THE SHARP SOBOLEV INEQUALITY IN QUANTITATIVE FORM

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Abstract. A quantitative version of the sharp Sobolev inequality in $W^{1,p}(\mathbb{R}^n)$, $1 < p < n$, is established with a remainder term involving the distance from extremals.

1. Introduction and main result

A sharp form of the standard Sobolev inequality in $\mathbb{R}^n$, $n \geq 2$, tells us that if $1 < p < n$ and $p^* = np/(n-p)$, then

$$S(p,n)\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq \|\nabla f\|_{L^p(\mathbb{R}^n)} \tag{1.1}$$

for every function f from the homogeneous Sobolev space $W^{1,p}(\mathbb{R}^n)$ of functions $f \in L^{p^*}(\mathbb{R}^n)$ such that $\nabla f \in L^p(\mathbb{R}^n)$. Here

$$S(p,n) = \sqrt{\pi}\, n^{1/p} \left(\frac{n-p}{p-1}\right)^{(p-1)/p} \left(\frac{\Gamma(n/p)\Gamma(1+n-n/p)}{\Gamma(1+n/2)\Gamma(n)}\right)^{1/n}$$

is the best possible constant in (1.1), and $\|\nabla f\|_{L^p(\mathbb{R}^n)}$ stands for the $L^p(\mathbb{R}^n)$ norm of the gradient ([Au, Ta]). A family of extremals in (1.1) is given by the functions $g_{a,b,x_0} : \mathbb{R}^n \to [0,+\infty)$ defined as

$$g_{a,b,x_0}(x) = \frac{a}{\left(1+b|x-x_0|^{p'}\right)^{(n-p)/p}} \qquad \text{for } x \in \mathbb{R}^n \tag{1.2}$$

for some $a \neq 0$, $b > 0$, $x_0 \in \mathbb{R}^n$. Here, $p' = p/(p-1)$, the Hölder conjugate of p. In fact, as pointed out by the recent contribution [CNV], functions having the form (1.2) are the only ones attaining equality in (1.1) for every $p \in (1,n)$. Incidentally, note that, when $p = 2$, the classical result of [GNN], applied to the Euler equation of the functional $\|\nabla f\|_{L^2(\mathbb{R}^n)}/\|f\|_{L^{2^\star}(\mathbb{R}^n)}$, can alternatively be used to derive this characterization of the extremals in (1.1).

The objective of the present paper is to strengthen inequality (1.1) by an additional term on the left-hand side which accounts for the deviation of f from extremals. More precisely, we establish a quantitative version of inequality (1.1), with a remainder term depending on the (normalized) distance of f from the family of extremals (1.2) given by

$$\lambda(f) = \inf_{a,b,x_0} \left\{ \frac{\|f - g_{a,b,x_0}\|^{p^*}_{L^{p^*}(\mathbb{R}^n)}}{\|f\|^{p^*}_{L^{p^*}(\mathbb{R}^n)}} : \|g_{a,b,x_0}\|_{L^{p^*}(\mathbb{R}^n)} = \|f\|_{L^{p^*}(\mathbb{R}^n)} \right\} \tag{1.3}$$

if $f \neq 0$, and $\lambda(0) = 0$.

Theorem 1. *Let $n \geq 2$ and let $1 < p < n$. Then, positive constants α and κ, depending only on p and n, exist such that*

$$S(p,n)\|f\|_{L^{p^*}(\mathbb{R}^n)}\left(1 + \kappa\lambda(f)^\alpha\right) \leq \|\nabla f\|_{L^p(\mathbb{R}^n)} \tag{1.4}$$

for every $f \in W^{1,p}(\mathbb{R}^n)$.

In analogy with the terminology of [Fu, Ha, HHW, FMP1, FMP2], we will refer to $\lambda(f)$ as the *asymmetry of* f. Notice that one could alternatively consider the quantity defined as

$$d(f) = \inf_{a,\,b,\,x_0} \frac{\|f - g_{a,b,x_0}\|_{L^{p^*}(\mathbb{R}^n)}}{\|f\|_{L^{p^*}(\mathbb{R}^n)}}$$

if $f \neq 0$, and $d(0) = 0$. It is obvious that $d(f) \leq \lambda(f)^{1/p^*}$; on the other hand, one can check that $\lambda(f)^{1/p^*} \leq 2d(f)$. Therefore, inequality (1.4) is equivalent to

$$S(p,n)\|f\|_{L^{p^*}(\mathbb{R}^n)}\left(1 + \kappa d(f)^\theta\right) \leq \|\nabla f\|_{L^p(\mathbb{R}^n)} \tag{1.5}$$

with $\theta = p^*\alpha$, and up to changing the value of κ.

Inequality (1.5) gives a positive answer to a question raised by Brezis and Lieb in [BL], which has been settled in [BE] in the special case when $p = 2$ in the even stronger form with $\|f - g_{a,b,x_0}\|_{L^{2^\star}(\mathbb{R}^n)}$ replaced by $\|\nabla f - \nabla g_{a,b,x_0}\|_{L^2(\mathbb{R}^n)}$ in (1.3). The method of [BE] heavily rests upon the Hilbert space structure of $W^{1,2}(\mathbb{R}^n)$ and on eigenvalue properties of a weighted Laplacian in $\mathbb{R}^n$. Such an approach, which has been employed to deal with other related problems involving Sobolev spaces endowed with a Hilbert space structure ([Lo, LW]), does not seem suitable for extensions to the general case where $p \neq 2$. Following the lines traced in [Au] and [Ta], we have instead to resort to certain methods of geometric flavour, exploiting such tools as isoperimetric inequalities and symmetrizations. Developments of these results led to quantitative forms of isoperimetric ([Fu, Ha, FMP1]), isocapacitary ([HHW, FMP3]) and Sobolev inequalities ([Ci1, FMP2, Ci2]) in the spirit of (1.4).

To be more specific, the proof of Theorem 1 basically consists of three steps, each step amounting to an extension of inequality (1.4) to a broader class of functions. After starting with spherically symmetric functions, we proceed with n-symmetric functions, namely functions which are symmetric about n orthogonal hyperplanes, and we eventually conclude with arbitrary Sobolev functions. This strategy can be clarified by the following considerations.

The operation of *spherically symmetric rearrangement*, which associates with any nonnegative function $f \in W^{1,p}(\mathbb{R}^n)$ the spherically symmetric equidistributed function $f^\star \in W^{1,p}(\mathbb{R}^n)$ (see (3.2)), satisfies

$$\|f^\star\|_{L^{p^*}(\mathbb{R}^n)} = \|f\|_{L^{p^*}(\mathbb{R}^n)}$$

and

$$\|\nabla f^\star\|_{L^p(\mathbb{R}^n)} \leq \|\nabla f\|_{L^p(\mathbb{R}^n)} \tag{1.6}$$

([BZ, Ka, Ta]). As a consequence,

$$\|\nabla f^\star\|_{L^p(\mathbb{R}^n)} - S(p,n)\|f^\star\|_{L^p(\mathbb{R}^n)} \leq \|\nabla f\|_{L^p(\mathbb{R}^n)} - S(p,n)\|f\|_{L^{p^*}(\mathbb{R}^n)} \tag{1.7}$$

and

$$\|\nabla f\|_{L^p(\mathbb{R}^n)} - \|\nabla f^\star\|_{L^p(\mathbb{R}^n)} \leq \|\nabla f\|_{L^p(\mathbb{R}^n)} - S(p,n)\|f\|_{L^{p^*}(\mathbb{R}^n)}\,. \tag{1.8}$$

for every $f \in W^{1,p}(\mathbb{R}^n)$. In view of (1.7) and (1.8), the underlying idea in the proof of inequality (1.4) is to split the problem: first, establishing the inequality in the class of spherically symmetric functions; second, estimating the L^{p^*} distance of f from (a suitable translated of) $f^\star$ in terms of $\|\nabla f\|_{L^p(\mathbb{R}^n)} - \|\nabla f^\star\|_{L^p(\mathbb{R}^n)}$.

Even in the special class of spherically symmetric functions, the derivation of (1.4) is not straightforward. Actually, standard proofs of the one-dimensional Bliss inequality, to which (1.1) reduces when restricted to spherically symmetric functions, do not seem suitable for modifications yielding stability results. A more flexible approach to the relevant one-dimensional inequality, which can be successfully augmented to provide a quantitative version, follows instead on specializing a mass transportation technique employed in [CNV] (see also [LYZ]). The resulting estimate, whose proof also requires a sharp version of a trace Sobolev inequality form [MV], is contained in Theorem 2, and settles the first of the two steps outlined above.

Major problems arise in the attempt at estimating the asymmetry of f in terms of the left-hand side of (1.8). Indeed, this is just impossible, without additional assumptions on f, as demonstrated by simple examples where $\|\nabla f\|_{L^p(\mathbb{R}^n)}$ almost agrees with $\|\nabla f^\star\|_{L^p(\mathbb{R}^n)}$, without f being close to any translated of $f^\star$. The presence of plateaus in the graph of f, or more generally, of large sets where $|\nabla f^\star|$ is small, is responsible of this phenomenon (see e.g. [BZ, CF2]). A key observation to overcome this obstacle is that a bound for the distance of f from $f^\star$ via $\|\nabla f\|_{L^p(\mathbb{R}^n)} - \|\nabla f^\star\|_{L^p(\mathbb{R}^n)}$ can be restored if f is already known to enjoy certain partial symmetry properties. It is at this stage that the class of n-symmetric functions comes into play. Indeed, on the one hand, the distance of f from $f^\star$ can actually be estimated by $\|\nabla f\|_{L^p(\mathbb{R}^n)} - \|\nabla f^\star\|_{L^p(\mathbb{R}^n)}$ if f is a priori assumed to be n-symmetric (Theorem 3), thus enabling us to establish (1.4) in this class of functions (Corollary 4). On the other hand, any function $f \in W^{1,p}(\mathbb{R}^n)$ can be replaced, through a careful construction exploiting reflection arguments, by a suitable n-symmetric function in such a way that $\|\nabla f\|_{L^p(\mathbb{R}^n)} - S(p,n)\|f\|_{L^{p^*}(\mathbb{R}^n)}$ and $\lambda(f)$ do not increase and decrease, respectively, too much (Theorem 6). This fact, combined with the former step, easily leads to the conclusion of Theorem 1. Let us emphasize that the reduction to n-symmetric functions, although related to a similar construction employed in [FMP1, FMP2], entails the overcoming of new serious obstacles in the present setting, mainly due to the nonlinear growth of the functional $\|\nabla f\|^p_{L^p(\mathbb{R}^n)}$.

We conclude this section by noting that, in view of the results of [BE] and [FMP2], the question arises of the optimal exponent α in equality (1.4). Furthermore, the result of [BE] also leaves open the problem of whether the distance of f from the family of extremals in $L^{p^*}(\mathbb{R}^n)$ can be replaced by the distance in $W^{1,p}(\mathbb{R}^n)$ in Theorem 1.

2. A quantitative Bliss inequality

In the present section, Theorem 1 will be established in the special class of spherically symmetric functions. Notice that the Sobolev inequality (1.1), restricted to this class of functions, is equivalent to the one-dimensional Bliss inequality

$$S(p,n)\left(n\omega_n\int_0^\infty u(r)^{p^*}r^{n-1}\,dr\right)^{1/p^*} \le \left(n\omega_n\int_0^\infty (-u'(r))^p r^{n-1}\,dr\right)^{1/p} \tag{2.1}$$

for every decreasing, locally absolutely continuous function $u : [0,\infty) \to [0,\infty)$, where ω_n is the measure of the unit ball in $\mathbb{R}^n$. The extremals in (2.1) have the form

$$v_{a,b}(r) = \frac{a}{(1+br^{p'})^{(n-p)/p}} \qquad \text{for } r \ge 0, \tag{2.2}$$

for some $a > 0$, $b > 0$ ([Bl, CNV, LYZ, Ta]). Thus, on setting, with a slight abuse of notation,

$$\lambda(u) = \inf\left\{\frac{\int_0^\infty |u(r)-v_{a,b}|^{p^*}r^{n-1}\,dr}{\int_0^\infty u(r)^{p^*}r^{n-1}\,dr} : \int_0^\infty v_{a,b}(r)^{p^*}r^{n-1}\,dr = \int_0^\infty u(r)^{p^*}r^{n-1}\,dr,\ a,\ b>0\right\},$$

Theorem 1 for spherically symmetric functions is equivalent to the following quantitative Bliss inequality.

Theorem 2. *Let $n \geq 2$ and let $1 < p < n$. Then there exist constants β and κ such that*

$$S(p,n)\left(n\omega_n \int_0^\infty u(r)^{p^*} r^{n-1}\,dr\right)^{1/p^*}\left(1+\kappa\lambda(u)^\beta\right) \leq \left(n\omega_n \int_0^\infty (-u'(r))^p r^{n-1}\,dr\right)^{1/p} \tag{2.3}$$

for every decreasing, locally absolutely continuous function $u : [0,\infty) \to [0,\infty)$.

In the proof of Theorem 2, we shall make use of the notation

$$\delta(u) = \frac{\left(n\omega_n \int_0^\infty (-u'(r))^p r^{n-1}\,dr\right)^{1/p}}{S(p,n)\left(n\omega_n \int_0^\infty u(r)^{p^*} r^{n-1}\,dr\right)^{1/p^*}} - 1\,, \tag{2.4}$$

so that (2.3) can be rewritten as

$$\lambda(u) \leq C\delta(u)^{1/\beta}\,, \tag{2.5}$$

where $C = \kappa^{-1/\beta}$.

