

A SLICING APPROACH TO STRESS-STRAIN DUALITY

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ABSTRACT. The classical Kohn-Temam stress-strain pairing $(\mathbf{A} : E\mathbf{u})$ for symmetric tensors $\mathbf{A}$ and $\mathbf{u} \in BD$ is typically formulated under summability assumptions on the divergence of $\mathbf{A}$. This excludes stress fields whose divergence has singular surface contributions, as occurs at cracks and material interfaces in continuum mechanics. We define and study stress-strain pairings for bounded symmetric divergence-measure tensor fields. For general $\mathbf{u} \in BD$, we introduce a slicing pairing $((\mathbf{A} : E\mathbf{u}))_{\Xi}$ for tensor fields satisfying a directional BV -type condition with respect to a finite frame Ξ . The definition is based on a one-dimensional disintegration strategy, and despite this construction, the new pairing enjoys analogous properties of the usual pairing $(\mathbf{A} : E\mathbf{u})$, such as the absolute continuity with respect to $|E\mathbf{u}|$ and the Gauss-Green formulas. We also identify several situations in which the pairing is independent of the choice of frame Ξ , including the relevant case in which the stress field $\mathbf{A}$ belongs to BV . While a distributional stress-strain pairing can be defined naturally for bounded BD functions, it cannot be extended to the unbounded setting, since the truncation techniques available in BV fail in BD . The slicing pairing is consistent with the distributional one whenever the latter is defined, while being more general even for bounded $\mathbf{u}$. Indeed, its existence does not require the compatibility condition $|\text{Div } \mathbf{A}|(S_{\mathbf{u}} \setminus J_{\mathbf{u}}) = 0$ which is necessary for the distributional definition. This allows the treatment of stress fields interacting with diffuse micro-cracking.

1. INTRODUCTION

The mathematical formulation of the duality between stress and strain tensor fields represents a cornerstone in the variational analysis of inelastic behaviors, such as perfect plasticity, damage, and fracture mechanics. In the classical framework established by Kohn and Temam in [29], the definition of a pairing $(\mathbf{A} : E\mathbf{u})$ between a bounded symmetric stress tensor $\mathbf{A} \in L^{\infty}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and a strain tensor $E\mathbf{u}$ associated with a displacement $\mathbf{u} \in BD(\Omega)$ crucially requires that the divergence of the stress tensor satisfies $\text{Div } \mathbf{A} \in L^N(\Omega; \mathbb{R}^N)$. The only available notion is the classical distributional definition of pairing (see [30, Ch. II, Sec. 7.3] and [29])

$$(1.1) \quad \langle (\mathbf{A} : E\mathbf{u}), \varphi \rangle := - \int_{\Omega} \varphi \mathbf{u} \cdot \text{Div } \mathbf{A} \, dx - \int_{\Omega} \mathbf{A} : [\mathbf{u} \odot \nabla \varphi] \, dx,$$

for every $\varphi \in C_c^{\infty}(\Omega)$, where $\odot$ denotes the symmetrized tensor product between vectors. Most developments in elasto-plasticity have been formulated under assumptions ensuring that the admissible stresses remain within the scope of the classical theory, in particular requiring summable divergence. Within the framework of functions with bounded deformation, these issues have been extensively investigated through variational approaches. In the context of plasticity, a foundational contribution is provided by Dal Maso, De Simone and Mora [24], who developed a general theory of quasistatic evolution for linearly elastic-perfectly plastic materials.

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Nevertheless, in several areas of applied sciences and engineering, tensor fields naturally arise whose divergence contains jump (surface) terms. For instance, in composite elastic bodies, jumps in the stress across interfaces describe surface forces associated with adhesion and debonding phenomena, and the corresponding pairing with the displacement field represents interfacial work. In fracture mechanics, cracks induce discontinuities in the stress field, while the pairing $(\mathbf{A} : E\mathbf{u})$ accounts for bulk elastic energy as well as fracture and crack-opening energy (see [10, 26, 27]). In fluid dynamics, shocks in conservation laws produce concentration effects in the divergence of stress-flux fields (see [12, 13]). These examples show that the symmetric stress-strain pairing in BD is not only mathematically natural, but also essential for modeling realistic systems involving cracks, interfaces, plasticity, and concentrated forces. In particular, they illustrate how jump contributions in the divergence operator naturally encode surface and interface forces within a unified variational framework.

On the other hand, starting from the seminal paper [5], the pairing theory in BV has undergone significant developments in recent years, driven by applications in several contexts ([12, 14, 15, 19–22, 25]), which suggest possible extensions of the classical BD theory to more irregular stresses and divergences. Indeed, the original motivation of [5] was the variational approach to elasto-plasticity (see [4, 6, 7]).

Inspired by these recent advances, we address the problem of defining a general pairing $(\mathbf{A} : E\mathbf{u})$. The ideas developed in the aforementioned works allow one to treat the case of bounded BD functions rather naturally through a distributional formulation. The extension to arbitrary BD functions is more delicate, as the available techniques break down in the absence of truncation arguments and must be replaced by a different approach.

Our results. Motivated by this perspective, we aim to extend the classical definition (1.1) in order to include bounded symmetric tensor fields $\mathbf{A}$ with divergence measure.

We start by introducing a pairing in BD for *bounded* functions (see, for instance, [18, Theorem 3.1] for an example of a function of this type). In this case, a natural definition of pairing between $\mathbf{A}$ and $E\mathbf{u}$ would be the distribution $(\mathbf{A} : E\mathbf{u}) \in \mathcal{D}'(\Omega)$ acting as

$$(1.2) \quad \langle (\mathbf{A} : E\mathbf{u}), \varphi \rangle := - \int_{\Omega} \varphi \mathbf{u}^* \cdot d \operatorname{Div} \mathbf{A} - \int_{\Omega} \mathbf{A} : (\mathbf{u} \odot \nabla \varphi) dx,$$

for every $\varphi \in C_c^\infty(\Omega)$, where $\mathbf{u}^*$ is the precise representative of $\mathbf{u}$ defined on $\Omega \setminus (S_{\mathbf{u}} \setminus J_{\mathbf{u}})$ by (2.7). However, although $|\operatorname{Div} \mathbf{A}| \ll \mathcal{H}^{N-1}$, the term $\int_{\Omega} \varphi \mathbf{u}^* \cdot d \operatorname{Div} \mathbf{A}$ may fail to be well defined, since it is not known whether $\mathcal{H}^{N-1}(S_{\mathbf{u}} \setminus J_{\mathbf{u}}) = 0$ (see Section 2 (§ *Functions of bounded deformation*)). To overcome this issue, we require the assumption

$$(H_*) \quad |\operatorname{Div}^s \mathbf{A}|(S_{\mathbf{u}} \setminus J_{\mathbf{u}}) = 0,$$

where $\operatorname{Div}^s \mathbf{A}$ denotes the singular part of the measure $\operatorname{Div} \mathbf{A}$, so that $\mathbf{u}^*$ turns out to be defined $|\operatorname{Div} \mathbf{A}|$ -a.e. (see discussion in Section 5). From a physical standpoint, condition (H_*) stipulates that any singular internal or external force concentrations (represented by the measure $\operatorname{Div} \mathbf{A}$) cannot be distributed on the diffuse, irregular boundary regions where $\mathbf{u}$ lacks a clean jump. Instead, singular forces must be strictly concentrated along well-defined interfaces or crack surfaces. This assumption is perfectly compatible with the modern variational models of fracture and cohesive zones, but it excludes the case of diffuse micro-cracking.

The distribution (1.2) is shown to be a Radon measure (see Theorem 5.3) and to satisfy the Gauss-Green formula (see Theorem 5.8).

On the other hand, the boundedness of $\mathbf{u}$ is a rather restrictive modeling assumption, especially in brittle fracture mechanics, so that it seems natural to treat the case of *unbounded* BD functions. However, a further difficulty arises in this setting, since the distributional definition (1.2) is no longer directly applicable, as it would require a truncation argument which is not available in BD . The use of slicing techniques to overcome this type of difficulty is reminiscent of the definitions of the spaces of generalized functions of bounded deformation GBD and $GSBD$ introduced by Dal Maso [23], where generalized deformations are characterized through suitable one-dimensional sections.

This observation naturally leads us to introduce, for general BD functions, a slicing-based definition of pairing. We impose a structural assumption ensuring one-dimensional BV regularity of the components of a symmetric matrix $\mathbf{A}$ along a “frame” - that is, a suitable finite set of directions $\boldsymbol{\xi} \in \Xi$ - namely, $\mathbf{A} \in BV_{\Xi}^{\infty}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ (see (6.4)). This allows us to establish a Fubini-type formula that reduces the pairing to one-dimensional contributions. The argument relies on slicing along directions $\boldsymbol{\xi} \in \Xi$ and on the fact that, for a.e. $y \in \boldsymbol{\xi}^{\perp}$, the scalar projections $\hat{u}_y^{\boldsymbol{\xi}}(t) := \mathbf{u}(y + t\boldsymbol{\xi}) \cdot \boldsymbol{\xi}$ belong to $BV_{\text{loc}}(\Omega_y^{\boldsymbol{\xi}})$, where $\Omega_y^{\boldsymbol{\xi}}$ is a one dimensional slice of Ω .

A preliminary, but crucial, result is a slicing formula for the distributional pairing (1.2) for $\mathbf{u} \in L^{\infty}(\Omega; \mathbb{R}^N)$, as

$$(1.3) \quad (\mathbf{A} : E\mathbf{u}) = \sum_{\boldsymbol{\xi} \in \Xi} \mathcal{L}^{N-1} \llcorner \Omega^{\boldsymbol{\xi}} \otimes_{\mathcal{M}} (a_y^{\boldsymbol{\xi}}, D\hat{u}_y^{\boldsymbol{\xi}}) \llcorner \Omega_y^{\boldsymbol{\xi}},$$

in the sense of measures (see Theorem 6.5), where $\otimes_{\mathcal{M}}$ denotes the generalized product of measures, see Section 2 (§ *Measures*). Here, $a_y^{\boldsymbol{\xi}}(s) := a^{\boldsymbol{\xi}}(y + s\boldsymbol{\xi})$ and $a^{\boldsymbol{\xi}}$ are the coefficients of $\mathbf{A}$ with respect to the dual base of $\{\boldsymbol{\xi} \otimes \boldsymbol{\xi}\}_{\boldsymbol{\xi} \in \Xi}$. In fact, the proof is obtained by disintegrating separately the Lebesgue, Cantor and jump parts. The handling of the Cantor part is more delicate and requires that $|E^c \mathbf{u}|(S_{\mathbf{A}}) = 0$; i.e. the Cantor part of $E\mathbf{u}$ does not charge the set of the essential discontinuities of $\mathbf{A}$.

Now, for a general BD function, we introduce a new pairing, denoted by $((\mathbf{A} : E\mathbf{u}))_{\Xi}$, taking (1.3) as the starting point. More precisely, given $\mathbf{A} \in BV_{\Xi}^{\infty}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in BD(\Omega)$, we define the linear functional $((\mathbf{A} : E\mathbf{u}))_{\Xi} : C_c^{\infty}(\Omega) \rightarrow \mathbb{R}$ by

$$(1.4) \quad \langle ((\mathbf{A} : E\mathbf{u}))_{\Xi}, \varphi \rangle := \sum_{\boldsymbol{\xi} \in \Xi} \int_{\boldsymbol{\xi}^{\perp}} \left(\int_{\Omega_y^{\boldsymbol{\xi}}} \varphi_y^{\boldsymbol{\xi}}(s) d(a_y^{\boldsymbol{\xi}}, D\hat{u}_y^{\boldsymbol{\xi}})(s) \right) d\mathcal{L}^{N-1}(y).$$

The resulting object is well defined, depends on the frame Ξ and is consistent with the bounded case. This is the content of the following theorem (see Theorem 7.2 and Theorem 7.9 for more precise statements), which is the main result of the paper.

Theorem 1.1. *Let $\mathbf{A} \in BV_{\Xi}^{\infty}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in BD(\Omega)$. Then the slicing pairing $((\mathbf{A} : E\mathbf{u}))_{\Xi}$ is a Radon measure in Ω , absolutely continuous with respect to $|E\mathbf{u}|$, and a Gauss-Green formula holds on sets of finite perimeter. Moreover, if $\mathbf{u} \in BD(\Omega) \cap L^{\infty}(\Omega; \mathbb{R}^N)$ such that $|E^c \mathbf{u}|(S_{\mathbf{A}}) = 0$, and (H_*) holds, then $((\mathbf{A} : E\mathbf{u}))_{\Xi}$ coincides with the pairing defined in (1.2); i.e.,*

$$((\mathbf{A} : E\mathbf{u}))_{\Xi} = (\mathbf{A} : E\mathbf{u}),$$

in the sense of measures.

The construction is not intrinsic in this level of generality, as the pairing $((\mathbf{A} : E\mathbf{u}))_{\Xi}$ depends on the choice of the finite frame Ξ . This dependence is an unavoidable consequence of the slicing approach, since the directional BV assumptions are imposed only along the

directions of Ξ . Although such coordinate dependence may at first seem at odds with the objectivity expected in continuum mechanics, it provides the flexibility needed to accommodate singularities of $\text{Div } \mathbf{A}$. We identify several situations in which the pairing becomes intrinsic and is therefore independent of the chosen frame (see Remarks 7.4 and 7.5, and Appendix A). In particular, this occurs when $\mathbf{A} \in BV(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \cap L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$, in which case the slicing construction reconstructs the intrinsic measure $\mathbf{A}^* : E\mathbf{u}$.

Notice that the slicing pairing is well-defined without requiring the compatibility assumption (H_*) , even for bounded $\mathbf{u}$. This allows the treatment of diffuse micro-cracking, which is excluded from the framework of the distributional formulation.

A particularly illuminating step for developing the previous analysis, which is of independent interest, was a preliminary slicing decomposition of the pairing $(\mathbf{A}, D\mathbf{u})$ between bounded tensor fields with measure-valued divergence and $\mathbf{u} \in BV(\Omega; \mathbb{R}^N)$, as studied in [25]. This is achieved by exploiting the slicing theory in BV , which characterizes BV functions through their one-dimensional sections and allows for a corresponding disintegration of the measure $D\mathbf{u}$. This study relies on several key features of BV functions, which are not available in BD : the coarea formula, the chain rule together with truncation arguments, the fact that a vector-valued BV function can be characterized componentwise in terms of scalar BV functions (which in turn implies that the pairing can be decomposed as the sum of the corresponding scalar pairings), and the property $\mathcal{H}^{N-1}(S_{\mathbf{u}} \setminus J_{\mathbf{u}}) = 0$.

In the last part of the paper, we introduce a new notion of pairing $((\mathbf{A}^D : E^D\mathbf{u}))_\Xi$ between the deviatoric parts of $\mathbf{A}$ and $E\mathbf{u}$, which can be compared with the one studied in [30, Lemma 7.3] and [29, Theorem 3.2]. The main difficulty is that, to the best of the authors' knowledge, no slicing result is currently available for $E^D\mathbf{u}$ (see [8, Example 10.1]). This lack of a slicing theory is consistent with the fact that the deviatoric part of the strain does not satisfy the rank-one property, which is essential for slicing arguments. Nevertheless, we are able to give a “mixed” definition of $((\mathbf{A}^D : E^D\mathbf{u}))_\Xi$ for $\mathbf{u} \in BD(\Omega)$ and $\mathbf{A} \in L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ with divergence measure (see (8.3)).

Finally, in Section 9, we discuss a simple two-dimensional example of Griffith-type Mode I opening configuration, modeling a pre-existing crack in the reference configuration. The stress tensor considered therein has a singular $\text{Div } \mathbf{A}$ with respect to $\mathcal{L}^2$, hence the classical Kohn–Temam pairing theory does not apply and it does not allow one to compute the mechanical work across the interface. By contrast, the framework developed in this paper naturally captures the interfacial contribution.

Organization of the paper. In Section 2 we collect preliminaries on BV , BD , and divergence measure fields, including slicing and disintegration results. Section 3 is devoted to slicing formulas for divergence and normal traces. In Section 4 we establish the slicing representation of the pairing in the BV setting for scalar and vector-valued functions. Sections 5–7 develop the extension to tensor fields and BD functions, including the definition of the pairing via slicing and the analysis of the unbounded case. In Section 8 we compare our results with the classical theory, and extend it to (bounded) tensor fields with divergence measure. Finally, in Section 9 we discuss an example.

2. PRELIMINARIES

Vectors and matrices. If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$, we write $\mathbf{a} \cdot \mathbf{b}$ for the Euclidean scalar product, and we denote by $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$ the associated norm.

We write $\mathbb{R}^{N \times N}$ for the set of real $N \times N$ matrices, and denote by $\mathbb{R}_{\text{sym}}^{N \times N}$ that of all real symmetric $N \times N$ matrices. The components A_{ij} of a matrix $\mathbf{A}$ are defined by

$$A_{ij} := \mathbf{e}_i \cdot \mathbf{A} \mathbf{e}_j.$$

Given two matrices $\mathbf{A}$ and $\mathbf{B} \in \mathbb{R}^{N \times N}$, we consider the Frobenius scalar product $\mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{A}^T \mathbf{B})$, where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$, $\text{tr} \mathbf{A}$ is its trace, i.e. $\text{tr} \mathbf{A} = \sum_{i=1}^N A_{ii}$, and we denote by $|\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A}}$ the associated norm. Note that

$$(2.1) \quad |\text{tr} \mathbf{A}| \leq \sqrt{N} |\mathbf{A}|.$$

We recall that for any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$, $\mathbf{a} \otimes \mathbf{b} := \mathbf{a} \mathbf{b}^T \in \mathbb{R}^{N \times N}$ stands for the tensor product, while $\mathbf{a} \odot \mathbf{b}$ denotes the symmetric matrix $\frac{1}{2}(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})$. We may also write

$$\mathbf{A} = \sum_{i,j=1}^N A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j.$$

The following identities (see, e.g., [28, Section 3]) easily follow from the definitions and the properties of the tensor and scalar products:

$$(2.2) \quad \mathbf{A} : (\mathbf{v} \odot \mathbf{w}) = \mathbf{A} : (\mathbf{v} \otimes \mathbf{w}) = \mathbf{v} \cdot (\mathbf{A} \mathbf{w}) = \mathbf{w} \cdot (\mathbf{A} \mathbf{v}),$$

for every $\mathbf{v}, \mathbf{w} \in \mathbb{R}^N$ and $\mathbf{A} \in \mathbb{R}_{\text{sym}}^{N \times N}$.

For every $\mathbf{A} \in \mathbb{R}^{N \times N}$ let us denote by $\mathbf{A}^D$ the deviatoric part of $\mathbf{A}$, defined by

$$(2.3) \quad \mathbf{A}^D = \mathbf{A} - \frac{1}{N} (\text{tr} \mathbf{A}) \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. Recalling that the Frobenius scalar product is bilinear, and that $\mathbf{A} : \mathbf{I} = \text{tr} \mathbf{A}$ (so that, in particular, $\mathbf{I} : \mathbf{I} = N$), we easily deduce from (2.3) the well-known identity (see, e.g., [30, eq. (7.41)])

$$(2.4) \quad \mathbf{A}^D : \mathbf{B}^D = \mathbf{A} : \mathbf{B} - \frac{1}{N} \text{tr} \mathbf{A} \text{tr} \mathbf{B}.$$

Measures. The following definitions and basic facts about measures can be found, e.g., in [3, Chapters 1 and 2].

We denote by $\mathcal{L}^N$ and $\mathcal{H}^\alpha$ the Lebesgue measure and the α -dimensional Hausdorff measure in $\mathbb{R}^N$ for some $\alpha \in [0, N]$, respectively. Unless otherwise stated, a measurable set is a $\mathcal{L}^N$ -measurable set. It is well known that $\mathcal{H}^{N-1} = \mathcal{L}^{N-1}$ on each hyperplane $\Omega^\xi \subset \mathbb{R}^N$ (see [3, eq. (2.37)]). Unless otherwise specified, $\Omega \subseteq \mathbb{R}^N$ is an open set.

Following the notation of [3], we denote by $\mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^m)$ and $\mathcal{M}(\Omega; \mathbb{R}^m)$ the spaces of vector-valued Radon measures and finite Radon measures on Ω , respectively. In the case of scalar measures ($m = 1$), we will shorten the notation and use the symbols $\mathcal{M}_{\text{loc}}(\Omega)$ and $\mathcal{M}(\Omega)$.

For matrix valued measures we will use the notation $\mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^{N \times N})$, $\mathcal{M}(\Omega; \mathbb{R}^{N \times N})$, $\mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathcal{M}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$.

We recall the definition of *generalized product* of two measures (see, e.g., [3, Definition 2.27]).

Definition 2.1. Let $E \subset \mathbb{R}^N$, $F \subset \mathbb{R}^M$ be open sets, let μ be a positive Radon measure on E , and let $x \mapsto \nu_x$ be a family of finite $\mathbb{R}^m$ -valued Radon measures on F such that $x \mapsto \nu_x(B)$ is μ -measurable for every Borel set $B \subset F$. Assume that

$$\int_{E'} |\nu_x|(F) d\mu(x) < +\infty \quad \forall E' \Subset E \text{ open}.$$

We define the $\mathbb{R}^m$ -valued Radon measure $\mu \otimes_{\mathcal{M}} \nu_x$ on $E \times F$ by setting

$$(\mu \otimes_{\mathcal{M}} \nu_x)(B) := \int_E \left(\int_F \chi_B(x, y) d\nu_x(y) \right) d\mu(x), \quad \text{for all Borel sets } B \subset K \times F,$$

where $K \subset E$ is compact.

Moreover, if f is a positive or negative Borel function, or if $f \in L^1(E \times F, \mu \otimes_{\mathcal{M}} \nu_x)$, then the integration formula

$$(2.5) \quad \int_{E \times F} f(x, y) d(\mu \otimes_{\mathcal{M}} \nu_x)(x, y) = \int_E \left(\int_F f(x, y) d\nu_x(y) \right) d\mu(x),$$

holds.

We denote by $L^1_\mu(\Omega; \mathbb{R}^N)$ the space of $\mathbb{R}^N$ -valued functions that are integrable with respect to μ on Ω , and by $L^1_{\mu, \text{loc}}(\Omega; \mathbb{R}^N)$ the space of functions that belong to $L^1_\mu(K; \mathbb{R}^N)$ for every compact set $K \subset \Omega$. In the scalar-valued case, we omit the target space and simply write $L^1(\Omega, \mu)$ and $L^1_{\text{loc}}(\Omega, \mu)$.

Approximate limits. The following basic definitions and results can be found, e.g., in [3, Sections 3.6 and 4.5].

We say that a function $\mathbf{u} \in L^1_{\text{loc}}(\Omega; \mathbb{R}^N)$ has an *approximate limit* $\mathbf{z} \in \mathbb{R}^N$ at $x \in \Omega$ if

$$\lim_{r \rightarrow 0^+} \frac{1}{\mathcal{L}^N(B_r(x))} \int_{B_r(x)} |\mathbf{u}(y) - \mathbf{z}| dy = 0;$$

in this case we say that x is a *Lebesgue point* of $\mathbf{u}$. The set $S_{\mathbf{u}} \subset \Omega$ of points where this property does not hold is called the *approximate discontinuity set* of $\mathbf{u}$, and, thanks to Lebesgue's differentiation theorem, we know that $\mathcal{L}^N(S_{\mathbf{u}}) = 0$. For any $x \in \Omega \setminus S_{\mathbf{u}}$ the approximate limit $\mathbf{z}$ is uniquely determined and is denoted by $\mathbf{z} =: \tilde{\mathbf{u}}(x)$.

Given $\mathbf{u} \in L^1_{\text{loc}}(\Omega; \mathbb{R}^N)$, we say that $x \in \Omega$ is an *approximate jump point* of $\mathbf{u}$ if there exist $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$, $\mathbf{a} \neq \mathbf{b}$, and a unit vector $\boldsymbol{\nu} \in \mathbb{R}^N$ such that

$$(2.6) \quad \begin{aligned} \lim_{r \rightarrow 0^+} \frac{1}{\mathcal{L}^N(B_r^+(x))} \int_{B_r^+(x)} |\mathbf{u}(y) - \mathbf{a}| dy &= 0, \\ \lim_{r \rightarrow 0^+} \frac{1}{\mathcal{L}^N(B_r^-(x))} \int_{B_r^-(x)} |\mathbf{u}(y) - \mathbf{b}| dy &= 0, \end{aligned}$$

where $B_r^\pm(x) := \{y \in B_r(x) : \pm(y - x) \cdot \boldsymbol{\nu} > 0\}$. The triplet $(\mathbf{a}, \mathbf{b}, \boldsymbol{\nu})$, uniquely determined by (2.6) up to a permutation of $(\mathbf{a}, \mathbf{b})$ and a change of sign of $\boldsymbol{\nu}$, is denoted by $(\mathbf{u}^+(x), \mathbf{u}^-(x), \boldsymbol{\nu}_{\mathbf{u}}(x))$. The set of approximate jump points of $\mathbf{u}$ is denoted by $J_{\mathbf{u}}$, and it clearly satisfies $J_{\mathbf{u}} \subset S_{\mathbf{u}}$. It is easy to check that $J_{\mathbf{u}}$ and $S_{\mathbf{u}}$ are Borel sets, and $\tilde{\mathbf{u}}$, $\mathbf{u}^+$ and $\mathbf{u}^-$ are Borel functions.

Finally, for $\mathbf{u} \in L^1_{\text{loc}}(\Omega; \mathbb{R}^N)$ we define the *precise representative* of $\mathbf{u}$ in $x \in \Omega$ as

$$\mathbf{u}^*(x) := \lim_{r \rightarrow 0^+} \frac{1}{\mathcal{L}^N(B_r(x))} \int_{B_r(x)} \mathbf{u}(y) dy,$$

whenever the limit exists. It is then clear that

$$(2.7) \quad \mathbf{u}^*(x) = \begin{cases} \tilde{\mathbf{u}}(x) & x \in \Omega \setminus S_{\mathbf{u}}, \\ \frac{\mathbf{u}^+(x) + \mathbf{u}^-(x)}{2} & x \in J_{\mathbf{u}}. \end{cases}$$

A priori, it is not clear whether $\mathbf{u}^*$ is well posed in $S_{\mathbf{u}} \setminus J_{\mathbf{u}}$, in general. However, for sufficiently regular functions it is known that $S_{\mathbf{u}} \setminus J_{\mathbf{u}}$ is suitably small (for instance, when $\mathbf{u}$ is a function of bounded variation, see [3, Theorem 3.78]).

Let $\mathbf{u} \in L^1(\Omega; \mathbb{R}^N)$, and consider the sequence

$$(2.8) \quad \mathbf{u}_\varepsilon := \eta_\varepsilon * \mathbf{u} = (\eta_\varepsilon * u_1, \eta_\varepsilon * u_2, \dots, \eta_\varepsilon * u_N),$$

where $\eta_\varepsilon(x) := \frac{1}{\varepsilon^N} \eta(\frac{x}{\varepsilon})$ and η is a positive symmetric mollifier. Then, combining [3, Proposition 3.64(b) and Proposition 3.69(b)] we get

$$(2.9) \quad \mathbf{u}_\varepsilon \rightarrow \mathbf{u}^* \quad \text{pointwise in } \Omega \setminus (S_{\mathbf{u}} \setminus J_{\mathbf{u}}) \text{ as } \varepsilon \rightarrow 0.$$

All the previous definitions extend verbatim to the setting of matrix-valued functions.

Slicing of functions of bounded variation. For a detailed treatment of the theory of BV functions, we refer the reader to the monograph [3]. The vector space of all functions of bounded variation in Ω will be denoted by $BV(\Omega; \mathbb{R}^N)$. In addition, $BV_{\text{loc}}(\Omega; \mathbb{R}^N)$ will denote the class of functions belonging to $BV(\Omega'; \mathbb{R}^N)$ for every $\Omega' \Subset \Omega$.

