Integral inequalities for the thin-film equation in multiple space dimension: improvements and applications

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Abstract. We extend the validity of well-known integral inequalities for the thin-film equation to the range $n \in (\frac{1}{2}, 3)$ in any space dimension N. Consequently, known existence results on strong solutions to the thin-film equation for spatial dimensions $N < 4$ as well as recent results by Cornalba, Fischer and Maringová-Kokavcová on their Hölder continuity for $N = 2$ are extended to the range $n \in (\frac{1}{2}, 3)$. We also show that the above-mentioned range is optimal and extend some dissipation relations involving weighted Dirichlet energies to arbitrary spatial dimensions.

Mathematics Subject Classification (2010). 35A23; 35K25; 35K55; 35K65; 35Q35; 35R35; 76D08.

Keywords. Integral inequalities, thin-film equation, higher-order degenerate parabolic equations, free boundary problems, lubrication theory.

1. Introduction and results

We are concerned with the integral inequalities

$$\int_\Omega h^{n-4}\left(h^2|\nabla h|^2|D^2h|^2 + |\nabla h|^6\right) + \int_{\partial\Omega} h^{n-2}|\nabla h|^2 I\!I(\nabla h, \nabla h) \\ \le C(n)\int_\Omega h^n|\nabla\Delta h|^2 \tag{1.1}$$

on an open, bounded, and convex domain $\Omega \subset \mathbb{R}^N$ with smooth boundary, where $I\!I(.,.)$ is the second fundamental form of $\partial\Omega$. Such inequalities have been derived by the second author in [17] with values of n in the range

$$2 - \sqrt{\tfrac{8}{N+8}} < n < 3 \tag{1.2}$$

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and space dimensions $N < 6$ for strictly positive $h \in H^2(\Omega)$, satisfying $\nabla h \cdot \mathbf{n} = 0$ on $\partial\Omega$. Like the celebrated Bernis inequalities [4]

$$\int_I h^{b-q}|h_x|^{2q} \le C_1(b,q)\int_I h^b|h_{xx}|^q \le C_2(b,q)\int_I h^{b+q/3}|h_{xxx}|^{2q/3} \tag{1.3}$$

with $b, q \in \mathbb{R}$, $q \ge 2$, $h \in C^3(\bar{I}, \mathbb{R}^+)$ with $h_x = 0$ on ∂I, and $-\frac{1}{2} < b < \frac{q-1}{2q}$ for $q \ne 3$ or $-\frac{1}{2} < b < 2$ if $q = 3$, these inequalities allow one to control third-order derivatives of appropriate powers of h in $L^2(\Omega)$ (or $L^2(I)$, respectively) and first-order derivatives of powers of h in $L^6(\Omega)$ (or $L^6(I)$, respectively). As an application, weighted versions of these inequalities have been extensively used in the analysis of the thin-film equation

$$h_t + \nabla\cdot(h^n\nabla\Delta h) = 0 \quad \text{in } (0,\infty)\times\Omega \tag{1.4}$$

with initial data $h_0 \in H^1(\Omega; \mathbb{R}_0^+)$ and boundary conditions

$$\nabla h\cdot\mathbf{n} = h^n\nabla\Delta h\cdot\mathbf{n} = 0 \quad \text{on } (0,\infty)\times\partial\Omega, \tag{1.5}$$

where $\mathbf{n}$ denotes the outer normal vector field on $\partial\Omega$. Note that solutions to (1.4)-(1.5) satisfy the energy estimate

$$\tfrac{1}{2}\int_\Omega |\nabla h(x,t)|^2 \mathrm{d}x + \int_0^t\int_\Omega h^n|\nabla\Delta h|^2 \mathrm{d}x\mathrm{d}t \le \tfrac{1}{2}\int_\Omega |\nabla h_0|^2 \mathrm{d}x\ . \tag{1.6}$$

This estimate implies time-integrability of the left-hand side of (1.1), a regularity property which carries several important implications – even beyond existence of strong solutions to the Cauchy problem in the parameter range $n \in [2,3)$ in the multi-dimensional case [21]. In the same range, which corresponds to the assumption of a Navier or weaker slip condition at the liquid-solid interface, results on finite speed of propagation [3, 19, 18, 16, 28, 11, 13] and on waiting-time phenomena [9, 14, 12, 13, 10, 20] are based on weighted versions of (1.1) or (1.3), respectively.

Inequalities (1.3) were later extended to nonlinearities of the form $(h^3 + h^n)h_{xxx}^2$ by Otto and the first author in [15], where they were used to derive intermediate scaling laws. Another extension of (1.3), due to Ansini and the first author in [1], pertained to nonlinearities of the form $h^n|h_{xxx}|^q$, $q \ge \frac{4}{3}$, where they were used to establish existence of solutions and quantitative estimates on support propagation for thin-film equations with non-Newtonian rheologies in one spatial dimension. For an application of weighted versions of (1.3) to stochastic thin-film equations, we refer the reader to [22, 23]. In the recent preprint [26], Lienstromberg and Nik generalize inequalities (1.1) to right-hand sides $\int_\Omega h^n|\nabla\Delta h|^q$ for $2 \le q < \frac{19}{3}$.

While in the parameter regime $0 < n < 2$ basic existence results for zero contact angle solutions with compactly supported initial data to (1.4)-(1.5) do not rely on the inequalities (1.1) or (1.3) (see [2, 5, 8, 6]), the use of these inequalities is apparently indispensable in the parameter regime $[2,3)$ for proving existence of zero-contact-angle solutions to the Cauchy problem (see [3, 1, 21]). In this light, the restriction $n \in (2-\sqrt{\frac{8}{8+N}}, 3)$ in [17] seemed to be satisfactory with respect to applications. Very recently, however, Cornalba, Fischer and Maringová-Kokavcová [7] solved a longstanding open problem in the theory of (1.4) when they proved Hölder-continuity of solutions in the physically relevant case $N = 2$. They widely used weighted versions of the inequalities (1.1) in their proof, and the parameter range for n in their result coincides with the validity range of the inequalities (1.1). Therefore, it is the main goal of this manuscript to extend the validity range of (1.1) to the interval $n \in (\frac{1}{2}, 3)$ which had been obtained in the case $N = 1$ by Bernis [4]. In addition, we discuss the (very few) changes in the proof of the existence result of [21] for the Cauchy problem to (1.4) which are needed to extend it in space dimensions $N \in \{2, 3\}$ to the parameter regime $n \in (\frac{1}{2}, 3)$. As a consequence, the Hölder-regularity result of [7] now holds for all $n \in (\frac{1}{2}, 3)$.

Our first main result is the following one.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be an open, bounded, and convex domain with smooth boundary. If $n \in (\frac{1}{2}, 3)$, then $C(n) > 0$ exists such that* (1.1) *holds for any positive function $h \in C^\infty(\overline{\Omega})$ such that $\nabla h \cdot \mathbf{n} = 0$ on $\partial\Omega$.*

The lower bound $n > \frac{1}{2}$ has been known to be sharp since [4] for $N = 1$. Up to now, the sharpness of the condition $n < 3$ was conjectured but unconfirmed. The next result states that the range $(\frac{1}{2}, 3)$ is sharp for any N.