Proof of Theorem 2. Approximation, rescaling and normalization arguments allow us to assume that u is continuously differentiable, with support equal to $[0,1]$, and that

$$n\omega_n \int_0^\infty u(r)^{p^*} r^{n-1}\,dr = 1\,.$$

Moreover, for the time being, we assume that

$$\delta(u) \leq \varepsilon(p,n) \tag{2.6}$$

for some positive constant $\varepsilon(p,n) < 1$, to be chosen later.
Let us set

$$v(r) = v_{a,1}(r) \qquad \text{for } r > 0\,,$$

where a is such that

$$n\omega_n \int_0^\infty v(r)^{p^*} r^{n-1}\,dr = 1\,. \tag{2.7}$$

Owing to (2.6) and (2.7), the equation

$$\int_0^r u(s)^{p^*} s^{n-1} ds = \int_0^{T(r)} v(s)^{p^*} s^{n-1} ds \tag{2.8}$$

implicitly defines a strictly increasing function $T : [0,1) \to [0,\infty)$ such that $T \in C^1(0,1)$, $T(0) = 0$, $\lim_{r\to 1^-} T(r) = \infty$, and

$$u(r)^{p^*} = v(T(r))^{p^*} M(r)^{n-1} T'(r) \qquad \text{for } r \in (0,1)\,, \tag{2.9}$$

where $M : (0,1) \to (0,+\infty)$ is given by

$$M(r) = \frac{T(r)}{r} \qquad \text{for } r \in (0,1)\,.$$

In particular, equation (2.9) entails that

$$\int_0^1 h(T(r))u(r)^{p^*} r^{n-1}\,dr = \int_0^\infty h(r)v(r)^{p^*} r^{n-1}\,dr\,, \tag{2.10}$$

for every Borel function $h : [0,\infty) \to [0,\infty]$. In the terminology of the theory of mass transportation, to which the present proof is inspired, the function T can be regarded as a *transport map* carrying the density $u(r)^{p^*} r^{n-1}$ into $v(r)^{p^*} r^{n-1}$.
Notice that when $T(r) = kr$ for some $k > 0$, one has $u(r) = k^{(n-p)/p} v(kr)$, namely u is an extremal function in the Bliss inequality (2.1). Thus, our plan is to show that, if $\delta(u)$ is small,

then an interval $[r_1, r_2] \subseteq [0,1]$ can be chosen in such a way that $T(r)$ is close to some linear function kr for $r \in [r_1, r_2]$, and simultaneously the integral of $u(r)^{p^*} r^{n-1}$ outside $[r_1, r_2]$ is small. These facts will enable us to conclude that u is close to $k^{(n-p)/p} v(kr)$ in $L^{p^*}(r^{n-1}\,dr)$.

For ease of presentation, we accomplish the proof in steps.

Step I. *Mass transportation proof of Bliss inequality.*

We begin by giving a proof of the Bliss inequality relying on the mass transportation approach of [CNV] (see also [LYZ]). Set $p^\sharp = p(n-1)/(n-p)$, the optimal exponent in the trace inequality in $\mathbb{R}^n$. Owing to (2.10) and (2.9) we have

$$\begin{aligned}\int_0^\infty v^{p^\sharp} r^{n-1}\,dr &= \int_0^\infty v(r)^{-p^*/n} v(r)^{p^*} r^{n-1}\,dr = \int_0^1 v(T(r))^{-p^*/n} u(r)^{p^*} r^{n-1}\,dr \\ &= \int_0^1 \left(\frac{M(r)^{n-1}T'(r)}{u(r)^{p^*}}\right)^{1/n} u(r)^{p^*} r^{n-1}\,dr \\ &= \int_0^1 M(r)^{1/n'} T'(r)^{1/n} u(r)^{p^\sharp} r^{n-1}\,dr\,.\end{aligned} \tag{2.11}$$

By Young's inequality

$$\begin{aligned}\int_0^1 M(r)^{1/n'} T'(r)^{1/n} u(r)^{p^\sharp} r^{n-1}\,dr &\le \frac{1}{n}\int_0^1 (T'(r) + (n-1)M(r)) u(r)^{p^\sharp} r^{n-1}\,dr \\ &= \frac{1}{n}\int_0^1 (r^{n-1}T(r))' u(r)^{p^\sharp}\,dr \\ &= \frac{p^\sharp}{n}\int_0^1 T(r)(-u'(r)) u(r)^{p^\sharp-1} r^{n-1}\,dr\,.\end{aligned} \tag{2.12}$$

The last equality can be justified as follows. By Hölder inequality and by (2.10),

$$\begin{aligned}&\int_0^1 T(r)(-u'(r)) u(r)^{p^\sharp-1} r^{n-1}\,dr \\ &\qquad \le \left(\int_0^1 (-u'(r))^p r^{n-1}\,dr\right)^{1/p} \left(\int_0^1 T(r)^{p'} u(r)^{p^*} r^{n-1}\,dr\right)^{1/p'} \\ &\qquad = \left(\int_0^1 (-u'(r))^p r^{n-1}\,dr\right)^{1/p} \left(\int_0^\infty v(r)^{p^*} r^{p'+n-1}\,dr\right)^{1/p'}.\end{aligned} \tag{2.13}$$

In particular,

$$\int_0^1 T(r)(-u'(r)) u(r)^{p^\sharp-1} r^{n-1}\,dr < +\infty\,. \tag{2.14}$$

Since u is bounded, an integration by parts yields

$$\int_0^R (r^{n-1}T(r))' u(r)^{p^\sharp}\,dr = R^{n-1}T(R)u(R)^{p^\sharp} + p^\sharp \int_0^R T(r)(-u'(r)) u(r)^{p^\sharp-1} r^{n-1}\,dr \tag{2.15}$$

for $0 < R < 1$. Observe now that, since $u(1) = 0$,

$$\int_R^1 T(r)\big(-u'(r)\big) u(r)^{p^\sharp-1} r^{n-1}\,dr \ge T(R)R^{n-1} \int_R^1 -u'(r) u(r)^{p^\sharp-1}\,dr = \frac{T(R)R^{n-1}u(R)^{p^\sharp}}{p^\sharp}\,.$$

Hence by (2.14) it follows that $T(R)u(R)^{p^\sharp} \to 0$ for $R \to 1$, so that the last inequality in (2.12) follows on passing to the limit in (2.15).

Now, define $\zeta : [0,\infty) \to [0,\infty)$ as

$$\zeta(t) = t + (n-1) - nt^{1/n} \qquad \text{for } t \ge 0\,,$$

and set

$$C_0 = p^\sharp \left(\int_0^\infty v(r)^{p^*} r^{p'+n-1}\, dr \right)^{1/p'},$$

a constant depending only on p and n. Inequalities (2.11)–(2.13) entail that

$$\int_0^1 \zeta\left(\frac{T'(r)}{M(r)}\right) M(r) u(r)^{p^\sharp} r^{n-1}\, dr \le C_0 \left(\int_0^1 (-u'(r))^p r^{n-1}\, dr \right)^{1/p} - n \int_0^\infty v^{p^\sharp} r^{n-1}\, dr\,. \quad (2.16)$$

One can easily verify that

$$C_0 \left(\int_0^\infty (-v'(r))^p r^{n-1}\, dr \right)^{1/p} = n \int_0^\infty v^{p^\sharp} r^{n-1}\, dr\,.$$

Consequently, recalling (2.7), a direct calculation shows that

$$n \int_0^\infty v^{p^\sharp} r^{n-1}\, dr = \frac{C_0 S(p,n)}{(n\omega_n)^{1/p}}\,.$$

In conclusion, (2.16) tells us that

$$\int_0^1 \zeta\left(\frac{T'(r)}{M(r)}\right) M(r) u(r)^{p^\sharp} r^{n-1}\, dr \le \frac{C_0 S(p,n)}{(n\omega_n)^{1/p}}\, \delta(u)\,, \quad (2.17)$$

for every u as in the statement. Notice that, if $\delta(u) = 0$ then (2.17) gives $T'(r)/M(r) = 1$ for all $r \in (0,1)$, hence $T(r) = kr$ and as underlined before this implies that u is as in (2.2). This observation was a crucial point in [CNV]. In our case, instead, we have to extract a quantitative information from (2.17) by proving that if $\delta(u)$ is small then $T(r)$ is close to a suitable linear function of r.

Step II. *A lower bound for* $u(r)^{p^\sharp} r^{n-1}$.
We prove now a bound for $u(r)^{p^\sharp} r^{n-1}$ from below in a suitable subinterval of $(0,1)$, and we combine it with (2.9) to derive an integral estimate on such intervals involving T and T'. A key ingredient here is a trace inequality from [MV], Theorem 1.3, which, in the one-dimensional case, tells us that

$$\left(n\omega_n \int_0^r u(s)^{p^*} s^{n-1} ds \right)^{p/p^*} \le \frac{n\omega_n}{S(p,n)^p} \int_0^r (-u'(s))^p s^{n-1} ds + C_1 \left(u(r)^{p^\sharp} r^{n-1} \right)^{p/p^\sharp} \quad (2.18)$$

and

$$\left(n\omega_n \int_r^1 u(s)^{p^*} s^{n-1} ds \right)^{p/p^*} \le \frac{n\omega_n}{S(p,n)^p} \int_r^1 (-u'(s))^p s^{n-1} ds + C_1 \left(u(r)^{p^\sharp} r^{n-1} \right)^{p/p^\sharp} \quad (2.19)$$

for every $0 < r < 1$, for some constant $C_1 > 0$. Set

$$\gamma(r) = n\omega_n \int_0^r u(s)^{p^*} s^{n-1} ds \qquad \text{for } r \in [0,1]\,.$$

Adding up inequalities (2.18) and (2.19) implies that

$$\begin{aligned} \gamma(r)^{p/p^*} + (1-\gamma(r))^{p/p^*} &\le (1+\delta(u))^p + 2C_1 \left(u(r)^{p^\sharp} r^{n-1} \right)^{p/p^\sharp} \\ &\le 1 + C_2 \delta(u) + 2C_1 \left(u(r)^{p^\sharp} r^{n-1} \right)^{p/p^\sharp}, \end{aligned} \quad (2.20)$$

for some positive constant C_2. Notice that the second inequality holds owing to (2.6). On setting

$$\psi(t) = t^{p/p^*} + (1-t)^{p/p^*} - 1 \qquad \text{for } t \in [0,1]\,,$$

inequality (2.20) reads

$$\Big(\psi(\gamma(r)) - C_2\delta(u)\Big)^{p^\sharp/p} \le C_3 u(r)^{p^\sharp} r^{n-1} \qquad \text{for } r \in [0,1]\,, \tag{2.21}$$

with $C_3 = (2C_1)^{p^\sharp/p}$. It is easily seen that a positive constant C_4 exists such that, if $0 < \varepsilon < 1/C_4$, then

$$\psi(t) \ge 3\varepsilon \qquad \text{for } t \in [(4\varepsilon)^{p^*/p}, 1 - (4\varepsilon)^{p^*/p}]\,. \tag{2.22}$$

Hence, given any $\varepsilon \in (0, 1/C_4)$, on denoting by r_1 and r_2 the positive numbers satisfying

$$\gamma(r_1) = (4\varepsilon)^{p^*/p}\,, \qquad \gamma(r_2) = 1 - (4\varepsilon)^{p^*/p}\,, \tag{2.23}$$

and assuming that

$$\delta(u) \le \frac{\varepsilon}{C_2}\,, \tag{2.24}$$

we get that

$$(2\varepsilon)^{p^\sharp/p} \le C_3 u(r)^{p^\sharp} r^{n-1} \qquad \text{for } r \in [r_1, r_2]\,. \tag{2.25}$$

On the other hand, owing to (2.22) and to (2.24), inequality (2.21) entails that

$$\left(\frac{\psi(\gamma(r))}{2}\right)^{p^\sharp/p} \le C_3 u(r)^{p^\sharp} r^{n-1} \qquad \text{for } r \in [r_1, r_2]\,. \tag{2.26}$$

Since $\gamma'(r) = n\omega_n u(r)^{p^*} r^{n-1}$ for $r > 0$, we infer from (2.26) and (2.9) that

$$C_5 \frac{\gamma'(r)}{\psi(\gamma(r))^{p^\sharp/p}} \ge u(r)^{p^*-p^\sharp} = u(r)^{p^*/n} = v(T(r))^{p^*/n} M(r)^{1/n'} T'(r)^{1/n} \quad \text{for every } r \in [r_1, r_2]\,,$$

for some positive constant C_5. Hence,

$$\int_{r_1}^{r_2} v(T(r))^{p^*/n} M(r)^{1/n'} T'(r)^{1/n}\, dr \le C_6\,, \tag{2.27}$$

for some constant C_6.

Step III. *An integral bound for $|T' - M|$.*
The task of the present step is to provide an estimate for $\int_{r_1}^{r_2} |T'(r) - M(r)|\, dr$. Our starting point is the inequality

$$\int_{r_1}^{r_2} M(r)\zeta\left(\frac{T'(r)}{M(r)}\right) dr \le C_7 \frac{\delta(u)}{\varepsilon^{p^\sharp/p}}\,, \tag{2.28}$$

which follows from (2.17) and (2.25) and holds for some positive constant C_7. Since $\zeta'(1) = \zeta(1) = 0$ and $\zeta''(t) = (1/n')t^{-2+1/n}$, a decreasing function, by Taylor's formula we have

$$\zeta(t) \ge \frac{1}{2n'} \min\left\{1, t^{-2+1/n}\right\}(t-1)^2 \qquad \text{for } t \ge 0\,.$$

Thus, inequality (2.28) tells us that

$$2C_7 n' \frac{\delta(u)}{\varepsilon^{p^\sharp/p}} \ge \int_{r_1}^{r_2} \frac{(T'(r) - M(r))^2}{M(r)} \min\left\{1, \left(\frac{M(r)}{T'(r)}\right)^{2-1/n}\right\} dr\,. \tag{2.29}$$

Define

$$I = \left\{r \in [r_1, r_2] : T'(r) \le M(r)\right\}\,, \qquad J = [r_1, r_2] \setminus I\,.$$

By (2.29), Hölder inequality and (2.27),

$$
\begin{aligned}
2C_7 n' \frac{\delta(u)}{\varepsilon^{p^\sharp/p}} &\geq \int_J \frac{(T'(r)-M(r))^2}{T'(r)^2 v(T(r))^{p^*/n}} v(T(r))^{p^*/n} T'(r)^{1/n} M(r)^{1/n'}\, dr \\
&\geq \frac{1}{C_6}\left(\int_J \frac{|T'(r)-M(r)|}{T'(r) v(T(r))^{p^*/2n}} v(T(r))^{p^*/n} T'(r)^{1/n} M(r)^{1/n'}\, dr\right)^2 \\
&= \frac{1}{C_6}\left(\int_J |T'(r)-M(r)| v(T(r))^{p^*/2n} \left(\frac{M(r)}{T'(r)}\right)^{1/n'} dr\right)^2 .
\end{aligned} \tag{2.30}
$$

