Regularity of minimizers of convex functionals with linear growth and small fidelity term

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Abstract

We study the regularity of minimizers of one-dimensional convex functionals $\overline{\mathcal{F}}(u) : BV(I;\mathbb{R}^k) \to \mathbb{R}$ with linear growth with respect to u' and an L^1 fidelity term $\int_I |u-w|\, dx$. We prove that minimizers are of class $C^1(I;\mathbb{R}^k)$ whenever w is sufficiently small either in L^∞ or in L^1, with thresholds independent of the length of the interval. In the case of the relaxed length functional, this provides a positive answer to De Giorgi's conjecture [8] under the weaker assumption of smallness in the L^1 norm.

Key words: Relaxation, convexity, regularity of minimizers, Γ-convergence, approximation.

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1 Introduction

Variational problems involving convex functionals with linear growth arise in several contexts, ranging from image processing to free-discontinuity problems. In this setting, one of the main issues concerns the regularity of minimizers of relaxed energies defined on spaces of functions of bounded variation.
In this paper we consider the functional

$$\overline{\mathcal{F}}(u) = \int_a^b \phi(|u'|)dx + |D_s u|(a,b) + \int_a^b |u-w|\, dx,$$

defined for $u \in BV(I;\mathbb{R}^k)$, where w is a prescribed datum, and $\phi : [0,+\infty) \to (0,+\infty)$ is a strictly convex function of class C^1 with linear growth at infinity. Here u' denotes the absolutely continuous part of the distributional derivative of u, while $D_s u$ denotes its singular part. The functional $\overline{\mathcal{F}}$ is convex (although not strictly convex), coercive and lower semicontinuous with respect to the L^1-topology. Therefore, minimizers exist in $BV(I;\mathbb{R}^k)$, although uniqueness cannot in general be expected. A natural question is whether minimizers exhibit a stronger regularity. Our first result shows that the C^1 regularity holds whenever the datum w is sufficiently small in L^∞, with a threshold depending only on the integrand ϕ and independent of the length of the interval $I = (a,b)$. More precisely,

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Theorem 1.1. *There exists a constant $\gamma > 0$ independent of I and depending only on ϕ such that the following holds: for all $w \in L^\infty(I;\mathbb{R}^k)$ with $\|w\|_{L^\infty} \le \gamma$ minimizers of $\overline{\mathcal{F}}$ are of class $C^1(I;\mathbb{R}^k)$.*

The independence of the constant γ from the interval I is particularly relevant, as we will see in the following.
Our second result further weakens the assumptions on the datum. Specifically, we prove that it is sufficient to require the smallness of w in the L^1-norm.

Theorem 1.2. *There exists a constant $\widehat{\gamma} > 0$ independent of I and depending only on ϕ such that the following holds: for all $w \in L^1(I;\mathbb{R}^k)$ with $\|w\|_{L^1} \le \widehat{\gamma}$ minimizers of $\overline{\mathcal{F}}$ are of class $C^1(I;\mathbb{R}^k)$.*

In the particular case $\phi(t) = \sqrt{1+t^2}$, $\overline{\mathcal{F}}$ is related to the area functional. Namely, given Ω an open bounded subset of $\mathbb{R}^n$ and a map $u \in C^1(\Omega;\mathbb{R}^k)$, the area functional measures the area of the graph of u. The relaxation of this functional has been extensively studied in the literature, mainly due to its application to the Cartesian Plateau problem; see, for instance, [1, 6, 8]. In the one-dimensional setting, the area functional naturally reduces to the length functional $\mathcal{A}_w(u)$, measuring the length of the generalized graph of a given curve u in $\mathbb{R}^k$. Specifically, given an interval $I = (a,b)$, for every $u \in BV(I;\mathbb{R}^k)$, the relaxed length functional with the fidelity term in L^1 is

$$\overline{\mathcal{A}}_w(u) = \int_a^b \sqrt{1+|u'|^2}\,dx + |D_s u|(a,b) + \int_a^b |u-w|\,dx,$$

where $w \in L^\infty(I;\mathbb{R}^k)$ is a given map acting as fidelity term. As already observed, multiple minimizers may exist. According to De Giorgi [8], its conjecture asserts that minimizers are of class C^1 if the L^∞-norm of w is sufficiently small. More precisely, the conjecture reads as follows:

Conjecture 1.3. *There exists a constant $\gamma > 0$ independent of I such that the following holds: for all $w \in L^\infty(I;\mathbb{R}^k)$ with $\|w\|_{L^\infty} \le \gamma$ minimizers of $\overline{\mathcal{A}}_w$ are of class $C^1(I;\mathbb{R}^k)$.*

In [5] we addressed a variant of this conjecture: starting from an Ambrosio–Tortorelli-type functional, and under suitable conditions on the fidelity term, we proved that its minimizers are Sobolev-regular and that the same regularity property is inherited by its Γ-limit. As a corollary, we obtained the existence of an explicit constant $\beta = \min\{\sqrt{\frac{1}{128}}, \frac{1}{68(b-a)^{1/2}}\}$ such that for every $w \in L^2(I;\mathbb{R}^n)$ with $\|w\|_{L^2} \le \beta$, the unique minimizer of

$$\int_a^b \sqrt{1+|u'|^2}\,dx + |D_s u|(a,b) + \int_a^b |u-w|^2\,dx$$

is of class $C^1(I;\mathbb{R}^k)$.
We emphasize that this variant differs from the original conjecture in two respects: first, the presence of the exponent 2 in the fidelity term (namely, the squared L^2-norm of $w-u$ in place of the L^1 term), and second, the explicit dependence of β on the interval I for large domains (specifically, for $b-a > \frac{128}{68^2}$). On the other hand, the result contains the weaker assumption that w has a small L^2-norm, rather than a small L^∞-norm.
We also point out that Conjecture 1.3 has recently been tackled in [4] with techniques different from those in [5], although the proof still requires the constant β to depend on the interval I.
In this work we make an incremental step toward the original De Giorgi conjecture in the one-dimensional case, obtaining a result that removes the dependence of β on I and thereby refines the contributions in [5, 4]. Furthermore, Theorem 1.2 shows that the regularity conclusion remains valid under a different smallness assumption on the fidelity datum: it is enough to require the L^1-norm of w to be sufficiently small, instead of assuming smallness in the L^∞

topology as in Conjecture 1.3. Therefore, our result provides a positive answer to De Giorgi's conjecture under a weaker norm assumption on the datum. On the other hand, we emphasize that Theorem 1.1 does not follow from Theorem 1.2 since for large domain $b - a >> 1$ the smallness in L^∞ does not imply the smallness in L^1.

In order to prove our main results, we first select a minimizer $\widetilde{u}$ of $\overline{\mathcal{F}}$ and introduce the functional $\overline{\mathcal{F}}_\lambda$ given by $\overline{\mathcal{F}}_\lambda(u) = \overline{\mathcal{F}}(u) + \lambda \int_a^b |u - \widetilde{u}|\,dx$. This energy has as unique minimizer $\widetilde{u}$; in order to prove regularity of $\widetilde{u}$ we regularize $\overline{\mathcal{F}}_\lambda$ introducing, for a parameter $\varepsilon \in (0,1)$, the functional $\mathcal{F}_{\varepsilon,\lambda}$ in (2.2), which we will prove to Γ-converge to $\overline{\mathcal{F}}_\lambda$ (as $\varepsilon \to 0$). Hence we prove uniformly a-priori estimates on the $W^{1,\infty}$-norm of minimizers of $\mathcal{F}_{\varepsilon,\lambda}$ to obtain that similar estimates are satisfied by $\widetilde{u}$. In this strategy, some additional effort is due to the presence of ϕ, which is not C^2 and for this reason has to be regularized as well (see Section 2.3).

The paper is organized as follows: in Section 2 we introduce the setting of the problem and prove, using the regularizations of $\overline{\mathcal{F}}_\lambda$, the aforementioned a-priori estimates needed to prove Theorem 1.1. In Section 3 we assume w merely in L^1, and following a similar strategy (which differs a bit from the one in Section 2), we show the counterpart a-priori estimates. With these at our disposal, we can conclude the proofs of our main results in Section 4.

2 Preliminary results

This section collects the notation, assumptions, and preliminary results that will be used throughout the paper.
We consider a one-dimensional setting, where $I := (a, b)$ represents the domain of interest. We use standard notation for Sobolev and Lebesgue spaces. If μ is a Radon measure with bounded total variation, we denote by $|\mu|(a, b)$ its total variation on (a, b).
Let $u \in BV(I; \mathbb{R}^k)$ be a map. We denote by Du the distributional derivative of u, which splits as

$$Du = \nabla u + D_s u,$$

where ∇u is the absolutely continuous part of Du with respect to the Lebesgue measure $\mathcal{L}^1$, and $D_s u$ is the singular part of Du. For a detailed study of the properties of BV functions, we refer to [2]. Moreover, a comprehensive introduction to Γ-convergence can be found in [7].

2.1 The function ϕ

We now discuss the main properties of the integrand ϕ and establish the auxiliary estimates needed in the sequel (see Theorem 2.7).
Let $k \geq 1$ be an integer and $w \in L^\infty(I; \mathbb{R}^k)$. We assume that $\phi : [0, +\infty) \to (0, +\infty)$ is a strictly convex function of class C^1 satisfying $\phi'(0) = 0$ and $\lim_{t\to+\infty} \phi(t)/t =: \sigma \in (0, +\infty)$.

Lemma 2.1. *Let $\phi : [0, +\infty) \to (0, +\infty)$ be a strictly convex function of class C^1 such that $\phi'(0) = 0$. Then*

$$\theta(t) = t\phi'(t) - \phi(t) + \phi(0)$$

is increasing and $\theta(0) = 0$. In particular,

$$\lim_{t\to+\infty} \theta(t) =: \ell > 0.$$

Proof. The identity $\theta(0) = 0$ follows immediately from the definition of θ. Let $0 \leq s < t$. To prove that $\theta(t) - \theta(s) > 0$, it is sufficient to combine the inequality $\phi(s) - \phi(t) > \phi'(t)(s - t)$ with the relation $\phi'(t) > \phi'(s)$, which follows directly from the strict convexity of ϕ. Consequently, since θ is increasing on $[0, +\infty)$ and $\theta(0) = 0$, using the strict monotonicity of ϕ and that $\phi'(0) = 0$, it follows that $\theta(t) > 0$ for all $t > 0$, which directly implies that $\lim_{t\to+\infty} \theta(t) > 0$. $\square$

We observe that the hypothesis that $\lim_{t\to+\infty}\phi(t)/t = \sigma < +\infty$ implies that ϕ has a linear growth, i.e.,

$$\phi(t) \le \phi(0) + \sigma t,$$

for all $t \ge 0$. The hypothesis that $\phi(0) > 0$ also implies that there is a constant $\hat\sigma \in (0,1]$ such that

$$\phi(t) \ge \hat\sigma t$$

for all $t \ge 0$. The prototype energy we have in mind is

$$\phi(t) = \sqrt{1+t^2},$$

for which one takes $\sigma = \hat\sigma = 1$. Eventually, we notice that it can be also $\ell = +\infty$ and we set

$$c_\ell := \min\{\ell, 1\}.$$

In the case $\phi(t) = \sqrt{1+t^2}$ it is easily seen that $\ell = c_\ell = 1$.