Let us fix $\boldsymbol{\xi} \in \mathbb{R}^N$ and $\boldsymbol{\xi}^\perp$ the hyperplane orthogonal to $\boldsymbol{\xi}$. For every $E \subset \mathbb{R}^N$ and for every $y \in \boldsymbol{\xi}^\perp$ we denote $E_y^\boldsymbol{\xi} := \{s \in \mathbb{R} : y + s\boldsymbol{\xi} \in E\}$ and by $E^\boldsymbol{\xi} := \{y \in \boldsymbol{\xi}^\perp : E_y^\boldsymbol{\xi} \neq \emptyset\}$. Given $u : \Omega \rightarrow \mathbb{R}$, for any $\boldsymbol{\xi} \in \mathbb{R}^N \setminus \{0\}$ and $y \in \boldsymbol{\xi}^\perp$, we define

$$u_y^\boldsymbol{\xi}(s) := u(y + s\boldsymbol{\xi}) \quad \text{for } s \in \Omega_y^\boldsymbol{\xi}.$$

If $\mathbf{u} : \Omega \rightarrow \mathbb{R}^N$, for any $\boldsymbol{\xi} \in \mathbb{R}^N \setminus \{0\}$ and $y \in \boldsymbol{\xi}^\perp$, we define

$$\hat{u}_y^\boldsymbol{\xi}(s) := \mathbf{u}(y + s\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \quad \text{for } s \in \Omega_y^\boldsymbol{\xi}.$$

We first state a well-known result on the slicing of directional derivatives (see [3, Theorem 3.103 and 3.107]).

Proposition 2.2. *Let $v \in L^1(\Omega)$ and let $\boldsymbol{\xi} \in \mathbb{S}^{N-1}$. The following conditions are equivalent:*

- (a) $D_{\boldsymbol{\xi}} v \in \mathcal{M}(\Omega)$;
- (b) for $\mathcal{L}^{N-1}$ -a.e. $y \in \Omega^\boldsymbol{\xi}$ the function $v_y^\boldsymbol{\xi}$ belongs to $BV(\Omega_y^\boldsymbol{\xi})$ and

$$\int_{\Omega^\boldsymbol{\xi}} |Dv_y^\boldsymbol{\xi}|(\Omega_y^\boldsymbol{\xi}) d\mathcal{L}^{N-1}(y) < \infty.$$

If these conditions are satisfied, then

$$(2.10) \quad D_{\boldsymbol{\xi}} v = Dv \cdot \boldsymbol{\xi} = \mathcal{L}^{N-1} \llcorner \Omega^\boldsymbol{\xi} \otimes_{\mathcal{M}} Dv_y^\boldsymbol{\xi};$$

i.e.,

$$(2.11) \quad \int_{\Omega} \varphi dD_{\boldsymbol{\xi}} v = \int_{\Omega^\boldsymbol{\xi}} \left(\int_{\Omega_y^\boldsymbol{\xi}} \varphi_y^\boldsymbol{\xi} dDv_y^\boldsymbol{\xi} \right) d\mathcal{L}^{N-1}(y), \quad \text{for every } \varphi \in C_c(\Omega).$$

Moreover, if $f \in L^1(\Omega, |D_{\boldsymbol{\xi}} v|)$, then

$$(2.12) \quad \int_{\Omega} f dD_{\boldsymbol{\xi}} v = \int_{\Omega^\boldsymbol{\xi}} \left(\int_{\Omega_y^\boldsymbol{\xi}} f_y^\boldsymbol{\xi} dDv_y^\boldsymbol{\xi} \right) d\mathcal{L}^{N-1}(y).$$

Functions of bounded deformation. We collect some definitions and basic results about functions of bounded deformation. We refer the reader to [30] for a comprehensive treatment on the topic.

We denote by $BD(\Omega)$ the space of *functions of bounded deformation* in Ω , i.e., $\mathbf{u} \in BD(\Omega)$ if $\mathbf{u} \in L^1(\Omega; \mathbb{R}^N)$ and $E\mathbf{u} \in \mathcal{M}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$, where $E\mathbf{u} := (D\mathbf{u} + D\mathbf{u}^T)/2$ and $D\mathbf{u}$ is the distributional derivative of $\mathbf{u}$. In addition, $BD_{\text{loc}}(\Omega)$ will denote the class of functions belonging to $BD(\Omega')$ for every $\Omega' \Subset \Omega$. In general we have that $J_{\mathbf{u}} \subset S_{\mathbf{u}}$. While for a function $\mathbf{u} \in BV(\Omega; \mathbb{R}^N)$ we have $\mathcal{H}^{N-1}(S_{\mathbf{u}} \setminus J_{\mathbf{u}}) = 0$, the inclusion can be strict for a function $\mathbf{u} \in BD(\Omega)$ (see [1, Remark 6.3]). However, by [1, Theorem 6.1], the set $S_{\mathbf{u}} \setminus J_{\mathbf{u}}$ of discontinuity points of $\mathbf{u}$ which are not jump points is $|E\mathbf{v}|$ -negligible for every $\mathbf{v} \in BD(\Omega)$; i.e., $|E\mathbf{v}|$ -almost every $x \in \Omega$ is either a Lebesgue point or a jump point for $\mathbf{u}$ (see [1, Proposition 6.1]).

A sequence $(\mathbf{u}_k)_{k \in \mathbb{N}}$ in $BD(\Omega)$ converges *strictly* to a function $\mathbf{u} \in BD(\Omega)$ if

$$\begin{cases} \mathbf{u}_k \rightarrow \mathbf{u} & \text{in } L^1(\Omega, \mathbb{R}^N), \\ E\mathbf{u}_k \xrightarrow{*} E\mathbf{u} & \text{in the sense of measures,} \\ |E\mathbf{u}_k|(\Omega) \rightarrow |E\mathbf{u}|(\Omega). \end{cases}$$

The following result (see [7, Theorem 1.3] and [30, II-Theorem 3.3]) concerns the approximation of a function $\mathbf{u} \in BD(\Omega)$ by a sequence of smooth functions.

Theorem 2.3. *For every $\mathbf{u} \in BD(\Omega)$ there exists a sequence $(\mathbf{u}_k) \subset C^\infty(\Omega; \mathbb{R}^N) \cap BD(\Omega)$ such that $\mathbf{u}_k \rightarrow \mathbf{u}$ strictly. In particular, if $\mathbf{u} \in BD(\Omega) \cap L^\infty(\Omega; \mathbb{R}^N)$, then $\|\mathbf{u}_k\|_{L^\infty(\Omega; \mathbb{R}^N)} \leq 3\|\mathbf{u}\|_{L^\infty(\Omega; \mathbb{R}^N)}$ for every k . If, in addition, Ω is bounded and with Lipschitz boundary, the same result holds with $C^\infty(\overline{\Omega}; \mathbb{R}^N) \cap BD(\Omega)$ in place of $C^\infty(\Omega; \mathbb{R}^N) \cap BD(\Omega)$.*

The properties of the trace operator on open sets with Lipschitz boundaries (see also, e.g., [30, Chapter II] for the analogous classical result on C^1 boundaries) are collected in the theorem below, which is due to Babadjian [9].

Theorem 2.4. [9, Theorem 3.2 and Proposition 3.4] *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. There exists a unique linear continuous mapping*

$$\gamma : BD(\Omega) \rightarrow L^1(\partial\Omega; \mathbb{R}^N)$$

such that the following integration by parts formula holds: for every $\mathbf{u} \in BD(\Omega)$ and $\varphi \in C_c^1(\mathbb{R}^N)$,

$$(2.13) \quad \int_{\Omega} \mathbf{u} \odot \nabla \varphi \, dx + \int_{\Omega} \varphi \, dE\mathbf{u} = \int_{\partial\Omega} \gamma(\mathbf{u}) \odot \boldsymbol{\nu}_{\Omega} \varphi \, d\mathcal{H}^{N-1},$$

where $\boldsymbol{\nu}_{\Omega}$ is the outer unit normal to $\partial\Omega$. In addition,

(i) it holds that

$$\gamma(\mathbf{u}) = \mathbf{u}|_{\partial\Omega} \quad \text{for all } \mathbf{u} \in C^0(\overline{\Omega}; \mathbb{R}^N) \cap BD(\Omega);$$

(ii) the trace is continuous with respect to the strict convergence of $BD(\Omega)$: if $\mathbf{u} \in BD(\Omega)$ and $(\mathbf{u}_k) \subset BD(\Omega)$ is such that $\mathbf{u}_k \rightarrow \mathbf{u}$ strictly in $BD(\Omega)$, then $\gamma(\mathbf{u}_k) \rightarrow \gamma(\mathbf{u})$ strongly in $L^1_{\mathcal{H}^{N-1} \llcorner \partial\Omega}(\partial\Omega; \mathbb{R}^N)$.

Remark 2.5. Notice that formula (2.13) implies the following

$$(2.14) \quad \int_{\Omega} \mathbf{u} \cdot \text{Div } \boldsymbol{\Phi} \, dx + \int_{\Omega} \boldsymbol{\Phi} : dE\mathbf{u} = \int_{\partial\Omega} \boldsymbol{\Phi} : (\gamma(\mathbf{u}) \odot \boldsymbol{\nu}_{\Omega}) \, d\mathcal{H}^{N-1},$$

with $\Phi \in C^1(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$. Indeed, let $\Phi = (\varphi_{ij})$, and note that each φ_{ij} is a test function in (2.13). We then have, writing (2.13) componentwise with $\varphi = \varphi_{ij}$, summing over the indices i, j and using $\varphi_{ij} = \varphi_{ji}$,

$$\begin{aligned} \sum_{i,j} \int_{\partial\Omega} \varphi_{ij} (\gamma(\mathbf{u}) \odot \nu_\Omega)_{ij} d\mathcal{H}^{N-1} &= \frac{1}{2} \sum_{i,j} \int_{\Omega} (u_i \partial_j \varphi_{ij} + u_j \partial_i \varphi_{ij}) dx + \sum_{i,j} \int_{\Omega} \varphi_{ij} d(E\mathbf{u})_{ij} \\ &= \sum_i \int_{\Omega} u_i \sum_j \partial_j \varphi_{ij} dx + \sum_{i,j} \int_{\Omega} \varphi_{ij} d(E\mathbf{u})_{ij} \\ &= \sum_i \int_{\Omega} u_i (\text{Div } \Phi)_i dx + \sum_{i,j} \int_{\Omega} \varphi_{ij} d(E\mathbf{u})_{ij}, \end{aligned}$$

which corresponds to (2.14).

The existence of one-sided Lebesgue limits on countably $\mathcal{H}^{N-1}$ -rectifiable subsets of Ω for functions of bounded deformation is proved in the following result.

Proposition 2.6. [9, Proposition 4.1] *Let $\mathbf{u} \in BD(\Omega)$, and Σ be an oriented countably $\mathcal{H}^{N-1}$ -rectifiable subset of Ω . Then, for $\mathcal{H}^{N-1}$ -almost every $x \in \Sigma$, there exist one-sided Lebesgue limits $\mathbf{u}_\Sigma^\pm(x)$ with respect to the approximate unit normal $\nu_\Sigma(x)$ to Σ ; that is,*

$$\lim_{\rho \rightarrow 0^+} \frac{1}{\rho^N} \int_{B_\rho^\pm(x, \nu_\Sigma(x))} |\mathbf{u}(y) - \mathbf{u}_\Sigma^\pm(x)| dy = 0,$$

where $B_\rho^\pm(x, \nu_\Sigma(x)) := \{y \in B_\rho(x) : \pm(y - x) \cdot \nu_\Sigma(x) > 0\}$. In addition, we have the representation

$$E\mathbf{u} \llcorner \Sigma = (\mathbf{u}_\Sigma^+ - \mathbf{u}_\Sigma^-) \odot \nu_\Sigma \mathcal{H}^{N-1} \llcorner \Sigma.$$

Since for every $\mathbf{u} \in BD(\Omega)$ the jump set $J_\mathbf{u}$ is a countably $\mathcal{H}^{N-1}$ -rectifiable Borel set (see [1, Proposition 3.5]), by Proposition 2.6 applied with $\Gamma = J_\mathbf{u}$, we obtain the following decomposition of the Radon measure $E\mathbf{u}$

$$(2.15) \quad E\mathbf{u} = e(\mathbf{u})\mathcal{L}^N + E^s\mathbf{u} = e(\mathbf{u})\mathcal{L}^N + ([\mathbf{u}^+ - \mathbf{u}^-] \odot \nu_\mathbf{u}) \mathcal{H}^{N-1} \llcorner J_\mathbf{u} + E^c\mathbf{u},$$

where $e(\mathbf{u})$ is the density of the absolutely continuous part of $E\mathbf{u}$ with respect to $\mathcal{L}^N$, $E^s\mathbf{u}$ is the singular part, $E^c\mathbf{u}$ is the Cantor part and vanishes on Borel sets B with $\mathcal{H}^{N-1}(B) < +\infty$. We also introduce the space $LD(\Omega)$ of such functions $\mathbf{u} \in BD(\Omega)$ complying with

$$(2.16) \quad E\mathbf{u} = e(\mathbf{u})\mathcal{L}^N.$$

Recalling the notation of Section 2 (§ *Slicing of functions of bounded variation*), the space $BD(\Omega)$ can be characterized using one-dimensional sections (see [1, Section 3 and Prop. 3.2], [30, Chap. II, Section 2.2] and [3, Proposition 2.26 and Theorem 2.28]).

Proposition 2.7. *Let $\mathbf{u} \in BD(\Omega)$ and let $\xi \in \mathbb{R}^N$ with $\xi \neq \mathbf{0}$. Then the following two conditions hold for $\mathcal{H}^{N-1}$ -almost every $y \in \Omega^\xi$:*

- (i) $\widehat{u}_y^\xi$ is defined and coincides with $\hat{u}_y^\xi \mathcal{L}^1$ -almost everywhere in Ω_y^ξ ;
- (ii) $\hat{u}_y^\xi \in BV(\Omega_y^\xi)$ (and so $\hat{u}_y^\xi \in L^\infty_{\text{loc}}(\Omega_y^\xi)$).

Moreover,

$$(2.17) \quad E\mathbf{u}\xi \cdot \xi = \int_{\Omega^\xi} D\hat{u}_y^\xi d\mathcal{H}^{N-1}(y), \quad |E\mathbf{u}\xi \cdot \xi| = \int_{\Omega^\xi} |D\hat{u}_y^\xi| d\mathcal{H}^{N-1}(y),$$

as measures in Ω . In particular, for every bounded Borel function $g : \Omega \rightarrow \mathbb{R}$ the map

$$(2.18) \quad y \mapsto \int_{\Omega^\xi} g_y^\xi(s) dD\hat{u}_y^\xi(s)$$

is $\mathcal{H}^{N-1}$ -measurable on Ω^ξ .

We recall the Structure Theorem for BD functions, see [1, Theorem 4.5].

Theorem 2.8. *Let $\mathbf{u} \in BD(\Omega)$ and let $\xi \in \mathbb{R}^N$ with $\xi \neq \mathbf{0}$. Then*

(i)

$$E^a \mathbf{u} \xi \cdot \xi = \int_{\Omega^\xi} D^a \hat{u}_y^\xi d\mathcal{H}^{N-1}(y), \quad |E^a \mathbf{u} \xi \cdot \xi| = \int_{\Omega^\xi} |D^a \hat{u}_y^\xi| d\mathcal{H}^{N-1}(y);$$

(ii) *for $\mathcal{H}^{N-1}$ -almost every $y \in \Omega^\xi$, the functions $\hat{u}_y^\xi$ and $\tilde{u}_y^\xi$ belong to $BV(\Omega_y^\xi)$ and coincide $\mathcal{L}^1$ -almost everywhere on Ω_y^ξ ; the measures $|D\hat{u}_y^\xi|$ and $|D\tilde{u}_y^\xi|$ coincide on Ω_y^ξ , and*

$$e(\mathbf{u})(y + t\xi) \xi \cdot \xi = \nabla \hat{u}_y^\xi(t) = (\tilde{u}_y^\xi)'(t) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \Omega_y^\xi$$

(iii)

$$E^j \mathbf{u} \xi \cdot \xi = \int_{\Omega^\xi} D^j \hat{u}_y^\xi d\mathcal{H}^{N-1}(y), \quad |E^j \mathbf{u} \xi \cdot \xi| = \int_{\Omega^\xi} |D^j \hat{u}_y^\xi| d\mathcal{H}^{N-1}(y);$$

(iv)

$$(J_{\mathbf{u}}^\xi)_y = J_{\hat{u}_y^\xi} \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } y \in \Omega^\xi,$$

$$\mathbf{u}^+(y + t\xi) \cdot \xi = (\hat{u}_y^\xi)^+(t) = \lim_{s \rightarrow t^+} \tilde{u}_y^\xi(s),$$

$$\mathbf{u}^-(y + t\xi) \cdot \xi = (\hat{u}_y^\xi)^-(t) = \lim_{s \rightarrow t^-} \tilde{u}_y^\xi(s),$$

and for every $t \in (J_{\mathbf{u}}^\xi)_y^\xi$ where the normals to $J_{\mathbf{u}}$ and $J_{\hat{u}_y^\xi}$ are oriented so that $\nu_{\mathbf{u}} \cdot \xi \geq 0$ and $\nu_{\hat{u}_y^\xi} = 1$;

(v)

$$(2.19) \quad E^c \mathbf{u} \xi \cdot \xi = \int_{\Omega^\xi} D^c \hat{u}_y^\xi d\mathcal{H}^{N-1}(y), \quad |E^c \mathbf{u} \xi \cdot \xi| = \int_{\Omega^\xi} |D^c \hat{u}_y^\xi| d\mathcal{H}^{N-1}(y).$$

We say that $\mathbf{u} \in BD(\Omega)$ is a *special function with bounded deformation*, and we write $\mathbf{u} \in SBD(\Omega)$, if the measure $E^c \mathbf{u}$ is zero; that is, $|E^s \mathbf{u}|(\Omega \setminus J_{\mathbf{u}}) = 0$. $SBD^p(\Omega)$, $p > 1$, is defined as the space of SBD functions such that $e(\mathbf{u}) \in L^p(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathcal{H}^{N-1}(J_{\mathbf{u}}) < +\infty$.

Vector and matrix fields with divergence measure. We introduce the spaces of bounded vector and matrix fields whose distributional divergence is a finite Radon measure in Ω . Namely,

$$\mathcal{DM}^\infty(\Omega; \mathbb{R}^N) := \{ \mathbf{A} \in L^\infty(\Omega; \mathbb{R}^N) : \text{div } \mathbf{A} \in \mathcal{M}(\Omega) \},$$

and

$$(2.20) \quad \begin{aligned} \mathcal{DM}^\infty(\Omega; \mathbb{R}^{N \times N}) &:= \{ \mathbf{A} \in L^\infty(\Omega; \mathbb{R}^{N \times N}) : \text{Div } \mathbf{A} \in \mathcal{M}(\Omega; \mathbb{R}^N) \}, \\ \mathcal{DM}^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) &:= \{ \mathbf{A} \in L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) : \text{Div } \mathbf{A} \in \mathcal{M}(\Omega; \mathbb{R}^N) \}. \end{aligned}$$

Here, $\text{Div } \mathbf{A}$ denotes the distributional divergence of $\mathbf{A}$, defined componentwise by

$$\text{Div } \mathbf{A} := (\text{div } \mathbf{A}_1, \dots, \text{div } \mathbf{A}_N),$$

where $\mathbf{A}_j$ is the j -th column of $\mathbf{A}$ and

$$\operatorname{div} \mathbf{A}_j := \sum_{i=1}^N \frac{\partial A_{ij}}{\partial x_i}, \quad j = 1, \dots, N.$$

We will denote by $\mathcal{DM}_{\text{loc}}^\infty(\Omega; \mathbb{R}^N)$, $\mathcal{DM}_{\text{loc}}^\infty(\Omega; \mathbb{R}^{N \times N})$ and $\mathcal{DM}_{\text{loc}}^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ the corresponding local spaces.

Let $\mathbf{A} \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^N)$, and let $\Theta_{\mathbf{A}}$ be the jump set of the measure $|\operatorname{div} \mathbf{A}|$, i.e.

$$\Theta_{\mathbf{A}} := \left\{ x \in \Omega : \limsup_{r \rightarrow 0^+} \frac{|\operatorname{div} \mathbf{A}|(B_r(x))}{r^{N-1}} > 0 \right\};$$

it is a Borel set, σ -finite with respect to $\mathcal{H}^{N-1}$. Moreover, the following decomposition for the measure $\operatorname{div} \mathbf{A}$ holds:

$$\operatorname{div} \mathbf{A} = \operatorname{div}^a \mathbf{A} + \operatorname{div}^c \mathbf{A} + \operatorname{div}^j \mathbf{A},$$

where $\operatorname{div}^a \mathbf{A} \ll \mathcal{L}^N$, $\operatorname{div}^c \mathbf{A}(B) = 0$ for every set $B \subset \Omega$ with $\mathcal{H}^{N-1}(B) < +\infty$, $\operatorname{div}^j \mathbf{A} \ll \mathcal{H}^{N-1} \llcorner \Theta_{\mathbf{A}}$. Finally, $\operatorname{div} \mathbf{A}(B) = 0$ for every Borel set $B \subset \Omega$ with $\mathcal{H}^{N-1}(B) = 0$.

Now, we turn to matrix-valued fields. We recall that, for every Borel set B , the estimate

$$(2.21) \quad |\operatorname{Div} \mathbf{A}|(B) \leq \sum_{i=1}^N |\operatorname{div} \mathbf{A}_i|(B)$$

holds (see, e.g., [3, p. 4]).

The following result shows that the divergence measure of $\mathbf{A}$ is absolutely continuous with respect to $\mathcal{H}^{N-1}$, see [25, Proposition 2.5].

Proposition 2.9. *Let $\mathbf{A} \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^{N \times N})$, and let $B \subset \Omega$ be a Borel measurable set such that $\mathcal{H}^{N-1}(B) = 0$. Then $|\operatorname{Div} \mathbf{A}|(B) = 0$.*

As a consequence, the set

$$\Theta_{\mathbf{A}} := \left\{ x \in \Omega : \limsup_{r \rightarrow 0^+} \frac{|\operatorname{Div} \mathbf{A}|(B_r(x))}{r^{N-1}} > 0 \right\},$$

is a Borel set, σ -finite with respect to $\mathcal{H}^{N-1}$. Moreover, the measure $\operatorname{Div} \mathbf{A}$ admits the decomposition

$$\operatorname{Div} \mathbf{A} = \operatorname{Div}^a \mathbf{A} + \operatorname{Div}^c \mathbf{A} + \operatorname{Div}^j \mathbf{A},$$

where $\operatorname{Div}^a \mathbf{A} \ll \mathcal{L}^N$, $\operatorname{Div}^c \mathbf{A}(B) = 0$ for every Borel set B with $\mathcal{H}^{N-1}(B) < +\infty$, and $\operatorname{Div}^j \mathbf{A} \ll \mathcal{H}^{N-1} \llcorner \Theta_{\mathbf{A}}$.

The following lemma provides an approximation result for stress fields.

Lemma 2.10. *Given $\mathbf{A} \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^{N \times N})$, there exists $\{\mathbf{A}_k\} \subset C^\infty(\Omega; \mathbb{R}^{N \times N})$ such that*

- (i) $\mathbf{A}_k \rightarrow \mathbf{A}$ in $L^1(\Omega; \mathbb{R}^{N \times N})$;
- (ii) $\|\mathbf{A}_k\|_{L^\infty(\Omega; \mathbb{R}^{N \times N})} \leq 3\|\mathbf{A}\|_{L^\infty(\Omega; \mathbb{R}^{N \times N})}$ for every k ;
- (iii) $\sup_{k \in \mathbb{N}} \int_{\Omega} |\operatorname{Div} \mathbf{A}_k| dx < +\infty$;
- (iv) $\operatorname{Div} \mathbf{A}_k \rightharpoonup^* \operatorname{Div} \mathbf{A}$ as Radon measures on Ω .