From (2.8) we deduce that

$$
a^{p^*} \frac{T(r)^n}{n} \geq \int_0^{T(r)} v(s)^{p^*} s^{n-1} ds = \int_0^r u(s)^{p^*} s^{n-1} ds \geq u(r)^{p^*} \frac{r^n}{n} \qquad \text{for } r \geq 0 ,
$$

whence, by (2.9),

$$
T'(r) v(T(r))^{p^*} \leq a^{p^*} M(r) \qquad \text{for } r \geq 0 . \tag{2.31}
$$

Coupling (2.30) and (2.31) implies that

$$
\begin{aligned}
C_8 \sqrt{\frac{\delta(u)}{\varepsilon^{p^\sharp/p}}} &\geq \int_J |T'(r)-M(r)| v(T(r))^{p^*(1-1/2n)}\, dr \\
&\geq v(T(r_2))^{p^*(1-1/2n)} \int_J |T'(r)-M(r)|\, dr .
\end{aligned} \tag{2.32}
$$

Now, observe that

$$
v(T(r_2))^{p^*} = \frac{a^{p^*}}{(1+T(r_2)^{p'})^n} \geq \frac{a^{p^*}}{2^n} \min\left\{1, \frac{1}{T(r_2)^{p'n}}\right\} . \tag{2.33}
$$

Equation (2.23) can be used to deduce that

$$
\begin{aligned}
(4\varepsilon)^{n/(n-p)} &= n\omega_n \int_{r_2}^1 u(s)^{p^*} s^{n-1} ds = n\omega_n \int_{T(r_2)}^\infty v(s)^{p^*} s^{n-1} ds = n\omega_n a^{p^*} \int_{T(r_2)}^\infty \frac{s^{n-1}}{(1+s^{p'})^n} ds \\
&\leq n\omega_n a^{p^*} \int_{T(r_2)}^\infty s^{n-1-np'} ds = (p-1)\omega_n a^{p^*} T(r_2)^{-n/(p-1)} ,
\end{aligned}
$$

whence

$$
T(r_2) \leq \frac{C_9}{\varepsilon^{(p-1)/(n-p)}} , \tag{2.34}
$$

for some constant C_9. From (2.33), combined with (2.34), we infer that

$$
v(T(r_2))^{p^*} \geq \frac{a^{p^*} \varepsilon^{p^*}}{2^n\, C_9^{p'n}} ,
$$

provided that $\varepsilon < \varepsilon(p,n)$ for a sufficiently small $\varepsilon(p,n)$. From this inequality and (2.32) one gets

$$
\int_J |T'(r)-M(r)|\, dr \leq C_{10} \sqrt{\frac{\delta(u)}{\varepsilon^{p^\sharp/p + (2-1/n)p^*}}} . \tag{2.35}
$$

As far as $\int_I |T'(r)-M(r)|\,dr$ is concerned, by (2.29) and (2.27) again one has

$$\begin{aligned} 2C_7 n' \frac{\delta(u)}{\varepsilon^{p^\sharp/p}} &\geq \int_I \frac{(T'(r)-M(r))^2}{T'(r)^2 v(T(r))^{p^*/n}} \left(\frac{T'(r)}{M(r)}\right)^{2-1/n} v(T(r))^{p^*/n} T'(r)^{1/n} M(r)^{1/n'}\,dr \\ &\geq \frac{1}{C_6}\left(\int_I |T'(r)-M(r)| \left(\frac{v(T(r))^{p^*}T'(r)}{M(r)}\right)^{1/2n} dr\right)^2 \\ &= \frac{1}{C_6}\left(\int_I |T'(r)-M(r)| \left(\frac{u(r)^{p^*}r^n}{T^n(r)}\right)^{1/2n} dr\right)^2 . \end{aligned} \tag{2.36}$$

Note that in the last inequality we have made use of (2.9). Inasmuch as $T(r)\leq T(r_2)$ for $r\in I$, inequalities (2.36), (2.34) and (2.25) ensure that

$$\varepsilon^{p^*/2n}\int_I |T'(r)-M(r)|\,dr \leq C_{11}\sqrt{\frac{\delta(u)}{\varepsilon^{p^\sharp/p}}} \tag{2.37}$$

for some positive constant C_{11}. Coupling (2.35) and (2.37) yields

$$\int_{r_1}^{r_2} |T'(r)-M(r)|\,dr \leq C_{12}\sqrt{\frac{\delta(u)}{\varepsilon^{\omega_0}}} \tag{2.38}$$

for some positive C_{12} where

$$\omega_0 = \frac{p^\sharp}{p} + \left(2-\frac{1}{n}\right)p^* .$$

Step IV. *Conclusion.*
Here, we single out the extremal (2.2) to be used in estimating $\lambda(u)$. Set

$$k = M(r_2)$$

and define $v_0 = [0,\infty)\to[0,\infty)$ as

$$v_0(r) = k^{(n-p)/p} v(kr) \qquad\qquad \text{for } r\geq 0 .$$

Clearly, v_0 is an extremal function in the Bliss inequality, still fulfilling $n\omega_n\int_0^\infty v_0(r)^{p^*}r^{n-1}\,dr = 1$. Consequently, by (2.23),

$$\begin{aligned} \lambda(u) &\leq n\omega_n \int_0^\infty |u(r)-v_0(r)|^{p^*} r^{n-1}\,dr \\ &\leq C_{13}\Big(\varepsilon^{n/(n-p)} + \int_{r_2}^\infty v_0(r)^{p^*}r^{n-1}\,dr + \int_0^{r_1} v_0(r)^{p^*}r^{n-1}\,dr \\ &\qquad\qquad + \int_{r_1}^{r_2} |u(r)-v_0(r)|^{p^*}r^{n-1}\,dr\Big). \end{aligned} \tag{2.39}$$

The point is to estimate the last three integrals. As far as the first one is concerned, owing to (2.23) one has

$$\begin{aligned} \int_{r_2}^\infty v_0(r)^{p^*}r^{n-1}\,dr &= k^n\int_{r_2}^\infty v(kr)^{p^*}r^{n-1}\,dr = \int_{kr_2}^\infty v(r)^{p^*}r^{n-1}\,dr \\ &= \int_{T(r_2)}^\infty v(r)^{p^*}r^{n-1}\,dr = \frac{1}{n\omega_n} - \int_0^{r_2} u(r)^{p^*}r^{n-1}\,dr = \frac{(4\varepsilon)^{n/(n-p)}}{n\omega_n} . \end{aligned} \tag{2.40}$$

Next, we have

$$\int_0^{r_1} v_0(r)^{p^*}r^{n-1}\,dr = \int_0^{r_1 M(r_2)} v(r)^{p^*}r^{n-1}\,dr . \tag{2.41}$$

If $M(r_1) \le M(r_2)$, then

$$\begin{aligned} M(r_2)r_1 &= T(r_1) + r_1(M(r_2) - M(r_1)) = T(r_1) + r_1 \int_{r_1}^{r_2} \frac{T'(r) - M(r)}{r}\,dr \\ &\le T(r_1) + \int_{r_1}^{r_2} |T'(r) - M(r)|\,dr \le T(r_1) + C_{12}\sqrt{\frac{\delta(u)}{\varepsilon^{\omega_0}}}\,, \end{aligned} \tag{2.42}$$

where we have exploited (2.38) in the last inequality. Consequently, (2.41), (2.42) and (2.23) tell us that

$$\begin{aligned} \int_0^{r_1} v_0(r)^{p^*} r^{n-1}\,dr &\le (4\varepsilon)^{p^*/p} + a^{p^*} \int_{T(r_1)}^{T(r_1)+C_{12}\sqrt{\delta(u)/\varepsilon^{\omega_0}}} \frac{r^{n-1}}{(1+r^{p'})^n}\,dr \\ &\le C_{14}\left(\varepsilon^{p^*/p} + \sqrt{\frac{\delta(u)}{\varepsilon^{\omega_0}}}\right) \end{aligned} \tag{2.43}$$

for some positive constant C_{14}. Inequality (2.43) continues to hold even if $M(r_2) \le M(r_1)$, since $M(r_2)r_1 = T(r_1)M(r_2)/M(r_1)$, and hence, by (2.41) and (2.23),

$$n\omega_n \int_0^{r_1} v_0(r)^{p^*} r^{n-1}\,dr \le n\omega_n \int_0^{T(r_1)} v(r)^{p^*} r^{n-1}\,dr = n\omega_n \int_0^{r_1} u(r)^{p^*} r^{n-1}\,dr = (4\varepsilon)^{p^*/p}\,.$$

The estimate for the last integral in (2.39) is the most delicate. Thanks to (2.9),

$$\begin{aligned} &\int_{r_1}^{r_2} |u(r) - v_0(r)|^{p^*} r^{n-1}\,dr = \int_{r_1}^{r_2} |u(r) - k^{(n-p)/p} v(kr)|^{p^*} r^{n-1}\,dr \\ &= \int_{r_1}^{r_2} \left| \Big(v(T(r))^{p^*} M(r)^{n-1} T'(r)\Big)^{1/p^*} - \Big(k^n v(kr)^{p^*}\Big)^{1/p^*} \right|^{p^*} r^{n-1}\,dr \\ &\le \int_{r_1}^{r_2} \left| v(T(r))^{p^*} M(r)^{n-1} T'(r) - k^n v(kr)^{p^*} \right| r^{n-1}\,dr \\ &\le \int_{r_1}^{r_2} v(T(r))^{p^*} T(r)^{n-1} |T'(r) - M(r)|\,dr + \int_{r_1}^{r_2} v(T(r))^{p^*} |M(r)^n - k^n| r^{n-1}\,dr \\ &\qquad + k^n \int_{r_1}^{r_2} |v(T(r))^{p^*} - v(kr)^{p^*}| r^{n-1}\,dr\,. \end{aligned} \tag{2.44}$$

Since $v(T(r))^{p^*} T(r)^{n-1}$ is bounded from above in terms of p and n only, one has by (2.38)

$$\int_{r_1}^{r_2} v(T(r))^{p^*} T(r)^{n-1} |T'(r) - M(r)|\,dr \le C_{15}\sqrt{\frac{\delta(u)}{\varepsilon^{\omega_0}}}\,. \tag{2.45}$$

for some constant C_{15}. The boundedness of $v(T(r))^{p^*} T(r)^{n-1}$ again implies that

$$\begin{aligned} \int_{r_1}^{r_2} v(T(r))^{p^*} |M(r)^n - k^n| r^{n-1}\,dr &\le C_{16} \int_{r_1}^{r_2} \frac{|M(r)^n - k^n|}{M(r)^{n-1}}\,dr \\ &\le nC_{16} \int_{r_1}^{r_2} |M(r) - M(r_2)| \frac{\max\left\{M(r)^{n-1}, M(r_2)^{n-1}\right\}}{M(r)^{n-1}}\,dr \end{aligned} \tag{2.46}$$

for some constant C_{16}. By (2.34),

$$\frac{M(r_2)}{M(r)} = \frac{T(r_2)}{T(r)}\,\frac{r}{r_2} \le \frac{T(r_2)}{T(r)} \le \frac{T(r_2)}{T(r_1)} \le \frac{C_9}{\varepsilon^{(p-1)/(n-p)} T(r_1)} \qquad \text{for } r \in [r_1, r_2]\,,$$

and, by (2.23),

$$\frac{(4\varepsilon)^{p^*/p}}{n\omega_n}=\int_0^{T(r_1)} v(r)^{p^*}r^{n-1}\,dr=a^{p^*}\int_0^{T(r_1)}\frac{r^{n-1}}{(1+r^{p'})^n}\,dr\le\frac{a^{p^*}}{n}\,T(r_1)^n\,,$$

whence

$$\frac{M(r_2)}{M(r)}\le\frac{C_{17}}{\varepsilon^{p/(n-p)}}\qquad\text{for } r\in[r_1,r_2]\,,\tag{2.47}$$

for some constant C_{17}. Combining (2.46) and (2.47) yields

$$\int_{r_1}^{r_2} v(T(r))^{p^*}|M(r)^n-k^n|r^{n-1}\,dr\le\frac{C_{18}}{\varepsilon^{p^\sharp}}\int_{r_1}^{r_2}|M(r)-M(r_2)|\,dr\,,\tag{2.48}$$

for some constant C_{18}. On the other hand,

$$\begin{aligned}\int_{r_1}^{r_2}|M(r)-M(r_2)|\,dr&\le\int_{r_1}^{r_2}dr\int_r^{r_2}\frac{|T'(t)-M(t)|}{t}\,dt\\&=\int_{r_1}^{r_2}\frac{|T'(t)-M(t)|}{t}\,dt\int_{r_1}^{t}dr\le\int_{r_1}^{r_2}|T'(t)-M(t)|\,dt\,.\end{aligned}\tag{2.49}$$

Thus, thanks to (2.48), (2.49) and (2.38),

$$\int_{r_1}^{r_2} v(T(r))^{p^*}|M(r)^n-k^n|r^{n-1}\,dr\le C_{19}\sqrt{\frac{\delta(u)}{\varepsilon^{\omega_0+2p^\sharp}}}\tag{2.50}$$

for some constant C_{19}. Finally, since the function v^{p^*} is Lipschitz continuous in $[0,\infty)$ (with Lipschitz constant not exceeding $np'a^{p^*}$),

$$|v(T(r))^{p^*}-v(M(r_2)r)^{p^*}|\le np'a^{p^*}r|M(r)-M(r_2)|\qquad\text{for } r\ge0\,.$$