2.2 Perturbation and regularization of the functional $\overline{\mathcal{F}}$

In this section, we introduce the functional $\overline{\mathcal{F}}$, whose minimizers are the main object of our regularity analysis. To this end, we first define the perturbed functional $\overline{\mathcal{F}}_\lambda$, whose unique minimizer is a given minimizer $\widetilde{u}$ of $\overline{\mathcal{F}}$, and then approximate it by the regularized functionals $\mathcal{F}_{\varepsilon,\lambda}$. Uniform $W^{1,\infty}$ a priori estimates for the minimizers of $\mathcal{F}_{\varepsilon,\lambda}$, together with Γ-convergence, allow us to deduce the regularity of $\widetilde{u}$.
Let us introduce the functional

$$\overline{\mathcal{F}}(u) = \int_a^b \phi(|u'|)\;dx + |D_s u|(a,b) + \int_a^b |u-w|\,dx$$

for $u \in BV(I;\mathbb{R}^k)$. By standard arguments, it is straightforward that $\overline{\mathcal{F}}$ admits minimizers in $BV(I;\mathbb{R}^k)$. By a truncation argument we have:

Lemma 2.2. *Any minimizer $\widetilde{u}$ of $\overline{\mathcal{F}}$ belongs to $L^\infty(I;\mathbb{R}^k)$, and satisfies $\|\widetilde{u}\|_{L^\infty} \le \|w\|_{L^\infty}$. In particular*

$$\min\{\overline{\mathcal{F}}(u) : u \in BV(I;\mathbb{R}^k)\} = \min\{\overline{\mathcal{F}}(u) : u \in BV(I;\mathbb{R}^k),\ \|u\|_{L^\infty} \le \|w\|_{L^\infty}\}.$$

Proof. Let $\widetilde{u} \in BV(I;\mathbb{R}^k)$ be a minimizer of $\overline{\mathcal{F}}$. We set $M := \|w\|_{L^\infty}$ and define the truncated function $v \in BV(I;\mathbb{R}^k)$ as:

$$v := \begin{cases} T_M(|\widetilde{u}|)\frac{\widetilde{u}}{|\widetilde{u}|}, & \text{if } \widetilde{u} \ne 0,\\ 0 & \text{otherwise,}\end{cases}$$

where $T_M(\rho) = \min\{\rho, M\}$. Observe that $v' = 0$ a.e. on $E_M := \{|\widetilde{u}| > M\}$ and moreover

$$|D_s v|(I) \le |D_s\widetilde{u}|(I); \tag{2.1}$$

furthermore it is easy to check that, since $|w| \le M$ a.e., it follows

$$\int_a^b |v-w|\;dx \le \int_a^b |\widetilde{u}-w|\;dx, \tag{2.2}$$

and strict inequality holds if $|E_M| > 0$. Finally, recalling that ϕ is strictly increasing, we get

$$\int_a^b \phi(|v'|)\;dx \le \int_a^b \phi(|\widetilde{u}'|)\;dx. \tag{2.3}$$

Hence, combining (2.1), (2.2) and (2.3), we obtain

$$\overline{\mathcal{F}}(v) < \overline{\mathcal{F}}(\widetilde{u}),$$

if $|E_M| > 0$. This contradicts the hypothesis that $\widetilde{u}$ is a minimizer; thus, $|E_M| = 0$ and $\widetilde{u} \in L^\infty(I;\mathbb{R}^k)$. In particular,

$$\|\widetilde{u}\|_{L^\infty} \leq \|w\|_{L^\infty}. \tag{2.4}$$

□

Let $\widetilde{u}$ be a minimizer of $\overline{\mathcal{F}}$ and, for a real number $\lambda > 0$, consider the perturbed functional

$$\overline{\mathcal{F}}_\lambda(u) = \overline{\mathcal{F}}(u) + \lambda \int_a^b |u - \widetilde{u}|\,dx.$$

Notice that $\widetilde{u}$ is the unique minimizer of $\overline{\mathcal{F}}_\lambda$. Indeed, we have

$$\overline{\mathcal{F}}_\lambda(\widetilde{u}) = \overline{\mathcal{F}}(\widetilde{u}) \leq \overline{\mathcal{F}}(u) < \overline{\mathcal{F}}_\lambda(u) \text{ for all } u \neq \widetilde{u}.$$

Furthermore, given a parameter $\varepsilon \in (0,1)$, we also consider the regularized functional $\mathcal{F}_{\varepsilon,\lambda} : L^1(I;\mathbb{R}^k) \to [0,+\infty]$ defined, for all $u \in H^1(I;\mathbb{R}^k)$ by

$$\mathcal{F}_{\varepsilon,\lambda}(u) = \int_a^b \phi(|u'|)\,dx + \frac{\varepsilon}{2}\int_a^b |u'|^2\,dx + \int_a^b |u-w|^{1+\varepsilon}\,dx + \lambda\int_a^b |u-\widetilde{u}|^{1+\varepsilon}\,dx,$$

and defined as $+\infty$ on $L^1(I;\mathbb{R}^k) \setminus H^1(I;\mathbb{R}^k)$. As we will prove, the functional $\mathcal{F}_{\varepsilon,\lambda}$ Γ-converges with respect to $L^1((a,b))$ to the functional $\overline{\mathcal{F}}_\lambda$ as $\varepsilon \to 0^+$. Hence, we proceed to study the minimizers $u_{\varepsilon,\lambda}$ of $\mathcal{F}_{\varepsilon,\lambda}$ and then pass to the limit to recover properties of the minimizer $\widetilde{u}$ of $\overline{\mathcal{F}}_\lambda$.

2.3 A regularization of $\mathcal{F}_{\varepsilon,\lambda}$: the functional $\mathcal{F}_{\varepsilon,\lambda,h}$

In this section, we introduce a further regularization of $\mathcal{F}_{\varepsilon,\lambda}$, denoted as $\mathcal{F}_{\varepsilon,\lambda,h}$, by replacing the integrand ϕ with a suitable C^2 approximation. This additional regularization is required since ϕ is only of class C^1.
We extend ϕ to $\mathbb{R}$ by symmetry with respect to the y-axis and define

$$\widetilde{\phi}_h(z) := \frac{1}{h}\int_z^{z+h} \phi(s)\,ds.$$

Observe that

$$\widetilde{\phi}_h'(z) = \frac{\phi(z+h)-\phi(z)}{h}, \qquad \widetilde{\phi}_h''(z) = \frac{\phi'(z+h)-\phi'(z)}{h}.$$

Moreover, $\widetilde{\phi}_h$ is increasing on $(-\frac{h}{2},+\infty)$, while for $z < -\frac{h}{2}$ it is decreasing. Furthermore

$$\widetilde{\phi}_h'(-\tfrac{h}{2}) = 0.$$

We also define

$$\phi_h(z) := \widetilde{\phi}_h\Big(z - \frac{h}{2}\Big),$$

and observe that $\phi_h \in C^2(\mathbb{R})$ and $\phi_h'(0) = 0$. Moreover, $\phi_h \to \phi$ in $C^1_{\mathrm{loc}}(\mathbb{R})$ as $h \to 0^+$.
We introduce

$$\mathcal{F}_{\varepsilon,\lambda,h}(u) = \int_a^b \phi_h(|u'|)\,dx + \frac{\varepsilon}{2}\int_a^b |u'|^2\,dx + \int_a^b |u-w|^{1+\varepsilon}\,dx + \lambda\int_a^b |u-\widetilde{u}|^{1+\varepsilon}\,dx,$$

defined for $u \in H^1(I;\mathbb{R}^k)$.

Theorem 2.3. *The functional $\mathcal{F}_{\varepsilon,\lambda,h}$ admits minimizers u_h in $W^{2,\infty}(I;\mathbb{R}^k)$ for every $\varepsilon, \lambda, h > 0$, with $\|u_h\|_{L^\infty} \le \|w\|_{L^\infty}$, and each of it satisfies the Euler–Lagrange equation*

$$\frac{d}{dx}\left(\phi_h'(|u_h'|)\frac{u_h'}{|u_h'|} + \varepsilon\, u_h'\right) = (1+\varepsilon)\left[\frac{u_h - w}{|u_h - w|}\,|u_h - w|^\varepsilon + \lambda\,\frac{u_h - \widetilde{u}}{|u_h - \widetilde{u}|}\,|u_h - \widetilde{u}|^\varepsilon\right] \tag{2.5}$$

with Neumann boundary conditions

$$u_h'(x) = 0 \qquad \text{for } x \in \{a, b\}. \tag{2.6}$$

Furthermore, there exists a constant $C > 0$ independent of h such that

$$\|u_h\|_{W^{2,\infty}} \le C. \tag{2.7}$$

Proof. Let $(u_j) \in H^1(I;\mathbb{R}^k)$ be a minimizing sequence and assume without loss of generality that for all $j \ge 1$

$$\mathcal{F}_{\varepsilon,\lambda,h}(u_j) \le m.$$

Thus from $\frac{\varepsilon}{2}\int_I |u_j'|^2\,dx \le m$ we get

$$\|u_j'\|_{L^2} \le C.$$

Using that $\widetilde{u}, w \in L^\infty(I;\mathbb{R}^k)$ together with $\|\widetilde{u}\|_{L^\infty} \le \|w\|_{L^\infty}$, by a truncation argument similar to the one applied in Lemma 2.2, we might assume

$$\|u_j\|_{L^\infty} \le \|w\|_{L^\infty}.$$

Therefore,

$$\|u_j\|_{H^1} \le C \quad \forall j$$

and up to a subsequence, $u_j \rightharpoonup u$ weakly in $H^1(I;\mathbb{R}^k)$. Weak convergence in $H^1(I;\mathbb{R}^k)$ ensures

$$\|u'\|_{L^2} \le \liminf_{j\to+\infty} \|u_j'\|_{L^2}$$

and, as a consequence, we have

$$\frac{\varepsilon}{2}\int_I |u'(x)|^2\,dx \le \liminf_{j\to+\infty} \frac{\varepsilon}{2}\int_I |u_j'(x)|^2\,dx. \tag{2.8}$$

Moreover, by the Rellich Theorem, $u_j \to u$ strongly in $L^{1+\varepsilon}(I;\mathbb{R}^k)$ which in turn implies

$$\begin{aligned} \int_a^b |u - w|^{1+\varepsilon}\,dx &= \lim_{j\to+\infty}\int_a^b |u_j - w|^{1+\varepsilon}\,dx, \\ \lambda\int_a^b |u - \widetilde{u}|^{1+\varepsilon}\,dx &= \lambda\lim_{j\to+\infty}\int_a^b |u_j - \widetilde{u}|^{1+\varepsilon}\,dx. \end{aligned} \tag{2.9}$$

Finally, since $u_j' \rightharpoonup u'$ in $L^1(I;\mathbb{R}^k)$ and $\phi_h(u')$ is convex with respect to u', we obtain

$$\int_a^b \phi_h(|u'|)\,dx \le \liminf_{j\to+\infty}\int_a^b \phi_h(|u_j'|)\,dx. \tag{2.10}$$

