exists a uniform constant $\delta > 0$ such that for all $x \in [\partial\{u > 0\} \cap K] \setminus \mathcal{N}$ and $r > 0$ small enough, we have

$$0 < \delta \leq \frac{\mathcal{L}^N(B(x,r)\cap\{u>0\})}{\mathcal{L}^N(B(x,r))} \longrightarrow \chi_{\{u>0\}}(x) \quad \text{when} \quad r \to 0.$$

Hence, this yields that $u(x) > 0$. But, this is a contradiction as $x \in \partial\{u > 0\}$ and u is continuous. $\square$

Finally, we conclude this paper by the following

Remark 4.2. *Our work can be extended to the case of a general obstacle ψ (i.e., $u \geq \psi$), provided that $\psi \in C^{1,\lambda}(\Omega)$ and that some suitable assumptions are imposed on $\Delta_p\psi$. In this setting, one can show that the solution u of the obstacle problem belongs to $C^{1,\alpha'}_{loc}(\Omega)$ but with $\alpha' = \min\{\alpha, \lambda\}$, where α is the same exponent obtained for the solution u in the case $\psi = 0$.*

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