Hence, via (2.34), (2.49) and (2.38), we get

$$\begin{aligned}\int_{r_1}^{r_2}|v(T(r))^{p^*}-v(kr)^{p^*}|k^nr^{n-1}\,dr&\le np'a^{p^*}T(r_2)^n\int_{r_1}^{r_2}|M(r)-M(r_2)|\,dr\\&\le\frac{np'a^{p^*}C_9^n}{\varepsilon^{n(p-1)/(n-p)}}\int_{r_1}^{r_2}|M(r)-M(r_2)|\,dr\le\frac{np'a^{p^*}C_9^n}{\varepsilon^{n(p-1)/(n-p)}}\int_{r_1}^{r_2}|T'(r)-M(r)|\,dr\\&\le C_{20}\sqrt{\frac{\delta(u)}{\varepsilon^{\omega_0+2p^\sharp}}}\end{aligned}\tag{2.51}$$

for some constant C_{20}. Combining (2.44), (2.45), (2.50) and (2.51) tells us that

$$\int_{r_1}^{r_2}|u(r)-v_0(r)|^{p^*}r^{n-1}\,dr\le C_{21}\sqrt{\frac{\delta(u)}{\varepsilon^{\omega_1}}}\,,\tag{2.52}$$

where

$$\omega_1=\omega_0+2p^\sharp\,.$$

From (2.39), (2.40), (2.43) and (2.52) we conclude that

$$\lambda(u)\le C_{22}\left\{\varepsilon^{n/(n-p)}+\sqrt{\frac{\delta(u)}{\varepsilon^{\omega_1}}}\right\}$$

for some constant C_{22}. The choice

$$\varepsilon^{\omega_1+(2n)/(n-p)}=\delta(u)\,,$$

which is compatible with (2.24) provided that (2.6) holds for a sufficiently small $\varepsilon(p,n)$, yields

$$\lambda(u)\le C_{23}\delta(u)^{\omega}\,,\tag{2.53}$$

where

$$\omega = \frac{n}{(n-p)\omega_1 + 2n}$$

and C_{23} is a suitable constant. Obviously, inequality (2.53) continues to hold for some constant even if (2.6) is violated, namely if $\delta(u) \geq \varepsilon(p,n)$. Actually, since $\lambda(u) \leq 2^{p^*}$,

$$\delta(u) \geq \varepsilon(p,n) \geq \varepsilon(p,n) 2^{-p^\star/\omega} \lambda(u)^{1/\omega} .$$

This proves (2.5) with

$$\beta = \frac{1}{\omega} = 3 + 4p - \frac{3p+1}{n} . \tag{2.54}$$

□

3. The case of n-symmetric functions

As recalled in the Introduction, the Pólya–Szegö inequality (1.6) does not enjoy the stability property which would immediately imply Theorem 1 via the one dimensional Theorem 2 and from inequality (1.8). Indeed, although equality trivially holds in (1.6) whenever f is spherically symmetric, the sole gap between $\|\nabla f\|_{L^p(\mathbb{R}^n)}$ and $\|\nabla f^\star\|_{L^p(\mathbb{R}^n)}$ is not sufficient to estimate the asymmetry of f measured as a distance (in some integral norm) of f from a (translated of) $f^\star$. Such a distance, in any L^q norm with $1 \leq q < p^*$, can be actually estimated, if information on the measure of the sets $\{|\nabla f^\star| < \varepsilon\}$ is also retained, as recently shown in [CEFT]. In fact, this result could be used to prove a weaker form of Theorem 1, for functions supported in sets of finite measure and with p^* in definition (1.3) replaced by any smaller exponent. The full version of Theorem 1 requires, instead, the quantitative Pólya–Szegö principle for n-symmetric functions contained in Theorem 3 below. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is $k-$*symmetric*, with $1 \leq k \leq n$, if there exist k mutually orthogonal hyperplanes such that f is symmetric with respect to each of them. Moreover, if $f : \mathbb{R}^n \to [0, \infty)$ is any measurable function satisfying

$$|\{x : \, f(x) > t\}| < \infty \qquad \text{for } t > 0 , \tag{3.1}$$

its *spherically symmetric rearrangement* $f^\star : \mathbb{R}^n \to [0,\infty)$ is given by

$$f^\star(x) = \sup\{t \geq 0 : \, |\{f > t\}| > \omega_n |x|^n\} \qquad \text{for } x \in \mathbb{R}^n . \tag{3.2}$$

Theorem 3. *Let $n \geq 2$ and let $1 < p < n$. Set $q = \max\{p, 2\}$. Then a positive constant C exists such that*

$$\int_{\mathbb{R}^n} |f - f^\star|^{p^*} \leq C \left(\int_{\mathbb{R}^n} |f|^{p^*} \right)^{p/n} \left(\int_{\mathbb{R}^n} |\nabla f^\star|^p \right)^{1/q'} \left(\int_{\mathbb{R}^n} |\nabla f|^p - \int_{\mathbb{R}^n} |\nabla f^\star|^p \right)^{1/q} \tag{3.3}$$

for every nonnegative $f \in W^{1,p}(\mathbb{R}^n)$ which is symmetric with respect to the coordinate hyperplanes.

It is clear that, up to a rigid motion, Theorem 3 holds for any $n-$symmetric function.

Thanks to inequalities (1.7) and (1.8), a combination of Theorems 2 and 3 easily yields inequality (1.4) for n-symmetric functions.

Corollary 4. *Let $n \geq 2$ and let $1 < p < n$. Then a constant $\kappa > 0$ exists such that* (1.4) *holds for every nonnegative n-symmetric function $f \in W^{1,p}(\mathbb{R}^n)$, with $\alpha = \beta$, β as in (2.54).*

In analogy with (2.4) (and again with a slight abuse of notation), we define

$$\delta(f) = \frac{\|\nabla f\|_{L^p(\mathbb{R}^n)}}{S(p,n)\|f\|_{L^{p^*}(\mathbb{R}^n)}} - 1$$

for $f \in W^{1,p}(R^n)$. Inequality (1.4) can then be written as

$$\lambda(f) \le C\delta(f)^{1/\alpha},$$

where $C = (1/\kappa)^{1/\alpha}$.

Proof of Corollary 4. We may assume, without loss of generality, that $\|f\|_{L^{p^*}(\mathbb{R}^n)} = 1$ and f is symmetric with respect to the coordinate hyperplanes; in fact, both $\delta(f)$ and $\lambda(f)$ are invariant by rescaling, multiplication by a constant and rigid motions. Suppose, for the time being, that $\delta(f) \le 1/S(p,n)$. Then

$$S(p,n) \le \|\nabla f^\star\|_{L^p(\mathbb{R}^n)} \le \|\nabla f\|_{L^p(\mathbb{R}^n)} \le 1 + S(p,n). \tag{3.4}$$

We have

$$\begin{aligned}\lambda(f)^{1/p^*} &\le \lambda(f^\star)^{1/p^*} + \|f - f^\star\|_{L^{p^*}(\mathbb{R}^n)} \\ &\le C\Big(\Big(\|\nabla f^\star\|_{L^{p^*}(\mathbb{R}^n)} - S(p,n)\Big)^{1/\beta p^*} \\ &\qquad + \|\nabla f^\star\|_{L^p(\mathbb{R}^n)}^{p/q'p^*}\Big(\|\nabla f\|^p_{L^p(\mathbb{R}^n)} - \|\nabla f^\star\|^p_{L^p(\mathbb{R}^n)}\Big)^{1/qp^*}\Big),\end{aligned} \tag{3.5}$$

for some constant C, where the first inequality is just a consequence of the triangle inequality, and the second one follows from Theorems 2 and 3. Inequalities (3.4) ensure that

$$\|\nabla f\|^p_{L^p(\mathbb{R}^n)} - \|\nabla f^\star\|^p_{L^p(\mathbb{R}^n)} \le C\big(\|\nabla f\|_{L^p(\mathbb{R}^n)} - \|\nabla f^\star\|_{L^p(\mathbb{R}^n)}\big) \tag{3.6}$$

for some constant C. Combining (3.5), (3.6), (1.7) and (1.8) yields

$$\lambda(f)^{1/p^*} \le C\left(\delta(f)^{1/\beta p^*} + \delta(f)^{1/qp^*}\right)$$

for some constant C. Hence, inequality (1.4) follows with $\alpha = \beta$, since by (2.54) it is $\beta > q$. If $\delta(f) > 1/S(p,n)$, the assertion is a straightforward consequence of the inequality $\lambda(f) \le 2^{p^*}$. □

The following estimate for the distance between functions in $L^q(\mathbb{R}^n)$ involving the measure of the symmetric difference of their level sets will be exploited in the proof of Theorem 3.

Lemma 5. *Let $q \ge 1$. Given any nonnegative functions $f, g \in L^q(\mathbb{R}^n)$, set*

$$E_t = \{f > t\}\Delta\{g > t\},$$

where Δ stands for the symmetric difference of sets. Then

$$\int_{\mathbb{R}^n} |f-g|^q \le q\int_0^\infty |E_t| t^{q-1}\,dt. \tag{3.7}$$

Proof. The layer-cake formula and Fubini's theorem yield

$$\begin{aligned}\int_{\mathbb{R}^n} |f-g|^q &= \int_{\mathbb{R}^n} |f(x)-g(x)|^{q-1}\left|\int_0^\infty \chi_{\{f>t\}}(x)\,dt - \int_0^\infty \chi_{\{g>t\}}(x)\,dt\right| dx \\ &\le \int_{\mathbb{R}^n} |f(x)-g(x)|^{q-1}\int_0^\infty |\chi_{\{f>t\}}(x) - \chi_{\{g>t\}}(x)|\,dt\,dx \\ &= \int_0^\infty \int_{E_t} |f(x)-g(x)|^{q-1}\,dx\,dt.\end{aligned}$$

Here χ_G denotes the characteristic function of the set G. Thus, since

$$\int_{E_t} |f-g|^{q-1} = \int_{\{g\le t<f\}} |f-g|^{q-1} + \int_{\{f\le t<g\}} |f-g|^{q-1} \le \int_{\{g\le t<f\}} f^{q-1} + \int_{\{f\le t<g\}} g^{q-1}$$

for any $t \ge 0$, one has

$$\int_{\mathbb{R}^n} |f-g|^q \le \int_0^\infty \int_{\{g\le t<f\}} f(x)^{q-1}\,dx\,dt + \int_0^\infty \int_{\{f\le t<g\}} g(x)^{q-1}\,dx\,dt\,. \tag{3.8}$$

Now,

$$\begin{aligned}\int_0^\infty \int_{\{g\le t<f\}} f(x)^{q-1}\,dx\,dt &= \int_0^\infty \int_{\mathbb{R}^n} \chi_{\{g\le t<f\}}(x) \int_0^\infty \chi_{[0,f(x)^{q-1}]}(s)\,ds\,dx\,dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \chi_{\{f>\max\{t,s^{1/(q-1)}\};\,t\ge g\}}(x)\,ds\,dx\,dt\,.\end{aligned} \tag{3.9}$$

Another application of Fubini's theorem ensures that

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \chi_{\{f>\max\{t,s^{1/(q-1)}\};\,t\ge g\}}(x)\,ds\,dx\,dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} \int_0^{t^{q-1}} \chi_{\{g\le t<f\}}(x)\,ds\,dx\,dt + \int_{\mathbb{R}^n} \int_0^\infty \int_{t^{q-1}}^\infty \chi_{\{f>s^{1/(q-1)};\,t\ge g\}}(x)\,ds\,dt\,dx \\ &= \int_0^\infty t^{q-1}|\{g\le t<f\}|\,dt + \int_{\mathbb{R}^n} \int_0^\infty \int_0^{s^{1/(q-1)}} \chi_{\{f>s^{1/(q-1)};\,t\ge g\}}(x)\,dt\,ds\,dx \\ &\le \int_0^\infty t^{q-1}|\{g\le t<f\}|\,dt + \int_{\mathbb{R}^n} \int_0^\infty \int_0^{s^{1/(q-1)}} \chi_{\{g\le s^{1/(q-1)}<f\}}(x)\,dt\,ds\,dx \\ &= \int_0^\infty t^{q-1}\big|\{g\le t<f\}\big|\,dt + \int_0^\infty s^{1/(q-1)}\big|\{g\le s^{1/(q-1)}<f\}\big|\,ds\,.\end{aligned} \tag{3.10}$$

The change of variable $s = \tau^{q-1}$ in the last integral yields

$$\int_0^\infty s^{1/(q-1)}\big|\{g\le s^{1/(q-1)}<f\}\big|\,ds = (q-1)\int_0^\infty \tau^{q-1}\big|\{g\le \tau<f\}\big|\,d\tau\,.$$

Thus, combining (3.9) and (3.10) entails that

$$\int_0^\infty \int_{\{g\le t<f\}} f(x)^{q-1}\,dx\,dt \le q\int_0^\infty t^{q-1}\big|\{g\le t<f\}\big|\,dt\,. \tag{3.11}$$

Since $|E_t| = \big|\{g\le t<f\}\big| + \big|\{f\le t<g\}\big|$ for $t\ge 0$, inequality (3.7) follows from (3.8), from (3.11) and from an analogous estimate for the second integral on the right-hand side of (3.8). □

Proof of Theorem 3. Assume, without loss of generality, that $\|f\|_{L^{p^*}(\mathbb{R}^n)} = 1$. By the coarea formula,

$$\mu(t) = \big|\{f>t\}\cap\{\nabla f = 0\}\big| + \int_t^\infty \int_{\{f=s\}} \frac{d\mathscr{H}^{n-1}}{|\nabla f|}\,ds \qquad \text{for } t>0\,,$$

where $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure (see e.g. [BZ], or [CF1]). Hence,

$$-\mu'(t) \ge \int_{\{f=t\}} \frac{d\mathscr{H}^{n-1}}{|\nabla f|} \qquad \text{for a.e. } t>0\,. \tag{3.12}$$

One has

$$\mathscr{H}^{n-1}\big(\{f=t\}\big) = P(\{f>t\}) \qquad \text{for a.e. } t>0\,,$$

where P stands for perimeter in the sense of geometric measure theory (see e.g. [BZ, equation (19)]). Then, an application of the coarea formula again, Hölder's inequality and (3.12) entail that

$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla f|^p &= \int_0^\infty \int_{\{f=t\}} |\nabla f|^{p-1} d\mathscr{H}^{n-1}\,dt \geq \int_0^\infty \frac{\mathscr{H}^{n-1}(\{f=t\})^p}{\left(\int_{\{f=t\}} \frac{d\mathscr{H}^{n-1}}{|\nabla f|}\right)^{p-1}}\,dt \\ &\geq \int_0^\infty \frac{\mathscr{H}^{n-1}(\{f=t\})^p}{(-\mu'(t))^{p-1}}\,dt = \int_0^\infty \frac{P(\{f>t\})^p}{(-\mu'(t))^{p-1}}\,dt\,.\end{aligned} \tag{3.13}$$

