SERRIN PROBLEMS WITH VERTICAL BOUNDARY BEHAVIOR

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Abstract. We study Serrin-type overdetermined problems for a class of possibly degenerate elliptic operators under a vertical boundary condition forcing gradient blow-up. We identify the precise regime in which the problem admits a solution on a ball. In this regime, we prove rigidity: every solution domain is a ball, and the solution is uniquely determined by an explicit radial profile.

1. Introduction and main results

In a celebrated paper, Serrin [16] showed that the *overdetermined boundary value problem*

$$-\mathcal{L}u = 1 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \qquad \frac{\partial u}{\partial \nu} = c \text{ on } \partial\Omega \text{ for some constant } c > 0, \tag{1.1}$$

when

$$\mathcal{L}u = \Delta u \tag{1.2}$$

and $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary and interior unit normal ν, is solvable in $C^2(\overline{\Omega})$ precisely when Ω is a ball. Besides (1.2), the results of [16] actually apply to much more general elliptic operators. Among them, a relevant instance is provided by the *prescribed mean curvature* operator

$$\mathcal{L}u = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{1.3}$$

Besides their intrinsic interest, both problems have a strong physical motivation: (1.2) arises in the study of the torsion of solid bars and viscous incompressible fluids, and (1.3) describes capillary surfaces which, due to the Neumann condition in (1.1), meet the boundary of a cylindrical container at a constant (non-vanishing) *wetting angle*. Serrin's original proof is based on the *moving planes method*, and has inspired many subsequent symmetry results [9, 13, 15]. Shortly after [16], Weinberger [17] provided a different proof of the rigidity of (1.1). The core idea of [17] consists in showing the constancy of the *P-function*

$$P(x) = |\nabla u(x)|^2 + \frac{2}{n}u(x)$$

by combining a maximum principle argument with appropriate integral identities (cf. also [11]). These approaches face serious difficulties in the case of *degenerate* elliptic operators. Among the obstructions, (weak) solutions in $C^1(\overline{\Omega})$ may fail to be C^2. A prototypical instance is given by the *p-Laplace operator*

$$\mathcal{L}u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \qquad 1 < p < \infty, \tag{1.4}$$

which is singular when $1 < p < 2$ and degenerate when $p > 2$. The first outcomes in the degenerate setting are available only under appropriate growth conditions on $\mathcal{L}$ [8], and may distinguish between singular [4] and degenerate [1] regime. A significant improvement has been achieved in [6, 7] for the fairly general class of operators

$$\mathcal{L}u = \operatorname{div}\left(A(|\nabla u|)\nabla u\right), \tag{1.5}$$

where $A \in C^2(0,\infty)$ satisfies the (possibly degenerate) ellipticity assumption

$$\lim_{t\to 0} B(t) = 0, \qquad \dot{B}(t) > 0 \text{ on } (0,\infty), \qquad B(t) := tA(t). \tag{A1}$$

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Clearly, (1.2), (1.3) and (1.4) fall within this class, which nevertheless allows for operators with more general growth and degeneracy conditions, including for instance the class

$$\mathcal{L}u = \operatorname{div}\left(\frac{|\nabla u|^{p-2}}{(1+|\nabla u|^2)^{q/2}}\nabla u\right), \qquad 1 < p < \infty, \qquad 0 \leqslant q \leqslant p-1. \tag{1.6}$$

In this paper, we investigate the singular case in which the graph meets the boundary cylinder with a vertical contact angle, forcing $|\nabla u|$ to blow up as $\partial\Omega$ is approached. More precisely, we consider the problem

$$\begin{cases} -\operatorname{div}\left(A(|\nabla u|)\nabla u\right) = 1 & \text{in } \Omega, & \text{(P1)} \\ u = 0 & \text{on } \partial\Omega, & \text{(P2)} \\ \lim\limits_{x\to x_0} \dfrac{\langle \nabla u(x), \nu(x_0)\rangle}{\sqrt{1+|\nabla u(x)|^2}} = 1 & \text{for every } x_0 \in \partial\Omega. & \text{(P3)} \end{cases}$$

In the following, $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with C^1 boundary, ν is the interior unit normal to $\partial\Omega$ and $A \in C^2(0,\infty)$ satisfies (A1). Heuristically, the Neumann condition (P3) arises by singularizing the equation

$$\frac{\langle \nabla u, \nu\rangle}{\sqrt{1+|\nabla u|^2}} = \frac{c}{\sqrt{1+c^2}} \qquad \text{on } \partial\Omega,$$

which is satisfied by solutions to (1.1). Indeed, (P3) forces the uniform vertical displacement

$$\lim_{x\to x_0} \frac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}} = \nu(x_0), \qquad \lim_{x\to x_0} |\nabla u(x)| = +\infty \qquad \text{uniformly for } x_0 \in \partial\Omega. \tag{1.7}$$

In the prototypical case of the prescribed mean curvature operator (1.3), (P3) describes the limiting case of *perfectly wetting* capillary surfaces (cf. [10]). Because of (1.7), the lack of C^1-boundary regularity of u is forced by the very structure of the problem. Accordingly, solutions to (P) are meant as follows.