Proposition 1.2. *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 1$. If $0 < n \leq \frac{1}{2}$ or $n \geq 3$, then Theorem* 1.1 *does not hold.*

Theorem 1.1 may be extended to strictly positive H^2-functions, provided that $N < 6$[1]:

Corollary 1.3. *Let $\Omega \subset \mathbb{R}^N$, $1 \leq N < 6$, be an open, bounded, and convex domain with smooth boundary. If $n \in (\frac{1}{2}, 3)$, then $C(n) > 0$ exists such that* (1.1) *holds for any strictly positive function $h \in H^2(\Omega)$ such that $\nabla h \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $\int_\Omega h^n |\nabla \Delta h|^2 < +\infty$.*

For the sake of completeness, we state the weighted version of Theorem 1.3, as it plays an important role in applications.

Corollary 1.4. *Under the assumptions of Corollary* 1.3*, there exists a constant $C = C(n) > 0$ such that*

$$\int_\Omega \phi^6 h^{n-4} \left(h^2 |\nabla h|^2 |D^2 h|^2 + |\nabla h|^6\right) + \int_{\partial\Omega} \phi^6 h^{n-2} |\nabla h|^2 I\!I(\nabla h, \nabla h) \leq C(n) \left(\int_\Omega \phi^6 h^n |\nabla \Delta h|^2 + \int_\Omega h^{n+2} |\nabla \phi|^6\right) \tag{1.7}$$

holds for any strictly positive function $h \in H^2(\Omega)$ such that $\nabla h \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $\int_\Omega h^n |\nabla \Delta h|^2 < +\infty$ and any nonnegative function $\phi \in C^1(\bar{\Omega})$.

Consequently, the existence results of [21] in spatial dimensions two and three may be extended to $n \in (\frac{1}{2}, 3)$:

[1] For $N = 1$, the result holds for nonnegative H^2-functions as well, see [15, Section 5].

Theorem 1.5. *Let $n \in (\frac{1}{2}, 3)$ and $N \in \{2,3\}$. Then:*

a) *If $\Omega \subset \mathbb{R}^N$ is an open, bounded, and convex domain with smooth boundary, for any non-negative $h_0 \in H^1(\Omega)$ there exists a solution h to the Neumann problem for* (1.4) *in the sense of* [21, Definition I.1 and Theorem V.3]*;*
b) *For any $h_0 \in H^1(\mathbb{R}^N)$ non-negative and compactly supported, there exists a solution h to the Cauchy problem for* (1.4) *in the sense of* [21, Theorem VI.1]*.*

Combining this result with [7, Theorem 1], whose range of validity directly depends on the range of validity of the inequalities (1.1) (see the statement at the beginning of Section 3 in [7] immediately before the presentation of Theorem 1), we obtain the following:

Corollary 1.6. *Let $n \in (\frac{1}{2}, 3)$, $N = 2$, and assume initial data $h_0 \in H^1(\mathbb{R}^N)$ to be non-negative and compactly supported. Let h be a weak solution to the thin-film equation in the sense of* [21, Theorem VI.1]*. Then the results of Theorem 1 in* [7] *hold, namely:*

a) *h is Hölder continuous for $t > 0$ in the sense $h \in C^\sigma_{\mathrm{loc}}(\mathbb{R}^2 \times (0,T))$ for some $\sigma = \sigma(n) > 0$;*
b) *if in addition $\nabla h_0 \in L^p(\mathbb{R}^2)$ for some $p > 2$, then h is Hölder continuous up to $t = 0$ in the sense $h \in C^\sigma_{\mathrm{loc}}(\mathbb{R}^2 \times [0,T))$ for some $\sigma = \sigma(n,p) > 0$.*

Our next result formally generalizes (1.6). In one space dimension independently introduced by Laugesen [25] and by Jüngel and Matthes [24], it suggests powers $h^{1+\frac{p}{2}}$ of solutions to the thin-film equation to be contained in $L^\infty(0,T; H^1(\Omega))$ for suitable values of $p \neq 0$. The starting point for such an estimate is the following identity, which is formally obtained using multiple integrations by parts:

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\int \tfrac{1}{2}h^p|\nabla h|^2 &= -\int \tfrac{p}{2}h^{p-1}|\nabla h|^2 h_t - \int h^p \nabla h \nabla h_t \\
&\overset{(1.5)}{=} -\int \left(\tfrac{p}{2}h^{p-1}|\nabla h|^2 - h^p\Delta h\right) h_t \\
&\overset{(1.4),(1.5)}{=} -\int h^n \nabla\left(\tfrac{p}{2}h^{p-1}|\nabla h|^2 - h^p\Delta h\right)\nabla\Delta h \\
&= \|R\|^2 + \tfrac{p(p-1)}{2}(R,T) + p(R, S+Q) =: D,
\end{aligned} \tag{1.8}$$

where $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega)$, $\|\cdot\|$ is the L^2-norm, and (in the spirit of [25])

$$\begin{aligned}
R &= h^{\frac{n+p}{2}}\nabla\Delta h, & S &= h^{\frac{n+p-2}{2}}\Delta h\nabla h, \\
Q &= h^{\frac{n+p-2}{2}}\nabla h D^2 h, & T &= h^{\frac{n+p-4}{2}}|\nabla h|^2\nabla h.
\end{aligned}$$

Note that (1.8) coincides with the standard energy estimate (1.6) if $p = 0$. In [24, 25], it is shown in one spatial dimension that the right-hand side of (1.8) is bounded from below by non-negative integral quantities in a certain range of values of p, which depends on n. We extend these results to any space dimension. We define the sets

$$\boldsymbol{E}_1 = \bigcup_{z>1/3} E_1(z) \quad \text{and} \quad \boldsymbol{E}_2 = \bigcup_{z>1/3} E_2(z), \tag{1.9a}$$

where

$$E_1(z) = \left\{(n,p):\ B(n,p,z) + A(z)\tfrac{p^2}{2} > 0 \ \wedge\ \Delta(n,p,z) > 0\right\}, \tag{1.9b}$$

$$E_2(z) = \left\{(n,p):\ \Delta(n,p,z) - \left(B(n,p,z) + A(z)\tfrac{p^2}{2}\right)^2 > 0\right\}, \tag{1.9c}$$

and

$$A(z) = -\,4(1+z)^2(8+3z), \tag{1.10a}$$

$$\begin{aligned} B(n,p,z) =& -16(1+z+z^2) - 4(1+z)(10+9z)p - 4(1+z)(8+3z)p^2 \\ &+ 4\left(8+4z(4+z)+(1+z)(10+3z)p\right)(n+p) \\ &- 4(2+z)^2(n+p)^2, \end{aligned} \tag{1.10b}$$

$$C(n,p,z) = -\,p^2\left(4(-1+n-p)(-2+2n+p) - 3(1+p)(-5+2n+p)z\right), \tag{1.10c}$$

$$\Delta(n,p,z) = (B(n,p,z))^2 - 4A(n,p,z)C(n,p,z). \tag{1.10d}$$

The result reads as follows:

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be an open, bounded, and convex domain with smooth boundary. If $(n,p) \in \boldsymbol{A} := \boldsymbol{E}_1 \cup \boldsymbol{E}_2$ (in particular, if $n \in (\frac{1}{2}, 3)$ for $p = 0$), then $C(n,p) > 1$ exists such that*

$$\int_\Omega h^{n+p-4}\left(h^4|\nabla\Delta h|^2 + h^2|\nabla h|^2|D^2h|^2 + |\nabla h|^6\right) + \int_{\partial\Omega} h^{n+p-2}|\nabla h|^2 I\!I(\nabla u,\nabla u) \leq C(n,p)D \tag{1.11}$$

holds for any positive function $h \in C^\infty(\overline{\Omega})$ such that $\nabla h \cdot \mathbf{n} = 0$ on $\partial\Omega$. If $N < 6$, then $h \in C^\infty(\overline{\Omega}; (0,\infty))$ is weakened to $h \in H^2(\Omega)$ strictly positive.

