THE ANTIFERROMAGNETIC XY-MODEL ON THE LAYERED TRIANGULAR LATTICE: CHIRALITY TRANSITIONS IN A RIGID REGIME AND DIMENSION REDUCTION

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Abstract. In this paper we carry out a discrete-to-continuum variational analysis of the antiferromagnetic XY-model on a layered triangular lattice. This system is frustrated in the in-plane interactions, while no frustration occurs in the out-of plane interactions. We show that for this system we have energy concentration on a surface scale that can be detected in terms of a chirality variable. We characterise the Γ-limit of the system in a regime that enforces rigidity of admissible limiting chiralities and apply this result to a dimension-reduction problem.

Keywords: Discrete-to-continuum limit; Γ-convergence; frustrated spin systems; chirality transitions
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1. Introduction

The complex behaviour of ordering phenomena in magnetism has drawn the attention of both the mathematics and the physics community. Magnetic systems which exhibit a particularly complex behaviour and which have received an increased interest in recent years are so-called *frustrated spin systems* (see [23] for an overview on this topic). Upon identifying magnets with spins (a vector in the unit sphere), frustration refers to the situation where the spins cannot attain an orientation that minimises all pairwise exchange interactions simultaneously. This effect prevents long-range magnetic ordering even at zero temperature and consequently frustrated systems typically show only partial ordering following complex geometric patterns and inducing intricate structural and magnetic effects. From a physical standpoint, understanding the behaviour of such systems is therefore crucial, e.g., for engineering of materials that display similar structural and magnetic properties. At the same time, these systems often give rise to challenging and interesting mathematical problems. One of those problems is the derivation of suitable continuum models that characterise the effective behaviour of lattice spin systems as the lattice spacing vanishes. Such continuum models can provide a macroscopic description of the partial ordering taking place at the microscopic level. Following a variational (or energetic) approach, these models can be obtained via Γ-convergence, a procedure that nowadays is often referred to as *variational coarse graining* (see [3], in particular [3, Chapter 7] for frustrated systems).

Background and setup of the model. In the last years, the variational coarse graining of systems where frustration stems from competing ferromagnetic interactions (favouring alignment) and antiferromagnetic interactions (favouring anti-alignment) has been addressed in several works (see [1] and [11] for Ising systems, [29], [20], [17] for XY spin systems, and [19] and [18] for first steps in the direction of $\mathbb{S}^2$-valued spin systems). However, for purely antiferromagnetic lattice spin systems, frustration can also stem from the geometry of the lattice. The variational coarse graining of such *geometrically frustrated* systems has only recently been initiated in [6] and [7]. Specifically, in [6] and [7] the authors consider the antiferromagnetic XY-model on the two-dimensional triangular lattice. Based on their result, in the present paper we initiate the analysis of a three-dimensional geometrically frustrated system, namely of the antiferromagnetic

XY-model on the layered triangular lattice. To be precise, we let $\varepsilon > 0$ be a small parameter and consider the layered triangular lattice $\mathcal{L}^{3d}_\varepsilon := \mathcal{L}^{2d}_\varepsilon \times \mathbb{Z}_\varepsilon$ where $\mathcal{L}^{2d}_\varepsilon$ and $\mathbb{Z}_\varepsilon$ are respectively the triangular lattice and the 1-dimensional square lattice with spacing ε (see Subsection 2.2 for the precise definition). To each spin field $u : \mathcal{L}^{3d}_\varepsilon \to \mathbb{S}^1$ we associate the energy

$$\sum_{\substack{\varepsilon\sigma,\varepsilon\sigma'\in\mathcal{L}^{3d}_\varepsilon \\ \sigma,\sigma' \text{ in-plane neighbours}}} \varepsilon^3\langle u(\varepsilon\sigma), u(\varepsilon\sigma')\rangle - R \sum_{\substack{\varepsilon\sigma,\varepsilon\sigma'\in\mathcal{L}^{3d}_\varepsilon \\ \sigma,\sigma' \text{ out-of-plane neighbours}}} \varepsilon^3\langle u(\varepsilon\sigma), u(\varepsilon\sigma')\rangle, \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^2$, the factor ε^3 simplifies with the number of lattice points contained in a bounded domain in $\mathbb{R}^3$, and R is a positive real number. Note that the in-plane interactions are indeed antiferromagnetic, as they favour anti-alignment of nearest neighbours, while the out-of-plane interactions favour alignment of nearest neighbours and are thus ferromagnetic. This structure is known to appear in the Caesium-Copper-Chlorine compound $CsCuCl_3$ and in the Caesium-Nickel-Iron compound $CsNiF_3$ (see [21, Section V]). However, concerning the analysis carried out here, the system in (1.1) is equivalent to a completely antiferromagnetic system. This is due to the fact that the frustration in the system only stems from the in-plane interactions (see Section 3.3 for more details). Indeed, on any triangular layer $\mathcal{L}^{2d}_\varepsilon \times \{\varepsilon z\}$ with $z \in \mathbb{Z}$ fixed, the lattice geometry prevents the system from minimising simultaneously all in-plane nearest neighbour interactions. On the contrary, the structure of the vertical interactions is that of the square lattice where the ferromagnetic and antiferromagnetic system share the same asymptotic behaviour (see [4]). In the setting of (1.1), the ferromagnetic out-of-plane interactions favour alignment of the spin field in the vertical direction and the coefficient $R > 0$ can be seen as a parameter penalising out-of-plane variations in the spin field.

Groundstates of the sytem. In this paper we are interested in characterising the asymptotic behaviour and structural properties of a certain class of so-called *low energy states* for the energies (1.1). Roughly speaking, those are states whose energy differs from the minimal energy of the system by an amount that vanishes as $\varepsilon \to 0$. This will be done by removing from (1.1) the energy of a ground state and determining the Γ-limit of (a scaled version of) the remaining *excess energy*. Following this track, we first characterise the groundstates of (1.1). The above considerations suggest that they are obtained by extending the ground state of the two-dimensional model [6] constantly in the vertical direction. In fact, following the approach of [6] we reorder the terms in (1.1) by summing up over triangular plaquettes $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$ with $\varepsilon i, \varepsilon j, \varepsilon k \in \mathcal{L}^{3d}_\varepsilon$ to rewrite the energy (1.1) as

$$\frac{1}{2}\sum_T \left(\varepsilon^3\Big(\big|u(\varepsilon i)+u(\varepsilon j)+u(\varepsilon k)\big|^2 - 3\Big) + R\sum_{\alpha\in\{i,j,k\}} \varepsilon^3\Big(\big|u(\varepsilon\alpha)-u(\varepsilon(\alpha+e_3))\big|^2 - 2\Big)\right). \tag{1.2}$$

(Note that throughout we will restrict our analysis to portions of the lattice $\mathcal{L}^{3d}_\varepsilon$ contained in a bounded domain $\Omega \subset \mathbb{R}^3$, which turns all sums above into finite sums.) On each triangle T the energy in (1.2) is minimised if and only if

$$u(\varepsilon i) + u(\varepsilon j) + u(\varepsilon k) = 0 \quad \text{and} \quad u(\varepsilon\alpha) = u(\varepsilon(\alpha+e_3)) \text{ for all } \alpha \in \{i,j,k\}\,. \tag{1.3}$$

The first condition in (1.3) is realised if the spin field u rotates by a fixed angle of $120°$ between $\varepsilon i, \varepsilon j$, and εk (either clockwise or counter-clockwise, see Figure 1). The second condition is then realised by indeed extending those configurations constantly in the e_3-direction. On any triangular layer $\mathcal{L}^{2d}_\varepsilon \times \{\varepsilon z\}$ with $z \in \mathbb{Z}$ fixed it is now possible to realise the first condition in (1.3) globally by decomposing $\mathcal{L}^{2d}_\varepsilon \times \{\varepsilon z\}$ into three sublattices according to Figure 1 (see Sections 2.2 and 3 for a precise definition). Thus, by putting the ground states $u^{\mathrm{pos}}_\varepsilon$ or $u^{\mathrm{neg}}_\varepsilon$ depicted in Figure 1 consistently on each layer of $\mathcal{L}^{3d}_\varepsilon$, we obtain two families of spin fields that satisfy both conditions of (1.3) and thus minimise the energy (1.2). Note that by the $\mathbb{S}^1$-symmetry of the system each

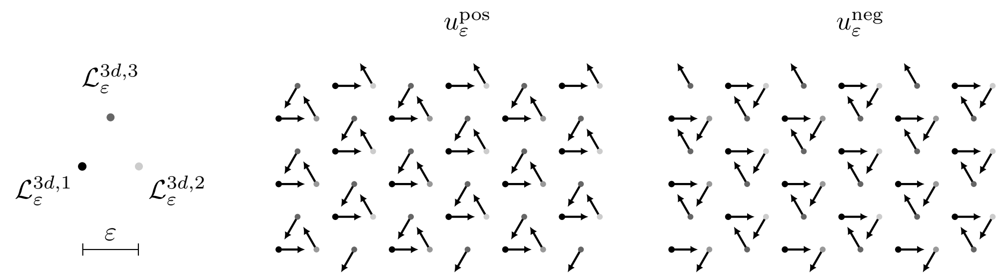

Figure 1. On the left: Three points of the sublattices $\mathcal{L}_\varepsilon^{3d,1}$, $\mathcal{L}_\varepsilon^{3d,2}$, and $\mathcal{L}^{3d,3}$ in black, light grey, and dark grey, respectively. In the centre and on the right: Ground states $u_\varepsilon^{\mathrm{pos}}$ and $u_\varepsilon^{\mathrm{neg}}$ on a triangular layer $\mathcal{L}_\varepsilon^{2d} \times \{\varepsilon z\}$.

global rotation of $u_\varepsilon^{\mathrm{pos}}$ and $u_\varepsilon^{\mathrm{neg}}$ is a minimiser as well, hence $u_\varepsilon^{\mathrm{pos}}$ and $u_\varepsilon^{\mathrm{neg}}$ indeed give rise to two whole families of ground states. At the same time, $u_\varepsilon^{\mathrm{pos}}$ and $u_\varepsilon^{\mathrm{neg}}$ are not global rotations of each other. In fact, they are distinguished by their handedness and correspond to different *chiral states* of the system.

Similar to [6] we can associate a *chirality variable* to the system that quantifies the above mentioned handedness. Namely, for any triangle $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$ with $\varepsilon i, \varepsilon j, \varepsilon k$ belonging to the three sublattices $\mathcal{L}_\varepsilon^{3d,1}$, $\mathcal{L}_\varepsilon^{3d,2}$, $\mathcal{L}_\varepsilon^{3d,3}$ (cf. Figure 1) we define

$$\chi(u,T) = \frac{2}{3\sqrt{3}}\left(u(\varepsilon i) \times u(\varepsilon j) + u(\varepsilon j) \times u(\varepsilon k) + u(\varepsilon k) \times u(\varepsilon i)\right),$$

where the symbol $\times$ is the vector product on $\mathbb{R}^2$. This variable is extended to almost all of $\mathbb{R}^3$ by setting $\chi(u)(x) = \chi(u,T)$ whenever x belongs to the interior of a prism $P = \mathrm{conv}\{T, T + \varepsilon e_3\}$. In this way, $\chi(u)$ takes values in $[-1,1]$ and $\chi(u) \equiv 1$ if and only if u is a rotation of $u_\varepsilon^{\mathrm{pos}}$, while $\chi(u) \equiv -1$ if and only if u is a rotation of $u_\varepsilon^{\mathrm{neg}}$. This characterisation of the ground states indicates that χ is the relevant variable for studying the asymptotic behaviour of low-energy states. Here we investigate a low-energy regime that allows for the coexistence of both families of ground states and gives rise to an interfacial energy between phases of chirality $\{\chi = 1\}$ and $\{\chi = -1\}$.

Main results of the paper. We consider the energy (1.2) restricted to a fixed open and bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ and remove the energy of a groundstate, that is, for any $u : \mathcal{L}_\varepsilon^{3d} \to \mathbb{S}^1$ we consider the excess energy

$$E_{\varepsilon,R}(u) = \sum_{T \subset \Omega} \varepsilon^3 \left(|u(\varepsilon i) + u(\varepsilon j) + u(\varepsilon k)|^2 + R \sum_{\alpha \in \{i,j,k\}} |u(\varepsilon \alpha) - u(\varepsilon(\alpha + e_3))|^2 \right). \tag{1.4}$$

Even though there is no frustration in the e_3-direction, we show that low energy states have a similar behaviour as in the two-dimensional case. To be precise, we show that for sequences of spin fields u_ε satisfying $E_{\varepsilon,R}(u_\varepsilon) \leq C\varepsilon$ for a constant $C > 0$, the chiralities $\chi(u_\varepsilon)$ converge (up to subsequences) to a function $\chi \in BV(\Omega; \{-1,1\})$ (see Proposition 5.1). Therefore, the continuum limit of the energies $E_{\varepsilon,R}$ only allows phases of chirality -1 or 1, partitioning the space in finitely many sets of finite perimeter in Ω. In this way, the model shows a similar behaviour as other discrete systems such as the Ising model or Potts model [1, 10]. However, for our model the phase transitions are observed via the chirality variable instead of the spin field itself, which reflects the fact that only partial ordering is expected for frustrated systems like (1.4).

Based on the previous compactness result we study the asymptotic behaviour of the scaled energies $\frac{1}{\varepsilon}E_{\varepsilon,R_\varepsilon}$ under the assumption that $R_\varepsilon \to +\infty$ as $\varepsilon \to 0$. We refer to this regime as a rigid regime, since heuristically one expects that the diverging parameters R_ε enforce alignment and thus an asymptotic rigidity condition in the spirit of [16] on admissible limiting configurations $\chi \in BV(\Omega;\{-1,1\})$. Namely, one expects that limiting configurations are constant in the e_3-direction. In Theorem 3.2 (i) we verify this in a measure-theoretic sense. To be precise, we show that admissible limiting chiralities $\chi \in BV(\Omega;\{-1,1\})$ satisfy $\langle \nu_\chi(x), e_3\rangle = 0$ $\mathcal{H}^2$-a.e. on S_χ, where S_χ is the jump set of χ and $\nu_\chi(x)$ the measure-theoretic normal to S_χ at $x \in S_\chi$. This finally allows us to show that for a cylindrical domain $\Omega \subset \mathbb{R}^3$ the Γ-limit of $\frac{1}{\varepsilon}E_{\varepsilon,R_\varepsilon}$ can be characterised in terms of χ as

$$E(\chi) = \int_{S_\chi} \varphi\big(\nu_\chi(x)\big)\, d\mathcal{H}^2(x)$$

for $\chi \in BV(\Omega;\{-1,1\})$ satisfying $\langle \nu_\chi(x), e_3\rangle = 0$ $\mathcal{H}^2$-a.e. on S_χ and equal to $+\infty$ otherwise. Here, the density φ corresponds to the 2-dimensional density obtained in [6] evaluated in the first two components of $\nu_\chi(x) \in \mathbb{S}^2$ (see Theorem 3.2 (ii) for a precise statement). It is worth mentioning that the assumption on Ω being cylindrical is only needed for the Γ-limsup inequality and we will comment on this in more detail below and in Section 6.2. At the same time, the diverging parameter R_ε naturally appears as a scaling parameter $R_\varepsilon = \frac{1}{\varepsilon}$ in the setting of dimension reduction, where Ω being cylindrical is a standard modelling assumption. Starting from the seminal work [25] on the dimension reduction for variational models in nonlinear elasticity, the coupling of dimension reduction with a passage from discrete to continuum has been investigated in the context of elasticity [2, 28] (see [14] for a coupling with homogenization) and in the context of Ising systems [12]. Here, we apply Theorem 3.2 to obtain a dimension-reduction result for frustrated spin systems (Theorem 3.4).

Main difficulties and proof strategy. We close this introduction by explaining the main difficulties encountered in proving our main result Theorem 3.2. In classical models in which finitely many phases arise in the limit such as the Modical-Mortola model [27, 26] (see [15] for a discrete version), the phase transition is observed via the function itself, whereas in this model, we observe the phase transition through the chirality variable. The chirality depends nonlinearly on the spin fields, which makes it difficult to transfer information obtained for u_ε (for example via energy bounds) to $\chi(u_\varepsilon)$ and vice versa. This was already a source of difficulty in [6] in the characterisation of φ, which required a careful adaption of De Giorgi's well-known averaging-slicing procedure. However, part of this adaption is intrinsically two-dimensional and extending it to a three-dimensional setting would require very careful modifications. By working within the rigid regime $R_\varepsilon \to +\infty$ we avoid this, but instead we need to establish the asymptotic rigidity condition for admissible limiting chiralities. Also here the main difficulty lies in the non-linear dependence of $\chi(u)$ on u, since the energy $E^{3d}_{\varepsilon,R_\varepsilon}$ only bounds terms of the form $R_\varepsilon|u_\varepsilon(\varepsilon\alpha)-u_\varepsilon(\varepsilon(\alpha+e_3))|^2$, while we would like to obtain an asymptotic rigidity condition on $\chi(u_\varepsilon)$. We approach this problem by studying an auxiliary one-dimensional model where we consider a suitable scaled version of the energies $E_{\varepsilon,R}$ for fixed $R > 0$ on a column of prisms along the e_3-axis (see Section 4). In this way we are able to quantify the energy induced by changes of chirality in the vertical direction and give a lower bound depending on R. Specifically, we show that the one-dimensional Γ-limit is of the form $c_R\#(S_\chi)$ for χ being a one-dimensional function of bounded variation and the constant c_R multiplying the number of jump points of χ is given by an optimal-profile problem. A crucial step then consists in showing that $c_R \geq C\sqrt{R}$ for any $R > 0$. Returning to the three-dimensional problem, the previous estimate together with well-known slicing properties of BV-functions allows us to provide an asymptotic lower bound of the form $C\sqrt{R}\int_{S_\chi}|\langle\nu_\chi, e_3\rangle|d\mathcal{H}^2$ (see Proposition 5.1), from which we finally obtain the asymptotic rigidity condition by letting $R \to +\infty$. Thanks to the rigidity condition, the Γ-liminf inequality follows directly from the two-dimensional result [6]. Here, a blow-up procedure allows us to establish the Γ-liminf inequality for non-cylindrical domains

$\Omega \subset \mathbb{R}^3$. As mentioned above we need to restrict ourselves to a cylindrical domain for the Γ-limsup inequality. This is due to the fact that to the best of our knowledge it is not possible to approximate the jump set of a function $\chi \in BV(\Omega\ \{-1,1\})$ satisfying the constraint $\langle \nu_\chi, e_3\rangle = 0$ with a polyhedral set as in [13] while keeping the constraint on the normal. Such a result however would be required to obtain the Γ-limsup inequality via a density argument. In the non-rigid regime $R_\varepsilon \to R$ for some $R > 0$ we could apply the density result provided by [13], but the study of (1.4) in this regime is left for future research.

2. Setting of the Problem and preliminary results

2.1. **General notation.** Throughout this paper $\Omega \subset \mathbb{R}^3$ is an open, bounded set with Lipschitz boundary. For $k \in \{1, 2, 3\}$ we let $\mathscr{L}^k$ denote the k-dimensional Lebesgue measure, while with $\mathcal{H}^k$ we indicate the k-dimensional Hausdorff measure in $\mathbb{R}^3$. We denote by $\{e_1, e_2, e_3\}$ the standard orthonormal basis of $\mathbb{R}^3$.

For $x, y \in \mathbb{R}^3$, $\langle x, y\rangle$ is the scalar product between x and y and $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$ is the standard euclidian norm in $\mathbb{R}^3$. The sets $\mathbb{S}^1 := \{\nu \in \mathbb{R}^2 : |\nu| = 1\}$ and $\mathbb{S}^2 := \{\xi \in \mathbb{R}^3 \colon |\xi| = 1\}$ are the sets of unit vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. For any $\xi \in \mathbb{R}^3$ we let

$$\Pi_\xi := \big\{x \in \mathbb{R}^3 \colon \langle x, \xi\rangle = 0\big\}$$

be the hyperplane orthogonal to ξ and passing through the origin.

Given a vector $\nu = (\nu_1, \nu_2) \in \mathbb{S}^1$ we denote by $\nu^\perp := (-\nu_2, \nu_1) \in \mathbb{S}^1$ the unit vector orthogonal to ν obtained by rotating ν counterclockwise by $\pi/2$. Moreover, given $v, w \in \mathbb{S}^1$ we denote by $v \times w := \langle v, w^\perp\rangle$ the cross product between v and w. Moreover, $\mathrm{d}_{\mathbb{S}^1}(v, w)$ is the geodesic distance on $\mathbb{S}^1$ between v and w. It satisfies

$$|v - w| \le \mathrm{d}_{\mathbb{S}^1}(v, w) \le \frac{\pi}{2}|v - w| \,. \tag{2.1}$$

Finally, it will often be convenient to write a vector in $v \in \mathbb{S}^1$ as $v = \exp(\iota\theta)$ with $\theta \in \mathbb{R}$. In this way, for $v = \exp(\iota\theta), w = \exp(\iota\phi) \in \mathbb{S}^1$ we have that

$$\mathrm{d}_{\mathbb{S}^1}(v, w) = \mathrm{dist}(\theta - \phi; 2\pi\mathbb{Z}) \,. \tag{2.2}$$

2.2. **The layered triangular lattice and unit cells.** In this paragraph we define the triangular lattice in $\mathbb{R}^2$ and the layered triangular lattice in $\mathbb{R}^3$. They are given by

$$\mathcal{L}^{2d} := \mathrm{span}_{\mathbb{Z}}\{\hat{e}_1, \hat{e}_2\} = \{\alpha = z_1\hat{e}_1 + z_2\hat{e}_2 : z_1, z_2 \in \mathbb{Z}\}$$

and

$$\mathcal{L}^{3d} := \mathcal{L}^{2d} \times \mathbb{Z} = \big\{(\alpha, z) \in \mathbb{R}^3 \colon \alpha \in \mathcal{L}^{2d}\,,\ z \in \mathbb{Z}\big\}\,,$$

where $\hat{e}_1 = (1, 0)$ and $\hat{e}_2 = \frac{1}{2}(1, \sqrt{3})$. Note that by setting $\hat{e}_3 := \frac{1}{2}(-1, \sqrt{3})$ we can equivalently write

$$\mathcal{L}^{2d} = \mathrm{span}_{\mathbb{Z}}\{\hat{e}_1, \hat{e}_3\} = \mathrm{span}_{\mathbb{Z}}\{\hat{e}_2, \hat{e}_3\} \,.$$

Moreover, using the vector $\hat{e}_3$ we can decompose $\mathcal{L}^{2d}$ into the three sublattices

$$\mathcal{L}^{2d,1} := \{z_1(\hat{e}_1 + \hat{e}_2) + z_2(\hat{e}_2 + \hat{e}_3) : z_1, z_2 \in \mathbb{Z}\}, \quad \mathcal{L}^{2d,2} := \mathcal{L}^{2d,1} + \hat{e}_1, \quad \mathcal{L}^{2d,3} := \mathcal{L}^{2d,1} + \hat{e}_2 \,.$$

Accordingly, we decompose $\mathcal{L}^{3d}$ into the three sublattices

$$\mathcal{L}^{3d,\ell} := \mathcal{L}^{2d,\ell} \times \mathbb{Z} \ \text{ for } \ell = 1, 2, 3 \,.$$

It will also be convenient to introduce the 1-dimensional lattice

$$\mathcal{L}^{1d} := \{0, \hat{e}_1, \hat{e}_2\} \times \mathbb{Z} = \big\{(\alpha, z) \colon \alpha \in \{0, \hat{e}_1, \hat{e}_2\}\,,\ z \in \mathbb{Z}\big\} \,.$$

We finally introduce the classes of unit cells subordinated to the lattice $\mathcal{L}^{2d}$ and $\mathcal{L}^{3d}$. For $\mathcal{L}^{2d}$ this is the family of equilateral triangles

$$\mathcal{T}^{2d} := \big\{T = \mathrm{conv}\{i,j,k\} : i,j,k \in \mathcal{L}^{2d},\ |i-j| = |j-k| = |k-i| = 1\big\}\ , \tag{2.3}$$

where $\mathrm{conv}\{i,j,k\}$ denotes the closed convex hull of i,j,k.

Remark 2.1 (Identification of elements in $\mathbb{R}^2$ with elements in $\mathbb{R}^3$). In (2.3) we use the notation $\mathcal{T}^{2d}$ for consistency to indicate that we consider unit cells of $\mathcal{L}^{2d}$ contained in $\mathbb{R}^2$. However, in all that follows we will embed $\mathbb{R}^2$ into $\mathbb{R}^3$ via the mapping $x = (x_1,x_2) \mapsto (x_1,x_2,0)$. In this way, we will frequently interpret elements of $\mathcal{L}^{2d}$ and unit cells in $\mathcal{T}^{2d}$ as elements and subsets of $\mathbb{R}^3$, respectively.