Definition 1.1. *$u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ is a weak solution to (P) if it satisfies (P2) and (P3), and moreover*

$$\int_\Omega A(|\nabla u|)\langle \nabla u, \nabla\varphi\rangle\, dx = \int_\Omega \varphi\, dx, \qquad \textit{for every } \varphi \in C_c^\infty(\Omega). \tag{1.8}$$

When (A1) holds, [7, Proposition] and [6, Theorem 1.1] ensure that (1.1) is solvable precisely when Ω is a ball, but regardless of further constraints on A. On the other hand, the existence of a weak solution to (P) forces *a priori* constraints not only on the domain, but also on the differential operator. A first heuristic indication of this phenomenon is that the verticality condition (P3) rules out any operator for which standard elliptic regularity yields C^1-boundary regularity for the Dirichlet problem (e.g. the Laplacian), since this would be incompatible with the required blow-up of $|\nabla u|$. More precisely, the following holds.

Theorem 1.2. *Assume that $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ is a weak solution to (P). Assume that (A1) holds. Then*

$$\lambda := \lim_{t\to\infty} B(t) = \frac{|\Omega|}{\mathcal{H}^{n-1}(\partial\Omega)}. \tag{1.9}$$

Moreover, Ω is a self-Cheeger set. Namely

$$\frac{\mathcal{H}^{n-1}(\partial\Omega)}{|\Omega|} = \min\left\{\frac{P(F)}{|F|} \;:\; F \subseteq \Omega \textit{ has finite perimeter}, |F| > 0\right\}.$$

For instance, when $\mathcal{L}$ is as in (1.6), (1.9) implies that (P) is solvable only if $q = p-1$, thus ruling out (1.2) and (1.4). In order to understand the correct regime for the solvability of (P), one may look at the boundary behavior of the relevant P-function associated with (1.5). According to [6, 7], we set

$$P(x) := \Phi\big(|\nabla u(x)|\big) + \frac{2}{n}u(x), \qquad x \in \Omega, \tag{1.10}$$

where

$$\Phi(t) := 2\int_0^t s\,\dot{B}(s)\, ds, \qquad t \in (0,\infty).$$

In the finite-slope setting of [6, 7], the boundary continuity of P is crucial to carry out the appropriate maximum-principle argument. Accordingly, we observe that, due to (P2), (P3) and (1.7), P extends continuously up to $\partial\Omega$ to the constant value

$$\Phi_\infty := \lim_{t\to\infty}\Phi(t) \in (0,\infty)$$

provided that

$$\int_0^\infty t\,\dot{B}(t)\,dt < \infty. \tag{A2}$$

As we show in the following result, this heuristic consideration indeed identifies the correct class of operators for which (P) is solvable in the model case of a spherical domain.

Theorem 1.3. *Problem* (P) *has weak solution u on some ball B_R if and only if* (A2) *holds. In such cases,* (1.9) *holds, $R = n\lambda$, and u is uniquely determined by*

$$u(x) = \int_{|x|}^{n\lambda} B^{-1}\left(\frac{t}{n}\right) dt. \tag{1.11}$$

Once Theorem 1.3 is established, the main result of this paper provides the rigidity of (P), which is indeed solvable *precisely* when Ω is a ball.

Theorem 1.4. *Assume that there is a weak solution $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ to* (P). *Assume that* (A1) *and* (A2) *hold. Then, up to translations, Ω is the ball $B_{n\lambda}$, and u is the radially symmetric function defined by* (1.11).

We conclude with a brief outline of the proofs. The essential difference from the finite-slope regime lies in the loss of boundary regularity. In the setting of [6, 7], the solution is C^1 up to $\partial\Omega$, so that the Neumann datum is attained pointwise. Accordingly, a combination of the maximum principle and of an integral identity à la Pohožaev-Pucci-Serrin [5, 12, 14] can be performed directly on Ω, yielding the constancy of the P-function. By contrast, (P3) forces $|\nabla u|$ to diverge as $\partial\Omega$ is approached, so that neither ∇u nor $A(|\nabla u|)\nabla u$ can be handled directly at the boundary. To overcome this obstruction, we replace $\partial\Omega$ with a family of regular interior level sets. More precisely, for sufficiently small $\delta > 0$, we construct domains $E_\delta \Subset \Omega$ whose boundaries Σ_δ lie in the level set $\{u = \delta\}$. The vertical boundary behavior implies that $\nabla u \neq 0$ in a collar neighborhood of $\partial\Omega$, and hence that the hypersurfaces Σ_δ are of class C^1. Crucially, although u is not of class C^1 up to $\partial\Omega$, (P3) still provides enough control to ensure that

$$|E_\delta| \to |\Omega|, \qquad \mathcal{H}^{n-1}(\Sigma_\delta) \longrightarrow \mathcal{H}^{n-1}(\partial\Omega), \qquad \text{as } \delta \to 0.$$