Collecting (2.8), (2.9) and (2.10), we conclude that

$$\mathcal{F}_{\varepsilon,\lambda,h}(u) \le \liminf_{j\to+\infty} \mathcal{F}_{\varepsilon,\lambda,h}(u_j)$$

and u is a minimizer of $\mathcal{F}_{\varepsilon,\lambda,h}$.
We now proceed to prove the second part of the statement. Let u_h be a minimizer of the functional. Again from a truncation argument, we get

$$\|u_h\|_{L^\infty} \le \|w\|_{L^\infty}.$$

By standard computations we obtain that the weak form of the Euler-Lagrange equations is

$$\begin{aligned}0 = &\int_a^b \Big(\phi_h'(|u_h'|)\frac{u_h'}{|u_h'|} + \varepsilon u_h'\Big) \cdot \varphi' \, dx \\ &+ (1+\varepsilon)\int_a^b \Big(\frac{u_h - w}{|u_h - w|}\,|u_h - w|^\varepsilon + \lambda\,\frac{u_h - \widetilde{u}}{|u_h - \widetilde{u}|}\,|u_h - \widetilde{u}|^\varepsilon\Big) \cdot \varphi \, dx,\end{aligned}$$

valid for all $\varphi \in H^1(I;\mathbb{R}^k)$. Define $\Psi_h : \mathbb{R}^k \to \mathbb{R}^k$ as $\Psi_h(p) := \phi_h'(|p|)\frac{p}{|p|} + \varepsilon p$ where $\Psi_h(0) = 0$. Since $\phi_h \in C^2$ and $\phi_h'(0) = 0$, Taylor's expansion gives $\phi_h'(|p|) = o(1)$ as $p \to 0$. Therefore, Ψ_h is continuous in 0. Moreover, writing $\Psi_h(p) = \frac{p}{|p|}\,(\phi_h'(|p|) + \varepsilon|p|)$ it is easy to see that Ψ_h is surjective, we get $\Psi_h \in C^1(\mathbb{R}^k \setminus \{0\})$ and

$$\nabla\Psi_h(p) = \phi_h''(|p|)\frac{p}{|p|} \otimes \frac{p}{|p|} + \phi_h'(|p|)\nabla\left(\frac{p}{|p|}\right) + \varepsilon\, \mathrm{Id}.$$

We show that Ψ_h is injective. If $\Psi_h(p_1) = \Psi_h(p_2)$, from the definition of Ψ_h, it is easy to verify that $p_2 = tp_1$, for some $t \in \mathbb{R}$. Hence, we deduce

$$\phi_h'(t|p_1|) + \varepsilon|p_1|t = \phi_h'(|p_1|) + \varepsilon|p_1|,$$

and this implies $t = 1$, because $|p| \mapsto \phi_h'(|p|) + |p|\varepsilon$ is strictly monotone. Moreover, $\Psi_h^{-1} \in C^1(\mathbb{R}^k \setminus \{0\})$, its derivative is bounded, and Ψ_h^{-1} is continuous in 0. Indeed, passing in polar coordinates, a straightforward computation shows that

$$\|\nabla\Psi_h(p)\| > \sqrt{2}\varepsilon,$$

and therefore, by the inverse function theorem, we get

$$\|\nabla(\Psi_h^{-1})(y)\| \le \frac{1}{\sqrt{2}\varepsilon} \qquad \forall\, y \in \mathbb{R}^k \setminus \{0\}. \tag{2.11}$$

Consequently, Ψ_h^{-1} is Lipschitz continuous with Lipschitz constant at most $1/\sqrt{2}\varepsilon$.
Now, from (2.3), we have

$$\Big\langle -\frac{d}{dx}\Psi_h(u_h'), \varphi\Big\rangle = (1+\varepsilon)\int_a^b F(u_h)\varphi \; dx \qquad \text{for all } \varphi \in H^1(I;\mathbb{R}^k),$$

where $F(u_h) := \frac{u_h - w}{|u_h - w|}|u_h - w|^\varepsilon + \lambda\,\frac{u_h - \widetilde{u}}{|u_h - \widetilde{u}|}|u_h - \widetilde{u}|^\varepsilon$. Using (2.3) and the assumption $w \in L^\infty(I;\mathbb{R}^k)$, we deduce that $F(u_h) \in L^\infty(I;\mathbb{R}^k)$ and, by comparison

$$\frac{d}{dx}[\Psi_h(u_h')] \in L^\infty(I;\mathbb{R}^k)$$

and hence

$$\Psi_h(u_h') \in W^{1,\infty}(I;\mathbb{R}^k). \tag{2.12}$$

Hence, from (2.12) and (2.11), we get $u_h' \in W^{1,\infty}(I;\mathbb{R}^k)$, and in particular

$$\|u_h''\|_{L^\infty} \le \frac{1+\varepsilon}{\sqrt{2}\varepsilon}\|F(u_h)\|_{L^\infty} \le 4\frac{1+\lambda}{\sqrt{2}\varepsilon}\|w\|_{L^\infty}^\varepsilon, \tag{2.13}$$

for all $\varepsilon, \lambda, h \in (0,1)$. This, together with (2.3), implies

$$u_h \in W^{2,\infty}(I;\mathbb{R}^k)$$

and it yields

$$\begin{aligned} &\frac{d}{dx}\left(\phi_h'(|u_h'|)\frac{u_h'}{|u_h'|} + \varepsilon\, u_h'\right) = (1+\varepsilon)\left[\frac{u_h - w}{|u_h - w|}\,|u_h - w|^\varepsilon + \lambda\,\frac{u_h - \widetilde{u}}{|u_h - \widetilde{u}|}\,|u_h - \widetilde{u}|^\varepsilon\right], \\ &\phi_h'(|u_h'(x)|)\frac{u_h'(x)}{|u_h'(x)|} + \varepsilon\, u_h'(x) = 0 \text{ for } x \in \{a,b\}. \end{aligned} \tag{2.14}$$

Observe that (2.6) follows from (2.14) since $\varepsilon > 0$ and $\phi_h' \geq 0$. The control on $\|u_h\|_{W^{2,\infty}}$ follows from (2.13) and (2.3). □

Proposition 2.4. *Let $\varepsilon, \lambda \in (0,1)$ be fixed. Then the family of functionals $\mathcal{F}_{\varepsilon,\lambda,h}$ Γ-converges to $\mathcal{F}_{\varepsilon,\lambda}$ with respect to $L^1(I;\mathbb{R}^k)$ as $h \to 0^+$. In particular, if $u_h \in W^{2,\infty}(I;\mathbb{R}^k)$ is a minimizer of $\mathcal{F}_{\varepsilon,\lambda,h}$, then,*

$$u_h \rightharpoonup u \text{ weakly* in } W^{2,\infty}(I;\mathbb{R}^k),$$

and u is a minimizer of $\mathcal{F}_{\varepsilon,\lambda}$. Moreover, u satisfies

$$-\int_a^b \Big(\phi'(|u'|)\frac{u'}{|u'|} + \varepsilon u'\Big)\cdot\varphi'\,dx = (1+\varepsilon)\int_a^b \Big(\frac{u-w}{|u-w|}\,|u-w|^\varepsilon + \lambda\,\frac{u-\widetilde{u}}{|u-\widetilde{u}|}\,|u-\widetilde{u}|^\varepsilon\Big)\cdot\varphi\,dx, \tag{2.15}$$

for all $\varphi \in H^1(I;\mathbb{R}^k)$, and $\|u\|_{L^\infty} \leq \|w\|_{L^\infty}$.

Notice that the strong form of the Euler-Lagrange equation holds true

$$\frac{d}{dx}\Big(\phi'(|u'|)\frac{u'}{|u'|}\Big) = -\varepsilon u'' + (1+\varepsilon)\Big(\frac{u-w}{|u-w|}\,|u-w|^\varepsilon + \lambda\,\frac{u-\widetilde{u}}{|u-\widetilde{u}|}\,|u-\widetilde{u}|^\varepsilon\Big) \tag{2.16}$$

a.e. on (a,b), and by comparison

$$\phi'(|u'|)\frac{u'}{|u'|} \in W^{1,\infty}(I;\mathbb{R}^k).$$

Proof. Let us prove the Γ-convergence result: let $v_h \to v$ in $L^1(I;\mathbb{R}^k)$ and assume that

$$\liminf_{h\to 0^+} \mathcal{F}_{\varepsilon,\lambda,h}(v_h) < +\infty.$$

Up to a subsequence, we may suppose that

$$\sup_{h\to 0^+} \mathcal{F}_{\varepsilon,\lambda,h}(v_h) < +\infty.$$

Since $\varepsilon > 0$, this implies

$$\sup_{h\to 0^+}\int_a^b |v_h'|^2\,dx < +\infty.$$

Moreover, the fidelity terms give a uniform bound of v_h in $L^{1+\varepsilon}(I;\mathbb{R}^k)$. Hence (v_h) is bounded in $H^1(I;\mathbb{R}^k)$ and, up to a subsequence,

$$v_h \rightharpoonup v \quad \text{weakly in } H^1(I;\mathbb{R}^k).$$

Therefore, by lower semicontinuity,

$$\frac{\varepsilon}{2}\int_a^b |v'|^2\,dx \leq \liminf_{h\to 0^+}\frac{\varepsilon}{2}\int_a^b |v_h'|^2\,dx.$$

Moreover, since $v_h \to v$ strongly in $L^{1+\varepsilon}(I;\mathbb{R}^k)$, we have

$$\int_a^b |v_h - w|^{1+\varepsilon}\,dx \to \int_a^b |v - w|^{1+\varepsilon}\,dx$$

and

$$\lambda \int_I |v_h - \widetilde{u}|^{1+\varepsilon}\,dx \to \lambda \int_I |v - \widetilde{u}|^{1+\varepsilon}\,dx.$$

Finally, using the convergence $\phi_h \to \phi$ in $C^1_{\rm loc}(\mathbb{R})$ and the convexity of ϕ_h, we obtain

$$\int_I \phi(v')\,dx \le \liminf_{h\to 0^+} \int_I \phi_h(v_h')\,dx.$$

Combining the previous inequalities gives

$$\mathcal{F}_{\varepsilon,\lambda}(v) \le \liminf_{h\to 0^+} \mathcal{F}_{\varepsilon,\lambda,h}(v_h).$$

We now prove the limsup inequality. Let $v \in L^1(I;\mathbb{R}^k)$. If $\mathcal{F}_{\varepsilon,\lambda}(v) = +\infty$, there is nothing to prove. Otherwise $v \in H^1(I;\mathbb{R}^k)$ and we choose $v_h := v$. Then $v_h \to v$ in $L^1(I;\mathbb{R}^k)$. The quadratic term and the fidelity terms are independent of h, while

$$\int_I \phi_h(|v'|)\,dx \to \int_I \phi(|v'|)\,dx$$

by the convergence $\phi_h \to \phi$ in $C^1_{\rm loc}(\mathbb{R})$. Therefore,

$$\lim_{h\to 0^+} \mathcal{F}_{\varepsilon,\lambda,h}(v_h) = \mathcal{F}_{\varepsilon,\lambda}(v).$$

This concludes the proof of the Γ-convergence.
Let now u_h be minimizers of $\mathcal{F}_{\varepsilon,\lambda,h}$; by (2.7) there exists $u \in W^{2,\infty}(I;\mathbb{R}^k)$ such that

$$u_h \rightharpoonup u \text{ weakly* in } W^{2,\infty}(I;\mathbb{R}^k),$$

and u is a minimizer of $\mathcal{F}_{\varepsilon,\lambda}$. We now pass to the limit in the weak form of the Euler-Lagrange equation (2.3) valid for all $\varphi \in H^1(I;\mathbb{R}^k)$, so that, exploiting again the convergence $\phi_h \to \phi$ in $C^1_{\rm loc}(\mathbb{R})$ and that $u_h' \to u'$ in $W^{1,\infty}(I;\mathbb{R}^k)$ we obtain (2.15). Finally $\|u\|_{L^\infty} \le \|w\|_{L^\infty}$ follows from (2.3). □

Lemma 2.5. *Let $u \in W^{2,\infty}(I;\mathbb{R}^k)$ be the minimizer of $\mathcal{F}_{\varepsilon,\lambda}$. Then*

$$\|u'\|_{L^1(I;\mathbb{R}^k)} \le (b-a)\,\Phi^{-1}\big(2\|w\|^{1+\varepsilon}_{L^\infty(I;\mathbb{R}^k)}\big), \tag{2.17}$$

where $\Phi(t) = \phi(t) - \phi(0)$.