The quantitative isoperimetric inequality of [FMP1] tells us that a constant κ_0, depending only on n, exists such that

$$n\omega_n|E|^{1/n'}\left[1+\kappa_0\left(\inf\left\{\frac{|E\Delta B|}{|E|} : B \text{ ball}, |B|=|E|\right\}\right)^2\right] \leq P(E) \tag{3.14}$$

for every measurable subset of $\mathbb{R}^n$ having finite measure and perimeter. If, in addition, E is symmetric about n orthogonal hyperplanes containing 0, then

$$\inf\left\{\frac{|E\Delta B|}{|E|} : B \text{ ball}, |B|=|E|\right\} \geq \frac{1}{2^n}\frac{|E\Delta E^\star|}{|E|}\,, \tag{3.15}$$

where $E^\star$ denotes the ball, centered at 0, and such that $|E^\star|=|E|$ (see [FMP1, Lemma 2.2]). Since f is n-symmetric, so are its level sets $\{f>t\}$ for $t>0$; moreover, the ball $\{f>t\}^\star$ agrees with $\{f^\star>t\}$ for every $t>0$. Consequently, from (3.13) and from (3.14) and (3.15) applied with $E=\{f>t\}$, we deduce that

$$\int_{\mathbb{R}^n}|\nabla f|^p \geq \left(n\omega_n^{1/n}\right)^p \int_0^\infty \frac{\mu(t)^{p/n'}}{(-\mu'(t))^{p-1}}\left(1+\frac{\kappa_0}{4^n}\left(\frac{|F_t|}{\mu(t)}\right)^2\right)^p dt\,, \tag{3.16}$$

where we have set $F_t=\{f>t\}\Delta\{f^\star>t\}$ for $t>0$. Now, note that when (3.13) is applied with f replaced by $f^\star$, equality holds in the first equality because $|\nabla f^\star|$ is constant on $\{f^\star=t\}$ for a.e. $t>0$, and equality holds in the second inequality since also (3.12) turns into an equality in this case (see e.g. [CF1, Lemma 3.2]). Thus, inasmuch as $P(\{f^\star>t\})=n\omega_n^{1/n}\mu(t)^{1/n'}$ for a.e. $t>0$, one has

$$\int_{\mathbb{R}^n}|\nabla f^\star|^p = \left(n\omega_n^{1/n}\right)^p\int_0^\infty \frac{\mu(t)^{p/n'}}{(-\mu'(t))^{p-1}}\,dt\,. \tag{3.17}$$

Since $(1+s)^p \geq 1+ps$ for $s\geq 0$, we infer from (3.16) and (3.17) that

$$\int_{\mathbb{R}^n}|\nabla f|^p - \int_{\mathbb{R}^n}|\nabla f^\star|^p \geq \kappa\int_0^\infty \left(\frac{|F_t|}{\mu(t)}\right)^2 \frac{\mu(t)^{p/n'}}{(-\mu'(t))^{p-1}}\,dt \tag{3.18}$$

for some positive constant κ. By Lemma 5,

$$\int_{\mathbb{R}^n}|f-f^\star|^{p^*} \leq p^*\int_0^\infty |F_t|t^{p^*-1}\,dt\,. \tag{3.19}$$

The point is thus to estimate the right-hand side of (3.19) in terms of the right hand side of (3.18). We have

$$1=\int_{\mathbb{R}^n}f^{p^*} \geq \int_{\{f>t\}}f^{p^*} \geq t^{p^*}\mu(t) \qquad \text{for } t>0\,,$$

whence $\mu(t)^{p/n}t^{p^*-p} \le 1$ for every $t > 0$. Thus, by (3.19) and by Hölder inequality,

$$\begin{aligned}\int_{\mathbb{R}^n} |f - f^\star|^{p^*} &\le p^* \int_0^\infty \frac{|F_t|}{\mu(t)}\, \mu(t)^{1-p/n} t^{p-1}\, dt \\ &\le p^* \left(\int_0^\infty \left(\frac{|F_t|}{\mu(t)}\right)^p \frac{\mu(t)^{p/n'}}{(-\mu'(t))^{p-1}}\, dt \right)^{1/p} \left(\int_0^\infty \frac{-\mu'(t)}{\mu(t)^{p/n}} t^p\, dt \right)^{1/p'} .\end{aligned} \tag{3.20}$$

We claim that a constant C exists such that

$$\int_0^\infty \frac{-\mu'(t)}{\mu(t)^{p/n}}\, t^p\, dt \le C \int_{\mathbb{R}^n} |\nabla f^\star|^p . \tag{3.21}$$

To verify (3.21), fix any $\vartheta \in (1/p', 1/n')$. Then,

$$\begin{aligned} t^p = \left(\int_0^t ds \right)^p &\le \left(\int_0^t \frac{(-\mu'(s))}{\mu(s)^{\vartheta p'}}\, ds \right)^{p-1} \left(\int_0^t \frac{\mu(s)^{\vartheta p}}{(-\mu'(s))^{p-1}}\, ds \right) \\ &\le \frac{1}{\big((\vartheta p' - 1)\mu(t)^{\vartheta p' - 1}\big)^{p-1}} \int_0^t \frac{\mu(s)^{\vartheta p}}{(-\mu'(s))^{p-1}}\, ds ,\end{aligned}$$

by Hölder inequality. Therefore,

$$\begin{aligned}\int_0^\infty \frac{(-\mu'(t))t^p}{\mu(t)^{p/n}}\, dt &\le \frac{1}{(\vartheta p' - 1)^{p-1}} \int_0^\infty \frac{(-\mu'(t))}{\mu(t)^{\vartheta p + 1 - p/n'}} \left(\int_0^t \frac{\mu(s)^{\vartheta p}}{(-\mu'(s))^{p-1}}\, ds \right) dt \\ &= \frac{1}{(\vartheta p' - 1)^{p-1}} \int_0^\infty \frac{\mu(s)^{\vartheta p}}{(-\mu'(s))^{p-1}} \left(\int_s^\infty \frac{(-\mu'(t))}{\mu(t)^{\vartheta p + 1 - p/n'}}\, dt \right) ds \\ &\le \frac{1}{(\vartheta p' - 1)^{p-1}(p/n' - \vartheta p)} \int_0^\infty \frac{\mu(s)^{p/n'}}{(-\mu'(s))^{p-1}}\, ds .\end{aligned} \tag{3.22}$$

Inequality (3.21) follows from (3.22) and (3.17). Combining (3.20) and (3.21) yields

$$\int_{\mathbb{R}^n} |f - f^\star|^{p^*} \le C \|\nabla f^\star\|_{L^p(\mathbb{R}^n)}^{p-1} \left(\int_0^\infty \left(\frac{|F_t|}{\mu(t)}\right)^p \frac{\mu(t)^{p/n'}}{(-\mu'(t))^{p-1}}\, dt \right)^{1/p} . \tag{3.23}$$

When $1 < p < 2$, by Hölder inequality

$$\int_0^\infty \left(\frac{|F_t|}{\mu(t)}\right)^p \frac{\mu(t)^{p/n'}}{(-\mu'(t))^{p-1}}\, dt \le \left(\int_0^\infty \left(\frac{|F_t|}{\mu(t)}\right)^2 \frac{\mu(t)^{p/n'}}{(-\mu'(t))^{p-1}}\, dt \right)^{p/2} \left(\int_0^\infty \frac{\mu(t)^{p/n'}}{(-\mu'(t))^{p-1}}\, dt \right)^{1-p/2} , \tag{3.24}$$

and (3.3) follows via (3.23), (3.24), (3.17) and (3.18). If, instead, $p \ge 2$, then

$$\left(\frac{|F_t|}{\mu(t)}\right)^p \le 2^{p-2} \left(\frac{|F_t|}{\mu(t)}\right)^2 \qquad \text{for } t > 0 , \tag{3.25}$$

and (3.3) follows from (3.23), (3.25) and (3.18). □

4. Proof of Theorem 1

The task of the present section is to accomplish the symmetrization process to which we alluded in Section 1, showing that the proof of inequality (1.4) can always be reduced to the special case of n-symmetric functions dealt with in Corollary 4. This is the content of the following result.

Theorem 6. *Let $n \geq 2$ and let $1 < p < n$. Then, a positive constant C exists such that for every nonnegative function $f \in W^{1,p}(\mathbb{R}^n)$ there exists a nonnegative n-symmetric function $\widehat{f}$ such that*

$$\lambda(f) \leq C\lambda(\widehat{f}) \qquad\qquad \delta(\widehat{f}) \leq C\delta(f)^{1/\beta p^*}, \tag{4.1}$$

where β is given by (2.54).

Once Theorem 6 is established, Theorem 1 quite easily follows from Corollary 4.

Proof of Theorem 1. Consider first the case where f is nonnegative. Then, by Theorem 6, a n-symmetric function $\widehat{f}$ exists satisfying (4.1). Inequality (1.4) holds with f replaced by $\widehat{f}$, by Corollary 4. Owing to (4.1), inequality (1.4) continues to hold with $\alpha = \beta^2 p^*$ even for f.
Let us now remove the sign assumption on f. Consider any function $f \in W^{1,p}(\mathbb{R}^n)$, which, without loss of generality, can be assumed to satisfy $\|f\|_{L^{p^*}(\mathbb{R}^n)} = 1$ and $\delta(f) \leq 1$. We claim that a constant C exists such that

$$\min\left\{\int_{\{f<0\}} |f|^{p^*}, \int_{\{f>0\}} |f|^{p^*}\right\} \leq C\delta(f). \tag{4.2}$$

Actually, the Sobolev inequality (1.1) applied to $\max\{f,0\}$ and $\min\{f,0\}$ yields

$$S(p,n)^p \left(\int_{\{f\gtrless 0\}} |f|^{p^*}\right)^{p/p^*} \leq \int_{\{f\gtrless 0\}} |\nabla f|^p,$$

whence

$$\left(\int_{\{f>0\}} |f|^{p^*}\right)^{p/p^*} + \left(\int_{\{f<0\}} |f|^{p^*}\right)^{p/p^*} \leq \frac{1}{S(p,n)^p}\int_{\mathbb{R}^n} |\nabla f|^p = (1+\delta(f))^p. \tag{4.3}$$

Since the function $s \mapsto (s^{p/p^*} + (1-s)^{p/p^*})^{1/p} - 1$ is concave in $[0,1]$, a constant κ exists such that

$$\left(s^{p/p^*} + (1-s)^{p/p^*}\right)^{1/p} - 1 \geq \kappa \min\{s, 1-s\}.$$

Thus, inasmuch as $\int_{\{f<0\}} |f|^{p^*} = 1 - \int_{\{f>0\}} |f|^{p^*}$, one can infer from (4.3) that

$$\delta(f) \geq \left[\left(\int_{\{f>0\}} |f|^{p^*}\right)^{p/p^*} + \left(\int_{\{f<0\}} |f|^{p^*}\right)^{p/p^*}\right]^{1/p} - 1 \geq \kappa \min\left\{\int_{\{f<0\}} |f|^{p^*}, \int_{\{f>0\}} |f|^{p^*}\right\},$$

namely, (4.2). Now, to fix the ideas, assume that the minimum in (4.2) agrees with $\int_{\{f<0\}} |f|^{p^*}$, the other case being completely analogous. By applying (1.4) to $|f|$ and observing that $\delta(f) = \delta(|f|)$, from (4.2) we have

$$\lambda(f) \leq 2^{p^*-1}\left(\lambda(|f|) + \int_{\mathbb{R}^n} \big|f - |f|\big|^{p^*}\right) \leq C\left(\delta(f)^{1/\alpha} + \delta(f)\right) \leq C\delta(f)^{1/\alpha},$$

for a suitable constant C. Hence, the result easily follows. □

The remaining part of the paper is devoted to the proof of Theorem 6. The argument relies upon some delicate constructions, and is split in separate lemmas. We begin by showing that, when dealing with functions f which are symmetric about orthogonal hyperplanes intersecting in some lower dimensional affine space S, the quantity $\lambda(f)$ can be essentially replaced by the expression

$$\lambda(f|S) = \inf\left\{ \frac{\|f - g_{a,b,x_0}\|^{p^*}_{L^{p^*}(\mathbb{R}^n)}}{\|f\|^{p^*}_{L^{p^*}(\mathbb{R}^n)}} : \|g_{a,b,x_0}\|_{L^{p^*}(\mathbb{R}^n)} = \|f\|_{L^{p^*}(\mathbb{R}^n)},\, a \in \mathbb{R}, b > 0, x_0 \in S \right\}.$$

Lemma 7. *Let $n \geq 2$, $1 < p < n$ and f be a nonnegative function from $W^{1,p}(\mathbb{R}^n)$. Assume that f is $k-$symmetric, and let S be the intersection of the k hyperplanes of symmetry. Then*

$$\lambda(f|S) \leq 3^{p^*} \lambda(f)\,.$$

The proof of Lemma 7 in turn relies upon the technical results contained in Lemmas 8–10 below.

Lemma 8. *Let $\varphi : \mathbb{R} \to [0, +\infty)$ be increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Define $\Phi : [0, \infty) \to [0, \infty]$ as*

$$\Phi(h) = \int_{\mathbb{R}} A(|\varphi(t) - \varphi(t-h)|)\, dt\,,$$

where $A : [0, \infty) \to [0, \infty)$ is a l.s.c. increasing function. Then Φ is decreasing.