This convergence plays the role of the boundary regularity available in the finite-slope regime. The proof of Theorem 1.2 follows by combining the coarea formula with a careful choice of test functions in (1.8), in such a way to exhaust Ω. When Ω is a ball, a suitable weak comparison principle yields radial symmetry of the solution, and reduces (P1) to an ODE. This gives both the explicit radial profile and the sharp solvability condition (A2), proving Theorem 1.3. Finally, the proof of Theorem 1.4 follows the P-function strategy, but once again only after localization to the interior sets E_δ. By (A2), P admits finite constant trace Φ_∞ at $\partial\Omega$. Applying a maximum principle argument on E_δ and then letting $\delta \to 0$, we obtain the upper bound $P \leqslant \Phi_\infty$. To prove the equality, we combine (1.8) with the Pohožaev-Pucci-Serrin identity to appropriate vertical shifts of u over E_δ. The aforementioned convergence properties of E_δ and Σ_δ provide enough control on the remainder terms to pass to the limit and conclude that $P \equiv \Phi_\infty$. Once the constancy of P is established, the very same rigidity argument as in [6] forces the level sets of u to be concentric spheres. Consequently, Ω is a ball, and u coincides with the radial solution described by (1.11).

2. Proofs

Preliminary results. In the following, $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ is a fixed weak solution to (P). We denote by d the (inner) signed distance from $\partial\Omega$. Since $\partial\Omega$ is of class C^1, d is Lipschitz continuous. Since $\partial\Omega$ is compact, there exists $\bar{\varepsilon} > 0$ such that $x_0 + s\nu(x_0) \in \Omega$ for any $s \in (0, \bar{\varepsilon})$. We define $\Psi : \partial\Omega \times (0, \bar{\varepsilon}) \longrightarrow \Omega$ by

$$\Psi(x_0, s) = x_0 + s\nu(x_0).$$

Although Ψ may not be injective, still $\Psi(\partial\Omega \times (0, \bar{\varepsilon})) = \{0 < d < \bar{\varepsilon}\}$. By (1.7), we choose $\bar{\varepsilon}$ small so that

$$\frac{\partial}{\partial s}\Big(u(\Psi(x_0, s))\Big) = \langle \nabla u(x_0 + s\nu(x_0)), \nu(x_0)\rangle > 1 \qquad \text{on } \partial\Omega \times (0, \bar{\varepsilon}). \tag{2.1}$$

Therefore, as $u \equiv 0$ on $\partial\Omega$, and if $x = \Psi(x_0, s)$ for some $(x_0, s) \in \partial\Omega \times (0, \bar{\varepsilon})$, (2.1) yields

$$u(x) = u(x_0 + s\nu(x_0)) - u(x_0) \geqslant s \geqslant d(x_0 + s\nu(x_0)) = d(x) \qquad \text{on } \{0 < d < \bar{\varepsilon}\}. \tag{2.2}$$

Fix $\bar{\delta} < \bar{\varepsilon}$. For any $0 < \delta < \bar{\delta}$, set

$$E_\delta = \{u > \delta\} \cup \{d > \bar{\delta}\}. \tag{2.3}$$

First, $E_\delta \Subset \Omega$ is open and bounded. Moreover, (2.2) grants that $\{d \geqslant \bar{\delta}\} \subseteq E_\delta$, whence $\partial E_\delta \subseteq \{0 < d < \bar{\delta}\}$. Therefore, since (2.1) implies that $\nabla u \neq 0$ on $\{0 < d < \bar{\delta}\}$, then

$$\Sigma_\delta := \partial E_\delta = \{u = \delta\} \cap \{0 < d < \bar{\delta}\},$$

and Σ_δ is a hypersurface of class C^1. In addition, (2.2) implies that $\bigcup_{0<\delta<\bar{\delta}} E_\delta = \Omega$, whence

$$\lim_{\delta \to 0^+} |E_\delta| = |\Omega|. \tag{2.4}$$

A crucial consequence for the subsequent analysis is the convergence of the measures of Σ_δ.

