Stability of optimal transport on metric measure spaces

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Abstract

We prove a quantitative stability of Kantorovich potentials on non-smooth metric measure spaces with synthetic lower Ricci curvature bound, thereby confirming a recent conjecture of Kitagawa, Letrouit and Mérigot. Our proof, which employs the heat kernel-regularized c-transform, does not rely on linear structure or sectional curvature bounds, is new even in the smooth setting. As a corollary, we get a quantitative stability of optimal transport maps on Alexandrov spaces with lower curvature bound.

Keywords: optimal transport, heat kernel, Kantorovich potential, quantitative stability, metric measure space, curvature-dimension condition, Ricci curvature

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Contents

1 Introduction

1.1 Background and Motivation

Optimal transport, initiated by Monge [35] and reframed by Kantorovich [24], seeks the most efficient way to redistribute mass between two probability distributions. Precisely, given two probability measures ρ and μ defined on Polish spaces X and Y respectively, and a cost function $c : X \times Y \to \mathbb{R}$, the Monge optimal transport problem aims to find

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a minimizer T, called an optimal transport map, of the following optimization problem among all measurable maps $H : X \to Y$ pushing ρ forward to μ:

$$\int_X c(x, T(x))\,\mathrm{d}\rho(x) = \inf_{H_\sharp \rho = \mu} \int_X c(x, H(x))\,\mathrm{d}\rho(x). \tag{MP}$$

The Monge problem is not always well-posed, since it prevents mass splitting. Its relaxation, called Kantorovich problem, optimizes over joint couplings instead of deterministic maps. By Kantorovich duality, the Kantorovich problem is equivalent to the following dual problem:

$$\sup_{\phi(x)+\psi(y)\le c(x,y)} \left\{ \int_X \phi(x)\,\mathrm{d}\rho(x) + \int_Y \psi(y)\,\mathrm{d}\mu(y) \right\}. \tag{KD}$$

The optimal functions ϕ, ψ, which always exist under standard assumptions, are called Kantorovich potentials [2, 45].

In the Euclidean space, for $c(x, y) = \frac{1}{2}|x-y|^2$, Brenier's landmark result [5] established that, for absolutely continuous source measures, the optimal transport map is unique and takes the form $T(x) = \nabla u(x)$ for a convex function u. This was later extended to Riemannian manifolds by McCann [32], who showed that for $c(x, y) = \frac{1}{2}\mathrm{d}^2(x, y)$, the optimal transport map is given by $T(x) = \exp_x(-\nabla\varphi(x))$ for a Kantorovich potential φ.

A fundamental question in both theoretical and applied contexts of optimal transport is the quantitative stability of optimal transport maps and Kantorovich potentials under perturbations of the target measure (see Letrouit's lecture note [28] and the references therein). Based on recent breakthroughs in the quantitative stability of optimal transport on Euclidean spaces [11, 30], on boundaries of convex bodies [20, 27] and on Riemannian manifolds [26], Kitagawa–Letrouit–Mérigot [26, §1.2] conjecture that:

the quantitative stability results are also true in more general metric measure spaces with synthetic curvature bounds.

1.2 General Setting

We confirm the conjecture of Kitagawa–Letrouit–Mérigot in the following setting.

A. Metric measure spaces: An Alexandrov space is a geodesic space of finite Hausdorff dimension and of curvature bounded from below (cf. [8]). An $\mathrm{RCD}(K, N)$ space is a metric measure spaces verifying the synthetic Riemannian curvature-dimension condition [3,12,31,43]. An n-dimensional Alexandrov space with curvature bounded from below by k, equipped with its n-Hausdorff measure, is an $\mathrm{RCD}(k(n-1), n)$ space [41, 46]. After [5, 32], Gigli–Rajala–Sturm [15, Theorem 1.1] proved the existence and uniqueness of the optimal transport map on an $\mathrm{RCD(K, N)}$ space $(X, \mathrm{d}, \mathfrak{m})$, for the quadratic cost $c(x, y) = \frac{1}{2}\mathrm{d}^2(x, y)$ and the source measure $\rho \ll \mathfrak{m}$.

B. Source measures: In [30, Theorem 1.9], it has been shown that when the source measure ρ is the uniform density on some non-John domain $S \subset \mathbb{R}^n$, then no quantitative stability estimates of the form

$$\|\phi_\mu - \phi_\nu\|_{L^2(\rho)} \le C W_p^\alpha(\mu, \nu).$$

A bounded open subset S of a metric space is called a *John domain* if there is a distinguished point $x_0 \in S$ and a constant $\eta > 0$ such that, for every $x \in S$, there is a rectifiable curve $\gamma \colon [0, \ell(\gamma)] \to S$ parametrized by arc length, such that $\gamma(0) = x$, $\gamma(\ell(\gamma)) = x_0$, and

$$\mathrm{d}(\gamma(t), S^c) \ge \eta t, \quad \forall t \in [0, \ell(\gamma)].$$

John domains encompass many cases of interest, such as bounded Lipschitz domains, bounded domains satisfying a cone condition and certain fractal domains (see [7] for more discussions).

Moreover, examples found by Letrouit [29] indicates that both unboundedness of the density of the source measure ρ, and the openness of S may cause instability of optimal transport maps. So we assume that $a_1\mathfrak{m}|_S \leq \rho \leq a_2\mathfrak{m}|_S$ for some John domain S and constants $a_1, a_2 > 0$.

C. Kantorovich potentials: It is also necessary to (zero-mean) normalize the Kantorovich potential ϕ from ρ such that $\mathbb{E}_\rho(\phi) = \int_S \phi \, \mathrm{d}\rho = 0$. Together with the uniqueness of the optimal transport map, such ϕ is unique and thus makes sense to talk about its stability.

1.3 Main Results

Our main theorem concerns the quantitative L^1 stability of Kantorovich potentials on $\mathrm{RCD}(K,N)$ spaces.

Theorem 1.1. *Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ metric measure space. Let $S \subseteq X$ be a John domain and $Y \subseteq X$ be compact with $\mathfrak{m}(Y) > 0$. Let $\rho \in \mathcal{P}(S)$ be with $a_1\mathfrak{m}|_S \leq \rho \leq a_2\mathfrak{m}|_S$ for some positive constants a_1, a_2. Then there exists a constant $C > 0$, depending on $K, N, a_1, a_2, S, \mathrm{diam}(S \cup Y)$, such that for any $\mu, \nu \in \mathcal{P}(Y)$,*

$$\|\phi_\mu - \phi_\nu\|_{L^1(\rho)} \leq C W_1^{\frac{1}{2}}(\mu, \nu), \tag{1.1}$$

where ϕ_μ and ϕ_ν are the Kantorovich potentials from ρ to μ and ρ to ν respectively.

In particular, if $(S, \mathrm{d}, \mathfrak{m})$ is a compact $\mathrm{RCD}(K,N)$ space, S is surely a John domain. So we have the following corollary.

Corollary 1.2. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a compact $\mathrm{RCD}(K,N)$ metric measure space. Let $\rho \in \mathcal{P}(X)$ be with $a_1\mathfrak{m} \leq \rho \leq a_2\mathfrak{m}$ for some positive constants a_1, a_2. Then the conclusion of Theorem 1.1 holds.*

Adapting the strategy of [26], we can also prove the stability of optimal transport maps on Alexandrov spaces.

Theorem 1.3. *Let (X, d) be an n-dimensional Alexandrov space with curvature bounded from below by k, $\mathfrak{m}$ be the n-Hausdorff measure. Under the same assumption for S, Y and ρ as in Theorem 1.1, and if S additionally has finite perimeter, then there exists a constant $C > 0$, depending on $k, n, a_1, a_2, \mathrm{diam}(S \cup Y), S$, such that for any $\mu, \nu \in \mathcal{P}(Y)$,*

$$\int_S \mathrm{d}^2(T_\mu(x), T_\nu(x)) \, \mathrm{d}\rho(x) \leq C W_1^{1/6}(\mu, \nu), \tag{1.2}$$

where T_μ and T_ν are the optimal transport maps from ρ to μ and ρ to ν respectively.

1.4 Strategy: heat kernel-regularized c-transform

Motivated by regularized c-transforms using Gibbs kernels $e^{-c(x,y)/\varepsilon}$ [26], entropic optimal transport [16, 17] and Varadhan's formula

$$\lim_{t \to 0} -t \log p_{t/2}(x,y) = \frac{1}{2}\mathrm{d}^2(x,y) = c(x,y),$$

we make use of the following heat kernel-regularized c-transform [1]:

$$\mathrm{Lip}(X,\mathrm{d}) \ni \psi \mapsto \Phi_t[\psi](x) = -t\log\int_X e^{\frac{\psi(y)}{t}} p_{t/2}(x,y)\,\mathrm{d}\mathfrak{m}(y).$$

This approach allows us to bypass the low regularity of Kantorovich potentials in the non-smooth setting.

We define the heat kernel regularized Kantorovich functional as

$$\mathrm{K}_t[\psi] := \int_S \Phi_t[\psi]\,\mathrm{d}\rho.$$

We first derive a local concavity estimate of the functional K_t using heat kernel estimate, then globalize the estimate on the support of the source measure using a Boman chain argument for John domains. Finally, letting $t \to 0$ we obtain the quantitative stability of Kantorovich potentials.

Unlike the regularized c-transform used in [26], the existence of the boundary of Y may lead to the failure in our heat kernel regularization argument. This possibility is ruled out by using the measure concentration property of the heat kernel and by making a careful choice of Lipschitz extension.

Organization. This paper is structured as follows. In Section 2, we prove the quantitative stability of Kantorovich potentials on $\mathrm{RCD}(K,N)$ spaces. In Section 3, we establish the stability of optimal transport maps on Alexandrov spaces. The Appendix A contains technical lemmas about Poincaré inequalities.

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2 Stability of Kantorovich potentials

2.1 Heat kernel estimate

We begin by recalling the short-time asymptotic behaviour of the heat kernel on RCD spaces. It was first studied by Varadhan on Riemannian manifolds [44], and is known as Varadhan formula today.

Lemma 2.1. *Let $p_t(x,y)$ be the heat kernel on an $\mathrm{RCD}(K,N)$ space $(X,\mathrm{d},\mathfrak{m})$. Then*

$$\lim_{t\to 0} -t\log p_{\frac{t}{2}}(x,y) = \frac{1}{2}\mathrm{d}^2(x,y), \quad \textit{uniformly in} \quad x\in S,\ y\in Y. \tag{2.1}$$

Proof. By the heat kernel estimate [22, Theorem 1.2], for any $\epsilon > 0$, it holds

$$\begin{aligned}&\frac{1}{C_1(\epsilon)\mathfrak{m}\left(B(y,\sqrt{\frac{t}{2}})\right)}\exp\left(-\frac{\mathrm{d}^2(x,y)}{(4-\epsilon)t} - C_2(\epsilon)t\right)\\ &\quad\le p_t(x,y) \le \frac{C_1(\epsilon)}{\mathfrak{m}\left(B(y,\sqrt{\frac{t}{2}})\right)}\exp\left(-\frac{\mathrm{d}^2(x,y)}{(4+\epsilon)t} + C_2(\epsilon)t\right).\end{aligned} \tag{2.2}$$

[1] We are told by Luca Taminini that in solving the Schrödinger equation with the Cole–Hopf transform, a similar formula will occur.