Proof. Applying the approximation theorem of Chen-Frid [12, Theorem 1.2] to each column $\mathbf{A}_j$ of $\mathbf{A}$, $j = 1, \dots, N$, we find $\mathbf{A}_{j,k} \in C^\infty(\Omega; \mathbb{R}^N)$ such that $\mathbf{A}_{j,k} \rightarrow \mathbf{A}_j$ in L^1 and

$\int_{\Omega} |\operatorname{div} \mathbf{A}_{j,k}| dx \rightarrow |\operatorname{div} \mathbf{A}_j|(\Omega)$ for each j . Define $\mathbf{A}_k := (\mathbf{A}_{1,k}, \dots, \mathbf{A}_{N,k})$. Then (i) and (ii) follow immediately, while (iii) is a consequence of (2.21) and the fact that

$$\sup_{k \in \mathbb{N}} \int_{\Omega} |\operatorname{div} \mathbf{A}_{j,k}| dx < +\infty.$$

For what concerns (iv), from (i) we have

$$\lim_{k \rightarrow +\infty} \int_{\Omega} \mathbf{A}_k : \nabla \varphi dx = \int_{\Omega} \mathbf{A} : \nabla \varphi dx \quad \text{for every } \varphi \in C_c^1(\Omega; \mathbb{R}^N),$$

so that the assertion follows from a standard density argument of $C_c^1(\Omega; \mathbb{R}^N)$ in $C_c(\Omega; \mathbb{R}^N)$ with respect to the $L^\infty(\Omega; \mathbb{R}^N)$ norm, and the bound (iii). $\square$

Normal traces. We first recall the definition of traces of the normal component of a vector field $\mathbf{C} \in \mathcal{DM}^\infty(\Omega; \mathbb{R})$, which can be defined as distributions $\operatorname{Tr}^\pm(\mathbf{C}, \Sigma)$ on every oriented countably $\mathcal{H}^{N-1}$ -rectifiable set $\Sigma \subset \Omega$ (see [2, 5, 12]). According to [2], given a domain $\Omega' \Subset \Omega$ of class C^1 , the trace of the normal component of $\mathbf{C}$ on $\partial\Omega'$ is defined in the distributional sense by

$$(2.22) \quad \langle \operatorname{Tr}(\mathbf{C}, \partial\Omega'), \varphi \rangle := \int_{\Omega'} \mathbf{C} \cdot \nabla \varphi dx + \int_{\Omega'} \varphi d \operatorname{div} \mathbf{C}, \quad \forall \varphi \in C_c^\infty(\Omega).$$

It is shown in [2, Proposition 3.2] that this distribution is in fact induced by an L^∞ function on $\partial\Omega'$, which we continue to denote by $\operatorname{Tr}(\mathbf{C}, \partial\Omega')$, i.e.,

$$(2.23) \quad \langle \operatorname{Tr}(\mathbf{C}, \partial\Omega'), \varphi \rangle = \int_{\partial\Omega'} \varphi \operatorname{Tr}(\mathbf{C}, \partial\Omega') d\mathcal{H}^{N-1},$$

and satisfies the estimate

$$\|\operatorname{Tr}(\mathbf{C}, \partial\Omega')\|_{L^\infty(\partial\Omega')} \leq \|\mathbf{C}\|_{L^\infty(\Omega'; \mathbb{R})}.$$

Now, since Σ is oriented and countably $\mathcal{H}^{N-1}$ -rectifiable, we can find countably many oriented C^1 hypersurfaces Σ_i , with classical normal vectors ν_{Σ_i} , and pairwise disjoint Borel sets $E_i \subseteq \Sigma_i$ such that $\mathcal{H}^{N-1}(\Sigma \setminus \bigcup_i E_i) = 0$. Moreover, we may assume, without loss of generality, that for each i there exist two open, bounded sets Ω_i, Ω'_i with C^1 boundaries and exterior normal vectors ν_{Ω_i} and $\nu_{\Omega'_i}$ respectively, such that $E_i \subseteq \partial\Omega_i \cap \partial\Omega'_i$ and

$$\nu_{\Sigma_i}(x) = \nu_{\Omega_i}(x) = -\nu_{\Omega'_i}(x) \quad \forall x \in E_i.$$

We then define the orientation on Σ by setting $\nu_\Sigma(x) := \nu_{\Sigma_i}(x)$ for $\mathcal{H}^{N-1}$ -a.e. $x \in E_i$. Using the localization property established in [2, Proposition 3.2], the normal traces of $\mathbf{C}$ on Σ are defined by

$$(2.24) \quad \operatorname{Tr}^-(\mathbf{C}, \Sigma) := \operatorname{Tr}(\mathbf{C}, \partial\Omega_i), \quad \operatorname{Tr}^+(\mathbf{C}, \Sigma) := -\operatorname{Tr}(\mathbf{C}, \partial\Omega'_i), \quad \mathcal{H}^{N-1} - \text{a.e. on } E_i.$$

These two normal traces belong to $L^\infty(\Sigma, \mathcal{H}^{N-1} \llcorner \Sigma)$ (see [2, Proposition 3.2]) and satisfy

$$(2.25) \quad \operatorname{div} \mathbf{C} \llcorner \Sigma = [\operatorname{Tr}^+(\mathbf{C}, \Sigma) - \operatorname{Tr}^-(\mathbf{C}, \Sigma)] \mathcal{H}^{N-1} \llcorner \Sigma.$$

For our purposes, we need to extend definition (2.22) to a matrix-valued field. Given a domain Ω' as above, we define the trace of the normal component of $\mathbf{A} \in \operatorname{DM}^\infty(\Omega; \mathbb{R}^{N \times N})$ on $\partial\Omega'$ in the distributional sense as the N -tuple of scalar distributions

$$\operatorname{Tr}(\mathbf{A}, \partial\Omega') := \left(\operatorname{Tr}(\mathbf{A}_1, \partial\Omega'), \operatorname{Tr}(\mathbf{A}_2, \partial\Omega'), \dots, \operatorname{Tr}(\mathbf{A}_N, \partial\Omega') \right).$$

Then, recalling (2.22), we have

$$\begin{aligned}
 \langle \mathbf{Tr}(\mathbf{A}, \partial\Omega'), \varphi \rangle &= \sum_{i=1}^N \langle \mathbf{Tr}(\mathbf{A}_i, \partial\Omega'), \varphi_i \rangle \\
 &= \int_{\Omega'} \mathbf{A} : \nabla \varphi \, dx + \int_{\Omega'} \varphi \cdot d \operatorname{Div} \mathbf{A}, \quad \forall \varphi \in C_c^\infty(\Omega; \mathbb{R}^N).
 \end{aligned}
 \tag{2.26}$$

Therefore, this distribution is induced by a vector-valued L^∞ function on $\partial\Omega'$ (which we still denote by $\mathbf{Tr}(\mathbf{A}, \partial\Omega')$), the estimate

$$\| \mathbf{Tr}(\mathbf{A}, \partial\Omega') \|_{L^\infty(\partial\Omega'; \mathbb{R}^N)} \leq \| \mathbf{A} \|_{L^\infty(\Omega'; \mathbb{R}^{N \times N})}$$

holds, together with the localization property

$$\mathbf{Tr}(\mathbf{A}, \partial\Omega_1) = \mathbf{Tr}(\mathbf{A}, \partial\Omega_2) \quad \mathcal{H}^{N-1} - \text{a.e. on } E,$$

if E is a Borel subset of $\partial\Omega_1 \cap \partial\Omega_2$ and $\nu_{\Omega_1} = \nu_{\Omega_2}$ on E . This allows us to define the normal traces of $\mathbf{A}$ on Σ by

$$\mathbf{Tr}^-(\mathbf{A}, \Sigma) := \mathbf{Tr}(\mathbf{A}, \partial\Omega_i), \quad \mathbf{Tr}^+(\mathbf{A}, \Sigma) := -\mathbf{Tr}(\mathbf{A}, \partial\Omega'_i), \quad \mathcal{H}^{N-1} - \text{a.e. on } E_i,$$

thus extending [2, Definition 3.3] to a matrix-valued field. These two normal traces belong to $L^\infty_{\mathcal{H}^{N-1} \llcorner \Sigma}(\Sigma; \mathbb{R}^N)$ and, arguing as for [2, Proposition 3.4(ii)] with minor changes, we have

$$\operatorname{Div}^j \mathbf{A} \llcorner \Sigma = [\mathbf{Tr}^+(\mathbf{A}, \Sigma) - \mathbf{Tr}^-(\mathbf{A}, \Sigma)] \mathcal{H}^{N-1} \llcorner \Sigma.$$

A brief review for the pairing. In [19] (see also [5, 12]), the pairing between a vector field $\mathbf{A} \in \mathcal{DM}^\infty(\Omega)$ and the measure gradient of a scalar function $u \in BV(\Omega)$ is defined as the linear functional $(\mathbf{A}, Du) : C_c^\infty(\Omega) \rightarrow \mathbb{R}$ given by

$$\langle (\mathbf{A}, Du), \varphi \rangle := - \int_{\Omega} u^* \varphi \, d \operatorname{div} \mathbf{A} - \int_{\Omega} u \mathbf{A} \cdot \nabla \varphi \, dx,$$

under the assumption $u^* \in L^1(\Omega, |\operatorname{div} \mathbf{A}|)$.

Remark 2.11. We observe that if $|\operatorname{div} \mathbf{A}| \in BV(\Omega)^*$, then for every $u \in BV(\Omega)$ we have $u^* \in L^1(\Omega, |\operatorname{div} \mathbf{A}|)$ (see [16, Lemma 4.7]).

The distribution $(\mathbf{A}, Du)$ is a Radon measure in Ω , absolutely continuous with respect to $|Du|$ (see [19, Theorem 4.12]), hence the identity

$$\operatorname{div}(u\mathbf{A}) = u^* \operatorname{div} \mathbf{A} + (\mathbf{A}, Du)$$

holds in the sense of measures in Ω .

The following representation result can be found in [19, Theorem 4.12 and Remark 3.4] and [14, Theorem 3.5].

Theorem 2.12. *Let $\mathbf{A} \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^N)$ and $u \in BV_{\operatorname{loc}}(\Omega)$ with $u^* \in L^1_{\operatorname{loc}}(\Omega, |\operatorname{div} \mathbf{A}|)$. Then the measure $(\mathbf{A}, Du)$ admits the following decomposition:*

- (i) *absolutely continuous part:* $(\mathbf{A}, Du)^a = \mathbf{A} \cdot \nabla u \mathcal{L}^N$;
- (ii) *jump part:* $(\mathbf{A}, Du)^j = \frac{\operatorname{Tr}^+(\mathbf{A}, J_u) + \operatorname{Tr}^-(\mathbf{A}, J_u)}{2} (u^+ - u^-) \mathcal{H}^{N-1} \llcorner J_u$;
- (iii) *Cantor part:*

$$(\mathbf{A}, Du)^c \llcorner (\Omega \setminus S_{\mathbf{A}}) = \tilde{\mathbf{A}} \cdot D^c u \llcorner (\Omega \setminus S_{\mathbf{A}}),$$

where $S_{\mathbf{A}}$ is the approximate discontinuity set of $\mathbf{A}$.

In particular, if

$$(2.31) \quad |D^c u|(S_{\mathbf{A}}) = 0,$$

then

$$(\mathbf{A}, Du)^c = \tilde{\mathbf{A}} \cdot D^c u, \quad \text{in } \Omega.$$

Moreover, if $N = 1$, the above assumption is not needed. Finally, if $\mathbf{A} \in BV(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$, (and so (2.31) is satisfied), then

$$(\mathbf{A}, Du) = \mathbf{A}^* \cdot Du,$$

in the sense of measures.

Remark 2.13. Assumption (2.31) excludes the possibility that the Cantor part of the measure Du charges the essential discontinuity set of the field, precisely the situation in which no canonical pointwise representative of $\mathbf{A}$ is available. We also point out that this assumption is automatically satisfied in the one-dimensional setting. Indeed, if $N = 1$ and $a \in \mathcal{DM}^\infty(\Omega)$, then $a \in BV(\Omega) \cap L^\infty(\Omega)$, and the set S_a reduces to the jump set J_a , which is at most countable. Since the Cantor part $D^c u$ of the derivative of a one-dimensional BV function is a non-atomic measure, it does not charge countable sets, and therefore (2.31) holds. Consequently, if $N = 1$ the identity

$$(a, Du)^c = \tilde{a} D^c u$$

holds without any additional assumption.

The following Gauss–Green theorem is proved in [19, Theorem 5.1]

Theorem 2.14. *Let $\mathbf{A} \in \mathcal{DM}_{\text{loc}}^\infty(\mathbb{R}^N)$, $u \in BV_{\text{loc}}(\mathbb{R}^N)$, and assume that $u^* \in L_{\text{loc}}^1(\mathbb{R}^N, |\operatorname{div} \mathbf{A}|)$. Let $E \subset \mathbb{R}^N$ be a bounded set of finite perimeter. Then the following Gauss–Green formulas hold:*

$$\begin{aligned} \int_{E^1} u^* d\operatorname{div} \mathbf{A} + \int_{E^1} (\mathbf{A}, Du) &= - \int_{\partial^* E} \operatorname{Tr}^+(\mathbf{A}, \partial^* E) u^+ d\mathcal{H}^{N-1}, \\ \int_{E^1 \cup \partial^* E} u^* d\operatorname{div} \mathbf{A} + \int_{E^1 \cup \partial^* E} (\mathbf{A}, Du) &= - \int_{\partial^* E} \operatorname{Tr}^-(\mathbf{A}, \partial^* E) u^- d\mathcal{H}^{N-1}, \end{aligned}$$

where E^1 denotes the measure-theoretic interior of E , and $\operatorname{Tr}^\pm(\mathbf{A}, \partial^* E)$ are the normal traces of $\mathbf{A}$ when $\partial^* E$ is oriented with respect to the interior unit normal vector.

In particular for $N = 1$ we have the following corollary.

Corollary 2.15. *Let $a, u \in BV_{\text{loc}}(\mathbb{R})$ and assume that $u^* \in L_{\text{loc}}^1(\mathbb{R}, |Da|)$. Let $F \subset \mathbb{R}$ be a bounded set of finite perimeter (i.e., a finite union of bounded intervals). Then the following Gauss–Green formulas hold:*

$$\begin{aligned} \int_{F^1} u^* dDa + \int_{F^1} (a, u') &= - \sum_{s \in \partial^* F} a^+(s) \nu(s) u^+(s), \\ \int_{F^1 \cup \partial^* F} u^* dDa + \int_{F^1 \cup \partial^* F} (a, u') &= \sum_{s \in \partial^* F} a^-(s) \nu(s) u^-(s). \end{aligned}$$

In this case, $\operatorname{Tr}^+(a, \partial^* F) = a^+(s) \nu(s)$ and $\operatorname{Tr}^-(a, \partial^* F) = -a^-(s) \nu(s)$ are the normal traces of a and for every $s \in \partial^* F$, $\nu(s)$ is the interior unit normal.

We recall the definition of pairing between a matrix-valued field $\mathbf{A}$ and the matrix derivative of a function in $BV(\Omega; \mathbb{R}^N)$, recently introduced in [25].

Let $\mathbf{A} \in \text{DM}^\infty(\Omega; \mathbb{R}^{N \times N})$ and $\mathbf{u} \in BV(\Omega; \mathbb{R}^N)$ be such that $\mathbf{u}^* \in L^1_{|\text{Div } \mathbf{A}|}(\Omega; \mathbb{R}^N)$. Then the linear functional $(\mathbf{A} : D\mathbf{u}) : C_c^\infty(\Omega) \rightarrow \mathbb{R}$ defined by

$$(2.32) \quad \langle (\mathbf{A} : D\mathbf{u}), \varphi \rangle := - \int_{\Omega} \varphi \mathbf{u}^* \cdot d\text{Div } \mathbf{A} - \int_{\Omega} \mathbf{A} : [\mathbf{u} \otimes \nabla \varphi] dx$$

is well-defined. Moreover, it is straightforward to check that

$$(2.33) \quad (\mathbf{A} : D\mathbf{u}) = \sum_{j=1}^N (\mathbf{A}_j, Du_j),$$

where for every $j \in \{1, \dots, N\}$ $(\mathbf{A}_j, Du_j)$ is the pairing (2.29) between the j -th column $\mathbf{A}_j$ of $\mathbf{A}$ and the j -th component u_j of $\mathbf{u}$. Note that each scalar pairing $(\mathbf{A}_j, Du_j)$ on the right-hand side is well defined, since the assumption $\mathbf{u}^* \in L^1_{|\text{Div } \mathbf{A}|}(\Omega; \mathbb{R}^N)$ implies $u_j^* \in L^1(\Omega; |\text{div } \mathbf{A}_j|)$ for every $j \in \{1, \dots, N\}$.

Remark 2.16. We notice that if $|\text{div } \mathbf{A}_j| \in BV(\Omega)^*$ for every $j \in \{1, \dots, N\}$, then for every $\mathbf{u} \in BV(\Omega; \mathbb{R}^N)$ by [16, Lemma 4.7] we have $u_j^* \in L^1(\Omega, |\text{div } \mathbf{A}_j|)$ and so by (2.21)

$$\left| \int_{\Omega} \mathbf{u}^* \cdot d\text{Div } \mathbf{A} \right| \leq \sum_{j=1}^N \int_{\Omega} |u_j^*| d|\text{div } \mathbf{A}_j| < +\infty.$$

As an immediate consequence of the representation (2.33), the pairing (2.32) inherits the main properties of the scalar pairing (2.29). These properties are summarized in the following theorem (see [25, Theorem 3.2]).

Theorem 2.17. *Let $\mathbf{A} \in \text{DM}^\infty(\Omega; \mathbb{R}^{N \times N})$ and $\mathbf{u} \in BV(\Omega; \mathbb{R}^N)$ with $\mathbf{u}^* \in L^1_{|\text{Div } \mathbf{A}|}(\Omega; \mathbb{R}^N)$. Then, the distribution $(\mathbf{A} : D\mathbf{u})$ defined in (2.32) is a Radon measure in Ω , absolutely continuous with respect to $|D\mathbf{u}|$, and it satisfies for every open set $U \Subset \Omega$*

$$|(\mathbf{A} : D\mathbf{u})|(B) \leq \|\mathbf{A}\|_{L^\infty(U)} |D\mathbf{u}|(B) \quad \text{for every Borel set } B \subset U.$$

Moreover, $\text{div}(\mathbf{A}^T \mathbf{u}) \in \mathcal{M}(\Omega; \mathbb{R}^N)$ and the identity

$$\text{div}(\mathbf{A}^T \mathbf{u}) = \mathbf{u}^* \cdot \text{Div } \mathbf{A} + (\mathbf{A} : D\mathbf{u})$$

holds, in the sense of measures, in Ω .

The following is the main decomposition theorem for the pairing measure, see [25, Theorem 3.7].

Theorem 2.18. *Let $\mathbf{A} \in \text{DM}^\infty(\Omega; \mathbb{R}^{N \times N})$ and $\mathbf{u} \in BV(\Omega; \mathbb{R}^N)$ with $\mathbf{u}^* \in L^1_{|\text{Div } \mathbf{A}|}(\Omega; \mathbb{R}^N)$. Then the measure $(\mathbf{A} : D\mathbf{u})$ admits the following decomposition:*

- (i) *absolutely continuous part:* $(\mathbf{A} : D\mathbf{u})^a = \mathbf{A} : \nabla \mathbf{u} \mathcal{L}^N$;
- (ii) *jump part:* $(\mathbf{A} : D\mathbf{u})^j = \frac{\text{Tr}^+(\mathbf{A}, J_{\mathbf{u}}) + \text{Tr}^-(\mathbf{A}, J_{\mathbf{u}})}{2} \cdot (\mathbf{u}^+ - \mathbf{u}^-) \mathcal{H}^{N-1} \llcorner J_{\mathbf{u}}$.
- (iii) *Cantor part:* $(\mathbf{A} : D\mathbf{u})^c \llcorner (\Omega \setminus S_{\mathbf{A}}) = \tilde{\mathbf{A}} : D^c \mathbf{u} \llcorner (\Omega \setminus S_{\mathbf{A}})$, where $S_{\mathbf{A}}$ is the approximate discontinuity set of $\mathbf{A}$. In particular, if

$$(2.34) \quad |D^c \mathbf{u}|(S_{\mathbf{A}}) = 0,$$

then

$$(\mathbf{A} : D\mathbf{u})^c = \tilde{\mathbf{A}} : D^c \mathbf{u}, \quad \text{in } \Omega.$$

3. SLICING PROPERTIES OF VECTOR FIELDS WITH MEASURE-VALUED DIAGONAL DERIVATIVES

In this section we introduce the class of essentially bounded vector fields with measure-valued diagonal derivatives, namely vector fields whose diagonal components of the distributional Jacobian are Radon measures. We investigate their slicing properties and show that the measure-valued diagonal derivatives admit a one-dimensional representation in terms of the distributional derivatives of the corresponding slices. This yields a slicing formula for the divergence measure of the vector field and, as a consequence, a representation formula for the normal trace on oriented rectifiable sets.

Let $\mathbf{A} \in L^1(\Omega; \mathbb{R}^N)$, and assume that

$$(H) \quad D_i A^i \in \mathcal{M}(\Omega) \text{ for every } i \in \{1, \dots, N\}.$$

In this case we have

$$\operatorname{div} \mathbf{A} = \sum_{i=1}^N D_i A^i \in \mathcal{M}(\Omega).$$

Remark 3.1. By Proposition 2.2, applied with $v = A^i$ and $\boldsymbol{\xi} = \mathbf{e}_i$, condition (H) is equivalent to the following slicing characterization: for every $i \in \{1, \dots, N\}$

$$(\tilde{H}) \quad \begin{cases} (A^i)_y^{\mathbf{e}_i} \in BV(\Omega_y^{\mathbf{e}_i}) \text{ for } \mathcal{L}^{N-1}\text{-a.e. } y \in \Omega^{\mathbf{e}_i}, \\ \int_{\Omega^{\mathbf{e}_i}} |D(A^i)_y^{\mathbf{e}_i}|(\Omega_y^{\mathbf{e}_i}) d\mathcal{L}^{N-1}(y) < +\infty. \end{cases}$$

This implies that $(A^i)_y^{\mathbf{e}_i} \in L_{\text{loc}}^\infty(\Omega_y^{\mathbf{e}_i})$ for $\mathcal{L}^{N-1}$ -a.e. $y \in \Omega^{\mathbf{e}_i}$.

We introduce the class of essentially bounded vector fields satisfying (H):

$$(3.1) \quad BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^N) := \{\mathbf{A} \in L^\infty(\Omega; \mathbb{R}^N) : D_i A^i \in \mathcal{M}(\Omega) \text{ for every } i \in \{1, \dots, N\}\}.$$

The notation “diag” emphasizes that only the diagonal entries of the distributional Jacobian are required to be Radon measures, while no information is assumed on the off-diagonal derivatives $D_j A^i$, $i \neq j$.

It is immediate to check that

$$BV(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N) \subset BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^N) \subset \mathcal{DM}^\infty(\Omega; \mathbb{R}^N).$$

Remark 3.2. The second inclusion above is, in general, strict. For instance, let $\Omega := (-1, 1)^2$ and $\mathbf{A} = \mathbf{A}(x_1, x_2) := (f(x_1 - x_2), f(x_1 - x_2))$, where

$$f(s) := \begin{cases} \sin(\frac{1}{s}), & \text{if } s \neq 0, \\ 0, & \text{if } s = 0. \end{cases}$$

Then, $\operatorname{div} \mathbf{A} = 0 \in \mathcal{M}(\Omega)$, whereas $D_1 A^1 \notin \mathcal{M}(\Omega)$ and $D_2 A^2 \notin \mathcal{M}(\Omega)$. Indeed, $f \notin BV(-1, 1)$. Moreover, for every $x_2 \in (-1, 1)$, $(A^1)_{x_2}^{\mathbf{e}_1}(t) = f(t - x_2) = \sin(\frac{1}{t - x_2})$ for $t \neq x_2$, hence $(A^1)_{x_2}^{\mathbf{e}_1} \notin BV(-1, 1)$. Therefore, the first condition in $(\tilde{H})$ fails.

We first prove the following representation formula for the divergence measure.

Proposition 3.3. *Let $\mathbf{A} \in BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^N)$. Then for every $\varphi \in C_c(\Omega)$ we have*

$$(3.2) \quad \int_{\Omega} \varphi d \operatorname{div} \mathbf{A} = \sum_{i=1}^N \int_{\Omega^{\mathbf{e}_i}} \left(\int_{\Omega_y^{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) d D((A^i)_y^{\mathbf{e}_i})(s) \right) d \mathcal{L}^{N-1}(y);$$

i.e.,

$$(3.3) \quad \operatorname{div} \mathbf{A} = \sum_{i=1}^N \mathcal{L}^{N-1} \llcorner \Omega^{\mathbf{e}_i} \otimes_{\mathcal{M}} D((A^i)_y^{\mathbf{e}_i}) \llcorner \Omega_y^{\mathbf{e}_i}.$$

as measures on Ω .

Proof. For every $i = 1, \dots, N$, by assumption (H) and (2.10), applied with $v = A^i$ and $\xi = \mathbf{e}_i$, we get

$$D_i A^i = \mathcal{L}^{N-1} \llcorner \Omega^{\mathbf{e}_i} \otimes_{\mathcal{M}} D((A^i)_y^{\mathbf{e}_i}) \llcorner \Omega_y^{\mathbf{e}_i}$$

as measures on Ω . Summing over $i = 1, \dots, N$, and recalling that $\operatorname{div} \mathbf{A} = \sum_{i=1}^N D_i A^i$, we deduce (3.3). Formula (3.2) follows by testing against $\varphi \in C_c(\Omega)$. $\square$

We next derive a slicing formula for the normal trace of fields in $BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^N)$.

Proposition 3.4. *Let $\mathbf{A} \in BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^N)$. Let $\Omega' \Subset \Omega$ be an open set with C^1 boundary. Then for every test function $\varphi \in C_c(\Omega)$*

$$(3.4) \quad \int_{\partial\Omega'} \varphi \operatorname{Tr}(\mathbf{A}, \partial\Omega') d\mathcal{H}^{N-1} = - \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\sum_{s \in \partial(\Omega')_y^{\mathbf{e}_i}} ((A^i)_y^{\mathbf{e}_i})^+(s) \nu_{(\Omega')_y^{\mathbf{e}_i}}(s) \varphi_y^{\mathbf{e}_i}(s) \right) d\mathcal{L}^{N-1}(y),$$

where $\nu_{(\Omega')_y^{\mathbf{e}_i}} \in \{-1, 1\}$ is the normal of $\partial(\Omega')_y^{\mathbf{e}_i}$ oriented with respect to the interior unit normal vector and satisfies the relation

$$(3.5) \quad \nu_{(\Omega')_y^{\mathbf{e}_i}}(s) = -\operatorname{sign}(\boldsymbol{\nu}_{\Omega'}(y + s\mathbf{e}_i) \cdot \mathbf{e}_i).$$

The sign convention for a one-dimensional slice interval $(a, b) \subset (\Omega')_y^{\mathbf{e}_i}$ is the following: the interior normal is 1 at a and -1 at b .

Proof. First, we assume that $\varphi \in C_c^1(\Omega)$. In order to prove (3.4), we recall that by (2.22) and (2.23) we have

$$(3.6) \quad \int_{\partial\Omega'} \varphi \operatorname{Tr}(\mathbf{A}, \partial\Omega') d\mathcal{H}^{N-1} = \int_{\Omega'} \mathbf{A} \cdot \nabla \varphi dx + \int_{\Omega'} \varphi d \operatorname{div} \mathbf{A}.$$

For every fixed $i \in \{1, 2, \dots, N\}$ and for $\mathcal{L}^{N-1}$ -a.e. $y \in \mathbf{e}_i^\perp$, integrating by parts (see [21, Theorem 6.3]) we have

$$\begin{aligned} \int_{(\Omega')_y^{\mathbf{e}_i}} (\varphi_y^{\mathbf{e}_i})'(s) (A^i)_y^{\mathbf{e}_i}(s) ds &= - \int_{(\Omega')_y^{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) dD((A^i)_y^{\mathbf{e}_i})(s) \\ &\quad - \sum_{s \in \partial(\Omega')_y^{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) ((A^i)_y^{\mathbf{e}_i})^+(s) \nu_{(\Omega')_y^{\mathbf{e}_i}}(s). \end{aligned}$$

Furthermore, by Fubini's Theorem,

$$(3.7) \quad \int_{\Omega'} A^i \partial_i \varphi dx = \int_{\mathbf{e}_i^\perp} \left(\int_{(\Omega')_y^{\mathbf{e}_i}} (A^i)_y^{\mathbf{e}_i}(s) (\varphi_y^{\mathbf{e}_i})'(s) ds \right) d\mathcal{L}^{N-1}(y).$$

Now, integrating over $\mathbf{e}_i^\perp$ and summing up i , taking into account (3.7) and (3.2) we obtain

$$\int_{\Omega'} \mathbf{A} \cdot \nabla \varphi dx = - \int_{\Omega'} \varphi d \operatorname{div} \mathbf{A} - \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\sum_{s \in \partial(\Omega')_y^{\mathbf{e}_i}} ((A^i)_y^{\mathbf{e}_i})^+(s) \nu_{(\Omega')_y^{\mathbf{e}_i}}(s) \varphi_y^{\mathbf{e}_i}(s) \right) d\mathcal{L}^{N-1}(y),$$

whence, in view of (3.6), (3.4) follows. For a general $\varphi \in C_c(\Omega)$, using coarea formula [3, Theorem 2.93], we find

$$\begin{aligned}
 (3.8) \quad & \left| \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\sum_{s \in \partial(\Omega')_y^{\mathbf{e}_i}} ((A^i)_y^{\mathbf{e}_i})^+(s) \nu_{(\Omega')_y^{\mathbf{e}_i}}(s) \varphi_y^{\mathbf{e}_i}(s) \right) d\mathcal{L}^{N-1}(y) \right| \\
 & \leq \|\mathbf{A}\|_{L^\infty(\Omega; \mathbb{R}^N)} \|\varphi\|_{L^\infty(\Omega)} \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \#(\partial(\Omega')_y^{\mathbf{e}_i}) d\mathcal{L}^{N-1}(y) \\
 & = \|\mathbf{A}\|_{L^\infty(\Omega; \mathbb{R}^N)} \|\varphi\|_{L^\infty(\Omega)} \sum_{i=1}^N \int_{\partial\Omega'} |\nu_{\Omega'} \cdot \mathbf{e}_i| d\mathcal{H}^{N-1}(y) \\
 & \leq N \|\mathbf{A}\|_{L^\infty(\Omega; \mathbb{R}^N)} \mathcal{H}^{N-1}(\partial\Omega') \|\varphi\|_{L^\infty(\Omega)},
 \end{aligned}$$

so that the conclusion follows by a standard density argument. $\square$

Proposition 3.5. *Let $\mathbf{A} \in BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^N)$ and let $\Sigma \subset \Omega$ be an oriented countably $\mathcal{H}^{N-1}$ -rectifiable set with unit normal ν_Σ . Then for every test function $\varphi \in C_c(\Omega)$ we have*

$$(3.9) \quad \int_\Sigma \varphi \operatorname{Tr}^\pm(\mathbf{A}, \Sigma) d\mathcal{H}^{N-1} = \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\sum_{s \in \Sigma_y^{\mathbf{e}_i}} ((A^i)_y^{\mathbf{e}_i})^\pm(s) \nu_{\Sigma_y^{\mathbf{e}_i}}(s) \varphi_y^{\mathbf{e}_i}(s) \right) d\mathcal{L}^{N-1}(y),$$

where $\nu_{\Sigma_y^{\mathbf{e}_i}} \in \{-1, 1\}$ is the normal of $\Sigma_y^{\mathbf{e}_i}$ oriented with respect to ν_Σ and satisfies the relation

$$\nu_{\Sigma_y^{\mathbf{e}_i}}(s) = \operatorname{sign}(\nu_\Sigma(y + s\mathbf{e}_i) \cdot \mathbf{e}_i).$$

Proof. The assertion (3.9) follows by combining (3.4) and (2.24). $\square$

We now extend the previous results to matrix-valued fields. Since the divergence of a matrix field is defined columnwise, the relevant assumption is that each column satisfies the condition (H). More precisely, given $\mathbf{A} \in L^1(\Omega; \mathbb{R}^{N \times N})$, we assume that

$$(\text{H}') \quad D_i A_{ij} \in \mathcal{M}(\Omega) \text{ for every } i, j \in \{1, \dots, N\}.$$

In this case we have

$$(\operatorname{Div} \mathbf{A})_j = \sum_{i=1}^N D_i A_{ij} \in \mathcal{M}(\Omega).$$

Remark 3.6. Let $\mathbf{A} \in L^1(\Omega; \mathbb{R}^{N \times N})$. Then, by Proposition 2.2 applied with $v = A_{ij}$ and $\xi = \mathbf{e}_i$, condition (H') holds if and only if for every $i, j \in \{1, \dots, N\}$

$$(\tilde{\text{H}}') \quad \begin{cases} (A_{ij})_y^{\mathbf{e}_i} \in BV(\Omega_y^{\mathbf{e}_i}) \text{ for } \mathcal{L}^{N-1}\text{-a.e. } y \in \mathbf{e}_i^\perp, \\ \int_{\mathbf{e}_i^\perp} |D(A_{ij})_y^{\mathbf{e}_i}|(\Omega_y^{\mathbf{e}_i}) d\mathcal{L}^{N-1}(y) < +\infty. \end{cases}$$

This implies that $(A_{ij})_y^{\mathbf{e}_i} \in L_{\text{loc}}^\infty(\Omega_y^{\mathbf{e}_i})$ for $\mathcal{L}^{N-1}$ -a.e. $y \in \Omega^{\mathbf{e}_i}$.