With the convention of Remark 2.1 we can extend the class $\mathcal{T}^{2d}$ via periodicity to the class of triangles subordinated to $\mathcal{L}^{3d}$ by setting

$$\mathcal{T}^{3d} := \big\{T + ze_3 \colon T \in \mathcal{T}^{2d}\ ,\ z \in \mathbb{Z}\big\}\,.$$

In this way, we can finally express the family of unit cells of $\mathcal{L}^{3d}$ as

$$\mathcal{P}^{3d} := \big\{P = \mathrm{conv}\{T, T+e_3\} : T \in \mathcal{T}^{3d}\big\}\ .$$

For any $\varepsilon > 0$ and any Borel set $A \subset \mathbb{R}^2$, $B \subset \mathbb{R}^3$ the rescaled and localised versions of $\mathcal{T}^{2d}$, $\mathcal{T}^{3d}$, and $\mathcal{P}^{3d}$ are given by

$$\begin{aligned}
\mathcal{T}^{2d}_\varepsilon(A) &:= \big\{\varepsilon T \in \varepsilon\mathcal{T}^{2d} \colon \varepsilon T \subset A\big\}\,,\\
\mathcal{T}^{3d}_\varepsilon(B) &:= \big\{\varepsilon T \in \varepsilon\mathcal{T}^{3d} \colon \varepsilon T \subset B\big\}\,,\\
\mathcal{P}^{3d}_\varepsilon(B) &:= \{\varepsilon P \in \varepsilon\mathcal{P}^{3d} \colon \varepsilon P \subset B\}\,.
\end{aligned}$$

Similarly, we set

$$\mathcal{L}^{2d}_\varepsilon(A) := \varepsilon\mathcal{L}^{2d} \cap A \quad \text{and} \quad \mathcal{L}^{3d}_\varepsilon(B) := \varepsilon\mathcal{L}^{3d} \cap B\,.$$

If $A = \mathbb{R}^2$ or $B = \mathbb{R}^3$ we simply write $\mathcal{L}^{2d}_\varepsilon$ and $\mathcal{L}^{3d}_\varepsilon$ instead of $\mathcal{L}^{2d}_\varepsilon(\mathbb{R}^2)$ and $\mathcal{L}^{3d}_\varepsilon(\mathbb{R}^3)$, as well as $\mathcal{T}^{2d}_\varepsilon$, $\mathcal{T}^{3d}_\varepsilon$, and $\mathcal{P}^{3d}_\varepsilon$. It is also convenient to associate to any fixed triangle $T_0 \in \mathcal{T}^{2d}_\varepsilon(\mathbb{R}^2)$ the column of scaled prisms

$$\mathcal{C}_\varepsilon(T_0) := \big\{P = \mathrm{conv}\{T_0 + \varepsilon z e_3, T_0 + \varepsilon(z+1)e_3\} \in \mathcal{P}^{3d}_\varepsilon(\mathbb{R}^3) \colon z \in \mathbb{Z}\big\}\,. \tag{2.4}$$

Finally, we fix

$$T_{\mathrm{ref}} := \mathrm{conv}\{0, \hat{e}_1, \hat{e}_2\} \quad \text{and} \quad P_{\mathrm{ref}} := \mathrm{conv}\{T_{\mathrm{ref}}, T_{\mathrm{ref}} + ze_3\}$$

as a reference triangle and a reference prism and we set

$$\mathcal{T}^{1d} := \big\{T = T_{\mathrm{ref}} + ze_3 \colon z \in \mathbb{Z}\big\} \quad \text{and} \quad \mathcal{P}^{1d} := \{P = \mathrm{conv}\{T, T+e_3\} \colon T \in \mathcal{T}^{1d}\}\,.$$

We conclude this paragraph by introducing the set of so-called spin fields defined on $\mathcal{L}^{3d}_\varepsilon$. More precisely, we set

$$\mathcal{SF}_\varepsilon := \big\{u : \mathcal{L}^{3d}_\varepsilon \to \mathbb{S}^1\big\}\,.$$

If $\varepsilon = 1$ we simply write $\mathcal{SF}$ in place of $\mathcal{SF}_1$.

Remark 2.2 (Extending maps from $\mathcal{L}^{2d}_\varepsilon$ and from $\mathcal{L}^{1d}_\varepsilon$ to $\mathcal{L}^{3d}_\varepsilon$). We will frequently identify a map $u_\varepsilon : \mathcal{L}^{2d}_\varepsilon \to \mathbb{S}^1$ with an element $u_\varepsilon \in \mathcal{SF}_\varepsilon$ by periodically extending u_ε from $\mathcal{L}^{2d}_\varepsilon$ to $\mathcal{L}^{3d}_\varepsilon$, i.e., by setting $u_\varepsilon(\varepsilon\alpha, \varepsilon z) := u_\varepsilon(\varepsilon\alpha)$ for every $(\alpha, z) \in \mathcal{L}^{3d}$. Similarly, we will identify a map $u_\varepsilon : \mathcal{L}^{1d}_\varepsilon \to \mathbb{S}^1$ with the spin field $u_\varepsilon \in \mathcal{SF}_\varepsilon$ obtained by repeating u_ε according to the sublattices $\mathcal{L}^{3d,\beta}_\varepsilon$, that is, by setting $u_\varepsilon(\varepsilon i, \varepsilon z) := u_\varepsilon(0, \varepsilon z)$ for $(i,z) \in \mathcal{L}^{3d,1}$, $u_\varepsilon(\varepsilon j, \varepsilon z) := u_\varepsilon(\varepsilon\hat{e}_1, \varepsilon z)$ for $(j,z) \in \mathcal{L}^{3d,2}$, and $u_\varepsilon(\varepsilon k, \varepsilon z) := u_\varepsilon(\varepsilon\hat{e}_2, \varepsilon z)$ for $(k,z) \in \mathcal{L}^{3d,3}$.

2.3. **Definition of the discrete energies.** We are now in a position to introduce the discrete energies that we will consider in this paper. To this end, let $u \in \mathcal{SF}_\varepsilon$ and $n \in \{1,2,3\}$. For any $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon^{3d}$ we set

$$E_\varepsilon^{nd}(u,T) := \varepsilon^{n-1}|u(\varepsilon i) + u(\varepsilon j) + u(\varepsilon k)|^2 \,. \tag{2.5}$$

Let now $R > 0$ and suppose that $P = \mathrm{conv}\{T, T + \varepsilon e_3\} \in \mathcal{P}_\varepsilon^{3d}$ with $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$. We then set

$$E_{\varepsilon,R}^{nd}(u,P) := E_\varepsilon^{nd}(u,T) + E_\varepsilon^{nd}(u,T+\varepsilon e_3) + \varepsilon^{n-1} R \sum_{\beta \in \{i,j,k\}} |u(\varepsilon\beta) - u\big(\varepsilon(\beta + e_3)\big)|^2 \,. \tag{2.6}$$

In the case $n = 3$ we obtain the main energies considered in this paper by setting

$$E_{\varepsilon,R}^{3d}(u,B) := \sum_{P \in \mathcal{P}_\varepsilon^{3d}(B)} E_{\varepsilon,R}^{3d}(u,P) \tag{2.7}$$

for any Borel set $B \subset \mathbb{R}^3$.

The main result of this paper characterises the Γ-limit of $E_{\varepsilon,R_\varepsilon}^{3d}$ when $\varepsilon \to 0$ and at the same time $R_\varepsilon \to \infty$ (see Theorem 3.2). To obtain this result we will on the one hand rely on the result established in [6] for the antiferromagnetic XY-model energy on $\mathcal{L}_\varepsilon^{2d}$ and on the other hand we will make use of an auxiliary one-dimensional energy. Both energies can be conveniently defined based on (2.5) and (2.6) in the case $n = 1, 2$. Specifically, for $n = 2$ we recover the energies considered in [6] by extending E_ε^{2d} to any Borel set $A \subset \mathbb{R}^2$ via

$$E_\varepsilon^{2d}(u,A) := \sum_{T \in \mathcal{T}_\varepsilon^{2d}(A)} E_\varepsilon^{2d}(u,T) \,. \tag{2.8}$$

Finally, to define the 1-dimensional auxiliary energies we proceed as follows. For any $I \subset \mathbb{R}$ Borel we set

$$\mathcal{P}_\varepsilon^{1d}(I) = \big\{P = \varepsilon\,\mathrm{conv}\{T_{\mathrm{ref}} + ze_3, T_{\mathrm{ref}} + (z+1)e_3\} \colon z \in \mathbb{Z}\,,\ \varepsilon[z, z+1] \subset I\big\} \,.$$

For any spin field $u \in \mathcal{SF}_\varepsilon(\mathbb{R}^3)$ we then define

$$E_{\varepsilon,R}^{1d}(u,I) := \sum_{P \in \mathcal{P}_\varepsilon^{1d}(I)} E_{\varepsilon,R}^{1d}(u,P) \,. \tag{2.9}$$

Remark 2.3 ((Anti-)ferromagnetic out-of-plane interactions). In the definition of the energies $E_{\varepsilon,R}^{3d}$ according to (2.6)–(2.7) we consider ferromagnetic out-of-plane interactions instead of antiferromagnetic ones. This is to simplify the exposition in the following sections. Similar to [4, Remark 4.6], the asymptotic analysis of the fully antiferromagnetic energies can be obtained from the one we carry out for the energies in (2.7) via a change of variables (see Section 3.3). This change of variables does not affect the chirality variable introduced in the following section, which turns out to be the relevant variable to characterise the asymptotic behaviour of $E_{\varepsilon,R}^{3d}$.

2.4. **Chirality.** In this section we associate to any spin field $u \in \mathcal{SF}_\varepsilon$ a so-called chirality variable in a similar way as in [6]. Specifically, for every $u \in \mathcal{SF}_\varepsilon$ and $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}_\varepsilon^{3d}$ with $i \in \mathcal{L}^{3d,1}$, $j \in \mathcal{L}^{3d,2}$, and $k \in \mathcal{L}^{3d,3}$ we set

$$\chi(u,T) := \frac{2}{3\sqrt{3}}\big(u(\varepsilon i) \times u(\varepsilon j) + u(\varepsilon j) \times u(\varepsilon k) + u(\varepsilon k) \times u(\varepsilon i)\big) \,. \tag{2.10}$$

Moreover, we extend χ to a function $\chi(u) : \mathbb{R}^3 \to \mathbb{R}$ by setting $\chi(u)(x) := \chi(u,T)$ whenever $x \in \mathrm{int}\,P \cup \mathrm{int}\,T$ for some prism $P = \mathrm{conv}\{T, T + \varepsilon e_3\} \in \mathcal{P}_\varepsilon^{3d}$. In this way, $\chi(u)$ is defined $\mathscr{L}^3$-almost everywhere in $\mathbb{R}^3$ and $\mathscr{L}^2$-almost everywhere on every horizontal layer, i.e., on every slice $\{x \in \mathbb{R}^3 \colon x \cdot e_3 = t\}$ with $t \in \mathbb{R}$.

Below we collect a couple of useful observations on the chirality variable $\chi(u,T)$ introduced in (2.10). Throughout the remainder of this paragraph we use the following conventions. For $\varepsilon>0$ and $T,T'\in\mathcal{T}^{3d}_\varepsilon$ we consistently write

$$T=\mathrm{conv}\{\varepsilon i,\varepsilon j,\varepsilon k\}\ \text{ and }\ T'=\mathrm{conv}\{\varepsilon i',\varepsilon j',\varepsilon k'\}\ \text{ with }\ i,i'\in\mathcal{L}^{3d,1}\,,\ j,j'\in\mathcal{L}^{3d,2}\,,\ k,k'\in\mathcal{L}^{3d,3}\,.$$

Moreover, whenever $u\in\mathcal{SF}_\varepsilon$ we use θ to refer to the angular lifting of u, i.e., $\theta:\mathcal{L}^{3d}_\varepsilon\to\mathbb{R}$ is a function satisfying

$$u(\varepsilon\beta)=\exp\big(\iota\theta(\varepsilon\beta)\big)\ \text{ for every } \beta\in\mathcal{L}^{3d}\,. \tag{2.11}$$

Remark 2.4 (Expressing the chirality in the angular variable). It is sometimes convenient to express both the chirality and the energy associated to a spin field $u\in\mathcal{SF}_\varepsilon$ in terms of its angular lifting θ. Staying within the above convention we have that

$$\chi(u,T)=\frac{2}{3\sqrt{3}}\Big(\sin\big(\theta(\varepsilon j)-\theta(\varepsilon i)\big)+\sin\big(\theta(\varepsilon k)-\theta(\varepsilon j)\big)+\sin\big(\theta(\varepsilon i)-\theta(\varepsilon k)\big)\Big)\,, \tag{2.12}$$

$$E^{nd}_\varepsilon(u,T)=\varepsilon^{n-1}\bigg(3+2\Big(\cos\big(\theta(\varepsilon j)-\theta(\varepsilon i)\big)+\cos\big(\theta(\varepsilon k)-\theta(\varepsilon j)\big)+\cos\big(\theta(\varepsilon i)-\theta(\varepsilon k)\big)\Big)\bigg)\,, \tag{2.13}$$

as well as

$$\big|u(\varepsilon\beta')-u(\varepsilon\beta)\big|^2=2\Big(1-\cos\big(\theta(\varepsilon\beta')-\theta(\varepsilon\beta)\big)\Big)\,. \tag{2.14}$$

Remark 2.5 (Vanishing energy). Let $\varepsilon>0$, $u\in\mathcal{SF}_\varepsilon$ and $T\in\mathcal{T}^{3d}_\varepsilon$. Then [6, Lemma 2.1 and Remark 2.2] imply that

$$E^{nd}_\varepsilon(u,T)=0\iff\chi(u,T)\in\{-1,1\}\,. \tag{2.15}$$

Since both $E^{nd}_\varepsilon(u,T)$ and $\chi(u,T)$ depend in a continuous way on u we thus deduce that for every $\delta>0$ there exists $C_\delta>0$ such that for all $\varepsilon>0$, $u\in\mathcal{SF}_\varepsilon$, and $T\in\mathcal{T}^{3d}_\varepsilon$ the implication

$$\chi(u,T)\in[-1+\delta,1-\delta]\implies E^{nd}_\varepsilon(u,T)\geq\varepsilon^{n-1}C_\delta \tag{2.16}$$

holds (see [6, Remark 2.2]).

Remark 2.6 (Optimal angles). Let $\varepsilon>0$, let $u\in\mathcal{SF}_\varepsilon$, and let $T=\mathrm{conv}\{\varepsilon i,\varepsilon j,\varepsilon k\}\in\mathcal{T}^{3d}_\varepsilon$. Moreover, let $\theta:\mathcal{L}^{3d}_\varepsilon\to\mathbb{R}$ be an angular lifting of u and set

$$\theta_1:=\theta(\varepsilon j)-\theta(\varepsilon i)\,,\qquad\theta_2:=\theta(\varepsilon k)-\theta(\varepsilon i)\,. \tag{2.17}$$

Then we have that $\chi(u,T)=\frac{2}{3\sqrt{3}}\big(\sin(\theta_1)+\sin(\theta_2-\theta_1)-\sin(\theta_2)\big)$ and thus

$$\chi(u,T)=1\iff(\theta_1,\theta_2)=\left(\frac{2\pi}{3},\frac{4\pi}{3}\right)+2\pi(z_1,z_2) \tag{2.18}$$

for some $z_1,z_2\in\mathbb{Z}$ and

$$\chi(u,T)=-1\iff(\theta_1,\theta_2)=\left(\frac{4\pi}{3},\frac{2\pi}{3}\right)+2\pi(z_1,z_2) \tag{2.19}$$

for some $z_1,z_2\in\mathbb{Z}$ (see [6, Lemma 2.1]). Since the chirality continuously depends on the angular lifting θ this in turn implies that for every $\eta>0$ there exists a $\delta_\eta>0$ such that for all $\varepsilon>0$, $u\in\mathcal{SF}_\varepsilon$ and $T\in\mathcal{T}^{3d}_\varepsilon$ the implications

$$\begin{aligned}\chi(u,T)\in(1-\delta_\eta,1]&\implies\mathrm{dist}\big(\theta_1-\tfrac{2\pi}{3};2\pi\mathbb{Z}\big)+\mathrm{dist}\big(\theta_2-\tfrac{4\pi}{3};2\pi\mathbb{Z}\big)<\eta\,,\\ \chi(u,T)\in[-1,-1+\delta_\eta)&\implies\mathrm{dist}\big(\theta_1-\tfrac{4\pi}{3};2\pi\mathbb{Z}\big)+\mathrm{dist}\big(\theta_2-\tfrac{2\pi}{3};2\pi\mathbb{Z}\big)<\eta\end{aligned} \tag{2.20}$$

hold.

Remark 2.7 (Change of Chirality). In a similar fashion as in Remark 2.6 the continuous dependence of the chirality variable on the angular variable leads to the following observation. For every $\delta > 0$ there exists $\lambda_\delta > 0$ such that for all $\varepsilon > 0$, $u \in \mathcal{SF}_\varepsilon$ and $T, T' \in \mathcal{T}_\varepsilon^{3d}$ the implication

$$\operatorname{dist}\big(\theta(\varepsilon\beta') - \theta(\varepsilon\beta); 2\pi\mathbb{Z}\big) < \lambda_\delta \ \text{ for all } \beta \in \{i,j,k\} \implies |\chi(u,T) - \chi(u,T')| < \delta \tag{2.21}$$

holds. To see this, let $\delta > 0$ be arbitrary, let u, T, T' be as above, let θ_1, θ_2 be the relative angles as in (2.17), and define θ_1', θ_2' accordingly with i, j, k replaced by i', j', k'. Then there exists $\lambda_\delta > 0$ such that $|\chi(u,T) - \chi(u,T')| < \delta$ whenever $\operatorname{dist}(\theta_1' - \theta_1; 2\pi\mathbb{Z}) < 2\lambda_\delta$ and $\operatorname{dist}(\theta_2' - \theta_2; 2\pi\mathbb{Z}) < 2\lambda_\delta$. For $\beta \in \{i,j,k\}$ let $z_\beta \in \mathbb{Z}$ be chosen such that $\operatorname{dist}(\theta(\varepsilon\beta') - \theta(\varepsilon\beta); 2\pi\mathbb{Z}) = |\theta(\varepsilon\beta') - \theta(\varepsilon\beta) - 2\pi z_\beta|$. Suppose now that the left-hand side of (2.21) holds. Then we have that

$$\begin{aligned}\operatorname{dist}\big(\theta_1' - \theta_1; 2\pi\mathbb{Z}\big) &\le |\theta_1' - \theta_1 - 2\pi(z_j - z_i)| \le |\theta(\varepsilon j') - \theta(\varepsilon j) - 2\pi z_j| + |\theta(\varepsilon i') - \theta(\varepsilon i) - 2\pi z_i| \\ &= \operatorname{dist}\big(\theta(\varepsilon j') - \theta(\varepsilon j); 2\pi\mathbb{Z}\big) + \operatorname{dist}\big(\theta(\varepsilon i') - \theta(\varepsilon i); 2\pi\mathbb{Z}\big) < 2\lambda_\delta\end{aligned}$$

and similarly $\operatorname{dist}(\theta_2' - \theta_2; 2\pi\mathbb{Z}) < 2\lambda_\delta$. By the above considerations this implies that indeed $|\chi(u,T) - \chi(u,T')| < \delta$, i.e., (2.21) is satisfied.

Remark 2.8 (Energy barrier for chirality changes). For $\delta > 0$ arbitrary let λ_δ be as in Remark 2.7. Let moreover $\varepsilon > 0$ and suppose that $u \in \mathcal{SF}_\varepsilon$ and $T, T' \in \mathcal{T}_\varepsilon^{3d}$ are such that $|\chi(u,T) - \chi(u,T')| \ge \delta$. Then (2.21) implies that there exists $\beta \in \{i,j,k\}$ such that

$$\operatorname{dist}\big(\theta(\varepsilon\beta') - \theta(\varepsilon\beta); 2\pi\mathbb{Z}\big) \ge \lambda_\delta\,.$$

Thanks to (2.2) and (2.1) this implies that

$$\big|u(\varepsilon\beta') - u(\varepsilon\beta)\big|^2 \ge \frac{4}{\pi^2}\operatorname{dist}^2\big(\theta(\varepsilon\beta') - \theta(\varepsilon\beta); 2\pi\mathbb{Z}\big) \ge \frac{4\lambda_\delta^2}{\pi^2}\,. \tag{2.22}$$

In particular, in the case $T' = T + \varepsilon e_3$ we can estimate $E^{nd}_{\varepsilon,R}$ on the prism $P = \operatorname{conv}\{T, T'\}$ via

$$E^{nd}_{\varepsilon,R}(u,P) \ge \varepsilon^{n-1}\frac{4R\lambda_\delta^2}{\pi^2}\,. \tag{2.23}$$

3. Statement of the main results

3.1. **Chirality transitions in a rigid regime.** The main result of this section is a Γ-convergence result for the discrete energies $E^{3d}_{\varepsilon,R_\varepsilon}$ in the case that (R_ε) is an increasing sequence of parameters $R_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Since the relevant variable to track the asymptotic behaviour of $E^{3d}_{\varepsilon,R_\varepsilon}$ is the chirality variable, we first express our discrete energies in terms of the latter. Specifically, we recall that for any $\Omega \subset \mathbb{R}^3$ open and bounded and for any spin field $u \in \mathcal{SF}_\varepsilon$ the chirality $\chi(u)$ belongs to $L^1(\Omega)$. Thus for every $R > 0$ we can extend $E^{3d}_{\varepsilon,R}$ to a function $\mathcal{E}^{3d}_{\varepsilon,R} : L^1(\Omega) \to [0, +\infty]$ by setting

$$\mathcal{E}^{3d}_{\varepsilon,R}(\chi,\Omega) := \inf\big\{E^{3d}_{\varepsilon,R}(u,\Omega) \colon u \in \mathcal{SF}_\varepsilon\,,\ \chi(u) = \chi \ \ \mathscr{L}^3\text{-a.e. in } \Omega\big\}\,, \tag{3.1}$$

with the convention $\inf\varnothing = +\infty$. Similarly, for $\omega \subset \mathbb{R}^2$ open and bounded we define $\mathcal{E}^{2d}_\varepsilon : L^1(\omega) \to [0, +\infty]$ by setting

$$\mathcal{E}^{2d}_\varepsilon(\chi,\omega) := \inf\big\{E^{2d}_\varepsilon(u,\omega) \colon u \in \mathcal{SF}_\varepsilon\,,\ \chi(u) = \chi \ \ \mathscr{L}^2\text{-a.e. in } \omega\big\}\,. \tag{3.2}$$

To state the Γ-convergence result for the energies $\mathcal{E}^{3d}_{\varepsilon,R_\varepsilon}$, it is convenient to first recall the Γ-limit obtained in [6] for the energies $\mathcal{E}^{2d}_\varepsilon$. To recall this result, we start by fixing two ground states $u^{\mathrm{pos}}_\varepsilon, u^{\mathrm{neg}}_\varepsilon \in \mathcal{SF}_\varepsilon$ whose chirality is globally equal to 1 and -1, respectively. These ground states can be conveniently defined via their angular lifting as

$$u^{\mathrm{pos}}_\varepsilon(\varepsilon\alpha) := \exp(\iota\theta^{\mathrm{pos}}_\varepsilon(\varepsilon\alpha)) \quad \text{and} \quad u^{\mathrm{neg}}_\varepsilon(\varepsilon\alpha) := \exp(\iota\theta^{\mathrm{neg}}_\varepsilon(\varepsilon\alpha))\,, \tag{3.3}$$

where for every $\alpha \in \mathcal{L}^{2d}$ we set

$$\theta_\varepsilon^{\mathrm{pos}}(\varepsilon\alpha) := \begin{cases} 0 & \text{if } \alpha \in \mathcal{L}^{2d,1}\,, \\ 2\pi/3 & \text{if } \alpha \in \mathcal{L}^{2d,2}\,, \\ 4\pi/3 & \text{if } \alpha \in \mathcal{L}^{2d,3}\,, \end{cases} \qquad \theta_\varepsilon^{\mathrm{neg}}(\varepsilon\alpha) := \begin{cases} 0 & \text{if } \alpha \in \mathcal{L}^{2d,1}\,, \\ 4\pi/3 & \text{if } \alpha \in \mathcal{L}^{2d,2}\,, \\ 2\pi/3 & \text{if } \alpha \in \mathcal{L}^{2d,3}\,. \end{cases} \tag{3.4}$$

We also set $u^{\mathrm{pos}} := u_1^{\mathrm{pos}}$, $u^{\mathrm{neg}} := u_1^{\mathrm{neg}}$, $\theta^{\mathrm{pos}} := \theta_1^{\mathrm{pos}}$, $\theta^{\mathrm{neg}} := \theta_1^{\mathrm{neg}}$. In this way, $u_\varepsilon^{\mathrm{pos}}$ and $u_\varepsilon^{\mathrm{neg}}$ are the two ground states depicted in Figure 1.