Then we have

$$
\begin{aligned}
&-t\log\frac{C_1(\epsilon)}{\mathfrak{m}\left(B(y,\sqrt{\frac{t}{2}})\right)}+\frac{2\mathrm{d}^2(x,y)}{4+\epsilon}-\frac{C_2(\epsilon)}{2}t^2\\
&\le -t\log p_{\frac{t}{2}}(x,y)\le t\log C_1(\epsilon)\mathfrak{m}\left(B(y,\sqrt{\frac{t}{2}})\right)+\frac{2\mathrm{d}^2(x,y)}{4-\epsilon}+\frac{C_2(\epsilon)}{2}t^2.
\end{aligned}
\tag{2.3}
$$

By Bishop–Gromov inequality [43, Theorem 2.3], we have

$$
\lim_{t\to 0} t\log\mathfrak{m}\left(B(y,\sqrt{\frac{t}{2}})\right)=0,\quad \text{uniformly in } y\in Y. \tag{2.4}
$$

Letting $t\to 0$ in (2.3), we get the following uniform estimate:

$$
\frac{2\mathrm{d}^2(x,y)}{4+\epsilon}\le \varliminf_{t\to 0} -t\log p_{\frac{t}{2}}(x,y)\le \varlimsup_{t\to 0} -t\log p_{\frac{t}{2}}(x,y)\le \frac{2\mathrm{d}^2(x,y)}{4-\epsilon}. \tag{2.5}
$$

Letting $\epsilon\to 0$, we prove the lemma. □

2.2 Heat kernel regularization

To establish the quantitative stability, we adopt a regularization technique similar to those in [26,36] and [16,17] (see also [9,10,34]).

For $\psi\in \mathrm{Lip}(Y,\mathrm{d})$, define

$$
\bar\psi(y)=\sup_{z\in Y}\left\{\psi(z)-\mathrm{Lip}(\psi)\,\mathrm{d}(z,y)\right\},\quad y\in X.
$$

By MacShane's Lemma, $\mathrm{Lip}(\bar\psi)=\mathrm{Lip}(\psi)$ and $\bar\psi|_Y=\psi|_Y$. Denote $D=\mathrm{diam}(S\cup Y)$, $\Lambda_\psi=D+\mathrm{Lip}(\psi)+1$ and define $\psi_*=\bar\psi-\Lambda_\psi \mathrm{d}(\cdot,Y)$.

For $t>0$, we define the heat kernel regularized c-transform as

$$
\mathrm{Lip}(X,\mathrm{d})\ni\varphi\mapsto \Phi_t[\varphi](x):=-t\log\int_X e^{\frac{\varphi(y)}{t}}p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y),
$$

and define the heat kernel regularized Kantorovich functional as

$$
\mathrm{K}_t[\varphi]:=\int_S \Phi_t[\varphi]\,\mathrm{d}\rho.
$$

For $\varphi\in\mathrm{Lip}(X,\mathrm{d})$ and $x\in X$, we associate a probability measure

$$
\mathrm{d}\mu_x^t[\varphi](y):=\frac{e^{\frac{\varphi(y)}{t}}p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}{\int_X e^{\frac{\varphi(y)}{t}}p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}
$$

and define $\mu^t[\varphi]:=\int_X\mu_x^t[\varphi]\,\mathrm{d}\rho(x)$. This means, for any $v\in\mathrm{Lip}(X,\mathrm{d})$, it holds

$$
\mathbb{E}_{\mu^t[\varphi]}(v)=\int_X \mathbb{E}_{\mu_x^t[\varphi]}(v)\,\mathrm{d}\rho(x). \tag{2.6}
$$

Next we prove the Sobolev regularity of $x\mapsto\mathbb{E}_{\mu_x^t[\varphi]}(v)$. We refer to [3] for Sobolev calculus on metric measure spaces.

Lemma 2.2. *For* $t \in \big(0, \frac{D+1}{\sqrt{(N-1)|K|}} \wedge 1\big)$*,* $\psi \in \mathrm{Lip}(Y, \mathrm{d})$ *and* $v \in \mathrm{Lip}(X, \mathrm{d})$*, there holds* $\mathbb{E}_{\mu_x^t[\psi_*]}(v) \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$*.*

Proof. Denote by $H_t f(x) = \int_X f(y) p_t(x,y)\,\mathrm{d}\mathfrak{m}(y)$ the heat flow from f. Then

$$\mathbb{E}_{\mu_x^t[\psi_*]}(v) = \frac{\int_X v(y) e^{\frac{\psi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}{\int_X e^{\frac{\psi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)} = \frac{H_{\frac{t}{2}}(v e^{\frac{\psi_*}{t}})(x)}{H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}})(x)}.$$

Claim: $v e^{\frac{\psi_*}{t}}, e^{\frac{\psi_*}{t}} \in L^2(\mathfrak{m}) \cap L^\infty(\mathfrak{m})$.

Note that $|v(y)| \le \sup_Y |v| + \mathrm{Lip}(v)\,\mathrm{diam}(Y) + \mathrm{Lip}(v)\mathrm{d}(y, Y)$ and $\psi_*(y) \le \sup_Y \psi - (D+1)\mathrm{d}(y,Y)$. So $v e^{\frac{\psi_*}{t}}, e^{\frac{\psi_*}{t}} \in L^\infty(\mathfrak{m})$.

Fix $y_0 \in Y$ and denote $B_i = \{y \in X : i \le \mathrm{d}(y,Y) < i+1\}, i \in \mathbb{N}$. It holds

$$\begin{aligned} \int_X e^{\frac{2\psi_*}{t}}\,\mathrm{d}\mathfrak{m} &\le \sum_{i=0}^{+\infty} e^{\frac{2\sup_Y \psi}{t}} e^{\frac{-2(D+1)i}{t}} \mathfrak{m}(B_i) \\ &\le \sum_{i=0}^{+\infty} e^{\frac{2\sup_Y \psi}{t}} e^{\frac{-2(D+1)i}{t}} \mathfrak{m}(B(y_0, \mathrm{diam}(Y) + i + 1)). \end{aligned} \tag{2.7}$$

Without loss of generality, we may assume $K < 0$. By Bishop–Gromov inequality [43, Theorem 2.3], we have

$$\mathfrak{m}\big(B(y_0, \mathrm{diam}(Y) + i + 1)\big) \le c_1 \int_0^{\mathrm{diam}(Y)+i+1} \sinh^{N-1}\Big(\sqrt{\frac{-K}{N-1}}s\Big)\,\mathrm{d}s$$

where $c_1 = \frac{\mathfrak{m}(B(y_0,\mathrm{diam}(Y)))}{\int_0^{\mathrm{diam}(Y)} \sinh^{N-1}(\sqrt{\frac{-K}{N-1}}s)\mathrm{d}s}$. Note that $\sinh^{N-1}(\sqrt{\frac{-K}{N-1}}s) \le \frac{e^{\sqrt{(N-1)|K|}s}}{2^{N-1}}$, then

$$\mathfrak{m}\big(B(y_0, \mathrm{diam}(Y) + i + 1)\big) \le c_1 \int_0^{\mathrm{diam}(Y)+i+1} \frac{e^{\sqrt{(N-1)|K|}s}}{2^{N-1}}\,\mathrm{d}s \le c_2 e^{\sqrt{(N-1)|K|}i} \tag{2.8}$$

where $c_2 = c_1 \frac{e^{\sqrt{(N-1)|K|}(\mathrm{diam}(Y)+1)}}{2^{N-1}\sqrt{(N-1)|K|}}$.

For $t < \frac{D+1}{\sqrt{(N-1)|K|}}$, it holds $\sqrt{(N-1)|K|} - \frac{2(D+1)}{t} < -\frac{D+1}{t}$, so by (2.7) and (2.8),

$$\int_X e^{\frac{2\psi_*}{t}}\,\mathrm{d}\mathfrak{m} \le \sum_{i=0}^{+\infty} e^{\frac{2\sup_Y \psi}{t}} c_2 e^{\frac{-2(D+1)i}{t}} e^{\sqrt{(N-1)|K|}i} \le \sum_{i=0}^{+\infty} e^{\frac{2\sup_Y \psi}{t}} c_2 e^{\frac{-(D+1)i}{t}} < +\infty. \tag{2.9}$$

Similarly, we can prove $\int_X v^2 e^{\frac{2\psi_*}{t}}\,\mathrm{d}\mathfrak{m} < +\infty$ and we prove the claim.

By the regularization of heat flow [3, Theorem 6.5], we have $H_{\frac{t}{2}}(v e^{\frac{\psi_*}{t}}), H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}}) \in W^{1,2}(X, \mathrm{d}, \mathfrak{m}) \cap \mathrm{Lip}(X, \mathrm{d}) \cap L^\infty(\mathfrak{m})$. Moreover, by the heat kernel estimate (2.2),

$$H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}})(x) \ge \int_Y e^{\frac{\psi(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y) > 0. \tag{2.10}$$

So $\frac{1}{H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}})} \in W^{1,2}(X, \mathrm{d}, \mathfrak{m}) \cap L^\infty(\mathfrak{m})$ and by chain rule (see [14, Theorem 4.3.3]), we know the function $x \mapsto \mathbb{E}_{\mu_x^t[\psi_*]}(v) \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$. □

Lemma 2.3. *For any $v \in \mathrm{Lip}(X, \mathrm{d})$ and $\psi \in \mathrm{Lip}(Y, \mathrm{d})$, we have*

- $\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\mathrm{K}_t[\psi_* + sv] = -\mathbb{E}_{\mu^t[\psi_*]}(v)$;
- $\frac{\mathrm{d}^2}{\mathrm{d}s^2}|_{s=0}\mathrm{K}_t[\psi_* + sv] = -\frac{1}{t}\int_S \mathrm{Var}_{\mu^t_x[\psi_*]}(v)\,\mathrm{d}\rho(x)$.

For abbreviation, we can write

$$\nabla\mathrm{K}_t[\psi_*] = -\mu^t[\psi_*], \quad \langle D^2\mathrm{K}_t[\psi_*]v, v\rangle = -\frac{1}{t}\int_S \mathrm{Var}_{\mu^t_x[\psi_*]}(v)\,\mathrm{d}\rho(x).$$

Proof. Let $|s| \le 1$. By direct computation,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi_t[\psi_* + sv] = -\frac{\int_X v(y)e^{\frac{\psi_*(y)+sv(y)}{t}}p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}{\int_X e^{\frac{\psi_*(y)+sv(y)}{t}}p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}.$$

Similar to Lemma 2.2, we can see that $\left|\frac{\mathrm{d}}{\mathrm{d}s}\Phi_t[\psi_* + sv]\right|$ is uniformly bounded in s.

By differentiating under the integral defining $\mathrm{K}_t[\psi_*]$ and letting $s = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\mathrm{K}_t[\psi_* + sv] = \int_S \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\Phi_t[\psi_* + sv]\,\mathrm{d}\rho(x) = -\mathbb{E}_{\mu^t[\psi_*]}(v).$$

Similarly, we can prove

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi_t[\psi_* + sv]\\ &= -\frac{1}{t}\left(\frac{\int_X v^2(y)e^{\frac{\psi_*(y)+sv(y)}{t}}p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}{\int_X e^{\frac{\psi_*(y)+sv(y)}{t}}p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)} - \left|\frac{\int_X v(y)e^{\frac{\psi_*(y)+sv(y)}{t}}p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}{\int_X e^{\frac{\psi_*(y)+sv(y)}{t}}p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}\right|^2\right)\end{aligned}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}|_{s=0}\mathrm{K}_t[\psi_* + sv] = \int_S \frac{\mathrm{d}^2}{\mathrm{d}s^2}|_{s=0}\Phi_t[\psi_* + sv]\,\mathrm{d}\rho(x) = -\frac{1}{t}\int_S \mathrm{Var}_{\mu^t_x[\psi_*]}(v)\,\mathrm{d}\rho(x).$$

□

2.3 Gradient estimate

To leverage the Hessian formula in Lemma 2.3, we need to bound the variance term from below; this amounts to proving strong concavity of the regularised Kantorovich functional. We achieve this by establishing a gradient estimate for the marginal density $x \mapsto \mathbb{E}_{\mu^t_x[\psi_*]}(v)$.

