

COMPACTNESS IN DIMENSION FIVE AND EQUIVARIANT NONCOMPACTNESS FOR THE CR YAMABE PROBLEM

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ABSTRACT. We study compactness and noncompactness phenomena for the CR Yamabe equation on compact strictly pseudoconvex CR manifolds. First, in dimension five, we establish uniform *a priori* estimates for families of positive solutions of subcritical equations for the conformal CR sub-Laplacian

$$L_J u = u^p,$$

with p bounded away from the critical exponent, assuming positivity of the CR Yamabe constant and positivity of the p -mass at every point. As a consequence, the corresponding set of solutions is precompact in Hölder topologies. Secondly, we consider the equivariant CR Yamabe problem for a compact subgroup G of pseudo-Hermitian transformations. We construct a G -invariant CR structure on S^3 , not equivalent to the standard one, for which the associated CR Yamabe equation admits a sequence of G -invariant solutions whose maxima diverge, thereby proving noncompactness in the equivariant setting. The arguments combine a Pohozaev-type identity in pseudohermitian normal coordinates with a blow-up analysis and Liouville-type classification results on the Heisenberg group.

1. INTRODUCTION

Suppose (M, g_0) is a compact n -dimensional Riemannian manifold of dimension $n \geq 3$. As a generalization of the Uniformization Theorem, the *Yamabe problem* is to find a metric g conformal to g_0 such that the scalar curvature R_g of g is constant. If we write $g = u^{\frac{4}{n-2}} g_0$ for some $0 < u \in C^\infty(M)$, then the scalar curvatures of g_0 and g are related by

$$(1.1) \quad -\frac{4(n-1)}{n-2} \Delta_{g_0} u + R_{g_0} u = R_g u^{\frac{n+2}{n-2}},$$

where Δ_{g_0} is the Laplacian with respect to the metric g_0 . In view of (1.1), the Yamabe problem is equivalent to finding $0 < u \in C^\infty(M)$ such that

$$(1.2) \quad -\frac{4(n-1)}{n-2} \Delta_{g_0} u + R_{g_0} u = c u^{\frac{n+2}{n-2}}$$

for some constant c . The Yamabe problem was solved in a series of work by Aubin [6], Trudinger [41], and Schoen [39]. In other words, (1.2) has at least one solution.

When (M, g_0) has negative or zero Yamabe constant (which corresponds to the case when $c < 0$ or $c = 0$ in (1.2) respectively), it is easy to see that the solution to (1.2) is unique up to normalization. In the case when (M, g_0) has positive Yamabe constant (which corresponds to the case when $c > 0$ in (1.2)), Schoen raised the compactness conjecture in a topics course at Stanford in 1988: the set of solutions of the Yamabe equation (1.2) is compact except when the manifold is conformally equivalent to the standard unit sphere S^n .

The compactness conjecture was proved by Khuri, Marques and Schoen [34] in dimension $n \leq 24$ (see also [19, 35, 36, 38] for previous results). It turns out that the compactness conjecture is false when $n \geq 25$; the counterexample was constructed by Brendle for $n \geq 52$ in [8] and by Brendle and Marques for $25 \leq n \leq 51$ in [9].

Since then, various compactness and noncompactness results have been proved in different contexts. For example, compactness and noncompactness results were obtained for the Yamabe-type problem on manifolds with boundary [3, 4, 5, 13, 16, 17, 24, 27, 33].

The Yamabe problem can also be posed in the context of CR manifolds. Given a compact strictly pseudoconvex CR manifold (M, J, θ) of real dimension $2n + 1$, the *CR Yamabe problem* is to find a contact form conformal to θ such that its Webster scalar curvature is constant. This was first introduced by Jerison and Lee in [30]. If we write $\tilde{\theta} = u^{\frac{2}{n}}\theta$ for some $0 < u \in C^\infty(M)$, then the Webster scalar curvature of θ and $\tilde{\theta}$ are related by

$$(1.3) \quad L_J u = \tilde{R} u^{1 + \frac{2}{n}}$$

where L_J is the conformal CR sub-Laplacian given by

$$L_J = -b_n \Delta_b + R, \quad b_n := 2 + \frac{2}{n}.$$

Here, R and \tilde{R} are the Webster scalar curvatures of θ and $\tilde{\theta}$, respectively. It follows from (1.3) that the CR Yamabe problem is equivalent to finding $0 < u \in C^\infty(M)$ such that

$$(1.4) \quad L_J u = c u^{1 + \frac{2}{n}}$$

for some constant c . After the work of Jerison and Lee in [31, 32, 30], the CR Yamabe problem was studied in [14, 21, 22].

Inspired by Schoen's compactness conjecture stated above, it is natural to consider the compactness and noncompactness of solutions to the CR Yamabe equation (1.4). The following theorem was proved in [1] by the first author.

Theorem 1.1 (Theorem 1.1 in [1]). *Let (M, J, θ) be a compact 3-dimensional strictly pseudoconvex CR manifold of positive CR Yamabe constant such that, for every $x \in M$, its p -mass at x is positive, i.e. $m_x > 0$. Then, for every $\epsilon > 0$ and $k \in \mathbb{N}$, there exists a constant C such that*

$$\frac{1}{C} \leq u \leq C, \quad \|u\|_{\Gamma^{k,\alpha}} \leq C$$

for every $u \in \cup_{1+\epsilon \leq p \leq 3} \mathcal{M}_p$ and $0 < \alpha < 1$. Here

$$\mathcal{M}_p = \{u > 0 : L_J u = u^p\},$$

where $\Gamma_{k,\alpha}$ is the Hölder space. In particular, $\cup_{1+\epsilon \leq p \leq 3} \mathcal{M}_p$ is compact in the $\Gamma^{k,\alpha}$ topology.

As pointed out in [1], the assumption that p -mass is positive at every point $x \in M$ is difficult to check. Fortunately, we have the results by Takeuchi in [40] and by Cheng, Malchiodi, and Yang in [14], which say that any embeddable 3-dimensional CR manifold M which is not CR-equivalent to S^3 with the standard CR structure must have positive p -mass at every point $x \in M$. Combining these with Theorem 1.1, we have the following:

Corollary 1.2 (Corollary 1.2 in [1]). *Suppose that (M, J, θ) is an embeddable 3-dimensional CR manifold which has positive CR Yamabe constant and is not CR-equivalent to S^3 with the standard CR structure. Then the statement of Theorem 1.1 holds.*

In this paper, we prove the following theorem, which is the corresponding case of Theorem 1.1 for dimension 5.

Theorem 1.3. *Let (M, J, θ) be a compact 5-dimensional strictly pseudoconvex CR manifold of positive CR Yamabe constant such that, for every $x \in M$ such that the Chern tensor satisfies $S(x) = 0$, its p -mass at x is positive, i.e. $m_x > 0$. Then, for every $\epsilon > 0$ and $k \in \mathbb{N}$, there exists a constant C such that*

$$\frac{1}{C} \leq u \leq C, \quad \|u\|_{\Gamma^{k,\alpha}} \leq C$$

for every $u \in \cup_{1+\epsilon \leq p \leq 2} \mathcal{M}_p$ and $0 < \alpha < 1$. Here

$$\mathcal{M}_p = \{u > 0 : L_J u = u^p\},$$

where $\Gamma_{k,\alpha}$ is the Hölder space. In particular, $\cup_{1+\epsilon \leq p \leq 2} \mathcal{M}_p$ is compact in the $\Gamma^{k,\alpha}$ topology.