Remark 1.8. *A simple computation of $\nabla\Delta h^{\frac{n+p+2}{2}}$ shows that*

$$C(n,p)|\nabla\Delta h^{\frac{n+p+2}{2}}|^2 \leq h^{n+p-4}\left(h^4|\nabla\Delta h|^2 + h^2(|\nabla h|^2|\Delta h|^2 + |\nabla h D^2 h|^2) + |\nabla h|^6\right).$$

Hence (1.11) *implies the following estimate for some positive constant $C(n,p)$:*

$$\int \left|\nabla\Delta h^{\frac{n+p+2}{2}}\right|^2 \leq C(n,p)D.$$

Remark 1.9. *A weighted version of Theorem* 1.7 *analogous to Corollary* 1.4 *also holds true: with*

$$\begin{aligned} R &= \phi^3 h^{\frac{n+p}{2}}\nabla\Delta h, & S &= \phi^3 h^{\frac{n+p-2}{2}}\Delta h\nabla h, \\ Q &= \phi^3 h^{\frac{n+p-2}{2}}\nabla h D^2 h, & T &= \phi^3 h^{\frac{n+p-4}{2}}|\nabla h|^2\nabla h \end{aligned} \tag{1.12}$$

and D *as in* (1.8)*, one obtains*

$$\int_\Omega \phi^6 h^{n+p-4}\left(h^4|\nabla\Delta h|^2 + h^2|\nabla h|^2|D^2h|^2 + |\nabla h|^6\right)$$
$$+\int_{\partial\Omega}\phi^6 h^{n+p-2}|\nabla h|^2 I\!I(\nabla u,\nabla u) \le C(n,p)\left(D + \int_\Omega h^{n+p+2}|\nabla\phi|^6\right).$$

The paper is organized as follows: The integral estimates of Theorem 1.1, Corollary 1.3, Corollary 1.4, Theorem 1.7, and Remark 1.9 are proven in Section 2. In Section 3, we show optimality of the range $n \in (\frac{1}{2}, 3)$ as stated in Proposition 1.2. In Section 4, we give the details which permit to prove Theorem 1.5, following the arguments of [21] and using the extended range $n \in \left(\frac{1}{2}, 3\right)$ of Corollary 1.4. Finally, Section 5 discusses the admissible set $\boldsymbol{A}$ which appears in Theorem 1.7.

2. Proofs of the integral inequalities

In this section, we start from equation (1.8) to prove the integral estimates of Theorems 1.1 and 1.7. Our approach, while phrased in the notation of [25], is analogous to that of [24] in one space dimension[4]. In higher dimensions, we are able to obtain an algebraically tractable system by allowing formula (2.3) below to be an inequality rather than an equality.

Proof of the Integral Inequalities. We begin with (1.8):

$$D := \|R\|^2 + \tfrac{p(p-1)}{2}(R,T) + p(R, S+Q)$$

where it is now our goal to show that $D \ge 0$. Recall that

$$R = h^{\frac{n+p}{2}}\nabla\Delta h, \qquad S = h^{\frac{n+p-2}{2}}\Delta h\nabla h,$$
$$Q = h^{\frac{n+p-2}{2}}\nabla h D^2 h, \qquad T = h^{\frac{n+p-4}{2}}|\nabla h|^2\nabla h.$$

There are three linearly independent relations between R, Q, S, T:

$$0 =(S,T) + (n+p-3)\|T\|^2 + 4(T,Q) \tag{2.1}$$
$$0 =(R,T) + (n+p-2)(S,T) + 2(S,Q) + \|S\|^2 \tag{2.2}$$
$$0 \ge(n+p-2)(T,Q) + 3\|Q\|^2 + (R,T) + V^2, \tag{2.3}$$

where

$$V^2 = \int_{\partial\Omega} h^{n+p-2}|\nabla h|^2 I\!I(\nabla h,\nabla h).$$

The relations follow from integrations by parts. For (2.1), we write

$$(S,T) = \int h^{n+p-3}|\nabla h|^4\Delta h$$
$$= -(n+p-3)\int h^{n+p-4}|\nabla h|^6 - 4\int h^{n+p-3}|\nabla h|^2\nabla h D^2 h\nabla h.$$

[4]In fact, D. Matthes mentioned in [27] that he had set up computations to prove the multi-dimensional estimate (1.1) in the parameter regime $\left(\frac{1}{2}, 2-\sqrt{\frac{8}{8+N}}\right]$. To the best of our knowledge, these computations have never been published.

For (2.2), we write

$$
\begin{aligned}
(R,T) =& \int h^{n+p-2}|\nabla h|^2 \nabla h \nabla \Delta h \\
=& -(n+p-2)\int h^{n+p-3}|\nabla h|^4 \Delta h - 2\int h^{n+p-2}\Delta h \nabla h D^2 h \nabla h \\
& - \int h^{n+p-2}|\nabla h|^2 (\Delta h)^2.
\end{aligned}
$$

For (2.3), we write

$$
\begin{aligned}
(n+p-2)(T,Q) =& (n+p-2)\int h^{n+p-3}|\nabla h|^2 \nabla h D^2 h \nabla h \\
=& \tfrac{n+p-2}{2}\int h^{n+p-3}|\nabla h|^2 \nabla h \nabla |\nabla h|^2 \\
=& \tfrac{1}{2}\int_{\partial\Omega} h^{n+p-2}|\nabla h|^2 \nabla |\nabla h|^2 \cdot \mathbf{n} \\
& - \tfrac{1}{2}\int h^{n+p-2}|\nabla|\nabla h|^2|^2 - \tfrac{1}{2}\int h^{n+p-2}|\nabla h|^2 \Delta |\nabla h|^2.
\end{aligned}
$$

Arguing as in [17] we see that $\nabla|\nabla h|^2 \cdot \mathbf{n} = -2I\!I(\nabla h, \nabla h)$. We also use the identity $\Delta|\nabla h|^2 = 2|D^2h|^2 + 2\nabla h \nabla \Delta h$ and write

$$
\begin{aligned}
(n+p-2)(T,Q) =& -\int_{\partial\Omega} h^{n+p-2}|\nabla h|^2 I\!I(\nabla u, \nabla u) - 2\int h^{n+p-2}|\nabla h D^2 h|^2 \\
& - \int h^{n+p-2}|\nabla h|^2 (|D^2 h|^2 + \nabla h \nabla \Delta h).
\end{aligned} \tag{2.4}
$$

Then (2.3) follows noting that $|\nabla h|^2|D^2h|^2 \geq |\nabla h D^2 h|^2$.