The ground states $u_\varepsilon^{\mathrm{pos}}$, $u_\varepsilon^{\mathrm{neg}}$ will be used as boundary data in minimisation problems for the energy E_ε^{2d} on suitably rotated (two-dimensional) cubes. Specifically, for $\nu \in \mathbb{S}^1$ we denote by

$$Q^{\nu,2d} := \{x \in \mathbb{R}^2 \colon |\langle x, \nu\rangle| < 1 \text{ and } |\langle x, \nu^\perp\rangle| < 1\} \tag{3.5}$$

the open unit cube centred at the origin with two sides orthogonal to ν. For every $\varepsilon > 0$ we then define the 'upper' and 'lower' discrete boundary of $Q^{\nu,2d}$ as

$$\partial_\varepsilon^\pm Q^{\nu,2d} = \{\alpha \in \mathcal{L}_\varepsilon^{2d} \colon \pm\langle \nu, \alpha\rangle \geq 3\varepsilon\,,\ \mathrm{dist}(\alpha, \partial Q^{\nu,2d}) \leq 3\varepsilon\}\,. \tag{3.6}$$

As in [6] we then define $\varphi^{2d} : \mathbb{S}^1 \to [0, +\infty)$ as

$$\varphi^{2d}(\nu) := \lim_{\varepsilon\to 0} \min\left\{E_\varepsilon^{2d}(u, Q^\nu) \colon u = u_\varepsilon^{\mathrm{pos}} \text{ on } \partial_\varepsilon^+ Q^{\nu,2d},\ u = u_\varepsilon^{\mathrm{neg}} \text{ on } \partial_\varepsilon^- Q^{\nu,2d}\right\}\,. \tag{3.7}$$

We are now in a position to formulate the following result which was proven in [6, Theorem 2.5].

Theorem 3.1 (Γ-limit of $\mathcal{E}_\varepsilon^{2d}$)**.** *Let $\omega \subset \mathbb{R}^2$ be open, bounded, and with Lipschitz boundary. The energies $\mathcal{E}_\varepsilon^{2d}(\,\cdot\,, \omega)$ defined as in (3.2) Γ-converge in the strong $L^1(\omega)$-topology to the functionals $\mathcal{E}^{2d} : L^1(\omega) \to [0, +\infty]$ given by*

$$\mathcal{E}^{2d}(\chi) := \begin{cases} \displaystyle\int_{S_\chi \cap \omega} \varphi^{2d}(\nu_\chi)\mathrm{d}\mathcal{H}^1 & \text{if } \chi \in BV(\omega; \{-1, 1\})\,, \\ +\infty & \text{otherwise in } L^1(\omega)\,. \end{cases} \tag{3.8}$$

Throughout the remainder of this section we assume that $\omega \subset \mathbb{R}^2$ is an open bounded set with Lipschitz boundary, $(a, b) \subset \mathbb{R}$ with $a < b$ is a bounded open interval, and $\Omega \subset \mathbb{R}^3$ is the cylindrical domain

$$\Omega = \omega \times (a, b)\,. \tag{3.9}$$

Our main result states that on cylindrical domains Ω as in (3.9) the Γ-limit of the three-dimensional energies $\mathcal{E}_{\varepsilon,R_\varepsilon}^{3d}$ essentially coincides with the Γ-limit of $\mathcal{E}_\varepsilon^{2d}$ on ω, provided $R_\varepsilon \to \infty$ as $\varepsilon \to 0$. As a key ingredient, we establish an asymptotic rigidity result for admissible limits of chiralities $\chi(u_\varepsilon)$ associated to spin fields u_ε with equi-bounded energy $E_{\varepsilon,R_\varepsilon}^{3d}(u_\varepsilon, \Omega)$. Specifically, we will show that the following holds true.

Theorem 3.2 (Compactness, Rigidity and Γ-limit for $\mathcal{E}_{\varepsilon,R_\varepsilon}^{3d}$)**.** *Let $\Omega \subset \mathbb{R}^3$ be as in (3.9) and let (R_ε) be a sequence of increasing parameters with $R_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Then the following holds true.*

(i) (Compactness and Rigidity) Suppose that $(\chi_\varepsilon) \subset L^1(\Omega)$ satisfies $\sup_\varepsilon \mathcal{E}_{\varepsilon,R_\varepsilon}^{3d}(\chi_\varepsilon, \Omega) < +\infty$. Then there exist a subsequence (not relabelled) and a function $\chi \in BV(\Omega; \{-1, 1\})$ such that $\chi_\varepsilon \to \chi$ in $L^1(\Omega)$ as $\varepsilon \to 0$. Moreover, χ satisfies the rigidity condition $\langle \nu_\chi(y), e_3\rangle = 0$ for $\mathcal{H}^2$-a.e. $y \in S_\chi$.

(ii) *(Γ-limit) The functionals $\mathcal{E}^{3d}_{\varepsilon,R_\varepsilon}$ defined according to* (3.1) *Γ-converge in the strong $L^1(\Omega)$-topology to the functional $\mathcal{E}^{3d}: L^1(\Omega) \to [0,+\infty]$ given by*

$$\mathcal{E}^{3d}(\chi) = \begin{cases} \displaystyle\int_{S_\chi} \varphi(\nu_\chi)\, d\mathcal{H}^2 & \text{if } \chi \in BV(\Omega;\{-1,1\}) \text{ and } \langle \nu_\chi(y), e_3\rangle = 0 \text{ for } \mathcal{H}^2\text{-a.e. } y \in S_\chi\,, \\ +\infty & \text{otherwise in } L^1(\Omega)\,, \end{cases}$$

where for any $\nu = (\nu', 0) \in \mathbb{S}^2 \cap \Pi_{e_3}$, $\varphi(\nu)$ is given by

$$\varphi(\nu) = 2\varphi^{2d}(\nu')$$

with φ^{2d} as in (3.7).

Remark 3.3 (Assumption on Ω). The assumption that Ω is a cylindrical domain is only needed in the construction of a recovery sequence. Instead, the compactness in the chirality variable, the rigidity of the limiting chirality, and the lower bound can be obtained for any bounded open set $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary (see Section 5.2 and Proposition 6.1).

3.2. **Application to dimension reduction.** We finally apply Theorem 3.2 to establish a dimension reduction result for frustrated spin systems. In this setting, the diverging parameters R_ε enforcing the rigidity of admissible limiting chiralities naturally appear as a scaling factor $\frac{1}{\varepsilon}$.

To set up the problem, we let $M \in \mathbb{N}$ be fixed and we consider the parameter-dependent family of thin domains

$$\Omega^M_\varepsilon := \omega \times (0, (M+1)\varepsilon)\,. \tag{3.10}$$

Then, for any $u \in \mathcal{SF}_\varepsilon$ we set

$$E^M_\varepsilon(u,\omega) := \sum_{P \in \mathcal{P}^{3d}_\varepsilon(\Omega^M_\varepsilon)} E^{2d}_{\varepsilon,1}(u,P)\,,$$

where $E^{2d}_{\varepsilon,1}$ is defined according to (2.6) with $n = 2$ and $R = 1$. As in (3.1)–(3.2) we will extend the functionals E^M_ε to the chirality variable. This will be done by adopting some conventions of [2]. Namely, to any $u : \mathcal{L}^{3d}_\varepsilon(\Omega^M_\varepsilon) \to \mathbb{S}^1$ we associate a chirality in $[L^1(\omega)]^M$ by setting for any $\ell \in \{1,\dots,M\}$

$$\chi^\ell(u) := \chi(u)_{|\omega\times\varepsilon[\ell,\ell+1)}\,. \tag{3.11}$$

We then extend $E^M_\varepsilon(\,\cdot\,,\omega)$ to a function $\mathcal{E}^M_\varepsilon(\,\cdot\,,\omega) : [L^1(\omega)]^M \to [0,+\infty]$ by setting for each $\chi = \big(\chi^1,\dots,\chi^M\big) \in [L^1(\omega)]^M$

$$\mathcal{E}^M_\varepsilon(\chi,\omega) := \inf\big\{E^M_\varepsilon(u,\omega)\colon u \in \mathcal{SF}_\varepsilon\,,\ \chi^\ell(u) = \chi^\ell \ \text{ for all } \ell = 1,\dots,M\big\}\,. \tag{3.12}$$

The following dimension-reduction result can be obtained as a consequence of Theorem 3.2 and will be proved in Section 7.

Theorem 3.4 (Dimension Reduction)**.** *Let $M \in \mathbb{N}$ be fixed and for any $\varepsilon > 0$ let Ω^M_ε be as in* (3.10) *and let $\mathcal{E}^M_\varepsilon$ be as in* (3.12)*. Then the following holds true.*

(i) *(Compactness) Suppose that $(\chi_\varepsilon) \subset [L^1(\omega)]^M$ is such that $\sup_\varepsilon \mathcal{E}^M_\varepsilon(\chi_\varepsilon,\omega) < +\infty$. Then up to subsequences (not relabelled) $\chi_\varepsilon \to \chi$ in $[L^1(\omega)]^M$ for some $\chi \in [L^1(\omega)]^M$, $\chi = (\chi^1,\dots,\chi^M)$ satisfying*

$$\chi^1 = \chi^2 = \ldots = \chi^M \in BV\big(\omega;\{-1,1\}\big)\,.$$

(ii) *(Γ-limit) the functionals $\mathcal{E}^M_\varepsilon(\,\cdot\,,\omega)$ Γ-converge in the strong $[L^1(\omega)]^M$-topology to the functional $\mathcal{E}^M : [L^1(\omega)]^M \to [0,+\infty]$ given by*

$$\mathcal{E}^M(\chi) := \begin{cases} 2(M-1)\displaystyle\int_{S_{\chi^1}\cap\omega} \varphi^{2d}(\nu_{\chi^1})\,\mathrm{d}\mathcal{H}^1 & \text{if } \chi^1 = \ldots = \chi^M \in BV\big(\omega;\{-1,1\}\big)\,, \\ \\ +\infty & \text{otherwise in } [L^1(\omega)]^M\,. \end{cases}$$

3.3. **Antiferromagnetic out-of-plane interactions.** As pointed out in Remark 2.3 the energies $\mathcal{E}^{3d}_{\varepsilon,R_\varepsilon}$ whose Γ-limit is characterised in Theorem 3.2 take into account ferromagnetic out-of-plane interactions instead of antiferromagnetic interactions. However, in this section we show that in terms of the chirality variable these two energies are equivalent.

To introduce the fully antiferromagnetic energies, it is convenient to first rewrite for any spin field $u \in \mathcal{SF}_\varepsilon$ and any prism $P = \mathrm{conv}\{T, T+\varepsilon e_3\} \in \mathcal{P}^{3d}_\varepsilon$ the energy $E^{3d}_{\varepsilon,R_\varepsilon}(u,P)$ as

$$E^{3d}_{\varepsilon,R_\varepsilon}(u,P) = E^{3d}_\varepsilon(u,T) + E^{3d}_\varepsilon(u,T+\varepsilon e_3) + \varepsilon^2 R_\varepsilon E^{\mathrm{ver,ferro}}_\varepsilon(u,P)$$

with

$$E^{\mathrm{ver,ferro}}_\varepsilon(u,P) := \sum_{\beta\in\{i,j,k\}} \big|u\big(\varepsilon(\beta+e_3)\big) - u(\varepsilon\beta)\big|^2\,,$$

where we used the convention $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$. In analogy to this, we define

$$E^{\mathrm{ver,anti}}_\varepsilon(u,P) := \sum_{\beta\in\{i,j,k\}} \big|u\big(\varepsilon(\beta+e_3)\big) + u(\varepsilon\beta)\big|^2$$

and we set

$$F^{3d}_{\varepsilon,R_\varepsilon}(u,P) := E^{3d}_\varepsilon(u,T) + E^{3d}_\varepsilon(u,T+\varepsilon e_3) + \varepsilon^2 R_\varepsilon E^{\mathrm{ver,anti}}_\varepsilon(u,P)\,.$$

Finally, in analogy to (2.7) and (3.1) we set

$$F^{3d}_{\varepsilon,R_\varepsilon}(u,\Omega) := \sum_{P\in\mathcal{P}^{3d}_\varepsilon(\Omega)} F^{3d}_{\varepsilon,R_\varepsilon}(u,P)$$

and we extend $F^{3d}_{\varepsilon,R_\varepsilon}$ to $L^1(\Omega)$ by setting

$$\mathcal{F}^{3d}_{\varepsilon,R_\varepsilon}(\chi,\Omega) := \inf\big\{F^{3d}_{\varepsilon,R_\varepsilon}(u,\Omega)\colon u\in\mathcal{SF}_\varepsilon\,, \chi(u) = \chi\ \ \mathscr{L}^3\text{-a.e. in }\Omega\big\}$$

for any $\chi \in L^1(\Omega)$. In this way, we have that

$$\mathcal{E}^{3d}_{\varepsilon,R_\varepsilon}(\chi,\Omega) = \mathcal{F}^{3d}_{\varepsilon,R_\varepsilon}(\chi,\Omega) \ \text{ for all } \chi\in L^1(\Omega)\,. \tag{3.13}$$

To see this, let us associate to any $u \in \mathcal{SF}_\varepsilon$ a spin field $v \in \mathcal{SF}_\varepsilon$ by setting

$$v(\varepsilon\alpha,\varepsilon z) := (-1)^z u(\varepsilon\alpha,\varepsilon z) \ \text{ for every } (\alpha,z)\in\mathcal{L}^{2d}_\varepsilon\times\mathbb{Z}\,.$$

In this way, u and v coincide on even layers of $\mathcal{L}^{3d}_\varepsilon$, while on odd layers v is obtained by rotating u globally by an angle π. Since the chirality variable is defined layer-wise and does not depend on global rotations, this in particular implies that $\chi(u) = \chi(v)$. Since moreover $E^{3d}_{\varepsilon,R_\varepsilon}(u,P) = F^{3d}_{\varepsilon,R_\varepsilon}(v,P)$, we obtain (3.13) by passing to the infimum over all admissible spin fields.

The identity in (3.13) shows that in terms of the chirality variable the discrete energies considered here do not distinguish between ferromagnetic and antiferromagnetic out-of-plane interactions. Since the chirality variable is the relevant variable to characterise the asymptotic behaviour of $E^{3d}_{\varepsilon,R_\varepsilon}$ and $F^{3d}_{\varepsilon,R_\varepsilon}$, this in turn implies that they share the same asymptotic behaviour. This is consistent with the fact that the only frustration in the system is due to the in-plane interactions on each triangular layer, while no additional frustration occurs in the out-of-plane interaction. In fact, a similar phenomenon occurs for the XY-model on the square lattice where the ferromagnetic and the antiferromagnetic models share the same asymptotic behaviours (see [4, Remark 4.6]). Instead, on the triangular lattice itself the asymptotic behaviour of the antiferromagnetic model and the ferromagnetic model differ (see [6] and [7] in contrast to [22]).

4. 1-dimensional result

In this section we characterise the asymptotic behaviour as $\varepsilon \to 0$ of the one-dimensional energies $E^{1d}_{\varepsilon,R}$ introduced in (2.9). Throughout this section $I \subset \mathbb{R}$ is a bounded open subset. Moreover, we associate to any $u \in \mathcal{SF}_\varepsilon$ a chirality variable $\chi(u) : \mathbb{R} \to \mathbb{R}$ by setting for all $z \in \mathbb{Z}$ and all $t \in \big[\varepsilon z, \varepsilon(z+1)\big)$

$$\chi(u)(t) := \chi\big(u, \varepsilon(T_{\mathrm{ref}} + z e_3)\big)\,,$$

where we recall that

$$T_{\mathrm{ref}} = \mathrm{conv}\,\big\{0, \hat{e}_1, \hat{e}_2\big\}\,.$$

As in (3.1) we then extend $E^{1d}_{\varepsilon,R}$ to a function $\mathcal{E}^{1d}_{\varepsilon,R} : L^1(I) \to [0,+\infty]$ by setting

$$\mathcal{E}^{1d}_{\varepsilon,R}(\chi, I) := \inf\big\{E^{1d}_{\varepsilon,R}(u, I) \colon u \in \mathcal{SF}_\varepsilon\,,\ \chi(u)(t) = \chi(t) \text{ for a.e. } t \in I\big\}\,. \tag{4.1}$$

Theorem 4.1 (Compactness and Γ-convergence in 1d)**.** *Let $R > 0$ be arbitrary, let $I \subset \mathbb{R}$ be open and bounded, and for every $\varepsilon > 0$ let $E^{1d}_{\varepsilon,R}(I)$ be as in (2.9). Then the following holds true.*

(i) (Compactness) Let (u_ε) be a sequence of spin fields $u_\varepsilon \in \mathcal{SF}_\varepsilon$ such that

$$\sup_{\varepsilon > 0} E^{1d}_{\varepsilon,R}(u_\varepsilon, I) < +\infty\,. \tag{4.2}$$

Then (up to subsequences) $\chi(u_\varepsilon) \to \chi$ in $L^1(I)$ for some $\chi \in BV(I;\{-1,1\})$.

(ii) The sequence of energies $\mathcal{E}^{1d}_{\varepsilon,R}$ defined in (4.1) Γ-converge in the strong $L^1(I)$-topology to the functional $\mathcal{E}^{1d}_R : L^1(I) \to [0,+\infty]$ given by

$$\mathcal{E}^{1d}_R(\chi) := \begin{cases} c_R \#(S_\chi \cap I) & \text{if } \chi \in BV(I;\{-1,1\})\,, \\ +\infty & \text{otherwise in } L^1(I)\,, \end{cases}$$

where c_R is defined as

$$c_R = \inf\Big\{E^{1d}_{1,R}(v, \mathbb{R}) \colon v \in \mathcal{SF}\,,\ \lim_{t \to +\infty} \chi(v)(t) = 1,\ \lim_{t \to -\infty} \chi(v)(t) = -1\ \Big\}\,. \tag{4.3}$$

(iii) There exists a constant $c_0 > 0$ such that $c_R \geq c_0\sqrt{R}$ for every $R > 0$.

Throughout this section we will consistently use the labelling $\alpha \in \{0, \hat{e}_1, \hat{e}_2\}$, so that $(\alpha, z) \in \mathcal{L}^{1d}$ for $z \in \mathbb{Z}$.

4.1. Properties of c_R.

To prove Theorem 4.1 it will be convenient to provide equivalent characterisations of the value c_R obtained by the optimal profile problem in (4.3). Establishing these characterisations and proving Theorem 4.1 (iii) is the purpose of this section.

Lemma 4.2. *For $R > 0$ let c_R be as in (4.3); let moreover*

$$\widetilde{c}_R := \inf_{S>0} \inf\Big\{E^{1d}_{1,R}\big(v, \mathbb{R}\big) \colon v \in \mathcal{SF},\ \chi(v)(t) = 1 \text{ if } t \geq S,\ \chi(v)(t) = -1 \text{ if } t \leq -S\Big\}\,.$$

Then we have that $c_R = \widetilde{c}_R$ for every $R > 0$.

Proof. For every $S>0$ we set

$$\widetilde{c}_R(S):=\inf\Big\{E^{1d}_{1,R}\big(v,\mathbb{R}\big)\colon v\in\mathcal{SF},\ \chi(v)(t)=1 \text{ if } t\geq S,\ \chi(v)(t)=-1 \text{ if } t\leq -S\Big\}\,. \tag{4.4}$$

Then any $v\in\mathcal{SF}$ which is admissible for the minimisation problem defining $\widetilde{c}_R(S)$ is admissible for the minimisation problem defining c_R in (4.3). Passing to the infimum over $S>0$ we thus obtain that $c_R\leq\widetilde{c}_R$.

It remains to show that $\widetilde{c}_R\leq c_R$. To this end, let $\eta>0$ and let v_η be a candidate for the minimisation problem defining c_R such that

$$E^{1d}_{1,R}(v_\eta,\mathbb{R})\leq c_R+\eta\,.$$

Moreover, let δ_η be chosen according to Remark 2.6 satisfying (2.20). Since

$$\lim_{t\to+\infty}\chi(v_\eta)(t)=1 \text{ and } \lim_{t\to-\infty}\chi(v_\eta)(t)=-1\,,$$

there exists $z_\eta\in\mathbb{N}$ such that

$$\chi(v_\eta)(t)\geq 1-\delta_\eta \text{ for all } t\geq z_\eta \quad\text{and}\quad \chi(v_\eta)(t)\leq -1+\delta_\eta \text{ for all } t\leq -z_\eta. \tag{4.5}$$

We now modify v_η to a function $\widetilde{v}_\eta$ admissible for $\widetilde{c}_R(z_\eta+1)$. To this end, we let θ_η be an angular lifting of v_η and we define $\widetilde{\theta}_\eta:\mathcal{L}^{1d}\to\mathbb{R}$ by setting $\widetilde{\theta}_\eta((\alpha,z)):=\theta_\eta((\alpha,z))$ for all $\alpha\in\{0,\hat{e}_1,\hat{e}_2\}$ and $z\in\mathbb{Z}$ with $|z|\leq z_\eta$, while for $z\in\mathbb{Z}$ with $z\geq z_\eta+1$ we set

$$\widetilde{\theta}_\eta((0,z)):=\theta_\eta((0,z_\eta))\,,\quad \widetilde{\theta}_\eta((\hat{e}_1,z)):=\theta_\eta((0,z))+\tfrac{2\pi}{3}\,,\quad \widetilde{\theta}_\eta((\hat{e}_2,z)):=\theta_\eta((0,z))+\tfrac{4\pi}{3}\,,$$

and for $z\in\mathbb{Z}$ with $z\leq-(z_\eta+1)$ we set

$$\widetilde{\theta}_\eta((0,z)):=\theta_\eta((0,-z_\eta))\,,\quad \widetilde{\theta}_\eta((\hat{e}_1,z)):=\theta_\eta((0,z))+\tfrac{4\pi}{3}\,,\quad \widetilde{\theta}_\eta((\hat{e}_2,z)):=\theta_\eta((0,z))+\tfrac{2\pi}{3}\,.$$

By construction, the function $\widetilde{v}_\eta:=\exp(\iota\widetilde{\theta}_\eta)\in\mathcal{SF}$ satisfies $\chi(\widetilde{v}_\eta)(t)=1$ for $t\geq z_\eta+1$ and $\chi(\widetilde{v}_\eta)(t)=-1$ for $t\leq-(z_\eta+1)$. In particular, $\widetilde{v}_\eta$ is admissible for $\widetilde{c}_R(z_\eta+1)$ and thus

$$\widetilde{c}_R\leq E^{1d}_{1,R}(\widetilde{v}_\eta,\mathbb{R})\,. \tag{4.6}$$

Moreover, Remark 2.5 implies that $E^{1d}_{1,R}(\widetilde{v}_\eta,T_{\mathrm{ref}}+ze_3)=0$ for all $z\in\mathbb{Z}$ with $|z|\geq z_\eta+1$. Let now $T^\pm_\eta:=T_{\mathrm{ref}}\pm z_\eta e_3$, $P^+_\eta=\mathrm{conv}\{T^+_\eta,T^+_\eta+e_3\}$, and $P^-_\eta:=\mathrm{conv}\{T^-_\eta-e_3,T^-_\eta\}$. Then the previous consideration yields

$$\begin{aligned}E^{1d}_{1,R}(\widetilde{v}_\eta,\mathbb{R})&=E^{1d}_{1,R}\big(\widetilde{v}_\eta,[-z_\eta-1,z_\eta+1]\big)\\&=E^{1d}_{1,R}\big(v_\eta,[-z_\eta,z_\eta]\big)+E^{1d}_{1,R}(\widetilde{v}_\eta,P^+_\eta)+E^{1d}_{1,R}(\widetilde{v}_\eta,P^-_\eta)\,.\end{aligned} \tag{4.7}$$

Finally, the construction of $\widetilde{v}_\eta$ together with (4.5), (2.1)–(2.2), and (2.20) ensure that

$$\begin{aligned}E^{1d}_{1,R}(\widetilde{v}_\eta,P^+_\eta)&=E^{1d}_{1,R}(v_\eta,T^+_\eta)\\&\qquad+R\big(|\widetilde{v}_\eta((\hat{e}_1,z_\eta+1))-v_\eta((\hat{e}_1,z_\eta))|^2+|\widetilde{v}_\eta((\hat{e}_2,z_\eta+1))-v_\eta((\hat{e}_2,z_\eta))|^2\big)\\&\leq E^{1d}_{1,R}(v_\eta,T^+_\eta)+R\,\mathrm{dist}^2\Big(\theta_\eta\big((0,z_\eta)\big)+\tfrac{2\pi}{3}-\theta_\eta\big((\hat{e}_1,z_\eta)\big);2\pi\mathbb{Z}\Big)\\&\qquad+R\,\mathrm{dist}^2\Big(\theta_\eta\big((0,z_\eta)\big)+\tfrac{4\pi}{3}-\theta_\eta\big((\hat{e}_2,z_\eta)\big);2\pi\mathbb{Z}\Big)\\&\leq E^{1d}_{1,R}(v_\eta,T^+_\eta)+2R\eta^2\end{aligned}$$

Since an analogue estimate holds for $E^{1d}_{1,R}(\widetilde{v}_\eta,P^-_\eta)$, we thus deduce from (4.6)–(4.7) that

$$\widetilde{c}_R\leq E^{1d}_{1,R}(v_\eta,\mathbb{R})+2R\eta^2\leq c_R+\eta+4R\eta^2\,,$$

and we conclude by the arbitrariness of $\eta>0$. □

Based on Lemma 4.2 we now establish a further equivalent characterisation of c_R which will be convenient to construct a recovery sequence.