The Chern tensor in the above theorem is the analogue of the Weyl tensor in Riemannian geometry and will be defined in Section 2.

As in the three-dimensional case, the verification of the hypothesis of positivity of the mass can be checked by the positive mass theorems available in the literature: the one in [11] is valid for spherical CR manifolds satisfying an analytical condition in dimension five, the one in [12] is valid for five-dimensional spherical spin manifolds. Many conjecture that in dimensions higher than three, a CR Positive Mass Theorem without additional hypotheses, similar to the Riemannian one, should hold, but the problem is currently completely open.

Our proof of Theorem 1.3 is inspired by the work of Marques [38]; in particular, we adapt to the CR case the technique of *symmetry estimates*.

The *equivariant CR Yamabe problem* was first introduced and studied by the second author in [26]. To state it, we recall that a *CR automorphism* of (M, J, θ) is a diffeomorphism $f : M \rightarrow M$ such that its differential at any point maps horizontal vectors to horizontal vectors, i.e. $f^*(\ker \theta) \subseteq \ker \theta$. Note that f is a CR automorphism if and only if

$$f^* \theta = u \theta \quad \text{for some } u \in C^\infty(M).$$

Let $Aut_{CR}(M, J, \theta)$ be the group of all CR automorphisms of (M, J, θ) . On the other hand, let $I(M, J, \theta)$ be the group of all pseudo-Hermitian transformations f of (M, θ) that preserve the associated contact Riemannian metric

$$g = \theta \cdot \theta + d\theta \circ J.$$

Note that $I(M, J, \theta)$ is a subgroup of $Aut_{CR}(M, J, \theta)$ (cf. [7]).

Conjecture 1.4 (Equivariant CR Yamabe problem). *Given a compact strictly pseudoconvex CR manifold (M, J, θ) of real dimension $2n+1$, and a compact subgroup G of $I(M, \theta)$, there exists a G -invariant contact form conformal to θ such that its Webster scalar curvature is constant.*

Here, a contact form $\tilde{\theta}$ is said to be G -invariant if $f^* \tilde{\theta} = \tilde{\theta}$ for all $f \in G$. We remark that the classical CR Yamabe problem is the special case of Conjecture 1.4 when $G = \{id_M\}$.

In [2], the first and third authors proved the following:

Theorem 1.5 (Theorem 1.1 in [2]). *There exists a CR structure on S^3 , not equivalent to the standard one, such that the associated CR Yamabe equation*

$$L_J u = 2u^3$$

has a set of solutions $\{u_k\}_{k \in \mathbb{N}}$ with $\max u_k \rightarrow \infty$.

Here, $L_J = -4\Delta_b + R$ is the conformal CR sub-Laplacian when $n = 1$.

It is natural to ask whether the result corresponding to Theorem 1.5 is true in the equivariant case. To answer this, we let $f : S^3 \rightarrow S^3$ be given by

$$f(w_1, w_2) = (-w_1, w_2) \text{ for } (w_1, w_2) \in S^3,$$

and let $G = \{f, id_{S^3}\}$. We have the following:

Theorem 1.6. *There exists a CR structure on S^3 which is G -invariant and not equivalent to the standard one, such that the associated CR Yamabe equation*

$$(1.5) \quad L_J u = 2u^3$$

has a set of G -invariant solutions $\{u_k\}_{k \in \mathbb{N}}$ with $\max u_k \rightarrow \infty$.

Here, a function u is G -invariant if $\phi^* u = u$ for all $\phi \in G$, and a CR structure is G -invariant if $\phi^* J = J$ for all $\phi \in G$. Therefore, Theorem 1.6 proves the noncompactness of the equivariant CR Yamabe problem. We remark that the noncompactness of the equivariant Yamabe problem and the equivariant Yamabe problem with boundary were obtained in [25] and [28], respectively.

We also remark that the compactness and noncompactness of the equivariant CR Yamabe equation can be subtle. Indeed, if we take H to be the group $\{h, id_{S^3}\}$, where

$$h(w_1, w_2) = (-w_1, -w_2) \text{ for } (w_1, w_2) \in S^3.$$

On the Rossi sphere (S^3, J_s) , all H -invariant solutions u of the CR Yamabe equation (1.5) can be viewed as a solution of the CR Yamabe equation (1.5) on the quotient $(S^3/H, J_s)$. It is known that $(S^3/H, J_s)$ is embeddable and has positive CR Yamabe constant. In particular, it follows from Corollary 1.2 stated above that the set of all solutions to the CR Yamabe equation (1.5) on $(S^3/H, J_s)$ is compact. Hence, the set of all H -invariant solutions to the CR Yamabe equation (1.5) on the Rossi sphere (S^3, J_s) is compact.

Organization of the paper: In Section 2 we collect the basic pseudohermitian and CR-geometric preliminaries, fix notation, and recall the CR Yamabe operator and its main analytic properties. In Section 3 we prove Theorem 1.6. Section 4 is dedicated to the blow-up analysis and the proof of Theorem 1.3. We conclude the paper with an Appendix containing some technical computations.

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2. PRELIMINARIES

We briefly recall some basic notions about CR geometry, referring to the monograph [18] for a complete introduction. A CR structure on a $2n + 1$ -dimensional manifold M is an n -dimensional complex subbundle \mathcal{H} of $TM \otimes \mathbb{C}$ such that $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$ and $[\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H})$. The real part of \mathcal{H} , $H(M) = \text{Re}(\mathcal{H} + \overline{\mathcal{H}})$, is called the Levi distribution, and it carries the natural complex structure J defined by $J(Z + \overline{Z}) = i(Z - \overline{Z})$. $H(M)$ and J determine the CR structure. The CR structure is said to be nondegenerate if $H(M)$ is a contact distribution; it is said to be pseudoconvex if for some contact form θ the bilinear form $L_\theta(Z, \overline{W}) = -d\theta(Z, \overline{W})$ is positive definite. In the following we will always assume the hypothesis of pseudoconvexity. In such a case, the choice of a contact form θ determines a rich geometric structure, including a subriemannian metric on $H(M)$ (which induces a subriemannian distance d), a measure, and a connection called the Tanaka-Webster connection. By contracting twice the associated curvature tensor through the metric, a scalar curvature invariant known as Webster scalar curvature is obtained. Any other contact form for a given CR structure is of the form $\tilde{\theta} = u^{\frac{2}{n}}\theta$ for some smooth positive function u , and the Webster scalar curvature associated to $\tilde{\theta}$ is given by the formula

$$\tilde{R} = u^{-\frac{n+2}{n}}(-b_n \Delta_b + R)u$$

where $\Delta_b = \text{div} \circ \nabla_b$ (where ∇_b is the subriemannian gradient) is a second-order operator known as the sub-Laplacian, and $b_n = 2 + \frac{2}{n}$. Equivalently

$$\Delta_b u = u_{\alpha\bar{\alpha}} + u_{\bar{\alpha}\alpha}.$$

The most important CR manifold is the Heisenberg group, which is the Lie group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, t) \cdot (w, s) = (z + w, t + s + 2\text{Im}(z\bar{w})).$$

\mathbb{H}^n is endowed with the left-invariant CR structure \mathcal{H} generated by the left-invariant vector fields