Let $a, b, z \in \mathbb{R}$ with $zb \geq 0$. Multiplying (2.1)-(2.3) respectively by a, b, z and plugging into (1.8), we obtain

$$
\begin{aligned}
D \geq& a\left((S,T) + (n+p-3)\|T\|^2 + 4(T,Q)\right) \\
& + b\left((R,T) + (n+p-2)(S,T) + 2(S,Q) + \|S\|^2\right) \\
& + zb\left((n+p-2)(T,Q) + 3\|Q\|^2 + (R,T) + V^2\right) \\
& + \|R\|^2 + \tfrac{p(p-1)}{2}(R,T) + p(R,S+Q) \\
=& A_1^2 + \left(b - \tfrac{p^2}{4}\right) A_2^2 + b(3z-1)A_3^2 + \tilde{f}(n,p,a,b,z)\|T\|^2 + zbV^2,
\end{aligned} \tag{2.5}
$$

where

$$\begin{aligned}
A_1 &= \left\| R + \tfrac{p}{2}(S+Q) + \tfrac{1}{4}(2b(z+1)+p(p-1))T \right\|^2, \\
A_2 &= \left\| S + Q + \tfrac{2}{4b-p^2}\left(a + b(n+p-2) - \tfrac{p}{4}(2b(z+1)+p(p-1))\right) T \right\|^2, \\
A_3 &= \left\| Q + \tfrac{2}{2b(3z-1)}\left(3a + b(z-1)(n+p-2)\right) T \right\|^2, \\
\tilde f &= a(n+p-3) - \tfrac{1}{16}(2b(z+1)+p(p-1))^2 \\
&\quad - \tfrac{1}{4b(3z-1)}(3a + b(z-1)(n+p-2))^2 \\
&\quad - \tfrac{1}{4b-p^2}\left(a + b(n+p-2) - \tfrac{p}{4}(2b(z+1)+p(p-1))\right)^2.
\end{aligned}$$

The last equality in (2.5) is obtained by first completing the square in R, then in S, and then in Q. We now look for a choice of a, b, z such that $\tilde f > 0$ while subject to the constraints

$$b > \tfrac{p^2}{4} \quad \text{and} \quad 3z > 1 \tag{2.6}$$

(which entail $zb > 0$). We note that $\tilde f$ is quadratic and concave with respet to a, hence its maximum is attained at

$$\bar a(n,p,b,z) = \left(\tfrac{\partial \tilde f}{\partial a}\right)^{-1}\{0\} = \tfrac{b^2(4z(-6+p)+6z^2p-2(4-4n-3p))+bp^2(1-2p-n+3z(3-n))}{4b(8+3z)-9p^2}$$

(note that the denominator is not zero in view of (2.6)). Since (2.6) is independent of a, maximizing with respect to a we see that

$$D \geq A_1^2 + \left(b - \tfrac{p^2}{4}\right) A_2^2 + b(3z-1)A_3^2 + f(n,p,b,z)\|T\|^2 + zbV^2, \tag{2.7}$$

where (after algebraic manipulations)

$$\begin{aligned}
f(n,p,b,z) &= \tilde f(n,p,\bar a(n,p,b,z),b,z) \\
&= \tfrac{b}{16b(8+3z)-36p^2}\left(A(z)b^2 + B(n,p,z)b + C(n,p,z)\right)
\end{aligned}$$

with A, B, C as given in (1.10).

The special case $p = 0$. Then $C \equiv 0$ and

$$f(n,0,b,z) = \tfrac{b}{16(8+3z)}\left(A(z)b + B(n,0,z)\right).$$

Noting that $A(z) < 0$, under the constraint (2.6) we have $f > 0$ if and only if there exists $z > 1/3$ such that

$$B(n,0,z) = -(n-2)^2z^2 - 4(n^2-4n+1)z - 4(n^2-2n+1) > 0$$

(indeed, in that case one can choose $b = b(n,z) > 0$ sufficiently small). For $n \neq 2$, it is easily seen that such z exists if and only if the discriminant in z is positive and the largest root z_+ exceeds $\frac{1}{3}$; that is,

$$(n+1)(3-n)(-1+2n) > 0, \quad z_+ = \tfrac{2(-1-(-4+n)n+\sqrt{(n+1)(3-n)(-1+2n)})}{(-2+n)^2} > \tfrac{1}{3}.$$

The former yields $n \in (\frac{1}{2}, 3)$ and the latter is easily seen to hold in this range. For $n = 2$ we have $B(2,0,z) = 12z - 4 > 0$, hence any choice of $z > 1/3$ works.

The general case $p \neq 0$. We look for b, z such that $f > 0$ under the constraints in (2.6). Let $\Delta(n,p,z)$ as in (1.10d). Since $A(z) < 0$ and $\frac{b}{16b(8+3z)-36p^2} > 0$ under the constraints (2.6), $f > 0$ if and only if $\Delta(n,p,z) > 0$ and $b \in (b_-(n,p,z), b_+(n,p,z))$, where

$$b_\pm(n,p,z) = -\frac{1}{2A(n,p,z)}\left(B(n,p,z) \pm \sqrt{\Delta(n,p,z)}\right).$$

Hence $f > 0$ under the constraints (2.6) if and only if there exists $z > \frac{1}{3}$ such that

$$\Delta(n,p,z) > 0 \quad \text{and} \quad b_+(n,p,z) > \frac{p^2}{4},$$

which is equivalent to

$$\begin{cases} B(n,p,z) + A(z)\frac{p^2}{2} > 0 \\ \Delta(n,p,z) > 0 \end{cases} \quad \text{or} \quad \Delta(n,p,z) > \left(B(n,p,z) + A(z)\frac{p^2}{2}\right)^2.$$

Hence for any (n,p) as in the statement there exists $\hat{z} = z(n,p) > 1/3$ such that $b_+ = b_+(n,p,z(n,p)) > p^2/4$ and $f(n,p,b_+,\hat{z}) > 0$. Recalling the definitions of A_i, (2.7) implies that $C(n,p) > 1$ exists such that

$$\begin{aligned} V^2 + \int_\Omega h^{n+p-4}\left(h^4|\nabla\Delta h|^2 + h^2(|\nabla h|^2|\Delta h|^2 + |\nabla h D^2 h|^2) + |\nabla h|^6\right) \\ \leq C(n,p)D. \end{aligned} \tag{2.8}$$

Finally, it follows from (2.4) that $C > 0$ exists such that

$$\begin{aligned} V^2 &+ \int_\Omega h^{n+p-2}|\nabla h|^2|D^2 h|^2 \\ &\leq C\left(\int_\Omega h^{n+p-4}\left(h^4|\nabla\Delta h|^2 + h^2|\nabla h D^2 h|^2 + |\nabla h|^6\right)\right), \end{aligned} \tag{2.9}$$

hence (1.11) follows from (2.8) and (2.9). This completes the proof of Theorem 1.1 and the first part of Theorem 1.7.

Arguing as in [17, Proof of Theorem 1.1], we obtain Corollary 1.3 and the second part of Theorem 1.7. Corollary 1.4 and Remark 1.9 require only minor modifications of the argument: with R, S, Q, T as in (1.12), the three relations (2.1)-(2.3) and hence (2.5) contain further terms depending on derivatives of ϕ. These extra terms can be absorbed into the right-hand side of (2.5) via Hölder's inequality at the cost of $\int_\Omega h^{n+p+2}|\nabla\phi|^6$.

□

3. Optimality of the range $n \in (\frac{1}{2}, 3)$

In this short section, we prove Proposition 1.2. For the counterexamples, we will use radially symmetric functions:

$$\begin{aligned} h(x) = f(r),\ r = |x|, \\ |\nabla h| = |f'(r)|,\ |\nabla\Delta h| = \left|\left(f''(r) + \tfrac{N-1}{r}f'(r)\right)'\right|, \end{aligned} \tag{3.1}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}r}$. In what follows, C will denote a generic positive constant independent of δ.