Lemma 4.3. *For $R>0$ let c_R be as in (4.3); let moreover*

$$\widehat{c}_R := \inf_{S>0} \inf \Big\{ E^{1d}_{1,R}\big(v,\mathbb{R}\big) \colon v \in \mathcal{SF},\ v_{|T_{\mathrm{ref}}+ze_3} = u^{\mathrm{pos}}_{|T_{\mathrm{ref}}} \text{ and } v_{|T_{\mathrm{ref}}-ze_3} = u^{\mathrm{neg}}_{|T_{\mathrm{ref}}} \text{ if } z \geq S \Big\}\,,$$

where u^{pos} and u^{neg} are as in (3.3). Then we have that $c_R = \widehat{c}_R$ for every $R>0$.

Proof. Thanks to Lemma 4.2 it suffices to show that $\widehat{c}_R = \widetilde{c}_R$ for every $R>0$. In analogy to (4.4) we set

$$\widehat{c}_R(S) := \inf \Big\{ E^{1d}_{1,R}\big(v,\mathbb{R}\big) \colon v \in \mathcal{SF},\ v_{|T_{\mathrm{ref}}+ze_3} = u^{\mathrm{pos}}_{|T_{\mathrm{ref}}} \text{ and } v_{|T_{\mathrm{ref}}-ze_3} = u^{\mathrm{neg}}_{|T_{\mathrm{ref}}} \text{ if } z \geq S \Big\} \tag{4.8}$$

for every $S>0$. Suppose now that $v \in \mathcal{SF}$ is a candidate for the minimisation problem defining $\widehat{c}_R(S)$. Then $\chi(v)(t) = 1$ for $t \geq \lceil S \rceil$ and $\chi(v)(t) = -1$ for $t \leq -\lceil S \rceil$. Thus, v is a candidate for the minimisation problem defining $\widetilde{c}_R(\lceil S \rceil)$ as in (4.4). Passing to the infimum over $S>0$ we obtain $\widetilde{c}_R \leq \widehat{c}_R$.

It remains to show that $\widehat{c}_R \leq \widetilde{c}_R$. To this end, let $S>0$ and $N \in \mathbb{N}$ be arbitrary and let $v_N \in \mathcal{SF}$ be a candidate for the minimisation problem defining $\widetilde{c}_R(S)$ satisfying

$$E^{1d}_{1,R}(v_N,\mathbb{R}) \leq \widetilde{c}_R(S) + \frac{1}{N}\,. \tag{4.9}$$

Let moreover $\theta_N : \mathcal{L}^{1d} \to \mathbb{R}$ be an angular lifting of v_N and let $\hat{z} := \lceil S \rceil$ and $T_+ := T_{\mathrm{ref}} + \hat{z}e_3$, $T_- := T_{\mathrm{ref}} - \hat{z}e_3$. Then we know that $\chi(v_N,T_+) = 1$ and $\chi(v_N,T_-) = -1$. Thanks to Remark 2.6 it is thus not restrictive to assume that the angular lifting θ_N satisfies

$$\begin{aligned}
&\theta_N\big((\hat{e}_1,\hat{z})\big) = \theta_N\big((0,\hat{z})\big) + \frac{2\pi}{3} \quad \text{and} \quad \theta_N\big((\hat{e}_2,\hat{z})\big) = \theta_N\big((0,\hat{z})\big) + \frac{4\pi}{3}\,,\\
&\theta_N\big((\hat{e}_1,-\hat{z})\big) = \theta_N\big((0,-\hat{z})\big) + \frac{4\pi}{3} \quad \text{and} \quad \theta_N\big((\hat{e}_2,-\hat{z})\big) = \theta_N\big((0,-\hat{z})\big) + \frac{2\pi}{3}\,.
\end{aligned}$$

We now modify the function v using a linear interpolation on the angle variable between the given ground state reached in T_+ (resp. T_-) and the ground state u^{pos} (resp. u^{neg}). To achieve this, we define $\widehat{\theta}_N : \mathcal{L}^{1d} \to \mathbb{R}$ in the following way. For $\alpha \in \{0,\hat{e}_1,\hat{e}_2\}$ and $z \in \{-\hat{z},\dots,\hat{z}\}$ we set $\widehat{\theta}_N((\alpha,z)) := \theta_N((\alpha,z))$. Moreover, for $z \in \{0,\dots,N\}$ we set

$$\widehat{\theta}_N\big((\alpha,\hat{z}+z)\big) := \begin{cases} \frac{N-z}{N}\theta_N\big((0,\hat{z})\big) & \text{if } \alpha = 0\,,\\ \frac{N-z}{N}\theta_N\big((\hat{e}_2,\hat{z})\big) + \frac{2\pi z}{3N} & \text{if } \alpha = \hat{e}_1\,,\\ \frac{N-z}{N}\theta_N\big((\hat{e}_2,\hat{z})\big) + \frac{4\pi z}{3N} & \text{if } \alpha = \hat{e}_2, \end{cases}$$

and

$$\widehat{\theta}_N\big(\alpha,-(\hat{z}+z)\big)\big) := \begin{cases} \frac{N-z}{N}\theta_N\big((0,-\hat{z})\big) & \text{if } \alpha = 0\,,\\ \frac{N-z}{N}\theta_N\big((\hat{e}_1,-\hat{z})\big) + \frac{4\pi z}{3N} & \text{if } \alpha = \hat{e}_1\,,\\ \frac{N-z}{N}\theta_N\big((\hat{e}_2,-\hat{z})\big) + \frac{2\pi z}{3N} & \text{if } \alpha = \hat{e}_2\,. \end{cases}$$

Finally for $z \in \mathbb{N}$ with $z \geq N+1$ we set $\widehat{\theta}_N((\alpha,\hat{z}+z)) := \theta^{\mathrm{pos}}(\alpha)$ and $\widehat{\theta}_N(\alpha,-\hat{z}-z) := \theta^{\mathrm{neg}}(\alpha)$ for any $\alpha \in \{0,\hat{e}_1,\hat{e}_2\}$, where θ^{pos} and θ^{neg} are as in (3.4). In this way, the function $\widehat{v}_N := \exp(\iota\widehat{\theta}_N)$ is admissible for the minimisation problem defining $\widehat{c}_R(\hat{z}+N)$, which implies that

$$\widehat{c}_R \leq E^{1d}_{1,R}(\widehat{v}_N,\mathbb{R})\,. \tag{4.10}$$

Moreover, the construction of $\widehat{\theta}_N$ implies that $\chi(\widehat{v}_N, T_{\rm ref} \pm ze_3) = \pm 1$ for $z \in \mathbb{N}$ with $z \geq \hat{z}$. This in turn yields

$$E^{1d}_{1,R}(\widehat{v}_N, \mathbb{R}) \leq E^{1d}_{1,R}\big(v_N, \big[-\hat{z}, \hat{z}\big]\big) + R\sum_{z=0}^{N-1} \sum_{\alpha \in \{0,\hat{e}_1,\hat{e}_2\}} \big|\widehat{v}_N\big((\alpha, \hat{z}+z+1)\big) - \widehat{v}_N\big((\alpha, \hat{z}+z)\big)\big|^2 + R\sum_{z=0}^{N-1} \sum_{\alpha \in \{0,\hat{e}_1,\hat{e}_2\}} \big|\widehat{v}_N\big((\alpha, -\hat{z}-z-1)\big) - \widehat{v}_N\big((\alpha, -\hat{z}-z)\big)\big|^2 . \tag{4.11}$$

It remains to estimate the last two terms on the right-hand side of (4.11). For any $\alpha \in \{0, \hat{e}_1, \hat{e}_2\}$ and $z \in \{0, \dots, N-1\}$ we deduce from (2.1)–(2.2) that

$$\big|\widehat{v}_N\big((\alpha, \hat{z}+z+1)\big) - \widehat{v}_N\big((\alpha, \hat{z}+z)\big)\big|^2 \leq \Big|\widehat{\theta}_N\big((\alpha, \hat{z}+z+1)\big) - \widehat{\theta}_N\big((\alpha, \hat{z}+z)\big)\Big|^2 .$$

The definition of $\widehat{\theta}_N$ then gives

$$\sum_{\alpha \in \{0,\hat{e}_1,\hat{e}_2\}} \big|\widehat{v}_N\big((\alpha, \hat{z}+z+1)\big) - \widehat{v}_N\big((\alpha, \hat{z}+z)\big)\big|^2 \leq \frac{1}{N^2}\left(\widehat{\theta}_N\big((0,\hat{z})\big)^2 + \left|\frac{2\pi}{3} - \widehat{\theta}_N\big((\hat{e}_1,\hat{z})\big)\right|^2 + \left|\frac{4\pi}{3} - \widehat{\theta}_N\big((\hat{e}_2,\hat{z})\big)\right|^2\right) \leq \frac{C}{N^2}.$$

Summing over $z = 0, ..., N-1$, we obtain

$$\sum_{z=0}^{N-1} \sum_{\alpha \in \{0,\hat{e}_1,\hat{e}_2\}} \big|\widehat{v}_N\big((\alpha, \hat{z}+z+1)\big) - \widehat{v}_N\big((\alpha, \hat{z}+z)\big)\big|^2 \leq \frac{C}{N}.$$

Since an analogue estimate holds for the last term in (4.11), we finally deduce from (4.9) together with (4.10)–(4.11) that

$$\widehat{c}_R \leq E^{1d}_{1,R}(v_N, \mathbb{R}) + \frac{CR}{N} \leq \widetilde{c}_R(S) + \frac{CR}{N} .$$

Letting $N \to +\infty$, the above estimate yields $\widehat{c}_R \leq \widetilde{c}_R(S)$ for all $S > 0$. Passing to the infimum over $S > 0$, we finally conclude that $\widehat{c}_R \leq \widetilde{c}_R$. □

We close this section by proving Theorem 4.1 (iii).

Proof of Theorem 4.1(iii). Let $v \in \mathcal{SF}$ a candidate for the minimisation problem defining c_R and let $\theta : \mathcal{L}^{3d} \to \mathbb{R}$ be an angular lifting of v as in (2.11). The fact that $\chi(v)(t) \to \pm 1$ as $t \to \pm\infty$ allows us to choose integers $z_+, z_- \in \mathbb{Z}$ as follows. Since $\chi(v)(t) \to 1$ as $t \to +\infty$, we can choose z_+ such that

$$\chi(v)(z_+) = \chi(v, T_{\rm ref} + z_+e_3) \geq \frac{1}{2} \tag{4.12}$$

and $\chi(v)(z) < \frac{1}{2}$ for all $z < z_+$, i.e., z_+ is the smallest integer satisfying (4.12). Since moreover $\chi(v)(t) \to -1$ as $t \to -\infty$, we can choose $z_- < z_+$ such that

$$\chi(v)(z_-) = \chi(v, T_{\rm ref} + z_-e_3) \leq -\frac{1}{2} . \tag{4.13}$$

and $\chi(v)(z) > -\frac{1}{2}$ for all $z > z_-$, i.e., z_- is the largest integer in $(-\infty, z_+)$ satisfying (4.13). In this way, we have that

$$-\frac{1}{2} \leq \chi(v)(z) = \chi(v, T_{\rm ref} + ze_3) \leq \frac{1}{2} \text{ for all } z \in \{z_-, \dots, z_+\} .$$

In particular, for each prism $P = \mathrm{conv}\{T_{\mathrm{ref}} + ze_3, T_{\mathrm{ref}} + (z+1)e_3\} \in \mathcal{P}^{1d}\big([z_-, z_+]\big)$ the chirality of v lies in $(-\frac{1}{2}, \frac{1}{2})$ on at least one of the triangles $T_{\mathrm{ref}} + ze_3$ and $T_{\mathrm{ref}} + (z+1)e_3$. Remark 2.5 together with Jensen's inequality Young's inequality thus implies that

$$\begin{aligned} E^{1d}_{1,R}(v, P) &\geq C_{\frac{1}{2}} + R \sum_{\alpha \in \{0, \hat{e}_1, \hat{e}_2\}} \big|v\big((\alpha, z+1)\big) - v\big((\alpha, z)\big)\big|^2 \\ &\geq C_{\frac{1}{2}} + \frac{R}{3}\Bigg(\sum_{\alpha \in \{0, \hat{e}_1, \hat{e}_2\}} \big|v\big((\alpha, z+1)\big) - v\big((\alpha, z)\big)\big|\Bigg)^2 \\ &\geq \frac{2\sqrt{RC_{\frac{1}{2}}}}{\sqrt{3}} \sum_{\alpha \in \{0, \hat{e}_1, \hat{e}_2\}} \big|v\big((\alpha, z+1)\big) - v\big((\alpha, z)\big)\big| \,, \end{aligned} \tag{4.14}$$

where $C_{\frac{1}{2}} > 0$ is as in (2.16) with $\delta = \frac{1}{2}$. Moreover, since (4.12) ensures that

$$|\chi(v, T_{\mathrm{ref}} + z_+e_3) - \chi(v, T_{\mathrm{ref}} + z_-e_3)| \geq 1 \,,$$

we deduce from (2.22) that

$$\max_{\alpha \in \{0, \hat{e}_1, \hat{e}_2\}} \big|v\big((\alpha, z_+)\big) - v\big((\alpha, z_-)\big)\big|^2 \geq \frac{4\lambda_1^2}{\pi^2} \,, \tag{4.15}$$

where λ_1 is as in Remark 2.8 with $\delta = 1$. Summing up (4.14) over all prisms $P \in \mathcal{P}^{1d}\big([z_-, z_+]\big)$ and using (4.15) finally gives

$$\begin{aligned} E^{1d}_{1,R}(v, \mathbb{R}) &\geq \sum_{P \in \mathcal{P}^{1d}([z_-, z_+])} E^{1d}_{1,R}(v, P) \geq \frac{2\sqrt{RC_{\frac{1}{2}}}}{\sqrt{3}} \sum_{z=z_-}^{z_+-1} \sum_{\alpha \in \{0, \hat{e}_1, \hat{e}_2\}} \big|v\big((\alpha, z+1)\big) - v\big((\alpha, z)\big)\big| \\ &\geq \frac{2\sqrt{RC_{\frac{1}{2}}}}{\sqrt{3}} \sum_{\alpha \in \{0, \hat{e}_1, \hat{e}_2\}} \Bigg|\sum_{z=z_-}^{z_+-1} v\big((\alpha, z+1)\big) - v\big((\alpha, z)\big)\Bigg| \\ &\geq \frac{2\sqrt{RC_{\frac{1}{2}}}}{\sqrt{3}} \max_{\alpha \in \{0, \hat{e}_1, \hat{e}_2\}} \big|v\big((\alpha, z_+)\big) - v\big((\alpha, z_-)\big)\big| \geq \frac{4\lambda_1\sqrt{RC_{\frac{1}{2}}}}{\sqrt{3}\pi} \,, \end{aligned}$$

and we conclude by passing to the infimum over all admissible $v \in \mathcal{SF}$. □

4.2. **Proof of Compactness and Liminf-inequality.** In this section we prove Theorem 4.1 (i) and the lower bound of Theorem 4.1 (ii). To prove the compactness result we will make use of the following auxiliary statement which holds in any dimension $n \in \{1, 2, 3\}$ and provides a lower bound on the energy that is necessary to switch the sign of the chirality within a prism.

Lemma 4.4. *Let $n \in \{1, 2, 3\}$; for every $R > 0$ there exists a constant $\gamma_R > 0$ such that for all $\varepsilon > 0$, all $u \in \mathcal{SF}_\varepsilon(I)$ and all $P = \mathrm{conv}\{T, T + \varepsilon e_3\} \in \mathcal{P}^{nd}_\varepsilon$ with $\chi(u, T)\chi(u, T + \varepsilon e_3) \leq 0$ we have that*

$$E^{nd}_{\varepsilon,R}(u, P) \geq \varepsilon^{n-1}\gamma_R \,. \tag{4.16}$$

Moreover, γ_R is bounded and increasing in R.

Proof. Let $R > 0$, $\varepsilon > 0$, $u \in \mathcal{SF}_\varepsilon$ and $P = \mathrm{conv}\{T, T + \varepsilon e_3\}$ be as in the statement. To obtain (4.16) we distinguish between the two exhaustive cases $|\chi(u, T) - \chi(u, T + \varepsilon e_3)| < \frac{1}{2}$ or $|\chi(u, T) - \chi(u, T + \varepsilon e_3)| \geq \frac{1}{2}$. In the first case the requirement $\chi(u, T)\chi(u, T + \varepsilon e_3) \leq 0$ enforces that $\chi(u,T)$ and $\chi(u,T+\varepsilon e_3)$ belong to the interval $\big(-\frac{1}{2},\frac{1}{2}\big)$. In view of Remark 2.5 this implies that

$$E^{nd}_{\varepsilon,R}(u,P)\geq E^{nd}_{\varepsilon}(u,T)+E^{nd}_{\varepsilon}(u,T+\varepsilon e_3)\geq 2\varepsilon^{n-1}C_{\frac{1}{2}}\,, \tag{4.17}$$

where $C_{\frac{1}{2}}$ is as in (2.16) with $\delta=\frac{1}{2}$. In the second case, applying Remark 2.8 with $\delta=\frac{1}{2}$ gives

$$E^{nd}_{\varepsilon,R}(u,P)\geq \varepsilon^{n-1}\frac{4R\lambda^2_{\frac{1}{2}}}{\pi^2}\,. \tag{4.18}$$

Combining (4.17)–(4.18) we obtain (4.16) by setting $\gamma_R:=\min\big\{2C_{\frac{1}{2}},\frac{4R\lambda^2_{\frac{1}{2}}}{\pi^2}\big\}$. □

Based on Lemma 4.4 we can now prove Theorem 4.1 (i).

Proof of Theorem 4.1(i). We prove the assertion in two step. First, we construct a sequence $(\widehat{\chi}_\varepsilon)$ of auxiliary functions $\widehat{\chi}_\varepsilon : I\to\{-1,1\}$ such that the level sets $\{\widehat{\chi}_\varepsilon=1\}$ have uniformly bounded perimeter. Then, we show that for $\varepsilon\to 0$, the auxiliary functions and the original chirality $\chi(u_\varepsilon)$ are close with respect to the $L^1(I)$ topology.

Step 1. (Compactness of the auxiliary functions)

Let $(u_\varepsilon)_\varepsilon$ be a sequence of spin fields $u_\varepsilon\in\mathcal{SF}_\varepsilon$ such that $\sup_\varepsilon E^{1d}_{\varepsilon,R}(u_\varepsilon,I)<+\infty$. We construct a sequence $(\widehat{\chi}_\varepsilon)_\varepsilon$ of auxiliary functions defined on $\mathbb{R}^3$ by setting

$$\widehat{\chi}_\varepsilon:=\begin{cases}1 & \text{if } \chi(u_\varepsilon)>0\\ -1 & \text{otherwise.}\end{cases}$$

Let $I'\subset\subset I$. For all $\varepsilon>0$, the function $\widehat{\chi}_\varepsilon$ is constant by part on each prism $P\in\mathcal{P}^{1d}_\varepsilon(I)$ and takes values in $\{-1,1\}$. Therefore, for all $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}$ and $z^1_\varepsilon,...,z^{N_\varepsilon}_\varepsilon\in I'\cap\varepsilon\mathbb{Z}$ such that

$$\{S_{\chi_\varepsilon}\cap I'\}=\{z^1_\varepsilon,\dots,z^{N_\varepsilon}_\varepsilon\}\,. \tag{4.19}$$

By construction, this implies that $\chi(u_\varepsilon,\varepsilon T_{\rm ref}+z^n_\varepsilon e_3)\chi(u_\varepsilon,\varepsilon T_{\rm ref}+(z^n_\varepsilon+1)e_3)\leq 0$ for $n\in\{1,\dots,N_\varepsilon\}$. From Lemma 4.4 we thus deduce that

$$\gamma_R N_\varepsilon=\gamma_R\#(S_{\chi_\varepsilon}\cap I')\leq E^{1d}_{\varepsilon,R}(u_\varepsilon,I')\,, \tag{4.20}$$

and the uniform bound on $E^{1d}_{\varepsilon,R}(u_\varepsilon)$ along with the arbitrariness of I' allows to conclude that there exists $\widehat{\chi}\in BV(I;\{-1,1\})$ such that

$$\widehat{\chi}_\varepsilon\overset{L^1(I)}{\to}\widehat{\chi}$$

(see, e.g., [9, Proposition 5.3])

Step 2. (Closeness of $\hat{\chi}_\varepsilon$ and $\chi(u_\varepsilon)$)

From continuity of χ, we have for $P\in\mathcal{P}^{1d}_\varepsilon(\mathbb{R})$, that

$$\chi(u_\varepsilon,P)\in(-1+\delta,1-\delta)\implies E^{1d}_{\varepsilon,R}(u_\varepsilon,P)\geq C_\delta\,,$$

where C_δ a constant depending on δ. Let $I'\subset\subset I$, $\delta>0$ and C_δ given by the previous assertion. We define

$$P^\delta_\varepsilon:=\big\{P\in\mathcal{P}^{1d}_\varepsilon(I)\,|\,\chi(u_\varepsilon,P)\in(-1+\delta,1-\delta)\big\}\ .$$

For ε sufficiently small, we have

$$|\{|\widehat{\chi}_\varepsilon-\chi(u_\varepsilon)|>\delta\}\cap I'|\leq\varepsilon\#\mathcal{P}^\delta_\varepsilon\leq\varepsilon C_\delta^{-1}\sum_{P\in\mathcal{P}^\delta_\varepsilon}E^{1d}_{\varepsilon,R}(u_\varepsilon,P)\leq\varepsilon C_\delta^{-1}E^{1d}_{\varepsilon,R}(u_\varepsilon,I)\,.$$

Letting $\varepsilon\to 0$ and with the uniform bound (4.2), we obtain the local convergence in measure

$$\lim_{\varepsilon\to 0}|\{|\widehat{\chi}_\varepsilon-\chi(u_\varepsilon)|>\delta\}\cap I'|=0\,.$$

Additionally, we have $\|\widehat{\chi}_\varepsilon - \chi(u_\varepsilon)\|_\infty \leq 2$, therefore

$$\widehat{\chi}_\varepsilon - \chi(u_\varepsilon) \to 0 \text{ in } L^1(I)\,.$$

□

We close this section by proving the lower-bound inequality of Theorem 4.1 (ii). Specifically, we prove the following result.