Proposition 2.1. *Let $S = \{(x, u(x)) \,:\, x \in \Omega\}$. Then $\overline{S}$ is a hypersurface of class C^1. Moreover,*

$$\lim_{\delta \to 0^+} \mathcal{H}^{n-1}(\Sigma_\delta) = \mathcal{H}^{n-1}(\partial\Omega). \tag{2.5}$$

Proof. The regularity of $\overline{S}$ may be proved locally near $\partial\Omega$. Fix $\hat{x} \in \partial\Omega$. As Definition 1.1 is invariant under isometries, we assume that $\hat{x} = 0$ and $\nu(\hat{x}) = -\frac{\partial}{\partial x_1}$. We denote points in $\mathbb{R}^n$ by (x_1, x'). Since $\partial\Omega$ is of class C^1, there exist a neighborhood U' of $0 \in \mathbb{R}^{n-1}$, a neighborhood U_1 of $0 \in \mathbb{R}$ and $\omega \in C^1(\overline{U'})$ such that

$$\Omega \cap (U_1 \times U') = \{(x_1, x') \in \mathbb{R}^n \,:\, x_1 < \omega(x')\}, \qquad \partial\Omega \cap (U_1 \times U') = \{(x_1, x') \in \mathbb{R}^n \,:\, x_1 = \omega(x')\}.$$

In particular, denoting by ∇' the gradient with respect to x',

$$\nu\big((\omega(x'), x')\big) = -\frac{1}{\sqrt{1 + |\nabla'\omega(x')|^2}}\frac{\partial}{\partial x_1} + \sum_{i=2}^{n} \frac{\partial_i \omega(x')}{\sqrt{1 + |\nabla'\omega(x')|^2}}\frac{\partial}{\partial x_i}.$$

Therefore, by (1.7) and for any $\tilde{x} \in \partial\Omega \cap (U_1 \times U')$,

$$\lim_{x \to \tilde{x}} \frac{\partial_1 u}{|\nabla u|} = -\frac{1}{\sqrt{1 + |\nabla'\omega(\tilde{x}')|^2}}, \qquad \lim_{x \to \tilde{x}} \frac{\partial_i u}{|\nabla u|} = \frac{\partial_i \omega(\tilde{x}')}{\sqrt{1 + |\nabla'\omega(\tilde{x}')|^2}} \qquad \text{for } i = 2, \dots, n. \tag{2.6}$$

Up to smaller neighborhoods, (P3) implies that

$$\partial_1 u < 0 \text{ on } (U_1 \times U') \cap \Omega, \qquad \lim_{x \to x_0} \partial_1 u = -\infty \text{ for any } x_0 \in (U_1 \times U') \cap \partial\Omega. \tag{2.7}$$

In particular, $u > 0$ on $(U_1 \times U') \cap \Omega$. Therefore, up to smaller neighborhoods, the implicit function theorem yields the existence of $\gamma > 0$ and $\psi \in C^1(U' \times (0, \gamma))$ such that

$$u\left(\psi(x', x_{n+1}), x'\right) = x_{n+1} \qquad \text{on } U' \times (0, \gamma). \tag{2.8}$$

Therefore, $S \cap (U_1 \times U' \times (0, \gamma))$ is the x_1-graph of ψ. Notice that ψ can be extended with continuity up to $U' \times \{0\}$ by setting $\psi(x', 0) = \omega(x')$. Indeed, assume that $((x_k)', (x_k)_{n+1}) \to (x', 0)$ as $k \to \infty$. Chose an arbitrary subsequence, without relabeling it. Notice that $\big\{\psi\big((x_k)', (x_k)_{n+1}\big)\big\}_k$ is bounded: up to a further subsequence, it converges to some $\tilde{\psi} \in \mathbb{R}$. Passing to the limit in (2.8), we infer that $u(\tilde{\psi}, x') = 0$. Since $u > 0$ locally by (2.2), the unique possibility is that $\tilde{\psi} = \omega(x')$. Therefore ψ is continuous up to $U' \times \{0\}$. We show that ψ is of class C^1 up to $U' \times \{0\}$. Differentiating (2.8) at $(x', x_{n+1}) \in U' \times (0, \gamma)$ yields

$$\partial_i \psi(x', x_{n+1}) = -\frac{\partial_i u\left(\psi(x', x_{n+1}), x'\right)}{\partial_1 u\left(\psi(x', x_{n+1}), x'\right)} \quad i = 2, \dots, n, \quad \partial_{n+1}\psi(x', x_{n+1}) = \frac{1}{\partial_1 u\left(\psi(x', x_{n+1}), x'\right)} \tag{2.9}$$