We introduce the class

$$BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^{N \times N}) := \{ \mathbf{A} \in L^\infty(\Omega; \mathbb{R}^{N \times N}) : D_i A_{ij} \in \mathcal{M}(\Omega) \text{ for every } i, j \in \{1, \dots, N\} \}.$$

In view of definition (3.1), $\mathbf{A} \in BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^{N \times N})$ if and only if each column $\mathbf{A}_j \in BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^N)$, $j = 1, \dots, N$.

It is immediate to check that

$$BV(\Omega; \mathbb{R}^{N \times N}) \cap L^\infty(\Omega; \mathbb{R}^{N \times N}) \subset BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^{N \times N}) \subset DM^\infty(\Omega; \mathbb{R}^{N \times N}).$$

We conclude this section by noting that slicing formulas for the divergence and traces of matrix fields in $BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^{N \times N})$ can be directly obtained from the corresponding results for vector fields.

Proposition 3.7. *Let $\mathbf{A} \in BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^{N \times N})$. For every $\varphi \in C_c(\Omega; \mathbb{R}^N)$ we have that*

$$\int_{\Omega} \varphi \cdot d\text{Div } \mathbf{A} = \sum_{i,j=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{\Omega_y^{\mathbf{e}_i}} (\varphi_j)_y^{\mathbf{e}_i}(s) dD(A_{ij})_y^{\mathbf{e}_i}(s) \right) d\mathcal{L}^{N-1}(y).$$

Proof. It is sufficient to note that

$$\int_{\Omega} \varphi \cdot d\text{Div } \mathbf{A} = \sum_{j=1}^N \int_{\Omega} \varphi_j d\text{div } \mathbf{A}_j$$

and then use (3.2). $\square$

Proposition 3.8. *Let $\mathbf{A} \in BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^{N \times N})$. Then, for every test function $\varphi \in C_c(\Omega; \mathbb{R}^N)$ and every oriented countably $\mathcal{H}^{N-1}$ -rectifiable set $\Sigma \subset \Omega$,*

$$(3.10) \quad \langle \text{Tr}^\pm(\mathbf{A}, \Sigma), \varphi \rangle = \sum_{i,j=1}^N \int_{\mathbf{e}_i^\perp} \left(\sum_{s \in \Sigma_y^{\mathbf{e}_i}} ((A_{ij})_y^{\mathbf{e}_i})^\pm(s) \nu_{\Sigma_y^{\mathbf{e}_i}}(s) (\varphi_j)_y^{\mathbf{e}_i}(s) \right) d\mathcal{L}^{N-1}(y),$$

where $\nu_{\Sigma_y^{\mathbf{e}_i}} \in \{-1, 1\}$ is the normal of $\Sigma_y^{\mathbf{e}_i}$ oriented as

$$\nu_{\Sigma_y^{\mathbf{e}_i}}(s) = \text{sign}(\nu_\Sigma(y + s\mathbf{e}_i) \cdot \mathbf{e}_i).$$

Proof. Assertion (3.10) follows from (3.9), once we recall that, by definition,

$$\langle \text{Tr}^\pm(\mathbf{A}, \Sigma), \varphi \rangle = \sum_{j=1}^N \langle \text{Tr}^\pm(\mathbf{A}_j, \Sigma), \varphi_j \rangle.$$

$\square$

4. SLICING FORMULAS FOR PAIRINGS WITH BV FUNCTIONS

In this section we establish slicing formulas for the pairing measure between divergence-measure fields and BV functions. First, for vector fields belonging to $BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^N)$, we show that the pairing measure $(\mathbf{A}, Du)$ admits a representation in terms of the one-dimensional pairings associated with the corresponding slices.

Theorem 4.1. *Let $\mathbf{A} \in BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^N)$. If $u \in BV(\Omega)$ with $u^* \in L^1(\Omega, |\text{div } \mathbf{A}|)$ and (2.31) holds, then we have*

$$\int_{\Omega} \varphi d(\mathbf{A}, Du) = \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{\Omega_y^{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) d((A^i)_y^{\mathbf{e}_i}, Du_y^{\mathbf{e}_i})(s) \right) d\mathcal{L}^{N-1}(y),$$

for every $\varphi \in C_c(\Omega)$, i.e.

$$(\mathbf{A}, Du) = \sum_{i=1}^N (\mathcal{L}^{N-1} \llcorner \Omega^{\mathbf{e}_i}) \otimes_{\mathcal{M}} ((A^i)_y^{\mathbf{e}_i}, Du_y^{\mathbf{e}_i}) \llcorner (\Omega_y^{\mathbf{e}_i})$$

as measures in Ω .

Since the pairing measure admits the standard decomposition into absolutely continuous, Cantor, and jump parts, it suffices to establish the result separately for each component. We observe that the only point where the additional assumption (2.31) is required is the slicing formula for the Cantor part.

Proposition 4.2. *Let $\mathbf{A} \in BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^N)$. If $u \in BV(\Omega)$ with $u^* \in L^1(\Omega, |\operatorname{div} \mathbf{A}|)$, we have*

$$(4.1) \quad \int_{\Omega} \varphi d(\mathbf{A}, Du)^a = \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{\Omega_y^{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) ((A^i)_y^{\mathbf{e}_i}, Du_y^{\mathbf{e}_i})^a ds \right) d\mathcal{L}^{N-1}(y),$$

for every $\varphi \in C_c(\Omega)$.

Proof. From Theorem 2.12(i) and Fubini's Theorem we have

$$\begin{aligned} \int_{\Omega} \varphi d(\mathbf{A}, Du)^a &= \int_{\Omega} \varphi \mathbf{A} \cdot \nabla u \, dx = \sum_{i=1}^N \int_{\Omega} \varphi A^i \partial_i u \, dx \\ &= \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{\Omega_y^{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) (A^i)_y^{\mathbf{e}_i} (u_y^{\mathbf{e}_i})' ds \right) d\mathcal{L}^{N-1}(y), \end{aligned}$$

and (4.1) follows again from Theorem 2.12 (i) with $N = 1$. $\square$

Proposition 4.3. *Let $\mathbf{A} \in BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^N)$. If $u \in BV(\Omega)$ with $u^* \in L^1(\Omega, |\operatorname{div} \mathbf{A}|)$, then for every $\varphi \in C_c(\Omega)$ we have*

$$\int_{\Omega} \varphi d(\mathbf{A}, Du)^j = \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{J_{u_y}^{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) d((A^i)_y^{\mathbf{e}_i}, Du_y^{\mathbf{e}_i})^j(s) \right) d\mathcal{L}^{N-1}(y).$$

Proof. We subdivide the proof into steps.

Step 1. We first consider the case $u = \chi_F$, where $F \subset \Omega$ is a set of finite perimeter. Then $\chi_F \in BV(\Omega) \cap L^\infty(\Omega)$. Using Theorem 2.12(ii) with $u = \chi_F$, we get

$$(\mathbf{A}, D\chi_F)^j = \frac{\operatorname{Tr}^+(\mathbf{A}, \partial^* F) + \operatorname{Tr}^-(\mathbf{A}, \partial^* F)}{2} \mathcal{H}^{N-1} \llcorner \partial^* F,$$

since $J_u = \partial^* F$, $\chi_F^+ = 1$ and $\chi_F^- = 0$ $\mathcal{H}^{N-1}$ -a.e. in $\partial^* F$. Then, in view of (3.9) and using that $\partial^*(F_y^{\mathbf{e}_i}) = (\partial^* F)_y^{\mathbf{e}_i}$, we obtain

$$\begin{aligned} \int_{\Omega} \varphi d(\mathbf{A}, D\chi_F)^j &= \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\sum_{s \in (\partial^* F)_y^{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) \frac{((A^i)_y^{\mathbf{e}_i})^+(s) + ((A^i)_y^{\mathbf{e}_i})^-(s)}{2} \nu_{(\partial^* F)_y^{\mathbf{e}_i}}(s) \right) d\mathcal{L}^{N-1}(y) \\ &= \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\sum_{s \in \partial^*(F_y^{\mathbf{e}_i})} \varphi_y^{\mathbf{e}_i}(s) d((A^i)_y^{\mathbf{e}_i}, D\chi_{F_y^{\mathbf{e}_i}})^j(s) \right) d\mathcal{L}^{N-1}(y), \end{aligned}$$

where in the latter equality we invoked Theorem 2.12(ii) for $N = 1$.

Step 2. We now address the general case $u \in BV(\Omega)$ with $u^* \in L^1(\Omega, |\operatorname{div} \mathbf{A}|)$.

First, it is easy to verify that for $\mathcal{L}^1$ -a.e. $t \in \mathbb{R}$

$$(4.2) \quad \{u > t\}_y^{\mathbf{e}_i} = \{u_y^{\mathbf{e}_i} > t\} \quad \text{for } \mathcal{L}^{N-1}\text{-a.e. } y \in \mathbf{e}_i^\perp.$$

Now, setting $F_t := \{u > t\}$, by *Step 1*, coarea formula [19, Theorem 4.2] and (4.2) we obtain

$$\begin{aligned}
\int_{\Omega} \varphi d(\mathbf{A}, Du)^j &= \int_{J_u} \varphi d(\mathbf{A}, Du) = \int_{\mathbb{R}} \int_{\partial^* F_t} \varphi(\mathbf{A}, D\chi_{F_t}) dt \\
&= \int_{\mathbb{R}} \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{\partial^*(F_t)_{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) d((A^i)_y^{\mathbf{e}_i}, D\chi_{(F_t)_{\mathbf{e}_i}})^j(s) \right) d\mathcal{L}^{N-1}(y) dt \\
&= \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{\mathbb{R}} \int_{\partial^*(F_t)_{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) d((A^i)_y^{\mathbf{e}_i}, D\chi_{(F_t)_{\mathbf{e}_i}})^j(s) dt \right) d\mathcal{L}^{N-1}(y) \\
&= \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{\mathbb{R}} \int_{\partial^*\{u_y^{\mathbf{e}_i} > t\}} \varphi_y^{\mathbf{e}_i}(s) d((A^i)_y^{\mathbf{e}_i}, D\chi_{\{u_y^{\mathbf{e}_i} > t\}})^j(s) dt \right) d\mathcal{L}^{N-1}(y) \\
&= \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{J_{u_y^{\mathbf{e}_i}}} \varphi_y^{\mathbf{e}_i}(s) d((A^i)_y^{\mathbf{e}_i}, Du_y^{\mathbf{e}_i})^j(s) \right) d\mathcal{L}^{N-1}(y).
\end{aligned}$$

This concludes the proof. $\square$

As a final step, we derive a slicing formula for the Cantor part of the pairing measure.

Proposition 4.4. *Let $\mathbf{A} \in BV_{\text{diag}}^\infty(\Omega; \mathbb{R}^N)$. If $u \in BV(\Omega)$ with $u^* \in L^1(\Omega, |\operatorname{div} \mathbf{A}|)$ and (2.31) holds, then for every $\varphi \in C_c(\Omega)$ we have*

$$(4.3) \quad \int_{\Omega} \varphi d(\mathbf{A}, Du)^c = \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{\Omega_y^{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) d((A^i)_y^{\mathbf{e}_i}, Du_y^{\mathbf{e}_i})^c(s) \right) d\mathcal{L}^{N-1}(y).$$

Proof. We recall that, since (2.31) holds, one has $(\mathbf{A}, Du)^c = \tilde{\mathbf{A}} \cdot D^c u$ on Ω , as stated in Theorem 2.12(iii). Then, by [3, Theorem 3.108] for $D^c u \cdot \mathbf{e}_i$, we obtain

$$\begin{aligned}
\int_{\Omega} \varphi d(\mathbf{A}, Du)^c &= \int_{\Omega} \varphi \tilde{\mathbf{A}} \cdot dD^c u = \sum_{i=1}^N \int_{\Omega} \varphi(\tilde{\mathbf{A}} \cdot \mathbf{e}_i) d(D^c u \cdot \mathbf{e}_i) \\
&= \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{\Omega_y^{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) \widetilde{(A^i)_y^{\mathbf{e}_i}}(s) dD^c(u_y^{\mathbf{e}_i})(s) \right) d\mathcal{L}^{N-1}(y),
\end{aligned}$$

so that applying Theorem 2.12(iii) in dimension one and Remark 2.13, we conclude that

$$\begin{aligned}
&\sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{\Omega_y^{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) \widetilde{(A^i)_y^{\mathbf{e}_i}}(s) dD^c(u_y^{\mathbf{e}_i})(s) \right) d\mathcal{L}^{N-1}(y) \\
&= \sum_{i=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{\Omega_y^{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) d((A^i)_y^{\mathbf{e}_i}, Du_y^{\mathbf{e}_i})^c(s) \right) d\mathcal{L}^{N-1}(y).
\end{aligned}$$

This proves (4.3). $\square$

We now address the pairing $(\mathbf{A} : Du)$, defined in (2.32), and investigate whether an analogous slicing representation can be obtained. Formula (2.33) plays a fundamental role in this analysis, as it allows us to reduce the vectorial pairing to a finite sum of scalar pairings. Consequently, the desired slicing formula follows from Theorem 4.1, provided that each column of $\mathbf{A}$ satisfies assumption (H) stated above.

By applying Theorem 4.1 to every vector field $\mathbf{A}_j$ and using (2.33), we obtain the following result.

Theorem 4.5. *Let $\mathbf{A} \in \text{BV}_{\text{diag}}^\infty(\Omega; \mathbb{R}^{N \times N})$. If $\mathbf{u} \in \text{BV}(\Omega; \mathbb{R}^N)$ with $\mathbf{u}^* \in L^1_{|\text{Div } \mathbf{A}|}(\Omega; \mathbb{R}^N)$ and $|D^c \mathbf{u}|(S_{\mathbf{A}}) = 0$, then we have*

$$\int_{\Omega} \varphi d(\mathbf{A} : D\mathbf{u}) = \sum_{i,j=1}^N \int_{\mathbf{e}_i^\perp} \left(\int_{\Omega_y^{\mathbf{e}_i}} \varphi_y^{\mathbf{e}_i}(s) ((A_{ij})_y^{\mathbf{e}_i}, D(u_j)_y^{\mathbf{e}_i}) ds \right) d\mathcal{L}^{N-1}(y),$$

for every $\varphi \in C_c(\Omega)$.

Remark 4.6. We observe that, if $\mathbf{A} \in \text{BV}(\Omega; \mathbb{R}^{N \times N}) \cap L^\infty(\Omega; \mathbb{R}^{N \times N})$ (and so (2.34) is satisfied), then

$$(\mathbf{A} : D\mathbf{u}) = \mathbf{A}^* : D\mathbf{u},$$

in the sense of measures.

5. A PAIRING FOR TENSOR FIELDS AND BOUNDED BD FUNCTIONS

We start by introducing a pairing in BD for bounded functions. Given a symmetric matrix field $\mathbf{A} \in \text{DM}^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in \text{BD}(\Omega) \cap L^\infty(\Omega; \mathbb{R}^N)$, a natural definition would be the analogue of (2.32), where the term $\mathbf{A} : [\mathbf{u} \otimes \nabla \varphi]$ is replaced by $\mathbf{A} : [\mathbf{u} \odot \nabla \varphi]$ in view of the symmetry of $\mathbf{A}$. However, since it is not known whether $\mathcal{H}^{N-1}(S_{\mathbf{u}} \setminus J_{\mathbf{u}}) = 0$ for every $\mathbf{u} \in \text{BD}(\Omega)$, the term $\int_{\Omega} \varphi \mathbf{u}^* \cdot d\text{Div } \mathbf{A}$ may fail to be well-defined. This motivates the assumption

$$(H_*) \quad |\text{Div}^s \mathbf{A}|(S_{\mathbf{u}} \setminus J_{\mathbf{u}}) = 0.$$

Here, $\text{Div}^s \mathbf{A}$ denotes the singular part of the measure $\text{Div } \mathbf{A}$. Since $\mathcal{L}^N(S_{\mathbf{u}} \setminus J_{\mathbf{u}}) = 0$, condition (H_*) is equivalent to

$$|\text{Div } \mathbf{A}|(S_{\mathbf{u}} \setminus J_{\mathbf{u}}) = 0.$$

Remark 5.1. Assumption (H_*) can be interpreted as a natural compatibility condition between the singular part of the divergence of the stress tensor $\mathbf{A}$ and the fine structure of the displacement field $\mathbf{u}$. Indeed, a characteristic property of BD functions ensures that $|E\mathbf{u}|(S_{\mathbf{u}} \setminus J_{\mathbf{u}}) = 0$, so that the strain measure does not detect the non-jump part of the singular set of $\mathbf{u}$. In this perspective, condition (H_*) requires that the singular part of $\text{Div } \mathbf{A}$ does not charge the same set, namely the portion of $S_{\mathbf{u}}$ that is invisible to the strain. This ensures that concentrated forces are confined to the physically relevant jump set $J_{\mathbf{u}}$, where a well-defined notion of interfacial interaction is available, thereby guaranteeing a consistent coupling between kinematic and static singularities.

Assumption (H_*) is satisfied, for instance, in the following cases:

- (i) $\mathbf{u} \in \text{BV}(\Omega; \mathbb{R}^N)$, by Proposition 2.9, since $\mathcal{H}^{N-1}(S_{\mathbf{u}} \setminus J_{\mathbf{u}}) = 0$;
- (ii) $|\text{Div } \mathbf{A}| \ll \mathcal{L}^N$;
- (iii) $\mathbf{u} \in \text{SBD}^p(\Omega; \mathbb{R}^N)$ for some $p > 1$, since also in this case $\mathcal{H}^{N-1}(S_{\mathbf{u}} \setminus J_{\mathbf{u}}) = 0$ (see [18, Theorem 4.1]).

Under assumption (H_*) , the precise representative $\mathbf{u}^*$ of $\mathbf{u}$ (see (2.7)) is well defined $|\text{Div } \mathbf{A}|$ -a.e. in Ω .

This allows us to introduce the following definition. For every $\mathbf{A} \in \text{DM}^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in \text{BD}(\Omega) \cap L^\infty(\Omega; \mathbb{R}^N)$ satisfying (H_*) we define the linear functional $(\mathbf{A} : E\mathbf{u}) : C_c^\infty(\Omega) \rightarrow$

$\mathbb{R}$ by

$$(5.1) \quad \langle (\mathbf{A} : E\mathbf{u}), \varphi \rangle := - \int_{\Omega} \varphi \mathbf{u}^* \cdot d \operatorname{Div} \mathbf{A} - \int_{\Omega} \mathbf{A} : [\mathbf{u} \odot \nabla \varphi] dx.$$

The definition above is consistent with the classical notion of pairing introduced by Temam in the case of fields with summable divergence, as shown in the following remark.

Remark 5.2. If $\operatorname{Div} \mathbf{A} \ll \mathcal{L}^N$ we retrieve Temam's definition of pairing

$$(5.2) \quad \langle (\mathbf{A} : E\mathbf{u}), \varphi \rangle := - \int_{\Omega} \varphi \mathbf{u} \cdot \operatorname{Div} \mathbf{A} dx - \int_{\Omega} \mathbf{A} : [\mathbf{u} \odot \nabla \varphi] dx,$$

(see [30, Ch. II, Sec. 7.3] and [29]). Indeed, in this case $\mathbf{u}^* = \mathbf{u}$ $\mathcal{L}^N$ -a.e. and $d \operatorname{Div} \mathbf{A} = \operatorname{Div} \mathbf{A} dx$, so that (5.1) reduces to (5.2). Moreover, the distribution is well defined under the assumptions that Ω is a bounded Lipschitz subset of $\mathbb{R}^N$ and $\operatorname{Div} \mathbf{A} \in L^N(\Omega; \mathbb{R}^N)$, since $\mathbf{u} \in BD(\Omega) \subset L^{\frac{N}{N-1}}(\Omega; \mathbb{R}^N)$ (see [30, Ch. II, Sec. 2.4, Theorem 2.2]). Hence, by Hölder's inequality, $\mathbf{u} \in L^1_{|\operatorname{Div} \mathbf{A}|}(\Omega; \mathbb{R}^N)$. Temam proved that this distribution is a bounded Radon measure on Ω (see [29, Proposition 1.1] and [30, Lemma 7.3]).

Theorem 5.3. *Let $\mathbf{A} \in \operatorname{DM}^\infty(\Omega; \mathbb{R}_{\operatorname{sym}}^{N \times N})$ and $\mathbf{u} \in BD(\Omega) \cap L^\infty(\Omega; \mathbb{R}^N)$ satisfy (H_*) . Then the distribution $(\mathbf{A} : E\mathbf{u})$ defined in (5.1) is a Radon measure in Ω , absolutely continuous with respect to $|E\mathbf{u}|$, and it holds that*

$$(5.3) \quad |(\mathbf{A} : E\mathbf{u})|(B) \leq \|\mathbf{A}\|_{L^\infty(U)} |E\mathbf{u}|(B) \quad \text{for every Borel set } B \subset U \Subset \Omega.$$

Moreover, $\operatorname{div}(\mathbf{A}\mathbf{u}) \in \mathcal{M}(\Omega)$ and the following integration by parts formula holds

$$(5.4) \quad \operatorname{div}(\mathbf{A}\mathbf{u}) = \mathbf{u}^* \cdot \operatorname{Div} \mathbf{A} + (\mathbf{A} : E\mathbf{u})$$

in the sense of measures in Ω .

Proof. In this case, we follow the approach of [12, Theorem 3.1]. Let $(\mathbf{A}_k)$ be the sequence of Lemma 2.10, and let $(\mathbf{u}_k)$ be the sequence approximating $\mathbf{u}$ given in Theorem 2.3. Since

$$\left(\frac{\mathbf{A}_k + \mathbf{A}_k^T}{2} \mathbf{u}_k \right) \cdot \nabla \varphi = \mathbf{A}_k : e(\varphi \mathbf{u}_k) - \varphi \mathbf{A}_k : e(\mathbf{u}_k) \quad \text{a.e. in } \Omega, \text{ for every } \varphi \in C_c^1(\Omega),$$

arguing as in [12] we can prove that

$$(5.5) \quad \begin{aligned} & \int_{\Omega} \left| \operatorname{div} \left(\frac{\mathbf{A}_k + \mathbf{A}_k^T}{2} \mathbf{u}_k \right) \right| dx \\ &= \sup \left\{ \int_{\Omega} \left(\frac{\mathbf{A}_k + \mathbf{A}_k^T}{2} \mathbf{u}_k \right) \cdot \nabla \varphi dx : \varphi \in C_c^1(\Omega), \|\varphi\|_\infty \leq 1 \right\} \\ &\leq 3 \|\mathbf{u}\|_{L^\infty(\Omega; \mathbb{R}^N)} \sup \left\{ \int_{\Omega} \mathbf{A}_k : e(\psi) dx : \psi \in C_c^1(\Omega; \mathbb{R}^N), \|\psi\|_\infty \leq 1 \right\} \\ &\quad + 3 \|\mathbf{A}\|_{L^\infty(\Omega; \mathbb{R}_{\operatorname{sym}}^{N \times N})} \sup \left\{ \int_{\Omega} e(\mathbf{u}_k) : \phi dx : \phi \in C_c^1(\Omega; \mathbb{R}_{\operatorname{sym}}^{N \times N}), \|\phi\|_\infty \leq 1 \right\} \\ &\leq 3 \left\{ \|\mathbf{u}\|_{L^\infty(\Omega; \mathbb{R}^N)} \int_{\Omega} |\operatorname{Div} \mathbf{A}_k| dx + \|\mathbf{A}\|_{L^\infty(\Omega; \mathbb{R}_{\operatorname{sym}}^{N \times N})} \|e(\mathbf{u}_k)\|_{L^1(\Omega; \mathbb{R}_{\operatorname{sym}}^{N \times N})} \right\}, \end{aligned}$$

where in the first inequality we used as test functions $\varphi = \varphi \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|_{L^\infty(\Omega; \mathbb{R}^N)}}$ and $\phi = \varphi \frac{\mathbf{A}_k}{\|\mathbf{A}_k\|_{L^\infty(\Omega; \mathbb{R}_{\operatorname{sym}}^{N \times N})}}$. We note that by Lemma 2.10 the sequence $\mathbf{v}_k := \frac{\mathbf{A}_k + \mathbf{A}_k^T}{2} \mathbf{u}_k$ converges to

$\mathbf{v} := \mathbf{A}\mathbf{u}$ in the L^1 -convergence. Moreover, the functional $\mathbf{v} \mapsto \int_{\Omega} \mathbf{v} \cdot \nabla \varphi \, dx$ is continuous with respect to the L^1 -convergence. Hence, we get

$$\begin{aligned} |\operatorname{div}(\mathbf{A}\mathbf{u})|(\Omega) &\leq \liminf_{k \rightarrow +\infty} \int_{\Omega} \left| \operatorname{div} \left(\frac{\mathbf{A}_k + \mathbf{A}_k^T}{2} \mathbf{u}_k \right) \right| dx \\ &\leq 3 \left\{ \|\mathbf{u}\|_{L^\infty(\Omega; \mathbb{R}^N)} \sup_{k \in \mathbb{N}} \int_{\Omega} |\operatorname{Div} \mathbf{A}_k| \, dx + \|\mathbf{A}\|_{L^\infty(\Omega; \mathbb{R}^{N \times N}_{\text{sym}})} |E\mathbf{u}|(\Omega) \right\} < +\infty, \end{aligned}$$

where in the latter we used Lemma 2.10(iii) and Theorem 2.3. Since $\mathbf{A}\mathbf{u} \in L^\infty(\Omega; \mathbb{R}^N)$, we conclude that $\mathbf{A}\mathbf{u} \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^N)$. We subdivide into steps the rest of the proof.