The following auxiliary lemma will be used throughout this section. We refer to [13, 14] for a comprehensive introduction to L^2-normed tangent module $L^2(TX)$. By [19, Proposition 2.9] and [13, Theorem 1.4.11], elements of $L^2(TX)$ admit local-coordinate representations. Readers unfamiliar with non-smooth calculus may safely interpret the proof in the language of Riemannian geometry.

Lemma 2.4. *Let $f \in L^2(\mathfrak{m}) \cap L^\infty(\mathfrak{m})$, and $g \in L^0(X \times X)$ be with $g(\cdot, y), g(x, \cdot) \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ for $\mathfrak{m}$-a.e. $x, y \in X$. Then $\int_X f(y)g(x,y)\,\mathrm{d}\mathfrak{m}(y) \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ and for any $\varphi \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$, we have*

$$\int \langle \nabla_x \int_X f(y)g(x,y)\,\mathrm{d}\mathfrak{m}(y), \varphi(x)\rangle\,\mathrm{d}\mathfrak{m}(x) = \int_X f(y)\int \langle \nabla g(\cdot, y), \varphi\rangle\,\mathrm{d}\mathfrak{m}\,\mathrm{d}\mathfrak{m}(y) \qquad (2.11)$$

and

$$\left|\nabla_x \int_X f(y)g(x,y)\,\mathrm{d}\mathfrak{m}(y)\right| \le \int_X |f(y)||\nabla_x g(x,y)|\,\mathrm{d}\mathfrak{m}(y). \tag{2.12}$$

In particular, we can write

$$\int_X f(y)\nabla_x g(x,y)\,\mathrm{d}\mathfrak{m}(y) = \nabla_x \int_X f(y)g(x,y)\,\mathrm{d}\mathfrak{m}(y) \in L^2(TX). \tag{2.13}$$

Proof. For $\varphi \in \mathrm{TestF} := \{\phi \in \mathrm{Lip}(X,\mathrm{d}) \cap L^\infty(\mathfrak{m}) \cap \mathrm{D}(\Delta) : \Delta\phi \in W^{1,2}(X,\mathrm{d},\mathfrak{m}) \cap L^\infty(\mathfrak{m})\}$, by integration by parts and Fubini theorem, we get

$$\begin{aligned} &\int_X \Delta\varphi(x) \left(\int_X f(y)g(x,y)\,\mathrm{d}\mathfrak{m}(y)\right)\,\mathrm{d}\mathfrak{m}(x) \\ =& \int_X f(y) \int_X g(x,y)\Delta\varphi(x)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) \\ =& -\int_X f(y) \int_S \langle \nabla_x g(x,y), \nabla\varphi(x)\rangle\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y). \end{aligned} \tag{2.14}$$

By density of TestF (cf. [13, §3.2]) and the Riesz representation theorem for Hilbert module (cf. [14, Theorem 3.2.14, Example 3.2.15]), we prove the lemma. □

Lemma 2.5. *For any* $\psi \in \mathrm{Lip}(Y,\mathrm{d})$, *it holds*

$$|\nabla_x \mathbb{E}_{\mu_x^t[\psi_*]}(v)|^2 \le \mathrm{Var}_{\mu_x^t[\psi_*]}(v)\mathrm{Var}_{\mu_x^t[\psi_*]}\big(\nabla_x \log p_{\frac{t}{2}}(x,\cdot)\big) \tag{2.15}$$

for $\mathfrak{m}$*-a.e.* $x \in S$, *where*

$$\mathrm{Var}_{\mu_x^t[\psi_*]}\big(\nabla_x \log p_{\frac{t}{2}}(x,\cdot)\big) := \mathbb{E}_{\mu_x^t[\psi_*]}\left(\left|\nabla_x \log p_{\frac{t}{2}}(x,\cdot) - \mathbb{E}_{\mu_x^t[\psi_*]}(\nabla_x \log p_{\frac{t}{2}}(x,\cdot))\right|^2\right)$$

and

$$\mathbb{E}_{\mu_x^t[\psi_*]}\big(\nabla_x \log p_{\frac{t}{2}}(x,\cdot)\big) = \frac{\int_X e^{\frac{\psi_*(z)}{t}} \nabla_x p_{\frac{t}{2}}(x,z)\,\mathrm{d}\mathfrak{m}(z)}{\int_X e^{\frac{\psi_*(z)}{t}} p_{\frac{t}{2}}(x,z)\,\mathrm{d}\mathfrak{m}(z)}.$$

Proof. First of all, by [22], $p_s(\cdot,y) \in W^{1,2}(X,\mathrm{d},\mathfrak{m})$ for $\mathfrak{m}$-a.e. $y \in X$ so all the formulas above are well-defined. By Lemma 2.2, Lemma 2.4 and the chain rule for L^2-normed modules (cf. [14, Theorem 4.3.3]), we have

$$\begin{aligned} \nabla_x \mathbb{E}_{\mu_x^t[\psi_*]}(v) &= \frac{H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}})\nabla H_{\frac{t}{2}}(ve^{\frac{\psi_*}{t}}) - H_{\frac{t}{2}}(ve^{\frac{\psi_*}{t}})\nabla H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}})}{\big(H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}})\big)^2}(x) \\ &= \int_X v(y)\nabla_x \log p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mu_x^t[\psi_*](y) - \mathbb{E}_{\mu_x^t[\psi_*]}(v)\int_X \nabla_x \log p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mu_x^t[\psi_*](y) \\ &= \int_X \big(v(y) - \mathbb{E}_{\mu_x^t[\psi_*]}(v)\big)\left(\nabla_x \log p_{\frac{t}{2}}(x,y) - \mathbb{E}_{\mu_x^t[\psi_*]}\big(\nabla_x \log p_{\frac{t}{2}}(x,\cdot)\big)\right)\,\mathrm{d}\mu_x^t[\psi_*](y), \end{aligned}$$

where in the last equality we use the identity

$$\begin{aligned} &\int_X \big(v(y) - \mathbb{E}_{\mu_x^t[\psi_*]}(v)\big)\mathbb{E}_{\mu_x^t[\psi_*]}\big(\nabla_x \log p_{\frac{t}{2}}(x,\cdot)\big)\,\mathrm{d}\mu_x^t[\psi_*](y) \\ =\;& \mathbb{E}_{\mu_x^t[\psi_*]}\big(\nabla_x \log p_{\frac{t}{2}}(x,\cdot)\big)\int_X \big(v(y) - \mathbb{E}_{\mu_x^t[\psi_*]}(v)\big)\,\mathrm{d}\mu_x^t[\psi_*](y) = 0. \end{aligned}$$

By Lemma 2.4 and Hölder inequality we get (2.15). □

Lemma 2.6. *The following identity holds:*

$$\mathrm{Var}_{\mu_x^t[\psi_*]}\big(\nabla_x \log p_{\frac{t}{2}}(x,\cdot)\big) = \Delta_x \log\Big(H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}})(x)\Big) - \mathbb{E}_{\mu_x^t[\psi_*]}\big(\Delta_x \log p_{\frac{t}{2}}(x,\cdot)\big). \tag{2.16}$$

Proof. By chain rule of Laplacian (cf. [14, Theorem 5.2.3]), it holds

$$\Delta \log u = \frac{\Delta u}{u} - |\nabla \log u|^2, \tag{2.17}$$

for $u = p_{\frac{t}{2}}(x,y)$ and $H_{\frac{t}{2}}(e^{\frac{b}{t}})$. By Lemma 2.4, we have

$$\begin{aligned}
&\mathrm{Var}_{\mu_x^t[\psi_*]}\big(\nabla_x \log p_{\frac{t}{2}}(x,\cdot)\big)\\
&= \int_X |\nabla_x \log p_{\frac{t}{2}}(x,y)|^2 \,\mathrm{d}\mu_x^t[\psi_*](y) - \left|\int_X \nabla_x \log p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mu_x^t[\psi_*](y)\right|^2\\
&\overset{(2.17)}{=} \int_X \frac{\Delta_x p_{\frac{t}{2}}(x,y)}{p_{\frac{t}{2}}(x,y)}\,\mathrm{d}\mu_x^t[\psi_*](y) - \mathbb{E}_{\mu_x^t[\psi_*]}\big(\Delta_x \log p_{\frac{t}{2}}(x,\cdot)\big)\\
&\quad - \left|\int_X \nabla_x \log p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mu_x^t[\psi_*](y)\right|^2\\
&= \frac{\Delta_x H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}})(x)}{H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}})(x)} - |\nabla \log H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}})(x)|^2 - \mathbb{E}_{\mu_x^t[\psi_*]}\big(\Delta_x \log p_{\frac{t}{2}}(x,\cdot)\big)\\
&\overset{(2.17)}{=} \Delta_x \log\Big(H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}})(x)\Big) - \mathbb{E}_{\mu_x^t[\psi_*]}\big(\Delta_x \log p_{\frac{t}{2}}(x,\cdot)\big)
\end{aligned} \tag{2.18}$$

which is the thesis. □

Lemma 2.7. *For $t > 0$ small enough, it holds*

$$J_t(x) := \mathbb{E}_{\mu_x^t[\psi_*]}\big(|\nabla_x \log p_{\frac{t}{2}}(x,\cdot)|^2\big) \le \frac{C_1(K,N,\Lambda_\psi)}{t^2}. \tag{2.19}$$

Proof. Integrating the following Li–Yau type estimate (cf. [21, Theorem 1.2])

$$|\nabla_x \log p_{\frac{t}{2}}(x,y)|^2 \le e^{-\frac{Kt}{3}} \frac{\Delta_x p_{\frac{t}{2}}(x,y)}{p_{\frac{t}{2}}(x,y)} + \frac{NK}{3}\frac{e^{-\frac{2Kt}{3}}}{1-e^{-\frac{Kt}{3}}} \tag{2.20}$$

with respect to $\mu_x^t[\psi_*]$, we obtain

$$J_t(x) \le e^{-\frac{Kt}{3}} \int_X \frac{\Delta_x p_{\frac{t}{2}}(x,y)}{p_{\frac{t}{2}}(x,y)}\,\mathrm{d}\mu_x^t[\psi_*] + \frac{NK}{3}\frac{e^{-\frac{2Kt}{3}}}{1-e^{-\frac{Kt}{3}}}. \tag{2.21}$$

By symmetry of the heat kernel, we have

$$\int_X \Delta_x p_{\frac{t}{2}}(x,y) e^{\frac{\psi_*(y)}{t}}\,\mathrm{d}\mathfrak{m}(y) = \int_X \Delta_y p_{\frac{t}{2}}(x,y) e^{\frac{\psi_*(y)}{t}}\,\mathrm{d}\mathfrak{m}(y).$$

Then by integration by parts formula,

$$\begin{aligned}\int_X \frac{\Delta_x p_{\frac{t}{2}}(x,y)}{p_{\frac{t}{2}}(x,y)}\,\mathrm{d}\mu_x^t[\psi_*](y) =& \frac{\int_X \Delta_x p_{\frac{t}{2}}(x,y) e^{\frac{\psi_*(y)}{t}}\,\mathrm{d}\mathfrak{m}(y)}{\int_X e^{\frac{\psi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)} \\ =& \frac{\int_X \Delta_y p_{\frac{t}{2}}(x,y) e^{\frac{\psi_*(y)}{t}}\,\mathrm{d}\mathfrak{m}(y)}{\int_X e^{\frac{\psi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)} \\ =& \frac{-\int_X \langle \nabla_y p_{\frac{t}{2}}(x,y), \nabla_y e^{\frac{\psi_*(y)}{t}}\rangle\,\mathrm{d}\mathfrak{m}(y)}{\int_X e^{\frac{\psi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)} \\ \le& \int_X |\nabla_y \log p_{\frac{t}{2}}(x,y)| \left| \frac{\nabla_y \bar\psi(y) - \Lambda_\psi \nabla_y \mathrm{d}(y,Y)}{t} \right| \mathrm{d}\mu_x^t[\psi_*](y) \\ \le& \frac{2\Lambda_\psi}{t} \Bigg(\underbrace{\int_X |\nabla_y \log p_{\frac{t}{2}}(x,y)|^2 \mathrm{d}\mu_x^t[\psi_*](y)}_{:=\tilde J_t(x)} \Bigg)^{\frac{1}{2}}. \end{aligned} \tag{2.22}$$