If $0 < n \leq \frac{1}{2}$, the counterexample known for $N = 1$ [4] may be easily extended to any N: Choosing $h_\delta(x) = f_\delta(r) = \delta + (R^2 - r^2)^2$, from (3.1) we see that $|\nabla h_\delta| = -4r(R^2 - r^2)$ and $|\nabla \Delta h_\delta| = 8(N+2)r$, so that

$$\int_\Omega h_\delta^n |\nabla \Delta h_\delta|^2 = C \int_0^R (\delta + (R^2 - r^2)^2)^n r^{N+1} \mathrm{d}r \leq C \qquad \text{for } 0 < \delta \leq 1,$$

whereas if $2n - 2 \leq -1$, that is, $n \leq \frac{1}{2}$, we have by monotone convergence

$$\int_\Omega h_\delta^{n-4} |\nabla h_\delta|^6 = C \int_0^R (\delta + (R^2 - r^2)^2)^{n-4} (R^2 - r^2)^6 r^{N+5} \mathrm{d}r \overset{\delta \to 0}{\to} +\infty.$$

If $n \geq 3$, the counterexample was instead unknown so far even for $N = 1$: we take smooth approximations of a paraboloid (for which the r.h.s. of (1.1) would be zero). For $\delta \leq R/2$, we let $I_\delta = \{r \in (0, R) : \ R^2 - r^2 \leq \delta\}$ and $B_\delta = (0, R) \setminus I_\delta$, and we choose

$$f_\delta(r) = \delta + \frac{(R^2 - r^2)^2}{\delta + R^2 - r^2} = O(1) \begin{cases} \delta & \text{in } I_\delta \\ R^2 - r^2 & \text{in } B_\delta, \end{cases}$$

where here and after, $O(1)$ denotes a generic function of r which is bounded above and below by positive constants independent of δ. By (3.1)

$$|\nabla h_\delta| = -2r(R^2 - r^2) \frac{2\delta + R^2 - r^2}{(\delta + R^2 - r^2)^2} = O(1) \begin{cases} -\delta^{-1} r (R^2 - r^2) & \text{in } I_\delta \\ -r & \text{in } B_\delta, \end{cases}$$

and

$$\begin{aligned} |\nabla \Delta h_\delta| &= 8\delta^2 r \left(\frac{N+2}{(\delta + R^2 - r^2)^3} + \frac{6r^2}{(\delta + R^2 - r^2)^4} \right) \\ &= \begin{cases} O(1)\delta^{-1} r + O(1)\delta^{-2} r^3 & \text{in } I_\delta \\ O(1)\delta^2 r (R^2 - r^2)^{-4} (R^2 - r^2 + O(1) r^2) & \text{in } B_\delta, \end{cases} \end{aligned}$$

Therefore

$$\begin{aligned} \int_\Omega h_\delta^n |\nabla \Delta h_\delta|^2 \leq & C \int_{I_\delta} \delta^{n-4} \mathrm{d}r + C \int_{B_\delta} \delta^4 (R^2 - r^2)^{n-8} r \mathrm{d}r \\ \leq & C\delta^{n-3} + C \left\{ \begin{array}{ll} \delta^{n-3} & \text{if } 3 \leq n < 7 \\ \delta^4 |\ln \delta| & \text{if } n = 7 \\ \delta^4 & \text{if } n > 7 \end{array} \right\}, \end{aligned}$$

whereas

$$\begin{aligned} \int_\Omega h_\delta^{n-4} |\nabla h_\delta|^6 \geq & C \int_{I_\delta} \delta^{n-10} (R^2 - r^2)^6 r \mathrm{d}r + C \int_{B_\delta} (R^2 - r^2)^{n-4} r \mathrm{d}r \\ \geq & C\delta^{n-3} + C \left\{ \begin{array}{ll} |\ln \delta| & \text{if } n = 3 \\ 1 & \text{if } n > 3 \end{array} \right\}. \end{aligned}$$

In the last lines, $C = C(r, R, N)$ are positive constants which may become different at each occurrence and which provide estimates for the product of the $\mathcal{H}^{N-1}$-measure of $\mathbb{S}^{N-1}$ – denoted by $\omega_N := \mathcal{H}^{N-1}(\mathbb{S}^{N-1})$ – multiplied by appropriate powers of r. For instance, for the first term on the right-hand side of the first inequality in the last line, we get – using $\delta \le R/2$ –

$$\begin{aligned}\int_{B_R(0)\setminus B_{\sqrt{R^2-\delta^2}}(0)} h_\delta^{n-4}|\nabla h_\delta|^6 &\ge O(1)\omega_N \int_{I_\delta} \delta^{n-10}(R^2-r^2)^6 r^{6+N-1}dr\\ &\ge O(1)\omega_N\left(\tfrac{\sqrt{3}}{2}R\right)^{N+4}\int_{I_\delta}\delta^{n-10}(R^2-r^2)^6 r dr\\ &=: C(R,r,N)\int_{I_\delta}\delta^{n-10}(R^2-r^2)^6 r dr.\end{aligned}$$

4. Proof of Theorem 1.5

For $n \ge 1$, the proofs of Theorem [21, Theorem V.3 and Theorem VI.1] can be repeated word by word, as the constraint $n > 2-\sqrt{\frac{8}{N+8}}$ only comes from the range of validity of Theorem 1.1 which was known at that time (see [21, Remark (1), p. 1734]). For $n < 1$, instead, a few observation are in order, in which we denote $\Omega_T = (0,T)\times\Omega$.

1. [21, Theorem VI.1] follows from [21, Theorem V.3] via an exhaustion argument which can be repeated word by word. Hence it suffices to look at the latter.

2. The solution [21, Theorem V.3] is obtained as limit of solutions $\tilde{u} = u_{\sigma_1,\delta}$ to obstacle problems related to the thin-film equation, more precisely passing first with σ_1 and then with δ to zero. The functions $\tilde{u}$ are known to have the following properties (see [21, Theorem II.8]), which we shall use later:

$$\tilde{u} \in L^2_{loc}([0,\infty);H^2(\Omega)) \;\text{ with }\; \nabla u(t)\cdot\mathbf{n} = 0 \;\text{ on }\; \partial\Omega \text{ for a.e. } t; \tag{4.1}$$

$$\tilde{u}(x,t) \in [\delta,\delta^{-1}] \qquad \text{for all } (x,t)\in\Omega_T; \tag{4.2}$$

$$\frac{1}{2}\int_\Omega |\nabla\tilde{u}(t)|^2 + \sigma_1\iint_{\Omega_t}\tilde{u}_t^2 + \iint_{\Omega_t}\tilde{u}^n|\nabla\Delta\tilde{u}-\sigma_1\nabla\tilde{u}_t|^2 \le \frac{1}{2}\int_\Omega|\nabla\tilde{u}(0)|^2. \tag{4.3}$$

3. The proof of [21, Theorem V.3] uses [21, Lemma IV.4], where $n \ge 1$ is assumed. However, Lemma IV.4 is applied to the functions $\tilde{u}$, which are strictly positive in view of (4.2). For that reason, it is straightforward to check that the additional assumption $n \ge 1$ is not necessary and Lemma IV.4 extends to all n for which Theorem 1.1 is valid (i.e. $n \in (\frac{1}{2},3)$).