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} + i\bar{z}^\alpha \frac{\partial}{\partial t}$$

and the contact form

$$(2.1) \quad \theta = dt + i \sum_{\alpha=1}^n (z^\alpha d\bar{z}^\alpha - \bar{z}^\alpha dz^\alpha).$$

Every CR manifold has local coordinates around any point with values in \mathbb{H}^n , called pseudohermitian normal coordinates (see [32]). We call $H(\mathbb{H}^1) = \text{Re}(\mathcal{H} + \overline{\mathcal{H}})$ the Levi distribution associated to this CR structure, and J_0 the complex structure on it. The Reeb vector field corresponding to this contact structure is $T = \frac{\partial}{\partial t}$. \mathbb{H}^n is endowed with the one-parameter group of CR and group automorphisms

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 t),$$

for $\lambda \in (0, \infty)$, called dilations. The pseudohermitian measure (which is obviously a Haar measure) satisfies $(\delta_\lambda)_\# dx = \lambda^{2n+2} dx$; for this reason the number $Q = 2n + 2$ is

called the homogeneous dimension of \mathbb{H}^n . The generator of the group of dilations is the vector field

$$(2.2) \quad \Xi = \sum_{\alpha=1}^n (zZ_\alpha + \bar{z}Z_{\bar{\alpha}}) + 2tT,$$

where $Z_{\bar{\alpha}} = \overline{Z_\alpha}$. On \mathbb{H}^n the sub-Laplacian is equal to $\frac{1}{2}\nabla_b = \sum_{\alpha=1}^n (Z_\alpha Z_{\bar{\alpha}} + Z_{\bar{\alpha}} Z_\alpha)$. The CR Yamabe equation in \mathbb{H}^n

$$(2.3) \quad -b_n \Delta_b u = u^{\frac{n+2}{n}}$$

has the solution

$$U(z, t) = c_n \frac{1}{(t^2 + (1 + |z|^2)^2)^{n/2}}$$

which geometrically corresponds to the standard contact form of the CR sphere S^{2n+1} pulled back to \mathbb{H}^n through the Cayley transform, a CR equivalence analogous to the stereographic projection. If $L_x(y) = x^{-1}y$ denotes left translation, then, by the invariance properties of the sub-Laplacian, the functions

$$(2.4) \quad U_{x,\lambda} = \lambda U \circ \delta_\lambda \circ L_x$$

for $x \in \mathbb{H}^n$ and $\lambda \in (0, \infty)$ form a family of solutions.

Let $S^1(\mathbb{H}^n)$ be the completion of $C_c^\infty(\mathbb{H}^n)$ with respect to the product

$$\langle u, v \rangle = \int_{\mathbb{H}^n} \nabla_b u \cdot \nabla_b v$$

which, by the Folland-Stein embeddings, is a subset of $L^{\frac{2Q}{Q-2}}$.

Then the family of solutions $\{U_{x,\lambda}\}_{x \in \mathbb{H}^n, \lambda > 0}$ is stable in $S^1(\mathbb{H}^n)$ in the following sense.

Theorem 2.1 (Lemma 5 from [37]). *$u \in S^1(\mathbb{H}^n)$ solves the linearized equation of (2.3) in U*

$$-b_n \Delta_b u = \frac{n+2}{n} U^{\frac{2}{n}} u$$

if and only if there exist $a, \gamma \in \mathbb{R}$ and $\mu_\alpha \in \mathbb{C}$ such that

$$u = a\Xi U + \gamma T U + \sum_{\alpha=1}^n (\mu_\alpha Z_\alpha U + \bar{\mu}_\alpha Z_{\bar{\alpha}} U).$$

Similarly to Riemannian geometry, there exists a tensor which characterizes conformal flatness (except in the lowest dimensions). We recall that a CR manifold is called spherical if it is locally CR equivalent to \mathbb{H}^n (or equivalently to S^{2n+1}). The Chern tensor is defined as the tensor

$$S_\beta^\alpha{}_{\lambda\bar{\sigma}} = R_\beta^\alpha{}_{\lambda\bar{\sigma}} - \frac{1}{n+2} (R_\beta^\alpha h_{\lambda\bar{\sigma}} + R_\lambda^\alpha h_{\beta\bar{\sigma}} + \delta_\beta^\alpha R_{\lambda\bar{\sigma}} + \delta_\lambda^\alpha R_{\beta\bar{\sigma}}) + \\ + \frac{R}{(n+1)(n+2)} (\delta_\beta^\alpha h_{\lambda\bar{\sigma}} + \delta_\lambda^\alpha h_{\beta\bar{\sigma}})$$

where $R_\beta^\alpha{}_{\lambda\bar{\sigma}}$ is the curvature tensor, $h_{\lambda\bar{\sigma}}$ is the metric and $R_\beta^\alpha = R_\beta^\sigma{}_{\sigma\bar{\gamma}} h^{\alpha\bar{\gamma}}$ is the pseudohermitian Ricci tensor.

Then the following theorem of Chern and Moser [15] holds (see Section 7.3 in [29] for a more modern presentation and an alternative proof).

Theorem 2.2. *If $n \geq 2$ then M is spherical if and only if $S = 0$.*

3. PROOF OF THEOREM 1.6

In this section we will work in the first Heisenberg group \mathbb{H}^1 . We will consider various CR structures whose Levi distribution coincides with $H(\mathbb{H}^1)$, and we will fix the contact form θ defined in equation (2.1). Therefore the CR structure will be uniquely determined by the complex structure J on $H(\mathbb{H}^1)$, and we will denote by ∇_J , Δ_J and so forth the various pseudohermitian quantities when the dependence on J is relevant.

We define $\widehat{f} : \mathbb{H}^1 \rightarrow \mathbb{H}^1$ by

$$\widehat{f}(z, t) = (-z, t).$$

Let $\widehat{G} = \{\widehat{f}, id_{\mathbb{H}^1}\}$.

Let X be the Hilbert space

$$X = \{u \in L^4(\mathbb{H}^1) : \|\nabla_{J_0} u\|_{L^2} < \infty\}$$

with the inner product

$$\langle u, v \rangle_X = \int_{\mathbb{H}^1} \nabla_{J_0} u \cdot \nabla_{J_0} v.$$

Let $X_{\widehat{G}}$ be the subspace of X which contains all the \widehat{G} -invariant functions in X , i.e.

$$X_{\widehat{G}} = \left\{ u \in X : u \text{ is } \widehat{G}\text{-invariant} \right\}.$$

The Folland-Stein embedding asserts that there exists K such that

$$\|u\|_{L^4} \leq K \|u\|_X.$$

We define

$$\mathcal{M} = \{U_{x,\lambda} : x \in \mathbb{H}^1, \lambda \in (0, \infty)\}$$

where $U_{x,\lambda}$ is given as in (2.4). We also let

$$\mathcal{E}_{(x,\lambda)} = \text{span}\{Z_1 U_{x,\lambda}, Z_{\bar{1}} U_{x,\lambda}, T U_{x,\lambda}, \Xi U_{x,\lambda}\}^\perp.$$

That is to say,

$$\mathcal{E}_{(x,\lambda)} = \{u \in X : \langle u, Z_1 U_{x,\lambda} \rangle_X = \langle u, Z_{\bar{1}} U_{x,\lambda} \rangle_X = \langle u, T U_{x,\lambda} \rangle_X = \langle u, \Xi U_{x,\lambda} \rangle_X = 0\}.$$

We also let

$$\mathcal{M}_0 = \{U_{x,\lambda} : x \in \mathbb{H}_0^1, \lambda \in (0, \infty)\}$$

where

$$\mathbb{H}_0^1 = \{x = (z, t) \in \mathbb{H}^1 : z = 0\},$$

and

$$\mathcal{E}_{(x,\lambda),\widehat{G}} = \{u \in \mathcal{E}_{(x,\lambda)} : u \text{ is } \widehat{G}\text{-invariant}\}.$$

In general, $U_{x,\lambda} \notin \mathcal{E}_{(x,\lambda),\widehat{G}}$. But $U_{x,\lambda} \in \mathcal{E}_{(x,\lambda),\widehat{G}}$ whenever $x \in \mathbb{H}_0^1$.