Proof. Since u is a minimizer, $\mathcal{F}_{\varepsilon,\lambda}(u) \le \mathcal{F}_{\varepsilon,\lambda}(0)$; hence

$$\begin{aligned}\int_a^b \phi(|u'|)\,dx &\le \mathcal{F}_{\varepsilon,\delta,\lambda}(u)\\ &\le \mathcal{F}_{\varepsilon,\delta,\lambda}(0) = \int_a^b \phi(0)\,dx + \int_a^b |w|^{1+\varepsilon}\,dx + \lambda\int_a^b |\widetilde{u}|^{1+\varepsilon}\,dx.\end{aligned}$$

Recalling the results achieved in Lemma 2.2 and using (2.3), we obtain

$$\int_a^b (\phi(|u'|) - \phi(0))\,dx \le \int_a^b (1+\lambda)\|w\|^{1+\varepsilon}_{L^\infty}\,dx.$$

Dividing by the length of the interval we get

$$\frac{1}{b-a}\int_a^b (\phi(|u'|)-\phi(0))\,dx \le (1+\lambda)\|w\|_{L^\infty}^{1+\varepsilon} \le 2\|w\|_{L^\infty}^{1+\varepsilon}. \tag{2.18}$$

Taking $d\mu = \frac{dx}{b-a}$ and $\Phi(p)=\phi(p)-\phi(0)$, we can rewrite (2.18) as

$$\int_a^b \Phi(|u'(x)|)\,d\mu(x) \le 2\|w\|_{L^\infty}^{1+\varepsilon}.$$

Applying now Jensen's inequality we have

$$\Phi\Big(\int_a^b |u'(x)|\,d\mu(x)\Big) \le \int_a^b \Phi(|u'(x)|)\,d\mu(x) \le 2\|w\|_{L^\infty}^{1+\varepsilon},$$

that is,

$$\Phi\Big(\frac{1}{b-a}\int_a^b |u'(x)|\,dx\Big) \le 2\|w\|_{L^\infty}^{1+\varepsilon}.$$

Since Φ is strictly increasing, convex, continuous, and invertible, with continuous inverse Φ^{-1}, from the previous inequality we have

$$\int_a^b |u'(x)|\,dx = \|u'\|_{L^1} \le (b-a)\,\Phi^{-1}\big(2\|w\|_{L^\infty}^{1+\varepsilon}\big).$$

□

Corollary 2.6. *Under the same assumptions of Lemma 2.5*

$$\int_a^b \Phi(|u'(x)|)\,dx \le \int_a^b |w|^{1+\varepsilon}\,dx + \lambda\int_a^b |\widetilde{u}|^{1+\varepsilon}\,dx.$$

Theorem 2.7. *Let $u\in W^{2,\infty}(I;\mathbb{R}^k)$ be the minimizer of $\mathcal{F}_{\varepsilon,\lambda}$. There exist positive constants γ and C, depending only on ϕ, such that if $\|w\|_{L^\infty}\le\gamma$ then*

$$\|u'\|_{L^\infty(I;\mathbb{R}^k)} \le C < +\infty, \tag{2.19}$$

for all $\varepsilon,\lambda\in(0,1)$.

Proof. We split the proof into two cases, depending on whether the length of the interval $b-a$ is sufficiently small or not.

Case $b-a\le 1$. In (2.15) we choose $\varphi=u'$, and integrating over (a,x) we obtain

$$\begin{aligned}
&(1+\varepsilon)\int_a^x u'\cdot\left[\frac{u-w}{|u-w|}\,|u-w|^\varepsilon + \lambda\,\frac{u-\widetilde{u}}{|u-\widetilde{u}|}\,|u-\widetilde{u}|^\varepsilon\right]dt\\
&\quad= |u'(x)|\phi'(|u'(x)|) - \int_a^x u''\cdot\Big(\phi'(|u'|)\frac{u'}{|u'|}\Big)\,dt + \frac{\varepsilon}{2}|u'(x)|^2\\
&\quad= |u'(x)|\phi'(|u'(x)|) - \phi(|u'(x)|) + \phi(0) + \frac{\varepsilon}{2}|u'(x)|^2\\
&\quad\ge |u'(x)|\phi'(|u'(x)|) - \phi(|u'(x)|) + \phi(0).
\end{aligned} \tag{2.20}$$

The left-hand side can be estimated as

$$\begin{aligned}
&(1+\varepsilon)\left|\int_a^x u'\cdot\left[\frac{u-w}{|u-w|}\,|u-w|^\varepsilon + \lambda\,\frac{u-\widetilde{u}}{|u-\widetilde{u}|}\,|u-\widetilde{u}|^\varepsilon\right]dt\right|\\
&\le 2^\varepsilon(1+\varepsilon)(1+\lambda)\|w\|_{L^\infty}^\varepsilon\|u'\|_{L^1}\\
&\le 2^\varepsilon(1+\varepsilon)(1+\lambda)\|w\|_{L^\infty}^\varepsilon\Phi^{-1}\big(2\|w\|_{L^\infty}^{1+\varepsilon}\big),
\end{aligned}$$

where we have used (2.17) together with $b-a \leq 1$. Hence, we arrive to

$$|u'(x)|\phi'(|u'(x)|) - \phi(|u'(x)|) + \phi(0) \leq 2^{\varepsilon}(1+\varepsilon)(1+\lambda)\|w\|_{L^\infty}^{\varepsilon}\Phi^{-1}\big(2\|w\|_{L^\infty}^{1+\varepsilon}\big)$$

Assuming, without loss of generality, that $\|w\|_{L^\infty} \leq 1$ and using the monotonicity of Φ we infer

$$|u'(x)|\phi'(|u'(x)|) - \phi(|u'(x)|) + \phi(0) \leq 8\Phi^{-1}\big(2\|w\|_{L^\infty}\big).$$

Now $\Phi(s) \to 0$ as $s \to 0^+$, and so there is a constant $\gamma_1 > 0$ such that if $s \leq \gamma_1$ it holds $8\Phi^{-1}(2s) \leq \frac{c_\ell}{2}$, where $\ell = \lim_{t\to\infty}\theta(t)$ is the constant appearing in Lemma 2.1, $\theta(t) = t\phi'(t) - \phi(t) + \phi(0)$, and $c_\ell := \min\{\ell, 1\}$. We obtain the inequality

$$\theta(|u'(x)|) = \frac{c_l}{2},$$

that, since θ is strictly increasing and invertible, implies

$$|u'(x)| \leq \theta^{-1}\big(\frac{c_\ell}{2}\big) < +\infty. \tag{2.21}$$

Case $b-a>1$. Let α, β be such that $a \leq \alpha < \beta \leq b$ and

$$\beta - \alpha = 1. \tag{2.22}$$

Our goal is to obtain, for $\|w\|_{L^\infty}$ sufficiently small, an upper bound for $\|u'\|_{L^1((\alpha,\beta);\mathbb{R}^k)}$. This local estimate will then allow us to derive an upper bound for $\|u'\|_{L^\infty((a,b);\mathbb{R}^k)}$.
Denote by $r_{\alpha,\beta}$ the affine function passing through the points $(\alpha, u(\alpha))$ and $(\beta, u(\beta))$ and let $m_{\alpha,\beta}$ denote its slope. Since $\|u\|_{L^\infty} \leq \|w\|_{L^\infty}$, on (α, β) we have

$$|r_{\alpha,\beta}| \leq \max\big(|u(\alpha)|, |u(\beta)|\big) \leq \|w\|_{L^\infty}, \tag{2.23}$$

$$|m_{\alpha,\beta}| = |u(\alpha) - u(\beta)| \leq 2\|w\|_{L^\infty}. \tag{2.24}$$

Since $u|_{(\alpha,\beta)}$ is a minimizer of

$$\int_\alpha^\beta \phi(|v'|)\,dx + \frac{\varepsilon}{2}\int_\alpha^\beta |v'|^2\,dx + \int_\alpha^\beta |w-v|^{1+\varepsilon}\,dx + \lambda\int_\alpha^\beta |\widetilde{u}-v|^{1+\varepsilon}\,dx,$$

among all the functions $v \in H^1((\alpha,\beta);\mathbb{R}^k)$ such that $v(\alpha) = u(\alpha)$ and $v(\beta) = u(\beta)$, it follows that

$$\int_\alpha^\beta \phi(|u'|)\,dx \leq \int_\alpha^\beta \phi(|m_{\alpha,\beta}|)\,dx + \frac{\varepsilon}{2}\int_\alpha^\beta |m_{\alpha,\beta}|^2\,dx + \int_\alpha^\beta \big|w - r_{\alpha,\beta}\big|^{1+\varepsilon}\,dx + \lambda\int_\alpha^\beta \big|\widetilde{u} - r_{\alpha,\beta}\big|^{1+\varepsilon}\,dx.$$

From (2.23) and (2.24), and using that $\phi(m_{\alpha,\beta}) \leq \phi(0) + \phi'(m_{\alpha,\beta})m_{\alpha,\beta} \leq \phi(0) + \sigma m_{\alpha,\beta}$, we deduce that

$$\int_\alpha^\beta \phi(|u'|) - \phi(0)\,dx \leq 2\sigma\|w\|_{L^\infty} + 2\varepsilon\|w\|_{L^\infty}^2 + 4\|w\|_{L^\infty} \leq 8\|w\|_{L^\infty}.$$

Setting $\Phi(p) = \phi(p) - \phi(0)$, we use Jensen inequality to obtain

$$\Phi\left(\int_\alpha^\beta |u'|\,dx\right) \leq 8\|w\|_{L^\infty},$$

that is

$$\|u'\|_{L^1((\alpha,\beta);\mathbb{R}^k)} \leq \Phi^{-1}\left(8\|w\|_{L^\infty}\right). \tag{2.25}$$

Since $u' \in W^{1,\infty}(I;\mathbb{R}^k)$, the function $x \mapsto |u'(x)|$ is continuous. Therefore, there exists a point $\overline{x}_{\alpha,\beta} \in (\alpha,\beta)$ such that

$$|u'(\overline{x}_{\alpha,\beta})| \le \int_\alpha^\beta |u'(t)|\,dt \le \Phi^{-1}\left(8\|w\|_{L^\infty}\right), \tag{2.26}$$

where we used (2.22).
Multiplying (2.16) by u' and integrating over $(\overline{x}_{\alpha,\beta}, x)$, arguing as in (2.20), we obtain

$$\begin{aligned}(1+\varepsilon)&\int_{\overline{x}_{\alpha,\beta}}^x u'\cdot\left[\frac{u-w}{|u-w|}|u-w|^\varepsilon + \lambda\frac{u-\widetilde{u}}{|u-\widetilde{u}|}|u-\widetilde{u}|^\varepsilon\right]dt \\ &= \theta(|u'(x)|) - \theta(|u'(\overline{x}_{\alpha,\beta})|) + \frac{\varepsilon}{2}\left(|u'(x)|^2 - |u'(\overline{x}_{\alpha,\beta})|^2\right).\end{aligned} \tag{2.27}$$