4. In addition to what is stated in [21, Theorem II.8], for all $T > 0$ the functions $\tilde{u}$ satisfy

$$\iint_{\Omega_T}|\nabla\tilde{u}^{\frac{\alpha+n+1}{4}}|^4 \le C(\alpha,n)\int_\Omega G_\alpha(u(0)) + \sigma_1 C(\delta,\alpha,n)\int_\Omega|\nabla\tilde{u}(0)|^2 \tag{4.4}$$

for all $\alpha \in \left(\frac{1}{2}-n, 2-n\right) \setminus \{0\}$, where $G_\alpha''(u) = u^{\alpha-1}$ with $G(1) = G'(1) = 0$. Indeed, it follows from [21, Formula (IV.9)] (with $\psi = 1$) that

$$\frac{1}{C(\alpha,n)} \iint_{\Omega_T} |\nabla \tilde{u}^{\frac{\alpha+n+1}{4}}|^4 \le \int_\Omega G_\alpha(u(0)) + \underbrace{\left| \iint_{\Omega_T} \sigma_1 \tilde{u}^{\alpha+n-1} \nabla \tilde{u}_t \nabla \tilde{u} \right|}_{=:R}. \quad (4.5)$$

We rewrite R as[2]

$$\begin{aligned} 2R &= \sigma_1 \iint_{\Omega_T} \tilde{u}^{\alpha+n-1} \partial_t |\nabla \tilde{u}|^2 \\ &= \sigma_1 \int_\Omega \tilde{u}^{\alpha+n-1} |\nabla \tilde{u}(t)|^2 \Big|_{t=0}^{t=T} - \sigma_1(\alpha+n-1) \iint_{\Omega_T} \tilde{u}^{\alpha+n-2} \tilde{u}_t |\nabla \tilde{u}|^2 \\ &=: R_1 + R_2 \end{aligned}$$

and we estimate using (4.2) and (4.3)

$$\begin{aligned} |R_1| &\le \sigma_1 C(\delta,\alpha,n) \int_\Omega |\nabla \tilde{u}(0)|^2, \\ |R_2| &\le \varepsilon \iint_{\Omega_T} \tilde{u}^{\alpha+n-3} |\nabla \tilde{u}|^4 + \sigma_1 C(\varepsilon,\delta,\alpha,n) \int_\Omega |\nabla \tilde{u}(0)|^2 \end{aligned}$$

for all $\varepsilon > 0$. Plugging these estimates into (4.5) and choosing $\varepsilon = \varepsilon(\alpha,n)$ sufficiently small we obtain (4.4).

5. The proof of [21, Theorem V.3] uses [21, Formula (V.2)], where a σ_1-independent and δ-uniform bound on the mass of $\tilde{u}$ is obtained for $n \in (\frac{3}{4}, 3)$. We now work out an analogous bound which actually holds in the whole range $n \in (\frac{1}{2}, 3)$. More precisely, we show that the following holds for all $\sigma_1, \delta \in (0,1]$:

$$\begin{aligned} (1-\delta) \int_\Omega \tilde{u}(T) &\le \int_\Omega \tilde{u}(0) + \delta |\ln \delta| \\ &\quad + \left(C(\alpha,n)\delta^{-\alpha} + \sqrt{\sigma_1} C(\delta,n,\alpha)\right) \left(\int_\Omega G_\alpha(\tilde{u}(0)) + \int_\Omega |\nabla \tilde{u}(0)|^2 \right) \quad (4.6a) \end{aligned}$$

$$\text{for all } \alpha \in (\tfrac{1}{2}-n, 2-n) \cap (-1,0), \text{ with } G_\alpha \text{ as in (4.4).} \quad (4.6b)$$

Note that the range of admissible α is not empty for any $n \in (\frac{1}{2}, 3)$. In order to show (4.6), we start as in the proof of [21, Lemma V.1] from

$$\begin{aligned} \int_\Omega (\tilde{u}(t) - \delta \log \tilde{u}(t)) \Big|_{t=0}^{t=T} &\le \delta \iint_{\Omega_T} \tilde{u}^n \nabla(\Delta u - \sigma_1 \tilde{u}_t)(-\nabla \tilde{u}^{-1}) \\ &= \delta \iint_{\Omega_T} \nabla(\Delta u - \sigma_1 \tilde{u}_t) \tilde{u}^{n-2} \nabla \tilde{u} =: \tilde{R}, \end{aligned}$$

[2]Here and in formula (4.8), too, we use that (4.2) and (4.3) imply $\nabla\Delta\tilde{u}$ and $\nabla\partial_t\tilde{u}$ to be contained in $L^2((0,\infty)\times\Omega)$ and $\Delta\tilde{u}$ to be element of $L^\infty((0,\infty);L^2(\Omega))$, all with bounds depending in particular on δ due to (4.2).

which in view of (4.2) means that

$$\int_\Omega \tilde{u}(T) \le \int_\Omega \tilde{u}(0) + \delta|\ln\delta||\Omega| + \delta\int_{[\tilde{u}(T)>1]} \log\tilde{u}(T) + \tilde{R}, \tag{4.7}$$

Integrating once by parts, we get

$$\begin{aligned}\tilde{R} &\overset{(4.1)}{=} -\delta\iint_{\Omega_T} \tilde{u}^{n-2}\left(|\Delta\tilde{u}|^2 + (n-2)\tilde{u}^{-1}|\nabla\tilde{u}|^2\Delta\tilde{u}\right)\\ &\qquad + \delta\sigma_1\iint_{\Omega_T} \tilde{u}^{n-2}\tilde{u}_t\left(\Delta\tilde{u} + (n-2)\tilde{u}^{-1}|\nabla\tilde{u}|^2\right)\\ &=: -\delta\iint_{\Omega_T}\tilde{u}^{n-2}|\Delta\tilde{u}|^2 + \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3.\end{aligned} \tag{4.8}$$

With α as in (4.4) such that $\alpha\in(-1,0)$, we estimate R_i via Cauchy-Schwarz and Young inequalities together with (4.2), (4.3) and (4.4):

$$\begin{aligned}\tilde{R}_1 &\le \frac{\delta}{2}\iint_{\Omega_T}\tilde{u}^{n-2}|\Delta\tilde{u}|^2 + C(n)\delta\|\tilde{u}^{-\alpha-1}\|_\infty\iint_{\Omega_T}\tilde{u}^{\alpha+n-3}|\nabla\tilde{u}|^4\\ &\le \frac{\delta}{2}\iint_{\Omega_T}\tilde{u}^{n-2}|\Delta\tilde{u}|^2\\ &\quad + C(\alpha,n)\delta^{-\alpha}\left(\int_\Omega G_\alpha(\tilde{u}(0)) + \sigma_1 C(\delta,\alpha,n)\int_\Omega|\nabla\tilde{u}(0)|^2\right),\\ \tilde{R}_2 &\le \frac{\delta}{2}\iint_{\Omega_T}\tilde{u}^{n-2}|\Delta\tilde{u}|^2 + C\sigma_1\delta\|\tilde{u}^{n-2}\|_\infty\left(\sigma_1\iint_{\Omega_T}\tilde{u}_t^2\right)\\ &\le \frac{\delta}{2}\iint_{\Omega_T}\tilde{u}^{n-2}|\Delta\tilde{u}|^2 + \sigma_1 C(\delta,n,\alpha)\int_\Omega|\nabla\tilde{u}(0)|^2,\\ \tilde{R}_3 &\le \sqrt{\sigma_1}\delta C(n)\|\tilde{u}^{\frac{n-3-\alpha}{2}}\|_\infty\left(\sigma_1\iint_{\Omega_T}\tilde{u}_t^2\right)^{\frac12}\left(\iint_{\Omega_T}\tilde{u}^{\alpha+n-3}|\nabla\tilde{u}|^4\right)^{\frac12}\\ &\le \sqrt{\sigma_1}C(\delta,n,\alpha)\left(\int_\Omega|\nabla\tilde{u}(0)|^2 + \int_\Omega G_\alpha(u(0)) + \sigma_1\int_\Omega|\nabla\tilde{u}(0)|^2\right).\end{aligned}$$

Collecting these estimates into (4.7) we obtain (4.6) for all $\sigma_1\le 1$ and $\delta\le 1$.