Similarly, we can prove

$$\bar J_t(x) \le e^{-\frac{Kt}{3}} \frac{2\Lambda_\psi}{t} \bar J_t^{\frac{1}{2}}(x) + \frac{NK}{3} \frac{e^{-\frac{2Kt}{3}}}{1-e^{-\frac{Kt}{3}}}. \tag{2.23}$$

Since $1-e^{-\frac{Kt}{3}} = \frac{K}{3}t + o(t)$, we have $\bar J_t(x) \lesssim \frac{1}{t^2}$. Then (2.21) and (2.22) implies

$$J_t(x) \le e^{-\frac{Kt}{3}} \frac{2\Lambda_\psi}{t} \tilde J_t^{\frac{1}{2}}(x) + \frac{NK}{3} \frac{e^{-\frac{2Kt}{3}}}{1-e^{-\frac{Kt}{3}}} \le \frac{C_1(K,N,\Lambda_\psi)}{t^2}. \tag{2.24}$$

□

Lemma 2.8. *Let $x_0 \in S$, $0 < r_0 \le 1$ and $B = B(x_0, r_0) \subseteq S$. For $t > 0$ small enough, it holds*

$$\int_B \mathrm{Var}_{\mu_x^t[\psi_*]}\big(\nabla_x \log p_{\frac{t}{2}}(x,\cdot)\big)\,\mathrm{d}\rho(x) \le \frac{C_2 \mathfrak{m}(B(x_0, 2r_0))}{r_0 t}, \tag{2.25}$$

where C_2 depends on K, N, Λ_ψ, a_2.

Proof. Take a cut-off function η satisfying $\eta \equiv 1$ on B, $\eta \equiv 0$ on $X \setminus B(x_0, 2r_0)$, $0 \le \eta \le 1$, and $|\nabla \eta| \le \frac{10}{r_0}$. Recall that $a_1 \mathfrak{m}|_S \le \rho \le a_2 \mathfrak{m}|_S$, by Lemma 2.6, we have

$$\begin{aligned} &\int_B \mathrm{Var}_{\mu_x^t[\psi_*]}\big(\nabla_x \log p_{\frac{t}{2}}(x,\cdot)\big)\,\mathrm{d}\rho(x) \\ \le& a_2 \int_X \eta(x) \mathrm{Var}_{\mu_x^t[\psi_*]}\big(\nabla_x \log p_{\frac{t}{2}}(x,\cdot)\big)\,\mathrm{d}\mathfrak{m}(x) \\ =& a_2 \left(\int_X \eta(x) \Delta_x \log \left(H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}})(x) \right) \mathrm{d}\mathfrak{m}(x) - \int_X \eta(x) \mathbb{E}_{\mu_x^t[\psi_*]}\big(\Delta_x \log p_{\frac{t}{2}}(x,\cdot)\big)\,\mathrm{d}\mathfrak{m}(x) \right) \\ :=& a_2(I_1 + I_2). \end{aligned} \tag{2.26}$$

For I_1, by integration by parts formula and Hölder inequality,

$$\begin{aligned} I_1 &= -\int_X \langle \nabla \eta, \nabla \log \left(H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}}) \right) \rangle \,\mathrm{d}\mathfrak{m} \\ &\le \left(\int_{B(x_0,2r_0)\setminus B} |\nabla \eta|^2 \,\mathrm{d}\mathfrak{m} \right)^{\frac{1}{2}} \left(\int_{B(x_0,2r_0)\setminus B} \left| \nabla \log \left(H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}}) \right) \right|^2 \,\mathrm{d}\mathfrak{m} \right)^{\frac{1}{2}}. \end{aligned} \tag{2.27}$$

By Lemma 2.4, it holds

$$\left| \nabla \log \left(H_{\frac{t}{2}}(e^{\frac{\psi_*}{t}}) \right) \right| (x) \le \int_X |\nabla_x \log p_{\frac{t}{2}}(x,y)| \,\mathrm{d}\mu_x^t[\psi_*](y). \tag{2.28}$$

Combining with Lemma 2.7, we get

$$I_1 \le \frac{10\sqrt{C_1}\,\mathfrak{m}(B(x_0,2r_0))}{r_0 t}. \tag{2.29}$$

For I_2, by (2.17) and the Li–Yau type estimate (2.20), we have

$$\begin{aligned} -\mathbb{E}_{\mu_x^t[\psi_*]}(\Delta_x \log p_{\frac{t}{2}}(x,\cdot)) &= \mathbb{E}_{\mu_x^t[\psi_*]} \left(|\nabla_x \log p_{\frac{t}{2}}(x,\cdot)|^2 - \frac{\Delta_x p_{\frac{t}{2}}(x,\cdot)}{p_{\frac{t}{2}}(x,\cdot)} \right) \\ &\le (e^{-\frac{Kt}{3}} - 1) \int_X \frac{\Delta_x p_{\frac{t}{2}}(x,y)}{p_{\frac{t}{2}}(x,y)} \,\mathrm{d}\mu_x^t[\psi_*](y) + \frac{NK}{3} \frac{e^{-\frac{2Kt}{3}}}{1-e^{-\frac{Kt}{3}}}. \end{aligned} \tag{2.30}$$

By (2.22) and (2.23), for t small enough, there is $c_1 = c_1(K, N, \Lambda_\psi)$ so that

$$-\mathbb{E}_{\mu_x^t[\psi_*]}(\Delta_x \log p_{\frac{t}{2}}(x,y)) \le \left| e^{-\frac{Kt}{3}} - 1 \right| \frac{2\Lambda_\psi}{t} \bar{J}_t^{\frac{1}{2}}(x) + \frac{NK}{3} \frac{e^{-\frac{2Kt}{3}}}{1-e^{-\frac{Kt}{3}}} \le \frac{c_1}{t}.$$

So $I_2 \le \frac{c_1 \mathfrak{m}(B(x_0,2r_0))}{t}$. Combining with (2.26), (2.29), we obtain (2.25) □

Proposition 2.9. *It holds that*

$$\int_B |\nabla_x \mathbb{E}_{\mu_x^t[\psi_*]}(v)| \,\mathrm{d}\rho(x) \le \frac{\sqrt{C_2 \mathfrak{m}(B(x_0,2r_0))}}{\sqrt{r_0 t}} \left(\int_B \mathrm{Var}_{\mu_x^t[\psi_*]}(v) \,\mathrm{d}\rho(x) \right)^{\frac{1}{2}}. \tag{2.31}$$

Proof. It follows from Lemma 2.5, Lemma 2.8, and Hölder inequality. □

2.4 Global concavity estimate

We improve the estimate (2.31) in Proposition2.9 to a global one. The proof is similar to [30, Lemma 3.3], which involves two key lemmas in Appendix A.

Proposition 2.10. *It holds that*

$$\int_S \left| \mathbb{E}_{\mu_x^t[\psi_*]}(v) - \mathbb{E}_{\mu^t[\psi_*]}(v) \right| \,\mathrm{d}\rho(x) \le \frac{\kappa}{\sqrt{t}} \left(\int_S \mathrm{Var}_{\mu_x^t[\psi_*]}(v) \,\mathrm{d}\rho(x) \right)^{\frac{1}{2}}, \tag{2.32}$$

where $\mu^t[\psi_*]$ is defined as in (2.6) and κ depends on $K, N, \Lambda_\psi, a_1, a_2, S$.

Proof. Since $a_1\mathfrak{m}|_S \le \rho \le a_2\mathfrak{m}|_S$, by [43, Corollary 2.4], ρ is a doubling measure. Moreover, since S is a John domain, by [26, Proposition 3.7], ρ satisfies the Boman chain condition (see Definition A.2) and we can choose a covering $\mathcal{F}$, such that for any $B \in \mathcal{F}$, $r_B \le 1$. Then

$$\begin{aligned}
&\int_S \left|\mathbb{E}_{\mu_x^t[\psi_*]}(v) - \mathbb{E}_{\mu^t[\psi_*]}(v))\right| \mathrm{d}\rho(x) \\
&\overset{\text{Lemma A.3}}{\le} C_4 \sum_{B\in\mathcal{F}} \rho(B) \int_B \left|\mathbb{E}_{\mu_x^t[\psi_*]}(v) - \rho(B)^{-1}\mathbb{E}_{\mu^t[\psi_*]}(v))\right| \mathrm{d}\rho_B(x) \\
&\overset{\text{Lemma A.1}}{\le} C_4 \sum_{B\in\mathcal{F}} \rho(B) C_3 r_B \int_B \left|\nabla_x \mathbb{E}_{\mu_x^t[\psi_*]}(v)\right| \mathrm{d}\rho_B(x) \\
&\overset{\text{Proposition 2.9}}{\le} \frac{\sqrt{C_2}C_3C_4}{\sqrt{t}} \sum_{B\in\mathcal{F}} \sqrt{\mathfrak{m}(B_{2r_B})}\sqrt{r_B} \left(\int_B \mathrm{Var}_{\mu_x^t[\psi_*]}(v)\,\mathrm{d}\rho(x)\right)^{\frac12} \\
&\overset{\text{Hölder}}{\le} \frac{\sqrt{C_2}C_3C_4}{\sqrt{t}} \left(\sum_{B\in\mathcal{F}} \mathfrak{m}(B_{2r_B})\right)^{\frac12} \left(\sum_{B\in\mathcal{F}} \int_B \mathrm{Var}_{\mu_x^t[\psi_*]}(v)\,\mathrm{d}\rho(x)\right)^{\frac12} \\
&\overset{\text{Boman chain}}{\le} \frac{\kappa}{\sqrt{t}} \left(\int_S \mathrm{Var}_{\mu_x^t[\psi_*]}(v)\,\mathrm{d}\rho(x)\right)^{\frac12},
\end{aligned} \tag{2.33}$$

which is the thesis. □

2.5 Stability estimate for Kantorovich functionals

Proposition 2.11. *Let $\varphi, \psi \in \mathrm{Lip}(Y, \mathrm{d})$ and $t > 0$. Then*

$$\begin{aligned}
&\int_S \left|\Big(\Phi_t[\varphi_*](x) - \mathrm{K}_t[\varphi_*]\Big) - \Big(\Phi_t(\psi_*)(x) - \mathrm{K}_t[\psi_*]\Big)\right| \mathrm{d}\rho(x) \\
\le& C_5 \left|\mathbb{E}_{\mu^t[\varphi_*]-\mu^t[\psi_*]}(\varphi_* - \psi_*)\right|^{\frac12},
\end{aligned} \tag{2.34}$$

where C_5 depends on $K, N, a_1, a_2, \mathrm{diam}(S\cup Y), S, \mathrm{Lip}(\varphi), \mathrm{Lip}(\psi)$.