Hence,

$$\theta(|u'(x)|) \le \theta(|u'(\overline{x}_{\alpha,\beta})|) + \frac{\varepsilon}{2}|u'(\overline{x}_{\alpha,\beta})|^2 + 8\|u'\|_{L^1((\overline{x}_{\alpha,\beta},x);\mathbb{R}^k)}. \tag{2.28}$$

Using (2.25) and (2.26), we get

$$\theta(|u'(x)|) \le \theta\left(\Phi^{-1}\left(8\|w\|_{L^\infty}\right)\right) + \frac{1}{2}\left(\Phi^{-1}\left(8\|w\|_{L^\infty}\right)\right)^2 + 8\Phi^{-1}\left(8\|w\|_{L^\infty}\right).$$

Recalling that $\theta(s) \to 0$ as $s \to 0^+$ and that $\Phi^{-1}(s) \to 0$ as $s \to 0^+$, there exists a constant $\gamma_2 > 0$ such that, if $s \le \gamma_2$, then

$$\theta\left(\Phi^{-1}(8s)\right) + \frac{1}{2}\left(\Phi^{-1}(8s)\right)^2 + 8\Phi^{-1}(8s) \le \frac{c_\ell}{2}.$$

Hence, as in the previous case, we obtain (2.21). The proof is complete in all cases, by setting $\gamma := \min\{\gamma_1, \gamma_2, 1\}$ and $C := \theta^{-1}\left(\frac{c_\ell}{2}\right)$. □

2.4 Approximation of $\overline{\mathcal{F}}_\lambda$

In this section we prove the Γ-convergence of $\mathcal{F}_{\varepsilon,\lambda}$ to $\overline{\mathcal{F}}_\lambda$ as $\varepsilon \to 0^+$.

Theorem 2.8. *For all $\varepsilon \in (0,1)$ let $v_\varepsilon \in H^1(I;\mathbb{R}^k)$ and suppose that there exists a constant $C > 0$ such that*

$$\sup_{\varepsilon\in(0,1)} \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) < C. \tag{2.29}$$

Then there exists $C' > 0$, independent of ε, such that

$$\sup_{\varepsilon\in(0,1)} \|v_\varepsilon\|_{W^{1,1}} < C'. \tag{2.30}$$

Furthermore, the functional $\mathcal{F}_{\varepsilon,\lambda}$ Γ-converges with respect to $L^1(I;\mathbb{R}^k)$ to the functional $\overline{\mathcal{F}}_\lambda$ as $\varepsilon \to 0^+$.

Proof. Let us prove the first part of the statement. Let v_ε satisfy (2.29); we readily see that

$$\int_a^b \phi(|v_\varepsilon'|)dx + \int_a^b |v_\varepsilon - w|^{1+\varepsilon}dx < C.$$

By the linear growth condition of ϕ it follows that

$$\|v_\varepsilon'\|_{L^1} \le C,$$

and the L^1 control of v_ε easily follows from $\int_a^b |v_\varepsilon - w|^{1+\varepsilon} dx < C$. Indeed, notice that for any $t \geq 0$ and $\alpha > 0$,

$$t \leq t^{1+\alpha} + \alpha.$$

Applying this to $t = |v_\varepsilon(x) - w(x)|$ and $\alpha = \varepsilon$, we obtain

$$|v_\varepsilon(x) - w(x)| \leq |v_\varepsilon(x) - w(x)|^{1+\varepsilon} + \varepsilon. \tag{2.31}$$

We conclude that $\|v_\varepsilon\|_{L^1}$ is uniformly bounded. The inequality (2.30) is then obtained.
It remains to address the Γ-convergence, proving the $\Gamma-\liminf$ and the $\Gamma-\limsup$ inequalities. Let $v_{\varepsilon_j} \to v$ in $L^1(I;\mathbb{R}^k)$ and assume that

$$\liminf_{j\to+\infty} \mathcal{F}_{\varepsilon,\lambda}(v_{\varepsilon_j}) < +\infty.$$

Up to a subsequence, we may suppose that

$$\sup_{j\to+\infty} \mathcal{F}_{\varepsilon,\lambda}(v_{\varepsilon_j}) < C,$$

which from (2.30) implies $v_{\varepsilon_j} \rightharpoonup v$ weakly* in $BV(I;\mathbb{R}^k)$. Given the assumptions on the function ϕ, it is well known that the functional

$$v \mapsto \int_a^b \phi(|v'|)\ dx + |D_s v|(a,b)$$

is L^1-lower semicontinuous. Consequently,

$$\begin{aligned}\int_a^b \phi(|v'|)\ dx + |D_s v|(a,b) &\leq \liminf_{j\to+\infty} \int_a^b \phi(|v'_{\varepsilon_j}|)\ dx + |D_s v_{\varepsilon_j}|(a,b)\\ &= \liminf_{j\to+\infty} \int_a^b \phi(|v'_{\varepsilon_j}|)\ dx,\end{aligned} \tag{2.32}$$

where we have used that each v_{ε_j} belongs to $H^1(I;\mathbb{R}^k)$. Moreover, using the same inequality as in (2.31) we pass to the lim inf in both sides and we get

$$|v(x) - w(x)| \leq \liminf_{j\to\infty} |v_{\varepsilon_j}(x) - w(x)|^{1+\varepsilon_j}.$$

The same argument applies to $|v(x) - \widetilde{u}(x)|$. Applying Fatou's Lemma, we obtain

$$\int_a^b |v-w|\, dx \leq \liminf_{j\to+\infty} \int_a^b |v_{\varepsilon_j} - w|^{1+\varepsilon_j}\, dx, \tag{2.33}$$

$$\lambda \int_a^b |v-\widetilde{u}|\, dx \leq \liminf_{j\to+\infty} \lambda \int_a^b |v_{\varepsilon_j} - \widetilde{u}|^{1+\varepsilon_j}\, dx. \tag{2.34}$$

Combining (2.32), (2.33), (2.34), we complete the Γ-liminf inequality. Thus, we are left with the construction of a recovery sequence (v_{ε_j}). If $v \in H^1(I;\mathbb{R}^k)$, we simply set $v_{\varepsilon_j} := v$.
On the other hand, we first introduce

$$I(v) := \int_a^b \phi(|v'|)\ dx + |D_s v|(a,b).$$

If $v \in BV(I;\mathbb{R}^k) \setminus H^1(I;\mathbb{R}^k)$, we have to prove that:

$$\lim_{j\to+\infty} \int_a^b \phi(|v'_{\varepsilon_j}|)\, dx = I(v), \tag{2.35}$$

$$\lim_{j\to+\infty} \int_a^b |v_{\varepsilon_j} - w|^{1+\varepsilon_j}\, dx = \int_a^b |v-w|\, dx, \tag{2.36}$$

$$\lim_{j\to+\infty} \lambda \int_a^b |v_{\varepsilon_j} - \widetilde{u}|^{1+\varepsilon_j}\, dx = \lambda \int_a^b |v-\widetilde{u}|\, dx, \tag{2.37}$$

$$\lim_{j\to+\infty} \frac{\varepsilon_j}{2} \int_a^b |v'_{\varepsilon_j}|^2\, dx = 0. \tag{2.38}$$

We begin by noting that

$$I(v) = \inf \Big\{ \liminf_{n\to+\infty} \int_I \phi(|v'_n|)\, dx : v_n \to v \text{ in } L^1(I;\mathbb{R}^k), v_n \in H^1(I;\mathbb{R}^k) \Big\},$$

i.e., $I(v)$ is the relaxation of $\displaystyle\int_I \phi(|v'_n|)\, dx$. Thus, there exists a sequence $v_n \in H^1(I;\mathbb{R}^k)$, $n > 0$, such that

$$\lim_{n\to+\infty} \int_I \phi(|v'_n|)\, dx = I(v). \tag{2.39}$$

For $n = 0$ set $v_0 \equiv 0$, and for $n > 0$ consider the sequence v_n such that (2.39) holds. For each j consider the indices in

$$J_j := \Big\{ n \in \mathbb{N} : \|v_n\|^2_{H^1} \le \frac{1}{\sqrt{\varepsilon_j}} \Big\}.$$

Note that $J_j \neq \emptyset$ because $v_0 \in J_j$, and at the same time J_j is finite, thanks to the assumption that $u \notin H^1(I;\mathbb{R}^k)$. For each j, choose $n_j := \max\{n : n \in J_j\}$ and set $v_{\varepsilon_j} := v_{n_j}$.
We now prove (2.38): indeed

$$\varepsilon_j \|v'_{\varepsilon_j}\|^2_{L^2} \le \varepsilon_j \|v_{\varepsilon_j}\|^2_{H^1} = \varepsilon_j \|v_{n_j}\|^2_{H^1} \le \sqrt{\varepsilon_j}.$$

Therefore (2.36) follows from $v_{\varepsilon_j} \to v$ weakly* in $BV(I;\mathbb{R}^k)$, which implies $v_{\varepsilon_j} \to v$. Moreover, as our domain is bounded and $1 + \epsilon_j \le 2$ for large j, the strong convergence $v_{\epsilon_j} \to v$ in $L^{1+\epsilon_j}(I;\mathbb{R}^k)$ follows immediately from the strong L^2-convergence via Hölder's inequality. The same holds for (2.37).
To establish (2.35), it is enough to check that $v_{\varepsilon_j} = v_{n_j}$ is still a recovery sequence for the relaxation, i.e.,

$$\lim_{j\to+\infty} \int_I \phi(|v'_{n_j}|)\, dx = I(v).$$

By (2.39), this would be true if we prove that

$$n_j \to \infty \quad \text{as } j \to \infty.$$

To see this, we observe that $J_j \subseteq J_{j+1}$ and $\bigcup_j J_j = \mathbb{N}$, since as $j \to +\infty$, $\varepsilon_j \to 0$ and $\frac{1}{\sqrt{\varepsilon_j}} \to \infty$, meaning that every $n \in \mathbb{N}$ belongs to some J_j. Moreover n_j is a non-decreasing sequence of natural numbers that cannot stabilize; if it did, there would exist a constant C such that $|J_j| < C$ for all j, contradicting the fact that $\bigcup_j J_j = \mathbb{N}$. We have concluded. □

3 The case $w \in L^1((a,b);\mathbb{R}^k)$

In this section we assume that $w \in L^1(I;\mathbb{R}^k)$. For $\varepsilon \in (0,1)$ we define the truncation

$$w_\varepsilon := (-\frac{1}{\varepsilon}) \vee w \wedge \frac{1}{\varepsilon}.$$

which belongs to $L^\infty(I;\mathbb{R}^k)$. Moreover, the functional

$$\overline{\mathcal{F}}(u) = \int_a^b \phi(|u'|)\, dx + |D_s u|(a,b) + \int_a^b |u-w|\, dx$$

for $u \in BV(I;\mathbb{R}^k)$ admits, by standard arguments, minima in $BV(I;\mathbb{R}^k)$. Denoting, as before, by $\widetilde{u}$ a minimum of it, we define

$$\widetilde{u}_\varepsilon := (-\frac{1}{\varepsilon}) \vee \widetilde{u} \wedge \frac{1}{\varepsilon}.$$