Proposition 4.5 (Liminf-inequality). *Let $I \subset \mathbb{R}$ be open and bounded. For every $\chi \in L^1(I)$ and every sequence $(\chi_\varepsilon) \subset L^1(I)$ with $\chi_\varepsilon \to \chi$ in $L^1(I)$, we have that*

$$\liminf_{\varepsilon\to 0} \mathcal{E}^{1d}_{\varepsilon,R}(\chi_\varepsilon, I) \geq \mathcal{E}^{1d}_R(\chi)\,. \tag{4.21}$$

Proof. Let $(\chi_\varepsilon) \subset L^1(I)$ and $\chi \in L^1(I)$ be as in the statement. Upon extracting a subsequence it is not restrictive to assume that

$$\liminf_{\varepsilon\to 0} \mathcal{E}^{1d}_{\varepsilon,R}(\chi_\varepsilon, I) = \lim_{\varepsilon\to 0} \mathcal{E}^{1d}_{\varepsilon,R}(\chi_\varepsilon, I)\,. \tag{4.22}$$

Then, to prove (4.21) it suffices to consider the case $\sup_{\varepsilon>0} \mathcal{E}^{1d}_{\varepsilon,R}(\chi_\varepsilon, I) < +\infty$, since otherwise (4.21) trivially holds. This ensures in particular that for every $\varepsilon > 0$ we can find $u_\varepsilon \in \mathcal{SF}_\varepsilon$ satisfying $\chi(u_\varepsilon) = \chi_\varepsilon$ a.e. on I and

$$E^{1d}_{\varepsilon,R}(u_\varepsilon, I) \leq \mathcal{E}^{1d}_{\varepsilon,R}(\chi_\varepsilon, I) + \varepsilon\,. \tag{4.23}$$

In this way, we have that $\sup_{\varepsilon>0} E^{1d}_{\varepsilon,R}(u_\varepsilon, I) < +\infty$, which together with Theorem 4.1 (i) implies that $\chi \in BV\big(I; \{-1,1\}\big)$. In view of (4.22)–(4.23) it thus remains to show that

$$\lim_{\varepsilon\to 0} E^{1d}_{\varepsilon,R}(u_\varepsilon, I) \geq c_R \#\big(S_\chi \cap I\big)\,. \tag{4.24}$$

To this end, write $S_\chi \cap I = \{t_1, \ldots, t_N\}$ for some $N \in \mathbb{N}$. Up to passing to a further subsequence (not relabelled) we can assume that $\chi(u_\varepsilon) \to \chi$ pointwise a.e. in I. This ensures that for each $1 \leq m \leq N$, there exist $a^-_m, a^+_m \in I \setminus S_\chi$ with $a^-_m < t_m < a^+_m$ such that $\chi(a^-_m) \neq \chi(a^+_m)$ and $\chi(u_\varepsilon)(a^\pm_m) \to \chi(a^\pm_m) \in \{-1,1\}$ as $\varepsilon \to 0$. Moreover, for $m \in \{1, \ldots, N-1\}$ we have $a^+_m < a^-_{m+1}$. Set now $z^\pm_{\varepsilon,m} := \varepsilon\big\lfloor \frac{a^\pm_m}{\varepsilon} \big\rfloor \in \varepsilon\mathbb{Z}$. Then we have that $\chi(u_\varepsilon)(a^\pm_m) = \chi(u_\varepsilon)(z^\pm_{\varepsilon,m})$ and hence

$$\lim_{\varepsilon\to 0} \chi(u_\varepsilon)(z^\pm_{\varepsilon,m}) = \chi(a^\pm_m) \in \{-1,1\}\,. \tag{4.25}$$

Moreover, for $\varepsilon > 0$ sufficiently small we have that $z^-_{\varepsilon,m} < z^+_{\varepsilon,m} < z^-_{\varepsilon,m+1}$ for all $m \in \{1, \ldots, N-1\}$, which yields

$$E^{1d}_{\varepsilon,R}(u_\varepsilon, I) \geq \sum_{m=1}^N E^{1d}_{\varepsilon,R}\big(u_\varepsilon, [z^-_{\varepsilon,m}, z_{\varepsilon,m^+}]\big)\,. \tag{4.26}$$

We now fix $m \in \{1, \ldots, N\}$ and estimate $E^{1d}_{\varepsilon,R}\big(u_\varepsilon, [z^-_{\varepsilon,m}, z_{\varepsilon,m^+}]\big)$. To this end, we set

$$z_{\varepsilon,m} := z^-_{\varepsilon,m} + \varepsilon\bigg\lfloor \frac{z^+_{\varepsilon,m} - z^-_{\varepsilon,m}}{2\varepsilon} \bigg\rfloor\,, \quad S^+_{\varepsilon,m} := \frac{z^+_{\varepsilon,m} - z_{\varepsilon,m}}{\varepsilon}\,, \quad S^-_{\varepsilon,m} := \frac{z^-_{\varepsilon,m} - z_{\varepsilon,m}}{\varepsilon}\,,$$

and we define $v^m_\varepsilon \in \mathcal{SF}$ by setting $v^m_\varepsilon\big((\alpha, z)\big) := u_\varepsilon\big((\varepsilon\alpha, \varepsilon(z + z_{\varepsilon,m}))\big)$ for every $(\alpha, z) \in \mathcal{L}^{1d}$. In this way, we get that $\chi(v^m_\varepsilon)(S^+_{\varepsilon,m}) = \chi(u_\varepsilon)(z^+_{\varepsilon,m})$ and $\chi(v^m_\varepsilon)(S^-_{\varepsilon,m}) = \chi(z^-_{\varepsilon,m})$. Moreover,

$$E^{1d}_{\varepsilon,R}\big(u_\varepsilon, [z^-_{\varepsilon,m}, z^+_{\varepsilon,m}]\big) = E^{1d}_{1,R}\big(v^m_\varepsilon, [S^-_{\varepsilon,m}, S^+_{\varepsilon,m}]\big)\,. \tag{4.27}$$

Finally, (4.25) allows us to construct for every $\varepsilon > 0$ a function $\widetilde{v}^m_\varepsilon$ as in the proof of Lemma 4.2 satisfying

$$\begin{aligned}
\chi(\widetilde{v}^m_\varepsilon)(t) &= \chi(a^+_m) \;\text{ for all } t \geq S^+_{\varepsilon,m} + 1,\\
\chi(\widetilde{v}^m_\varepsilon)(t) &= \chi(a^-_m) \;\text{ for all } t \leq S^-_{\varepsilon,m} - 1
\end{aligned}$$

and such that

$$E^{1d}_{1,R}\big(v^m_\varepsilon,[S^-_{\varepsilon,m},S^+_{\varepsilon,m}]\big)\geq E^{1d}_{1,R}\big(\widetilde{v}^m_\varepsilon,\mathbb{R}\big)+r(\varepsilon) \tag{4.28}$$

with $r(\varepsilon)\to 0$ as $\varepsilon\to 0$. Since each $\widetilde{v}^m_\varepsilon$ is admissible for the minimisation problem defining c_R, a combination of (4.27)–(4.28) yields

$$\liminf_{\varepsilon\to 0}E^{1d}_{\varepsilon,R}\big(u_\varepsilon,[z^-_{\varepsilon,m},z^+_{\varepsilon,m}]\big)\geq c_R$$

for every $m\in\{1,\dots,N\}$. Thanks to (4.26) and the superadditivity of the liminf we thus obtain (4.24) by summing up over all $m\in\{1,\dots,N\}$.

□

4.3. **Proof of the Limsup-inequality.**

We finally establish the limsup-inequality of Theorem 4.1, that is we prove the following result.

Proposition 4.6 (Limsup-inequality)**.** *Let $I\subset\mathbb{R}$ be open and bounded. For every $\chi\in L^1(I)$ there exists a sequence of spin fields $u_\varepsilon\in\mathcal{SF}_\varepsilon$ with $\chi(u_\varepsilon)\to\chi$ in $L^1(I)$ and such that*

$$\limsup_{\varepsilon\to 0}E^{1d}_{\varepsilon,R}(u_\varepsilon,I)\leq\mathcal{E}^{1d}_R(\chi)\,. \tag{4.29}$$

Proof. Let $\chi\in L^1(I)$ be arbitrary. If $\chi\notin BV(I;\{-1,1\})$, then $\mathcal{E}^{1d}_R(\chi)=+\infty$ and (4.29) is trivially satisfied. Suppose now that $\chi\in BV(I;\{-1,1\})$ and write $S_\chi\cap I=\{t_1,\dots,t_M\}$ with $M\in\mathbb{N}$, $t_1,\dots,t_M\in I$ and $t_{m-1}<t_m$ for $m=2,\dots,M$. We first consider the case where $I=(a,b)\subset\mathbb{R}$ is an open interval and we set $t_0:=a$, $t_{M+1}:=b$. Without loss of generality we assume that $\chi=-1$ a.e. on (t_0,t_1). Since I is connected and the t_m are ordered, this implies that $\chi=1$ a.e. on the subintervals (t_m,t_{m+1}) with m odd and $\chi=-1$ a.e. on the subintervals (t_m,t_{m+1}) with m even.

For every $\varepsilon>0$ we set $z_\varepsilon:=\Big\lfloor\frac{1}{\sqrt{\varepsilon}}\Big\rfloor$ and we choose $v_\varepsilon\in\mathcal{SF}$ with

$$v_{\varepsilon|T_{\mathrm{ref}}+ze_3}=u^{\mathrm{pos}}_{|T_{\mathrm{ref}}}\text{ if }z\geq z_\varepsilon\quad\text{and}\quad v_{\varepsilon|T_{\mathrm{ref}}+ze_3}=u^{\mathrm{neg}}_{|T_{\mathrm{ref}}}\text{ if }z\leq -z_\varepsilon$$

such that

$$E^{1d}_{1,R}(v_\varepsilon,\mathbb{R})\leq\widehat{c}_R(z_\varepsilon)+\varepsilon\,, \tag{4.30}$$

where $\widehat{c}_R(z_\varepsilon)$ is as in (4.8). Moreover, for each $m\in\{1,\dots,M\}$ we set $z^m_\varepsilon:=\lfloor\frac{t_m}{\varepsilon}\rfloor\in\mathbb{Z}$ and we define $u^m_\varepsilon:\mathcal{L}^{1d}_\varepsilon\to\mathbb{S}^1$ by setting for every $\alpha\in\{0,\hat{e}_1,\hat{e}_2\}$ and every $z\in\mathbb{Z}$

$$u^m_\varepsilon\big(\varepsilon(\alpha,z)\big):=\begin{cases}v_\varepsilon\big((\alpha,z-z^m_\varepsilon)\big) & \text{if } m \text{ is odd},\\ v_\varepsilon\big((\alpha,z^m_\varepsilon-z)\big) & \text{if } m \text{ is even}.\end{cases}$$

In this way, each u^m_ε satisfies

$$E^{1d}_{\varepsilon,R}\big(u^m_\varepsilon,\varepsilon[z^m_\varepsilon-z_\varepsilon,z^m_\varepsilon+z_\varepsilon]\big)=E^{1d}_{1,R}\big(v_\varepsilon,[-z_\varepsilon,z_\varepsilon]\big)=E^{1d}_{1,R}(v_\varepsilon,\mathbb{R})\leq\widehat{c}_R(z_\varepsilon)+\varepsilon\,. \tag{4.31}$$

Moreover, for m odd the implications

$$\begin{aligned}&\varepsilon z\geq t_m+\sqrt{\varepsilon}\implies z\geq z^m_\varepsilon+z_\varepsilon\implies u^m_\varepsilon\big(\varepsilon(\alpha,z)\big)=u^{\mathrm{pos}}\big(\varepsilon(\alpha,z)\big)\\&\varepsilon z\leq t_m-\sqrt{\varepsilon}-\varepsilon\implies z\leq z^m_\varepsilon-z_\varepsilon\implies u^m_\varepsilon\big(\varepsilon(\alpha,z)\big)=u^{\mathrm{neg}}\big(\varepsilon(\alpha,z)\big)\end{aligned} \tag{4.32}$$

hold. Similar, for m even we obtain that

$$\begin{aligned}&\varepsilon z\geq t_m+\sqrt{\varepsilon}\implies u^m_\varepsilon\big(\varepsilon(\alpha,z)\big)=u^{\mathrm{neg}}\big(\varepsilon(\alpha,z)\big)\\&\varepsilon z\leq t_m-\sqrt{\varepsilon}-\varepsilon\implies u^m_\varepsilon\big(\varepsilon(\alpha,z)\big)=u^{\mathrm{pos}}\big(\varepsilon(\alpha,z)\big)\,.\end{aligned} \tag{4.33}$$

For ε sufficiently small such that

$$\sqrt{\varepsilon}+\varepsilon<\frac{1}{2}\min\big\{t_{m+1}-t_m\,:\,m\in\{0,\dots,M\}\big\}$$

this motivates to define $u_\varepsilon : \mathcal{L}^{1d}_\varepsilon \to \mathbb{S}^1$ by setting for every $\alpha \in \{0, \hat{e}_1, \hat{e}_2\}$ and every $z \in \mathbb{Z}$

$$u_\varepsilon\big(\varepsilon(\alpha,z)\big) := \begin{cases} u^m_\varepsilon\big(\varepsilon(\alpha,z)\big) & \text{if } \varepsilon z \in [t_m - \sqrt{\varepsilon} - \varepsilon, t_{m+1} - \sqrt{\varepsilon} - \varepsilon) \text{ for some } m \in \{1,\dots,M\}\,, \\ u^1_\varepsilon\big(\varepsilon(\alpha,z)\big) & \text{if } \varepsilon z \in (t_0, t_1 - \sqrt{\varepsilon} - \varepsilon)\,, \\ u^M_\varepsilon\big(\varepsilon(\alpha,z)\big) & \text{if } \varepsilon z \in [t_M - \sqrt{\varepsilon} - \varepsilon, t_{M+1})\,. \end{cases}$$

In this way, (4.32)–(4.33) ensures that

$$E^{1d}_{\varepsilon,R}(u_\varepsilon, I) = \sum_{m=1}^M E^{1d}_{\varepsilon,R}\big(u^m_\varepsilon, \varepsilon[z^m_\varepsilon - z_\varepsilon, z^m_\varepsilon + z_\varepsilon]\big)\,. \tag{4.34}$$

Together with (4.31) and Lemma 4.3 this implies that

$$\limsup_{\varepsilon\to 0} E^{1d}_{\varepsilon,R}(u_\varepsilon, I) \le \sum_{m=1}^M \limsup_{\varepsilon \to 0} \big(\widehat{c}_R(z_\varepsilon) + \varepsilon\big) = M c_R\,.$$

Finally, from (4.32)–(4.33) together with the fact that $\chi(u_\varepsilon)(t), \chi(t) \in [-1,1]$ we also deduce that

$$\|\chi(u_\varepsilon) - \chi\|_{L^1(I)} \le 4M\varepsilon z_\varepsilon \le 4M(\sqrt{\varepsilon} + \varepsilon) \to 0 \ \text{ as } \varepsilon \to 0\,,$$

which concludes the proof. Since the above construction of the recovery sequence is local, in the case that $I \subset \mathbb{R}^2$ is not an interval, we can repeat the above construction on the connected components of I. □

5. Compactness and rigidity for $E^{3d}_{\varepsilon,R_\varepsilon}$

5.1. **Proof of compactness and rigidity.** As a next step towards the proof of Theorem 3.2 we establish a compactness and rigidity result in three dimensions for the chirality variable of spin fields with equi-bounded energy. We start by proving a compactness result together with an auxiliary lower bound for the sequences $E^{3d}_{\varepsilon,R}$. Since the lower bound will be obtained from the 1-dimensional result Theorem 4.1 via a slicing procedure, it is convenient to first introduce the following notation.

For $\xi \in \mathbb{S}^2$ we recall that

$$\Pi_\xi = \{y \in \mathbb{R}^3 \,|\, \langle y, \xi\rangle = 0\} \tag{5.1}$$

is the hyperplane orthogonal to ξ and passing through the origin. Moreover, for any $U \subset \mathbb{R}^3$ open and $y \in \Pi_\xi$, we define

$$U_{\xi,y} := \{t \in \mathbb{R} \,|\, y + t\xi \in U\}\ .$$

Finally, for any $w : U \to \mathbb{R}$, we define its section $w^{\xi,y} : U_{\xi,y} \to \mathbb{R}$ by setting

$$w^{\xi,y}(t) := w(y + t\xi) \ \text{ for all } t \in U_{\xi,y}\,. \tag{5.2}$$

We recall that if $w \in BV(U;\{-1,1\})$, then for every $\xi \in \mathbb{S}^2$ and $\mathcal{H}^2$-a.e. $y \in \Pi_\xi$ we have that $w^{\xi,y} \in BV(U_{\xi,y};\{-1,1\})$ with $S_{w^{\xi,y}} = \{t \in \mathbb{R}\colon y + t\xi \in S_w\}$. Moreover,

$$\int_{\Pi_\xi} \#(S_{w^{\xi,y}} \cap U_{\xi,y})\mathrm{d}\mathcal{H}^2(y) = \int_{S_w \cap U} |\langle \nu_w(y), \xi\rangle| \mathrm{d}\mathcal{H}^2(y) \tag{5.3}$$

(see [8, Theorem 4.1]). In analogy to (5.2), for any $u_\varepsilon : \mathcal{L}^{3d}_\varepsilon \to \mathbb{S}^1$ and for $\mathcal{H}^2$-a.e. $y \in \Pi_{e_3}$ we define a function $u^{e_3,y}_\varepsilon : \mathcal{L}^{1d}_\varepsilon \to \mathbb{S}^1$ as follows. If $y \in \operatorname{int} T$ for some $T = \operatorname{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}^{2d}_\varepsilon$ with $\varepsilon i \in \mathcal{L}^{2d,1}_\varepsilon$, $\varepsilon j \in \mathcal{L}^{2d,2}_\varepsilon$, $\varepsilon k \in \mathcal{L}^{2d,3}_\varepsilon$, we set

$$u^{e_3,y}_\varepsilon(0, \varepsilon z) := u_\varepsilon(\varepsilon i, \varepsilon z)\,, \quad u^{e_3,y}_\varepsilon(\varepsilon\hat{e}_1, \varepsilon z) := u_\varepsilon(\varepsilon j, \varepsilon z)\,, \quad u^{e_3,y}_\varepsilon(\varepsilon\hat{e}_2, \varepsilon z) := u_\varepsilon(\varepsilon k, \varepsilon z) \tag{5.4}$$

for any $z \in \mathbb{Z}$. We are now in a position to state and proof the following result.

Proposition 5.1 (Compactness and auxiliary lower bound)**.** *Let $\Omega \subset \mathbb{R}^3$ be open, bounded, and with Lipschitz boundary. Let moreover $R > 0$ and let (u_ε) be a sequence of spin fields $u_\varepsilon \in \mathcal{SF}_\varepsilon$ satisfying*

$$\sup_{\varepsilon>0} E^{3d}_{\varepsilon,R}(u_\varepsilon,\Omega) < +\infty\,. \tag{5.5}$$

Then (up to subsequences) $\chi(u_\varepsilon) \to \chi$ in $L^1(\Omega)$ for some $\chi \in BV(\Omega;\{-1,1\})$. Moreover,

$$\liminf_{\varepsilon\to 0} E^{3d}_{\varepsilon,R}(u_\varepsilon,\Omega) \geq \frac{4c_R}{\sqrt{3}} \int_{S_\chi\cap\Omega} |\langle \nu_\chi(y), e_3\rangle| \mathrm{d}\mathcal{H}^2(y)\,, \tag{5.6}$$

where c_R is as in (4.3)*.*

Proof. The proof will be divided into three steps. In the first two steps we establish the compactness result in a similar manner as Theorem 4.1 (i), that is, by constructing an auxiliary chirality function and then proving closeness to the original chirality function at the limit. In the third step we will establish the lower bound (5.6).

Step 1. (Compactness of an auxiliary function) Let $R > 0$ and let $(u_\varepsilon)_\varepsilon$ a sequence of spin fields satisfying (5.5). As in the proof of Theorem 4.1 (i) we define the sequence $(\widehat{\chi}_\varepsilon)_\varepsilon$ of auxiliary functions $\widehat{\chi}_\varepsilon : \Omega \to [-1,1]$ by setting

$$\widehat{\chi}_\varepsilon(x) := \begin{cases} 1 & \text{if } \chi(u_\varepsilon)(x) > 0,\\ -1 & \text{otherwise.}\end{cases}$$

Let $\Omega' \subset\subset \Omega$, and define

$$\mathcal{P}^\partial_\varepsilon := \Big\{ P \in \mathcal{P}^{3d}_\varepsilon(\Omega') \colon \chi(u_\varepsilon,P) > 0 \text{ and } \exists\, P' \in \mathcal{N}_\varepsilon(P) \text{ with } \chi(u_\varepsilon,P') \leq 0 \Big\}\,,$$

where $\mathcal{N}_\varepsilon(P) := \{P' \in \mathcal{P}^{3d}_\varepsilon \colon \mathcal{H}^2(\overline{P} \cap \overline{P}') > 0\}$ denotes the set of neighbouring prisms for a given prism $P \in \mathcal{P}^{3d}_\varepsilon$. By definition of $\widehat{\chi}_\varepsilon$, for $\varepsilon > 0$ sufficiently small we have that

$$\partial\{\widehat{\chi}_\varepsilon = 1\} \cap \Omega' \subset \partial \left(\bigcup_{P\in\mathcal{P}^\partial_\varepsilon} P \right)\,. \tag{5.7}$$

Given that any prism $P \in \mathcal{P}^{3d}_\varepsilon(\mathbb{R}^3)$ has at most two out-of-plane and three in-plane neighbours, we can estimate the $\mathcal{H}^2$ measure of the last set in (5.7) by

$$\mathcal{H}^2\left(\partial\left(\bigcup_{P\in\mathcal{P}^\partial_\varepsilon} P\right)\right) \leq \left(\frac{\sqrt{3}}{2} + 3\right)\varepsilon^2 \#\mathcal{P}^\partial_\varepsilon\,. \tag{5.8}$$

We now estimate the cardinality of $\mathcal{P}^\partial_\varepsilon$. To this end, let $P \in \mathcal{P}^\partial_\varepsilon$ be arbitrary. By definition of $\mathcal{P}^\partial_\varepsilon$, there exists $P' \in \mathcal{N}_\varepsilon(P)$ with $\chi(u_\varepsilon,P)\chi(u_\varepsilon,P') \leq 0$. We claim that

$$E^{3d}_{\varepsilon,R}(u_\varepsilon, P\cup P') \geq \varepsilon^2 \min\left\{\tfrac{5}{3}, \gamma_R\right\} =: \varepsilon^2\gamma'_R\,, \tag{5.9}$$

where γ_R is defined in Lemma 4.4. To prove this, it suffices to consider the following two exhaustive cases. If $P = \mathrm{conv}\{T, T+\varepsilon e_3\}$ and $P' = \mathrm{conv}\{T', T'+\varepsilon e_3\}$ are in-plane neighbours, i.e., $\mathcal{H}^2(\overline{P} \cup \overline{P'}) = \varepsilon^2$, then by assumption $\chi(u_\varepsilon,T)\chi(u_\varepsilon,T') \leq 0$ and thus the 2-dimensional result obtained in [6, Lemma 3.2] implies that

$$E^{2d}_\varepsilon(u_\varepsilon, T\cup T') \geq \frac{5}{3}\varepsilon\,.$$

Therefore we directly obtain that

$$E^{3d}_{\varepsilon,R}(u_\varepsilon, P\cup P') \geq \varepsilon E^{2d}_\varepsilon(u_\varepsilon, T\cup T') \geq \frac{5}{3}\varepsilon^2\,.$$

Otherwise, P, P' are out-of-plane neighbours, i.e., $\mathcal{H}^2(\overline{P}\cap\overline{P'}) = \frac{\sqrt{3}}{2}\varepsilon^2$. Without loss of generality we assume that $P' = P + \varepsilon e_3$. Then $P = \mathrm{conv}\{T, T'\}$ and thus Lemma 4.4 yields

$$E^{3d}_{\varepsilon,R}(u_\varepsilon, P\cup P') \geq E^{3d}_{\varepsilon,R}(u_\varepsilon, P) \geq \varepsilon^2\gamma_R\,.$$

This concludes the proof of the claim. From (5.9) we now deduce that

$$\varepsilon^2\gamma'_R\#\mathcal{P}^\partial_\varepsilon \leq \sum_{P\in\mathcal{P}^\partial_\varepsilon}\sum_{P'\in\mathcal{N}_\varepsilon(P)} E^{3d}_{\varepsilon,R}(u_\varepsilon, P\cup P') \leq 5E^{3d}_{\varepsilon,R}(u_\varepsilon,\Omega')\,, \tag{5.10}$$

where the constant 5 comes from the fact that each couple of neighbouring prisms in $\mathcal{P}^\partial_\varepsilon$ is accounted for at most 5 times. Combining 5.10 and 5.8 then gives the estimate

$$\mathcal{H}^2\left(\partial\left(\bigcup_{P\in\mathcal{P}^\partial_\varepsilon} P\right)\right) \leq \frac{5(\frac{\sqrt{3}}{2}+3)}{\gamma'_R}E^{3d}_{\varepsilon,R}(u_\varepsilon,\Omega')\,. \tag{5.11}$$

Finally, combining (5.7), (5.11), and (5.5) we deduce that there exists $M > 0$ independent of Ω' such that

$$\mathcal{H}^2\left(\partial\{\widehat{\chi}_\varepsilon = 1\}\cap\Omega'\right) \leq \frac{M}{\gamma'_R}$$

for ε sufficiently small (depending on Ω'). Thus, [5, Theorem 3.39 and Remark 3.37] imply that there exists $\widehat{\chi}\in BV(\Omega;\{-1,1\})$ such that, up to a subsequence,

$$\widehat{\chi}_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \widehat{\chi} \text{ in } L^1(\Omega)\,.$$

Step 2. (Closeness of $\chi(u_\varepsilon)$ and $\widehat{\chi}_\varepsilon$)
In this step we show that

$$\|\chi(u_\varepsilon) - \widehat{\chi}_\varepsilon\|_{L^1(\Omega)} \to 0 \ \text{ as } \varepsilon\to 0\,, \tag{5.12}$$

which together with Step 1 implies that $\chi(u_\varepsilon)\to\chi$ in $L^1(\Omega)$. To achieve this, it suffices to show that for every $\delta > 0$ and every $\Omega'\subset\subset\Omega$, it holds

$$\lim_{\varepsilon\to 0}|\{|\widehat{\chi}_\varepsilon - \chi(u_\varepsilon)| > \delta\}\cap\Omega'| = 0\,, \tag{5.13}$$

then (5.12) follows from the fact that $\|\chi(u_\varepsilon) - \widehat{\chi}_\varepsilon\|_{L^\infty(\Omega)} \leq 2$. Let $\Omega'\subset\subset\Omega$ and $\delta > 0$ be arbitrary. We define the family of prisms

$$\mathcal{P}^\delta_\varepsilon = \left\{P\in\mathcal{P}^{3d}_\varepsilon(\Omega')\,|\,\chi(u_\varepsilon, P)\in(-1+\delta, 1-\delta)\right\}\,.$$

Since the volume of a prism P in $\mathcal{P}^{3d}_\varepsilon(\mathbb{R}^3)$ is $|P| = \varepsilon^3\frac{\sqrt{3}}{4}$, we have that

$$|\{|\widehat{\chi}_\varepsilon - \chi(u_\varepsilon)| > \delta\}\cap\Omega'| \leq \varepsilon^3\frac{\sqrt{3}}{4}\#\mathcal{P}^\delta_\varepsilon + r(\varepsilon)\,, \tag{5.14}$$

where the remainder r due to boundary effects is a non-negative function satisfying $r(\varepsilon)\to 0$ as $\varepsilon\to 0$. From Remark 2.5 we obtain the constant C_δ and the estimate

$$C_\delta\varepsilon^2\#\mathcal{P}^\delta_\varepsilon \leq \sum_{P\in\mathcal{P}^\delta_\varepsilon} E^{3d}_{\varepsilon,R}(u_\varepsilon, P)\,. \tag{5.15}$$