To conclude, in the proof of [21, Theorem V.3] the limit process is performed first as $\sigma_1\to 0$, with $\tilde{u} = u_{\sigma_1,\delta}\to\bar{u} = u_\delta$, turning (4.6a) into

$$\begin{aligned}\int_\Omega \bar{u}(T) &\le \int_\Omega u_0 + |\Omega||\delta\ln\delta| + \delta\int_\Omega\bar{u}(T)\\ &\qquad + C(\alpha,n)\delta^{-\alpha}\left(\int_\Omega G_\alpha(u_0) + \int_\Omega|\nabla\tilde{u}_0|^2\right) = R(\delta)\end{aligned} \tag{4.9}$$

under (4.6b) – using standard estimates for the logarithm. Since $\alpha\in(-1,0)$, we have $\delta^{-\alpha}\to 0$ as $\delta\to 0$ and $G_\alpha(u_0) = \frac{1}{\alpha(\alpha+1)}(u_0^{\alpha+1}-1) - \frac{1}{\alpha}(u_0-1)$, hence $\int_\Omega G_\alpha(u_0)$ is uniformly controlled by $\int_\Omega u_0$ and $|\Omega|$. Therefore $R(\delta)\to\int_\Omega u_0$ as $\delta\to 0$, which suffices to complete the argument exactly as in [21].

5. Discussion of Theorem 1.7

In this section we elaborate on the admissible set $\mathbf{A} = \boldsymbol{E}_1 \cup \boldsymbol{E}_2$ in Theorem 1.7, comparing with the case $N = 1$ (Remark 5.1) and highlighting some of its properties.

Remark 5.1. *In one space dimension the argument used in the proof of Theorem* 1.7 *yields the same admissible set* $\boldsymbol{L}$ *as in* [25]*:* $\boldsymbol{L} = \boldsymbol{P} \cap (\boldsymbol{E}_1^1 \cup \boldsymbol{E}_2^1)$, *where*

$$\begin{aligned}
\boldsymbol{P} &= \{(n,p) : \ 1 - 2n < 2p < 6 - 2n, \quad 2n - 6 < 3p < 2n - 1\}, \\
\boldsymbol{E}_1^1 &= \{(n,p) : \ 175p^2 + 3(p - 4n + 7)^2 \le 75\}, \\
\boldsymbol{E}_2^1 &= \{(n,p) : \ 50p^2 + 3(p - 4n + 7)^2 \le 30\}.
\end{aligned}$$

Indeed, in one space dimension $Q = S$ and the linearly independent relations reduce to (2.1)-(2.2). Then the same optimization argument applied to

$$D \ge a\left(5(S,T) + (n+p-3)\|T\|^2\right) + b\left((R,T) + (n+p-2)(S,T) - 3\|S\|^2\right)$$

with respect to a, b, yields the admissible set $\boldsymbol{L}$.

Graphical representations of the sets $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ in (1.9) are reported in Figure 1 (a)-(b). They are obtained by superposing $E_i(z)$ for varying $z > 1/3$. Figure 1 (c) shows that the admissible set $\boldsymbol{A} = \boldsymbol{E}_1 \cup \boldsymbol{E}_2$ is strictly contained in its one-dimensional counterpart $\boldsymbol{L}$ (cf. Remark 5.1). This discrepancy is due to the fact that Δh coincides with $D^2 h$ only in one space dimension. In the proof of Theorem 1.7 the only possible loss of information is the use of the inequality $|D^2 h \nabla h| \le |\nabla h||D^2 h|$. It is not clear to us if the discrepancy is indeed a technical loss of information or rather an intrinsic difference.

Figure 1 (a)-(b) also report the following observations, which we prove at the end of the section.

Proposition 5.2. *The following segments belong to* $\overline{\boldsymbol{E}}_1$*:*

- *from* $P_0 = (\frac{1}{2}, 0)$ *to* $P_1 = (\frac{23}{22}, -\frac{6}{11})$
- *from* $P_2 = (\frac{1}{5}(9 + \sqrt{21}), -\frac{2}{15}(6 - \sqrt{21}))$ *to* $P_3 = (3, 0)$
- *from* $P_4 = (\frac{19}{7}, \frac{2}{7})$ *to* $P_3 = (3, 0)$
- *from* $P_0 = (\frac{1}{2}, 0)$ *to* $P_5 = (\frac{7}{5}, \frac{3}{5})$

The four points P_1*,* P_2*,* P_4 *and* P_5 *also belong to* $\overline{\boldsymbol{E}}_2$*.*

It is apparent from Figure 1 (a)-(c) that $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ and $\boldsymbol{A}$ are convex and that $\boldsymbol{E}_1 \cup \boldsymbol{E}_2 \subset \boldsymbol{P}$. It is also apparent that $\boldsymbol{E}_1 \subset \boldsymbol{E}_2$ outside of the two triangles $P_0P_1P_5$ and $P_2P_3P_4$, hence $\boldsymbol{A}$ may be seen as the union of $\boldsymbol{E}_2$ with the two triangles $P_0P_1P_5$ and $P_2P_3P_4$. Unfortunately, we were unable to provide a complete analytical proof of these facts.

In Figure 1 (d) we highlight a few other approximate points in $\boldsymbol{A}$ (though near its boundary) for the typical values $n = \frac{3}{2}$, $n = 2$:

$$\begin{aligned}
&P_{10} = (3/2, -0.585407), \qquad &&P_{11} = (2, -0.476681), \\
&P_{12} = (2, 0.679612), &&P_{13} = (3/2, 0.653904).
\end{aligned} \tag{5.1}$$

Together with the points $P_0, \dots, P_5$ from Proposition 5.2, they identify a decagon, visualized in Figure 1 (d), which is contained in $\boldsymbol{A}$ granted its convexity.

The points $P_{10}, \dots, P_{13}$ are obtained as follows. A computer-algebra assisted computation shows that

$$\begin{aligned} g(n,p,z) = &- \frac{1}{4p^2(1+z)^2(8+3z)} \left(\Delta(n,p,z) - \left(B(n,p,z) + A(z)\tfrac{p^2}{2} \right)^2 \right) \\ :=& 3p^2 z(1+z)(5+z) + 4n^2(12+z(4+z)) - 4np(6+z(11+z)) \\ &- 8n(4+z)(3+2z) + 4p(6+5z(3+z)) + 4(4+z)(3+4z). \end{aligned} \tag{5.2}$$

Recalling the constraint $z > 1/3$, $\boldsymbol{E}_2$ is equivalently characterized by

$$\boldsymbol{E}_2 = \bigcup_{z>1/3} E_2(z), \quad E_2(z) = \{(n,p):\ g(n,p,z) < 0\}. \tag{5.3}$$

For $n = 2$ we then proceed as follows: we evaluate g at $n = 2$; we identify the interval $(p_-(z), p_+(z))$ in which $g(2,p,\cdot) < 0$ by exactly solving the corresponding quadratic equation with respect to p; we pick two values of z close to the ones maximizing p_+, resp. minimizing p_-; finally, we evaluate $p_\pm$ at those values. The procedure for $n = \frac{3}{2}$ is identical. The values of z for the points in (5.1) are, respectively, $z = 2.2797$, $z = 3.9666$, $z = 1.2600$, and $z = 0.4669$.

We conclude with the proof of Proposition 5.2.