The approximating functional is now

$$\widehat{\mathcal{F}}_{\varepsilon,\lambda,h}(u) = \int_a^b \phi_h(|u'|)\, dx + \frac{\varepsilon}{2}\int_a^b |u'|^2\, dx + \int_a^b |u-w_\varepsilon|^{1+\varepsilon}\, dx + \lambda \int_a^b |u-\widetilde{u}_\varepsilon|^{1+\varepsilon}\, dx,$$

defined for $u \in H^1(I;\mathbb{R}^k)$, where ϕ_h is the regularization of ϕ defined as in Section 2.3. The counterpart of Theorem 2.3 is the following:

Theorem 3.1. *The functional $\widehat{\mathcal{F}}_{\varepsilon,\lambda,h}$ admits minimizers in $W^{2,\infty}(I;\mathbb{R}^k)$ for every $\varepsilon,\lambda,h>0$. Moreover, every minimizer u_h belongs to $W^{2,\infty}(I;\mathbb{R}^k)$ with $\|u_h\|_{L^\infty} \le \|w_\varepsilon\|_{L^\infty}$, and satisfies the Euler–Lagrange equation*

$$\frac{d}{dx}\left(\phi_h'(|u_h'|)\frac{u_h'}{|u_h'|} + \varepsilon\, u_h'\right) = (1+\varepsilon)\left[\frac{u_h - w_\varepsilon}{|u_h - w_\varepsilon|}\,|u_h - w_\varepsilon|^\varepsilon + \lambda\, \frac{u_h - \widetilde{u}_\varepsilon}{|u_h - \widetilde{u}_\varepsilon|}\,|u_h - \widetilde{u}_\varepsilon|^\varepsilon\right]$$

with Neumann boundary conditions

$$u_h'(x) = 0 \qquad \text{for } x \in \{a,b\}.$$

Finally, there exists a constant $C>0$ independent of h such that

$$\|u_h\|_{W^{2,\infty}} \le C.$$

The Γ-convergence as $h \to 0$ of $\widehat{\mathcal{F}}_{\varepsilon,\lambda,h}$ is easily achieved, following Proposition 2.4:

Proposition 3.2. *Let $\varepsilon,\lambda \in (0,1)$ be fixed. Then $\widehat{\mathcal{F}}_{\varepsilon,\lambda,h}$ Γ-converges, with respect to $L^1(I;\mathbb{R}^k)$ as $h \to 0^+$, to*

$$\widehat{\mathcal{F}}_{\varepsilon,\lambda}(u) := \int_a^b \phi(|u'|)\, dx + \frac{\varepsilon}{2}\int_a^b |u'|^2\, dx + \int_a^b |u-w_\varepsilon|^{1+\varepsilon}\, dx + \lambda \int_a^b |u-\widetilde{u}_\varepsilon|^{1+\varepsilon}\, dx.$$

If $u_h \in W^{2,\infty}(I;\mathbb{R}^k)$ is a minimizer of $\widehat{\mathcal{F}}_{\varepsilon\lambda,h}$, then

$$u_h \rightharpoonup u \text{ weakly* in } W^{2,\infty}(I;\mathbb{R}^k),$$

u is a minimizer of $\widehat{\mathcal{F}}_{\varepsilon,\lambda}$, and u satisfies

$$-\int_a^b \left(\phi'(|u'|)\frac{u'}{|u'|} + \varepsilon u'\right)\cdot \varphi'\, dx = (1+\varepsilon)\int_a^b \left(\frac{u-w_\varepsilon}{|u-w_\varepsilon|}\,|u-w_\varepsilon|^\varepsilon + \lambda\,\frac{u-\widetilde{u}_\varepsilon}{|u-\widetilde{u}_\varepsilon|}\,|u-\widetilde{u}_\varepsilon|^\varepsilon\right)\cdot\varphi\, dx,$$

for all $\varphi \in H^1(I;\mathbb{R}^k)$. Finally $\|u\|_{L^\infty} \le \|w_\varepsilon\|_{L^\infty}$.

By Corollary 2.6 we see that the minimizer $u_{\varepsilon,\lambda}$ satisfies

$$\int_a^b \Phi(|u'_{\varepsilon,\lambda}(x)|)\,dx \le \int_a^b |w_\varepsilon|^{1+\varepsilon}\,dx + \lambda \int_a^b |\widetilde{u}_\varepsilon|^{1+\varepsilon}\,dx,$$

where $\Phi(p) = \phi(p) - \phi(0)$. From this and the hypothesis for which there exists $\widehat{\sigma} \in (0,\sigma)$ such that $\phi(p) \ge \widehat{\sigma} p$, we readily obtain

$$\widehat{\sigma}\|u'_{\varepsilon,\lambda}\|_{L^1} \le \int_a^b |w_\varepsilon|^{1+\varepsilon}\,dx + \lambda \int_a^b |\widetilde{u}_\varepsilon|^{1+\varepsilon}\,dx + (b-a)\phi(0).$$

Using now Lemma 3.4 and the definitions of w_ε and $\widetilde{u}_\varepsilon$ we conclude

$$\|u'_{\varepsilon,\lambda}\|_{L^1} \le \frac{1}{\widehat{\sigma}\varepsilon^\varepsilon}(1+2\lambda)\|w\|_{L^1} + \frac{(b-a)\phi(0)}{\widehat{\sigma}}. \tag{3.1}$$

We finally state the following crucial Γ-convergence result:

Theorem 3.3. *If u_{ε_k} is a sequence with $\varepsilon_k \to 0$ as $k \to +\infty$ and*

$$\sup_{k\in\mathbb{N}} \widehat{\mathcal{F}}_{\varepsilon_k}(u_{\varepsilon_k}) < +\infty,$$

then there exists $u \in BV(I;\mathbb{R}^k)$ such that, up to a subsequence,

$$u_{\varepsilon_k} \rightharpoonup u \text{ weakly* in } BV(I;\mathbb{R}^k),$$

as $k \to \infty$. Moreover, the functionals $\widehat{\mathcal{F}}_{\varepsilon,\lambda}$ Γ-converge in $L^1(I;\mathbb{R}^k)$ to $\overline{\mathcal{F}}_\lambda$.

Proof. The linear growth condition of ϕ yields

$$\sup_k \int_a^b |u'_{\varepsilon_k}|\,dx < +\infty,$$

and the L^1 control of u_{ε_k} follows from $\int_a^b |u_{\varepsilon_k} - w_\varepsilon|\,dx < C$, exactly as in the proof of Theorem 2.8. Exploiting the definitions of w_ε and $\widetilde{u}_\varepsilon$, it is easy to see that $|w_\varepsilon|^{1+\varepsilon} \to |w|$ and $|\widetilde{u}|^{1+\varepsilon} \to |\widetilde{u}|$ in $L^1(I)$, and in particular

$$\int_a^b |w_\varepsilon|^{1+\varepsilon}\,dx + \lambda\int_a^b |\widetilde{u}_\varepsilon|^{1+\varepsilon}\,dx \qquad \to \qquad \int_a^b |w|\,dx + \lambda\int_a^b |\widetilde{u}|\,dx.$$

Therefore, the proof of the Γ-convergence follows exactly the same arguments as in the proof of Theorem 2.8. Indeed, the lower bound is obtained by combining the lower semicontinuity of the relaxed functional with the above convergence, and also the recovery sequence is the same as in the previously cited theorem. □

The functional $\widehat{\mathcal{F}}_{\varepsilon,\lambda}$ is strictly convex, coercive and sequentially lower semicontinuous with respect to the weak topology. Hence it admits a unique minimizer, denoted by $u_{\varepsilon,\lambda}$, which belongs to $W^{2,\infty}(I;\mathbb{R}^k)$. From Theorem 3.3, $u_{\varepsilon,\lambda} \rightharpoonup \widetilde{u}$ weakly* in $BV(I;\mathbb{R}^k)$ for any $\lambda > 0$, where $\widetilde{u}$ is the unique minimizer of $\overline{\mathcal{F}}_\lambda$.
As a consequence of the previous result, we get the following a posteriori estimate:

Lemma 3.4. *One has*

$$\|\widetilde{u}\|_{L^1(I;\mathbb{R}^k)} \le 2\|w\|_{L^1(I;\mathbb{R}^k)}.$$

Proof. We show that $u_{\varepsilon,\lambda}$ satisfies

$$(1+\lambda)\int_a^b |u_{\varepsilon,\lambda}|^{1+\varepsilon}\,dt \le 2^{1+\varepsilon}\left(\int_a^b |w_\varepsilon|^{1+\varepsilon}\,dt + \lambda\int_a^b |\widetilde{u}_\varepsilon|^{1+\varepsilon}\,dt\right). \tag{3.2}$$

Passing to the limit as $\varepsilon \to 0$ and applying Fatou's lemma (possibly after extracting a subsequence), we obtain

$$(1+\lambda)\int_a^b |\widetilde{u}|\,dt \le 2\left(\int_a^b |w|\,dt + \lambda\int_a^b |\widetilde{u}|\,dt\right).$$

Since this holds for any $\lambda > 0$, the thesis follows.
Let us now prove (3.2). Using $\varphi = u_{\varepsilon,\lambda}$ in (3.2), we get

$$\begin{aligned}
0 \ge (1+\varepsilon)&\left[\int_a^b \frac{u_{\varepsilon,\lambda}-w_\varepsilon}{|u_{\varepsilon,\lambda}-w_\varepsilon|}|u_{\varepsilon,\lambda}-w_\varepsilon|^\varepsilon \cdot u_{\varepsilon,\lambda}\,dt\right.\\
&\left.+\lambda\int_a^b \frac{u_{\varepsilon,\lambda}-\bar{u}_\varepsilon}{|u_{\varepsilon,\lambda}-\bar{u}_\varepsilon|}|u_{\varepsilon,\lambda}-\widetilde{u}_\varepsilon|^\varepsilon \cdot u_{\varepsilon,\lambda}\,dt\right]\\
= (1+\varepsilon)&\left[\int_a^b |u_{\varepsilon,\lambda}-w_\varepsilon|^{1+\varepsilon}\,dt + \int_a^b \frac{u_{\varepsilon,\lambda}-w_\varepsilon}{|u_{\varepsilon,\lambda}-w_\varepsilon|}|u_{\varepsilon,\lambda}-w_\varepsilon|^\varepsilon \cdot w_\varepsilon\,dt\right.\\
&\left.+\lambda\int_a^b |u_{\varepsilon,\lambda}-\widetilde{u}_\varepsilon|^{1+\varepsilon}\,dt + \lambda\int_a^b \frac{u_{\varepsilon,\lambda}-\bar{u}_\varepsilon}{|u_{\varepsilon,\lambda}-\bar{u}_\varepsilon|}\,|u_{\varepsilon,\lambda}-\widetilde{u}_\varepsilon|^\varepsilon \cdot \widetilde{u}_\varepsilon\,dt\right].
\end{aligned}$$

Applying Young inequality in the second and fourth integral of the previous equality with conjugate exponents $p = \frac{1+\varepsilon}{\varepsilon}$ and $q = 1+\varepsilon$, we infer

$$\begin{aligned}
0 \ge{}& (1+\varepsilon)\int_a^b |u_{\varepsilon,\lambda}-w_\varepsilon|^{1+\varepsilon}\,dt - \varepsilon\int_a^b |u_{\varepsilon,\lambda}-w_\varepsilon|^{1+\varepsilon}\,dt - \int_a^b |w_\varepsilon|^{1+\varepsilon}\,dt\\
&+\lambda(1+\varepsilon)\int_a^b |u_{\varepsilon,\lambda}-\widetilde{u}_\varepsilon|^{1+\varepsilon}\,dt - \lambda\varepsilon\int_a^b |u_{\varepsilon,\lambda}-\widetilde{u}_\varepsilon|^{1+\varepsilon}\,dt - \lambda\int_a^b |\widetilde{u}_\varepsilon|^{1+\varepsilon}\,dt\\
={}& \int_a^b |u_{\varepsilon,\lambda}-w_\varepsilon|^{1+\varepsilon}\,dt - \int_a^b |w_\varepsilon|^{1+\varepsilon}\,dt + \lambda\left[\int_a^b |u_{\varepsilon,\lambda}-\widetilde{u}_\varepsilon|^{1+\varepsilon}\,dt - \int_a^b |\widetilde{u}_\varepsilon|^{1+\varepsilon}\,dt\right].
\end{aligned}$$