Combining 5.14 and 5.15 we get

$$|\{|\widehat{\chi}_\varepsilon - \chi(u_\varepsilon)| > \delta\}\cap\Omega'| \leq \frac{\varepsilon}{C_\delta}\sum_{P\in\mathcal{P}^\delta_\varepsilon} E^{3d}_{\varepsilon,R}(u_\varepsilon, P) + r(\varepsilon)\,.$$

Together with (5.5) this yields (5.13)
Step 3. (Proof of (5.6))

Let $\Omega' \subset\subset \Omega$ be an open set, let Π_{e_3} be as in (5.1) and for $\mathcal{H}^2$-a.e. $y \in \Pi_{e_3}$ let $u_\varepsilon^{y,\xi} : \mathcal{L}_\varepsilon^{1d} \to \mathbb{S}^1$ be as in (5.4). Then for $\varepsilon > 0$ sufficiently small (depending on Ω') the following holds true. Let $T \in \mathcal{T}_\varepsilon^{2d}$ and $b_T \in \operatorname{int} T$ its barycentre; then

$$E_{\varepsilon,R}^{3d}\Bigg(u_\varepsilon, \bigcup_{P\in\mathcal{C}_\varepsilon(T)} P\cap\Omega\Bigg) \geq \varepsilon^2 E_{\varepsilon,R}^{1d}(u_\varepsilon^{e_3,b_T}, (\Omega')_{e_3,b_T})\,. \tag{5.16}$$

Moreover, since $y \mapsto E_{\varepsilon,R}^{1d}(u_\varepsilon^{y,e_3}, (\Omega')_{e_3,y})$ is constant on T we have that

$$E_{\varepsilon,R}^{1d}(u_\varepsilon^{e_3,b_T}, (\Omega')_{e_3,b_T}) = \frac{1}{|\varepsilon T_{\mathrm{ref}}|}\int_T E_{\varepsilon,R}^{1d}(u_\varepsilon^{e_3,y}, (\Omega')_{e_3,y})d\mathcal{H}^2(y)\,, \tag{5.17}$$

where $|\varepsilon T_{\mathrm{ref}}| = \frac{\varepsilon^2\sqrt{3}}{4}$ is the area of the scaled reference triangle. By decomposing $\mathcal{P}_\varepsilon^{3d}$ into columns $\mathcal{C}_\varepsilon(T)$ with T varying in $\mathcal{T}_\varepsilon^{2d}$ we finally deduce from (5.16)–(5.17) that

$$E_{\varepsilon,R}^{3d}(u_\varepsilon,\Omega) = \sum_{T\in\mathcal{T}_\varepsilon^{2d}} E_{\varepsilon,R}^{3d}\Bigg(u_\varepsilon, \bigcup_{P\in\mathcal{C}_\varepsilon(T)} P\cap\Omega\Bigg) \geq \frac{4}{\sqrt{3}}\int_{\Pi_{e_3}} E_{\varepsilon,R}^{1d}(u_\varepsilon^{e_3,y}, (\Omega')_{e_3,y})d\mathcal{H}^2(y)\,.$$

Together with Fatou's Lemma this implies that

$$\liminf_{\varepsilon\to0} E_{\varepsilon,R}^{3d}(u_\varepsilon,\Omega) \geq \frac{4}{\sqrt{3}}\int_{\Pi_{e_3}} \liminf_{\varepsilon\to0} E_{\varepsilon,R}^{1d}(u_\varepsilon^{e_3,y}, (\Omega')_{e_3,y})d\mathcal{H}^2(y)\,. \tag{5.18}$$

Moreover Fubini's Theorem implies that for $\mathcal{H}^2$-a.e. $y \in \Pi_{e_3}$ we have that $\chi(u_\varepsilon^{e_3,y}) = (\chi(u_\varepsilon))^{e_3,y} \to \chi^{e_3,y}$ in $L^1(\Omega_{e_3,y})$. Thus, an application of Theorem 4.1 (ii) yields

$$\liminf_{\varepsilon\to0} E_{\varepsilon,R}^{1d}(u_\varepsilon^{e_3,y}, (\Omega')_{e_3,y}) \geq c_R\#\big(S_{\chi^{e_3,y}}\cap(\Omega')_{e_3,y}\big)$$

for $\mathcal{H}^2$-a.e. $y \in \Pi_{e_3}$. Together with (5.18) and (5.3) this gives

$$\liminf_{\varepsilon\to0} E_{\varepsilon,R}^{3d}(u_\varepsilon,\Omega) \geq \frac{4c_R}{\sqrt{3}}\int_{S_\chi\cap\Omega'} |\langle\nu_\chi(y), e_3\rangle| \mathrm{d}\mathcal{H}^2(y)$$

and we conclude by letting $\Omega' \nearrow \Omega$. □

Based on Proposition 5.1 we can now prove Theorem 3.2(i).

Proof of Theorem 3.2(i). Let $\Omega \subset \mathbb{R}^3$ be open bounded and with Lipschitz boundary (not necessarily cylindrical) and let $(\chi_\varepsilon) \subset L^1(\Omega)$ be such that $\sup_\varepsilon \mathcal{E}_{\varepsilon,R_\varepsilon}^{3d}(\chi_\varepsilon,\Omega) < +\infty$. By definition of $\mathcal{E}_{\varepsilon,R_\varepsilon}^{3d}$ this implies that for every $\varepsilon > 0$ there exists $u_\varepsilon \in \mathcal{SF}_\varepsilon$ with $\chi(u_\varepsilon) = \chi_\varepsilon$ $\mathscr{L}^3$-a.e. on Ω and $E_{\varepsilon,R_\varepsilon}^{3d}(u_\varepsilon,\Omega) \leq \mathcal{E}_{\varepsilon,R_\varepsilon}^{3d}(\chi_\varepsilon,\Omega) + \varepsilon$. This in particular implies that

$$M := \sup_{\varepsilon>0} E_{\varepsilon,R_\varepsilon}^{3d}(u_\varepsilon,\Omega) < +\infty\,. \tag{5.19}$$

Let moreover $R > 0$ be arbitrary. Then $E_{\varepsilon,R_\varepsilon}^{3d}(u_\varepsilon,\Omega) \geq E_{\varepsilon,R}^{3d}(u_\varepsilon,\Omega)$ for ε sufficiently small and thus (5.19) together with Proposition 5.1 yields the existence of a subsequence (not relabelled) and $\chi \in BV(\Omega;\{-1,1\})$ such that $\chi(u_\varepsilon) \to \chi$ in $L^1(\Omega)$. Moreover, the auxiliary lower bound (5.6) in Proposition (5.1) ensures that

$$M \geq \liminf_{\varepsilon\to0} E_{\varepsilon,R}^{3d}(u_\varepsilon,\Omega) \geq \frac{4c_R}{\sqrt{3}}\int_{S_\chi} |\langle\nu_\chi(y), e_3\rangle| \mathrm{d}\mathcal{H}^2(y)\,.$$

Together with Theorem 4.1 (iii) this implies that

$$M \geq \frac{4c_0}{\sqrt{3}}\sqrt{R}\int_{S_\chi} |\langle\nu_\chi(y), e_3\rangle| \mathrm{d}\mathcal{H}^2(y)\,. \tag{5.20}$$

Since $R > 0$ is arbitrary this implies that $\langle \nu_\chi(y), e_3\rangle = 0$ for $\mathcal{H}^2$-a.e. $y \in S_\chi$, which concludes the proof. $\square$

5.2. **Rigidity in a cylindrical domain.** The proof of Theorem 3.2(i) does not require Ω to be cylindrical. However, in the case that Ω is cylindrical, the condition $\langle \nu_\chi(y), e_3\rangle = 0$ enforces $\chi \in BV(\Omega; \{-1, 1\})$ to be constant (or rigid) in the e_3-direction. This result together with some further useful properties of such rigid functions is contained in Lemma 5.2. To prove it, it is convenient to introduce some notation that we will employ also in Section 6.

For $\nu \in \mathbb{S}^2$ we let $Q^{\nu,3d} \subset \mathbb{R}^3$ be a cube centred at zero with side-length 1 and two faces orthogonal to ν. Here we use the the convention that for $\nu = (\nu_1, \nu_2, 0) = (\nu', 0)$ with $\nu' \in \mathbb{S}^1$ we choose

$$Q^{\nu,3d} = \left\{x \in \mathbb{R}^3 \colon |\langle x, \nu\rangle| < \tfrac{1}{2},\ |\langle x, \nu^\perp\rangle| < \tfrac{1}{2},\ |\langle x, e_3\rangle| < \tfrac{1}{2}\right\},$$

where $\nu^\perp := (-\nu_2, \nu_1, 0) = ((\nu')^\perp, 0)$. In this way, we have that

$$Q^{\nu,3d} = Q^{\nu',2d} \times \big(-\tfrac{1}{2}, \tfrac{1}{2}\big) \tag{5.21}$$

with $Q^{\nu',2d}$ as in (3.5). For $\rho > 0$ and $x_0 \in \mathbb{R}^3$, we also write $Q^{\nu,3d}_\rho := \rho Q^{\nu,3d}$ and $Q^{\nu,3d}_\rho(x_0) := \rho Q^{\nu,3d} + x_0$ for the scaled cubes with side-length ρ centred at zero and x_0, respectively. Finally, we set

$$Q^{\nu,3d}_{\rho,\pm}(x_0) := Q^{\nu,3d}_\rho(x_0) \cap \{\pm\langle x - x_0, \nu\rangle \geq 0\}\,. \tag{5.22}$$

For $y_0 \in \mathbb{R}^2$ and $\nu' \in \mathbb{S}^1$ we use the analogue notation $Q^{\nu',2d}_\rho(y_0)$ and $Q^{\nu',2d}_{\rho,\pm}(y_0)$.

Lemma 5.2. *Let $\Omega = \omega \times (a, b)$ with $\omega \subset \mathbb{R}^2$ being an open, bounded set with Lipschitz boundary and $(a, b) \subset \mathbb{R}$ a non-empty open bounded interval. Suppose moreover that $\chi \in BV(\Omega; \{-1, 1\})$ satisfies $\langle \nu_\chi(x), e_3\rangle = 0$ for $\mathcal{H}^2$-a.e. $x \in S_\chi$ and define $\widetilde{\chi} : \omega \to \mathbb{R}$ by setting*

$$\widetilde{\chi}(y) := \frac{1}{b - a}\int_a^b \chi(y, s)\, ds$$

for every $y \in \omega$. Then $\widetilde{\chi} \in BV(\omega; \{-1, 1\})$. Moreover, we have that

$$\chi(y, t) = \widetilde{\chi}(y) \ \text{ for } \mathscr{L}^2\text{-a.e. } y \in \omega \text{ and } \mathscr{L}^1\text{-a.e. } t \in (a, b)\,. \tag{5.23}$$

Finally, for $\mathscr{L}^2$-a.e. $y \in \omega$ we have $y \in J_{\widetilde{\chi}}$ if and only if $(y, t) \in J_\chi$ for every $t \in (a, b)$. In this case, $\nu_\chi(y, t) = (\nu_{\widetilde{\chi}}(y), 0)$.

Proof. Let $\chi \in BV(\Omega; \{-1, 1\})$ be as in the statement. Then (5.3) ensures that for $\mathscr{L}^2$-a.e. $y \in \omega$ the function $\chi_y : (a, b) \to \mathbb{R}$, $\chi_y(t) := \chi(y, t)$ belongs to $BV\big((a, b); \{-1, 1\}\big)$ and satisfies $\#\big(S_{\chi_y} \cap (a, b)\big) = 0$. Since (a, b) is connected, thanks to the Poincaré inequality [5, Theorem 3.44] this implies that for $\mathcal{H}^2$-a.e. $y \in \omega$ the function χ_y is constant on (a, b) with $\chi_y(t) = \widetilde{\chi}(y)$ for $\mathcal{H}^1$-a.e. $y \in \omega$. This shows that $\widetilde{\chi}$ takes values in $\{-1, 1\}$ and (5.23).

To show that $\widetilde{\chi}$ is a BV-function, we first observe that $\widetilde{\chi} \in L^1(\omega)$. Thus, in view of [5, Theorem 3.9] it suffices to show that there exists a sequence $(\widetilde{\psi}_n) \subset C^\infty(\omega)$ with $\sup_n \|\nabla_y \widetilde{\psi}_n\|_{L^1(\omega)} < +\infty$ and $\|\widetilde{\psi}_n - \widetilde{\chi}\|_{L^1(\omega)} \to 0$ as $n \to \infty$. Since $\chi \in BV(\Omega; \{-1, 1\})$ and Ω has Lipschitz boundary, thanks to [5, Theorem 3.9 and Remark 3.22] there exists a sequence $(\psi_n) \subset C^\infty(\overline{\Omega})$ satisfying $N := \sup_n \|\nabla \psi_n\|_{L^1(\Omega)} < +\infty$ and $\|\chi - \psi_n\|_{L^1(\Omega)} \to 0$ as $n \to +\infty$. We then define $\widetilde{\psi}_n : \omega \to \mathbb{R}$ by setting $\widetilde{\psi}_n(y) := \frac{1}{b-a}\int_a^b \psi_n(y, s)\, ds$ for every $y \in \omega$. Since $\psi_n \in C^\infty(\overline{\Omega})$, we have that $\widetilde{\psi}_n \in C^\infty(\omega)$ with

$$\frac{\partial \widetilde{\psi}_n}{\partial y_k}(y) = \frac{1}{b - a}\int_a^b \frac{\partial \psi_n}{\partial y_k}(y, s)\, ds \ \text{ for } k \in \{1, 2\}\,.$$

Together with Fubini's Theorem this implies that

$$\int_\omega |\nabla_y \widetilde{\psi}_n(y)|\, dy \le \frac{1}{b-a}\int_\Omega |\nabla_y \psi_n(y,s)|\, d(y,s) \le \frac{1}{b-a}\int_\Omega |\nabla \psi_n(x)|\, dx \le \frac{N}{b-a}$$

for every $n \in \mathbb{N}$. Finally, again by Fubini we have that $\|\widetilde{\chi} - \widetilde{\psi}_n\|_{L^1(\omega)} \le \frac{1}{b-a}\|\chi - \psi_n\|_{L^1(\Omega)} \to 0$ as $n \to +\infty$, and we conclude that $\widetilde{\chi} \in BV(\omega;\{-1,1\})$.

To establish the last part of the statement, we first observe that for $\mathscr{L}^2$-a.e. $y \in \omega$, every $t \in (a,b)$, and $\rho > 0$ sufficiently small such that $(t-\rho/2, t+\rho/2) \subset (a,b)$ we can write

$$\widetilde{\chi}(y) = \frac{1}{\rho}\int_{t-\rho/2}^{t+\rho/2} \chi(y,s)\, ds\,. \tag{5.24}$$

Let now $y_0 \in \omega$ be fixed such that (5.24) holds and let $t_0 \in (a,b)$. Let moreover $\nu' \in \mathbb{S}^1$, $\nu := (\nu', 0) \in \mathbb{S}^2$ and let $z \in \{-1,1\}$. From (5.24) we deduce for $\rho > 0$ sufficiently small

$$\begin{aligned}\frac{1}{\rho^2}\int_{Q^{\nu',2d}_{\rho,\pm}(y_0)} |\widetilde{\chi}(y) - z|\, dy &= \frac{1}{\rho^3}\int_{Q^{\nu',2d}_{\rho,\pm}(y_0)} \left| \int_{t_0-\rho/2}^{t_0+\rho/2} \big(\chi(y,t) - z\big)\, dt \right| dy \\ &= \frac{1}{\rho^3}\int_{Q^{\nu,3d}_{\rho,\pm}(y_0,t_0)} |\chi(x) - z|\, dx\,,\end{aligned}$$

where in the last step we used again that χ is constant in the e_3-direction. Letting $\rho \to 0$ and using the definition of approximate jump points [5, Definition 3.67] we conclude that $y_0 \in J_{\widetilde{\chi}}$ with normal $\nu_{\widetilde{\chi}}(y_0)$ if and only if $(y_0,t_0) \in J_\chi$ with normal $\nu_\chi(y_0,t_0) = (\nu_{\widetilde{\chi}}(y_0), 0)$. This concludes the proof. □

6. Proof Theorem 3.2(ii)

We finally prove Theorem 3.2(ii). The proof will be split into two parts establishing separately the lower and the upper bound for the Γ-limit. We start by fixing the following notation.

For $n \in \{2,3\}$ and $\nu \in \mathbb{S}^{n-1}$ we let $H^\nu_+ := \{x \in \mathbb{R}^n : \langle x, \nu\rangle \ge 0\}$ and $H^\nu_- := \mathbb{R}^n \setminus H^\nu_+$ be the half two spaces separated by the hyperplane Π_ν defined in (5.1). Moreover, we let $\chi^\nu \in BV_{\mathrm{loc}}(\mathbb{R}^n;\{-1,1\})$ defined via

$$\chi^\nu(x) := \begin{cases} 1 & \text{if } x \in H^\nu_+\,, \\ -1 & \text{if } x \in H^\nu_- \end{cases} \tag{6.1}$$

be the pure jump function with $S_{\chi^\nu} = \Pi_\nu$.

6.1. **Proof of the lower bound.**

Throughout this section we assume that $(R_\varepsilon)_\varepsilon$ is an increasing sequence of parameters $R_\varepsilon \to +\infty$ as $\varepsilon \to 0$ and we establish the following lower-bound inequality.

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^3$ be open, bounded, and with Lipschitz boundary. Then for every $\chi \in L^1(\Omega)$ we have*

$$\Gamma(L^1)\text{-}\liminf_{\varepsilon\to 0} \mathcal{E}^{3d}_{\varepsilon,R_\varepsilon}(\chi) \ge \mathcal{E}^{3d}(\chi)\,,$$

where the energies $\mathcal{E}^{3d}_{\varepsilon,R_\varepsilon}$ are given by (3.1) and $\mathcal{E}^{3d}$ is given in Theorem 3.2.

Proof. Let $\chi \in L^1(\Omega)$ and $(\chi_\varepsilon) \subset L^1(\Omega)$ with $\chi_\varepsilon \to \chi$ in $L^1(\Omega)$. Upon extracting a subsequence it is not restrictive to assume that $\liminf_\varepsilon \mathcal{E}^{3d}_{\varepsilon,R_\varepsilon}(\chi_\varepsilon,\Omega) = \lim_\varepsilon \mathcal{E}^{3d}_{\varepsilon,R_\varepsilon}(\chi_\varepsilon,\Omega)$. It then suffices to consider the case where $\sup_\varepsilon \mathcal{E}^{3d}_{\varepsilon,R_\varepsilon}(\chi_\varepsilon,\Omega) < +\infty$. In particular, for every $\varepsilon > 0$ we can find $u_\varepsilon \in \mathcal{SF}_\varepsilon$ satisfying $\chi(u_\varepsilon) = \chi_\varepsilon$ a.e. on Ω and $E^{3d}_{\varepsilon,R_\varepsilon}(u_\varepsilon,\Omega) \le \mathcal{E}^{3d}_{\varepsilon,R_\varepsilon}(\chi_\varepsilon,\Omega) + \varepsilon$. In this way, $(u_\varepsilon)_\varepsilon$ satisfies $\sup_\varepsilon E^{3d}_{\varepsilon,R_\varepsilon}(u_\varepsilon,\Omega) < +\infty$ and thus Theorem 3.2(i) implies that

$$\chi \in BV\big(\Omega;\{-1,1\}\big) \quad \text{and} \quad \langle \nu_\chi(y), e_3\rangle = 0\ \mathcal{H}^2\text{-a.e. on } S_\chi\,. \tag{6.2}$$

For $\mathcal{H}^2$-a.e. $y \in S_\chi$ we can thus write $\nu_\chi(y) = (\nu'_\chi(y), 0)$ with $\nu_\chi(y)' \in \mathbb{S}^1$ and it remains to show that

$$\lim_{\varepsilon\to 0} E^{3d}_{\varepsilon,R_\varepsilon}(u_\varepsilon,\Omega) \geq 2\int_{S_\chi\cap\Omega} \varphi^{2d}(\nu'_\chi(y))\mathrm{d}\mathcal{H}^2(y)\,, \tag{6.3}$$

where φ^{2d} is as in (3.7). To this end, we define the non-negative finite Radon measures

$$\mu^{3d}_\varepsilon := \sum_{P\in\mathcal{P}^{3d}_\varepsilon(\Omega)} E^{3d}_{\varepsilon,R_\varepsilon}(u_\varepsilon,P)\delta_{b(T)}\,,$$

where for any $P = \mathrm{conv}\{T, T + ze_3\} \in \mathcal{P}^{3d}_\varepsilon$ we denote by $\delta_{b(T)}$ the Dirac delta in the barycentre $b(T)$ of T. In this way, we have that

$$\mu^{3d}_\varepsilon(\Omega) = E^{3d}_{\varepsilon,R_\varepsilon}(u_\varepsilon,\Omega)\,, \tag{6.4}$$

hence the equi-boundedness of $E^{3d}_{\varepsilon,R_\varepsilon}(u_\varepsilon,\Omega)$ ensures that $\sup_\varepsilon|\mu_\varepsilon|(\Omega) = \sup_\varepsilon \mu_\varepsilon(\Omega) < +\infty$. Thus, there exist a subsequence (not relabelled) and a non-negative finite Radon measure μ such that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ as $\varepsilon \to 0$. Thanks to the Besicovitch derivation theorem [24, Theorem 1.153], there exist two non-negative and mutually singular measures μ_j and μ_s such that

$$\mu = \mu_j\mathcal{H}^2 \llcorner S_\chi + \mu_s\,.$$

In view of (6.4), to establish (6.3) it then suffices to show that

$$\mu_j(x_0) \geq 2\varphi^{2d}\big(\nu_\chi(x_0)'\big) \ \text{ for } \mathcal{H}^2\text{-a.e. } x_0 \in S_\chi\,. \tag{6.5}$$

To this end, we choose $x_0 \in S\chi$ satisfying

(i) $\langle\nu_\chi(x_0), e_3\rangle = 0$,

(ii) $\mu_j(x_0) = \dfrac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^2\llcorner S_\chi}(x_0) = \lim\limits_{\rho\to 0}\dfrac{\mu(Q^{\nu,3d}_\rho(x_0))}{\rho^2}$,

(iii) $\lim\limits_{\rho\to 0}\dfrac{1}{\rho^3}\displaystyle\int_{Q^{\nu,3d}_{\rho,+}(x_0)}|\chi(x)-1|\,\mathrm{d}x = 0 = \lim\limits_{\rho\to 0}\dfrac{1}{\rho^3}\displaystyle\int_{Q^{\nu,3d}_{\rho,-}(x_0)}|\chi(x)+1|\,\mathrm{d}x$,

where $\nu := \nu_\chi(x_0)$ and $Q^{\nu,3d}_{\rho,\pm}(x_0)$ is as in (5.22). Note that (ii) and (iii) are satisfied for $\mathcal{H}^2$-a.e. $x_0 \in S_\chi$ thanks to the Besicovitch derivation Theorem and the definition of approximate jump point, respectively, while (i) is satisfied for $\mathcal{H}^2$-a.e. $x_0 \in S_\chi$ thanks to (6.2). Moreover, since μ is a finite Radon measure, we can choose a sequence $\rho_n \to 0$ along which $\mu(\partial Q^{\nu,3d}_{\rho_n}(x_0)) = 0$. It is not restrictive to assume that $\rho_n \in (0,1)$ for every $n \in \mathbb{N}$. Applying [5, Proposition 1.62 (a)] to the upper-semicontinuous function with compact support $\mathbb{1}_{\overline{Q^{\nu,3d}_{\rho_n}(x_0)}}$ and using the convergence $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ then yields

$$\mu\big((Q^{\nu,3d}_{\rho_n}(x_0)\big) = \mu\big(\overline{Q^{\nu,3d}_{\rho_n}(x_0)}\big) \geq \limsup_{\varepsilon\to 0}\mu_\varepsilon\big(\overline{Q^{\nu,3d}_{\rho_n}(x_0)}\big) \geq \limsup_{\varepsilon\to 0} E^{3d}_{\varepsilon,R_\varepsilon}\big(u_\varepsilon, Q^{\nu,3d}_{\rho_n}(x_0)\big)\,.$$

Together with (ii) this implies that

$$\mu_j(x_0) = \lim_{n\to+\infty}\frac{\mu(Q^{\nu,3d}_{\rho_n}(x_0))}{\rho_n^2} \geq \limsup_{n\to+\infty}\limsup_{\varepsilon\to 0}\frac{1}{\rho_n^2}E^{3d}_{\varepsilon,R_\varepsilon}\big(u_\varepsilon, Q^{\nu,3d}_{\rho_n}(x_0)\big)\,. \tag{6.6}$$