Proof of Proposition 5.2. Let us first consider the segment $\overline{P_0P_1}$. On the line $p(n) = \frac{1}{2} - n$, we have

$$\begin{aligned} &\Delta(n,p(n),z) = (3z-2)^2 \hat{\Delta}(n,z), \\ &\qquad \hat{\Delta}(n,z) := 49 + 20n^2(1+z)(5+3z) - 20n(1+z)(7+6z) + 6z(13+9z). \end{aligned}$$

We therefore choose $z_* = \frac{2}{3}$, so that $\Delta(n,p,z_*) = 0$ along the segment, and consider the set $E_1(z_*)$ in (1.9b). We have in particular

$$\begin{aligned} &\Delta(n,p,z_*) = 2(p+n-\tfrac{1}{2})\tilde{\Delta}(n,p), \\ &\qquad \tilde{\Delta}(n,p) := (-19+8n+8p)\left(76+64n^2-8n(23+29p)+p(296+229p)\right). \end{aligned}$$

In particular $\tilde{\Delta}(n, \frac{1}{2} - n) = -\frac{1125}{4}(15 - 44n + 28n^2)$, which is positive if (and only if) $n \in (\frac{1}{2}, \frac{15}{14})$. Hence in any sufficiently small neighbourhood of each point of the segment the sign of $\Delta(n,p,z_*)$ is determined by that of $p+n-\frac{1}{2}$. In addition

$$\begin{aligned} B(n, \tfrac{1}{2} - n, z_*) + \tfrac{1}{2}A(z_*)p^2(n) &= -\tfrac{25}{4}(23 - 68n + 44n^2) > 0 \\ &\Leftrightarrow\ n \in (\tfrac{1}{2}, \tfrac{23}{22}) \subset (\tfrac{1}{2}, \tfrac{15}{14}). \end{aligned}$$

Therefore in a sufficiently small neighbourhood of each point of the segment the value of $B + \frac{1}{2}Ap^2$ remains positive. Hence the segment belongs to $\partial E_1(z_*)$. Since $\overline{E}_1(z_*) \subset \overline{\boldsymbol{E}}_1$, the segment belongs to $\overline{\boldsymbol{E}}_1$.

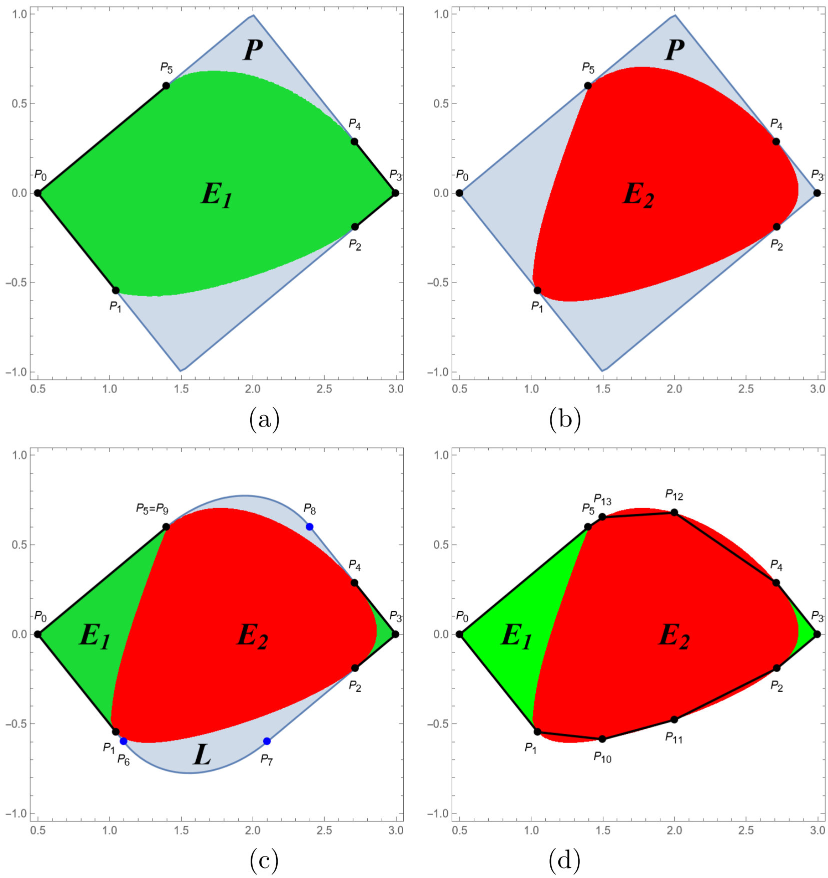

FIGURE 1. (a): $\boldsymbol{P}$, $\boldsymbol{E}_1$ and $\partial\boldsymbol{P}\cap\partial\boldsymbol{E}_1$. (b): $\boldsymbol{P}$ and $\boldsymbol{E}_2$. (c): $\boldsymbol{A}=\boldsymbol{E}_1\cup\boldsymbol{E}_2$ and the admissible set $\boldsymbol{L}$ in one space dimension (with its turning points $P_6,\dots,P_9$ from segments to ellipses, cf. [25]). (d): a decagon contained in $\boldsymbol{A}=\boldsymbol{E}_1\cup\boldsymbol{E}_2$.

The argument for the other three segments is analogous; the choices of z_* turn out to be, respectively: $z_*(n)=\frac{12}{5n-12}$ on the line $p(n)=\frac{2}{3}(n-3)$; $z_*=4$ on the line $p(n)=3-n$; $z_*(n)=\frac{3}{2n+5}$ on the line $p(n)=\frac{1}{3}(2n-1)$.

It remains to argue that P_1, P_2, P_4 and P_5 belong to $\overline{\boldsymbol{E}}_2$. We consider the function g in (5.2) and we recall that $\boldsymbol{E}_2$ is equivalently characterized by (5.3). For P_1 we evaluate g at $n=\frac{23}{22}$ and $z=\frac{2}{3}$, obtaining $g(\frac{23}{22},p,\frac{2}{3})=-\frac{2}{1089}(6+11p)(246+935p)$, which changes sign at $p=-\frac{6}{11}$. Hence $P_1\in\partial E_2(z)$, which implies that $P_1\in\overline{\boldsymbol{E}}_2$. The argument for P_2 and P_4 is analogous, evaluating g at, respectively: $n=\frac{1}{5}(9+\sqrt{21})$ and $z=3+\sqrt{21}$; $n=\frac{19}{7}$ and $z=4$. Note that in these three cases the value of z coincides with that of $z_*(n)$ in the argument for $\boldsymbol{E}_1$, the intuition being that at these points the corresponding three-dimensional sets $\{z\} \times \partial E_1(z)$ and $\{z\} \times \partial E_2(z)$ also intersect (tangentially). Instead, $P_5 \in \overline{\boldsymbol{E}}_2$ correspond to the limiting case $z = \frac{1}{3}$. Noting that $g(n, p, \frac{1}{3}) = (13 - 11n + 4p)^2$, we evaluate g along $p(n) = \frac{11n-13}{4}$: $g(n, p(n), z) = \frac{1}{16}(3z - 1)\tilde{g}(n, z)$. One checks that $\tilde{g}(n, \frac{1}{3})$ has two distinct roots at $n = \frac{247}{245}$ and $n = \frac{7}{5}$, hence $P_5 \in \overline{\boldsymbol{E}}_2$ follows by continuity. $\square$

Acknowledgment. Parts of this research have been done when L.G. visited Friedrich-Alexander-Universität Erlangen-Nürnberg. He is grateful for the kind hospitality of his hosts.

Author contributions. The authors contributed equally to this work.

Funding. L.G.'s visit and J.U.'s position as a doctoral researcher were funded by Graduiertenkolleg 2339 "Interfaces, Complex Structures, and Singular Limits (IntComSin)" of Deutsche Forschungsgemeinschaft (DFG, "German Research Foundation") under Project-ID 321821685. The support is gratefully acknowledged.

Conflict of interest. The authors declare that they have no conflict of interest.

Data availability. No dataset was generated or analyzed during this study.

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