Step 1. First, we assume that $\mathbf{u}$ is Lipschitz continuous on compact subsets of Ω . We claim that (5.4) holds in the form

$$(5.6) \quad \operatorname{div}(\mathbf{A}\mathbf{u}) = \mathbf{u} \cdot \operatorname{Div} \mathbf{A} + \mathbf{A} : e(\mathbf{u}) \mathcal{L}^N,$$

in the sense of measures on Ω . From Lemma 2.10(iv), we have $\operatorname{Div} \mathbf{A}_k \rightharpoonup^* \operatorname{Div} \mathbf{A}$ as Radon measures on Ω . Then

$$\mathbf{u} \cdot \operatorname{Div} \mathbf{A}_k + \mathbf{A}_k : \nabla \mathbf{u} \mathcal{L}^N \rightharpoonup^* \mathbf{u} \cdot \operatorname{Div} \mathbf{A} + \mathbf{A} : e(\mathbf{u}) \mathcal{L}^N \quad \text{in } \mathcal{M}(\Omega).$$

Moreover, $\operatorname{div}(\mathbf{A}_k^T \mathbf{u}) \rightharpoonup \operatorname{div}(\mathbf{A}\mathbf{u})$ in $\mathcal{D}'(\Omega)$. Collecting these facts, we can compute the limit, in the sense of distributions, of the identity

$$\operatorname{div}(\mathbf{A}_k^T \mathbf{u}) = \mathbf{u} \cdot \operatorname{Div} \mathbf{A}_k + \mathbf{A}_k : \nabla \mathbf{u},$$

so that

$$\operatorname{div}(\mathbf{A}\mathbf{u}) = \mathbf{u} \cdot \operatorname{Div} \mathbf{A} + \mathbf{A} : e(\mathbf{u})$$

holds in the sense of distributions. Since $C_c^1(\Omega)$ is dense in $C_c(\Omega)$, the same identity holds in the sense of measures.

Step 2. Let $\mathbf{u} \in BD(\Omega) \cap L^\infty(\Omega; \mathbb{R}^N)$, and let $(\mathbf{u}_\varepsilon)_\varepsilon$ be a sequence of mollifications of $\mathbf{u}$ as in (2.8). Then, by *Step 1*, we have

$$\operatorname{div}(\mathbf{A}\mathbf{u}_\varepsilon) = \mathbf{u}_\varepsilon \cdot \operatorname{Div} \mathbf{A} + \mathbf{A} : e(\mathbf{u}_\varepsilon) \mathcal{L}^N \quad \text{in } \mathcal{M}(\Omega).$$

Moreover, combining (2.9) with (H_{*}), we deduce $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}^* |\operatorname{Div} \mathbf{A}|$ -a.e. in Ω as $\varepsilon \rightarrow 0$.

By the Dominated Convergence Theorem applied with respect to the measure $\operatorname{Div} \mathbf{A}$, we obtain

$$(5.7) \quad \mathbf{u}_\varepsilon \cdot \operatorname{Div} \mathbf{A} \rightharpoonup^* \mathbf{u}^* \cdot \operatorname{Div} \mathbf{A} \quad \text{as } \varepsilon \rightarrow 0 \text{ in } \mathcal{M}(\Omega).$$

Arguing as for (5.5), we deduce that the sequence $(\operatorname{div}(\mathbf{A}\mathbf{u}_\varepsilon))_\varepsilon$ is equibounded in $\mathcal{M}(\Omega')$ for every $\Omega' \Subset \Omega$. Indeed,

$$\int_{\Omega'} |\operatorname{div}(\mathbf{A}\mathbf{u}_\varepsilon)| \, dx \leq 3 \left\{ \|\mathbf{u}\|_{L^\infty(\Omega; \mathbb{R}^N)} |\operatorname{Div} \mathbf{A}|(\Omega) + \|\mathbf{A}\|_{L^\infty(\Omega; \mathbb{R}^{N \times N}_{\text{sym}})} |E\mathbf{u}|(\Omega) \right\}.$$

Since $\operatorname{div}(\mathbf{A}\mathbf{u}_\varepsilon)$ converges to $\operatorname{div}(\mathbf{A}\mathbf{u})$ in the sense of distributions, it follows that

$$(5.8) \quad \operatorname{div}(\mathbf{A}\mathbf{u}_\varepsilon) \rightharpoonup^* \operatorname{div}(\mathbf{A}\mathbf{u}) \quad \text{as } \varepsilon \rightarrow 0 \text{ in } \mathcal{M}(\Omega).$$

Collecting (5.7) and (5.8), and using (5.6) with $\mathbf{u} = \mathbf{u}_\varepsilon$ together with the definition of pairing (5.1), we obtain

$$(5.9) \quad \mathbf{A} : e(\mathbf{u}_\varepsilon) \rightharpoonup^* \operatorname{div}(\mathbf{A}\mathbf{u}) - \mathbf{u}^* \cdot \operatorname{Div} \mathbf{A} = (\mathbf{A} : E\mathbf{u}) \quad \text{as } \varepsilon \rightarrow 0 \text{ in } \mathcal{M}(\Omega),$$

which shows that $(\mathbf{A} : E\mathbf{u})$ is a Radon measure on Ω . This also yields (5.4). The proof of (5.3) relies on an approximation argument based on the sequence $(\mathbf{u}_\varepsilon)_{\varepsilon > 0}$ introduced above. Let $U \subset \Omega$ be an open set, $K \Subset U$ be a compact set, and let $\varphi \in C_c(U)$ be a test function with $\operatorname{supp} \varphi \subset K$. There exists $r_0 > 0$ such that $K_r := K + B_r(0) \subset U$ for every

$r \in (0, r_0)$. Let $r \in (0, r_0)$ be such that $|E\mathbf{u}|(\partial K_r) = 0$. Indeed, this property holds for almost every r . Then, by (5.9),

$$\begin{aligned} |(\mathbf{A} : E\mathbf{u}), \varphi| &\leq \|\varphi\|_{L^\infty(K)} \|\mathbf{A}\|_{L^\infty(U; \mathbb{R}_{\text{sym}}^{N \times N})} \liminf_{\varepsilon \rightarrow 0} \int_{K_r} |e(\mathbf{u}_\varepsilon)| \, dx \\ &= \|\varphi\|_{L^\infty(K)} \|\mathbf{A}\|_{L^\infty(U; \mathbb{R}_{\text{sym}}^{N \times N})} |E\mathbf{u}|(K_r), \end{aligned}$$

whence, passing to the limit as $r \rightarrow 0$,

$$|(\mathbf{A} : E\mathbf{u})|(K) \leq \|\mathbf{A}\|_{L^\infty(U; \mathbb{R}_{\text{sym}}^{N \times N})} |E\mathbf{u}|(K).$$

Finally, (5.3) follows by the regularity of measures $|(\mathbf{A} : E\mathbf{u})|$ and $|E\mathbf{u}|$ for every Borel set $B \subset U$. $\square$

Remark 5.4. If $\mathbf{A} \in L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ with $\text{Div } \mathbf{A} \in L^2(\Omega; \mathbb{R}^N)$ and $\mathbf{u} \in LD(\Omega) \cap L^\infty(\Omega; \mathbb{R}^N)$ with $e(\mathbf{u}) \in L^2(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$, then

$$(\mathbf{A} : E\mathbf{u}) = \mathbf{A} : e(\mathbf{u}) \mathcal{L}^N,$$

in the sense of measures (see [30, Chap. II, Sect. 7, eq. (7.40)]).

Proposition 5.5. *Let $\mathbf{A} \in \mathcal{DM}^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$, $\mathbf{u} \in BD(\Omega) \cap L^\infty(\Omega; \mathbb{R}^N)$ such that (H_*) holds, and let $\Sigma \subset \Omega$ be an oriented countably $\mathcal{H}^{N-1}$ -rectifiable set. Then $\mathbf{A}\mathbf{u} \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^N)$ and the normal traces of $\mathbf{A}\mathbf{u}$ on Σ are given by*

$$(5.10) \quad \text{Tr}^\pm(\mathbf{A}\mathbf{u}, \Sigma) = \mathbf{u}^\pm \cdot \text{Tr}^\pm(\mathbf{A}, \Sigma), \quad \mathcal{H}^{N-1} - \text{a.e. in } \Sigma,$$

where $\mathbf{u}^\pm$ are the traces of $\mathbf{u}$ on Σ defined in Proposition 2.6. Moreover,

$$(5.11) \quad \text{div}(\mathbf{A}\mathbf{u}) \llcorner J_{\mathbf{u}} = [\mathbf{u}^+ \cdot \text{Tr}^+(\mathbf{A}, J_{\mathbf{u}}) - \mathbf{u}^- \cdot \text{Tr}^-(\mathbf{A}, J_{\mathbf{u}})] \mathcal{H}^{N-1} \llcorner J_{\mathbf{u}}.$$

Proof. The fact that $\mathbf{A}\mathbf{u} \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^N)$ is due to (5.4). We will show (5.10) only for Tr^- , since the argument for Tr^+ is analogous. Recalling the notation of Section 2 (§ *Normal traces*), we may assume that $J_{\mathbf{u}}$ is oriented with ν_Σ on $J_{\mathbf{u}} \cap \Sigma$. We adapt the argument of [25, Proposition 3.4] for $\mathbf{u} \in BV(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$.

Namely, we perform a blow-up argument around each point $x \in \Sigma$ such that

(a) $x \in \Omega \setminus (S_{\mathbf{u}} \setminus J_{\mathbf{u}})$, $x \in E_i$ for some i , the set E_i has density 1 at x , and x is a Lebesgue point of both $\text{Tr}^-(\mathbf{A}, \Sigma)$ and $\text{Tr}^-(\mathbf{A}\mathbf{u}, \Sigma)$ with respect to $\mathcal{H}^{N-1} \llcorner \partial\Omega_i$;

(b) $|\text{Div } \mathbf{A}| \llcorner \Omega_i(B_\varepsilon(x)) = o(\varepsilon^{N-1})$ and $|\text{div}(\mathbf{A}\mathbf{u})| \llcorner \Omega_i(B_\varepsilon(x)) = o(\varepsilon^{N-1})$ as $\varepsilon \rightarrow 0$.

Note that $\mathcal{H}^{N-1}$ -a.e. $x \in \Sigma \setminus (S_{\mathbf{u}} \setminus J_{\mathbf{u}})$ satisfies (a) and (b), as a consequence of [3, Theorem 2.56 and (2.41)] since $\mathbf{A}$ and $\mathbf{A}\mathbf{u}$ have measure divergence. In order not to overburden the notation, we set $\mathbf{u}^-(x) := \tilde{\mathbf{u}}(x)$ if $x \in \Omega \setminus S_{\mathbf{u}}$.

Let $\eta \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function, with $\text{supp } \eta \subseteq B_1(0)$, such that $0 \leq \eta \leq 1$, and for every $\varepsilon > 0$ set $\eta_\varepsilon(y) := \eta\left(\frac{y-x}{\varepsilon}\right)$. We will use both the definitions of trace (2.22) and (2.26). In particular, for ε small enough, we can choose η_ε and $\varphi = \mathbf{u}^-(x)\eta_\varepsilon$ as test functions in (2.22) and (2.26), respectively. Note that $\nabla \varphi(y) = \mathbf{u}^-(x) \otimes \nabla \eta_\varepsilon(y)$, hence by (2.2) we have

$$\mathbf{A} : \nabla \varphi(y) = \mathbf{A} : (\mathbf{u}^-(x) \odot \nabla \eta_\varepsilon(y)) = \mathbf{u}^-(x) \cdot (\mathbf{A} \nabla \eta_\varepsilon(y)) = \nabla \eta_\varepsilon(y) \cdot (\mathbf{A} \mathbf{u}^-(x)).$$

Then, for $\varepsilon > 0$ small enough, we can write

$$\begin{aligned}
(5.12) \quad & \frac{1}{\varepsilon^{N-1}} \int_{\partial\Omega_i} [\text{Tr}(\mathbf{A}\mathbf{u}, \partial\Omega_i) - \mathbf{u}^-(x) \cdot \mathbf{Tr}(\mathbf{A}, \partial\Omega_i)] \eta_\varepsilon(y) d\mathcal{H}^{N-1}(y) \\
&= \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \nabla \eta_\varepsilon(y) \cdot \mathbf{A}(y) \mathbf{u}(y) dy + \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \eta_\varepsilon(y) d \text{div}(\mathbf{A}\mathbf{u}) \\
&\quad - \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \mathbf{u}^-(x) \cdot (\mathbf{A}(y) \nabla \eta_\varepsilon(y)) dy - \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \eta_\varepsilon(y) \mathbf{u}^-(x) \cdot d \text{Div} \mathbf{A}(y) \\
&= \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \nabla \eta_\varepsilon(y) \cdot [\mathbf{A}(y) \mathbf{u}(y) - \mathbf{A}(y) \mathbf{u}^-(x)] dy \\
&\quad + \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \eta_\varepsilon(y) d[\text{div}(\mathbf{A}\mathbf{u}) - \mathbf{u}^-(x) \cdot \text{Div} \mathbf{A}](y) \\
&=: J_1(\varepsilon) + J_2(\varepsilon).
\end{aligned}$$

Using the change of variables $z = (y - x)/\varepsilon$, as $\varepsilon \rightarrow 0$ the left-hand side of this equality converges to

$$[\text{Tr}^-(\mathbf{A}\mathbf{u}, \Sigma)(x) - \mathbf{u}^-(x) \cdot \mathbf{Tr}^-(\mathbf{A}, \Sigma)] \int_{\Pi_x} \eta(z) d\mathcal{H}^{N-1}(z),$$

where Π_x is the tangent plane to Σ_i at x . Clearly η can be chosen in such a way that $\int_{\Pi_x} \eta d\mathcal{H}^{N-1} > 0$.

Therefore, in order to prove (5.10) for Tr^- it suffices to show that the two integrals $J_1(\varepsilon)$ and $J_2(\varepsilon)$ on the right hand side of (5.12) converge to 0 as $\varepsilon \rightarrow 0$.

With the change of variables $z = (y - x)/\varepsilon$ we have that

$$J_1(\varepsilon) = \int_{\Omega_i^\varepsilon} \mathbf{A}(x + \varepsilon z) [\mathbf{u}(x + \varepsilon z) - \mathbf{u}^-(x)] \cdot \nabla \eta(z) dz,$$

where

$$\Omega_i^\varepsilon := \frac{\Omega_i - x}{\varepsilon}.$$

As $\varepsilon \rightarrow 0$, these sets locally converge to the half-space $P_x := \{z \in \mathbb{R}^N : z \cdot \nu(x) < 0\}$, hence

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_i^\varepsilon \cap B_1} |\mathbf{u}(x + \varepsilon z) - \mathbf{u}^-(x)| dz = \lim_{\varepsilon \rightarrow 0} \int_{P_x \cap B_1} |\mathbf{u}(x + \varepsilon z) - \mathbf{u}^-(x)| dz = 0$$

(see [3, Remark 3.85]) so that

$$|J_1(\varepsilon)| \leq \|\mathbf{A}\|_{L^\infty(B_\varepsilon(x); \mathbb{R}_{\text{sym}}^{N \times N})} \|\nabla \eta\|_{L^\infty} \int_{\Omega_i^\varepsilon \cap B_1} |\mathbf{u}(x + \varepsilon z) - \mathbf{u}^-(x)| dz \rightarrow 0.$$

As for $J_2(\varepsilon)$, using (b) we have

$$\limsup_{\varepsilon \rightarrow 0} |J_2(\varepsilon)| \leq \limsup_{\varepsilon \rightarrow 0} |\mathbf{u}^-(x)| \frac{|\text{Div} \mathbf{A}|(B_\varepsilon(x))}{\varepsilon^{N-1}} + \limsup_{\varepsilon \rightarrow 0} \frac{|\text{div}(\mathbf{A}\mathbf{u})|(B_\varepsilon(x))}{\varepsilon^{N-1}} = 0,$$

so that $J_2(\varepsilon) \rightarrow 0$ and the proof is complete. The representation formula (5.11) is an immediate consequence of (5.10) and (2.28). $\square$

We are now ready to state the main decomposition theorem for the pairing measure.

Theorem 5.6. *Let $\mathbf{A} \in \text{DM}^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in \text{BD}(\Omega) \cap L^\infty(\Omega; \mathbb{R}^N)$ satisfying (H_*) . Then the measure $(\mathbf{A} : E\mathbf{u})$ admits the following decomposition:*

- (i) absolutely continuous part: $(\mathbf{A} : E\mathbf{u})^a = \mathbf{A} : e(\mathbf{u}) \mathcal{L}^N$;
 - (ii) jump part: $(\mathbf{A} : E\mathbf{u})^j = \frac{\mathbf{Tr}^+(\mathbf{A}, J_{\mathbf{u}}) + \mathbf{Tr}^-(\mathbf{A}, J_{\mathbf{u}})}{2} \cdot (\mathbf{u}^+ - \mathbf{u}^-) \mathcal{H}^{N-1} \llcorner J_{\mathbf{u}}$;
 - (iii) Cantor part: if, in addition,
- (H_c) $|E^c \mathbf{u}|(S_{\mathbf{A}}) = 0$,
- then $(\mathbf{A} : E\mathbf{u})^c = \tilde{\mathbf{A}} : E^c \mathbf{u}$.

Proof. For the proof, we adapt the arguments from [12, Theorem 3.2] and [19, Theorem 3.3].

We first prove assertion (i). Let $x \in \Omega$ be such that the limit

$$\lim_{r \rightarrow 0} \frac{1}{\mathcal{L}^N(B_r)} \int_{B_r(x)} d(\mathbf{A} : E\mathbf{u})(y)$$

exists, and

$$(5.13) \quad \begin{aligned} \lim_{r \rightarrow 0} \frac{|E^s \mathbf{u}|(B_r(x))}{\mathcal{L}^N(B_r)} &= 0, \\ \lim_{r \rightarrow 0} \frac{1}{\mathcal{L}^N(B_r)} \int_{B_r(x)} |\mathbf{A}(y) : e(\mathbf{u})(y) - \mathbf{A}(x) : e(\mathbf{u})(x)| dy &= 0. \end{aligned}$$

Note that $\mathcal{L}^N$ -almost every $x \in \Omega$ has this property. We fix $r > 0$ such that

$$(5.14) \quad |E^s \mathbf{u}|(\partial B_r(x)) = 0.$$

Let $\mathbf{u}_\varepsilon := \eta_\varepsilon * \mathbf{u}$ be as in (2.8). Then we have $e(\mathbf{u}_\varepsilon) = \eta_\varepsilon * e(\mathbf{u}) + \eta_\varepsilon * E^s \mathbf{u}$ and, using also the fact that $|\eta_\varepsilon * E^s \mathbf{u}| \leq \eta_\varepsilon * |E^s \mathbf{u}|$, for every $\varphi \in C_c(B_r(x))$ we can estimate

$$\begin{aligned} & \left| \int_{B_r(x)} \varphi(y) d(\mathbf{A} : E\mathbf{u})(y) - \int_{B_r(x)} \varphi(y) \mathbf{A}(x) : e(\mathbf{u})(x) dy \right| \\ & \leq \left| \int_{B_r(x)} \varphi(y) d(\mathbf{A} : E\mathbf{u})(y) - \int_{B_r(x)} \varphi(y) \mathbf{A}(y) : e(\mathbf{u}_\varepsilon)(y) dy \right| \\ & \quad + \int_{B_r(x)} |\varphi(y) [\mathbf{A}(y) : \eta_\varepsilon * e(\mathbf{u})(y) - \mathbf{A}(x) : e(\mathbf{u})(x)]| dy \\ & \quad + \|\varphi\|_{L^\infty(B_r(x))} \|\mathbf{A}\|_{L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})} \int_{B_r(x)} \eta_\varepsilon * |E^s \mathbf{u}| dy. \end{aligned}$$

Taking into account (5.9), and the fact that

$$\eta_\varepsilon * e(\mathbf{u}) \mathcal{L}^N \rightharpoonup^* e(\mathbf{u}) \mathcal{L}^N, \quad \eta_\varepsilon * |E^s \mathbf{u}| \mathcal{L}^N \rightharpoonup^* |E^s \mathbf{u}|$$

(see, e.g., [3, Theorem 2.2]), together with (5.14), passing to the limit as $\varepsilon \rightarrow 0$ in the estimate above we get

$$\begin{aligned} & \left| \int_{B_r(x)} \varphi(y) d(\mathbf{A} : E\mathbf{u})(y) - \int_{B_r(x)} \varphi(y) \mathbf{A}(x) : e(\mathbf{u})(x) dy \right| \\ & \leq \int_{B_r(x)} |\varphi(y) [\mathbf{A}(y) : e(\mathbf{u})(y) - \mathbf{A}(x) : e(\mathbf{u})(x)]| dy \\ & \quad + \|\varphi\|_{L^\infty(B_r(x))} \|\mathbf{A}\|_{L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})} |E^s \mathbf{u}|(B_r(x)). \end{aligned}$$

Since φ is arbitrary, we can choose a sequence (φ_k) , with $\|\varphi_k\|_{L^\infty(B_r(x))} \leq 1$ and $\varphi_k \rightarrow 1$ pointwise on $B_r(x)$, and write an estimate as above with φ_k in place of φ . Then, dividing

both the sides by $\mathcal{L}^N(B_r)$ and passing to the limit as $k \rightarrow +\infty$, by virtue of Lebesgue's Dominated Convergence Theorem we obtain

$$\begin{aligned} & \left| \frac{1}{\mathcal{L}^N(B_r)} \int_{B_r(x)} d(\mathbf{A} : E\mathbf{u})(y) - \mathbf{A}(x) : e(\mathbf{u})(x) \right| \\ & \leq \frac{1}{\mathcal{L}^N(B_r)} \int_{B_r(x)} |\mathbf{A}(y) : e(\mathbf{u})(y) - \mathbf{A}(x) : e(\mathbf{u})(x)| \, dy \\ & \quad + \frac{\|\mathbf{A}\|_{L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})}}{\mathcal{L}^N(B_r)} |E^s \mathbf{u}|(B_r(x)). \end{aligned}$$

Now, letting $r \rightarrow 0^+$ and using (5.13), we finally get

$$\lim_{r \rightarrow 0} \frac{1}{\mathcal{L}^N(B_r)} \int_{B_r(x)} d(\mathbf{A} : E\mathbf{u})(y) = \mathbf{A}(x) : e(\mathbf{u})(x),$$

whence (i) follows.

Let us prove (ii). Since, by virtue of Theorem 5.3, it holds that $(\mathbf{A} : E\mathbf{u}) \ll |E\mathbf{u}|$, it is clear that $(\mathbf{A} : E\mathbf{u})^j$ is supported in $J_{\mathbf{u}}$. From (5.4) and (5.11) we have that

$$\begin{aligned} (\mathbf{A} : E\mathbf{u})^j &= (\mathbf{A} : E\mathbf{u}) \lfloor J_{\mathbf{u}} = \operatorname{div}(\mathbf{A}\mathbf{u}) \lfloor J_{\mathbf{u}} - \mathbf{u}^* \cdot \operatorname{Div} \mathbf{A} \lfloor J_{\mathbf{u}} \\ &= [\mathbf{u}^+ \cdot \mathbf{Tr}^+(\mathbf{A}, J_{\mathbf{u}}) - \mathbf{u}^- \cdot \mathbf{Tr}^-(\mathbf{A}, J_{\mathbf{u}})] \mathcal{H}^{N-1} \lfloor J_{\mathbf{u}} \\ &\quad - \frac{\mathbf{u}^+ + \mathbf{u}^-}{2} \cdot [\mathbf{Tr}^+(\mathbf{A}, J_{\mathbf{u}}) - \mathbf{Tr}^-(\mathbf{A}, J_{\mathbf{u}})] \mathcal{H}^{N-1} \lfloor J_{\mathbf{u}} \\ &= \frac{\mathbf{Tr}^+(\mathbf{A}, J_{\mathbf{u}}) + \mathbf{Tr}^-(\mathbf{A}, J_{\mathbf{u}})}{2} \cdot (\mathbf{u}^+ - \mathbf{u}^-) \mathcal{H}^{N-1} \lfloor J_{\mathbf{u}}, \end{aligned}$$

and the proof is complete.

As for (iii), we may adapt the proof of [19, Theorem 3.3(iii)]. Let $E\mathbf{u} = \boldsymbol{\sigma}_{\mathbf{u}} |E\mathbf{u}|$ be the polar decomposition of $E\mathbf{u}$. By assumption (H_c) , the approximate limit $\tilde{\mathbf{A}}$ of $\mathbf{A}$ exists $|E^c \mathbf{u}|$ -a.e. in Ω . Hence, setting $\mu := (\mathbf{A} : E\mathbf{u})$, to prove the equality in (iii) we have to show that

$$\frac{d\mu}{d|E^c \mathbf{u}|}(x) = \frac{d\mu^c}{d|E^c \mathbf{u}|}(x) = \tilde{\mathbf{A}}(x) : \boldsymbol{\sigma}_{\mathbf{u}}(x) \quad \text{for } |E^c \mathbf{u}| \text{-a.e. } x \in \Omega.$$

We recall that $|E^c \mathbf{u}|$ -a.e. $x \in \Omega$ satisfies

- (h1) $|E^c \mathbf{u}|(B_r(x)) > 0$ for every $r > 0$;
- (h2) there exists $\lim_{r \rightarrow 0} \frac{\mu^c(B_r(x))}{|E^c \mathbf{u}|(B_r(x))}$;
- (h3) $\lim_{r \rightarrow 0} \frac{|E^j \mathbf{u}|(B_r(x))}{|E\mathbf{u}|(B_r(x))} = 0$, $\lim_{r \rightarrow 0} \frac{|E^a \mathbf{u}|(B_r(x))}{|E\mathbf{u}|(B_r(x))} = 0$;
- (h4) $\lim_{r \rightarrow 0} \frac{1}{|E^c \mathbf{u}|(B_r(x))} \int_{B_r(x)} \left| \tilde{\mathbf{A}}(y) : \boldsymbol{\sigma}_{\mathbf{u}}(y) - \tilde{\mathbf{A}}(x) : \boldsymbol{\sigma}_{\mathbf{u}}(x) \right| d|E^c \mathbf{u}|(y) = 0$.