Therefore, for $i = 2, \dots, n$, for any fixed $\tilde{x}' \in U'$ and by the continuity of ψ,

$$\lim_{(x', x_{n+1}) \to (\tilde{x}', 0)} \partial_i \psi(x', x_{n+1}) \overset{(2.9)}{=} \lim_{(x', x_{n+1}) \to (\tilde{x}', 0)} -\frac{\partial_i u\left(\psi(x', x_{n+1}), x'\right)}{\partial_1 u\left(\psi(x', x_{n+1}), x'\right)} \overset{(2.6)}{=} \partial_i \omega(\tilde{x}') = \partial_i \psi(\tilde{x}', 0). \tag{2.10}$$

Therefore, the tangential (with respect to $U' \times \{0\}$) derivatives of ψ are continuous up to $U' \times \{0\}$. Moreover,

$$\lim_{(x', x_{n+1}) \to (\tilde{x}', 0)} \partial_{n+1}\psi(x', x_{n+1}) \overset{(2.9)}{=} \lim_{(x', x_{n+1}) \to (\tilde{x}', 0)} \frac{1}{\partial_1 u\left(\psi(x', x_{n+1}), x'\right)} \overset{(2.7)}{=} 0. \tag{2.11}$$

By Lagrange mean value theorem, there exists $\psi(\tilde{x}', x_{n+1}) < \xi(\tilde{x}', x_{n+1}) < \psi(\tilde{x}', 0) = \omega(x')$ such that

$$x_{n+1} \overset{(2.8)}{=} u\big(\psi(\tilde{x}', x_{n+1}), \tilde{x}'\big) - u\big(\psi(\tilde{x}', 0), \tilde{x}'\big) = \partial_1 u\big(\xi(\tilde{x}', x_{n+1}), \tilde{x}'\big)\big(\psi(\tilde{x}', x_{n+1}) - \psi(\tilde{x}', 0)\big). \tag{2.12}$$

Therefore, noticing that $\xi(\tilde{x}', x_{n+1}) \to \omega(x')$ as $x_{n+1} \to 0^+$,

$$\partial^+_{n+1}\psi(\tilde{x}', 0) = \lim_{x_{n+1} \to 0^+} \frac{\psi(\tilde{x}, x_{n+1}) - \psi(\tilde{x}, 0)}{x_{n+1}} \overset{(2.12)}{=} \lim_{x_{n+1} \to 0^+} \frac{1}{\partial_1 u\big(\xi(\tilde{x}', x_{n+1}), \tilde{x}'\big)} \overset{(2.7)}{=} 0. \tag{2.13}$$

By (2.11) and (2.13), the normal derivative of v is continuous up to $U' \times \{0\}$. To prove (2.5), notice that

$$\Sigma_\delta \cap (U_1 \times U') = \{(\psi(x',\delta), x') \,:\, x' \in U'\}, \qquad \partial\Omega \cap (U_1 \times U') = \{(\omega(x'), x') \,:\, x' \in U'\}.$$

Therefore, for an arbitrary $\varphi \in C_c^\infty(U_1 \times U')$, the area formula yields

$$\begin{aligned} \lim_{\delta\to 0} \int_{\Sigma_\delta \cap (U_1\times U')} \varphi \, d\mathcal{H}^{n-1} &= \lim_{\delta \to 0} \int_{U'} \varphi(\psi(x',\delta),x')\sqrt{1+|\nabla'\psi(x',\delta)|^2}\, dx' \\ &\overset{(2.10)}{=} \int_{U'} \varphi(\omega(x'),x')\sqrt{1+|\nabla'\omega(x')|^2}\, dx' \\ &= \int_{\partial\Omega\cap(U_1\times U')} \varphi\, d\mathcal{H}^{n-1}. \end{aligned} \tag{2.14}$$

The choice of a finite partition of unity in a neighborhood of $\partial\Omega$ and (2.14) prove (2.5). □

We conclude this section with the following (weak) comparison principle.

Proposition 2.2. *Let $v, w \in C^1(\Omega) \cap C^0(\overline{\Omega})$ satisfy (1.8). If $v \leqslant w$ on $\partial\Omega$, then $v \leqslant w$ on $\overline{\Omega}$.*