Using now the convexity of $z \to |z|^{1+\varepsilon}$, we have

$$\begin{aligned}
0 \ge{}& 2^{-\varepsilon}\int_a^b |u_{\varepsilon,\lambda}|^{1+\varepsilon}\,dt - 2\int_a^b |w_\varepsilon|^{1+\varepsilon}\,dt\\
&+\lambda\left(2^{-\varepsilon}\int_a^b |u_{\varepsilon,\lambda}|^{1+\varepsilon}\,dt - 2\int_a^b |\widetilde{u}_\varepsilon|^{1+\varepsilon}\,dt\right)
\end{aligned}$$

which implies (3.2). □

Corollary 3.5. *We have*

$$\int_a^b |u_{\varepsilon,\lambda}|^{1+\varepsilon} \le 2^{1+\varepsilon}\varepsilon^{-\varepsilon}(1+2\lambda)\|w\|_{L^1}.$$

and so

$$\|u_{\varepsilon,\lambda}\|_{L^1} \le \varepsilon^{-\frac{\varepsilon}{1+\varepsilon}}\,2(b-a)^{\frac{\varepsilon}{1+\varepsilon}}(1+2\lambda)^{\frac{1}{1+\varepsilon}}\|w\|_{L^1}^{\frac{1}{1+\varepsilon}}$$

In particular, there exists $\varepsilon_0 \in (0,1)$ (depending on $b-a$), such that, for all $\varepsilon \in (0,\varepsilon_0)$ there holds

$$\|u_{\varepsilon,\lambda}\|_{L^1} \le 8(1+2\lambda)^{\frac{1}{1+\varepsilon}}\|w\|_{L^1}^{\frac{1}{1+\varepsilon}}$$

Assuming, without loss of generality, that $\lambda \in (0,1)$ and $\|w\|_{L^1} \le 1$, it follows that

$$\|u_{\varepsilon,\lambda}\|_{L^1} \le 24\|w\|_{L^1}^{\frac{1}{1+\varepsilon}}. \tag{3.3}$$

Theorem 3.6. *Let $u_{\varepsilon,\lambda} \in W^{2,\infty}(I;\mathbb{R}^k)$ be the minimizer of $\widehat{\mathcal{F}}_{\varepsilon,\lambda}$. There exist positive constants $\widehat{\gamma}$ and C depending only on ϕ such that, if $\|w\|_{L^1} \le \widehat{\gamma}$, then*

$$\|u'_{\varepsilon,\lambda}\|_{L^\infty(I;\mathbb{R}^k)} \le C < +\infty,$$

for all $\lambda \in (0,1)$ and ε small enough.

Proof. Step 1: Assume in this case

$$b - a \le k := \frac{c_\ell \widehat{\sigma}}{32\phi(0)}.$$

We follow the argument used in the proof of Theorem 2.7; in this case, the term in the left-hand side of (2.20) is estimated as

$$\begin{aligned}
&(1+\varepsilon)\left|\int_a^x u'_{\varepsilon,\lambda}\cdot\left[\frac{u_{\varepsilon,\lambda}-w_\varepsilon}{|u_{\varepsilon,\lambda}-w_\varepsilon|}\,|u_{\varepsilon,\lambda}-w_\varepsilon|^\varepsilon + \lambda\,\frac{u_{\varepsilon,\lambda}-\widetilde{u}_\varepsilon}{|u_{\varepsilon,\lambda}-\widetilde{u}_\varepsilon|}\,|u_{\varepsilon,\lambda}-\widetilde{u}_\varepsilon|^\varepsilon\right]dt\right| \\
&\le (1+\varepsilon)(1+\lambda)\|u'_{\varepsilon,\lambda}\|_{L^1}\frac{2^\varepsilon}{\varepsilon^\varepsilon} \\
&\le (1+\varepsilon)(1+\lambda)\left(\frac{1}{\hat{\sigma}\varepsilon^\varepsilon}(1+2\lambda)\|w\|_{L^1(I;\mathbb{R}^k)} + \frac{(b-a)\phi(0)}{\hat{\sigma}}\right)\frac{2^\varepsilon}{\varepsilon^\varepsilon} \\
&\le \frac{96}{\widehat{\sigma}}\|w\|_{L^1(I;\mathbb{R}^k)} + 16\frac{k\phi(0)}{\widehat{\sigma}},
\end{aligned}$$

where we have used (3.1) and that $\varepsilon^\varepsilon \in (\frac{1}{2},1)$ for ε small enough. We have obtained, recalling that $k = \frac{c_\ell \hat{\sigma}}{32\phi(0)}$,

$$\theta(|u'_{\varepsilon,\lambda}(x)|) \le \frac{96}{\widehat{\sigma}}\|w\|_{L^1} + \frac{c_\ell}{2}.$$

Therefore there exists a constant $\widehat{\gamma}_1 > 0$ (depending only on ϕ) such that if $\|w\|_{L^1} \le \widehat{\gamma}_1$ it holds $\theta(|u'_{\varepsilon,\lambda}(x)|) \le \frac{2}{3}c_\ell$ for all $x \in I$, and so

$$\|u'_{\varepsilon,\lambda}\|_{L^\infty} \le \theta^{-1}\Big(\frac{2}{3}c_\ell\Big) < +\infty, \tag{3.4}$$

for all ε small enough.
Step 2: Assume now $b - a > k$, let $h := \frac{k}{4} = \frac{c_\ell \hat{\sigma}}{128\phi(0)}$ and let $x, x+h \in (a,b)$. Consider the function $v_{h,x,\varepsilon}$ defined by

$$v_{h,x,\varepsilon}(t) := \begin{cases} u_{\varepsilon,\lambda}(t), & t \in [a,x]\cup[x+h,b], \\ r_{h,x,\varepsilon}(t), & t \in [x,x+h], \end{cases}$$

where $r_{h,x,\varepsilon}$ is the affine function interpolating the values of $u_{\varepsilon,\lambda}$ at the endpoints of the interval $[x,x+h]$, namely

$$r_{h,x,\varepsilon}(t) = u_\varepsilon(x) + \frac{u_\varepsilon(x+h)-u_\varepsilon(x)}{h}\,(t-x).$$

In the following, we simplify the notation writing $u = u_{\varepsilon,\lambda}$. Since

$$\widehat{\mathcal{F}}_{\varepsilon,\lambda}(u) \le \widehat{\mathcal{F}}_{\varepsilon,\lambda}(v_{h,x,\varepsilon}),$$

we obtain, recalling $\Phi(p) = \phi(p) - \phi(0) \le \phi'(p)p \le \sigma p$,

$$
\begin{aligned}
\int_x^{x+h} \Phi(u'(t))\,dt \le{}& \sigma \int_x^{x+h} |r'_{h,x,\varepsilon}(t)|\,dt + \frac{\varepsilon}{2}\int_x^{x+h} |r'_{h,x,\varepsilon}(t)|^2\,dt \\
&+ \int_x^{x+h} |r_{h,x,\varepsilon}(t) - w_\varepsilon(t)|^{1+\varepsilon}\,dt + \lambda \int_x^{x+h} |r_{h,x,\varepsilon}(t) - \widetilde{u}_\varepsilon(t)|^{1+\varepsilon}\,dt \\
\le{}& \sigma|u(x+h)| + \sigma|u(x)| + \frac{\varepsilon}{h}(|u(x+h)|^2 + |u(x)|^2) \\
&+ \int_x^{x+h} \Big[\frac{|u(x)| + |u(x+h)|}{h} + |w_\varepsilon(t)|\Big]^{1+\varepsilon} dt \\
&+ \lambda \int_x^{x+h} \Big[\frac{|u(x)| + |u(x+h)|}{h} + |\widetilde{u}_\varepsilon(t)|\Big]^{1+\varepsilon} dt
\end{aligned}
$$

Exploiting the convexity of $s \mapsto |s|^{1+\varepsilon}$ we infer

$$
\begin{aligned}
\int_x^{x+h} \Phi(u'(t))\,dt \le{}& \Big(|u(x+h)| + |u(x)|\Big)\left(\sigma + \frac{1}{h}\right) \\
&+ \frac{3^\varepsilon(1+\lambda)}{h^\varepsilon}\Big(|u(x)|^{1+\varepsilon} + |u(x+h)|^{1+\varepsilon}\Big) \\
&+ \frac{3}{\varepsilon^\varepsilon}\int_x^{x+h} \Big(|w(t)| + \lambda|\widetilde{u}(t)|\Big)dt.
\end{aligned}
$$

Now, let $(\alpha,\beta) \subset (a,b)$ be such that

$$\beta - \alpha = 2h.$$

If $(\alpha,\beta) \subset (a+h, b-h)$, integrating on $(\alpha - h, \beta)$ with respect to x we get

$$
\begin{aligned}
h\int_\alpha^\beta \Phi(u'(t))\,dt \le{}& \int_{\alpha-h}^\beta \Big(|u(x+h)| + |u(x)|\Big)\left(\sigma + \frac{1}{h}\right)\,dx \\
&+ \int_{\alpha-h}^\beta 6(1+\lambda)\Big(|u(x)|^{1+\varepsilon} + |u(x+h)|^{1+\varepsilon}\Big)\,dx \\
&+ \int_{\alpha-h}^\beta 6\int_x^{x+h}\Big(|w(t)| + \lambda|\widetilde{u}(t)|\Big)dt\ dx \\
\le{}& 2\|u\|_{L^1}\left(\sigma + \frac{1}{h} + 12(1+\lambda)\right) + 6(1+2\lambda)\|w\|_{L^1} \\
\le{}& 48\|w\|_{L^1}^{\frac{1}{1+\varepsilon}}\left(\sigma + \frac{1}{h} + 24\right) + 18\|w\|_{L^1} \\
\le{}& 48\|w\|_{L^1}^{\frac{1}{2}}\left(\sigma + \frac{128\phi(0)}{c_\ell\hat{\sigma}} + 24\right) + 18\|w\|_{L^1},
\end{aligned}
\tag{3.5}
$$

for ε small enough, where we have used (3.3).
Assume now $\alpha \in [a, a+h)$; for all $x \in [a+h, a+2h)$ we use the function

$$
v_{h,x,\varepsilon}(t) = \begin{cases} u(x) & \text{if } t \in (a,x), \\ u(t) & \text{if } t \in [x,b), \end{cases}
$$

and by minimality of u we infer

$$
\begin{aligned}
\int_a^{a+h} \Phi(u')dt \le \int_a^x \Phi(u')dt \le{}& \int_a^x |u(x) - w_\varepsilon(t)|^{1+\varepsilon}dt + \lambda\int_a^x |u(x) - \widetilde{u}_\varepsilon(t)|^{1+\varepsilon}dt \\
\le{}& 4\int_a^x |u(x)|dt + 2\int_a^x |w_\varepsilon| + \lambda|\widetilde{u}_\varepsilon(t)|dt \\
\le{}& 8h|u(x)| + 2\|w\|_{L^1} + 4\|w\|_{L^1}
\end{aligned}
$$