We fix any of such points $x \in \Omega$, and choose $r > 0$ such that

$$(5.15) \quad |E^j \mathbf{u}|(\partial B_r(x)) = 0 \quad \text{and} \quad |E^a \mathbf{u}|(\partial B_r(x)) = 0.$$

Let $\eta \in C_c^\infty(\mathbb{R}^N)$ be a symmetric convolution kernel with support in the unit ball, and denote by $\eta_\varepsilon(x) := \varepsilon^{-N} \eta(\frac{x}{\varepsilon})$. It is well known (see, e.g., [3]) that $\eta_\varepsilon * E\mathbf{u} = \eta_\varepsilon * E^a \mathbf{u} +$

$\eta_\varepsilon * E^j \mathbf{u} + \eta_\varepsilon * E^c \mathbf{u}$. Hence for every $\phi \in C_c(\mathbb{R}^N)$ with support in $B_r(x)$ it holds

$$\begin{aligned}
 (5.16) \quad & \left| \frac{1}{|E^c \mathbf{u}|(B_r(x))} \int_{B_r(x)} \phi(y) \mathbf{A}(y) : \eta_\varepsilon * E \mathbf{u}(y) dy \right. \\
 & \quad \left. - \frac{1}{|E^c \mathbf{u}|(B_r(x))} \int_{B_r(x)} \phi(y) \tilde{\mathbf{A}}(x) : \boldsymbol{\sigma}_{\mathbf{u}}(x) d|E^c \mathbf{u}|(y) \right| \\
 & \leq \left| \frac{1}{|E^c \mathbf{u}|(B_r(x))} \int_{B_r(x)} \phi(y) \mathbf{A}(y) : \eta_\varepsilon * E^c \mathbf{u}(y) dy \right. \\
 & \quad \left. - \frac{1}{|E^c \mathbf{u}|(B_r(x))} \int_{B_r(x)} \phi(y) \tilde{\mathbf{A}}(x) : \boldsymbol{\sigma}_{\mathbf{u}}(x) d|E^c \mathbf{u}|(y) \right| \\
 & \quad + \frac{1}{|E^c \mathbf{u}|(B_r(x))} \|\phi\|_\infty \|\mathbf{A}\|_{L^\infty(B_r(x); \mathbb{R}_{\text{sym}}^{N \times N})} \int_{B_r(x)} \eta_\varepsilon * |E^j \mathbf{u}| dy \\
 & \quad + \frac{1}{|E^c \mathbf{u}|(B_r(x))} \|\phi\|_\infty \|\mathbf{A}\|_{L^\infty(B_r(x); \mathbb{R}_{\text{sym}}^{N \times N})} \int_{B_r(x)} \eta_\varepsilon * |E^a \mathbf{u}| dy,
 \end{aligned}$$

where in the last inequality we used that $|\eta_\varepsilon * E^t \mathbf{u}| \leq \eta_\varepsilon * |E^t \mathbf{u}|$ for $t = a, j$. We note that by (5.15)

$$\lim_{\varepsilon \rightarrow 0} \int_{B_r(x)} \eta_\varepsilon * |E^j \mathbf{u}| dy = |E^j \mathbf{u}|(B_r(x)), \quad \lim_{\varepsilon \rightarrow 0} \int_{B_r(x)} \eta_\varepsilon * |E^a \mathbf{u}| dy = |E^a \mathbf{u}|(B_r(x)).$$

Furthermore,

$$\int_{B_r(x)} \phi(y) \mathbf{A}(y) : \eta_\varepsilon * E^c \mathbf{u}(y) dy = \int_{B_r(x)} [\eta_\varepsilon * (\phi \mathbf{A})](y) : \boldsymbol{\sigma}_{\mathbf{u}}(y) d|E^c \mathbf{u}|(y).$$

Hence by taking the limit as $\varepsilon \rightarrow 0$ in (5.16) we obtain

$$\begin{aligned}
 & \left| \frac{1}{|E^c \mathbf{u}|(B_r(x))} \int_{B_r(x)} \phi(y) d\mu(y) \right. \\
 & \quad \left. - \frac{1}{|E^c \mathbf{u}|(B_r(x))} \int_{B_r(x)} \phi(y) \tilde{\mathbf{A}}(x) : \boldsymbol{\sigma}_{\mathbf{u}}(x) d|E^c \mathbf{u}|(y) \right| \\
 & \leq \frac{1}{|E^c \mathbf{u}|(B_r(x))} \|\phi\|_\infty \int_{B_r(x)} \left| \tilde{\mathbf{A}}(y) : \boldsymbol{\sigma}_{\mathbf{u}}(y) - \tilde{\mathbf{A}}(x) : \boldsymbol{\sigma}_{\mathbf{u}}(x) \right| d|E^c \mathbf{u}|(y) \\
 & \quad + \frac{1}{|E^c \mathbf{u}|(B_r(x))} \|\phi\|_\infty \|\mathbf{A}\|_{L^\infty(B_r(x); \mathbb{R}_{\text{sym}}^{N \times N})} |E^j \mathbf{u}|(B_r(x)) \\
 & \quad + \frac{1}{|E^c \mathbf{u}|(B_r(x))} \|\phi\|_\infty \|\mathbf{A}\|_{L^\infty(B_r(x); \mathbb{R}_{\text{sym}}^{N \times N})} |E^a \mathbf{u}|(B_r(x)).
 \end{aligned}$$

Taking the supremum on ϕ , with $\|\phi\|_\infty \leq 1$, on the left hand side we get

$$\begin{aligned}
 & \left| \frac{\mu(B_r(x))}{|E^c \mathbf{u}|(B_r(x))} - \tilde{\mathbf{A}}(x) : \boldsymbol{\sigma}_{\mathbf{u}}(x) \right| \\
 & \leq \frac{1}{|E^c \mathbf{u}|(B_r(x))} \int_{B_r(x)} \left| \tilde{\mathbf{A}}(y) : \boldsymbol{\sigma}_{\mathbf{u}}(y) - \tilde{\mathbf{A}}(x) : \boldsymbol{\sigma}_{\mathbf{u}}(x) \right| d|E^c \mathbf{u}|(y) \\
 & \quad + \frac{|E^j \mathbf{u}|(B_r(x))}{|E^c \mathbf{u}|(B_r(x))} \|\mathbf{A}\|_{L^\infty(B_r(x); \mathbb{R}_{\text{sym}}^{N \times N})} + \frac{|E^a \mathbf{u}|(B_r(x))}{|E^c \mathbf{u}|(B_r(x))} \|\mathbf{A}\|_{L^\infty(B_r(x); \mathbb{R}_{\text{sym}}^{N \times N})}.
 \end{aligned}$$

Now, by taking $r \rightarrow 0$ and using (h3)-(h4) we get (iii). The proof is concluded. $\square$

Remark 5.7 (BV matrix-valued fields). Let $\mathbf{u} \in BD(\Omega) \cap L^\infty(\Omega; \mathbb{R}^N)$. If $\mathbf{A} \in BV_{\text{loc}}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \cap L^\infty_{\text{loc}}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$, then $\mathbf{A} \in DM^\infty_{\text{loc}}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and, by (2.26),

$$\mathbf{Tr}^\pm(\mathbf{A}, J_{\mathbf{u}}) = \mathbf{A}_{J_{\mathbf{u}}}^\pm \nu_{\mathbf{u}} \quad \mathcal{H}^{N-1} - \text{a.e. in } J_{\mathbf{u}}.$$

In addition, the jump part of $(\mathbf{A} : E\mathbf{u})$ can be written as

$$(\mathbf{A} : E\mathbf{u})^j = \frac{\mathbf{A}_{J_{\mathbf{u}}}^+ + \mathbf{A}_{J_{\mathbf{u}}}^-}{2} : E^j \mathbf{u} = \mathbf{A}_{J_{\mathbf{u}}}^* : E^j \mathbf{u}.$$

We notice that, if $\mathbf{A} \in BV(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \cap L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$, (hence (H_c) is satisfied), then

$$(\mathbf{A} : E\mathbf{u}) = \mathbf{A}^* : E\mathbf{u},$$

in the sense of measures.

Finally, the following theorem establishes the validity of the Gauss-Green formulas for the pairing on sets of finite perimeter.

Theorem 5.8. *Let $\mathbf{A} \in DM^\infty_{\text{loc}}(\mathbb{R}^N; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in BD_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N)$ satisfying (H_*) . Let $F \subset \mathbb{R}^N$ be a bounded set of finite perimeter. Then the following Gauss-Green formulas hold*

$$(5.17) \quad \int_{F^1} \mathbf{u}^* \cdot d\text{Div } \mathbf{A} + \int_{F^1} d(\mathbf{A} : E\mathbf{u}) = - \int_{\partial^* F} \mathbf{u}^+ \cdot \mathbf{Tr}^+(\mathbf{A}, \partial^* F) d\mathcal{H}^{N-1},$$

$$(5.18) \quad \int_{F^1 \cup \partial^* F} \mathbf{u}^* \cdot d\text{Div } \mathbf{A} + \int_{F^1 \cup \partial^* F} d(\mathbf{A} : E\mathbf{u}) = - \int_{\partial^* F} \mathbf{u}^- \cdot \mathbf{Tr}^-(\mathbf{A}, \partial^* F) d\mathcal{H}^{N-1},$$

where $\mathbf{u}^\pm$ are the traces of $\mathbf{u}$ on $\partial^* F$ defined in Proposition 2.6 and $\mathbf{Tr}^\pm(\mathbf{A}, \partial^* F)$ are the normal traces of $\mathbf{A}$ when $\partial^* F$ is oriented with respect to the interior unit normal vector.

Remark 5.9. Let $\mathbf{A} \in C^1(\mathbb{R}^N; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in BD_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N)$. Let $\Omega \subset \mathbb{R}^N$ be a set with Lipschitz boundary. Then (H_*) is satisfied, and $\mathbf{Tr}^+(\mathbf{A}, \partial\Omega) = -\mathbf{A}\nu_\Omega$. By (5.17) we obtain

$$\int_{\Omega} \mathbf{u} \cdot \text{Div } \mathbf{A} dx + \int_{\Omega} \mathbf{A} : dE\mathbf{u} = \int_{\partial\Omega} \gamma^+(\mathbf{u}) \cdot (\mathbf{A}\nu_\Omega) d\mathcal{H}^{N-1}.$$

Since by (2.2) we have

$$\gamma^+(\mathbf{u}) \cdot (\mathbf{A}\nu_\Omega) = \mathbf{A} : (\gamma^+(\mathbf{u}) \odot \nu_\Omega),$$

we conclude that

$$(5.19) \quad \int_{\Omega} \mathbf{u} \cdot \text{Div } \mathbf{A} dx + \int_{\Omega} \mathbf{A} : dE\mathbf{u} = \int_{\partial\Omega} \mathbf{A} : (\gamma^+(\mathbf{u}) \odot \nu_\Omega) d\mathcal{H}^{N-1}.$$

Thus, formula (5.17) yields the Gauss-Green formula (2.14) for DM^∞ fields and bounded BD functions, and consequently generalizes Babadjian's result (see Theorem 2.4 and formula (2.13)) as well as the Strang-Temam theorem (see Remark 2.5).

Proof of Theorem 5.8. We argue as in the proof of [25, Theorem 1.3]. Since F is bounded, up to consider a bounded extension domain compactly containing F , we may assume without loss of generality that $\mathbf{A} \in DM^\infty(\mathbb{R}^N; \mathbb{R}_{\text{sym}}^{N \times N})$ and that $\mathbf{u} \in BD(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N; \mathbb{R}^N)$.

Recall that $\chi_F \in BV(\mathbb{R}^N)$, that the reduced boundary $\partial^* F$ is countably $\mathcal{H}^{N-1}$ -rectifiable, and that

$$(5.20) \quad \chi_F^* = \chi_{F^1} + \frac{1}{2} \chi_{\partial^* F}.$$

Define the vector field $\mathbf{B} := \mathbf{A}\mathbf{u}$. Then $\chi_F \mathbf{B}$ has compact support, and therefore

$$\operatorname{div}(\chi_F \mathbf{B})(\mathbb{R}^N) = 0$$

(see [17, Lemma 3.1]).

Furthermore, by Theorem 5.3, the field $\mathbf{B}$ belongs to $\mathcal{DM}^\infty(\mathbb{R}^N; \mathbb{R}^N)$. Hence, applying the pairing theory for scalar BV functions recalled in Section 4, and taking χ_F and $\mathbf{B}$ in place of u and $\mathbf{A}$ in (2.30), we obtain

$$(5.21) \quad \int_{\mathbb{R}^N} \chi_F^* d \operatorname{div} \mathbf{B} = -(\mathbf{B}, D\chi_F)(\mathbb{R}^N).$$

By the definition of normal traces, together with (2.25) and (5.10), we have

$$(5.22) \quad \operatorname{div} \mathbf{B} \llcorner \partial^* F = (\mathbf{u}^+ \cdot \mathbf{Tr}^+(\mathbf{A}, \partial^* F) - \mathbf{u}^- \cdot \mathbf{Tr}^-(\mathbf{A}, \partial^* F)) \mathcal{H}^{N-1} \llcorner \partial^* F.$$

Combining (5.20) and (5.22), we can rewrite (5.21) as

$$(5.23) \quad \int_{\mathbb{R}^N} \chi_F^* d \operatorname{div} \mathbf{B} = \int_{F^1} d \operatorname{div} \mathbf{B} + \frac{1}{2} \int_{\partial^* F} (\mathbf{u}^+ \cdot \mathbf{Tr}^+(\mathbf{A}, \partial^* F) - \mathbf{u}^- \cdot \mathbf{Tr}^-(\mathbf{A}, \partial^* F)) d\mathcal{H}^{N-1}.$$

On the other hand, concerning the right-hand side of (5.21), by [19, Remark 4.4] and since $|D\chi_F| = \mathcal{H}^{N-1} \llcorner \partial^* F$, we obtain

$$(\mathbf{B}, D\chi_F) = \frac{\operatorname{Tr}^+(\mathbf{B}, \partial^* F) + \operatorname{Tr}^-(\mathbf{B}, \partial^* F)}{2} \mathcal{H}^{N-1} \llcorner \partial^* F.$$

Noting that $\mathbf{u}^\pm \in L^1_{\mathcal{H}^{N-1} \llcorner \partial^* F}(\partial^* F; \mathbb{R}^N)$, since $\mathbf{u} \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ implies that its traces are essentially bounded on $\partial^* F$, and applying (5.10), we deduce that

$$(5.24) \quad (\mathbf{B}, D\chi_F)(\mathbb{R}^N) = \int_{\partial^* F} \frac{1}{2} [\mathbf{u}^+ \cdot \mathbf{Tr}^+(\mathbf{A}, \partial^* F) + \mathbf{u}^- \cdot \mathbf{Tr}^-(\mathbf{A}, \partial^* F)] d\mathcal{H}^{N-1}.$$

Finally, inserting (5.23) and (5.24) into (5.21), simplifying the resulting expression, and recalling the definition of $\mathbf{B}$, we obtain

$$(5.25) \quad \int_{F^1} d \operatorname{div}(\mathbf{A}\mathbf{u}) = - \int_{\partial^* F} \mathbf{u}^+ \cdot \mathbf{Tr}^+(\mathbf{A}, \partial^* F) d\mathcal{H}^{N-1}.$$

Using now (5.4) together with the pairing $(\mathbf{A} : E\mathbf{u})$, the left-hand side of (5.25) can be rewritten as

$$\int_{F^1} d(\mathbf{u}^* \cdot \operatorname{Div} \mathbf{A} + (\mathbf{A} : E\mathbf{u})) = - \int_{\partial^* F} \mathbf{u}^+ \cdot \mathbf{Tr}^+(\mathbf{A}, \partial^* F) d\mathcal{H}^{N-1}.$$

This concludes the proof of (5.17). In order to prove (5.18) it suffices to note that, by (5.25)

$$\begin{aligned}
& \int_{F^1 \cup \partial^* F} d \operatorname{div}(\mathbf{A} \mathbf{u}) \\
&= \int_{F^1} d \operatorname{div}(\mathbf{A} \mathbf{u}) + \int_{\partial^* F} [\mathbf{u}^+ \cdot \mathbf{Tr}^+(\mathbf{A}, \partial^* F) - \mathbf{u}^- \cdot \mathbf{Tr}^-(\mathbf{A}, \partial^* F)] d\mathcal{H}^{N-1} \\
&= - \int_{\partial^* F} \mathbf{u}^+ \cdot \mathbf{Tr}^+(\mathbf{A}, \partial^* F) d\mathcal{H}^{N-1} \\
&\quad + \int_{\partial^* F} [\mathbf{u}^+ \cdot \mathbf{Tr}^+(\mathbf{A}, \partial^* F) - \mathbf{u}^- \cdot \mathbf{Tr}^-(\mathbf{A}, \partial^* F)] d\mathcal{H}^{N-1},
\end{aligned}$$

hence (5.18) follows. $\square$

6. SLICING FORMULAS FOR THE PAIRING WITH BOUNDED BD FUNCTIONS

In this section, we aim to derive a slicing formula for the pairing defined in (5.1). To this end, we introduce an assumption analogous to (H'), which is motivated by the representation of the distributional divergence of a symmetric matrix field as a linear combination of suitable directional derivatives.

We fix a set $\Xi \subset \mathbb{S}^{N-1}$ with $|\Xi| = \frac{N(N+1)}{2}$ such that $\{\boldsymbol{\xi} \otimes \boldsymbol{\xi} : \boldsymbol{\xi} \in \Xi\}$ is a basis of $\mathbb{R}_{\text{sym}}^{N \times N}$. We shall refer to such a set as a *frame*. A standard choice is

$$\Xi = \left\{ \mathbf{e}_i : i = 1, \dots, N \right\} \cup \left\{ \frac{\mathbf{e}_i + \mathbf{e}_j}{\sqrt{2}} : 1 \leq i < j \leq N \right\}.$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ is the canonical basis of $\mathbb{R}^N$.

Let $\{\mathbf{B}^\xi\}_{\xi \in \Xi} \subset \mathbb{R}_{\text{sym}}^{N \times N}$ denote the dual basis, defined by

$$\mathbf{B}^\xi : (\boldsymbol{\eta} \otimes \boldsymbol{\eta}) = \delta_{\xi \boldsymbol{\eta}} \quad \forall \boldsymbol{\xi}, \boldsymbol{\eta} \in \Xi.$$

Let $\mathbf{A} \in L_{\text{loc}}^1(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$. Then, for a.e. $x \in \Omega$, we have

$$\mathbf{A}(x) = \sum_{\xi \in \Xi} a^\xi(x) (\boldsymbol{\xi} \otimes \boldsymbol{\xi}),$$

where the coefficients a^ξ are given by

$$(6.1) \quad a^\xi(x) := \mathbf{A}(x) : \mathbf{B}^\xi, \quad \forall \boldsymbol{\xi} \in \Xi.$$

Lemma 6.1. *Let Ξ be a frame and $\mathbf{A} \in L_{\text{loc}}^1(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$. Then, the divergence of $\mathbf{A}$ admits the decomposition*

$$(6.2) \quad \operatorname{Div} \mathbf{A} = \sum_{\xi \in \Xi} (D_\xi a^\xi) \boldsymbol{\xi} \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^N),$$

where, for every $\boldsymbol{\xi} \in \Xi$, $D_\xi a^\xi$ denotes the distributional derivative of a^ξ in the direction $\boldsymbol{\xi}$.

Proof. For every $\varphi \in C_c^\infty(\Omega; \mathbb{R}^N)$, since the matrix $\mathbf{A}$ is symmetric, we have $\mathbf{A} : \nabla \varphi = \mathbf{A} : e(\varphi)$ and then, from the distributional definition of divergence, we have

$$\begin{aligned}
 \langle \operatorname{Div} \mathbf{A}, \varphi \rangle &= - \int_{\Omega} \mathbf{A} : e(\varphi) \, dx \\
 &= - \sum_{\xi \in \Xi} \int_{\Omega} a^\xi(x) (\xi \otimes \xi) : e(\varphi)(x) \, dx \\
 (6.3) \quad &= - \sum_{\xi \in \Xi} \int_{\Omega} a^\xi(x) (e(\varphi)(x) \xi \cdot \xi) \, dx \\
 &= - \sum_{\xi \in \Xi} \int_{\Omega} a^\xi(x) D_\xi \hat{\varphi}^\xi(x) \, dx = \left\langle \sum_{\xi \in \Xi} (D_\xi a^\xi) \xi, \varphi \right\rangle,
 \end{aligned}$$

since $D_\xi \hat{\varphi}^\xi = \xi \cdot (D\varphi(x)\xi) = \xi \cdot (D\varphi(x)^T \xi) = e(\varphi)\xi \cdot \xi$. $\square$

However, in order to give a meaning to a slicing formula for the pairing (5.1), we shall require a stronger regularity on the directional derivatives $D_\xi a^\xi$ of the coefficients, namely that they are Radon measures. This leads to the following assumption:

$$(H'') \quad D_\xi a^\xi \in \mathcal{M}(\Omega) \text{ for every } \xi \in \Xi.$$

By Proposition 2.2, this is equivalent to the following assumptions: for all $\xi \in \Xi$

$$(\tilde{H}'') \quad \begin{cases} a_y^\xi \in BV(\Omega_y^\xi) \text{ for } \mathcal{L}^{N-1}\text{-a.e. } y \in \xi^\perp, \\ \int_{\xi^\perp} |Da_y^\xi|(\Omega_y^\xi) \, d\mathcal{H}^{N-1}(y) < +\infty, \end{cases}$$

where, with a slight abuse of notation, we set

$$a_y^\xi(s) := a^\xi(y + s\xi), \quad s \in \Omega_y^\xi.$$

This implies that $a_y^\xi \in L_{\text{loc}}^\infty(\Omega_y^\xi)$.

We are led to introduce the class (depending on Ξ)

$$(6.4) \quad BV_\Xi^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) := \left\{ \mathbf{A} \in L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) : D_\xi a^\xi \in \mathcal{M}(\Omega) \text{ for every } \xi \in \Xi \right\}.$$

We notice that, if $\mathbf{A} \in BV(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \cap L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$, then the coefficients defined by (6.1) belong to $BV(\Omega) \cap L^\infty(\Omega)$, and $D_\xi a^\xi \in \mathcal{M}(\Omega)$ for every $\xi \in \mathbb{S}^{N-1}$ (see [3, Theorem 3.107]). Therefore, taking into account also (6.2) we deduce that

$$BV(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \cap L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \subset BV_\Xi^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \subset DM^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}).$$

Remark 6.2. The first inclusion is strict. To see this, consider $\Omega := (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$ the function $h : (-1, 1) \rightarrow \mathbb{R}$

$$h(t) := \begin{cases} \sin\left(\frac{1}{t}\right), & t \neq 0 \\ 0, & t = 0, \end{cases}$$

and the stress tensor field

$$\mathbf{A}(x) = \begin{pmatrix} h(x_2) & 0 \\ 0 & H(x_2 - \frac{1}{2}) \end{pmatrix},$$

where H denotes the Heaviside function. Then $\mathbf{A} \in L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$, but $\mathbf{A} \notin BV(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$. Indeed,

$$D_{\mathbf{e}_2} A_{11} = -\frac{\cos\left(\frac{1}{x_2}\right)}{x_2^2} \notin L^1_{\text{loc}}((-1, 1)), \quad \text{for } x_2 \neq 0$$

so this distributional derivative cannot be represented by a finite Radon measure. Let $\Xi := \{\mathbf{e}_1, \mathbf{e}_2, \frac{\mathbf{e}_1 + \mathbf{e}_2}{\sqrt{2}}\}$. A simple computation shows that

$$a^{\mathbf{e}_1} = \sin(1/x_2), \quad a^{\mathbf{e}_2} = H(x_2 - 1/2), \quad a^{\frac{\mathbf{e}_1 + \mathbf{e}_2}{\sqrt{2}}} = 0,$$

so that

$$D_{\mathbf{e}_1} a^{\mathbf{e}_1} = 0, \quad D_{\mathbf{e}_2} a^{\mathbf{e}_2} = \delta_{\{x_2=1/2\}}, \quad D_{\frac{\mathbf{e}_1 + \mathbf{e}_2}{\sqrt{2}}} a^{\frac{\mathbf{e}_1 + \mathbf{e}_2}{\sqrt{2}}} = 0.$$

Hence, $\mathbf{A} \in BV^\infty_\Xi(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$. Finally, note that

$$\text{Div } \mathbf{A} = (0, \delta_{\{x_2=1/2\}}).$$

Proposition 6.3. *Let $\mathbf{A} \in BV^\infty_\Xi(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$. For every $\varphi \in C_c(\Omega; \mathbb{R}^N)$ we have that*

$$(6.5) \quad \int_\Omega \varphi \cdot d\text{Div } \mathbf{A} = \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} \hat{\varphi}_y^\xi(s) dDa_y^\xi(s) \right) d\mathcal{L}^{N-1}(y),$$

and, for every $\mathbf{u} \in BD(\Omega) \cap L^\infty(\Omega; \mathbb{R}^N)$ and $\varphi \in C_c(\Omega)$, it holds

$$(6.6) \quad \int_\Omega \varphi \mathbf{u}^* \cdot d\text{Div } \mathbf{A} = \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} \varphi_y^\xi(s) (\mathbf{u}^*)_y^\xi(s) dDa_y^\xi(s) \right) d\mathcal{L}^{N-1}(y).$$

Proof. We first prove the identity (6.5). For every $\varphi \in C_c(\Omega; \mathbb{R}^N)$, by assumption $(\tilde{H}'')$ we may apply (2.11) with $\varphi = \varphi \cdot \xi$ and $v = a^\xi$ to obtain

$$\int_\Omega \varphi \cdot \xi dD_\xi a^\xi = \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} \hat{\varphi}_y^\xi(s) dDa_y^\xi(s) \right) d\mathcal{L}^{N-1}(y).$$

The claim follows by summing over all $\xi \in \Xi$. Finally, assertion (6.6) follows by

$$\int_\Omega \varphi \mathbf{u}^* \cdot d\text{Div } \mathbf{A} = \sum_{\xi \in \Xi} \int_\Omega \varphi (\mathbf{u}^* \cdot \xi) d(D_\xi a^\xi),$$

which is a consequence of (6.2) and the boundedness of $\mathbf{u}$, and by Proposition 2.2 applied with $f = \varphi(\mathbf{u}^* \cdot \xi)$ and $v = a^\xi$. $\square$

Now, we establish slicing formulas for the traces of DM^∞ -fields.

Proposition 6.4. *Let $\mathbf{A} \in BV^\infty_\Xi(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\Sigma \subset \Omega$ be an oriented countably $\mathcal{H}^{N-1}$ -rectifiable set. Then, for every test function $\varphi \in C_c(\Omega; \mathbb{R}^N)$, we have*

$$(6.7) \quad \langle \text{Tr}^\pm(\mathbf{A}, \Sigma), \varphi \rangle = \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\sum_{s \in \Sigma_y^\xi} (a_y^\xi)^\pm(s) \nu_{\Sigma_y^\xi}(s) \hat{\varphi}_y^\xi(s) \right) d\mathcal{L}^{N-1}(y);$$

i.e.,

$$(6.8) \quad \text{Tr}^\pm(\mathbf{A}, \Sigma) \mathcal{H}^{N-1} \llcorner \Sigma = \sum_{\xi \in \Xi} \mathcal{L}^{N-1} \llcorner \Omega^\xi \otimes_{\mathcal{M}} (a_y^\xi)^\pm(s) \nu_{\Sigma_y^\xi}(s) \xi \mathcal{H}^0 \llcorner \Sigma_y^\xi,$$

as measures on Σ , where $\nu_{\Sigma_y^\xi} \in \{-1, 1\}$ is the normal of Σ_y^ξ oriented as

$$(6.9) \quad \nu_{\Sigma_y^\xi}(s) = \text{sign}(\nu_\Sigma(y + s\xi) \cdot \xi).$$

Finally, for every $\mathbf{v} \in L^1_{\mathcal{H}^{N-1} \llcorner \Sigma}(\Omega; \mathbb{R}^N)$

$$(6.10) \quad \int_{\Sigma} \mathbf{Tr}^\pm(\mathbf{A}, \Sigma) \cdot \mathbf{v} d\mathcal{H}^{N-1} = \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\sum_{s \in \Sigma_y^\xi} (a_y^\xi)^\pm(s) \nu_{\Sigma_y^\xi}(s) \hat{v}_y^\xi(s) \right) d\mathcal{L}^{N-1}(y).$$

Proof. We first prove that for every open set $\Omega' \subset \mathbb{R}^N$ with C^1 boundary, and for every test function $\varphi \in C_c^1(\Omega; \mathbb{R}^N)$,

$$(6.11) \quad \int_{\partial\Omega'} \varphi \cdot \mathbf{Tr}(\mathbf{A}, \partial\Omega') d\mathcal{H}^{N-1} = - \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\sum_{s \in \partial(\Omega')_y^\xi} (a_y^\xi)^+(s) \nu_{(\Omega')_y^\xi}(s) \hat{\varphi}_y^\xi(s) \right) d\mathcal{L}^{N-1}(y).$$

In order to prove (6.11), we recall that by (2.26) and (2.23)

$$\int_{\partial\Omega'} \varphi \cdot \mathbf{Tr}(\mathbf{A}, \partial\Omega') d\mathcal{H}^{N-1} = \int_{\Omega'} \mathbf{A} : \nabla \varphi dx + \int_{\Omega'} \varphi \cdot d\text{Div } \mathbf{A}, \quad \forall \varphi \in C_c^1(\Omega; \mathbb{R}^N).$$

Then, by (6.5) and an analogous computation as in (6.3) we get

$$(6.12) \quad \begin{aligned} & \int_{\partial\Omega'} \varphi \cdot \mathbf{Tr}(\mathbf{A}, \partial\Omega') d\mathcal{H}^{N-1} \\ &= \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\int_{(\Omega')_y^\xi} \hat{\varphi}_y^\xi(s) dDa_y^\xi(s) + \int_{(\Omega')_y^\xi} a_y^\xi(s) (\hat{\varphi}_y^\xi)'(s) ds \right) d\mathcal{L}^{N-1}(y). \end{aligned}$$

Now, for every fixed $\xi \in \Xi$ and for $\mathcal{L}^{N-1}$ -a.e. $y \in \xi^\perp$, integrating by parts we have

$$\int_{(\Omega')_y^\xi} a_y^\xi(s) (\hat{\varphi}_y^\xi)'(s) ds = - \int_{(\Omega')_y^\xi} \hat{\varphi}_y^\xi(s) dDa_y^\xi(s) - \sum_{s \in \partial(\Omega')_y^\xi} (a_y^\xi)^+(s) \nu_{(\Omega')_y^\xi}(s) \hat{\varphi}_y^\xi(s),$$

which inserted in (6.12) gives (6.11). Arguing as in the proof of (3.8), we deduce the analogous identity for every $\varphi \in C_c(\Omega; \mathbb{R}^N)$. From this and a standard localization argument, see (2.27), we immediately get (6.7) which, in turn, implies (6.8). In particular, (6.10) follows from (6.8) and (2.5). $\square$

The following theorem provides a slicing representation for the pairing (5.1).