Proof. Denote $v = \varphi_* - \psi_*$, and $\phi_s = \psi_* + sv$ for $0 \le s \le 1$. By Lemma 2.3,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\Phi_t[\phi_s] - \mathrm{K}_t[\phi_s]\Big) = -\left(\mathbb{E}_{\mu_x^t[\phi_s]}(v) - \mathbb{E}_{\mu^t[\phi_s]}(v)\right). \tag{2.35}$$

By Proposition 2.10, we have

$$\begin{aligned}
&\int_S \left|\Big(\Phi_t[\varphi_*](x) - \mathrm{K}_t[\varphi_*]\Big) - \Big(\Phi_t[\psi_*](x) - \mathrm{K}_t[\psi_*]\Big)\right| \mathrm{d}\rho(x) \\
\le& \int_0^1 \int_S \left|\frac{\mathrm{d}}{\mathrm{d}s}\Big(\Phi_t[\phi_s] - \mathrm{K}_t[\phi_s]\Big)\right| \mathrm{d}\rho(x)\,\mathrm{d}s = \int_0^1 \int_S \left|\mathbb{E}_{\mu_x^t[\phi_s]}(v) - \mathbb{E}_{\mu^t[\phi_s]}(v)\right| \mathrm{d}\rho(x)\,\mathrm{d}s \\
\overset{*}{\le}& \kappa \left(-\int_0^1 \langle D^2\mathrm{K}_t[\phi_s]v, v\rangle\,\mathrm{d}s\right)^{\frac12} \\
=& \kappa \left|\langle \nabla\mathrm{K}_t[\varphi_*] - \nabla\mathrm{K}_t[\psi_*], \varphi_* - \psi_*\rangle\right|^{\frac12} = \kappa \left|\mathbb{E}_{\mu^t[\varphi_*]-\mu^t[\psi_*]}(\varphi_* - \psi_*)\right|^{\frac12},
\end{aligned}$$

where $(*)$ follows from Proposition 2.10, Lemma 2.3 and Hölder inequality, while the last two equalities follow from Lemma 2.3. This completes the proof. □

2.6 Passing to the limit

In the following lemma we pass to the limit of t to recover the Kantorovich potentials. We remark that, for $\psi \in C_b(X)$, this asymptotic formula has been proved by Gigli–Tamanini–Trevisan [17, Proposition 5.2] in $\mathrm{RCD}(K,\infty)$ spaces.

Lemma 2.12. *For any* $\psi \in \mathrm{Lip}(Y,\mathrm{d})$, *we have*

$$\lim_{t\to 0}\Phi_t[\psi_*](x) = \lim_{t\to 0} -t\log\int_X e^{\frac{\psi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y) = \psi^c(x), \tag{2.36}$$

for ρ*-a.e.* $x \in S$, *where* $\psi^c(x) = \inf_{y\in Y}\{c(x,y) - \psi(y)\}$.

Proof. For $\delta > 0$ and $Y_\delta := \{y \in X : \mathrm{d}(y,Y) \le \delta\}$, we denote

$$\int_X e^{\frac{\psi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y) = \underbrace{\int_{Y_\delta} e^{\frac{\psi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}_{I_1} + \underbrace{\int_{X\setminus Y_\delta} e^{\frac{\psi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}_{I_2}. \tag{2.37}$$

Step 1: Recall that $\psi_*(y) = \bar{\psi}(y) - \Lambda_\psi \mathrm{d}(y,Y)$, $\Lambda_\psi = D + \mathrm{Lip}(\psi) + 1$, $D = \mathrm{diam}(S \cup Y)$. We have

$$\frac{I_2}{I_1} \le \frac{e^{\frac{\sup_Y \psi_i - (D+1)\delta}{t}} \int_X p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}{e^{\frac{\inf_Y \psi_i}{t}} \int_Y p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}. \tag{2.38}$$

Combining with the stochastic completeness of the heat flow and the heat kernel estimate (2.10), we can find $\delta > 0$ large enough such that

$$\frac{I_2}{I_1} \lesssim e^{-\frac{D^2}{t}} \quad \text{as } t \to 0. \tag{2.39}$$

Step 2: We claim that

$$\inf_{y\in Y_\delta\setminus Y}\{c(x,y) - \psi_*(y)\} > \inf_{y\in Y}\{c(x,y) - \psi(y)\} = \psi^c(x). \tag{2.40}$$

Since Y is compact, there exists $y_1 \in \overline{Y_\delta \setminus Y}$ and $y_2 \in Y$, such that

$$c(x,y_1) - \psi_*(y_1) = \inf_{y\in Y_\delta\setminus Y}\{c(x,y) - \psi_*(y)\}, \quad \mathrm{d}(y_1,y_2) = \mathrm{d}(y_1,Y) =: s > 0.$$

Then (2.40) follows from the following estimate

$$\begin{aligned} &c(x,y_1) - \psi_*(y_1) - \big(c(x,y_2) - \psi(y_2)\big),\\ \ge&\frac{1}{2}\big(\mathrm{d}(x,y_2) - \mathrm{d}(y_1,y_2)\big)^2 - \frac{1}{2}\mathrm{d}^2(x,y_2) - \mathrm{Lip}(\bar{\psi})\mathrm{d}(y_1,y_2) + \Lambda_\psi \mathrm{d}(y_1,y_2)\\ \ge&\frac{1}{2}s^2 + (\Lambda_\psi - \mathrm{Lip}(\psi) - D)s = \frac{1}{2}s^2 + s > 0. \end{aligned} \tag{2.41}$$

Step 3: It holds that

$$\lim_{t\to 0} -t\log I_1 = \psi^c. \tag{2.42}$$

Then we obtain (2.36) by combing (2.39) and (2.42).

Lower bound:

$$\varliminf_{t\to 0} -t\log I_1 \geq \psi^c. \tag{2.43}$$

Denote $c_t(x,y) = -t\log p_{\frac{t}{2}}(x,y)$. By Lemma 2.1, we have $c_t(x,y) \to c(x,y)$ uniformly in $S\times Y_\delta$. So for any $\epsilon > 0$, there is $t_0 > 0$, such that for any $t < t_0$, $c_t(x,y) \geq c(x,y) - \epsilon$, and

$$\begin{aligned} -t\log\int_{Y_\delta} e^{\frac{\psi_*(y)-c_t(x,y)}{t}}\,\mathrm{d}\mathfrak{m}(y) \geq & -t\log\int_{Y_\delta} e^{\frac{\psi_*(y)-c(x,y)+\epsilon}{t}}\,\mathrm{d}\mathfrak{m}(y) \\ \overset{(2.40)}{\geq} & -t\log\int_{Y_\delta} e^{\frac{-\psi^c(x)+\epsilon}{t}}\,\mathrm{d}\mathfrak{m}(y) \\ = & -t\left(\log\mathfrak{m}(Y_\delta) + \frac{-\psi^c(x)+\epsilon}{t}\right) \\ = & \psi^c(x) - t\log\mathfrak{m}(Y_\delta) - \epsilon. \end{aligned} \tag{2.44}$$

Letting $t\to 0$, then letting $\epsilon\to 0$, we get (2.43).

Upper bound:

$$\varlimsup_{t\to 0} -t\log I_1 \leq \psi^c. \tag{2.45}$$

For any $x\in S$, by compactness of Y, there exists $T(x)\in Y$, such that

$$\psi(T(x)) + \psi^c(x) = c(x,T(x)).$$

For any $\epsilon > 0$, by continuity of ψ_* and $c(x,y)$, there is $r < \delta\wedge\epsilon$ such that

$$\psi_*(y) \geq \psi(T(x)) - \epsilon, \quad c(x,y) \leq c(x,T(x)) + \epsilon, \quad \forall y\in B(T(x),r).$$

Moreover, there exists $t_0 > 0$, such that for any $t < t_0$, there holds

$$c_t(x,y) \leq c(x,y) + \epsilon.$$

Then for any $t < t_0$, we have

$$\begin{aligned} -t\log\int_{Y_\delta} e^{\frac{\psi_*(y)-c_t(x,y)}{t}}\,\mathrm{d}\mathfrak{m}(y) \leq & -t\log\int_{B(T(x),r)} e^{\frac{\psi(T(x))-c(x,T(x))-3\epsilon}{t}}\,\mathrm{d}\mathfrak{m}(y) \\ = & -t\log\int_{B(T(x),r)} e^{\frac{-\psi^c(x)-3\epsilon}{t}}\,\mathrm{d}\mathfrak{m}(y) \\ = & -t\log\mathfrak{m}(B(T(x),r)) + \psi^c(x) + 3\epsilon. \end{aligned} \tag{2.46}$$

Letting $t\to 0$, then $\epsilon\to 0$, we get (2.45).

□

Lemma 2.13. *Let φ be a Kantorovich potentials from ν to ρ. Then $\mu^t[\varphi_*] \rightharpoonup \nu$, i.e., for any $v\in \mathrm{Lip}(X,\mathrm{d})$, it holds*

$$\lim_{t\to 0}\mathbb{E}_{\mu^t[\varphi_*]}(v) = \mathbb{E}_\nu(v). \tag{2.47}$$

Proof. We just need to show that

$$\lim_{t\to 0}\mathbb{E}_{\mu^t_x[\varphi_*]}(v) = v(T(x)), \quad \text{for } \rho\text{-a.e. } x\in S, \tag{2.48}$$

where T is the unique optimal transport map from ρ to ν. The proof is very similar to that of Lemma 2.12, so we only sketch it.

Step 1: Without loss of generality, we assume that $v \geq 0$. Similar to (2.39), there is $\delta > 0$ such that

$$\frac{\int_{X\setminus Y_\delta} v(y) e^{\frac{\varphi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}{\int_X e^{\frac{\varphi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)} \lesssim e^{-\frac{D^2}{t}} \quad \text{as} \quad t \to 0. \tag{2.49}$$

and

$$\frac{\int_{X\setminus Y_\delta} e^{\frac{\varphi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}{\int_X e^{\frac{\varphi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)} \lesssim e^{-\frac{D^2}{t}} \quad \text{as} \quad t \to 0. \tag{2.50}$$

Step 2: By uniqueness of optimal transport T and (2.40), for almost every $x \in S$, there is a unique $T(x) \in Y$ so that

$$\varphi^c(x) = c(x, T(x)) - \varphi_*(T(x)) = \inf_{y \in Y_\delta} \big\{c(x,y) - \varphi_*(y)\big\}.$$

Then for any $\epsilon > 0$, we can find $0 < r_2 < \delta$ such that

$$|v(y) - v(T(x)| < \epsilon, \qquad \forall y \in B\big(T(x), r_2\big) \tag{2.51}$$

and

$$\varphi_*(y) - c(x,y) < -\varphi^c - 4\delta_{r_2}, \qquad \forall y \in Y_\delta \setminus B\big(T(x), r_2\big) \tag{2.52}$$

for some $\delta_{r_2} > 0$. By Lemma 2.1, there is $t_0 > 0$ such that

$$|c_t(x,y) - c(x,y)| < \delta_{r_2}, \quad \forall (x,y) \in S \times Y \text{ and } t < t_0.$$

By the same argument as (2.44), for $t < t_0$ we have

$$e^{\frac{\varphi_*(y)}{t}} p_{\frac{t}{2}}(x,y) \leq e^{\frac{-\varphi^c(x) - 3\delta_{r_2}}{t}}, \quad \forall y \in Y_\delta \setminus B\big(T(x), r_2\big), \tag{2.53}$$

Furthermore, similar to (2.46), we can find $r_1 < r_2$ so that

$$e^{\frac{\varphi_*(y)}{t}} p_{\frac{t}{2}}(x,y) > e^{\frac{-\varphi^c(x) - 2\delta_{r_2}}{t}}, \quad \forall y \in B\big(T(x), r_1\big) \text{ and } t < t_0. \tag{2.54}$$

It follows from (2.53), (2.54) that

$$\frac{\int_{Y_\delta \setminus B(T(x), r_2)} e^{\frac{\varphi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}{\int_{B(T(x), r_2)} e^{\frac{\varphi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)} \to 0 \quad \text{as} \quad t \to 0. \tag{2.55}$$

Combining (2.49), (2.50), (2.51) and (2.55) we obtain

$$\begin{aligned}
& \lim_{t\to 0} |\mathbb{E}_{\mu_x^t[\varphi_*]}(v) - v(T(x))| \\
= & \lim_{t\to 0} \left| \frac{\int_X \big(v(y) - v(T(x))\big) e^{\frac{\varphi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}{\int_X e^{\frac{\varphi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)} \right| \\
= & \lim_{t\to 0} \left| \frac{\int_{B(T(x), r_2)} \big(v(y) - v(T(x))\big) e^{\frac{\varphi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)}{\int_{B(T(x), r_2)} e^{\frac{\varphi_*(y)}{t}} p_{\frac{t}{2}}(x,y)\,\mathrm{d}\mathfrak{m}(y)} \right| \\
\leq & \ \epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$ we get (2.48).