Proof. First of all, we may assume that A is continuous, since any l.s.c. increasing function can be approximated pointwise by an increasing sequence of continuous increasing functions. Then, using that for any $M > 0$ and any $s, t \in \mathbb{R}$, $|\min\{t, M\} - \min\{s, M\}| \leq |t - s|$, we may reduce to the case when φ is bounded. A simple approximation argument then shows that we may also assume that there exist $l_1 < 0 < l_2$ such that φ is constant both in $(-\infty, l_1]$ and in $[l_2, +\infty)$, and that φ is continuous. The function Φ is trivially affine and increasing in $[l_2 - l_1, +\infty)$, thus we may focus on the interval $[0, l_2 - l_1]$. For every $h_1 \in (0, l_2 - l_1)$, there exists $t_1 \in (0, h_1)$ satisfying $\varphi(t_1) = \varphi(t_1 - h_1)$; moreover, $\varphi(t) \geq \varphi(t - h_1)$ if $t < t_1$ and $\varphi(t) \leq \varphi(t - h_1)$ if $t > t_1$. Let $h_2 \in (h_1, l_2 - l_1]$. On the one hand,

$$\int_{-\infty}^{t_1} A(|\varphi(t) - \varphi(t-h_1)|)\, dt = \int_{-\infty}^{t_1} A(\varphi(t) - \varphi(t-h_1))\, dt \leq \int_{-\infty}^{t_1} A(|\varphi(t) - \varphi(t-h_2)|)\, dt\,,$$

since $\varphi(t - h_2) \leq \varphi(t - h_1) \leq \varphi(t)$ whenever $t \leq t_1$. On the other hand,

$$\begin{aligned}
\int_{t_1}^{\infty} A(|\varphi(t) - \varphi(t-h_1)|)\, dt &= \int_{t_1}^{\infty} A(\varphi(t-h_1) - \varphi(t))\, dt \\
&= \int_{t_1+h_2-h_1}^{\infty} A\big(\varphi(s-h_2) - \varphi(s+h_1-h_2)\big)\, ds \\
&\leq \int_{t_1+h_2-h_1}^{\infty} A\big(|\varphi(s-h_2) - \varphi(s)|\big)\, ds\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{\mathbb{R}} A(|\varphi(t) - \varphi(t-h_1)|)\, dt &\leq \int_{(-\infty,t_1)\cup(t_1+h_2-h_1,\infty)} A(|\varphi(t) - \varphi(t-h_2)|)\, dt \\
&\leq \int_{\mathbb{R}} A(|\varphi(t) - \varphi(t-h_2)|)\, dt
\end{aligned}$$

and the conclusion follows. □

Lemma 9. *Let $f:\mathbb{R}^n\to[0,\infty)$ be any spherically symmetric function. Given any $y\in\mathbb{R}^n$, define $f_y:\mathbb{R}^n\to[0,\infty)$ as*

$$f_y(x)=f(x-y) \qquad\qquad \text{for } x\in\mathbb{R}^n\,.$$

If A is as in Lemma 8, then

$$\int_{\mathbb{R}^n} A(|f_y(x)-f_w(x)|)\,dx\le\int_{\mathbb{R}^n} A(|f_y(x)-f_z(x)|)\,dx \tag{4.4}$$

for every $y,z\in\mathbb{R}^n$ and for every w lying on the segment joining y and z.

Proof. Without loss of generality, we may assume $y=0$ in (4.4); then, set $\nu=z/|z|$, whence $z=|z|\nu$ and $w=|w|\nu$. Denote by H the hyperplane orthogonal to ν and containing 0. Then

$$\begin{aligned}\int_{\mathbb{R}^n} A(|f(x)-f_w(x)|)\,dx&=\int_H\int_{\mathbb{R}} A(|f(x+t\nu)-f(x-w+t\nu)|)\,dt\,d\mathscr{H}^{n-1}(x)\\&=\int_H\int_{\mathbb{R}} A(|f(x+t\nu)-f(x+(t-|w|)\nu)|)\,dt\,d\mathscr{H}^{n-1}(x)\,.\end{aligned} \tag{4.5}$$

Fix any $x\in H$, and define $\varphi:\mathbb{R}\to[0,\infty)$ by

$$\varphi(t)=f(x+t\nu) \qquad\qquad \text{for } t\in\mathbb{R}\,.$$

Clearly, the function φ satisfies the assumptions of Lemma 8. Hence,

$$\int_{\mathbb{R}} A(|\varphi(t)-\varphi(t-h_1)|)\,dt\le\int_{\mathbb{R}} A(|\varphi(t)-\varphi(t-h_2)|)\,dt \qquad \text{if } 0<h_1\le h_2\,. \tag{4.6}$$

On applying (4.6) with $h_1=|w|$ and $h_2=|z|$ we get

$$\int_{\mathbb{R}} A\Big(\big|f(x+t\nu)-f(x+(t-|w|)\nu)\big|\Big)\,dt\le\int_{\mathbb{R}} A\Big(\big|f(x+t\nu)-f(x+(t-|z|)\nu)\big|\Big)\,dt\,.$$

Combining this inequality with (4.5) yields the conclusion. □

We want now to prove that, when f is positive, then the infima defining $\lambda(f)$ and $\lambda(f|S)$ are attained; this proof closely reminds the proof of Lemma B.1 in [FMP2], and we will obtain it in two steps.

Lemma 10. *Let $1<p<n$ and let f be any nonnegative function from $L^{p^*}(\mathbb{R}^n)$. Then $\lambda(f)$ is a minimum. The same holds true for $\lambda(f|S)$ with any affine subspace S of $\mathbb{R}^n$.*

Proof. We only give the proof for $\lambda(f)$, the other case being analogous; we also assume without loss of generality that $\|f\|_{L^{p^*}(\mathbb{R}^n)}=1$.

The proof is divided in two steps; notice that the sign assumption on f plays a role only in Step II.

Step I. *If $\lambda(f)<2$, then $\lambda(f)$ is a minimum.*
Let us consider a minimizing sequence for $\lambda(f)$, given by the functions

$$g_h(x)=\frac{a_h}{\big(1+b_h|x-x_h|^{p'}\big)^{(n-p)/p}}\,.$$

Up to a subsequence, we may assume that b_h converges to $b\in[0,\infty]$: our first goal is to check that $b\neq 0$, $b\neq\infty$. Indeed, chosen any $\varepsilon>0$ there is a positive constant $\rho=\rho(\varepsilon)$ converging to 0 for $\varepsilon\to 0$ such that for any $z\in\mathbb{R}^n$ one has

$$\int_{B(z,\varepsilon)}|f|^{p^*}\le\rho\,, \qquad\qquad \int_{B(0,1/\varepsilon)}|f|^{p^*}\ge 1-\rho\,.$$

Assume, by contradiction, that $b = +\infty$. Then,

$$\int_{\{|x-x_h|>\varepsilon\}} |g_h(x)|^{p^*}\,dx \le \varepsilon$$

provided that h is large enough depending on ε. Thus, for any such h,

$$\begin{aligned}\|f-g_h\|^{p^*}_{L^{p^*}(\mathbb{R}^n)} &= \|f-g_h\|^{p^*}_{L^{p^*}(\{|x-x_h|>\varepsilon\})} + \|g_h - f\|^{p^*}_{L^{p^*}(\{|x-x_h|\le\varepsilon\})}\\ &\ge \Big((1-\rho)^{1/p^*} - \varepsilon^{1/p^*}\Big)^{p^*} + \Big((1-\varepsilon)^{1/p^*} - \rho^{1/p^*}\Big)^{p^*}.\end{aligned} \tag{4.7}$$

Passing to the limit as $h \to \infty$ in (4.7) we would get

$$\lambda(f) \ge \Big((1-\rho)^{1/p^*} - \varepsilon^{1/p^*}\Big)^{p^*} + \Big((1-\varepsilon)^{1/p^*} - \rho^{1/p^*}\Big)^{p^*}.$$

Hence, taking a limit for $\varepsilon \to 0$, we get that $\lambda(f) \ge 2$, which contradicts the assumption. Next suppose, again by contradiction, that $b = 0$. Then,

$$\int_{\{|x|<1/\varepsilon\}} |g_h(x)|^{p^*}\,dx \le \int_{\{|x-x_h|<1/\varepsilon\}} |g_h(x)|^{p^*}\,dx \le \varepsilon$$

if h is large enough, depending on ε. Analogously to (4.7), we have

$$\begin{aligned}\|f-g_h\|^{p^*}_{L^{p^*}(\mathbb{R}^n)} &= \|f-g_h\|^{p^*}_{L^{p^*}(\{|x|<1/\varepsilon\})} + \|g_h - f\|^{p^*}_{L^{p^*}(\{|x|\ge1/\varepsilon\})}\\ &\ge \Big((1-\rho)^{1/p^*} - \varepsilon^{1/p^*}\Big)^{p^*} + \Big((1-\varepsilon)^{1/p^*} - \rho^{1/p^*}\Big)^{p^*},\end{aligned}$$

and we reach the same contradiction as above.

Now, since $b_h \to b \in (0,+\infty)$, and $\|g_h\|_{L^{p^*}(\mathbb{R}^n)} = 1$ for every h, we have that $a_h \to a$ for some $a \in \mathbb{R}\setminus\{0\}$. Let us now show that, again up to a subsequence, there exists $\bar x \in \mathbb{R}^n$ such that $x_h \to \bar x$. In order to prove this fact, it suffices to exclude that $|x_h| \to \infty$. We argue by contradiction again and observe that, if this is the case, then for every $L > 0$

$$\int_{\{|x-x_h|\le L\}} |f(x)|^{p^*}\,dx \le \frac{1}{L}$$

if h is large enough depending on L; fixed any $\varepsilon > 0$, since $b \in (0,\infty)$ we can choose L so large that

$$\int_{\{|x-x_h|\le L\}} |g_h(x)|^{p^*}\,dx \ge 1-\varepsilon$$

for every h. Therefore, similarly to (4.7), we deduce that

$$\|f-g_h\|^{p^*}_{L^{p^*}(\mathbb{R}^n)} \ge \Big((1-\varepsilon)^{1/p^*} - \frac{1}{L^{1/p^*}}\Big)^{p^*} + \Big(\Big(1-\frac{1}{L}\Big)^{1/p^*} - \varepsilon^{1/p^*}\Big)^{p^*},$$

whence we get the contradiction $\lambda(f) \ge 2$ on letting ε go to 0 and thus L to ∞. Since g_h converges to $g_{a,b,\bar x}$ in $L^{p^*}(\mathbb{R}^n)$, the latter function is a minimizer in the definition of $\lambda(f)$.

Step II. $\lambda(f) < 2$.

Set $g = g_{a,1,0}$, where a is the positive number such that $\|g\|_{L^{p^*}(\mathbb{R}^n)} = 1$. Set $F = \{f < g\}$ and $G = \{g < f\}$. Then

$$|f-g| = g - f \le g \quad \text{in } F, \qquad\qquad |f-g| = f-g < f \quad \text{in } G.$$

Note that strict inequality above holds since g is strictly positive. Thus

$$\lambda(f) \le \int_{\mathbb{R}^n} |f-g|^{p^*} = \int_F |f-g|^{p^*} + \int_G |f-g|^{p^*} < \int_F g^{p^*} + \int_G f^{p^*} < \int_{\mathbb{R}^n} g^{p^*} + \int_{\mathbb{R}^n} f^{p^*} = 2,$$

and the assertion follows. □

We are now finally ready to prove Lemma 7.

Proof of Lemma 7. We may assume, without loss of generality, that $\|f\|_{L^{p^*}(\mathbb{R}^n)} = 1$. Let a, b, x_0, according to Lemma 10, be such that $\lambda(f) = \|f - g_{a,b,x_0}\|^{p^*}_{L^{p^*}(\mathbb{R}^n)}$. Denote by z_0 be the orthogonal projection of x_0 on S, and let y_0 be the symmetral of x_0 about S. We have

$$\begin{aligned}\lambda(f|S)^{1/p^*} &\le \|f - g_{a,b,z_0}\|_{L^{p^*}(\mathbb{R}^n)} \le \|f - g_{a,b,x_0}\|_{L^{p^*}(\mathbb{R}^n)} + \|g_{a,b,x_0} - g_{a,b,z_0}\|_{L^{p^*}(\mathbb{R}^n)} \\ &= \lambda(f)^{1/p^*} + \|g_{a,b,x_0} - g_{a,b,z_0}\|_{L^{p^*}(\mathbb{R}^n)} .\end{aligned}$$

By Lemma 9, $\|g_{a,b,x_0} - g_{a,b,z_0}\|_{L^{p^*}(\mathbb{R}^n)} \le \|g_{a,b,x_0} - g_{a,b,y_0}\|_{L^{p^*}(\mathbb{R}^n)}$. On the other hand, the symmetries of f entail that $\lambda(f) = \|f - g_{a,b,y_0}\|_{L^{p^*}(\mathbb{R}^n)}$. Hence,

$$\|g_{a,b,x_0} - g_{a,b,y_0}\|_{L^{p^*}(\mathbb{R}^n)} \le \|f - g_{a,b,y_0}\|_{L^{p^*}(\mathbb{R}^n)} + \|g_{a,b,x_0} - f\|_{L^{p^*}(\mathbb{R}^n)} = 2\lambda(f)^{1/p^*} .$$

Therefore $\lambda(f|S)^{1/p^*} \le 3\lambda(f)^{1/p^*}$. □

The contribution of Lemma 11 below is in the same direction as Lemma 7, and provides an estimate for $\lambda(f|H)$ in terms of $\lambda(f)$, when H is a hyperplane splitting f in two functions having the same L^{p^*} norm. In what follows, we denote by H^+ and H^- the two halfspaces into which $\mathbb{R}^n$ is split by H. Moreover we denote by $T_H : \mathbb{R}^n \to \mathbb{R}^n$ the map which associates to any $x \in \mathbb{R}^n$ the point $T_H(x)$ obtained by reflecting x about H.