Note that the CR Yamabe equation for J is the Euler-Lagrange equation for the functional

$$\mathcal{I}_J(u) = \int_{\mathbb{H}^1} u L_J u - \int_{\mathbb{H}^1} u^4$$

in the space X .

Proposition 3.1. *There exists a constant α such that if J is a \widehat{G} -invariant CR structure on \mathbb{H}^1 coinciding with J_0 on $\mathbb{H}^1 \setminus B_1(0)$ and such that $\|J - J_0\|_{\Gamma^2} \leq \alpha$, then, for every $(x, \lambda) \in \mathbb{H}_0^1 \times (0, \infty)$, there exists a unique $v_{x,\lambda} \in \mathcal{E}_{(x,\lambda),\widehat{G}}$ with $\|v\| \lesssim \alpha$ satisfying*

$$\pi_{\mathcal{E}_{(x,\lambda),\widehat{G}}}(\nabla \mathcal{I}_J(U_{x,\lambda} + v_{x,\lambda})) = 0.$$

Proof. The proof is exactly the same as the proof of [2, Proposition 3.2]. \square

We have the following:

Proposition 3.2. *If $(x, \lambda) \in \mathbb{H}_0^1 \times (0, \infty)$ is a critical point of $\mathcal{I}_J(U_{x,\lambda} + v_{x,\lambda})$, then $\nabla \mathcal{I}_J(U_{x,\lambda} + v_{x,\lambda}) = 0$.*

Proof. By the Lyapunov-Schmidt method, we can conclude that

$$(3.1) \quad \int_{\mathbb{H}^1} (4\langle \phi, U_{x,\lambda} + v_{x,\lambda} \rangle_X + R(U_{x,\lambda} + v_{x,\lambda})\phi - 2(U_{x,\lambda} + v_{x,\lambda})^3\phi) = 0$$

for all $\phi \in X_{\widehat{G}}$. Let $w \in X$ be the unique solution to

$$(3.2) \quad L_J(w) = 2(U_{x,\lambda} + v_{x,\lambda})^3.$$

Since J is \widehat{G} -invariant and $U_{x,\lambda} + v_{x,\lambda}$ is \widehat{G} -invariant, we conclude that w is \widehat{G} -invariant. Putting $U_{x,\lambda} + v_{x,\lambda} - w \in X_{\widehat{G}}$ into (3.1) and using (3.2), we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{H}^1} \left(4\langle U_{x,\lambda} + v_{x,\lambda} - w, U_{x,\lambda} + v_{x,\lambda} \rangle_X \right. \\ &\quad \left. + R(U_{x,\lambda} + v_{x,\lambda})(U_{x,\lambda} + v_{x,\lambda} - w) - 2(U_{x,\lambda} + v_{x,\lambda})^3(U_{x,\lambda} + v_{x,\lambda} - w) \right) \\ &= \int_{\mathbb{H}^1} \left(4\langle U_{x,\lambda} + v_{x,\lambda} - w, U_{x,\lambda} + v_{x,\lambda} \rangle_X + 4(U_{x,\lambda} + v_{x,\lambda} - w)\Delta_b w \right) \\ &= \int_{\mathbb{H}^1} 4\langle U_{x,\lambda} + v_{x,\lambda} - w, U_{x,\lambda} + v_{x,\lambda} - w \rangle_X. \end{aligned}$$

From this, we conclude that $w = U_{x,\lambda} + v_{x,\lambda}$. In particular, it follows from (3.2) that $U_{x,\lambda} + v_{x,\lambda}$ satisfies

$$L_J(U_{x,\lambda} + v_{x,\lambda}) = 2(U_{x,\lambda} + v_{x,\lambda})^3.$$

This proves the assertion. \square

Let $x_k = (0, 0, \frac{1}{k}) \in \mathbb{H}_0^1$ and let r_k, R_k, s_k be sequences converging to zero such that the balls $B_{R_k}(x_k)$ are disjoint. Let J be a \widehat{G} -invariant CR structure coinciding with J_0 in $\mathbb{H}^1 \setminus \bigcup B_{Ar_k}(x_k)$, with J_{s_k} on $B_{r_k}(x_k)$, and with J_f on $B_{Ar_k}(x_k) \setminus B_{r_k}(x_k)$, where $|f| \leq s_k$.

Let us define

$$\Omega_k = \left\{ U_{x,\lambda} : x \in \mathbb{H}_0^1, |x - x_k| < R_k, \frac{\alpha}{R_k} < \lambda < \frac{\beta}{r_k} \right\} \subset \mathcal{M}_0$$

with α and β to be chosen later.

Then we want to show that, up to choosing the parameters appropriately, for every k there exists a critical point of $\mathcal{I}_J(U_{x,\lambda} + v_{x,\lambda})$ in Ω_k , that is, a \widehat{G} -invariant solution to the CR Yamabe equation for J that is approximately a bubble centered at x_k .

Note that Lemmas 4.4-4.7 in [2] still hold in our case.

Proof of Theorem 1.6. We choose the \widehat{G} -invariant CR structure as above, and $r_k = 2^{-k}$, $R_k = C2^{-k}$ with C large enough, $\alpha, \beta \gg 1$, $s_k = 2^{-2k}$. Thanks to Lemmas 4.4-4.7 in [2], we have

$$\max_{\Omega_k} \mathcal{I}_J(U_{x,\lambda} + v_{x,\lambda,s_k}) > \max_{\partial\Omega_k} \mathcal{I}_J(U_{x,\lambda} + v_{x,\lambda,s_k}),$$

and therefore there exists a critical point of $\mathcal{I}_J(U_{x,\lambda} + v_{x,\lambda})$ restricted to \mathcal{M}_0 in Ω_k . By Proposition 3.2, it is a free critical point, and therefore a \widehat{G} -invariant solution to (1.5). Finally, since

$$\begin{aligned} & |B_{r_k}(x_k)|^{\frac{1}{4}} \max_{B_{r_k}(x_k)} (U_{x,\lambda} + v_{x,\lambda,s_k}) \\ & \geq \|(U_{x,\lambda} + v_{x,\lambda,s_k})\|_{L^4(B_{r_k}(x_k))} \geq \|U_{x,\lambda}\|_{L^4(B_{r_k}(x_k))} - \|v_{x,\lambda,s_k}\|_X, \end{aligned}$$

we have $\max_{B_{r_k}(x_k)} (U_{x,\lambda} + v_{x,\lambda}) \rightarrow \infty$. This completes the proof of Theorem 1.6. \square

4. COMPACTNESS

We recall the Pohozaev identity for CR manifolds (see [1, Proposition 3.3]), which is the Pohozaev identity for \mathbb{H}^n by Garofalo and Lanconelli in [23] written in pseudohermitian normal coordinates. In the following, when working in such coordinates, we will use a circle superscript to indicate objects coming from \mathbb{H}^n through these coordinates ($\overset{\circ}{\Delta}_b, \overset{\circ}{\nabla}_b, \dots$).