Theorem 6.5. *Let $\mathbf{A} \in \text{BV}_\Xi^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in \text{BD}(\Omega) \cap L^\infty(\Omega; \mathbb{R}^N)$ be such that assumptions (H_*) and (H_c) hold. Then, for every $\varphi \in C_c(\Omega)$, we have*

$$(6.13) \quad \int_{\Omega} \varphi d(\mathbf{A} : E\mathbf{u}) = \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} \varphi_y^\xi(s) d(a_y^\xi, D\hat{u}_y^\xi)(s) \right) d\mathcal{L}^{N-1}(y).$$

Proof. We subdivide the proof into steps.

Step 1 (Lebesgue part). From Theorem 2.18(i) and Fubini's Theorem,

$$\begin{aligned}
\int_{\Omega} \varphi d(\mathbf{A} : E\mathbf{u})^a &= \int_{\Omega} \varphi \mathbf{A} : e(\mathbf{u}) d\mathcal{L}^N(x) \\
&= \sum_{\xi \in \Xi} \int_{\Omega} \varphi a^{\xi}(x) (e(\mathbf{u})(x) \xi \cdot \xi) d\mathcal{L}^N(x) \\
&= \sum_{\xi \in \Xi} \int_{\xi^{\perp}} \left(\int_{\Omega_y^{\xi}} a_y^{\xi}(s) (\hat{u}_y^{\xi})'(s) \varphi_y^{\xi}(s) ds \right) d\mathcal{L}^{N-1}(y) \\
&= \sum_{\xi \in \Xi} \int_{\xi^{\perp}} \left(\int_{\Omega_y^{\xi}} \varphi_y^{\xi}(s) (a_y^{\xi}, D\hat{u}_y^{\xi})^a ds \right) d\mathcal{L}^{N-1}(y),
\end{aligned}$$

where we used Theorem 2.12(i) for $N = 1$, since $\hat{u}_y^{\xi}$ is bounded and so it belongs to $L^1(\Omega_y^{\xi}, |Da_y^{\xi}|)$ and this concludes the proof of *Step 1*.

Step 2 (Jump part). From Theorem 5.6(ii), (6.10) with $\mathbf{v} := \varphi(\mathbf{u}^+ - \mathbf{u}^-)$, and Theorem 2.12(ii), we get

$$\begin{aligned}
\int_{\Omega} \varphi d(\mathbf{A} : E\mathbf{u})^j &= \int_{J_{\mathbf{u}}} \varphi \frac{\mathbf{Tr}^+(\mathbf{A}, J_{\mathbf{u}}) + \mathbf{Tr}^-(\mathbf{A}, J_{\mathbf{u}})}{2} \cdot (\mathbf{u}^+ - \mathbf{u}^-) d\mathcal{H}^{N-1} \\
&= \sum_{\xi \in \Xi} \int_{\xi^{\perp}} \left(\sum_{s \in J_{\hat{u}_y^{\xi}}} \varphi_y^{\xi}(s) (a_y^{\xi})^*(s) \nu_{\hat{u}_y^{\xi}}(s) \left((\hat{u}_y^{\xi})^+(s) - (\hat{u}_y^{\xi})^-(s) \right) \right) d\mathcal{L}^{N-1}(y) \\
&= \sum_{\xi \in \Xi} \int_{\xi^{\perp}} \left(\int_{J_{\hat{u}_y^{\xi}}} \varphi_y^{\xi}(s) d(a_y^{\xi}, D\hat{u}_y^{\xi})^j(s) \right) d\mathcal{L}^{N-1}(y),
\end{aligned}$$

where in the last equality we used again Theorem 2.12(ii) for $N = 1$.

Step 3 (Cantor part). We recall that, under assumption (H_c) , by Theorem 5.6(iii) the representation $(\mathbf{A} : E\mathbf{u})^c = \tilde{\mathbf{A}} : E^c \mathbf{u}$ holds. From this and (2.19), we have

$$\begin{aligned}
\int_{\Omega} \varphi d(\mathbf{A} : E\mathbf{u})^c &= \int_{\Omega} \varphi \tilde{\mathbf{A}} : dE^c \mathbf{u} = \sum_{\xi \in \Xi} \int_{\Omega} \varphi \tilde{a}^{\xi} d(E^c \mathbf{u} \xi \cdot \xi) \\
&= \sum_{\xi \in \Xi} \int_{\xi^{\perp}} \left(\int_{\Omega_y^{\xi}} \varphi_y^{\xi}(s) (\tilde{a}_y^{\xi})(s) dD^c \hat{u}_y^{\xi}(s) \right) d\mathcal{L}^{N-1}(y).
\end{aligned}$$

Now, by applying Theorem 2.12(iii) in dimension one and Remark 2.13, we obtain

$$\begin{aligned}
&\sum_{\xi \in \Xi} \int_{\xi^{\perp}} \left(\int_{\Omega_y^{\xi}} \varphi_y^{\xi}(s) (\tilde{a}_y^{\xi})(s) dD^c \hat{u}_y^{\xi}(s) \right) d\mathcal{L}^{N-1}(y) \\
&= \sum_{\xi \in \Xi} \int_{\xi^{\perp}} \left(\int_{\Omega_y^{\xi}} \varphi_y^{\xi}(s) d(a_y^{\xi}, D\hat{u}_y^{\xi})^c(s) \right) d\mathcal{L}^{N-1}(y),
\end{aligned}$$

and this concludes the proof of (6.13). $\square$

7. PAIRING FOR (POSSIBLY) UNBOUNDED BD FUNCTIONS

A natural extension of the pairing defined in (5.1) would be to consider possibly unbounded functions $\mathbf{u} \in BD(\Omega)$, requiring that $\mathbf{u}^* \in L^1_{|\text{Div } \mathbf{A}|}(\Omega; \mathbb{R}^N)$, still under assumption (H_*) . In this case, the functional in (5.1) would remain well defined; however, in general, it would fail to be a Radon measure. We recall that the strategy developed in [21, Proposition 4.4] to remove the assumption $\mathbf{u} \in L^\infty(\Omega; \mathbb{R}^N)$ - namely, passing to truncations - does not apply in the present setting, since BD is not closed under truncations, even smooth ones.

7.1. A new pairing by slicing. In this section, we introduce an alternative definition of pairing, which is given by slicing.

First, we study the measurability of the one-dimensional pairing. Let us fix a frame Ξ , a matrix $\mathbf{A} \in BV^\infty_\Xi(\Omega; \mathbb{R}^{N \times N}_{\text{sym}})$, and a function $\mathbf{u} \in BD(\Omega)$. For every $\xi \in \Xi$ and for $\mathcal{H}^{N-1}$ -a.e. $y \in \Omega^\xi$, let $(a_y^\xi, D\hat{u}_y^\xi)$ denote the one-dimensional pairing on Ω_y^ξ between the locally bounded BV function a_y^ξ and the measure derivative $D\hat{u}_y^\xi$ of the BV function $\hat{u}_y^\xi$. By Theorem 2.12, this pairing admits the representation

$$(7.1) \quad (a_y^\xi, D\hat{u}_y^\xi) = (a_y^\xi)^* D\hat{u}_y^\xi$$

in the sense of measures on Ω_y^ξ . However, in what follows, we prefer to keep the pairing notation.

Lemma 7.1. *For every $\xi \in \Xi$ and for every test function $\varphi \in C_c(\Omega)$, the function*

$$y \mapsto \int_{\Omega_y^\xi} \varphi_y^\xi(s) d(a_y^\xi, D\hat{u}_y^\xi)(s)$$

is $\mathcal{H}^{N-1}$ -measurable on Ω^ξ .

Proof. By modifying the function a^ξ on a set of Lebesgue measure zero, we may assume that a^ξ is a Borel function on Ω and that $a_y^\xi \in BV_{\text{loc}}(\Omega_y^\xi)$ for every $y \in \Omega^\xi$. For every $x \in \Omega$ we define

$$v_+^\xi(x) := \limsup_{\rho \rightarrow 0^+} \frac{1}{\rho} \int_0^\rho a^\xi(x + \tau\xi) d\tau \quad \text{and} \quad v_-^\xi(x) := \limsup_{\rho \rightarrow 0^+} \frac{1}{\rho} \int_{-\rho}^0 a^\xi(x + \tau\xi) d\tau.$$

By Fubini's theorem v_+^ξ and v_-^ξ are Borel functions on Ω . Therefore $F := \{x \in \Omega : v_+^\xi(x) \neq v_-^\xi(x)\}$ is a Borel set. For every $y \in \Omega^\xi$ we have for every $s \in \Omega_y^\xi$

$$(v_+^\xi)_y^\xi(s) = \limsup_{\rho \rightarrow 0^+} \frac{1}{\rho} \int_s^{s+\rho} a_y^\xi(\tau) d\tau \quad \text{and} \quad (v_-^\xi)_y^\xi(s) = \limsup_{\rho \rightarrow 0^+} \frac{1}{\rho} \int_{s-\rho}^s a_y^\xi(\tau) d\tau$$

and

$$(v_+^\xi)_y^\xi(s) = (v_-^\xi)_y^\xi(s) = a_y^\xi(s) \quad \mathcal{L}^1\text{-a.e. in } \Omega_y^\xi$$

thanks to Lebesgue's differentiation theorem. By elementary properties of BV functions of one variable, for every $y \in \Omega^\xi$ this implies

$$\lim_{\rho \rightarrow 0^+} \frac{1}{2\rho} \int_{s-\rho}^{s+\rho} |a_y^\xi(\tau) - (v_+^\xi)_y^\xi(s)| d\tau = 0, \quad \text{for every } s \in \Omega_y^\xi \setminus F_y^\xi,$$

$$(a_y^\xi)^+(s) = (v_+^\xi)_y^\xi(s) \quad \text{and} \quad (a_y^\xi)^-(s) = (v_-^\xi)_y^\xi(s), \quad \text{for every } s \in \Omega_y^\xi \cap F_y^\xi.$$

This implies that

$$(a_y^\xi)^*(s) = \begin{cases} (v_+^\xi)_y(s), & \text{for every } s \in \Omega_y^\xi \setminus F_y^\xi, \\ \frac{(v_+^\xi)_y(s) + (v_-^\xi)_y(s)}{2}, & \text{for every } s \in \Omega_y^\xi \cap F_y^\xi, \end{cases}$$

and $J_{a_y^\xi} = F_y^\xi$ for every $y \in \Omega^\xi$. Therefore, by (7.1), we have

$$\begin{aligned} \int_{\Omega_y^\xi \setminus J_{a_y^\xi}} \varphi_y^\xi(s) d(a_y^\xi, D\hat{u}_y^\xi)(s) &= \int_{\Omega_y^\xi \setminus F_y^\xi} \varphi_y^\xi(s) (v_+^\xi)_y dD\hat{u}_y^\xi(s) \\ \int_{\Omega_y^\xi \cap J_{a_y^\xi}} \varphi_y^\xi(s) d(a_y^\xi, D\hat{u}_y^\xi)(s) &= \int_{\Omega_y^\xi \cap F_y^\xi} \varphi_y^\xi(s) \frac{(v_+^\xi)_y(s) + (v_-^\xi)_y(s)}{2} dD\hat{u}_y^\xi(s) \end{aligned}$$

for every $y \in \Omega^\xi$. Hence, the map

$$y \mapsto \int_{\Omega_y^\xi} \varphi_y^\xi(s) \frac{(v_+^\xi)_y(s) + (v_-^\xi)_y(s)}{2} dD\hat{u}_y^\xi(s)$$

is $\mathcal{H}^{N-1}$ -measurable on Ω^ξ , by using (2.18) with the Borel function $g(x) := \varphi(x) \frac{v_+^\xi(x) + v_-^\xi(x)}{2}$. This concludes the proof. $\square$

Given a frame Ξ , a matrix $\mathbf{A} \in \text{BV}_\Xi^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and a function $\mathbf{u} \in BD(\Omega)$, we define the *slicing pairing* as the linear functional $((\mathbf{A} : E\mathbf{u}))_\Xi : C_c^\infty(\Omega) \rightarrow \mathbb{R}$ by

$$(7.2) \quad \langle ((\mathbf{A} : E\mathbf{u}))_\Xi, \varphi \rangle := \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} \varphi_y^\xi(s) d(a_y^\xi, D\hat{u}_y^\xi)(s) \right) d\mathcal{L}^{N-1}(y).$$

Theorem 7.2. *Let $\mathbf{A} \in \text{BV}_\Xi^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in BD(\Omega)$. Then*

(i) *the slicing pairing $((\mathbf{A} : E\mathbf{u}))_\Xi$ is a Radon measure in Ω and*

$$((\mathbf{A} : E\mathbf{u}))_\Xi = \sum_{\xi \in \Xi} \mathcal{L}^{N-1} \llcorner \Omega^\xi \otimes_{\mathcal{M}} (a_y^\xi, D\hat{u}_y^\xi) \llcorner \Omega_y^\xi$$

in the sense of measures;

(ii) *the measure $((\mathbf{A} : E\mathbf{u}))_\Xi$ is absolutely continuous with respect to $|E\mathbf{u}|$, and it holds that*

$$(7.3) \quad |((\mathbf{A} : E\mathbf{u}))_\Xi|(B) \leq C_\Xi \|\mathbf{A}\|_{L^\infty(U; \mathbb{R}^{N \times N})} |E\mathbf{u}|(B) \quad \text{for every Borel set } B \subset U \Subset \Omega,$$

where $C_\Xi := \left(\max_{\xi \in \Xi} \|\mathbf{B}^\xi\| \right) |\Xi|$;

(iii) *(consistency) if $\mathbf{u} \in BD(\Omega) \cap L^\infty(\Omega; \mathbb{R}^N)$ such that (H_c) and (H_*) hold, then $((\mathbf{A} : E\mathbf{u}))_\Xi$ coincides with the pairing defined in (5.1); i.e.,*

$$((\mathbf{A} : E\mathbf{u}))_\Xi = (\mathbf{A} : E\mathbf{u}),$$

in the sense of measures.

Finally, if $\mathbf{u} \in BD(\Omega)$ and $\mathbf{A} \in BV(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \cap L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$, then

$$((\mathbf{A} : E\mathbf{u}))_\Xi = \mathbf{A}^* : E\mathbf{u},$$

in the sense of measures.

Proof. As for (i), we note that for every $\xi \in \Xi$ and for $\mathcal{H}^{N-1}$ -a.e. $y \in \xi^\perp$, by (H'') we have $a_y^\xi \in BV_{\text{loc}}(\Omega_y^\xi) \cap L_{\text{loc}}^\infty(\Omega_y^\xi)$ and, by Proposition 2.7 (ii), $\hat{u}_y^\xi \in BV_{\text{loc}}(\Omega_y^\xi)$. Therefore, for such y , the pairing $(a_y^\xi, D\hat{u}_y^\xi)$ is well defined and is a Radon measure in Ω_y^ξ . Moreover, for every Borel set $B \subset U \Subset \Omega$ we have

$$|(a_y^\xi, D\hat{u}_y^\xi)|(B_y^\xi) \leq \|a_y^\xi\|_{L^\infty(U_y^\xi)} |D\hat{u}_y^\xi|(B_y^\xi) \leq \max_{\xi \in \Xi} \|\mathbf{B}^\xi\| \|\mathbf{A}\|_{L^\infty(U; \mathbb{R}_{\text{sym}}^{N \times N})} |D\hat{u}_y^\xi|(B_y^\xi).$$

Let $\varphi \in C_c^\infty(\Omega)$. For every $\xi \in \Xi$ and for $\mathcal{H}^{N-1}$ -a.e. $y \in \xi^\perp$, we estimate

$$\left| \int_{\Omega_y^\xi} \varphi_y^\xi(s) d(a_y^\xi, D\hat{u}_y^\xi)(s) \right| \leq \|\varphi\|_{L^\infty(\Omega)} |(a_y^\xi, D\hat{u}_y^\xi)|(\Omega_y^\xi).$$

Hence the map

$$y \mapsto \int_{\Omega_y^\xi} \varphi_y^\xi(s) d(a_y^\xi, D\hat{u}_y^\xi)(s)$$

(which is $\mathcal{H}^{N-1}$ -measurable on Ω^ξ by Lemma 7.1) belongs to $L^1(\Omega^\xi)$, thanks to the above estimate and to (2.17). Therefore, the linear functional on $C_c^\infty(\Omega)$ defined by (7.2) is well defined and satisfies the estimate

$$|\langle ((\mathbf{A} : E\mathbf{u}))_\Xi, \varphi \rangle| \leq \|\varphi\|_{L^\infty(\Omega)} \left(\max_{\xi \in \Xi} \|\mathbf{B}^\xi\| \right) \|\mathbf{A}\|_{L^\infty(U; \mathbb{R}_{\text{sym}}^{N \times N})} \sum_{\xi \in \Xi} \int_{\xi^\perp} |D\hat{u}_y^\xi|(\Omega_y^\xi) d\mathcal{L}^{N-1}(y).$$

Using (2.17), we obtain

$$|\langle ((\mathbf{A} : E\mathbf{u}))_\Xi, \varphi \rangle| \leq C_\Xi \|\varphi\|_{L^\infty(\Omega)} \|\mathbf{A}\|_{L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})} |E\mathbf{u}|(\Omega).$$

In particular, the functional is bounded with respect to the sup norm on $C_c^\infty(\Omega)$ and thus extends uniquely to a bounded linear functional on $C_c(\Omega)$. Therefore, it can be represented by a Radon measure, still denoted by $((\mathbf{A} : E\mathbf{u}))_\Xi$. By construction, its action on test functions is exactly given by (7.2), which directly translates to the representation formula in (i).

For what concerns (ii), for every Borel set $B \subset U \Subset \Omega$ we have

$$\begin{aligned} |((\mathbf{A} : E\mathbf{u}))_\Xi|(B) &\leq \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\int_{B_y^\xi} d|(a_y^\xi, D\hat{u}_y^\xi)|(s) \right) d\mathcal{L}^{N-1}(y) \\ &\leq \sum_{\xi \in \Xi} \int_{\xi^\perp} \|a_y^\xi\|_{L^\infty(U_y^\xi)} |D\hat{u}_y^\xi|(B_y^\xi) d\mathcal{L}^{N-1}(y) \leq C_\Xi \|\mathbf{A}\|_{L^\infty(U; \mathbb{R}_{\text{sym}}^{N \times N})} |E\mathbf{u}|(B). \end{aligned}$$

Assertion (iii) is an immediate consequence of the analysis developed in Section 6. In particular, it is a consequence of the boundedness of $\mathbf{u}$, of the assumptions (H_c) and (H_{*}), of Theorem 6.5 and definition (7.2). Finally, if $\mathbf{A} \in BV(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \cap L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$, then by Theorem 2.8 for every $\mathbf{u} \in BD(\Omega)$ and for every $\varphi \in C_c(\Omega)$ we get

$$\begin{aligned} \int_\Omega \varphi \mathbf{A}^* : dE\mathbf{u} &= \sum_{\xi \in \Xi} \int_\Omega \varphi (\mathbf{A}^* : \mathbf{B}^\xi) \xi \otimes \xi : dE\mathbf{u} = \sum_{\xi \in \Xi} \int_\Omega \varphi(x) (a_x^\xi)^*(x) d(E\mathbf{u}\xi \cdot \xi)(x) \\ &= \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} \varphi_y^\xi(s) (a_y^\xi)^*(s) dD\hat{u}_y^\xi(s) \right) d\mathcal{L}^{N-1}(y) = ((\mathbf{A} : E\mathbf{u}))_\Xi, \end{aligned}$$

where we used Theorem 2.12 for $N = 1$. $\square$

Remark 7.3. If $\mathbf{A} \in W^{1,1}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \cap L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in BD(\Omega)$, then

$$((\mathbf{A} : E\mathbf{u}))_\Xi = \tilde{\mathbf{A}} : E\mathbf{u},$$

in the sense of measures in Ω , since for every $\boldsymbol{\xi} \in \Xi$ and for $\mathcal{L}^{N-1}$ -a.e. $y \in \boldsymbol{\xi}^\perp$ we have $(a_y^\xi, D\hat{u}_y^\xi) = \tilde{a}_y^\xi dD\hat{u}_y^\xi$ in the sense of measures in Ω_y^ξ .

Moreover, if $\mathbf{A} \in BV_\Xi^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in LD(\Omega)$, it holds that

$$((\mathbf{A} : E\mathbf{u}))_\Xi = \mathbf{A} : e(\mathbf{u})\mathcal{L}^N,$$

in the sense of measures, since for every $\boldsymbol{\xi} \in \Xi$ and for $\mathcal{L}^{N-1}$ -a.e. $y \in \boldsymbol{\xi}^\perp$ we have $(a_y^\xi, D\hat{u}_y^\xi) = a_y^\xi (\hat{u}_y^\xi)' \mathcal{L}^1$ in the sense of measures in Ω_y^ξ (see (2.16)) (see [30, Remark 7.6]).

Remark 7.4 (Invariance of the slicing pairing for BV matrix fields). We emphasize that the previous definition depends on the particular choice of the set Ξ , which affects also the singularities of the chosen field $\mathbf{A}$. We recall that Ξ is a finite set of directions required to reconstruct a symmetric matrix from rank-one symmetric matrices. If we select an alternative frame Ξ' and reconstruct the functional, we obtain a different object. The invariance is recovered under suitable regularity assumptions. For instance, if $\mathbf{A} \in BV(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \cap L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$, then the definition (7.2) is independent of Ξ , since the slicing formula reconstructs the integral $\int_\Omega \varphi \mathbf{A}^* : dE\mathbf{u}$, which is intrinsically coordinate-independent (see the last assertion of Theorem 7.2). In this case, we use the notation $((\mathbf{A} : E\mathbf{u}))$ for the pairing. For non-smooth fields (such as those exhibiting fractures or microstructural shear bands aligned with the grid Ξ), the pairing (7.2) captures the concentrations of energy along the directions of Ξ .

Remark 7.5. We observe that, given two different sets of directions Ξ and Ξ' , the conditions

$$\mathbf{A} \in BV_\Xi^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \quad \text{and} \quad \mathbf{A} \in BV_{\Xi'}^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$$

do not, in general, imply that $\mathbf{A} \in BV(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$. However, in Appendix A we identify an algebraic condition ensuring that $\mathbf{A} \in BV(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ when $N = 2$. In this case, by Remark 7.4, we further have that

$$((\mathbf{A} : E\mathbf{u}))_\Xi = ((\mathbf{A} : E\mathbf{u}))_{\Xi'}.$$

The following result is a consequence of the standard decomposition of the pairing measure in its Lebesgue, jump and Cantor parts.

Proposition 7.6. *Given $\mathbf{A} \in BV_\Xi^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in BD(\Omega)$, for every $\varphi \in C_c(\Omega)$ we have*

$$\begin{aligned} \langle ((\mathbf{A} : E\mathbf{u}))_\Xi^a, \varphi \rangle &= \sum_{\boldsymbol{\xi} \in \Xi} \int_{\boldsymbol{\xi}^\perp} \left(\int_{\Omega_y^\xi} \varphi_y^\xi(s) a_y^\xi(s) (\hat{u}_y^\xi)'(s) ds \right) d\mathcal{L}^{N-1}(y) \\ &= \sum_{\boldsymbol{\xi} \in \Xi} \int_{\boldsymbol{\xi}^\perp} \left(\int_{\Omega_y^\xi} \varphi_y^\xi(s) (a_y^\xi, D\hat{u}_y^\xi)^a(s) ds \right) d\mathcal{L}^{N-1}(y); \\ \langle ((\mathbf{A} : E\mathbf{u}))_\Xi^j, \varphi \rangle &= \sum_{\boldsymbol{\xi} \in \Xi} \int_{\boldsymbol{\xi}^\perp} \left(\sum_{s \in J_{\hat{u}_y^\xi}} \varphi_y^\xi(s) (a_y^\xi)^*(s) \nu_{\hat{u}_y^\xi}(s) ((\hat{u}_y^\xi)^+(s) - (\hat{u}_y^\xi)^-(s)) \right) d\mathcal{L}^{N-1}(y) \\ &= \sum_{\boldsymbol{\xi} \in \Xi} \int_{\boldsymbol{\xi}^\perp} \left(\int_{\Omega_y^\xi} \varphi_y^\xi(s) d(a_y^\xi, D\hat{u}_y^\xi)^j(s) \right) d\mathcal{L}^{N-1}(y); \end{aligned}$$

$$\langle ((\mathbf{A} : E\mathbf{u}))_{\Xi}^c, \varphi \rangle = \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} \varphi_y^\xi(s) d(a_y^\xi, D\hat{u}_y^\xi)^c(s) \right) d\mathcal{L}^{N-1}(y).$$

7.2. The Gauss-Green formula for the slicing pairing. We begin by establishing a preliminary result that will be instrumental in proving the Gauss-Green formula for possibly unbounded BD functions. Specifically, we extend Proposition 6.3 to this setting.

Proposition 7.7. *Let $\mathbf{A} \in BV_{\Xi}^\infty(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$. For every $\mathbf{u} \in BD(\Omega)$, the assumptions*

$$(7.4) \quad \mathbf{u}^* \cdot \xi \in L^1(\Omega, |D_\xi a^\xi|) \text{ for every } \xi \in \Xi,$$

and

$$(7.5) \quad \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} |(\hat{u}_y^\xi)^*(s)| d|Da_y^\xi|(s) \right) d\mathcal{L}^{N-1}(y) < +\infty \text{ for every } \xi \in \Xi,$$

are equivalent. Moreover, under either of the equivalent conditions above, we have

$$(7.6) \quad \int_{\Omega} \mathbf{u}^* \cdot d\text{Div } \mathbf{A} = \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} (\hat{u}_y^\xi)^*(s) dDa_y^\xi(s) \right) d\mathcal{L}^{N-1}(y).$$

Proof. The equivalence between (7.4) and (7.5) is an immediate consequence of (2.10) combined with [3, Corollary 2.29], and (2.5) with $f = |\mathbf{u}^* \cdot \xi|$, which yield

$$\int_{\Omega} |\mathbf{u}^* \cdot \xi| d|D_\xi a^\xi| = \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} |(\hat{u}_y^\xi)^*(s)| d|Da_y^\xi|(s) \right) d\mathcal{L}^{N-1}(y) \text{ for every } \xi \in \Xi.$$

By (6.2), (7.4), (2.12) with $v = a^\xi$ and $f = \mathbf{u}^* \cdot \xi$, we get

$$\begin{aligned} \int_{\Omega} \mathbf{u}^* \cdot d\text{Div } \mathbf{A} &= \int_{\Omega} \sum_{\xi \in \Xi} (\mathbf{u}^* \cdot \xi) d(D_\xi a^\xi) = \sum_{\xi \in \Xi} \int_{\Omega} (\mathbf{u}^* \cdot \xi) d(D_\xi a^\xi) \\ &= \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\int_{\Omega_y^\xi} (\hat{u}_y^\xi)^*(s) dDa_y^\xi(s) \right) d\mathcal{L}^{N-1}(y). \end{aligned}$$

This concludes the proof of (7.6). $\square$

The next proposition provides a Gauss-Green formula by slicing, which will be the key ingredient in the proof of the Gauss-Green formula, see Theorem 7.9 below.