Lemma 11. *Let f be any nonnegative function from $W^{1,p}(\mathbb{R}^n)$, and let H be any hyperplane such that*

$$\int_{H^+} f^{p^*} = \int_{H^-} f^{p^*} = \frac{1}{2}\int_{\mathbb{R}^n} f^{p^*} .$$

Then a constant C exists such that

$$\lambda(f|H) \le C\lambda(f)^{1/p^*} \tag{4.8}$$

and

$$\int_{\mathbb{R}^n} |f \circ T_H - f|^{p^*} \le C\|f\|^{p^*}_{L^{p^*}(\mathbb{R}^n)}\lambda(f)^{1/p^*} . \tag{4.9}$$

Proof. We may assume, without loss of generality, that $\|f\|_{L^{p^*}(\mathbb{R}^n)} = 1$. By Lemma 10, let a, b, x_0 be such that $\lambda(f) = \|f - g_{a,b,x_0}\|^{p^*}_{L^{p^*}(\mathbb{R}^n)}$, and denote g_{a,b,x_0} simply by g_0. Call $\overline{x}$ the projection of x_0 on H, and set $\overline{g} = g_{a,b,\overline{x}}$. Then,

$$\lambda(f|H) \le \int_{\mathbb{R}^n} |f - \overline{g}|^{p^*} \le 2^{p^*-1}\left\{\lambda(f) + \int_{\mathbb{R}^n} |g_0 - \overline{g}|^{p^*}\right\} . \tag{4.10}$$

Let us now consider the half-spaces $K^\pm = (x_0 - \overline{x}) + H^\pm$. Clearly

$$\frac{1}{2} = \int_{K^\pm} g_0^{p^*} = \int_{H^\pm} f^{p^*} = \int_{H^\pm} \overline{g}^{p^*} .$$

On interchanging K^+ with K^-, if necessary, we may also assume that $K^+ \subseteq H^+$ and $H^- \subseteq K^-$. Thus

$$\int_{H^-} |g_0 - \overline{g}|^{p^*} = \int_{K^+} |g_0 - \overline{g}|^{p^*} \le \int_{H^+} |g_0 - \overline{g}|^{p^*} ,$$

whence

$$\int_{\mathbb{R}^n} |g_0 - \overline{g}|^{p^*} \le 2\int_{H^+} |g_0 - \overline{g}|^{p^*} . \tag{4.11}$$

One has $g_0(x) \geq \overline{g}(x)$ for $x \in K^+$, and hence $|g_0(x) - \overline{g}(x)|^{p^*} \leq g_0(x)^{p^*} - \overline{g}(x)^{p^*}$ for the same values of x. Thus,

$$\begin{aligned}\int_{K^+} |g_0 - \overline{g}|^{p^*} &\leq \int_{K^+} g_0^{p^*} - \int_{K^+} \overline{g}^{p^*} = \frac{1}{2} - \int_{H^-} g_0^{p^*} = \int_{H^-} f^{p^*} - \int_{H^-} g_0^{p^*} \\ &\leq C\left(\|f\|_{L^{p^*}(H^-)} - \|g_0\|_{L^{p^*}(H^-)}\right) \leq C\|f - g_0\|_{L^{p^*}(H^-)} \leq C\lambda(f)^{1/p^*},\end{aligned} \tag{4.12}$$

for some positive constant C. Note that we have made use of the fact that $\int_{K^+} \overline{g}^{p^*} = \int_{H^-} g_0^{p^*}$, by symmetry. On the other hand, by symmetry again,

$$\int_{H^+\setminus K^+} |g_0 - \overline{g}|^{p^*} \leq 2^{p^*-1} \int_{H^+\setminus K^+} (g_0^{p^*} + \overline{g}^{p^*}) = 2^{p^*} \int_{H^+\setminus K^+} g_0^{p^*}.$$

We have

$$\int_{H^+\setminus K^+} g_0^{p^*} = \int_{H^+} g_0^{p^*} - \frac{1}{2} = \int_{H^+} g_0^{p^*} - \int_{H^+} f^{p^*},$$

whence, similarly as in (4.12), one gets

$$\int_{H^+\setminus K^+} g_0^{p^*} \leq C\lambda(f)^{1/p^*}. \tag{4.13}$$

Thus, (4.8) follows from (4.10), (4.11), (4.12) and (4.13). As far as (4.9) is concerned, if a, b and $\hat{x} \in H$ are chosen, thanks to Lemma 10, in such a way that

$$\lambda(f|H) = \|f - g_{a,b,\hat{x}}\|_{L^{p^*}(\mathbb{R}^n)},$$

then, by (4.8),

$$\begin{aligned}\int_{H^\pm} |f \circ T_H - f|^{p^*} &\leq 2^{p^*-1}\left(\int_{H^+} |f \circ T_H - g_{a,b,\hat{x}}|^{p^*} + \int_{H^+} |f - g_{a,b,\hat{x}}|^{p^*}\right) \\ &= 2^{p^*-1} \int_{\mathbb{R}^n} |f - g_{a,b,\hat{x}}|^{p^*} = 2^{p^*-1}\lambda(f|H) \leq C\lambda(f)^{1/p^*}.\end{aligned}$$

□

Our next result can be regarded as a qualitative version of Theorem 1, and enables us to restrict our attention to the case where $\lambda(f)$ does not exceed some arbitrarily prescribed constant depending only on p and n.

Lemma 12. *Let $n \geq 2$ and let $1 < p < n$. For every $\varepsilon > 0$ there exists $\bar{\delta} > 0$ such that if $f \in W^{1,p}(\mathbb{R}^n)$, and $\delta(f) \leq \bar{\delta}$ then $\lambda(f) \leq \varepsilon$.*

Proof. Assume, by contradiction, that a sequence $\{f_h\} \subseteq W^{1,p}(\mathbb{R}^n)$ exists such that $\lim_{h\to\infty} \delta(f_h) = 0$ but $\lim_{h\to\infty} \lambda(f_h) > 0$. On normalizing, if necessary, we may assume that $\|f_h\|_{L^{p^*}(\mathbb{R}^n)} = 1$ for every $h \in \mathbb{N}$. Since $\lim_{h\to\infty} \|\nabla f_h\|_{L^{p^*}(\mathbb{R}^n)} = S(p,n)$, the concentration-compactness method of Lions ([Li]) can be applied (as in [St]) to show that there exists a subsequence of rescaled-translated functions $\tilde{f}_h(x) = r_h^{n/p^*} f_h\big(r_h(x - x_h)\big)$ such that $\tilde{f}_h \to f$ strongly in $L^{p^*}(\mathbb{R}^n)$ for some $f \in W^{1,p}(\mathbb{R}^n)$. Notice that $\lambda(\tilde{f}_h) = \lambda(f_h)$, $\delta(\tilde{f}_h) = \delta(f_h)$ and the functional $f \mapsto \lambda(f)$ is strongly continuous in $L^{p^*}(\mathbb{R}^n)$. Hence, $\lambda(f) = \lim_{h\to\infty} \lambda(f_h) > 0$. On the other hand, by lower semicontinuity, $0 = \lim_{h\to\infty} \delta(f_h) \geq \delta(f)$, namely $\delta(f) = 0$. Consequently, since the functions in (1.2) are the only optimal functions in (1.1) as proved in [CNV], one obtains $\lambda(f) = 0$, a contradiction. □

We are now at the core of the proof of Theorem 6, which basically consists of two steps. The first one amounts to an application of suitable reflections to the original function f, and results in a $(n-1)$-symmetric function $\widetilde{f}$ having the property that $\lambda(f)$ and $\delta(f)$ are controlled from above and from below, respectively, in terms of $\lambda(\widetilde{f})$ and $\delta(\widetilde{f})$. This will be accomplished in Lemma 13. In the second and final step, the passage from the $(n-1)$-symmetric function $\widetilde{f}$ to a n-symmetric function $\widehat{f}$ is performed. This requires a more delicate double reflection argument, and also involves the conclusion of Corollary 4. Its use, which seems indispensable at this stage, explains the presence of the exponent $1/\beta p^*$ in estimate (4.1).

Lemma 13. *Let $n \geq 2$ and let $1 < p < n$. Then a positive constant C exists having the following property. For every nonnegative function $f \in W^{1,p}(\mathbb{R}^n)$ there exists a $(n-1)$-symmetric function $\widetilde{f} \in W^{1,p}(\mathbb{R}^n)$ such that*

$$\lambda(f) \leq C\lambda(\widetilde{f}), \qquad \delta(\widetilde{f}) \leq 2^{n-1}\delta(f). \tag{4.14}$$

Proof. As usual, we may assume that $\|f\|_{L^{p^*}(\mathbb{R}^n)} = 1$. Moreover we may suppose that $\delta(f) \leq \bar{\delta}$ for some positive constant $\bar{\delta}$ (depending only on p and n) to be chosen later. Indeed, if $\delta(f) \geq \bar{\delta}$, one can pick a spherically symmetric function g, independent of f, such that $0 < \delta(g) \leq 2^{n-1}\bar{\delta}$. Thus $\delta(g) \leq 2^{n-1}\delta(f)$, and $\lambda(f) \leq 2^{p^*} \leq (2^{p^*}/\lambda(g))\lambda(g) \leq C\lambda(g)$, and hence the first inequality in (4.14) is fulfilled with $\widetilde{f} = g$.
If $\delta(f) \leq \bar{\delta}$, fixed any coordinate direction e_k, with $1 \leq k \leq n$, consider a hyperplane H_k orthogonal to e_k and the corresponding half-spaces H_k^+ and H_k^-, having the property that

$$\int_{H_k^+} f^{p^*} = \int_{H_k^-} f^{p^*} = \frac{1}{2}\,.$$

Denote, for simplicity, by $T_k : \mathbb{R}^n \to \mathbb{R}^n$ the reflection T_{H_k} about H_k, and define

$$f_k^+(x) = \begin{cases} f(x) & \text{if } x \in H_k^+\,, \\ f(T_k(x)) & \text{if } x \in H_k^-\,; \end{cases} \qquad f_k^-(x) = \begin{cases} f(T_k(x)) & \text{if } x \in H_k^+\,. \\ f(x) & \text{if } x \in H_k^-\,. \end{cases} \tag{4.15}$$

Clearly, $f_k^\pm$ are nonnegative functions from $W^{1,p}(\mathbb{R}^n)$, symmetric about H_k, and satisfying $\|f_k^\pm\|_{L^{p^*}(\mathbb{R}^n)} = 1$. Moreover,

$$\begin{aligned}\|\nabla f\|_{L^p(\mathbb{R}^n)} &= \left(\int_{H_k^+} |\nabla f_k^+|^p + \int_{H_k^-} |\nabla f_k^-|^p\right)^{1/p} = \left(\frac{1}{2}\int_{\mathbb{R}^n} |\nabla f_k^+|^p + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla f_k^-|^p\right)^{1/p} \\ &\geq \frac{1}{2}\left(\|\nabla f_k^+\|_{L^p(\mathbb{R}^n)} + \|\nabla f_k^-\|_{L^p(\mathbb{R}^n)}\right)\,.\end{aligned}$$

In particular,

$$\max\left\{\delta(f_k^+), \delta(f_k^-)\right\} \leq 2\delta(f)\,. \tag{4.16}$$

Denote by g_k^+ and g_k^- two functions realizing the minima in $\lambda(f_k^\pm|H_k)$, again thanks to Lemma 10. Then

$$\begin{aligned}\lambda(f) &\leq \int_{\mathbb{R}^n} |f - g_k^+|^{p^*} = \int_{H_k^+} |f_k^+ - g_k^+|^{p^*} + \int_{H_k^-} |f_k^- - g_k^+|^{p^*} \\ &\leq 2^{p^*-1}\left(\frac{\lambda(f_k^+|H_k) + \lambda(f_k^-|H_k)}{2} + \int_{H_k^-} |g_k^+ - g_k^-|^{p^*}\right) \\ &\leq 2^{p^*-2}3^{p^*}\left(\lambda(f_k^+) + \lambda(f_k^-) + \int_{H_k^-} |g_k^+ - g_k^-|^{p^*}\right)\,.\end{aligned} \tag{4.17}$$

Note that, in the last inequality, we have applied Lemma 7 to $f_k^\pm$. Now, we claim that positive constants C and $\bar\delta$ exist having the following property: whenever $\delta(f) \le \bar\delta$ and $1 \le i < j \le n$, there exists $k \in \{i, j\}$ such that

$$\int_{H_k^-} |g_k^+ - g_k^-|^{p^*} \le C \left(\int_{H_k^+} |f_k^+ - g_k^+|^{p^*} + \int_{H_k^-} |f_k^- - g_k^-|^{p^*} \right) . \tag{4.18}$$

Observe that once this claim is established, (4.14) will follow. Indeed, take $i = 1$ and $j = 2$ and suppose (up to relabelling the indices) that (4.18) holds with $k = 1$. Then from (4.17) and (4.16) applied with $k = 1$, we infer that

$$\lambda(f) \le C'(\lambda(f_1^+) + \lambda(f_1^-)) , \qquad \max\{\delta(f_1^+), \delta(f_1^-)\} \le 2\delta(f)$$

for some constant C'. In particular, at least one of the functions $f_1^\pm$, denote it by f_1, satisfies $\lambda(f) \le 2C'\lambda(f_1)$ and $\delta(f_1) \le 2\delta(f)$. Moreover, f_1 is 1-symmetric and satisfies $\|f_1\|_{L^{p^*}(\mathbb{R}^n)} = 1$. Then, one can repeat the argument starting from f_1, and obtain a 2-symmetric function f_2 fulfilling $\lambda(f) \le 4C'^2\lambda(f_2)$ and $\delta(f_2) \le 4\delta(f)$. On iterating this procedure, (4.14) follows.

We have now to prove our claim. The crucial observation is that, when δ is sufficiently small, all the functions $g_k^\pm$, $1 \le k \le n$, are close to each other in the L^{p^*} norm, in the sense that a constant C_0 exists such that

$$\int_{\mathbb{R}^n} |g_i^\sigma - g_j^\tau|^{p^*} \le C_0 \int_{H_i^\sigma \cap H_j^\tau} |g_i^\sigma - g_j^\tau|^{p^*} \quad \text{for every } 1 \le i < j \le n,\ \sigma, \tau \in \{+, -\} . \tag{4.19}$$

To verify (4.19), let us begin by noting that constants ϱ and C_1 exist such that if

(i) $\displaystyle\int_{\mathbb{R}^n} g_{a,b,x_0}^{p^*} = \int_{\mathbb{R}^n} g_{c,d,y_0}^{p^*} = 1$,

(ii) I and J are two orthogonal half-spaces with $x_0 \in \partial I$ and $y_0 \in \partial J$,

(iii) $\displaystyle\int_{I\cap J} g_{a,b,x_0}^{p^*} \ge \frac{1}{8}$ and $\displaystyle\int_{I\cap J} g_{c,d,y_0}^{p^*} \ge \frac{1}{8}$,

(iv) $\displaystyle\int_{\mathbb{R}^n} |g_{a,b,x_0} - g_{c,d,y_0}|^{p^*} \le \varrho$,

then

$$\int_{\mathbb{R}^n} |g_{a,b,x_0} - g_{c,d,y_0}|^{p^*} \le C_1 \int_{I\cap J} |g_{a,b,x_0} - g_{c,d,y_0}|^{p^*} .$$

Since (4.19) is a consequence of this assertion with the choice $g_{a,b,x_0} = g_i^\sigma$, $g_{c,d,y_0} = g_j^\tau$, $I = H_i^\sigma$ and $J = H_j^\tau$ we have only to check that (i)-(iv) are fulfilled in the present situation.