Proposition 4.1. *Let $\bar{x} \in M$ and u be a solution of*

$$-b_n \Delta_b u + Ru = \widetilde{R}u^p.$$

Then, in pseudohermitian normal coordinates around \bar{x} , the following holds

$$\begin{aligned} & \int_{B_r(\bar{x})} \left(\frac{1}{b_n} \left(\frac{2n+2}{p+1} - n \right) \widetilde{R}u^{p+1} - \frac{1}{b_n} Ru^2 + \frac{1}{b_n(p+1)} \Xi(\widetilde{R})u^{p+1} \right. \\ & \quad \left. - \frac{1}{2b_n} \Xi(R)u^2 - (\Xi u + nu)(\Delta_b u - \overset{\circ}{\Delta}_b u) \right) d\overset{\circ}{V} \\ & = \int_{\partial B_r(\bar{x})} \left(\left(\frac{1}{b_n(p+1)} \widetilde{R}u^{p+1} - \frac{1}{2b_n} Ru^2 \right) \Xi \cdot \overset{\circ}{\nu} + \mathcal{B}(x, u, \overset{\circ}{\nabla}_b u) \right) d\overset{\circ}{\sigma}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{B}(x, u, \overset{\circ}{\nabla}_b u) &= nu(x) \overset{\circ}{\nabla}_b u(x) \cdot \overset{\circ}{\nu}_{B_d(x,0)}(x) - \frac{1}{2} |\overset{\circ}{\nabla}_b u(x)|^2 \Xi(x) \cdot \overset{\circ}{\nu}_{B_d(x,0)}(x) \\ & \quad + \Xi u(x) \overset{\circ}{\nabla}_b u(x) \cdot \overset{\circ}{\nu}_{B_d(x,0)}(x) \end{aligned}$$

and B_r denotes the ball with respect to the Korányi norm.

4.1. Blow-up analysis. Let M be a $(2n+1)$ -dimensional CR manifold equipped with a pseudohermitian structure θ , p_i be a sequence with $1 < p_i \leq b_n - 1$ for all i and $p_i \rightarrow b_n - 1$ as $i \rightarrow \infty$, and $u_i \in C^2(M)$ a sequence of positive solutions of

$$(4.1) \quad L_J u_i = \widetilde{R}u_i^{p_i},$$

where \widetilde{R} is a positive function of class C^1 .

Definition 4.2. A point $\bar{x} \in M$ is called a *blow-up point* if there exists a sequence $x_i \rightarrow \bar{x}$ such that $M_i = u_i(x_i) \rightarrow \infty$.

Definition 4.3. A point $\bar{x} \in M$ is called an *isolated blow-up point* if there exist $\bar{r} > 0$, a constant C , and a sequence $x_i \rightarrow \bar{x}$ such that x_i is a local maximum of u_i , $M_i = u_i(x_i) \rightarrow \infty$, and

$$u_i(x) \leq Cd(x, x_i)^{-\frac{2}{p_i-1}}$$

for every $x \in B_{\bar{r}}(x_i)$.

Given an isolated blow-up point \bar{x} , we define

$$\bar{u}_i(r) = \int_{\partial B_1(x_i)} u_i \circ \delta_r d\bar{\sigma}$$

in pseudohermitian normal coordinates, and $\bar{w}_i(r) = r^{\frac{2}{p_i-1}} \bar{u}_i(r)$.

Definition 4.4. An isolated blow-up point \bar{x} is called an *isolated simple blow-up point* if there exists $\rho \in (0, \bar{r})$ independent of i such that \bar{w}_i has exactly one critical point in $(0, \rho)$.

We have the following lemma from [1].

Lemma 4.5 (Lemma 4.4 in [1]). *If \bar{x} is an isolated blow-up point, then there exists C such that, for $0 < r < \bar{r}/3$, there holds*

$$\max_{B_{2r}(\bar{x}) \setminus B_{r/2}(\bar{x})} u_i \leq C \min_{B_{2r}(\bar{x}) \setminus B_{r/2}(\bar{x})} u_i.$$

Consider the CR Yamabe equation on the Heisenberg group

$$(4.2) \quad -b_n \Delta_b u = u^{b_n-1} \quad \text{in } \mathbb{H}^n.$$

We have the following classification theorem proved by Flynn and Vétois; see also [10].

Theorem 4.6 (Theorem 1.1 in [20]). *Let $n \geq 2$ and u be a positive solution to (4.2) such that*

$$u(z, t) \leq C(|z|^2 + |t|)^{-\frac{n-2}{2}} \quad \text{for all } (z, t) \in \mathbb{H}^n \setminus \{(0, 0)\}$$

for some constant $C > 0$. Then u is of the form

$$(4.3) \quad U(z, t) = \frac{c_1}{(t^2 + (1 + |z|^2)^2)^{\frac{n}{2}}}$$

up to the left translation $L_{x_0}(x) = x_0^{-1}x$, for some constant $c_1 > 0$. In particular, when $n = 2$, any bounded positive solution to (4.2) must be of the form (4.3) up to the left translation.

In the following, given an isolated blow-up point $x_i \rightarrow \bar{x}$, in order to study the blow-up sequence of functions, we rescale by defining $M_i = u_i(x_i)$ and

$$v_i = \frac{1}{M_i} u_i \circ \delta_{M_i^{-\frac{p_i-1}{2}}} \circ L_{x_i}$$

defined on $B_{\frac{p_i-1}{2M_i}}(\bar{x})$. Note that v_i satisfies

$$L_{\theta_i} v_i = (\tilde{R} \circ \delta_{M_i^{-\frac{p_i-1}{2}}}) v_i^{p_i}$$

where L_{θ_i} is the conformal CR sub-Laplacian with respect to the rescaled contact form $\theta_i = M_i^{-p_i+1} \left(\delta_{M_i^{-\frac{p_i-1}{2}}} \circ L_{x_i} \right)^*$ on the rescaled CR structure.

In the following all covariant derivatives applied to v_i are meant with respect to this rescaled pseudohermitian structure.