□

Proof of Theorem 1.1. Let φ and ψ be Kantorovich potentials from μ to ρ and ν to ρ respectively. By the heat kernel lower bound (2.10) and the stochastic completeness of the heat flow, we can see that $|\Phi_t[\psi_*]|, |\Phi_t[\varphi_*]|$ are uniformly bounded for $t \in (0,1]$. So by the dominated convergence theorem and Lemma 2.12 we obtain

$$\begin{aligned}&\lim_{t\to 0}\int_S \left|\Big(\Phi_t[\varphi_*](x) - \mathrm{K}_t[\varphi_*]\Big) - \Big(\Phi_t[\psi_*](x) - \mathrm{K}_t[\psi_*]\Big)\right| \mathrm{d}\rho(x)\\ &= \int_S \left|\Big(\varphi^c(x) - \mathbb{E}_\rho(\varphi^c)\Big) - \Big(\psi^c(x) - \mathbb{E}_\rho(\psi^c)\Big)\right| \mathrm{d}\rho(x).\end{aligned} \tag{2.56}$$

Note that $(\varphi_* - \psi_*)_{|Y} = \varphi - \psi$, by Lemma 2.13, we have

$$\lim_{t\to 0}\mathbb{E}_{\mu^t[\varphi_*]-\mu^t[\psi_*]}(\varphi_* - \psi_*) = \mathbb{E}_{\mu-\nu}(\varphi - \psi). \tag{2.57}$$

Combining Proposition 2.11, (2.56) and (2.57), we obtain

$$\int_S \left|\Big(\varphi^c(x) - \mathbb{E}_\rho(\varphi^c)\Big) - \Big(\psi^c(x) - \mathbb{E}_\rho(\psi^c)\Big)\right| \mathrm{d}\rho(x) \le \sqrt{C\,|\mathbb{E}_{\mu-\nu}(\varphi-\psi)|} \tag{2.58}$$

for some $C > 0$. Note that $c(x,y) = \frac{1}{2}\mathrm{d}^2(x,y)$, thus the Kantorovich potential φ and ψ are $\mathrm{diam}(S\cup Y)$-Lipschitz, so C depends only on $K, N, a_1, a_2, \mathrm{diam}(S\cup Y), S$.

By assumption, $\phi_\nu = \varphi^c, \phi_\mu = \psi^c$ satisfies $\mathbb{E}_\rho(\phi_\mu) = \mathbb{E}_\rho(\phi_\nu) = 0$. Note also that $\mathrm{Lip}(\varphi - \psi) \le 2\,\mathrm{diam}(S\cup Y)$, by Kantorovich duality for W_1, we finally obtain

$$\|\phi_\mu - \phi_\nu\|^2_{L^1(\rho)} \le 2C\,\mathrm{diam}(S\cup Y) W_1(\mu,\nu). \tag{2.59}$$

□

3 Stability of optimal transport maps

In this section, we prove Theorem 1.3 concerning the quantitative stability of optimal transport maps on Alexandrov spaces. This is achieved by combining Theorem 1.1 and the following estimate.

Theorem 3.1. *Let (X,d) be an Alexandrov space with no boundary. Then under the same assumptions of Theorem 1.3, there exists a constant $\bar{C} > 0$, depending on k,n,a_1, a_2, $\mathrm{diam}(Y)$, S, such that for any $\mu,\nu \in \mathcal{P}(Y)$, we have*

$$\int_S |\nabla\phi_\mu(x) - \nabla\phi_\nu(x)|^2\,\mathrm{d}\rho(x) \le \bar{C}\left(\int_S |\phi_\mu(x) - \phi_\nu(x)|^2\,\mathrm{d}\rho(x)\right)^{\frac{1}{3}}, \tag{3.1}$$

where ϕ_μ and ϕ_ν are the Kantorovich potentials from ρ to μ and ρ to ν respectively.

We follow a strategy of [26] by lifting integrals from the base space to the unit tangent bundle, evolve the relevant quantities via the geodesic flow developed in [25], and then exploit the one-dimensional convexity of the potentials. For the reader's convenience, we recall the necessary Alexandrov geometry theory and provide a self-contained proof.

The Riemannian structure on the set of regular points X_{reg} (cf. [38]) endows the tangent bundle $\mathrm{T}X_{\mathrm{reg}}$ an Euclidean vector bundle structure. On this Euclidean vector bundle, one can define (cf. [25, Section3]) a canonical Liouville measure $\mathfrak{m}_{\mathrm{L}}$: it is the unique Borel measure on $\mathrm{T}X_{\mathrm{reg}}$ such that for any Borel set $A \subset \mathrm{T}X_{\mathrm{reg}}$, we have

$$\mathfrak{m}_{\mathrm{L}}(A) = \int_X \mathcal{H}^n\big(A \cap \mathrm{T}_x X\big)\,\mathrm{d}\mathcal{H}^n(x).$$

We extend $\mathfrak{m}_{\mathrm{L}}$ to a measure on the tangent bundle $\mathrm{T}X$ by setting $\mathfrak{m}_{\mathrm{L}}(\mathrm{T}X \setminus \mathrm{T}X_{\mathrm{reg}}) = 0$. Moreover, by [6, Theorem 1.4], if (X, d) is an Alexandrov space with no boundary, then the geodesic flow preserves the Liouville measure.

Let $\mathrm{S}X = \{v \in \mathrm{T}X : |v| = 1\}$ denote the unit tangent bundle (sphere bundle). There is a canonical Liouville measure $\mathfrak{m}_{\mathrm{S}}$ on $\mathrm{S}X$ (cf. [25, Section 3.6]), also called Liouville measure, such that if the geodesic flow defined on $\mathrm{T}X$ preserves $\mathfrak{m}_{\mathrm{L}}$, then the geodesic flow defined on $\mathrm{S}X$ preserves $\mathfrak{m}_{\mathrm{S}}$ as well. For any Borel set $A \subseteq \mathrm{S}X$, we have

$$\mathfrak{m}_{\mathrm{S}}(A) = \int_{\mathrm{S}X} \chi_A(x,v)\,\mathrm{d}\mathfrak{m}_{\mathrm{S}}(x,v) = c_n \int_X \int_{\Sigma_x(X)} \chi_A(x,v)\,\mathrm{d}\sigma_x(v)\,\mathrm{d}\mathcal{H}^n(x), \tag{3.2}$$

where $\Sigma_x(X)$ denotes the space of directions at x, $c_n = \mathcal{H}^{n-1}(\mathrm{S}^{n-1})$ and $\sigma_x \in \mathcal{P}(\Sigma_x(X))$ is the canonical probability measure on the fiber.

Lifting to the unit tangent bundle. Let $\varphi_s : \mathrm{S}X \to \mathrm{S}X$ be the geodesic flow at time $s \in [0,1]$ on $\mathrm{S}X$, and write $\varphi_s(x,v) = (b_s(x,v), t_s(x,v))$, where $b_s(x,v) \in X$ and $t_s(x,v) \in \Sigma_{b_s(x,v)}(X)$. For $\mathfrak{m}_{\mathrm{S}}$-a.e. $(x,v) \in \mathrm{S}X$ such that the curve $[0,1] \ni s \mapsto b_s(x,v)$ is a locally shortest path, denote by $I_S(x,v)$ the set of connected components of $\{s \in [0,1] : b_s(x,v) \in S\}$. Since S is an open set, we have

$$\{s \in [0,1] : b_s(x,v) \in S\} = \bigcup_{i \in I_S(x,v)} (\alpha_i(x,v), \beta_i(x,v)).$$

For $x \in S \cap X_{\mathrm{reg}}$, by [8, Theorem 10.8.4], $\Sigma_x(X)$ is isometric to S^{n-1}, then there is a universal constant d_n such that

$$\begin{aligned} &\int_{\Sigma_x(X)} \langle \nabla\phi_\mu(x) - \nabla\phi_\nu(x), v\rangle^2\,\mathrm{d}\sigma_x(v) \\ =&\int_{\mathrm{S}^{n-1}} \langle \nabla\phi_\mu(x) - \nabla\phi_\nu(x), v\rangle^2\,\mathrm{d}\sigma_x(v) \\ =&d_n |\nabla\phi_\mu(x) - \nabla\phi_\nu(x)|^2. \end{aligned} \tag{3.3}$$

By [25, Section 3.6], $\mathfrak{m}_{\mathrm{S}}$ is preserved by φ_s, so

$$\begin{aligned} &\int_S |\nabla\phi_\mu(x) - \nabla\phi_\nu(x)|^2\,\mathrm{d}\rho(x) \le a_2 \int_S |\nabla\phi_\mu(x) - \nabla\phi_\nu(x)|^2\,\mathrm{d}\mathcal{H}^n(x) \\ \overset{(3.3)}{=} &\frac{a_2}{d_n} \int_S \int_{\Sigma_x(X)} \langle \nabla\phi_\mu(x) - \nabla\phi_\nu(x), v\rangle^2\,\mathrm{d}\sigma_x(v)\,\mathrm{d}\mathcal{H}^n(x) \\ \overset{(3.2)}{=} &\frac{a_2}{c_n d_n} \int_{\mathrm{S}X} \langle \nabla\phi_\mu(x) - \nabla\phi_\nu(x), v\rangle^2 \chi_S(x)\,\mathrm{d}\mathfrak{m}_{\mathrm{S}}(x,v) \\ = &\frac{a_2}{c_n d_n} \int_{\mathrm{S}X} \int_0^1 \langle \nabla\phi_\mu(b_s(x,v)) - \nabla\phi_\nu(b_s(x,v)), t_s(x,v)\rangle^2 \chi_S(b_s(x,v))\,\mathrm{d}s\,\mathrm{d}\mathfrak{m}_{\mathrm{S}}(x,v) \\ = &\frac{a_2}{c_n d_n} \int_{\mathrm{S}X} \sum_{i \in I_S(x,v)} \int_{\alpha_i(x,v)}^{\beta_i(x,v)} \langle \nabla\phi_\mu(b_s(x,v)) - \nabla\phi_\nu(b_s(x,v)), t_s(x,v)\rangle^2\,\mathrm{d}s\,\mathrm{d}\mathfrak{m}_{\mathrm{S}}(x,v). \end{aligned} \tag{3.4}$$

One-dimensional convexity estimate. For $(x,v) \in \mathrm{S}X$ and $s \in [0,1]$, we denote $u_\mu^{(x,v)}(s) = \phi_\mu(b_s(x,v))$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s} u_\mu^{(x,v)}(s) = \langle \nabla\phi_\mu(b_s(x,v)), t_s(x,v)\rangle, \quad \text{for a.e. } s \in [0,1],$$

and an analogous formula holds for $u_\nu^{(x,v)}(s) = \phi_\nu(b_s(x,v))$.