Properties (i), (ii) hold by construction. The choice of $\bar\delta$ comes into play in connection with conditions (iii) and (iv). Actually condition (iii) is easily seen to hold provided that $\lambda(f)$ is sufficiently small, and we may suppose that this is the case, thanks to Lemma 12, since we are assuming that $\delta(f) \le \bar\delta$. Inequality (iv) relies upon Lemmas 11 and 12. Indeed,

$$\|g_i^\sigma - g_j^\tau\|_{L^{p^*}(\mathbb{R}^n)} \le \|g_i^\sigma - f_i^\sigma\|_{L^{p^*}(\mathbb{R}^n)} + \|f_i^\sigma - f\|_{L^{p^*}(\mathbb{R}^n)} + \|f - f_j^\tau\|_{L^{p^*}(\mathbb{R}^n)} + \|f_j^\tau - g_j^\tau\|_{L^{p^*}(\mathbb{R}^n)} . \tag{4.20}$$

By (4.9),

$$\int_{\mathbb{R}^n} |f_i^\sigma - f|^{p^*} = \frac{1}{2} \int_{\mathbb{R}^n} |f \circ T_i - f|^{p^*} \le C\lambda(f)^{1/p^*},$$

and hence the second and the third norm on the right-hand side of (4.20) can be made arbitrarily small, owing to Lemma 12, by a suitable choice of $\bar\delta$. The same assertion holds also for the other two norms, inasmuch as

$$\int_{\mathbb{R}^n} |g_i^\sigma - f_i^\sigma|^{p^*} \le 2^{p^*-1}\left(\lambda(f|H_i) + \|f - f_i^\sigma\|_{L^{p^*}(\mathbb{R}^n)}^{p^*}\right) \le C\lambda(f)^{1/p^*},$$

for some constant C, by (4.8). Inequality (4.19) is thus established. It remains only to make use of (4.19) to prove the claim concerning (4.18). To fix ideas, suppose that $i = 1$ and $j = 2$. Set, for $k = 1, 2$,

$$h_k = g_k^+ \chi_{H_k^+} + g_k^- \chi_{H_k^-}.$$

By (4.19)

$$\int_{\mathbb{R}^n} |h_1 - h_2|^{p^*} \ge \int_{H_1^+\cap H_2^+} |h_1 - h_2|^{p^*} = \int_{H_1^+\cap H_2^+} |g_1^+ - g_2^+|^{p^*} \ge \frac{1}{C_0}\int_{\mathbb{R}^n} |g_1^+ - g_2^+|^{p^*}.$$

A similar chain, with $H_1^+ \cap H_2^+$ replaced by $H_1^- \cap H_2^+$ yields

$$\int_{\mathbb{R}^n} |h_1 - h_2|^{p^*} \ge \frac{1}{C_0}\int_{\mathbb{R}^n} |g_1^- - g_2^+|^{p^*}.$$

In conclusion,

$$\int_{\mathbb{R}^n} |g_1^+ - g_1^-|^{p^*} \le 2^{p^*-1} C_0 \int_{\mathbb{R}^n} |h_1 - h_2|^{p^*}. \tag{4.21}$$

An analogous argument tells us that

$$\int_{\mathbb{R}^n} |g_2^+ - g_2^-|^{p^*} \le 2^{p^*-1} C_0 \int_{\mathbb{R}^n} |h_1 - h_2|^{p^*}.$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^n} |h_1 - h_2|^{p^*} &\le 2^{p^*-1}\left(\int_{\mathbb{R}^n} |h_1 - f|^{p^*} + \int_{\mathbb{R}^n} |h_2 - f|^{p^*}\right)\\ &= 2^{p^*-1}\left(\int_{H_1^+} |g_1^+ - f_1^+|^{p^*} + \int_{H_1^-} |g_1^- - f_1^-|^{p^*} + \int_{H_2^+} |g_2^+ - f_2^+|^{p^*} + \int_{H_2^-} |g_2^- - f_2^-|^{p^*}\right).\end{aligned} \tag{4.22}$$

Combining (4.21)–(4.22) ensures that (4.18) holds, for an appropriate constant C, with either $k = 1$ or $k = 2$. ☐

Proof of Theorem 6. We may assume that $\|f\|_{L^{p^*}(\mathbb{R}^n)} = 1$ and, by Lemma 13, that f is $(n-1)$-symmetric. As in the proof of that lemma, we may also suppose that $\delta(f)$ does not exceed a constant $\bar\delta$ to be chosen later. Finally, up to an isometry, we may assume that f is symmetric about the last $(n-1)$-coordinate hyperplanes and that

$$\int_{\{x_1>0\}} f^{p^*} = \frac{1}{2}.$$

Let f^+ and f^- be defined as in (4.15), with $H_1 = \{x_1 = 0\}$; denote by C_0 a (sufficiently large) positive constant to be chosen later. By (4.16),

$$\max\left\{\delta(f^+), \delta(f^-)\right\} \le 2\delta(f).$$

Thus, if either $\lambda(f) \le C_0\lambda(f^+)$, or $\lambda(f) \le C_0\lambda(f^-)$, inequality (4.1) immediately follows, with either $\widehat{f} = f^+$ or $\widehat{f} = f^-$, since both f^+ and f^- are n-symmetric. Therefore, we may focus on the case where

$$\lambda(f) \ge C_0 \max\{\lambda(f^+), \lambda(f^-)\}. \tag{4.23}$$

Consider the set $Q = \{x \in \mathbb{R}^n : |x_1| \le x_2\}$ and define the function $\widehat{f} : \mathbb{R}^n \to [0, \infty)$ as

$$\widehat{f}(x) = \begin{cases} f(x)\,, & \text{if } x \in Q\,, \\ f(R_1(x))\,, & \text{if } x \in R_1(Q)\,, \\ f(R_2(x))\,, & \text{if } x \in R_2(Q \cup R_1(Q))\,, \end{cases}$$

where $R_1, R_2 : \mathbb{R}^n \to \mathbb{R}^n$ denote the reflections about the hyperplanes $\{x \in \mathbb{R}^n : x_2 = x_1\}$ and $\{x \in \mathbb{R}^n : x_2 = -x_1\}$, respectively; notice that $\widehat{f}$ is symmetric with respect to the hyperplanes $\{x_1 = \pm x_2\}$ and $\{x_1 = 0\}$ for $3 \le i \le n$, hence $n-$symmetric. Moreover, on setting $Q^+ = \{x \in Q : x_1 > 0\}$ and $Q^- = \{x \in Q : x_1 < 0\}$, one has

$$\widehat{f} = f^+ \text{ in } Q^+\,, \qquad\qquad \widehat{f} = f^- \text{ in } Q^-\,.$$

We claim that, if C_0 is sufficiently large, then a constant C exists such that (4.1) holds. Let us begin by proving the first inequality: obviously $\int_{\mathbb{R}^n} \widehat{f}^{p^*} \le 2$ so that, on denoting by $\widehat{g}, g^+$ and g^- functions having the form (1.2), at which the infima in the definitions of $\lambda(\widehat{f}|\{0\}), \lambda(f^+|\{0\})$ and $\lambda(f^-|\{0\})$ are attained, we have

$$3^{p^*}\lambda(\widehat{f}) \ge \lambda(\widehat{f}|\{0\}) = \frac{\displaystyle\int_{\mathbb{R}^n} |\widehat{f} - \widehat{g}|^{p^*}}{\displaystyle\int_{\mathbb{R}^n} \widehat{f}^{p^*}} = 4\,\frac{\displaystyle\int_Q |f - \widehat{g}|^{p^*}}{\displaystyle\int_{\mathbb{R}^n} \widehat{f}^{p^*}} \ge 2\left(\int_{Q^+} |f^+ - \widehat{g}|^{p^*} + \int_{Q^-} |f^- - \widehat{g}|^{p^*}\right)$$
$$= 2\left(\int_{Q^+} |f^+ - \widehat{g}|^{p^*} + \int_{Q^+} |f^- - \widehat{g}|^{p^*}\right) \ge \frac{1}{2^{p^*-2}} \int_{Q^+} |f^+ - f^-|^{p^*}\,.$$

Observe that the first inequality holds thanks to Lemma 7. We are going to show that

$$\int_{Q^+} |f^+ - f^-|^{p^*} \ge \frac{\lambda(f)}{4^{p^*+2}}\,, \tag{4.24}$$

provided that C_0 is large enough, whence the first inequality in (4.1) follows. One has

$$\begin{aligned} \|f^+ - f^-\|_{L^{p^*}(Q^+)} &\ge \frac{1}{2}\|f^+ - f^-\|_{L^{p^*}(Q)} \\ &\ge \frac{1}{2}\left(\|g^+ - g^-\|_{L^{p^*}(Q)} - \|f^+ - g^+\|_{L^{p^*}(Q)} - \|f^- - g^-\|_{L^{p^*}(Q)}\right)\,. \end{aligned} \tag{4.25}$$

Moreover

$$\int_Q |f^\pm - g^\pm|^{p^*} \le \int_{\mathbb{R}^n} |f^\pm - g^\pm|^{p^*} = \lambda(f^\pm|\{0\}) \le 3^{p^*}\lambda(f^\pm) \le \frac{3^{p^*}}{C_0}\lambda(f)\,, \tag{4.26}$$

where we have exploited Lemma 7 and (4.23). From (4.25) and (4.26) we get

$$\|f^+ - f^-\|_{L^{p^*}(Q^+)} \ge \frac{1}{2}\left(\|g^+ - g^-\|_{L^{p^*}(Q)} - 2\left(\frac{3^{p^*}}{C_0}\lambda(f)\right)^{1/p^*}\right)\,. \tag{4.27}$$

On the other hand, owing to (4.26),

$$\begin{aligned} \lambda(f) &\le \int_{\mathbb{R}^n} |f - g^+|^{p^*} = \frac{1}{2}\int_{\mathbb{R}^n} |g^+ - f^+|^{p^*} + \frac{1}{2}\int_{\mathbb{R}^n} |g^+ - f^-|^{p^*} \\ &\le \frac{3^{p^*}}{2C_0}\lambda(f) + 2^{p^*-2}\left(\frac{3^{p^*}}{C_0}\lambda(f) + \int_{\mathbb{R}^n} |g^+ - g^-|^{p^*}\right)\,. \end{aligned}$$

Thus,

$$\int_{\mathbb{R}^n} |g^+ - g^-|^{p^*} \ge \frac{1}{2^{p^*-2}}\left(1 - \frac{3^{p^*}}{C_0}\left(\frac{1}{2} + 2^{p^*-2}\right)\right)\lambda(f) \ge \frac{1}{2^{p^*}}\lambda(f) \tag{4.28}$$

if C_0 is sufficiently large. Coupling (4.27) and (4.28) yields (4.24), provided that C_0 is large enough.

Let us now prove the second inequality in (4.1). One has

$$\left| \|f\|_{L^{p^*}(Q^+)} - \frac{1}{8^{1/p^*}} \right| = \left| \|f^+\|_{L^{p^*}(Q^+)} - \|g^+\|_{L^{p^*}(Q^+)} \right| \le \|f^+ - g^+\|_{L^{p^*}(Q^+)} .$$

Since a constant C exists such that $|s^{p^*} - r^{p^*}| \le C|s-r|$ if $r, s \in [0,1]$, we have

$$\begin{aligned} \left| \int_{Q^+} f^{p^*} - \frac{1}{8} \right| &\le C\|f^+ - g^+\|_{L^{p^*}(Q^+)} \le C\lambda(f^+|\{0\})^{1/p^*} \le C\lambda(f^+)^{1/p^*} \le C\delta(f^+)^{1/\beta p^*} \\ &\le C\delta(f)^{1/\beta p^*} \end{aligned} \tag{4.29}$$

for a suitable constant C. Note that the third inequality relies on Lemma 7, whereas Corollary 4 plays its role in the fourth one. The same estimate holds also in Q^-, $U^+ = \{x_2 > 0,\, x_1 > 0\} \setminus Q$ and $U^- = \{x_2 > 0,\, x_1 < 0\} \setminus Q$. As a consequence,

$$\left| \int_{\mathbb{R}^n} \widehat{f}^{p^*} - 1 \right| \le C\delta(f)^{1/\beta p^*} . \tag{4.30}$$

As far as the gradient of $\widehat{f}$ is concerned, we obviously have

$$\int_{\mathbb{R}^n} |\nabla \widehat{f}^p| = 4\int_Q |\nabla \widehat{f}|^p = 4\left(\int_{\{x_2>0\}} |\nabla f|^p - \int_{U^+\cup U^-} |\nabla f|^p \right) . \tag{4.31}$$

Since f is symmetric about the hyperplane $\{x_2 = 0\}$,

$$\int_{\{x_2>0\}} |\nabla f|^p = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla f|^p = S(p,n)^p\,\frac{(1+\delta(f))^p}{2} \le S(p,n)^p \frac{(1+C\delta(f))}{2} . \tag{4.32}$$

Here, we have made use of the fact that $\delta(f) \le \bar{\delta}$. Applying the Sobolev inequality (1.1) to the function obtained reflecting $f_{|U^+}$ first about $\{x_2 = x_1\}$, then about $\{x_1 = 0\}$, and finally about $\{x_2 = 0\}$, and keeping in mind (4.29), we have

$$\begin{aligned} \int_{U^+} |\nabla f|^p &\ge 8^{-1+p/p^*} S(p,n)^p \Big(\int_{U^+} f^{p^*} \Big)^{p/p^*} \ge 8^{-1+p/p^*} S(p,n)^p \left(\frac{1}{8} - C\delta(f)^{1/\beta p^*} \right)^{p/p^*} \\ &\ge S(p,n)^p \left(\frac{1}{8} - C\delta(f)^{1/\beta p^*} \right) , \end{aligned}$$

provided that $\bar{\delta}$ is small enough. An analogous estimate holds for $\int_{U^-} |\nabla f|^p$. Combining these estimates with (4.31) and (4.32) tells us that

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla \widehat{f}|^p &\le 4\Big(\frac{S(p,n)^p(1+C\delta(f))}{2} - 2S(p,n)^p\Big(\frac{1}{8} - C\delta(f)^{1/\beta p^*} \Big)\Big) \\ &\le S(p,n)^p(1 + C\delta(f)^{1/\beta p^*}) . \end{aligned}$$

Therefore, from (4.30) we conclude that

$$\delta(\widehat{f}) = \frac{\|\nabla \widehat{f}\|_{L^p(\mathbb{R}^n}}{S(p,n)\|\widehat{f}\|_{L^{p^*}(\mathbb{R}^n)}} - 1 \le \frac{S(p,n)(1 + C\delta(f)^{1/\beta p^*})}{S(p,n)(1 - C\delta(f)^{1/\beta p^*})} - 1 \le C\delta(f)^{1/\beta p^*} ,$$

namely the second inequality in (4.1). □

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