Proposition 4.7. *Suppose that $n = 2$. If \bar{x} is an isolated blow-up point, then for any $R_i \rightarrow \infty$, $\epsilon_i \rightarrow 0$ and $k \in \mathbb{N}$, after passing to a subsequence, in pseudohermitian normal coordinates around \bar{x} , there holds*

$$\left\| \frac{1}{M_i} u_i \left(\delta_{M_i^{-\frac{p_i-1}{2}}}(x_i^{-1} \cdot x) \right) - (U \circ \delta_{\tilde{R}(0)^{1/2}})(x) \right\| \leq \epsilon_i,$$

where U is defined as in (4.3), $M_i = u_i(x_i)$, and

$$\frac{R_i}{\log M_i} \rightarrow 0.$$

Proof. We just sketch the proof, since it is very similar to that of [1, Proposition 4.5]. Using the notation above, we note that v_i satisfies

$$\begin{cases} L_{\theta_i} v_i = (\tilde{R} \circ \delta_{M^{-i-\frac{p_i-1}{2}}}) v_i^{p_i}, \\ v_i(0) = 1, \\ \nabla_{b,\theta_i} v_i(0) = 0, \\ 0 < v_i(x) < C d(x, 0)^{-\frac{2}{p_i-1}}. \end{cases}$$

Following the argument of the proof of [1, Proposition 4.5], we find that, for any k and $R > 0$, after passing to a subsequence, v_i tends to some limit v in $C^{k,\alpha}(B_R)$. By a diagonal argument, after passing to a subsequence, we obtain a function v , defined and bounded on \mathbb{H}^n , satisfying

$$(4.4) \quad \begin{cases} L_{\theta} v = \tilde{R}(0) v^{b_n-1}, \\ v(0) = 1, \\ \nabla_{b,\theta_i} v(0) = 0, \\ v > 0. \end{cases}$$

Now Proposition 4.7 follows from Theorem 4.6. □

Lemma 4.8. *Let \bar{x} be an isolated simple blow-up point, $R_i \rightarrow \infty$, and suppose that Proposition 4.7 holds for some $\epsilon_i \rightarrow 0$. Then, given a fixed sufficiently small $\delta > 0$, there exists $\rho_1 \in (0, \rho)$ where ρ is the one from Definition 4.4 such that*

$$\begin{aligned} u_i(x) &\leq C M_i^{-\lambda_i} d(x, x_i)^{-2n+\delta}, \\ |(u_i)_{,\alpha}(x)| &\leq C M_i^{-\lambda_i} d(x, x_i)^{-2n-1+\delta} \quad \text{for } \alpha = 1, 2, \dots, n, \\ |(u_i)_{,\alpha\beta}(x)| &\leq C M_i^{-\lambda_i} d(x, x_i)^{-2n-2+\delta} \quad \text{for } \alpha, \beta = 1, 2, \dots, n, \\ |(u_i)_{,\alpha\bar{\beta}}(x)| &\leq C M_i^{-\lambda_i} d(x, x_i)^{-2n-2+\delta} \quad \text{for } \alpha, \beta = 1, 2, \dots, n, \\ |(u_i)_{,0}(x)| &\leq C M_i^{-\lambda_i} d(x, x_i)^{-2n-2+\delta}, \\ |(u_i)_{,0\alpha}(x)| &\leq C M_i^{-\lambda_i} d(x, x_i)^{-2n-3+\delta} \quad \text{for } \alpha = 1, 2, \dots, n, \\ |(u_i)_{,00}(x)| &\leq C M_i^{-\lambda_i} d(x, x_i)^{-2n-4+\delta}, \end{aligned}$$

for $R_i M_i^{-\frac{p_i-1}{2}} \leq d(x, x_i) \leq \rho_1$, where $\lambda_i = (2n - \delta) \frac{p_i-1}{2} - 1$.

Proof. The proof is analogous to the proof of [36, Lemma 3.3]. □

Lemma 4.9. *In the hypotheses of Lemma 4.8 for $|x| \leq \rho_1 M_i^{\frac{p_i-1}{2}}$, the following estimates hold:*

$$\begin{aligned}
v_i(x) &\leq C M_i^{\frac{p_i-1}{2}\delta} d(x, x_i)^{-2n}, \\
|(v_i)_{,\alpha}(x)| &\leq C M_i^{\frac{p_i-1}{2}\delta} d(x, x_i)^{-2n-1} \quad \text{for } \alpha = 1, 2, \dots, n, \\
|(v_i)_{,\alpha\beta}(x)| &\leq C M_i^{\frac{p_i-1}{2}\delta} d(x, x_i)^{-2n-2} \quad \text{for } \alpha, \beta = 1, 2, \dots, n, \\
|(v_i)_{,\alpha\bar{\beta}}(x)| &\leq C M_i^{\frac{p_i-1}{2}\delta} d(x, x_i)^{-2n-2} \quad \text{for } \alpha, \beta = 1, 2, \dots, n, \\
|(v_i)_{,0}(x)| &\leq C M_i^{\frac{p_i-1}{2}\delta} d(x, x_i)^{-2n-2}, \\
|(v_i)_{,0\alpha}(x)| &\leq C M_i^{\frac{p_i-1}{2}\delta} d(x, x_i)^{-2n-3} \quad \text{for } \alpha = 1, 2, \dots, n, \\
|(v_i)_{,00}(x)| &\leq C M_i^{\frac{p_i-1}{2}\delta} d(x, x_i)^{-2n-4},
\end{aligned}$$

Proof. This follows from Proposition 4.7 and Lemma 4.8. \square

Lemma 4.10. *If \bar{x} is an isolated blow-up point, then, with the notation of Lemma 4.8, in pseudohermitian normal coordinates around x_i*

$$\left| \int_{B_{\rho_1}(x_i)} (u_i + n\Xi u_i)(\mathring{\Delta}_b u_i - \Delta_b u_i) \right| \leq \begin{cases} C M_i^{-1+\delta+o(1)}, & \text{if } n = 2; \\ C M_i^{-\frac{11}{2}+\frac{2}{3}\delta+o(1)}, & \text{if } n = 3; \\ C M_i^{1+n-n^2+\frac{2}{n}\delta+o(1)}, & \text{if } n \geq 4. \end{cases}$$

Proof. Thanks to (2.2), (A.3), Lemma 4.9, and the fact that v_i is real, we have

$$\begin{aligned}
|v_i + n\Xi v_i| &\lesssim v_i + |x| |\mathring{Z}_\alpha v_i| + |x|^2 |\mathring{T} v_i| \lesssim v_i + |x| |(v_i)_{,\alpha}| + |x|^2 |(v_i)_{,0}| \\
&\lesssim M_i^{\frac{p_i-1}{2}\delta} (1 + |x|)^{-2n} + |x| M_i^{\frac{p_i-1}{2}\delta} (1 + |x|)^{-2n-1} + |x|^2 M_i^{\frac{p_i-1}{2}\delta} (1 + |x|)^{-2n-2} \\
&\lesssim M_i^{\frac{p_i-1}{2}\delta} (1 + |x|)^{-2n}.
\end{aligned}$$