Now, we estimate from below the energy in $Q^{\nu,3d}_{\rho_n}(x_0)$ by the energy in $Q^{\nu,3d}_{\rho^\varepsilon_n}(x^\varepsilon)$ for suitable sequences $(\rho^\varepsilon_n)_\varepsilon$ and $(x^\varepsilon)_\varepsilon$ such that $\lim_\varepsilon \rho^\varepsilon_n = \rho_n$, $\lim_\varepsilon x^\varepsilon = x_0$, $x^\varepsilon \in \mathcal{L}^{3d}_\varepsilon$ and

$$\mathcal{P}^{3d}_\varepsilon(Q^{\nu,3d}_{\rho^\varepsilon_n}(x^\varepsilon)) \subset \mathcal{P}^{3d}_\varepsilon(Q^{\nu,3d}_{\rho_n}(x_0))\,.$$

Specifically, writing x_0 in the basis $\hat{e}_1, \hat{e}_2, e_3$ as $x_0 = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3e_3$ for some $a_1, a_2, a_3 \in \mathbb{R}$ we consider

$$x^\varepsilon = \varepsilon\left\lfloor\frac{a_1}{\varepsilon}\right\rfloor\hat{e}_1 + \varepsilon\left\lfloor\frac{a_2}{\varepsilon}\right\rfloor\hat{e}_2 + \varepsilon\left\lfloor\frac{a_3}{\varepsilon}\right\rfloor e_3\,.$$

(Here we consider $\hat{e}_1$ and $\hat{e}_2$ as elements of $\mathbb{R}^3$ according to Remark 2.1). By construction, $x^\varepsilon \in \mathcal{L}^{3d}_\varepsilon$ and $|x^\varepsilon - x| \leq 3\varepsilon$. In particular, $x^\varepsilon \to x_0$ as $\varepsilon \to 0$. Moreover, for $\varepsilon > 0$ sufficiently small (depending on n) we have $\rho_n^\varepsilon := \rho_n - 9\varepsilon > 0$ and $\mathcal{P}^{3d}_\varepsilon(Q^{\nu,3d}_{\rho_n^\varepsilon}(x^\varepsilon)) \subset \mathcal{P}^{3d}_\varepsilon(Q^{\nu,3d}_{\rho_n}(x_0))$. By the non-negativity of the energy this implies that

$$E^{3d}_{\varepsilon,R_\varepsilon}\big(u_\varepsilon, Q^{\nu,3d}_{\rho_n}(x_0)\big) \geq E^{3d}_{\varepsilon,R_\varepsilon}\big(u_\varepsilon, Q^{\nu,3d}_{\rho_n^\varepsilon}(x^\varepsilon)\big) = \sum_{P\in\mathcal{P}^{3d}_\varepsilon(Q^{\nu,3d}_{\rho_n^\varepsilon}(x^\varepsilon))} E^{3d}_{\varepsilon,R_\varepsilon}\big(u_\varepsilon, P\big)\,. \tag{6.7}$$

Let now $P = \mathrm{conv}\{T, T + \varepsilon e_3\} \in \mathcal{P}^{3d}_\varepsilon\big(Q^{\nu,3d}_{\rho_n^\varepsilon}(x^\varepsilon)\big)$ with $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$ and set $\sigma_n^\varepsilon := \frac{\varepsilon}{\rho_n^\varepsilon}$. Then $\frac{1}{\rho_n^\varepsilon}\big(P - x^\varepsilon\big) \in \mathcal{P}^{3d}_{\sigma_n^\varepsilon}(Q^{\nu,3d})$. Moreover, since $\varepsilon = \rho_n^\varepsilon\sigma_n^\varepsilon < \sigma_n^\varepsilon$ for $\rho_n < 1$, we have $R_\varepsilon > R_{\sigma_n^\varepsilon}$. Let us set $v_{\varepsilon,n}(z) := u_\varepsilon(x^\varepsilon + \rho_n^\varepsilon z)$ for every $z \in \mathcal{L}^{3d}_{\sigma_n^\varepsilon}$ and $\beta' := \beta + e_3$ for $\beta \in \{i,j,k\}$, so that $T' := \mathrm{conv}\{\varepsilon i', \varepsilon j', \varepsilon k'\} = T + \varepsilon e_3$. Then we have

$$\begin{aligned}
E^{3d}_{\varepsilon,R_\varepsilon}(u_\varepsilon, P) &= \varepsilon^2\bigg(|u_\varepsilon(\varepsilon i) + u_\varepsilon(\varepsilon j) + u_\varepsilon(\varepsilon k)|^2 + |u_\varepsilon(\varepsilon i') + u_\varepsilon(\varepsilon j') + u_\varepsilon\big(\varepsilon k'\big)|^2 \\
&\qquad + R_\varepsilon \sum_{\beta\in\{i,j,k\}} |u_\varepsilon(\varepsilon\beta) - u_\varepsilon(\varepsilon\beta')|^2\bigg) \\
&\geq (\rho_n^\varepsilon\sigma_n^\varepsilon)^2\bigg(|u_\varepsilon(\varepsilon i) + u_\varepsilon(\varepsilon j) + u_\varepsilon(\varepsilon k)|^2 + |u_\varepsilon(\varepsilon i') + u_\varepsilon(\varepsilon j') + u_\varepsilon\big(\varepsilon k'\big)|^2 \\
&\qquad + R_{\sigma_n^\varepsilon} \sum_{\beta\in\{i,j,k\}} |u_\varepsilon(\varepsilon\beta) - u_\varepsilon(\varepsilon\beta')|^2\bigg) \\
&= (\rho_n^\varepsilon)^2 E^{3d}_{\sigma_n^\varepsilon,R_{\sigma_n^\varepsilon}}\big(v_{n,\varepsilon}, \tfrac{1}{\rho_n^\varepsilon}(P - x^\varepsilon)\big)\,.
\end{aligned} \tag{6.8}$$

Summing up (6.8) over all prisms $P \in \mathcal{P}^{3d}_\varepsilon(Q^{\nu,3d}_{\rho_n^\varepsilon}(x^\varepsilon))$ and using (6.7) yields

$$\frac{1}{\rho_n^2} E^{3d}_{\varepsilon,R_\varepsilon}\big(u_\varepsilon, Q^{\nu,3d}_{\rho_n}(x_0)\big) \geq \frac{(\rho_n^\varepsilon)^2}{\rho_n^2} E^{3d}_{\sigma_n^\varepsilon,R_{\sigma_n^\varepsilon}}\big(v_{n,\varepsilon}, Q^{\nu,3d}\big)\,. \tag{6.9}$$

Finally, thanks to (iii) and the fact that $\chi(u_\varepsilon) \to \chi$ in $L^1(\Omega)$ we have that $\chi(v_{\varepsilon,n}) \to \chi^\nu$ in $L^1(Q^{\nu,3d})$ as first $\varepsilon \to 0$ and then $n \to \infty$, where χ^ν is as in (6.1). Thus, combining (6.6) and (6.9), a diagonal argument provides us with a sequence $\sigma_m := \sigma_{n_m}^{\varepsilon_m} \to 0$ such that $v_m := v_{\varepsilon_m,n_m}$ satisfies $\chi(v_m) \to \chi^\nu$ in $L^1(Q^{\nu,3d})$ as $m \to +\infty$ and

$$\mu_j(x_0) \geq \liminf_{m\to\infty} E^{3d}_{\sigma_m,R_{\sigma_m}}(v_m, Q^{\nu,3d})\,.$$

In particular, we have that

$$\mu_j(x_0) \geq \inf\big\{\liminf_{\sigma\to 0} E^{3d}_{\sigma,R_\sigma}(v_\sigma, Q^{\nu,3d})\colon v_\sigma \in \mathcal{SF}_\sigma,\ \chi(v_\sigma) \to \chi^\nu \text{ in } L^1(Q^{\nu,3d})\big\}$$

and thus (6.5) follows from (i) together with Lemma 6.2 below. □

Lemma 6.2. *For any $\nu \in \mathbb{S}^2$ let χ^ν is be as in* (6.1) *and let*

$$\psi(\nu) := \inf\big\{\liminf_{\varepsilon\to 0} E^{3d}_{\varepsilon,R_\varepsilon}(u_\varepsilon, Q^{\nu,3d})\colon u_\varepsilon \in \mathcal{SF}_\varepsilon\,,\ \chi(u_\varepsilon) \to \chi^\nu \text{ in } L^1(Q^{\nu,3d})\big\}\,.$$

Then $\psi(\nu) \geq 2\varphi^{2d}(\nu')$ for all $\nu = (\nu', 0) \in \mathbb{S}^2 \cap \Pi_3$.

Proof. Let $\nu = (\nu', 0) \in \mathbb{S}^2 \cap \Pi_{e_3}$ and let us choose $u_\varepsilon \in \mathcal{SF}_\varepsilon$ with $\chi(u_\varepsilon) \to \chi^\nu$ in $L^1(Q^{\nu,3d})$ and

$$\psi(\nu) = \lim_{\varepsilon\to 0} E^{3d}_{\varepsilon,R_\varepsilon}\big(u_\varepsilon, Q^{\nu,3d}\big)\,, \tag{6.10}$$

which is always possible up to passing to a subsequence. Moreover, we choose a sequence σ_ε with $\sigma_\varepsilon \to +\infty$ as $\varepsilon \to 0$ and

$$\sigma_\varepsilon \|\chi^\nu - \chi(u_\varepsilon)\|_{L^1(Q^{\nu,3d})} \to 0 \quad \text{as } \varepsilon \to 0\,, \tag{6.11}$$

and we consider the perturbed energies

$$F_\varepsilon\big(u_\varepsilon, Q^{\nu,3d}\big) := E^{3d}_{\varepsilon,R_\varepsilon}\big(u_\varepsilon, Q^{\nu,3d}\big) + \sigma_\varepsilon \|\chi^\nu - \chi(u_\varepsilon)\|_{L^1(Q^{\nu,3d})}\,.$$

In this way, (6.10) and (6.11) ensure that

$$\psi(\nu) = \lim_{\varepsilon\to 0} F_\varepsilon\big(u_\varepsilon, Q^{\nu,3d}\big)\,, \tag{6.12}$$

and it remains to estimate $F_\varepsilon(u_\varepsilon, Q^{\nu,3d})$. This will be done by slicing $Q^{\nu,3d}$ vertically into shifted copies of $Q^{\nu',2d}$ (which is possible thanks to (5.21)) and select a slice of minimal energy. To this end, for every $m \in \mathbb{Z}$, we let

$$S_{\varepsilon,m} := \big\{x \in Q^{\nu,3d} \,|\, 0 \le \langle x, e_3\rangle \le \varepsilon\big\} + \varepsilon m e_3$$

denote the slice of $Q^{\nu,3d}$ of thickness ε between $\varepsilon m e_3$ and $\varepsilon(m+1)e_3$. Recalling that $Q^{\nu,3d} = Q^{\nu',2d} \times (-\frac{1}{2}, \frac{1}{2})$, we observe that for any $m \in \{-\lfloor\frac{1}{2\varepsilon}\rfloor, \lfloor\frac{1}{2\varepsilon}\rfloor\}$ the slice $S_{\varepsilon,m}$ contains exactly one layer of prisms in $\mathcal{P}^{3d}_\varepsilon(Q^{\nu,3d})$, hence

$$F_\varepsilon\big(u_\varepsilon, Q^{\nu,3d}\big) \ge \sum_{m=-\lfloor\frac{1}{2\varepsilon}\rfloor}^{\lfloor\frac{1}{2\varepsilon}\rfloor} \Big(E^{3d}_{\varepsilon,R_\varepsilon}\big(u_\varepsilon, S_{\varepsilon,m}\big) + \sigma_\varepsilon \|\chi^\nu - \chi(u_\varepsilon)\|_{L^1(S_{\varepsilon,m})}\Big)\,. \tag{6.13}$$

Moreover, we observe that for any $m \in \{-\lfloor\frac{1}{2\varepsilon}\rfloor, \lfloor\frac{1}{2\varepsilon}\rfloor\}$ and any prism $P = \operatorname{conv}\{T, T+\varepsilon e_3\} \in \mathcal{P}^{3d}_\varepsilon(S_{\varepsilon,m})$ with $T = \operatorname{conv}\{i,j,k\}$, we have

$$\begin{aligned} E^{3d}_{\varepsilon,R_\varepsilon}(u_\varepsilon, P) &\ge \varepsilon^2\Big(|u_\varepsilon(i)+u_\varepsilon(j)+u_\varepsilon(k)|^2 + |u_\varepsilon(i+\varepsilon e_3)+u_\varepsilon(j+\varepsilon e_3)+u_\varepsilon(k+\varepsilon e_3)|^2\Big) \\ &= \varepsilon E^{2d}_\varepsilon(u_\varepsilon, T) + \varepsilon E^{2d}_\varepsilon(u_\varepsilon, T+\varepsilon e_3)\,. \end{aligned}$$

Summing up over all prisms $P \in \mathcal{P}^{3d}_\varepsilon(S_{\varepsilon,m})$ this gives

$$E^{3d}_{\varepsilon,R_\varepsilon}(u_\varepsilon, S_{\varepsilon,m}) \ge \varepsilon E^{2d}_\varepsilon(u^m_\varepsilon, Q^{\nu',2d}) + \varepsilon E^{2d}_\varepsilon(u^{m+1}_\varepsilon, Q^{\nu',2d})\,, \tag{6.14}$$

where for any $z \in \mathbb{Z}$ we define $u^z_\varepsilon : \mathcal{L}^{2d}_\varepsilon \to \mathbb{S}^1$ by setting $u^z_\varepsilon(\varepsilon\alpha) := u_\varepsilon(\varepsilon\alpha, \varepsilon z)$ for every $\alpha \in \mathcal{L}^{2d}$. In addition, we know that the restrictions $\chi(u_\varepsilon)_{|S_{\varepsilon,m}}$ and $\chi^\nu_{|S_{\varepsilon,m}}$ are constant in the e_3-direction by definition. Consequently, using Fubini's theorem, we can write

$$\begin{aligned} \|\chi^\nu - \chi(u_\varepsilon)\|_{L^1(S_{\varepsilon,m})} &= \int_0^\varepsilon \bigg(\int_{Q^{\nu',2d}} |\chi^\nu(y+te_3) - \chi(u_\varepsilon)(y+te_3)|\, d\mathscr{L}^2(y)\bigg)\, dt \\ &= \varepsilon \int_{Q^{\nu',2d}} |\chi^{\nu'}(y) - \chi(u^m_\varepsilon)(y)|\, d\mathscr{L}^2(y) = \varepsilon \|\chi^{\nu'} - \chi(u^m_\varepsilon)\|_{L^1(Q^{\nu',2d})}\,, \end{aligned} \tag{6.15}$$

where $\chi^{\nu'}$ is defined according to (6.1) with ν replaced by ν'. Combining this with (6.13) and (6.14) yields

$$\begin{aligned} F_\varepsilon\big(u_\varepsilon, Q^{\nu,3d}\big) &\ge \varepsilon \sum_{m=-\lfloor\frac{1}{2\varepsilon}\rfloor}^{\lfloor\frac{1}{2\varepsilon}\rfloor} \Big(E^{2d}_\varepsilon\big(u^m_\varepsilon, Q^{\nu',2d}\big) + E^{2d}_\varepsilon\big(u^{m+1}_\varepsilon, Q^{\nu',2d}\big) + \sigma_\varepsilon \|\chi^{\nu'} - \chi(u^m_\varepsilon)\|_{L^1(Q^{\nu',2d})}\Big) \\ &\ge \varepsilon \sum_{m=-\lfloor\frac{1}{2\varepsilon}\rfloor+1}^{\lfloor\frac{1}{2\varepsilon}\rfloor-1} \Big(2E^{2d}_\varepsilon\big(u^m_\varepsilon, Q^{\nu',2d}\big) + \sigma_\varepsilon \|\chi^{\nu'} - \chi(u^m_\varepsilon)\|_{L^1(Q^{\nu',2d})}\Big)\,. \end{aligned} \tag{6.16}$$

Now, for each $\varepsilon > 0$ there exists $m_\varepsilon \in \{-\lfloor\frac{1}{2\varepsilon}\rfloor, ..., \lfloor\frac{1}{2\varepsilon}\rfloor\}$ such that

$$2E^{2d}_\varepsilon\big(u^m_\varepsilon, Q^{\nu' 2d}\big) + \sigma_\varepsilon\|\chi^{\nu'} - \chi(u^m_\varepsilon)\|_{L^1(Q^{\nu',2d})} \geq 2E^{2d}_\varepsilon\big(u^{m_\varepsilon}_\varepsilon, Q^{\nu',2d}\big) + \sigma_\varepsilon\|\chi^{\nu'} - \chi(u^{m_\varepsilon}_\varepsilon)\|_{L^1(Q^{\nu',2d})} \quad (6.17)$$

for all $m \in \{-\lfloor\frac{1}{2\varepsilon}\rfloor + 1, \dots, \lfloor\frac{1}{2\varepsilon}\rfloor - 1\}$. Note that

$$\#\{-\lfloor\tfrac{1}{2\varepsilon}\rfloor + 1, \dots, \lfloor\tfrac{1}{2\varepsilon}\rfloor - 1\} = 2\lfloor\tfrac{1}{2\varepsilon}\rfloor - 1 \geq \frac{1-3\varepsilon}{\varepsilon}\,.$$

Thus, a combination of (6.16) and (6.17) yields

$$F_\varepsilon\big(u_\varepsilon, Q^{\nu,3d}\big) \geq (1-3\varepsilon)\Big(2E^{2d}_\varepsilon\big(u^{m_\varepsilon}_\varepsilon, Q^{\nu',2d}\big) + \sigma_\varepsilon\|\chi^{\nu'} - \chi(u^{m_\varepsilon}_\varepsilon)\|_{L^1(Q^{\nu',2d})}\Big)\,. \quad (6.18)$$

Finally, thanks to (6.12), the non-negativity of E^{2d}_ε, and the fact that $\sigma_\varepsilon \to +\infty$ as $\varepsilon \to 0$, we deduce from (6.18) that

$$\|\chi^{\nu'} - \chi(u^{m_\varepsilon}_\varepsilon)\|_{L^1(Q^{\nu',2d})} \to 0 \text{ as } \varepsilon \to 0\,.$$

Consequently, Theorem 3.1 implies that

$$\liminf_{\varepsilon\to0} E^{2d}_\varepsilon\big(u^{m_\varepsilon}_\varepsilon, Q^{\nu',2d}\big) \geq \liminf_{\varepsilon\to0} \mathcal{E}^{2d}_\varepsilon\big(\chi(u^{m_\varepsilon}_\varepsilon), Q^{\nu',2d}\big) \geq \int_{S_{\chi^{\nu'}}\cap Q^{\nu',2d}} \varphi^{2d}(\nu')\, d\mathcal{H}^1 = \varphi^{2d}(\nu')\,. \quad (6.19)$$

Finally, combining (6.12), (6.18), and (6.19) gives $\psi(\nu) \geq 2\varphi^{2d}(\nu')$, which concludes the proof. □

6.2. **Proof of the upper bound.** It remains to prove the Γ-lim sup inequality in Theorem 3.2 in the case that Ω is a cylindrical domain as in (3.9).

Proposition 6.3. *Suppose that $\Omega = \omega \times (a,b) \subset \mathbb{R}^3$ is as in* (3.9)*. For every $\chi \in L^1(\Omega)$ we have*

$$\Gamma(L^1)\text{-}\limsup_{\varepsilon\to0} \mathcal{E}^{3d}_{\varepsilon,R_\varepsilon}(\chi) \leq \mathcal{E}^{3d}(\chi)\,,$$

where $\mathcal{E}^{3d}$ is given in Theorem 3.2 and $\mathcal{E}^{3d}_{\varepsilon,R_\varepsilon}$ is defined in (3.1)*.*

Proof. It is not restrictive to assume that $\chi \in BV(\Omega;\{-1,1\})$ and that $\langle\nu_\chi, e_3\rangle = 0$, since otherwise $\mathcal{E}^{3d}(\chi) = +\infty$ and there is nothing to prove. Let now $\widetilde{\chi}$ be as in Lemma 5.2. Since $\widetilde{\chi} \in BV(\omega;\{-1,1\})$, Theorem 3.1 provides us with a sequence of spin fields $\widetilde{u}_\varepsilon : \mathcal{L}^{2d}_\varepsilon \to \mathbb{S}^1$ satisfying $\chi(\widetilde{u}_\varepsilon) \to \widetilde{\chi}$ in $L^1(\omega)$ and

$$\lim_{\varepsilon\to0} E^{2d}_\varepsilon(\widetilde{u}_\varepsilon, \omega) = \int_{J_{\widetilde{\chi}}\cap\omega} \varphi^{2d}(\nu_{\widetilde{\chi}}(y))\,\mathrm{d}\mathcal{H}^1 = \frac{1}{2(b-a)}\int_{J_\chi\cap\Omega} \varphi(\nu_\chi)\mathrm{d}\mathcal{H}^2\,, \quad (6.20)$$

where the second equation follows again from Lemma 5.2 and the definition of $\varphi(\nu_\chi)$. We now define u_ε on $\mathcal{L}^{3d}_\varepsilon$ by setting $u_\varepsilon(\varepsilon\alpha, \varepsilon z) := \widetilde{u}_\varepsilon(\varepsilon\alpha)$ for any $(\alpha, z) \in \mathcal{L}^{2d} \times \mathbb{Z}$. In this way, for any $P = \mathrm{conv}\{T + \varepsilon z e_3, T + \varepsilon(z+1)e_3\}$ with $T \in \mathcal{T}^{2d}_\varepsilon(\omega)$ we have that $\chi(u_\varepsilon, P) = \chi(\widetilde{u}_\varepsilon, T)$. Since in addition for any $y \in T$ we can write

$$\widetilde{\chi}(y) = \frac{1}{b-a}\int_a^b \chi(y,t)\,dt = \frac{1}{\varepsilon}\int_{\varepsilon z}^{\varepsilon(z+1)} \chi(y,t)\,dt$$

we find that

$$\|\chi(u_\varepsilon) - \chi\|_{L^1(P)} = \varepsilon\|\chi(\widetilde{u}_\varepsilon) - \widetilde{\chi}\|_{L^1(T)}\,. \quad (6.21)$$

Moreover, by construction we have that

$$E^{3d}_{\varepsilon,R_\varepsilon}(u_\varepsilon, P) = \varepsilon\Big(E^{2d}_\varepsilon(\widetilde{u}_\varepsilon, T + \varepsilon z e_3) + E^{2d}_\varepsilon(\widetilde{u}_\varepsilon, T + \varepsilon(z+1)e_3)\Big)\,. \quad (6.22)$$

Since $\mathcal{P}^{3d}_\varepsilon(\Omega)$ is contained in the family of sets $\mathrm{conv}\{T+\varepsilon z e_3, T+\varepsilon(z+1)e_3\}$ with $T\in\mathcal{T}^{2d}_\varepsilon(\omega)$ and $z\in\big\{\lfloor\frac{a}{\varepsilon}\rfloor,\dots,\lfloor\frac{b}{\varepsilon}\rfloor\big\}$, summing up (6.22) over all prisms $P\in\mathcal{P}^{3d}_\varepsilon(\Omega)$ yields

$$\begin{aligned} E^{3d}_{\varepsilon,R_\varepsilon}(u_\varepsilon,\Omega) &\leq \varepsilon\sum_{z=\lfloor\frac{a}{\varepsilon}\rfloor}^{\lfloor\frac{b}{\varepsilon}\rfloor}\sum_{T\in\mathcal{T}^{2d}_\varepsilon(\omega)}\Big(E^{2d}_\varepsilon(\widetilde{u}_\varepsilon,T+\varepsilon z e_3)+E^{2d}_\varepsilon(\widetilde{u}_\varepsilon,T+\varepsilon(z+1)e_3)\Big)\\ &\leq 2\varepsilon\Big(\lfloor\tfrac{b}{\varepsilon}\rfloor-\lfloor\tfrac{a}{\varepsilon}\rfloor+1\Big)E^{2d}_\varepsilon(\widetilde{u}_\varepsilon,\omega)\,. \end{aligned}\tag{6.23}$$

Combining (6.20) and (6.23) and letting $\varepsilon\to0$ finally gives

$$\limsup_{\varepsilon\to0}E^{3d}_{\varepsilon,R_\varepsilon}(u_\varepsilon,\Omega)\leq 2(b-a)\lim_{\varepsilon\to0}E^{2d}_\varepsilon(\widetilde{u}_\varepsilon,\omega)=\int_{J_\chi\cap\Omega}\varphi(\nu_\chi)\mathrm{d}\mathcal{H}^2\,.\tag{6.24}$$

Thus, to conclude it suffices to show that $\chi(u_\varepsilon)\to\chi$ in $L^1(\Omega)$. Using that $\|\chi(u_\varepsilon)-\chi\|_{L^\infty}\leq 2$ we obtain

$$\|\chi(u_\varepsilon)-\chi\|_{L^1(\Omega)}\leq\sum_{P\in\mathcal{P}^{3d}_\varepsilon(\Omega)}\|\chi(u_\varepsilon)-\chi\|_{L^1(P)}+2\mathscr{L}^3\Big(\Omega\setminus\bigcup_{P\in\mathcal{P}^{3d}_\varepsilon(\Omega)}P\Big)\,.\tag{6.25}$$

Thanks to (6.21) the first term in (6.25) can be estimated via

$$\begin{aligned}\sum_{P\in\mathcal{P}^{3d}_\varepsilon(\Omega)}\|\chi(u_\varepsilon)-\chi\|_{L^1(P)}&\leq\varepsilon\Big(\lfloor\tfrac{b}{\varepsilon}\rfloor-\lfloor\tfrac{a}{\varepsilon}\rfloor+1\Big)\sum_{T\in\mathcal{T}^{2d}_\varepsilon(\omega)}\|\chi(\widetilde{u}_\varepsilon)-\widetilde{\chi}\|_{L^1(T)}\\ &\leq(b-a+2\varepsilon)\|\chi(\widetilde{u}_\varepsilon)-\widetilde{\chi}\|_{L^1(\omega)}\,.\end{aligned}\tag{6.26}$$

Moreover, since Ω is bounded and has Lipschitz boundary, the two-dimensional Minkowski content $\mathcal{M}^2(\partial\Omega)$ of $\partial\Omega$ coincides with $\mathcal{H}^2(\partial\Omega)$, which in turn implies that

$$\mathscr{L}^3\Big(\Omega\setminus\bigcup_{P\in\mathcal{P}^{3d}_\varepsilon(\Omega)}P\Big)\leq c\varepsilon\mathcal{M}^2(\partial\Omega)=c\varepsilon\mathcal{H}^2(\partial\Omega)\tag{6.27}$$

for some $c>0$ and $\varepsilon>0$ sufficiently small. Gathering (6.25)–(6.27) and recalling that $\chi(\widetilde{u}_\varepsilon)\to\widetilde{\chi}$ in $L^1(\omega)$, we finally conclude that $\chi(u_\varepsilon)\to\chi$ in $L^1(\Omega)$. □

Remark 6.4 (Lack of a density result for rigid jump functions). The assumption of Ω being cylindrical was essential to employ the two-dimensional result Theorem 3.1 to construct a recovery sequence. For general bounded Lipschitz domains $\Omega\subset\mathbb{R}^3$ a standard approach for constructing a recovery sequence would consist in first constructing a recovery sequence for functions $\chi\in BV(\Omega;\{-1,1\})$ whose jump set is polyhedral and then argue via density. However, even though [13, Theorem 2.1] provides an approximation result for finite partitions in terms of polyhedral partitions, keeping the rigidity constraint $\langle\nu_\chi,e_3\rangle=0$ in the approximation procedure is in general not possible.