Proof. Choose any family $(\Omega_\varepsilon)_\varepsilon$ of bounded domains, each of them compactly contained in Ω and with C^1 boundary, with the property that any converging sequence $(x_\varepsilon)_\varepsilon$ with $x_\varepsilon \in \partial\Omega_\varepsilon$ converges to a point in $\partial\Omega$. Set $M_\varepsilon = \max_{\partial\Omega_\varepsilon}(v-w)$ and $w_\varepsilon = w + M_\varepsilon$. Then w_ε satisfies (1.8). By construction, $v, w_\varepsilon \in C^1(\overline{\Omega}_\varepsilon)$ and $v \leqslant w_\varepsilon$ on $\partial\Omega_\varepsilon$. Then, by [7, Lemma 3.7], $v \leqslant w + M_\varepsilon$ on $\overline{\Omega}_\varepsilon$. Since $v \leqslant w$ on $\partial\Omega$ and both are continuous over $\overline{\Omega}$, the choice of $(\Omega_\varepsilon)_\varepsilon$ yields $\lim\limits_{\varepsilon\to 0} M_\varepsilon \leqslant 0$. Therefore, $v \leqslant w$ on $\overline{\Omega}$. □

Proof of Theorem 1.2. By (A1), $B \geqslant 0$, and it admits a limit $\lambda \in [0,\infty]$ as $t \to \infty$. Since $\partial\Omega$ is C^1, there is an open neighborhood $U \subseteq \overline{\Omega} \cap \{0 \leqslant d \leqslant \bar{\varepsilon}\}$ of $\partial\Omega$, and $\varrho \in C^1(U)$, such that $\partial\Omega = \{\varrho = 0\}$, $\varrho > 0$ on $U \cap \Omega$ and $\nabla\varrho \neq 0$ on U. Up to a smaller neighborhood, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $\varrho < \varepsilon_1$ on U, $|\nabla\varrho| > \varepsilon_2$ on U and $\varrho = \varepsilon_1$ on $\partial U \cap \Omega$. In particular, $\frac{\nabla\varrho}{|\nabla\varrho|} = \nu$ on $\partial\Omega$. By (1.7), we assume that $|\nabla u| \neq 0$ on U. Setting $\Gamma_\varepsilon = \{\varrho = \varepsilon\}$, one argues *verbatim* as in the proof of Proposition 2.1 and as in (2.2) to infer

$$\lim_{\varepsilon\to 0} \mathcal{H}^{n-1}(\Gamma_\varepsilon) = \mathcal{H}^{n-1}(\partial\Omega), \qquad \varepsilon_2 d(x) \leqslant \varepsilon \text{ on } \Gamma_\varepsilon. \tag{2.15}$$

Fix $\eta \in C^\infty\big([0,\infty)\big)$ such that $\eta \equiv 0$ on $\big[0,\frac{1}{3}\big]$, $\eta \equiv 1$ on $\big[\frac{2}{3},\infty\big)$ and $\dot\eta \geqslant 0$. For $0 < \varepsilon < \varepsilon_1$, define

$$\varphi_\varepsilon(x) = \begin{cases} \eta\left(\dfrac{\varrho(x)}{\varepsilon}\right) & \text{if } x \in U, \\ 1 & \text{if } x \in \Omega \setminus U. \end{cases}$$

Then $\varphi_\varepsilon \in C_c^1(\Omega)$. Therefore, (1.8) and the definition of φ_ε imply that

$$\int_\Omega \varphi_\varepsilon \, dx = \frac{1}{\varepsilon}\int_{\{0<\varrho<\varepsilon\}} B(|\nabla u|)\left\langle \frac{\nabla u}{|\nabla u|}, \frac{\nabla\varrho}{|\nabla\varrho|}\right\rangle \eta'\left(\frac{\varrho}{\varepsilon}\right)|\nabla\varrho|\, dx \tag{2.16}$$

On the one hand, the left hand side of (2.16) converges to $|\Omega|$ as $\varepsilon \to 0$. On the other hand, by the coarea formula and a change of variables, the right hand side of (2.16) reads as

$$\int_0^1 \eta'(\tau) \int_{\Gamma_{\tau\varepsilon}} B(|\nabla u|)\left\langle \frac{\nabla u}{|\nabla u|}, \frac{\nabla\varrho}{|\nabla\varrho|}\right\rangle d\mathcal{H}^{n-1}\, d\tau. \tag{2.17}$$

We claim that λ is finite. Assume not by contradiction. By (2.16), (2.17) is bounded above by $|\Omega|$. Therefore, by (1.7) and (2.15), and since $\frac{\nabla\varrho}{|\nabla\varrho|} = \nu$ on $\partial\Omega$, for any $M > 0$ there is $\varepsilon_M > 0$ such that

$$|\Omega| \geqslant \int_0^1 \eta'(\tau)\int_{\Gamma_{\tau\varepsilon}} B(|\nabla u|)\left\langle \frac{\nabla u}{|\nabla u|}, \frac{\nabla\varrho}{|\nabla\varrho|}\right\rangle d\mathcal{H}^{n-1}\, d\tau \geqslant M\int_0^1 \eta'(\tau)\, d\tau = M$$