Furthermore, using Lemmas A.5 and 4.8, we have

$$\begin{aligned}
&|\mathring{\Delta}_b u_i - \Delta_b u_i| \\
&\lesssim |x| |(u_i)_{,\alpha}| + |x|^2 |(u_i)_{,0}| + |x|^2 |(u_i)_{,\alpha\beta}| + |x|^2 |(u_i)_{,\alpha\bar{\beta}}| + |x|^3 |(u_i)_{,0\beta}| + |x|^6 |(u_i)_{,00}| \\
&\lesssim M_i \left(M_i^{-\frac{p_i-1}{2}} |x| M_i^{\frac{p_i-1}{2}} |(v_i)_{,\alpha}| + M_i^{-(p_i-1)} |x|^2 M_i^{p_i-1} |(v_i)_{,0}| \right. \\
&\quad + M_i^{-(p_i-1)} |x|^2 M_i^{p_i-1} |(v_i)_{,\alpha\beta}| + M_i^{-(p_i-1)} |x|^2 M_i^{p_i-1} |(v_i)_{,\alpha\bar{\beta}}| \\
&\quad \left. + M_i^{-3\frac{p_i-1}{2}} |x|^3 M_i^{3\frac{p_i-1}{2}} |(v_i)_{,0\alpha}| + M_i^{-3(p_i-1)} |x|^6 M_i^{2(p_i-1)} |(v_i)_{,00}| \right) \circ \delta_{M_i^{\frac{p_i-1}{2}}} \\
&\lesssim M_i M_i^{\frac{p_i-1}{2}\delta} \left(|x| (1 + |x|)^{-2n-1} + |x|^2 (1 + |x|)^{-2n-2} + |x|^3 (1 + |x|)^{-2n-3} \right. \\
&\quad \left. + M_i^{-(p_i-1)} |x|^6 (1 + |x|)^{-2n-4} \right) \circ \delta_{M_i^{\frac{p_i-1}{2}}} \\
&\lesssim M_i M_i^{\frac{p_i-1}{2}\delta} \left(|x| (1 + |x|)^{-2n-1} + M_i^{-(p_i-1)} |x|^6 (1 + |x|)^{-2n-4} \right) \circ \delta_{M_i^{\frac{p_i-1}{2}}}.
\end{aligned}$$

Combining these yields

$$\begin{aligned}
 & \left| \int_{B_{\rho_1}(x_i)} (u_i + n\Xi u_i)(\overset{\circ}{\Delta}_b u_i - \Delta_b u_i) \right| \\
 &= M_i^2 M_i^{-\frac{p_i-1}{2}(2n+2)} \left| \int_{B_{\frac{\rho_1 M_i^{\frac{p_i-1}{2}}}{\rho_1 M_i}}(0)} (v_i + n\Xi v_i)(\overset{\circ}{\Delta}_b u_i - \Delta_b u_i) \circ \delta_{M_i^{-\frac{p_i-1}{2}}} \right| \\
 &\lesssim M_i^2 M_i^{-\frac{p_i-1}{2}(2n+2)} M_i^{(p_i-1)\delta} \int_{B_{\frac{\rho_1 M_i^{\frac{p_i-1}{2}}}{\rho_1 M_i}}(0)} (1+|x|)^{-2n} \left(|x|(1+|x|)^{-2n-1} \right. \\
 &\quad \left. + M_i^{-(p_i-1)} |x|^6 (1+|x|)^{-2n-4} \right) \\
 &\lesssim \begin{cases} M_i^{n+3-(n+1)p_i} M_i^{(p_i-1)\delta} \left(\log M_i + M_i^{-(p_i-1)(n-2)} \right), & \text{if } n = 2; \\ M_i^{n+3-(n+1)p_i} M_i^{(p_i-1)\delta} \left(M_i^{-(p_i-1)(n-1)} + M_i^{-(p_i-1)} \log M_i \right), & \text{if } n = 3; \\ M_i^{n+3-(n+1)p_i} M_i^{(p_i-1)\delta} \left(M_i^{-(p_i-1)(n-1)} + M_i^{-(p_i-1)(n-2)} \right), & \text{if } n \geq 4. \end{cases}
 \end{aligned}$$

Note that the last terms are of order

$$\begin{cases} M_i^{-1+\delta+o(1)}, & \text{if } n = 2; \\ M_i^{-\frac{11}{2}+\frac{2}{3}\delta+o(1)}, & \text{if } n = 3; \\ M_i^{1+n-n^2+\frac{2}{n}\delta+o(1)}, & \text{if } n \geq 4, \end{cases}$$

This proves the assertion. \square

Lemma 4.11. *In the assumptions of Lemma 4.8, if $\tau_i := b_n - 1 - p_i$, then*

$$\tau_i = O(u_i(x_i)^{-1+\delta+o(1)}) \quad \text{whenever } n = 2,$$

and in particular, $u_i(x_i)^{\tau_i} \rightarrow 1$.

Proof. Applying the Pohozaev identity of Proposition 4.1 with respect to the base point x_i with $r = \rho_1$:

$$\begin{aligned}
 & \int_{B_{\rho_1}} \left(\frac{1}{b_n} \left(\frac{2n+2}{p_i+1} - n \right) \tilde{R} u_i^{p_i+1} - \frac{1}{b_n} R u_i^2 + \frac{1}{b_n(p_i+1)} \Xi(\tilde{R}) u_i^{p_i+1} \right. \\
 & \quad \left. - \frac{1}{2b_n} \Xi(R) u_i^2 - (\Xi u_i + n u_i)(\Delta_b u_i - \overset{\circ}{\Delta}_b u_i) \right) d\overset{\circ}{V} \\
 &= \int_{\partial B_{\rho_1}} \left(\left(\frac{1}{b_n(p_i+1)} \tilde{R} u_i^{p_i+1} - \frac{1}{2b_n} R u_i^2 \right) \Xi \cdot \overset{\circ}{\nu} + \mathcal{B}(x, u_i, \overset{\circ}{\nabla}_b u_i) \right) d\overset{\circ}{\sigma}.
 \end{aligned}$$

We are going to estimate each term in the above expression. By Lemma 4.9, we have

$$\begin{aligned}
& \int_{B_{\rho_1}} \Xi(\tilde{R})u_i^{p_i+1} \\
& \lesssim M_i^{p_i+1} M_i^{-(n+1)(p_i-1)} \int_B \left(M_i^{-\frac{p_i-1}{2}} |x| + M_i^{-(p_i-1)} |x|^2 \right) v_i^{p_i+1} \\
& \lesssim M_i^{-np_i+n+2} \left(\int_B M_i^{-\frac{p_i-1}{2}} |x| v_i^{p_i+1} + \int_B M_i^{-(p_i-1)} |x|^2 v_i^{p_i+1} \right) \\
& \lesssim M_i^{-\frac{(2n+1)p_i+2n+5}{2}} M_i^{\frac{p_i-1}{2}\delta} \int_B |x|(1+|x|)^{-2n(p_i+1)} \\
& \quad + M_i^{-(n+1)p_i+n+3} M_i^{\frac{p_i-1}{2}\delta} \int_B |x|^2(1+|x|)^{-2n(p_i+1)} \\
& \lesssim M_i^{-\frac{(2n+1)p_i+2n+5}{2}} M_i^{\frac{p_i-1}{2}\delta} M_i^{\frac{p_i-1}{2}(-2np_i+5)} + M_i^{-(n+1)p_i+n+3} M_i^{\frac{p_i-1}{2}\delta} M_i^{\frac{p_i-1}{2}(-2np_i+6)} \\
& = M_i^{-np_i^2+2p_i+n} M_i^{\frac{p_i-1}{2}\delta}.
\end{aligned}$$

Similarly, by Lemma 4.9, we have

$$\begin{aligned}
& \int_{B_{\rho_1}} \left(\frac{1}{b_n} R + \frac{1}{2b_n} \Xi(R) \right) u_i^2 \lesssim M_i^2 M_i^{-(n+1)(p_i-1)} \int_B v_i^2 \\
& \lesssim M_i^{n+3-(n+1)p_i} M_i^{(p_i-1)\delta} \int_B (1+|x|)^{-4n} \\
& \lesssim \begin{cases} M_i^{n+3-(n+1)p_i} M_i^{(p_i-1)\delta} \log M_i, & \text{if } n = 2; \\ M_i^{2n+2-2np_i} M_i^{(p_i-1)\delta}, & \text{if } n \geq 3. \end{cases}
\end{aligned}$$