Integrating both side on $(a+h, a+2h)$ in the x variable

$$h\int_a^{a+h} \Phi(u')dt \le 8h\|u\|_{L^1} + 2h\|w\|_{L^1} + 4h\|w\|_{L^1}$$
$$\le 192h\|w\|_{L^1}^{\frac{1}{2}} + 6h\|w\|_{L^1}$$

A similar argument shows that, if $\beta \in (b-h, b)$ then

$$h\int_{b-h}^{b} \Phi(u'_{\varepsilon,\lambda})dt \le 192h\|w\|_{L^1}^{\frac{1}{2}} + 6h\|w\|_{L^1}.$$

Using this together with (3.5) we finally obtain that, given any interval (α, β) of amplitude $2h$ it holds

$$h\int_\alpha^\beta \Phi(u')dt \le 48\|w\|_{L^1}^{\frac{1}{2}}\left(\sigma + \frac{128\phi(0)}{c_\ell\hat{\sigma}} + 24\right) + 18\|w\|_{L^1} + 192h\|w\|_{L^1}^{\frac{1}{2}} + 6h\|w\|_{L^1},$$

from which, using Jensen inequality and recalling that $h = \frac{k}{4} = \frac{c_\ell\hat{\sigma}}{128\phi(0)}$, we deduce that there exists a positive constant c' depending only on ϕ, such that

$$\Phi\left(\frac{1}{h}\int_\alpha^\beta |u'|dt\right) \le c'(\|w\|_{L^1} + \|w\|_{L^1}^{\frac{1}{2}})$$

for all $\varepsilon > 0$ small enough. Hence,

$$\int_\alpha^\beta |u'|dt \le h\Phi^{-1}\Big(c'\left(\|w\|_{L^1} + \|w\|_{L^1}^{\frac{1}{2}}\right)\Big). \tag{3.6}$$

whatever is the interval (α, β), with $\beta - \alpha = 2h$. In turn, this implies that for all such intervals there exists a point $\overline{x}_{\alpha,\beta} \in (\alpha, \beta)$ such that

$$|u'(\overline{x}_{\alpha,\beta})| \le \frac{1}{2}\Phi^{-1}\Big(c'\left(\|w\|_{L^1} + \|w\|_{L^1}^{\frac{1}{2}}\right)\Big).$$

We now go back to the proof of Theorem 2.7 and, arguing as in the case $b - a > 1$, we use $\varphi = u'$ in (3.2), integrate on $(\overline{x}_{\alpha,\beta}, x)$ to obtain (as in (2.27) and (2.28)) for ε small enough

$$\theta(|u'(x)|) \le \theta\Big(\frac{1}{2}\Phi^{-1}\Big(c'\Big(\|w\|_{L^1} + \|w\|_{L^1}^{\frac{1}{2}}\Big)\Big)\Big) + c'h\Phi^{-1}\Big(c'\Big(\|w\|_{L^1} + \|w\|_{L^1}^{\frac{1}{2}}\Big)\Big),$$

where now the last term is obtained by using (3.6).
We conclude that there exists a constant $\widehat{\gamma}_2 > 0$, depending only on ϕ, such that whenever $\|w\|_{L^1} \le \widehat{\gamma}_2$ there holds $\theta(|u'(x)|) \le \frac{2}{3}c_\ell$. From this (3.4) holds in any case as soon as $\|w\|_{L^1} \le \widehat{\gamma} := \min\{\widehat{\gamma}_1, \widehat{\gamma}_2, 1\}$. □

4 Proof of the main results

We begin by recalling a result on maximal monotone operators (see [3, Lemma 3.57] for a related statement).

Lemma 4.1. *Let A be a maximal monotone operator, $A : L^2(I;\mathbb{R}^k) \to L^2(I;\mathbb{R}^k)$ and $(y_\varepsilon, A(y_\varepsilon))$ a sequence in $L^2(I;\mathbb{R}^k) \times L^2(I;\mathbb{R}^k)$ which satisfies:*

(i) $y_\varepsilon \rightharpoonup y$ *weakly in* L^2, $A(y_\varepsilon) \rightharpoonup \eta$ *weakly in* L^2,

(ii) $\limsup_{\varepsilon\to 0} \langle A(y_\varepsilon), y_\varepsilon\rangle \le \langle \eta, y\rangle$.

Then $\eta = A(y)$.

We are now in a position to prove our main results.

Proof of Theorem 1.1. Let $u_{\varepsilon,\lambda} \in W^{2,\infty}(I;\mathbb{R}^k)$ be the minimizer of $\mathcal{F}_{\varepsilon,\lambda}$ and assume $\|w\|_{L^\infty} \leq \gamma$, so that by Theorem 2.7 we have that $u_{\varepsilon,\lambda}$ are uniformly bounded in $W^{1,\infty}(I;\mathbb{R}^k)$ with respect to $\varepsilon \in (0, \varepsilon_0)$ for a certain value $\varepsilon_0 > 0$. Further $u_{\varepsilon,\lambda}$ satisfies the Euler–Lagrange equation in (2.16) with Neumann boundary conditions, namely

$$\frac{d}{dx}\Big(\Psi_\varepsilon(u'_{\varepsilon,\lambda})\Big) = r_{\varepsilon,\lambda}(x), \qquad u'_{\varepsilon,\lambda}(a) = u'_{\varepsilon,\lambda}(b) = 0, \tag{4.1}$$

where

$$\Psi_\varepsilon(p) := \phi'(|p|)\frac{p}{|p|} + \varepsilon p, \quad r_{\varepsilon,\lambda}(x) := (1+\varepsilon)\left[\frac{u_{\varepsilon,\lambda}-w}{|u_{\varepsilon,\lambda}-w|}|u_{\varepsilon,\lambda}-w|^\varepsilon + \lambda\frac{u_{\varepsilon,\lambda}-\widetilde{u}}{|u_{\varepsilon,\lambda}-\widetilde{u}|}|u_{\varepsilon,\lambda}-\widetilde{u}|^\varepsilon\right].$$

After integration on (a, x), (4.1) can be rewritten as

$$\phi'(|u'_{\varepsilon,\lambda}(x)|)\frac{u'_{\varepsilon,\lambda}(x)}{|u'_{\varepsilon,\lambda}(x)|} + \varepsilon u'_{\varepsilon,\lambda}(x) = \int_a^x r_{\varepsilon,\lambda}(t)\; dt =: R_{\varepsilon,\lambda}(x). \tag{4.2}$$

Since $r_{\varepsilon,\lambda} \in L^\infty(I;\mathbb{R}^k)$ by definition and the boundedness of $u_{\varepsilon,\lambda}, w, \widetilde{u}$, it follows that

$$R_{\varepsilon,\lambda} \in W^{1,\infty}(I;\mathbb{R}^k). \tag{4.3}$$

Define the function $H : \mathbb{R}^k \to \mathbb{R}$ as

$$H(u'_{\varepsilon,\lambda}) := \phi(|u'_{\varepsilon,\lambda}|)$$

and observe that

$$|\nabla H(u'_{\varepsilon,\lambda})| = \left|\phi'(|u'_{\varepsilon,\lambda}|)\frac{u'_{\varepsilon,\lambda}}{|u'_{\varepsilon,\lambda}|}\right| = \phi'(|u'_{\varepsilon,\lambda}|) \leq C, \tag{4.4}$$

where the boundedness follows from the assumptions on the function ϕ.
Now, (4.2) reads as

$$\nabla H(u'_{\varepsilon,\lambda}) + \varepsilon u'_{\varepsilon,\lambda} = R_{\varepsilon,\lambda}. \tag{4.5}$$

Denoting by $\langle\cdot,\cdot\rangle$ the scalar product in $L^2(I;\mathbb{R}^k)$ and multiplying both sides by $u'_{\varepsilon,\lambda}$, we get

$$\langle\nabla H(u'_{\varepsilon,\lambda}), u'_{\varepsilon,\lambda}\rangle + \varepsilon\langle u'_{\varepsilon,\lambda}, u'_{\varepsilon,\lambda}\rangle = \langle R_{\varepsilon,\lambda}, u'_{\varepsilon,\lambda}\rangle. \tag{4.6}$$

By (4.4) we deduce that there exists some $\eta \in L^\infty(I;\mathbb{R}^k)$ such that, as $\varepsilon \to 0^+$,

$$\nabla H(u'_{\varepsilon,\lambda}) \rightharpoonup \eta \qquad \text{weakly in } L^2.$$

Moreover, from Theorem 2.7 and 2.8,

$$u'_{\varepsilon,\delta,\lambda} \rightharpoonup \widetilde{u}' \qquad \text{weakly in } L^2,$$

and from (4.3) it follows

$$R_{\varepsilon,\lambda}(x) \to R_\lambda(x) := \int_a^x r_\lambda(t)\; dt,$$

where $r_{\varepsilon,\lambda} \rightharpoonup r_\lambda$ weakly* in L^∞. As a consequence, passing to the limit as $\varepsilon \to 0^+$ in (4.6), we get

$$\lim_{\varepsilon\to 0^+} \langle \nabla H(u'_{\varepsilon,\lambda}), u'_{\varepsilon,\lambda}\rangle = \lim_{\varepsilon\to 0^+} (\langle R_{\varepsilon,\lambda}, u'_{\varepsilon,\lambda}\rangle - \varepsilon\langle u'_{\varepsilon,\lambda}, u'_{\varepsilon,\lambda}\rangle) = \langle R_\lambda, \widetilde{u}'\rangle.$$

In order to apply Lemma 4.1 we want to show that

$$\langle \eta, \widetilde{u}'\rangle = \langle R_\lambda, \widetilde{u}'\rangle. \tag{4.7}$$

To this end, we pass to the limit in (4.5) and we integrate by parts, obtaining

$$\int_a^b \eta \cdot \widetilde{u}'\, dx = \int_a^b R_\lambda \cdot \widetilde{u}',$$

where we have used that $R_\lambda(a) = R_\lambda(b) = 0$. Hence, (4.7) follows:

$$\eta = \nabla H(\widetilde{u}').$$

Thus

$$\phi'(|\widetilde{u}'|)\frac{\widetilde{u}'}{|\widetilde{u}'|} = R_\lambda,$$

where R_λ belongs to $W^{1,\infty}$ and, consequently, it is continuous. Let us now define $v := \phi'(|p|)\frac{p}{|p|}$ and $G : \mathbb{R}^k \to \mathbb{R}^k$ as

$$G(p) := v.$$

It is easy to check that

$$G^{-1}(v) = (\phi')^{-1}(|v|)\frac{v}{|v|},$$

which is a continuous function, as ϕ' is continuous and strictly increasing, and hence invertible with a continuous inverse. Thus, we can conclude that $\widetilde{u}' = G^{-1}(R_\lambda)$. Since it is the composition of continuous functions, $\widetilde{u}'$ is also continuous. □

Proof of Theorem 1.2. This can be done following the previous argument, from which the proof of Theorem 1.2 differs very slightly. The difference is that now

$$R_{\varepsilon,\lambda} \in W^{1,1}(I),$$

and the convergence $R_{\varepsilon,\lambda} \to R_\lambda$ takes place weakly in $W^{1,1}(I;\mathbb{R}^k)$ and strongly in $L^2(I;R^k)$. The rest of the proof is identical. □

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