Proposition 7.8. *Let $\mathbf{A} \in BV_{\Xi, \text{loc}}^\infty(\mathbb{R}^N; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in BD_{\text{loc}}(\mathbb{R}^N)$. Let $F \subset \mathbb{R}^N$ be a bounded set of finite perimeter. Assume that*

$$(7.7) \quad \int_{\xi^\perp} \left(\int_{(F^1)_y^\xi} |(\hat{u}_y^\xi)^*(s)| d|Da_y^\xi|(s) \right) d\mathcal{L}^{N-1}(y) < +\infty \text{ for every } \xi \in \Xi.$$

Then the following Gauss-Green formulas by slicing hold

$$\begin{aligned}
(7.8) \quad & \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\int_{(F^1)_y^\xi} (\hat{u}_y^\xi)^*(s) dDa_y^\xi(s) \right) d\mathcal{L}^{N-1}(y) + \int_{F^1} d((\mathbf{A} : E\mathbf{u}))_\Xi \\
&= - \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\sum_{s \in (\partial^* F)_y^\xi} (\hat{u}_y^\xi)^+(s) (a_y^\xi)^+(s) \nu_{F_y^\xi}(s) \right) d\mathcal{L}^{N-1}(y), \\
(7.9) \quad & \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\int_{(F^1)_y^\xi \cup (\partial^* F)_y^\xi} (\hat{u}_y^\xi)^*(s) dDa_y^\xi(s) \right) d\mathcal{L}^{N-1}(y) + \int_{F^1 \cup \partial^* F} d((\mathbf{A} : E\mathbf{u}))_\Xi \\
&= \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\sum_{s \in (\partial^* F)_y^\xi} (\hat{u}_y^\xi)^-(s) (a_y^\xi)^-(s) \nu_{F_y^\xi}(s) \right) d\mathcal{L}^{N-1}(y),
\end{aligned}$$

where

$$\nu_{F_y^\xi}(s) = -\operatorname{sign}(\nu_F(y + s\xi) \cdot \xi).$$

Proof. We only prove (7.8), the proof of the other one being similar. For every $\xi \in \Xi$ and for $\mathcal{L}^{N-1}$ -a.e. $y \in \xi^\perp$, by Corollary 2.15 we get

$$\int_{(F^1)_y^\xi} (\hat{u}_y^\xi)^*(s) dDa_y^\xi(s) + \int_{(F^1)_y^\xi} d(a_y^\xi, D\hat{u}_y^\xi)(s) = - \sum_{s \in (\partial^* F)_y^\xi} (a_y^\xi)^+(s) \nu_{(\partial^* F)_y^\xi}(s) (\hat{u}_y^\xi)^+(s),$$

where we used that, for $\mathcal{L}^{N-1}$ -a.e. $y \in \xi^\perp$, $\partial^*(F_y^\xi) = (\partial^* F)_y^\xi$ and $(F^1)_y^\xi = (F_y^\xi)^1$. We will integrate over $\xi^\perp$ and sum over $\xi \in \Xi$. Notice that the finiteness of the integral on the right hand side is ensured by the (7.7) and (7.2). By using (7.2) we have

$$\int_{F^1} d((\mathbf{A} : E\mathbf{u}))_\Xi = \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\int_{(F^1)_y^\xi} d(a_y^\xi, D\hat{u}_y^\xi)(s) \right) d\mathcal{L}^{N-1}(y).$$

This concludes the proof of (7.8). The second identity (7.9) follows analogously by using the second formula in Corollary 2.15. $\square$

We now combine the previous results to derive the Gauss-Green formula for possibly unbounded BD functions.

Theorem 7.9. *Let $\mathbf{A} \in BV_{\Xi, \text{loc}}^\infty(\mathbb{R}^N; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in BD_{\text{loc}}(\mathbb{R}^N)$. Let $F \subset \mathbb{R}^N$ be a bounded set of finite perimeter. Assume that*

$$(7.10) \quad \mathbf{u}^* \cdot \xi \in L_{\text{loc}}^1(\mathbb{R}^N, |D_\xi a^\xi|) \text{ for every } \xi \in \Xi, \quad \mathbf{u}^\pm \in L_{\mathcal{H}^{N-1} \llcorner \partial^* F, \text{loc}}^1(\mathbb{R}^N; \mathbb{R}^N),$$

where $\mathbf{u}^\pm$ are the traces of $\mathbf{u}$ on $\partial^* F$ defined in Proposition 2.6. Then

$$\begin{aligned}
& \int_{F^1} \mathbf{u}^* \cdot d \operatorname{Div} \mathbf{A} + \int_{F^1} d((\mathbf{A} : E\mathbf{u}))_\Xi = - \int_{\partial^* F} \mathbf{u}^+ \cdot \mathbf{Tr}^+(\mathbf{A}, \partial^* F) d\mathcal{H}^{N-1}, \\
& \int_{F^1 \cup \partial^* F} \mathbf{u}^* \cdot d \operatorname{Div} \mathbf{A} + \int_{F^1 \cup \partial^* F} d((\mathbf{A} : E\mathbf{u}))_\Xi = - \int_{\partial^* F} \mathbf{u}^- \cdot \mathbf{Tr}^-(\mathbf{A}, \partial^* F) d\mathcal{H}^{N-1},
\end{aligned}$$

where $\mathbf{Tr}^\pm(\mathbf{A}, \partial^* F)$ are the normal traces of $\mathbf{A}$ on $\partial^* F$ oriented by the interior unit normal vector.

Proof. By the second assumption in (7.10) and (6.10) with $\mathbf{v} = \mathbf{u}^\pm$, we obtain

$$\int_{\partial^* F} \mathbf{u}^\pm \cdot \mathbf{Tr}^\pm(\mathbf{A}, \partial^* F) d\mathcal{H}^{N-1} = \sum_{\xi \in \Xi} \int_{\xi^\perp} \left(\sum_{s \in \partial^* F_y^\xi} (\hat{u}_y^\xi)^\pm(s) (a_y^\xi)^\pm(s) \nu_{(\partial^* F)_y^\xi}(s) \right) d\mathcal{L}^{N-1}(y).$$

Combining these facts with Propositions 7.7 and 7.8, we obtain the two Gauss-Green formulas stated above. $\square$

Remark 7.10. If $F \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary, then the second assumption in (7.10) is satisfied (see Theorem 2.4) and it holds

$$\int_F \mathbf{u}^* \cdot d\text{Div } \mathbf{A} + \int_F d((\mathbf{A} : E\mathbf{u}))_\Xi = - \int_{\partial F} \gamma^+(\mathbf{u}) \cdot \mathbf{Tr}^+(\mathbf{A}, \partial F) d\mathcal{H}^{N-1}.$$

Remark 7.11. Let $\mathbf{A} \in C^1(\mathbb{R}^N; \mathbb{R}_{\text{sym}}^{N \times N})$, then (H_*) is satisfied, and $\mathbf{Tr}^+(\mathbf{A}, \partial\Omega) = -\mathbf{A}\nu_\Omega$. Let $\mathbf{u} \in BD_{\text{loc}}(\mathbb{R}^N)$, then the smoothness of $\mathbf{A}$ implies that $\mathbf{u} = \mathbf{u}^*$ $\mathcal{L}^N$ -a.e. and $\mathbf{u} \in L^1_{|\text{Div } \mathbf{A}|, \text{loc}}(\mathbb{R}^N; \mathbb{R}^N)$. By (5.17) and (2.2) we obtain that

$$(7.11) \quad \int_\Omega \mathbf{u} \cdot \text{Div } \mathbf{A} dx + \int_\Omega \mathbf{A} : dE\mathbf{u} = \int_{\partial\Omega} \mathbf{A} : (\gamma^+(\mathbf{u}) \odot \nu_\Omega) d\mathcal{H}^{N-1}.$$

Thus, identity (7.11) extends the Gauss-Green formula (2.14). Moreover, it provides an extension of (5.19) to BD functions that are not necessarily bounded.

8. COMPARISON WITH THE KOHN-TEMAM PAIRING IN PLASTICITY THEORY

We refer to [30, Ch. II, Sec. 7] and [29]. Let Ω be a bounded and Lipschitz subset of $\mathbb{R}^N$. Define

$$\Sigma(\Omega) := \left\{ \mathbf{A} \in L^2(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) : \text{Div } \mathbf{A} \in L^N(\Omega; \mathbb{R}^N), \mathbf{A}^D \in L^\infty(\Omega; \mathbb{R}^{N \times N}) \right\},$$

and

$$U(\Omega) := \left\{ \mathbf{u} \in BD(\Omega) : \text{div } \mathbf{u} \in L^2(\Omega) \right\}.$$

Recall that, by (2.4) for every $\mathbf{u} \in H^1(\Omega; \mathbb{R}^N)$,

$$\mathbf{A}^D : e^D(\mathbf{u}) = \mathbf{A} : e(\mathbf{u}) - \frac{1}{N} \text{tr}(e(\mathbf{u})) \text{tr } \mathbf{A}.$$

Observe that, for $\mathbf{u} \in U(\Omega)$,

$$E^D \mathbf{u} = E\mathbf{u} - \frac{1}{N} (\text{div } \mathbf{u}) I, \quad \text{tr } E\mathbf{u} = \text{div } \mathbf{u},$$

and that $\text{tr } \mathbf{A} \text{div } \mathbf{u} \in L^1(\Omega)$. Hence, for $\mathbf{u} \in U(\Omega)$ and $\mathbf{A} \in \Sigma(\Omega)$, one can define the following distributions (see [30, Definitions 7.42-7.43]):

$$(8.1) \quad \langle (\mathbf{A} : E\mathbf{u}), \varphi \rangle := - \int_\Omega \varphi \mathbf{u} \cdot \text{Div } \mathbf{A} dx - \int_\Omega \mathbf{A} : (\mathbf{u} \odot \nabla \varphi) dx,$$

and

$$(8.2) \quad \langle (\mathbf{A}^D : E^D \mathbf{u}), \varphi \rangle := - \frac{1}{N} \int_\Omega \varphi \text{tr } \mathbf{A} \text{div } \mathbf{u} dx + \langle (\mathbf{A} : E\mathbf{u}), \varphi \rangle,$$

for every $\varphi \in C_c^\infty(\Omega)$.

It is proved that, whenever $\mathbf{u} \in U(\Omega)$ and $\mathbf{A} \in \Sigma(\Omega)$, the distributions defined in (8.1)-(8.2) are bounded Radon measures on Ω (see [30, Lemma 7.3] and [29, Theorem 3.2]).

Now, we will extend the notion of the pairing (8.2) to stresses with divergence measure. We give a new definition of $(\mathbf{A}^D : E^D \mathbf{u})$ by using the slicing approach of the previous section only for the term $(\mathbf{A} : E \mathbf{u})$ and by assuming the minimal conditions for the well-posedness of the remaining term.

Let $\mathbf{u} \in BD(\Omega)$ (so that $\operatorname{div} \mathbf{u}$ is a Radon measure) and let $\mathbf{A} \in BV_{\Xi}^{\infty}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$. We define the following distribution: for every $\varphi \in C_c^{\infty}(\Omega)$,

$$(8.3) \quad \langle ((\mathbf{A}^D : E^D \mathbf{u}))_{\Xi}, \varphi \rangle := -\frac{1}{N} \int_{\Omega} \varphi \operatorname{tr} \mathbf{A} d(\operatorname{div} \mathbf{u}) + \sum_{\xi \in \Xi} \int_{\xi^{\perp}} \left(\int_{\Omega_y^{\xi}} \varphi_y^{\xi}(s) d(a_y^{\xi}, D\hat{u}_y^{\xi})(s) \right) d\mathcal{L}^{N-1}(y).$$

Theorem 8.1. *Let $\mathbf{A} \in BV_{\Xi}^{\infty}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$ and $\mathbf{u} \in BD(\Omega)$. Then the distribution $((\mathbf{A}^D : E^D \mathbf{u}))_{\Xi}$ is a Radon measure in Ω , absolutely continuous with respect to $|E \mathbf{u}|$, and there exists a positive constant $\tilde{C}_{\Xi}$ such that*

$$(8.4) \quad |((\mathbf{A}^D : E^D \mathbf{u}))_{\Xi}|(B) \leq \tilde{C}_{\Xi} \|\mathbf{A}\|_{L^{\infty}(U; \mathbb{R}^{N \times N})} |E \mathbf{u}|(B) \quad \text{for every Borel set } B \subset U \Subset \Omega.$$

Moreover, if Ω is a bounded and Lipschitz subset of $\mathbb{R}^N$, for every $\mathbf{u} \in U(\Omega) \cap L^{\infty}(\Omega; \mathbb{R}^N)$ and $\mathbf{A} \in BV_{\Xi}^{\infty}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \cap \Sigma(\Omega)$ (and so assumption (H_*) is satisfied) such that (H_c) holds, the measure $((\mathbf{A}^D : E^D \mathbf{u}))_{\Xi}$ coincides with the pairing defined in (8.2); i.e.,

$$(8.5) \quad ((\mathbf{A}^D : E^D \mathbf{u}))_{\Xi} = (\mathbf{A}^D : E^D \mathbf{u}),$$

in the sense of measures. Finally, if $\mathbf{u} \in C^1(\bar{\Omega}; \mathbb{R}^N)$ and $\mathbf{A} \in BV_{\Xi}^{\infty}(\Omega; \mathbb{R}_{\text{sym}}^{N \times N}) \cap \Sigma(\Omega)$, then

$$(8.6) \quad ((\mathbf{A}^D : E^D \mathbf{u}))_{\Xi} = (\mathbf{A} : e(\mathbf{u}) - \frac{1}{N} \operatorname{tr} \mathbf{A} \operatorname{tr}(e(\mathbf{u}))) \mathcal{L}^N = \mathbf{A}^D : e^D(\mathbf{u}) \mathcal{L}^N.$$

Proof. To prove condition (8.4), let us note that, by (7.3) and (2.1),

$$\begin{aligned} |((\mathbf{A}^D : E^D \mathbf{u}))_{\Xi}|(B) &\leq \frac{1}{N} \int_B |\operatorname{tr} \mathbf{A}| d|\operatorname{div} \mathbf{u}| + |((\mathbf{A} : E \mathbf{u}))_{\Xi}|(B) \\ &\leq \frac{1}{N} \|\operatorname{tr} \mathbf{A}\|_{L^{\infty}(U)} |\operatorname{div} \mathbf{u}|(B) + C_{\Xi} \|\mathbf{A}\|_{L^{\infty}(U; \mathbb{R}_{\text{sym}}^{N \times N})} |E \mathbf{u}|(B) \\ &\leq (1 + C_{\Xi}) \|\mathbf{A}\|_{L^{\infty}(U; \mathbb{R}_{\text{sym}}^{N \times N})} |E \mathbf{u}|(B). \end{aligned}$$

Assertion (8.5) follows by Theorem 7.2(iii), Remark 5.2 and (8.2).

On the other hand, by (8.5) we get

$$((\mathbf{A}^D : E^D \mathbf{u}))_{\Xi} = (\mathbf{A} : e(\mathbf{u}) - \frac{1}{N} \operatorname{tr} \mathbf{A} \operatorname{tr}(e(\mathbf{u}))) \mathcal{L}^N.$$

The second equality in (8.6) follows from (2.4). $\square$

9. AN EXAMPLE: A GRIFFITH-TYPE MODE I OPENING CONFIGURATION

To illustrate the physical relevance of our framework and to highlight how it overcomes the limitations of the classical Kohn-Temam theory, we consider a simple two-dimensional example describing a displacement jump across an interface, modeling a pre-existing crack (i.e., a displacement discontinuity) in the reference configuration.

Let $\Omega = (-1, 1)^2 \subset \mathbb{R}^2$ be the reference domain, divided into two subdomains by the interface $\Gamma = \{x \in \Omega : x_2 = 0\}$, with $\boldsymbol{\nu}_\Gamma = \mathbf{e}_2$. We define a displacement field $\mathbf{u} \in BD(\Omega) \cap L^\infty(\Omega; \mathbb{R}^2)$ representing a normal opening with amplitude $\gamma > 0$

$$\mathbf{u}(x) = \begin{pmatrix} 0 \\ \gamma H(x_2) \end{pmatrix},$$

where H denotes the Heaviside function. Note that $\mathbf{u} \in SBV(\Omega; \mathbb{R}^2)$, and

$$E\mathbf{u} = [\mathbf{u}^+ - \mathbf{u}^-] \odot \boldsymbol{\nu}_\Gamma \mathcal{H}^1 \llcorner \Gamma = (\gamma \mathcal{H}^1 \llcorner \Gamma) (\mathbf{e}_2 \otimes \mathbf{e}_2).$$

The jump of $\mathbf{u}$ across the interface is purely normal (Mode I opening, see [11]), since it is aligned with the normal direction to the interface and no tangential slip occurs. We consider a piecewise constant stress tensor, defined as

$$\mathbf{A}(x) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma(x_2) \end{pmatrix}, \quad \sigma(x_2) = \begin{cases} \sigma^+ & \text{if } x_2 > 0, \\ \sigma^- & \text{if } x_2 < 0, \end{cases}$$

with $\sigma^\pm \in \mathbb{R}$. This tensor corresponds to a Griffith-type normal traction state across the interface, with no shear components. Note that $\mathbf{A} \in BV(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2}) \cap L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$.

In this setting, the divergence of $\mathbf{A}$ is a vector-valued measure concentrated on the interface Γ , namely

$$\text{Div } \mathbf{A} = \begin{pmatrix} 0 \\ (\sigma^+ - \sigma^-) \mathcal{H}^1 \llcorner \Gamma \end{pmatrix}.$$

The singularity of $\text{Div } \mathbf{A}$ with respect to $\mathcal{L}^2$ prevents the application of the classical Kohn-Temam pairing theory, which consequently cannot be used to evaluate the mechanical work across the interface. The framework developed in this paper overcomes this limitation by naturally capturing the interfacial contribution.

Since $\mathbf{A} \in BV(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$, by Remark 5.7 we have

$$(\mathbf{A} : E\mathbf{u}) = \mathbf{A}^* : E\mathbf{u} = \frac{\sigma^+ + \sigma^-}{2} \gamma \mathcal{H}^1 \llcorner \Gamma.$$

This identity provides a consistent definition of the interfacial work associated with the discontinuity of the displacement field. Moreover, it shows that the generalized Gauss–Green formula (Theorem 5.8) remains valid even in the presence of singular divergences, thereby ensuring global energy balance on singular sets where classical divergence integrability fails.

Indeed, for the Lipschitz subdomain $F := (-\frac{1}{2}, \frac{1}{2})^2 \subset \Omega$ (see Figure 1), the generalized Gauss–Green formula reads

$$(9.1) \quad \int_F \mathbf{u}^* \cdot d\text{Div } \mathbf{A} + \int_F d(\mathbf{A} : E\mathbf{u}) = \int_{\partial F} \mathbf{u}^+ \cdot (\mathbf{A}^+ \boldsymbol{\nu}_F) d\mathcal{H}^1.$$

Now, evaluating both sides over the domain F yields a perfect identity. Specifically, the left-hand side combines the singular divergence and the interfacial work contributions along the crack line Γ :

$$(9.2) \quad \int_F \mathbf{u}^* \cdot d\text{Div } \mathbf{A} + \int_F d(\mathbf{A} : E\mathbf{u}) = \int_{F \cap \Gamma} \frac{\gamma}{2} (\sigma^+ - \sigma^-) d\mathcal{H}^1 + \int_{F \cap \Gamma} \frac{\sigma^+ + \sigma^-}{2} \gamma d\mathcal{H}^1 = \gamma \sigma^+,$$

which perfectly matches the right-hand side boundary integral, where a non-vanishing traction trace $\mathbf{u}^+ \cdot (\mathbf{A}^+ \boldsymbol{\nu}_F) = \gamma \sigma^+$ occurs exclusively along the upper edge of F . Indeed, on this portion of the boundary one has $\boldsymbol{\nu}_F = \mathbf{e}_2$, $\mathbf{u} = \gamma \mathbf{e}_2$, and $\mathbf{A}^+ \boldsymbol{\nu}_F = \sigma^+ \mathbf{e}_2$, while the contributions from the lower and the lateral edges of F vanish.

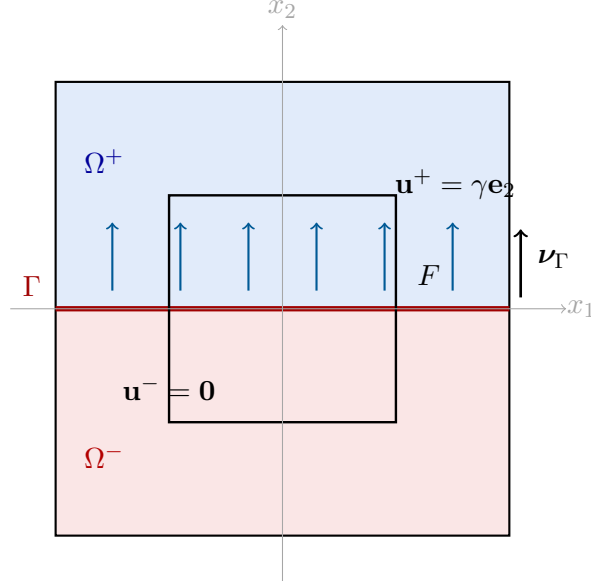


FIGURE 1. The reference configuration Ω with the interface Γ and the Lipschitz subdomain F crossing the interface.

APPENDIX A. A SUFFICIENT CONDITION FOR BV REGULARITY

We identify a sufficient condition, based on a control of the directional derivatives of $\mathbf{A}$ with respect to frames, that guarantees the BV regularity of $\mathbf{A}$.

Let $\Xi = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\Xi' = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ be two frames of $\mathbb{R}_{\text{sym}}^{N \times N}$, where $n = \frac{N(N+1)}{2}$. A symmetric tensor field $\mathbf{A}$ admits the representations

$$\mathbf{A} = \sum_{i=1}^n a^{\mathbf{v}_i} \mathbf{v}_i \otimes \mathbf{v}_i = \sum_{j=1}^n a^{\mathbf{w}_j} \mathbf{w}_j \otimes \mathbf{w}_j.$$

The change of frame is described by a matrix $\mathbf{G} \in \mathbb{R}^{n \times n}$ through the algebraic relation $\mathbf{v}_i \otimes \mathbf{v}_i = \sum_{j=1}^n G_{ij} \mathbf{w}_j \otimes \mathbf{w}_j$, which implies

$$(A.1) \quad a^{\mathbf{w}_j} = \sum_{k=1}^n G_{kj} a^{\mathbf{v}_k}.$$

Assume (H'') for both the frames, namely

$$(A.2) \quad D_{\mathbf{v}_i} a^{\mathbf{v}_i} =: \mu_i \in \mathcal{M}(\Omega), \quad D_{\mathbf{w}_j} a^{\mathbf{w}_j} =: \nu_j \in \mathcal{M}(\Omega) \quad \forall i, j = 1, 2, \dots, n.$$

Expanding the directional derivatives (A.2) in terms of the Cartesian basis $\{\mathbf{e}_\alpha\}_{\alpha=1}^N$ (using also (A.1)) yields a linear system of $2n$ equations for the $N \times n$ unknown distributional

derivatives $D_{\mathbf{e}_\alpha} a^{\mathbf{v}_k}$, $\alpha = 1, \dots, N$, $k = 1, \dots, n$:

$$\begin{cases} \sum_{\alpha=1}^N (\mathbf{v}_i \cdot \mathbf{e}_\alpha) D_{\mathbf{e}_\alpha} a^{\mathbf{v}_i} = \mu_i, & i = 1, \dots, n, \\ \sum_{k=1}^n \sum_{\beta=1}^N G_{kj} (\mathbf{w}_j \cdot \mathbf{e}_\beta) D_{\mathbf{e}_\beta} a^{\mathbf{v}_k} = \nu_j, & j = 1, \dots, n. \end{cases}$$

If the associated coefficient matrix has trivial kernel, then each distributional derivative $D_{\mathbf{e}_\alpha} a^{\mathbf{v}_k}$ can be written as a linear combination of the Radon measures μ_i and ν_j . In particular, $a^{\mathbf{v}_k} \in BV(\Omega)$ for every $k = 1, \dots, n$, and hence $\mathbf{A} \in BV(\Omega; \mathbb{R}_{\text{sym}}^{N \times N})$.

In particular, for $N = 2$, the system is square ($2n = Nn = 6$), and the condition reduces to the invertibility of the associated matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{e}_1 & \mathbf{v}_1 \cdot \mathbf{e}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{v}_2 \cdot \mathbf{e}_1 & \mathbf{v}_2 \cdot \mathbf{e}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{v}_3 \cdot \mathbf{e}_1 & \mathbf{v}_3 \cdot \mathbf{e}_2 \\ G_{11} \mathbf{w}_1 \cdot \mathbf{e}_1 & G_{11} \mathbf{w}_1 \cdot \mathbf{e}_2 & G_{21} \mathbf{w}_1 \cdot \mathbf{e}_1 & G_{21} \mathbf{w}_1 \cdot \mathbf{e}_2 & G_{31} \mathbf{w}_1 \cdot \mathbf{e}_1 & G_{31} \mathbf{w}_1 \cdot \mathbf{e}_2 \\ G_{12} \mathbf{w}_2 \cdot \mathbf{e}_1 & G_{12} \mathbf{w}_2 \cdot \mathbf{e}_2 & G_{22} \mathbf{w}_2 \cdot \mathbf{e}_1 & G_{22} \mathbf{w}_2 \cdot \mathbf{e}_2 & G_{32} \mathbf{w}_2 \cdot \mathbf{e}_1 & G_{32} \mathbf{w}_2 \cdot \mathbf{e}_2 \\ G_{13} \mathbf{w}_3 \cdot \mathbf{e}_1 & G_{13} \mathbf{w}_3 \cdot \mathbf{e}_2 & G_{23} \mathbf{w}_3 \cdot \mathbf{e}_1 & G_{23} \mathbf{w}_3 \cdot \mathbf{e}_2 & G_{33} \mathbf{w}_3 \cdot \mathbf{e}_1 & G_{33} \mathbf{w}_3 \cdot \mathbf{e}_2 \end{pmatrix},$$

i.e.,

$$\det(\mathbf{M}) \neq 0,$$

which can be interpreted as a joint non-degeneracy condition on the two frames Ξ and Ξ' , ensuring that all Cartesian partial derivatives are uniquely determined by the coupled system.

As a by-product of the previous argument, we obtain the following proposition.

Proposition A.1. *Let $\Xi = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\Xi' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be two frames of $\mathbb{R}_{\text{sym}}^{2 \times 2}$, and $\mathbf{M} \in \mathbb{R}^{6 \times 6}$ be defined as above. Assume that $\det(\mathbf{M}) \neq 0$. If $\mathbf{A} \in BV_{\Xi}^{\infty}(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2}) \cap BV_{\Xi'}^{\infty}(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$, then $\mathbf{A} \in BV(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2}) \cap L^{\infty}(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$.*

Remark A.2. For $N \geq 3$ one has $2n < Nn$, and the resulting system is underdetermined; hence two frames do not provide enough equations to determine all Cartesian partial derivatives. In general, one needs (H'') with respect to sufficiently many (at least N) frames in generic position so that the associated linear system becomes square, and its invertibility is ensured by a suitable non-degeneracy (rank) condition.

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