for any $0 < \varepsilon < \varepsilon_M$. Since $M > 0$ is arbitrary, a contradiction follows. Hence λ is finite. Therefore, by the dominated convergence theorem, (1.7) and (2.15) imply that (2.17) converges to $\lambda\mathcal{H}^{n-1}(\partial\Omega)$ as $\varepsilon \to 0$. This concludes the proof of (1.9). To conclude, notice that (1.9) implies that $x \mapsto A(|\nabla u(x)|)\nabla u(x)$ is a *bounded divergence-measure vector field* on Ω, e.g. in the sense of [2]. Therefore, (1.9) and the weak divergence theorem proved in [3, Theorem 2.9] ensures that Ω is self-Cheeger.

Proof of Theorem 1.3. Assume that (P) has a weak solution u over a ball B_R. Then u is radially symmetric. Otherwise, there would exist a rotation T such that $u \circ T \neq u$ over B_R. Since Definition 1.1 is invariant under isometries, $u \circ T$ solves (P) over $T^{-1}(B_R) = B_R$. But then, by Proposition 2.2, $u = u \circ T$, a contradiction. Then $u(x) = g(|x|)$ for some profile g. By the proof of [7, Proposition 2.2], $\dot{g} < 0$ on $(0, R)$, and

$$B(-\dot{g}(r)) = \frac{r}{n} \qquad \text{on } (0, R).$$

Since $B(0, \infty) = (0, \lambda)$ by (A1) and (1.9), then $R \leqslant n\lambda$. Moreover, B is invertible by (A1), whence

$$\dot{g}(r) = -B^{-1}\left(\frac{r}{n}\right) \qquad \text{on } (0, R).$$

By (P3), $|\nabla u(x)| = \left|B^{-1}(|x|/n)\right|$ blows up at ∂B_R, hence $R = n\lambda$. Therefore

$$g(r) = g(0) - \int_0^r B^{-1}\left(\frac{\tau}{n}\right) d\tau \qquad \text{on } (0, n\lambda). \tag{2.18}$$

By (P2), $\lim_{r \to n\lambda} g(r) = 0$. Therefore, by (2.18),

$$g(0) = \lim_{r \to n\lambda} \int_0^r B^{-1}\left(\frac{\tau}{n}\right) d\tau = \lim_{r \to n\lambda} n \int_0^{B^{-1}\left(\frac{r}{n}\right)} t\,\dot{B}(t)\,dt = n \int_0^\infty t\,\dot{B}(t)\,dt,$$

whence (A2) holds. Conversely, if (A2) holds, the above computation implies that u as in (1.11) solves (P).

Proof of Theorem 1.4. By (A2), (P2) and (P3), the P-function P defined in (1.10) extend continuously to $\partial\Omega$ by setting $P(x_0) = \Phi_\infty$ for any $x_0 \in \partial\Omega$. We claim that

$$P \equiv \Phi_\infty \qquad \text{on } \overline{\Omega}. \tag{2.19}$$

To this aim, fix $\delta \in (0, \bar{\delta})$, and let E_δ be as in (2.3). Since $u \in C^1(\Omega)$ and $E_\delta \Subset \Omega$, then $\nabla u \in C^0(\overline{E}_\delta, \mathbb{R}^n)$. Therefore, although P is not necessarily constant over $\partial E_\delta = \Sigma_\delta$, the maximum principle argument as in the proof of [7, Lemma 3.2] ensures that

$$P(x) \leqslant \max_{\Sigma_\delta} P = \max_{\Sigma_\delta} \Phi(|\nabla u|) + \frac{2\delta}{n} \qquad \text{for every } x \in E_\delta. \tag{2.20}$$

Let $x_\delta \in \Sigma_\delta$ be such that the maximum in the right hand side of (2.20) is assumed at x_δ. By (2.2), $d(x_\delta) \leqslant u(x_\delta) = \delta$, so that $\lim_{\delta \to 0} d(x_\delta) = 0$. In particular, letting $\delta \to 0$ in (2.20),

$$P(x) \leqslant \lim_{\delta \to 0} \Phi(|\nabla u(x_\delta)|) \overset{(1.7)}{=} \Phi_\infty \qquad \text{for every } x \in \Omega.$$