7. Proof of Theorem 3.4

This section is devoted to the proof of the dimension-reduction result. Recall that $M\in\mathbb{N}$ is fixed, $\omega\subset\mathbb{R}^2$ is a bounded open set with Lipschitz boundary, and for each $\varepsilon>0$ we have $\Omega^M_\varepsilon:=\omega\times\big(0,(M+1)\varepsilon\big)$. Also recall that for any $u\in\mathcal{SF}_\varepsilon$ we have set

$$E^M_\varepsilon(u,\omega):=\sum_{P\in\mathcal{P}^{3d}_\varepsilon(\Omega^M_\varepsilon)}\Big(E^{2d}_\varepsilon(u,T)+E^{2d}_\varepsilon(u,T+\varepsilon e_3)+\varepsilon\sum_{\beta\in\{i,j,k\}}|u(\varepsilon(\beta+e_3))-u(\varepsilon\beta)|^2\Big)\,,\tag{7.1}$$

where we use the convention $P = \mathrm{conv}\{T, T+\varepsilon e_3\}$ with $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\}$. Similar to [2], it will be convenient to associate to any function $u : \mathcal{L}^{3d}_\varepsilon(\Omega^M_\varepsilon) \to \mathbb{S}^1$ functions $u^\ell : \mathcal{L}^{2d}_\varepsilon(\omega) \to \mathbb{S}^1$ defined for $\ell \in \{1, \dots, M\}$ by setting

$$u^\ell(\alpha) := u(\alpha, \varepsilon\ell) \ \text{ for all } \alpha \in \mathcal{L}^{2d}_\varepsilon(\omega)\,. \tag{7.2}$$

Note that in this way we have that $\chi(u^\ell) = \chi^\ell(u)$ for each $\ell \in \{1, \dots, M\}$, where $\chi^\ell(u)$ is as in (3.11). Moreover, we can rewrite $E^M_\varepsilon(u, \omega)$ as

$$\begin{aligned} E^M_\varepsilon(u,\omega) = E^{2d}_\varepsilon(u^1,\omega) + E^{2d}_\varepsilon(u^M,\omega) + \sum_{\ell=2}^{M-1} 2E^{2d}_\varepsilon(u^\ell,\omega) \\ + \sum_{\ell=1}^{M-1} \sum_{T\in\mathcal{T}^{2d}_\varepsilon(\omega)} \sum_{\alpha\in\mathcal{L}^{2d}_\varepsilon(T)} \varepsilon|u^{\ell+1}(\alpha) - u^\ell(\alpha)|^2\,. \end{aligned} \tag{7.3}$$

Note that for $T = \mathrm{conv}\{\varepsilon i, \varepsilon j, \varepsilon k\} \in \mathcal{T}^{2d}_\varepsilon(\omega)$ we have $\mathcal{L}^{2d}_\varepsilon(T) = \{\varepsilon i, \varepsilon j, \varepsilon k\}$, so that (7.3) is indeed consistent with the definition of E^M_ε in (7.1). We now proceed with the proof of Theorem 3.4.

Proof of Theorem 3.4(i). Let $(\chi_\varepsilon) \subset [L^1(\omega)]^M$ be such that $\sup_\varepsilon \mathcal{E}^M_\varepsilon(\chi_\varepsilon, \omega) < +\infty$ and for each $\varepsilon > 0$ let $u_\varepsilon \in \mathcal{SF}_\varepsilon$ be such that $\chi^\ell(u_\varepsilon) = \chi^\ell_\varepsilon$ a.e. on ω for all $\ell \in \{1, \dots, M\}$ and such that $E^M_\varepsilon(u_\varepsilon, \omega) \le \mathcal{E}^M_\varepsilon(\chi_\varepsilon, \omega) + \varepsilon$. Then equi-boundedness in energy of χ_ε ensures that

$$\sup_{\varepsilon>0} E^M_\varepsilon(u_\varepsilon, \omega) < +\infty\,. \tag{7.4}$$

Below we first convert the energy $E^M_\varepsilon(u_\varepsilon, \omega)$ supported on the thin domain Ω^M_ε into a fully three-dimensional energy on the thick domain $\Omega^M := \omega \times (0, M)$ and we obtain a compactness result for an auxiliary chirality variable in $L^1(\Omega^M)$. Based on this, we will establish the compactness of χ_ε.

Step 1. (Compactness of an auxiliary variable in $L^1(\Omega^M)$)

As common in dimension-reduction problems, in this step we transform our energies E^M_ε defined on the thin domain Ω^M_ε into energies on the thick domain Ω^M by defining suitably rescaled spin fields v_ε. Specifically, we would like to obtain spin fields $v_\varepsilon \in \mathcal{SF}_\varepsilon$ satisfying for any $\ell \in \{1, \dots, M\}$ and any $\alpha \in \mathcal{L}^{2d}_\varepsilon(\omega)$

$$v_\varepsilon(\alpha, \varepsilon z) = u^\ell_\varepsilon(\alpha) \ \text{ for all } z \in \left\{\lfloor\tfrac{\ell-1}{\varepsilon}\rfloor + 1, \dots, \lfloor\tfrac{\ell}{\varepsilon}\rfloor\right\}. \tag{7.5}$$

This can be done by setting $v_\varepsilon(\alpha, \varepsilon z) := u_\varepsilon(\alpha, \varepsilon\lceil\varepsilon z\rceil)$ for every $(\alpha, \varepsilon z) \in \mathcal{L}^{3d}_\varepsilon$. For any $z \in \mathbb{Z}$ set $\omega^z_\varepsilon := \omega \times \varepsilon[z, z+1]$; then the definition of v_ε implies that for any $z \in \left\{\lfloor\frac{\ell-1}{\varepsilon}\rfloor + 1, \dots, \lfloor\frac{\ell}{\varepsilon}\rfloor\right\}$

$$\chi(v_\varepsilon, P) = \chi(u^\ell_\varepsilon, T) = \chi^\ell(u_\varepsilon, T) \ \text{ for any } P = \mathrm{conv}\{T, T+\varepsilon e_3\} \in \mathcal{P}^{3d}_\varepsilon(\omega^z_\varepsilon)\,. \tag{7.6}$$

We next show that

$$E^{3d}_{\varepsilon,\frac{1}{\varepsilon}}(v_\varepsilon, \Omega^M) \le 2E^M_\varepsilon(u_\varepsilon, \omega)\,. \tag{7.7}$$

In view of (7.5) it is convenient to split $E^{3d}_{\varepsilon,\frac{1}{\varepsilon}}(v_\varepsilon, \Omega^M)$ into the energetic contributions of v_ε on each layer ω^z_ε. Indeed, since $\mathcal{P}^{3d}_\varepsilon(\Omega^M_\varepsilon)$ can be decomposed into $\mathcal{P}^{3d}_\varepsilon(\omega^z_\varepsilon)$ with $z = 1, \dots, \lfloor\frac{M}{\varepsilon}\rfloor - 1$, we find that

$$\begin{aligned} E^{3d}_{\varepsilon,\frac{1}{\varepsilon}}\big(v_\varepsilon, \Omega^M\big) &= \sum_{z=1}^{\lfloor\frac{M}{\varepsilon}\rfloor-1} E^{3d}_{\varepsilon,\frac{1}{\varepsilon}}\big(v_\varepsilon, \omega^z_\varepsilon\big) \\ &= \sum_{\ell=1}^{M} \sum_{z=\lfloor\frac{\ell-1}{\varepsilon}\rfloor+1}^{\lfloor\frac{\ell}{\varepsilon}\rfloor-1} E^{3d}_{\varepsilon,\frac{1}{\varepsilon}}\big(v_\varepsilon, \omega^z_\varepsilon\big) + \sum_{\ell=1}^{M-1} E^{3d}_{\varepsilon,\frac{1}{\varepsilon}}\big(v_\varepsilon, \omega^{\lfloor\frac{\ell}{\varepsilon}\rfloor}_\varepsilon\big)\,, \end{aligned} \tag{7.8}$$

and we estimate the two terms on the right-hand side of (7.8) separately. To estimate the first term, let $\ell \in \{1,\dots,M\}$ be fixed and let $z \in \big\{\lfloor\frac{\ell-1}{\varepsilon}\rfloor+1,\dots,\lfloor\frac{\ell}{\varepsilon}\rfloor-1\big\}$ and $P=\mathrm{conv}\{T,T+\varepsilon e_3\}\in\mathcal{P}^{3d}_\varepsilon(\omega^z_\varepsilon)$ be arbitrary. Then (7.5) implies that

$$E^{3d}_{\varepsilon,\frac{1}{\varepsilon}}(v_\varepsilon,P)=E^{3d}_\varepsilon(v_\varepsilon,T)+E^{3d}_\varepsilon(v_\varepsilon,T+\varepsilon e_3)=2\varepsilon E^{2d}_\varepsilon\big(u^\ell_\varepsilon,T-ze_3\big)\,.$$

Summing up over all $P\in\mathcal{P}^{3d}_\varepsilon(\omega^z_\varepsilon)$ thus yields $E^{3d}_{\varepsilon,\frac{1}{\varepsilon}}(v_\varepsilon,\omega^z_\varepsilon)\le 2\varepsilon E^{2d}_\varepsilon(u^\ell_\varepsilon,\omega)$, hence

$$\sum_{z=\lfloor\frac{\ell-1}{\varepsilon}\rfloor+1}^{\lfloor\frac{\ell}{\varepsilon}\rfloor-1}E^{3d}_{\varepsilon,\frac{1}{\varepsilon}}\big(v_\varepsilon,\omega^z_\varepsilon\big)\le 2\varepsilon\Big(\big\lfloor\tfrac{\ell}{\varepsilon}\big\rfloor-\big\lfloor\tfrac{\ell-1}{\varepsilon}\big\rfloor-1\Big)E^{2d}_\varepsilon(u^\ell_\varepsilon,\omega)\le 2E^{2d}_\varepsilon(u^\ell_\varepsilon,\omega)\,. \tag{7.9}$$

To estimate the second term, let $\ell\in\{1,\dots,M-1\}$ and $P=\mathrm{conv}\{T,T+\varepsilon e_3\}\in\mathcal{P}^{3d}_\varepsilon(\omega^{\lfloor\frac{\ell}{\varepsilon}\rfloor}_\varepsilon)$ be fixed. In view of (7.5) we have that

$$E^{3d}_{\varepsilon,\frac{1}{\varepsilon}}(v_\varepsilon,P)=\varepsilon E^{2d}_\varepsilon\big(u^\ell_\varepsilon,T-\lfloor\tfrac{\ell}{\varepsilon}\rfloor e_3\big)+\varepsilon E^{2d}_\varepsilon\big(u^{\ell+1}_\varepsilon,T-\lfloor\tfrac{\ell}{\varepsilon}\rfloor e_3\big)+\varepsilon\sum_{\alpha\in\mathcal{L}^{2d}_\varepsilon(T-\lfloor\frac{\ell}{\varepsilon}\rfloor e_3)}|u^{\ell+1}_\varepsilon(\alpha)-u^\ell_\varepsilon(\alpha)|^2\,.$$

Summing up over all $P\in\mathcal{P}^{3d}_\varepsilon(\omega^{\lfloor\frac{\ell}{\varepsilon}\rfloor}_\varepsilon)$ this gives

$$E^{3d}_{\varepsilon,\frac{1}{\varepsilon}}\big(v_\varepsilon,\omega^{\lfloor\frac{\ell}{\varepsilon}\rfloor}_\varepsilon\big)\le\varepsilon E^{2d}_\varepsilon(u^\ell_\varepsilon,\omega)+\varepsilon E^{2d}_\varepsilon(u^{\ell+1}_\varepsilon,\omega)+\sum_{T\in\mathcal{T}^{2d}_\varepsilon(\omega)}\sum_{\alpha\in\mathcal{L}^{2d}_\varepsilon(T)}\varepsilon|u^{\ell+1}_\varepsilon(\alpha)-u^\ell_\varepsilon(\alpha)|^2\,. \tag{7.10}$$

Combining (7.8)–(7.10) and taking into account (7.3) we infer

$$E^{3d}_{\varepsilon,\frac{1}{\varepsilon}}(v_\varepsilon,\Omega^M)\le(1+\varepsilon)E^M_\varepsilon(u_\varepsilon,\omega)+2\varepsilon E^{2d}_\varepsilon(u^1_\varepsilon,\omega)+2\varepsilon E^{2d}_\varepsilon(u^M_\varepsilon,\omega)\le(1+2\varepsilon)E^M_\varepsilon(u_\varepsilon,\omega)\,.$$

In particular, we have that $\sup_\varepsilon E^{3d}_{\varepsilon,\frac{1}{\varepsilon}}(v_\varepsilon,\Omega^M)<+\infty$. Thus, applying (i) in Theorem 3.2 with the diverging sequence $R_\varepsilon=\frac{1}{\varepsilon}$ we deduce that up to a subsequence (not relabelled) $\chi(v_\varepsilon)\to\widetilde{\chi}$ for some $\widetilde{\chi}\in BV(\Omega^M;\{-1,1\})$ satisfying $\langle\nu_{\widetilde{\chi}}(y),e_3\rangle=0$ for $\mathcal{H}^2$-a.e. $y\in S_{\widetilde{\chi}}$. For every $\ell\in\{1,\dots,M\}$ we define $\chi^\ell:\omega\to\mathbb{R}$ by setting

$$\chi^\ell(y):=\int_{\ell-1}^{\ell}\widetilde{\chi}(y,t)\,dt \text{ for } \mathcal{H}^2\text{-a.e. } y\in\omega\,,$$

then Lemma 5.2 ensures that $\chi^1=\chi^2=\ldots=\chi^M\in BV(\omega;\{-1,1\})$ for each $\ell\in\{1,\dots,M\}$.

Step 2. (Convergence of χ^ℓ_ε to χ^ℓ)

We finally show that $\chi^\ell(u_\varepsilon)\to\chi^\ell$ in $L^1(\omega)$ for every $\ell\in\{1,\dots,M\}$. Since $\chi^\ell(u_\varepsilon)=\chi^\ell_\varepsilon$ a.e. on ω, this concludes the proof. Let $\ell\in\{1,\dots,M\}$ be fixed; by definition we have that

$$\begin{aligned}\|\chi^\ell(u_\varepsilon)-\chi^\ell\|_{L^1(\omega)}&=\int_\omega\Big|\chi^\ell(u_\varepsilon)(y)-\int_{\ell-1}^{\ell}\widetilde{\chi}(y,t)\,dt\Big|\,dy\le\int_\omega\bigg(\int_{\ell-1}^{\ell}\big|\chi^\ell(u_\varepsilon)(y)-\widetilde{\chi}(y,t)\big|\,dt\bigg)\,dy\\&\le\sum_{z=\lfloor\frac{\ell-1}{\varepsilon}\rfloor-1}^{\lfloor\frac{\ell}{\varepsilon}\rfloor}\int_\omega\bigg(\int_{\varepsilon z}^{\varepsilon(z+1)}|\chi(v_\varepsilon)(y,t)-\widetilde{\chi}(y,t)|\,dt\bigg)\,dy\\&\quad+\int_\omega\bigg(\int_{\ell-1}^{\varepsilon\lfloor\frac{\ell-1}{\varepsilon}\rfloor-1}|\chi^\ell(u_\varepsilon)(y)-\widetilde{\chi}(y,t)|\,dt\bigg)\,dy\,,\end{aligned} \tag{7.11}$$

where in the last step we used (7.6). Since both $\chi^\ell(u_\varepsilon)$ and $\widetilde{\chi}$ take values in $\{-1,1\}$, the last term on the right-hand side of (7.11) can be estimated via

$$\int_\omega \bigg(\int_{\ell-1}^{\varepsilon\lfloor\frac{\ell-1}{\varepsilon}\rfloor-1}|\chi^\ell(u_\varepsilon)(y)-\widetilde{\chi}(y,t)|\,dt\bigg)\,dy\leq 2\varepsilon\mathscr{L}^2(\omega)\,.$$

Thus, (7.11) implies that

$$\|\chi^\ell(u_\varepsilon)-\chi^\ell\|_{L^1(\omega)}\leq\|\chi(v_\varepsilon)-\widetilde{\chi}\|_{L^1(\omega\times(\ell-1,\ell))}+2\varepsilon\mathscr{L}^2(\omega)\ \to 0\ \text{ as }\varepsilon\to 0\,,$$

which concludes the proof. □

We close this section by proving Theorem 3.4(ii).

Proof of Theorem 3.4(ii). Let $\chi=(\chi^1,\dots,\chi^M)\in[L^1(\omega)]^M$ and $(\chi_\varepsilon)\subset[L^1(\omega)]^M$ with $\chi_\varepsilon\to\chi$ strongly in $[L^1(\omega)]^M$. We claim that

$$\liminf_{\varepsilon\to 0}\mathcal{E}^M_\varepsilon(\chi_\varepsilon,\omega)\geq\mathcal{E}^M(\chi)\,. \tag{7.12}$$

Upon passing to a subsequence (not relabelled) we can assume that the liminf in (7.12) is a limit and the sequence (χ_ε) satisfies $\sup_\varepsilon\mathcal{E}^M_\varepsilon(\chi_\varepsilon,\omega)<+\infty$, since otherwise (7.12) trivially holds. Then Theorem 3.4(i) implies that

$$\chi^1=\chi^2=\ldots=\chi^M\in BV(\omega;\{-1,1\})\,. \tag{7.13}$$

Moreover, for any $\varepsilon>0$ we can find $u_\varepsilon\in\mathcal{SF}_\varepsilon$ with $\chi^\ell(u_\varepsilon)=\chi(u^\ell_\varepsilon)=\chi^\ell_\varepsilon$ for all $\ell\in\{1,\dots,M\}$ and $E^M_\varepsilon(u_\varepsilon,\omega)\leq\mathcal{E}^M_\varepsilon(\chi_\varepsilon,\omega)+\varepsilon$. Together with (7.3) this implies that

$$\mathcal{E}^M_\varepsilon(\chi_\varepsilon,\omega)\geq E^{2d}_\varepsilon(u^1_\varepsilon,\omega)+E^{2d}_\varepsilon(u^M_\varepsilon)+2\sum_{\ell=2}^{M-1}E^{2d}_\varepsilon(u^\ell_\varepsilon,\omega)-\varepsilon\,, \tag{7.14}$$

where u^ℓ_ε is defined according to (7.2). For any $\ell\in\{1,\dots,M\}$ we can now apply Theorem 3.1 to obtain that

$$\liminf_{\varepsilon\to 0}E^{2d}_\varepsilon(u^\ell_\varepsilon,\omega)\geq\liminf_{\varepsilon\to 0}\mathcal{E}^{2d}_\varepsilon(\chi^\ell_\varepsilon,\omega)\geq\int_{S_{\chi^\ell}\cap\omega}\varphi^{2d}(\nu_{\chi^\ell})\,d\mathcal{H}^1(y)=\int_{S_{\chi^1}\cap\omega}\varphi^{2d}(\nu_{\chi^1})\,d\mathcal{H}^1(y)\,,$$

where the last step follows from (7.13). Summing up over all $\ell\in\{1,\dots,M\}$ and using (7.14) this finally yields

$$\liminf_{\varepsilon\to 0}\mathcal{E}^M_\varepsilon(\chi_\varepsilon,\omega)\geq 2(M-1)\int_{S_{\chi^1}\cap\omega}\varphi^{2d}(\nu_{\chi^1})\,d\mathcal{H}^1(y)\,,$$

which proves (7.12).

Let now $\chi\in[L^1(\omega)]^M$ be arbitrary and let us show that there exists $(\chi_\varepsilon)\subset[L^1(\omega)]^M$ with $\chi_\varepsilon\to\chi$ strongly in $[L^1(\omega)]^M$ and

$$\limsup_{\varepsilon\to 0}\mathcal{E}^M_\varepsilon(\chi_\varepsilon,\omega)\leq\mathcal{E}^M(\chi)\,. \tag{7.15}$$

It is not restrictive to assume that $\chi=(\chi^1,\dots,\chi^M)$ satisfies (7.13), since otherwise (7.15) is automatically satisfied. Moreover, in view of Theorem 3.1 there exists a sequence of spin fields $u_\varepsilon\in\mathcal{SF}_\varepsilon$ such that $\chi(u_\varepsilon)\to\chi^1$ in $L^1(\omega)$ and

$$\limsup_{\varepsilon\to 0}E^{2d}_\varepsilon(u_\varepsilon,\omega)\leq\int_{S_{\chi^1}\cap\omega}\varphi^{2d}(\nu_{\chi^1})\,d\mathcal{H}^1\,. \tag{7.16}$$

We now define $v_\varepsilon : \mathcal{L}^{3d}_\varepsilon \to \mathbb{S}^1$ by setting $v_\varepsilon(\varepsilon\alpha, \varepsilon z) := u_\varepsilon(\varepsilon\alpha, 0)$ for all $(\alpha, z) \in \mathcal{L}^{2d} \times \mathbb{Z}$. Moreover, for every $\ell \in \{1, \dots, M\}$ let $\chi^\ell_\varepsilon := \chi(u_\varepsilon)$. In this way, we have that $\chi^\ell_\varepsilon = \chi(v_\varepsilon)_{|\omega\times\varepsilon[\ell,\ell+1)}$ for every $\ell \in \{1, \dots, M\}$. Hence, by definition of $\mathcal{E}^M_\varepsilon$ in (3.12) we have

$$\mathcal{E}^M_\varepsilon(\chi_\varepsilon, \omega) \le E^M_\varepsilon(v_\varepsilon, \omega) = 2(M-1)E^{2d}_\varepsilon(u_\varepsilon, \omega)\,, \tag{7.17}$$

where the last step follows from (7.3) together with the fact that $v_\varepsilon(\varepsilon\alpha, \varepsilon(z+1)) = v_\varepsilon(\varepsilon\alpha, \varepsilon z)$ for all $(\alpha, z) \in \mathcal{L}^{2d} \times \mathbb{Z}$. Combining (7.16) and (7.17) finally gives

$$\limsup_{\varepsilon\to 0} \mathcal{E}^M_\varepsilon(\chi_\varepsilon, \omega) \le 2(M-1) \int_{S_{\chi^1}\cap\omega} \varphi^{2d}(\nu_{\chi^1})\, d\mathcal{H}^1 = \mathcal{E}^M(\chi)\,,$$

and we conclude by observing that $\chi^\ell_\varepsilon = \chi(u_\varepsilon) \to \chi^1 = \chi^\ell$ in $L^1(\omega)$ for all $\ell \in \{1, \dots, M\}$, hence $\chi_\varepsilon \to \chi$ in $[L^1(\omega)]^M$. □

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