Denote $S_1 = \{x \in X : \mathrm{d}(x,S) \le 1\}$. Note that in the last integral of (3.4), only the elements $(x,v) \in \mathrm{S}X$ for which $x \in S_1$ have a non-vanishing contribution. For $\mathfrak{m}_{\mathrm{S}}$-a.e. $(x,v) \in \mathrm{S}X$ with $x \in S_1$, the curve $s \mapsto b_s(x,v)$ is locally minimizing on $[0,1]$. Thus, applying Ohta's semiconcavity estimate [37, Lemma 3.2] on sufficiently small subsegments yields local concavity of $u_\mu^{(x,v)} - \zeta|s|^2$ with a uniform constant ζ. The local-to-global principle for concave functions then leads to the following lemma.

Lemma 3.2 ([37], Lemma 3.2)**.** *There exists ζ, depending on k and $\mathrm{diam}(S \cup Y)$, such that for $\mathfrak{m}_{\mathrm{S}}$-a.e. $(x,v) \in \mathrm{S}S_1$, the functions $u_\mu^{(x,v)} - \zeta|s|^2$, $u_\nu^{(x,v)} - \zeta|s|^2$ are concave on $s \in [0,1]$. Moreover, the modulus of their derivatives (which exist a.e. on $[0,1]$) is bounded above by $\mathrm{diam}(S \cup Y) + 2\zeta$.*

Applying Lemma 3.3 below to the functions $\zeta|s|^2 - u_\mu^{(x,v)}$ and $\zeta|s|^2 - u_\nu^{(x,v)}$ on each compact segment $[\alpha_i(x,v), \beta_i(x,v)]$ and using Hölder inequality, we obtain

$$\begin{aligned}
&\int_S |\nabla\phi_\mu(x) - \nabla\phi_\nu(x)|^2 \,\mathrm{d}\rho(x) \\
&\overset{(3.4)}{\le} \frac{a_2}{c_n d_n} \int_{\mathrm{S}X} \sum_{i \in I_S(x,v)} \int_{\alpha_i(x,v)}^{\beta_i(x,v)} \langle \nabla\phi_\mu(b_s(x,v)) - \nabla\phi_\nu(b_s(x,v)), t_s(x,v)\rangle^2 \,\mathrm{d}s \,\mathrm{d}\mathfrak{m}_{\mathrm{S}}(x,v) \\
&\le C_1 \int_{\mathrm{S}X} \sum_{i \in I_S(x,v)} \left(\int_{\alpha_i(x,v)}^{\beta_i(x,v)} |\phi_\mu(b_s(x,v)) - \phi_\nu(b_s(x,v))|^2 \,\mathrm{d}s \right)^{\frac{1}{3}} \mathrm{d}\mathfrak{m}_{\mathrm{S}}(x,v) \\
&\le C_1 \int_{\mathrm{S}X} (\# I_S)^{\frac{2}{3}} \left(\int_0^1 |\phi_\mu(b_s) - \phi_\nu(b_s)|^2 \chi_S(b_s) \,\mathrm{d}s \right)^{\frac{1}{3}} \mathrm{d}\mathfrak{m}_{\mathrm{S}} \\
&\le C_1 \left(\int_{\mathrm{S}X} \# I_S \,\mathrm{d}\mathfrak{m}_{\mathrm{S}} \right)^{\frac{2}{3}} \left(\int_{\mathrm{S}X} \int_0^1 |\phi_\mu(b_s) - \phi_\nu(b_s)|^2 \chi_S(b_s) \,\mathrm{d}s \,\mathrm{d}\mathfrak{m}_{\mathrm{S}} \right)^{\frac{1}{3}} \\
&\le C_1 \left(\frac{c_n}{a_1} \right)^{\frac{1}{3}} \left(\int_{\mathrm{S}X} \# I_S \,\mathrm{d}\mathfrak{m}_{\mathrm{S}} \right)^{\frac{2}{3}} \left(\int_S |\phi_\mu - \phi_\nu|^2 \,\mathrm{d}\rho \right)^{\frac{1}{3}},
\end{aligned} \tag{3.5}$$

where C_1 depends on $k, n, a_2, \mathrm{diam}(S \cup Y)$, and the last inequality follows from the invariance of $\mathfrak{m}_{\mathrm{S}}$ under the geodesic flow.

Lemma 3.3 ([11], Lemma 5.1)**.** *Let $I \subseteq \mathbb{R}$ be a compact segment and let $u, v : I \to \mathbb{R}$ be two convex functions such that $|u'|$ and $|v'|$ (defined a.e. on I) are uniformly bounded on I. Then*

$$\|u' - v'\|_{L^2(I)}^2 \le 8(\|u'\|_{L^\infty(I)} + \|v'\|_{L^\infty(I)})^{\frac{4}{3}} \|u - v\|_{L^2(I)}^{\frac{2}{3}}.$$

One-dimensional BV estimate. We now aim to control $\int_{\mathrm{S}X} \# I_S \,\mathrm{d}\mathfrak{m}_{\mathrm{S}}$, the average number of times a geodesic crosses the boundary of S. The argument of [26, Proposition 4.3] relies on the existence of $T > 0$, so that for $(x,v) \in \mathrm{S}X$, the geodesic $s \mapsto b_s(x,v)$ is minimizing on $[0,T]$, and hence in particular does not self-intersect. In an Alexandrov space, one cannot in general expect such a uniform injectivity radius bound. We use a localization method to overcome this difficulty.

In the following proposition, we denote by $|\mathrm{D}f|([0,1])$ the total variation of a function f on $[0,1]$ and denote by $\mathrm{Per}(S)$ the perimeter of a set $S \subset X$. We refer to [1,33] for BV functions and sets of finite perimeter in metric measure spaces.

Proposition 3.4. *Assume* $\mathrm{Per}(S) < +\infty$*, then*

$$\int_{\mathrm{S}X} \# I_S(x,v)\,\mathrm{d}\mathfrak{m}_{\mathrm{S}}(x,v) \le c_n\big(\mathcal{H}^n(S_1) + \mathrm{Per}(S)\big).$$

Proof. Let $(x,v) \in \mathrm{S}X$ be such that $s \mapsto b_s(x,v)$ is well-defined on $[0,1]$, and let $u \in \mathrm{Lip}(X,\mathrm{d})$. Note that $s \mapsto b_s(x,v)$ has unit speed, then

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}u(b_s(x,v))\right| \le |\nabla u|\,(b_s(x,v)), \quad \text{a.e. } s \in [0,1],$$

and

$$|\mathrm{D}u(b_s(x,v))|([0,1]) = \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}s}u(b_s(x,v))\right|\,\mathrm{d}s \le \int_0^1 |\nabla u|(b_s(x,v))\,\mathrm{d}s. \tag{3.6}$$

Let $(u_k)_{k\in\mathbb{N}} \subseteq \mathrm{Lip}(X,\mathrm{d})$ be such that $u_k \to \chi_S$ in $L^1(\mathcal{H}^n)$ and

$$\int_X |\nabla u_k|\,\mathrm{d}\mathcal{H}^n \to \mathrm{Per}(S). \tag{3.7}$$

Then

$$\begin{aligned} &\int_{\mathrm{S}X}\int_0^1 |u_k(b_s(x,v)) - \chi_S(b_s(x,v))|\,\mathrm{d}s\,\mathrm{d}\mathfrak{m}_{\mathrm{S}}(x,v) \\ =&\int_0^1\int_{\mathrm{S}X} |u_k(b_s(x,v)) - \chi_S(b_s(x,v))|\,\mathrm{d}\mathfrak{m}_{\mathrm{S}}(x,v)\,\mathrm{d}s \\ =&c_n\int_X |u_k(x) - \chi_S(x)|\,\mathrm{d}\mathcal{H}^n(x) \to 0. \end{aligned} \tag{3.8}$$

So up to taking a subsequence, it holds that for $\mathfrak{m}_{\mathrm{S}}$-a.e. $(x,v) \in \mathrm{S}X$,

$$u_k(b_s(x,v)) \to \chi_S(b_s(x,v)) \quad \text{in } L^1([0,1]) \text{ as } k \to \infty. \tag{3.9}$$

By lower semicontinuity of the total variation, we have

$$\begin{aligned} &\int_{\mathrm{S}X} |\mathrm{D}\chi_S(b_s(x,v))|([0,1])\,\mathrm{d}\mathfrak{m}_{\mathrm{S}}(x,v) \\ \le&\int_{\mathrm{S}X}\varliminf_{k\to\infty} |\mathrm{D}u_k(b_s(x,v))|([0,1])\,\mathrm{d}\mathfrak{m}_{\mathrm{S}}(x,v) \\ \overset{\text{Fatou}}{\le}&\varliminf_{k\to\infty}\int_{\mathrm{S}X} |\mathrm{D}u_k(b_s(x,v))|([0,1])\,\mathrm{d}\mathfrak{m}_{\mathrm{S}}(x,v) \\ \overset{*}{\le}&\varliminf_{k\to\infty} c_n\int_X |\nabla u_k|\,\mathrm{d}\mathcal{H}^n \overset{(3.7)}{=} c_n\mathrm{Per}(S), \end{aligned} \tag{3.10}$$

where $(*)$ follows from (3.6) and the invariance of $\mathfrak{m}_{\mathrm{S}}$ under the geodesic flow.

Finally, notice that

$$\# I_S(x,v) \le 1 + \frac{1}{2}|\mathrm{D}\chi_S(b_s(x,v))|([0,1]) \tag{3.11}$$

and that only the elements $(x,v) \in \mathrm{S}X$ for which $x \in S_1$ have a non-vanishing contribution. Combining with (3.10), we obtain

$$\int_{\mathrm{S}X} \# I_S(x,v)\,\mathrm{d}\mathfrak{m}_{\mathrm{S}}(x,v) \le c_n\mathcal{H}^n(S_1) + \frac{1}{2}c_n\mathrm{Per}(S). \tag{3.12}$$

□

Proof of the theorems.

Proof of Theorem 3.1. By (3.5) and Proposition 3.4, we obtain

$$\begin{aligned}
&\int_S |\nabla\phi_\mu - \nabla\phi_\nu|^2 \,\mathrm{d}\rho \\
\le& C_1 \left(\frac{c_n}{a_1}\right)^{\frac{1}{3}} \left(\int_{\mathrm{S}X} \# I_S \,\mathrm{d}\mathfrak{m}_{\mathrm{S}}\right)^{\frac{2}{3}} \left(\int_S |\phi_\mu - \phi_\nu|^2 \,\mathrm{d}\rho\right)^{\frac{1}{3}} \\
\le& \bar{C} \left(\int_S |\phi_\mu - \phi_\nu|^2 \,\mathrm{d}\rho\right)^{\frac{1}{3}},
\end{aligned} \tag{3.13}$$

where $\bar{C}$ depends on $k, n, a_1, a_2, \mathrm{diam}(S\cup Y), \mathrm{Per}(S), S$. □

Proof of Theorem 1.3. By Perelman's doubling theorem (or Petrunin's gluing theorem [39]), we may assume that (X, d) has no boundary.

Let ϕ_μ and ϕ_ν be the Kantorovich potentials from ρ to μ and ρ to ν respectively. From [4, 41] we know that $\nabla\phi_\mu(x), \nabla\phi_\nu(x) \in \mathrm{T}_x X$ and $T_\mu(x) = \exp_x(-\nabla\phi_\mu(x))$, $T_\nu(x) = \exp_x(-\nabla\phi_\nu(x))$ are well-defined for almost every $x \in S$. By triangle comparison condition (cf. [40]), it holds

$$\mathrm{d}\big(T_\mu(x), T_\nu(x)\big) \le c|\nabla\phi_\mu(x) - \nabla\phi_\nu(x)|, \tag{3.14}$$

for some constant $c > 0$ which depends on $k, \mathrm{diam}(S\cup Y)$.