The above inequality implies (2.19) provided that

$$\int_\Omega P(x)\,dx = \Phi_\infty |\Omega|. \tag{2.21}$$

Fix $0 < \delta < \bar{\delta}$. Define $u_\delta := u - \delta$ on $\overline{E}_\delta$. Then $u_\delta \in C^1(\overline{E}_\delta)$ and $u_\delta \equiv 0$ on ∂E_δ. Moreover, since $E_\delta \Subset \Omega$, u_δ satisfies (1.8) on E_δ. A standard approximation argument grants that u_δ satisfies (1.8) for every $\varphi \in C_0^1(\overline{E}_\delta)$. In particular, since $u_\delta \in C_0^1(\overline{E}_\delta)$ and $\nabla u_\delta = \nabla u$,

$$\int_{E_\delta} B(|\nabla u|)|\nabla u|\,dx = \int_{E_\delta} u_\delta\,dx. \tag{2.22}$$

On the other hand, notice that $A(|\nabla u|)\nabla u = \nabla \mathcal{L}(\nabla u)$, where

$$\mathcal{L}(p) := \int_0^{|p|} B(\tau)\,d\tau$$

is strictly convex in view of (A1). Therefore, applying [5, Theorem 2] with $a(x) \equiv 0$ and $h(x) \equiv x$ yields

$$n \int_{E_\delta} \mathcal{L}(\nabla u)\,dx - \int_{E_\delta} B(|\nabla u|)|\nabla u|\,dx + \int_{E_\delta} \langle x, \nabla u \rangle\,dx = \int_{\Sigma_\delta} \left(\mathcal{L}(\nabla u) - B(|\nabla u|)|\nabla u|\right) \left\langle x, \nu_\delta^e \right\rangle d\mathcal{H}^{n-1},$$

where ν_δ^e is the outer unit normal to E_δ. Noticing that

$$\Phi(|\nabla u|) = 2|\nabla u| B(|\nabla u|) - 2\mathcal{L}(\nabla u),$$

and since $u_\delta = 0$ on Σ_δ, the divergence theorem yields

$$-\frac{n}{2} \int_{E_\delta} \Phi(|\nabla u|)\,dx + (n-1) \int_{E_\delta} B(|\nabla u|)|\nabla u|\,dx - n \int_{E_\delta} u_\delta\,dx = -\frac{1}{2} \int_{\Sigma_\delta} \Phi\left(|\nabla u|\right) \left\langle x, \nu_\delta^e \right\rangle d\mathcal{H}^{n-1}.$$

Recalling the definition of P, exploiting (2.22) and multiplying by $-\frac{2}{n}$,

$$\int_{E_\delta} P(x)\,dx - \frac{2\delta}{n}|E_\delta| = \frac{1}{n}\int_{\Sigma_\delta} \Phi\left(|\nabla u|\right)\left\langle x, \nu^e_\delta\right\rangle\,d\mathcal{H}^{n-1}. \tag{2.23}$$

Since $P \in C^0(\overline{\Omega})$, (2.4) and the dominated convergence theorem imply that the left hand side of (2.23) converges to the left hand side of (2.21) as $\delta \to 0$. On the other hand, the divergence theorem grants that

$$\frac{1}{n}\int_{\Sigma_\delta} \Phi\left(|\nabla u|\right)\left\langle x, \nu^e_\delta\right\rangle\,d\mathcal{H}^{n-1} = \frac{1}{n}\int_{\Sigma_\delta} \left(\Phi\left(|\nabla u|\right) - \Phi_\infty\right)\left\langle x, \nu^e_\delta\right\rangle\,d\mathcal{H}^{n-1} + \Phi_\infty|E_\delta|. \tag{2.24}$$

Set $M = \max\{|x| \,:\, x \in \overline{\Omega}\}$. Then

$$\frac{1}{n}\left|\int_{\Sigma_\delta} \left(\Phi\left(|\nabla u|\right) - \Phi_\infty\right)\left\langle x, \nu^e_\delta\right\rangle\,d\mathcal{H}^{n-1}\right| \leqslant M\,\mathcal{H}^{n-1}(\Sigma_\delta)\,\max_{\Sigma_\delta}|\Phi - \Phi_\infty|. \tag{2.25}$$

Combining (1.7) with (2.5), (2.4) and with the boundary continuity of Φ, (2.25) and (2.24) yield

$$\lim_{\delta\to 0}\frac{1}{n}\int_{\Sigma_\delta} \Phi\left(|\nabla u|\right)\left\langle x, \nu^e_\delta\right\rangle\,d\mathcal{H}^{n-1} = \Phi_\infty|\Omega|,$$

whence (2.21) follows. By the above considerations, (2.19) holds. In order to conclude the proof, it suffices to observe that the second step of the proof of [6, Theorem 1.1] works regardless of the boundary regularity of u: (2.19) implies that the level sets $\{u = \delta\}$, for $\delta \in (0, \max u)$, are concentric spheres. This fact grants that Ω is a ball, and that u is radial. The rest of the thesis then follows by Theorem 1.3.

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