Since ϕ_μ, ϕ_ν are $\mathrm{diam}(S\cup Y)$-Lipschitz and $\mathbb{E}_\rho(\phi_\mu) = \mathbb{E}_\rho(\phi_\nu) = 0$, we have

$$\|\phi_\mu - \phi_\nu\|_{L^\infty(\rho)} \le \|\phi_\mu\|_{L^\infty(\rho)} + \|\phi_\nu\|_{L^\infty(\rho)} \le \mathrm{osc}(\phi_\mu) + \mathrm{osc}(\phi_\nu) \le 2\big(\mathrm{diam}(S\cup Y)\big)^2.$$

Recall that a finite dimensional Alexandrov space is also RCD (cf. [41, 46]). By Theorem 1.1, we get

$$\|\phi_\mu - \phi_\nu\|^2_{L^2(\rho)} \le \|\phi_\mu - \phi_\nu\|_{L^\infty(\rho)} \|\phi_\mu - \phi_\nu\|_{L^1(\rho)} \le C_3 W_1^{\frac{1}{2}}(\mu, \nu). \tag{3.15}$$

Combining Theorem 3.1, (3.14) and (3.15), we obtain

$$\int_S \mathrm{d}^2(T_\mu(x), T_\nu(x)) \,\mathrm{d}\rho(x) \le C W_1^{\frac{1}{6}}(\mu, \nu), \tag{3.16}$$

where C depends on $k, n, a_1, a_2, \mathrm{diam}(S\cup Y), \mathrm{Per}(S), S$. This complete the proof. □

Appendix A Poincaré inequality: local to global

Lemma A.1. *Let $x_0 \in S$, $0 < r_0 \le 1$ and $B = B(x_0, r_0) \subseteq S$, $\rho_B = \frac{\rho|_B}{\rho(B)}$. Then the following strong local $(1,1)$-Poincaré inequality holds for $f \in \mathrm{Lip}(B, \mathrm{d})$:*

$$\int_B |f(x) - \mathbb{E}_{\rho_B}(f)| \,\mathrm{d}\rho_B(x) \le C_3 r_0 \int_B |\nabla f| \,\mathrm{d}\rho_B, \tag{A.1}$$

where C_3 depends on K, N, a_1, a_2.

Proof. By [18, Chapter 9] and [42, Remark 3.3], $\mathfrak{m}$ satisfies the strong local $(1,1)$-Poincaré inequality:

$$\int_B |f - \mathbb{E}_{\mathfrak{m}_B}(f)| \,\mathrm{d}\mathfrak{m} \le c_1 r_0 \int_B |\nabla f| \,\mathrm{d}\mathfrak{m}, \tag{A.2}$$

where c_1 depends on K, N, $\mathfrak{m}_B = \frac{\mathfrak{m}|_B}{\mathfrak{m}(B)}$. Since $a_1 \mathfrak{m}|_S \le \rho \le a_2 \mathfrak{m}|_S$, we have

$$\begin{aligned}\int_B |f - \mathbb{E}_{\rho_B}(f)| \,\mathrm{d}\rho \le & \int_B |f - \mathbb{E}_{\mathfrak{m}_B}(f)| \,\mathrm{d}\rho + \int_B |\mathbb{E}_{\mathfrak{m}_B}(f) - \mathbb{E}_{\rho_B}(f)| \,\mathrm{d}\rho \\ \le & 2a_2 \int_B |f - \mathbb{E}_{\mathfrak{m}_B}(f)| \,\mathrm{d}\mathfrak{m} \\ \le & 2a_2 c_1 r_0 \int_B |\nabla f| \,\mathrm{d}\mathfrak{m} \le 2\frac{a_2}{a_1} c_1 r_0 \int_B |\nabla f| \,\mathrm{d}\rho,\end{aligned} \tag{A.3}$$

which is the thesis. □

Next we prove an L^1-variant of the gluing Lemma in [30, Lemma 3.3], following the same Boman-chain decomposition, utilizing the doubling property directly instead of the maximal function estimates.

Definition A.2 (Boman chain condition)**.** We say that a probability measure ρ on an open set S of a metric space satisfies *Boman chain condition* with parameters $E, F, G > 1$ if there is a covering $\mathcal{F}$ of S by open balls $B \in \mathcal{F}$ such that:

1. For any $x \in S$,
$$\sum_{B \in \mathcal{F}} \chi_{2B}(x) \le E \chi_S(x).$$
2. For some fixed ball B_0 in $\mathcal{F}$, called the *central ball*, and for every $B \in \mathcal{F}$, there exists a chain $B_0, B_1, \ldots, B_N = B$ of distinct balls from $\mathcal{F}$ such that
$$B \subset F B_j, \quad \forall j \in \{0, \ldots, N-1\}.$$
3. Consecutive balls of the above chain overlap quantitatively:
$$\rho(B_j \cap B_{j+1}) \ge G^{-1} \max(\rho(B_j), \rho(B_{j+1})), \quad \forall j \in \{0, \ldots, N-1\}.$$

Lemma A.3. *For any $f \in L^1(\rho)$, it holds that*

$$\int_S |f(x) - \mathbb{E}_\rho(f)| \,\mathrm{d}\rho(x) \le C_4 \sum_{B \in \mathcal{F}} \rho(B) \int_B |f(x) - \mathbb{E}_{\rho_B}(f)| \,\mathrm{d}\rho_B(x), \tag{A.4}$$

where C_4 depends on $K, N, \mathrm{diam}(S), S$.

Proof. Since S is a John domain, and ρ is a doubling measure on S, by [26, Proposition 3.7], ρ satisfies the Boman chain condition. Hence there exists a covering $\mathcal{F}$ of S satisfying Definition A.2.

For the central ball B_0, note that

$$\begin{aligned}\int_S |f(x) - \mathbb{E}_\rho(f)| \,\mathrm{d}\rho(x) \le & \int_S |f(x) - \mathbb{E}_{\rho_{B_0}}(f)| \,\mathrm{d}\rho(x) + \int_S |\mathbb{E}_{\rho_{B_0}}(f) - \mathbb{E}_\rho(f)| \,\mathrm{d}\rho(x) \\ \le & 2 \int_S |f(x) - \mathbb{E}_{\rho_{B_0}}(f)| \,\mathrm{d}\rho(x).\end{aligned} \tag{A.5}$$

For $B \in \mathfrak{F}$, denote $a_B = \int_B |f - \mathbb{E}_{\rho_B}(f)| \,\mathrm{d}\rho = \rho(B) \int_B |f - \mathbb{E}_{\rho_B}(f)| \,\mathrm{d}\rho_B$. Since $\mathfrak{F}$ is a covering of S, we have

$$\begin{aligned}
\int_S |f(x) - \mathbb{E}_{\rho_{B_0}}(f)| \,\mathrm{d}\rho(x) &\leq \sum_{B\in\mathfrak{F}} \int_B |f(x) - \mathbb{E}_{\rho_{B_0}}(f)| \,\mathrm{d}\rho(x) \\
&\leq \sum_{B\in\mathfrak{F}} \left(\int_B |f(x) - \mathbb{E}_{\rho_B}(f)| \,\mathrm{d}\rho(x) + \int_B |\mathbb{E}_{\rho_B}(f) - \mathbb{E}_{\rho_{B_0}}(f)| \,\mathrm{d}\rho(x) \right) \\
&\leq \sum_{B\in\mathfrak{F}} \left(a_B + \rho(B) |\mathbb{E}_{\rho_B}(f) - \mathbb{E}_{\rho_{B_0}}(f)| \right).
\end{aligned} \tag{A.6}$$

For any $B \in \mathfrak{F}$, by Boman chain condition, there exists a chain $B_0, B_1, \ldots, B_N = B$ of distinct balls from $\mathfrak{F}$, such that for any $j \in \{0, \ldots, N-1\}$,

$$\begin{aligned}
|\mathbb{E}_{\rho_{B_j}}(f) - \mathbb{E}_{\rho_{B_{j+1}}}(f)| &= \left| \frac{1}{\rho(B_j \cap B_{j+1})} \int_{B_j \cap B_{j+1}} \left(\mathbb{E}_{\rho_{B_j}}(f) - \mathbb{E}_{\rho_{B_{j+1}}}(f) \right) \mathrm{d}\rho \right| \\
&\leq \frac{1}{\rho(B_j \cap B_{j+1})} \int_{B_j \cap B_{j+1}} \left| \mathbb{E}_{\rho_{B_j}}(f) - \mathbb{E}_{\rho_{B_{j+1}}}(f) \right| \mathrm{d}\rho \\
&\leq \frac{1}{\rho(B_j \cap B_{j+1})} \left(\int_{B_j \cap B_{j+1}} \left| f - \mathbb{E}_{\rho_{B_j}}(f) \right| + \left| f - \mathbb{E}_{\rho_{B_{j+1}}}(f) \right| \mathrm{d}\rho \right) \\
&\leq \frac{a_{B_j} + a_{B_{j+1}}}{\rho(B_j \cap B_{j+1})} \overset{*}{\leq} G \left(\frac{a_{B_j}}{\rho(B_j)} + \frac{a_{B_{j+1}}}{\rho(B_{j+1})} \right),
\end{aligned} \tag{A.7}$$

where $(*)$ follows from the quantitative chain overlap of Boman chain condition 3.

Thus, we have

$$|\mathbb{E}_{\rho_B}(f) - \mathbb{E}_{\rho_{B_0}}(f)| \leq \sum_{j=0}^{N-1} |\mathbb{E}_{\rho_{B_j}}(f) - \mathbb{E}_{\rho_{B_{j+1}}}(f)| \leq 2G \sum_{j=0}^{N} \frac{a_{B_j}}{\rho(B_j)} \overset{*}{\leq} 2G \sum_{B \subset F\bar{B}} \frac{a_{\bar{B}}}{\rho(\bar{B})}, \tag{A.8}$$

where $\sum_{B\subset F\bar{B}}$ means that the sum runs over all $\bar{B} \in \mathfrak{F}$ satisfying $B \subset F\bar{B}$, and $(*)$ follows from the Boman chain condition 2. Then by Fubini–Tonelli theorem,

$$\sum_{B\in\mathfrak{F}} \rho(B) |\mathbb{E}_{\rho_B}(f) - \mathbb{E}_{\rho_{B_0}}(f)| \leq 2G \sum_{B\in\mathfrak{F}} \rho(B) \sum_{B\subset F\bar{B}} \frac{a_{\bar{B}}}{\rho(\bar{B})} \leq 2G \sum_{\bar{B}\in\mathfrak{F}} \frac{a_{\bar{B}}}{\rho(\bar{B})} \sum_{B\subset F\bar{B}} \rho(B). \tag{A.9}$$

By Boman chain condition 1 and the doubling property of ρ, we have

$$\sum_{B\subset F\bar{B}} \rho(B) \leq E\rho(F\bar{B}) \leq E\beta^2 F^{\frac{\log\beta}{\log 2}} \rho(\bar{B}), \tag{A.10}$$

where $\beta = \beta(K, N, \mathrm{diam}(S))$ is the doubling constant.

Combining (A.5), (A.6), (A.9) and (A.10), we obtain

$$\begin{aligned}
\int_S |f(x) - \mathbb{E}_\rho(f)| \,\mathrm{d}\rho(x) &\leq 2 \sum_{B\in\mathfrak{F}} \left(a_B + \rho(B) |\mathbb{E}_{\rho_B}(f) - \mathbb{E}_{\rho_{B_0}}(f)| \right) \\
&\leq 2(1 + 2\beta^2 E F^{\frac{\log\beta}{\log 2}} G) \sum_{B\in\mathfrak{F}} \rho(B) \int_B |f - \mathbb{E}_{\rho_B}(f)| \,\mathrm{d}\rho_B,
\end{aligned} \tag{A.11}$$

which is the thesis. □

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