Moreover, thanks to Lemma 4.8, we have

$$\begin{aligned}
& \int_{\partial B_{\rho_1}} \left(\left(\frac{1}{b_n(p+1)} \tilde{R}u_i^{p+1} - \frac{1}{2b_n} Ru_i^2 \right) \Xi \cdot \mathring{\nu} + \mathcal{B}(x, u_i, \mathring{\nabla}_b u_i) \right) d\mathring{\sigma} \\
& = O(M_i^{-2\lambda_i}) = O(M_i^{-(2n-\delta)(p_i-1)+2}).
\end{aligned}$$

By Proposition 4.7, there holds

$$\begin{aligned}
& \int_{B_{\rho_1}} u_i^{p_i+1} \gtrsim M_i^{\frac{p_i-1}{2}(2n+2)} M_i^{p_i+1} \int_B \left(\frac{1}{M_i} u_i \circ \delta_{M_i^{-\frac{p_i-1}{2}}} \right)^{p_i+1} \\
& \gtrsim M_i^{(n+2)p_i-n} \int_B \left(U \circ \delta_{\tilde{R}(0)^{1/2}} \right)^{p_i+1} \\
& \gtrsim M_i^{(n+2)p_i-n} M_i^{\frac{3n}{4}+4-\frac{n}{2}p_i} = M_i^{-\frac{n}{4}+2+(\frac{n}{2}+2)p_i} \gtrsim 1.
\end{aligned}$$

Now, combining these with Lemma 4.10 and the fact that

$$\frac{1}{b_n} \left(\frac{2n+2}{p_i+1} - n \right) = \frac{n^2}{(p_i+1)(2n+2)} \tau_i,$$

we prove the assertion. □

Lemma 4.12. *Suppose that $n = 2$. Under the hypotheses of Lemma 4.8, if \bar{x} is an isolated simple blow-up point, then for every $\sigma \in (0, \frac{r}{2})$*

$$\limsup_{i \rightarrow \infty} \max_{\partial B_\sigma} M_i u_i(x) \leq C(\sigma).$$

Proof. Thanks to Lemma 4.5, it is sufficient to prove the statement for σ small enough. In particular, as in the proof of Proposition 4.7, we can suppose that $R > 0$.

Let x_σ be such that $d(x_\sigma, x_i) = \sigma$, and define $w_i(x) = u_i(x_\sigma)^{-1} u_i(x)$. Then w_i satisfies

$$(4.5) \quad L_\theta w_i = u_i(x_\sigma)^{p_i-1} w_i^{p_i}.$$

Thanks to Lemmas 4.5 and 4.8, for every compact $K \subset B_{\rho_1}(\bar{x}) \setminus \{\bar{x}\}$, there exists C_K such that $C_K^{-1} \leq w_i \leq C_K$. Therefore, applying the regularity theory from [1, Theorem 2.3], we can deduce that, after passing to a subsequence, $w_i \rightarrow w$ in $C_{loc}^2(B_{\rho_1}(\bar{x}) \setminus \{\bar{x}\})$, and since, by Lemma 4.8, $u_i(x_\sigma) \rightarrow 0$, passing to the limit in (4.5), we get that $L_\theta w = 0$.

Since the blow-up is isolated simple, and since Proposition 4.7 implies that $r^{\frac{2}{p_i-1}} \bar{u}_i$ has a critical point in $(0, R_i M_i^{-\frac{p_i-1}{2}})$ after which it is decreasing, $r^{\frac{2}{p_i-1}} \bar{u}_i$ is decreasing in $(R_i M_i^{-\frac{p_i-1}{2}}, \rho)$, and because

$$u_i(x_\sigma)^{-1} r^{\frac{2}{p_i-1}} \bar{u}_i(r) = r^{\frac{2}{p_i-1}} \bar{w}_i(r) \rightarrow r^n \bar{w}(r),$$

$r^n \bar{w}(r)$ is decreasing on $(0, \rho)$. Since $w > 0$, w must be singular at \bar{x} . Corollary 9.1 in [36] can be extended to pseudohermitian geometry by repeating the proof with minor adaptations. Applying it, we get that

$$(4.6) \quad - \int_{B_\sigma(x_i)} \Delta_b w_i = - \int_{\partial B_\sigma(x_i)} \nabla_b w_i \cdot \nu = - \int_{\partial B_\sigma(\bar{x})} \nabla_b w \cdot \nu + o(1) = c + o(1) > 0,$$

while integrating (4.5) yields

$$(4.7) \quad -b_n \int_{B_\sigma(x_i)} \Delta_b w_i = \int_{B_\sigma(x_i)} (-R w_i + u_i(x_\sigma)^{p_i-1} w_i^{p_i}) \leq u_i(x_\sigma)^{-1} \int_{B_\sigma(x_i)} u_i^{p_i}.$$

But if we call $r_i = R_i M_i^{-\frac{p_i-1}{2}}$, it follows from Proposition 4.7, Lemma 4.8 and Lemma 4.11 that

$$(4.8) \quad \begin{aligned} \int_{B_\sigma(x_i)} u_i^{p_i} &= \left(\int_{B_{r_i}(x_i)} + \int_{B_\sigma(x_i) \setminus B_{r_i}(x_i)} \right) u_i^{p_i} \\ &\lesssim M_i^{-(n+1)(p_i-1)} M_i^{p_i} \int_{B_{R_i}(0)} (1+|x|)^{-2np_i} + M_i^{-\lambda_i p_i} \int_{B_\sigma(x_i) \setminus B_{r_i}(x_i)} |x|^{(-2n+\delta)p_i} \\ &\lesssim M_i^{-(n+1)(p_i-1)} M_i^{p_i} \log R_i + M_i^{-\lambda_i p_i} \left(R_i M_i^{-\frac{p_i-1}{2}} \right)^{-(2n-\delta)p_i+2n+4} \lesssim M_i^{-1} \end{aligned}$$

whenever $n = 2$. Combining (4.6)-(4.8), we see that $M_i u_i(x_\sigma)$ is a bounded sequence. Now the assertion follows from Lemma 4.5. □

Proposition 4.13. *If \bar{x} is an isolated simple blow-up point then there exists C such that*

$$M_i u_i(x) \leq C d(x, x_i)^{-2n}$$

if $d(x, x_i) \leq \frac{\rho}{2}$. Furthermore, after passing to a subsequence, there exists $a > 0$ such that

$$M_i u(x) \rightarrow a G_{\bar{x}}(x) + b$$

in $C_{loc}^2(B_{\frac{\rho}{2}}(\bar{x}) \setminus \{\bar{x}\})$, where $G_{\bar{x}}$ is the Green function of L_θ (which exists because M has positive CR Yamabe class) and $L_\theta b = 0$ on $B_{\frac{\rho}{2}}(\bar{x})$.

Proof. If this were not the case, then, after passing to a subsequence, there would exist a sequence \tilde{x}_i with $d(x_i, \tilde{x}_i) \leq \frac{\rho}{2}$ and

$$(4.9) \quad M_i u_i(\tilde{x}_i) d(x_i, \tilde{x}_i)^{2n-2} \rightarrow \infty.$$

Define $\tilde{r}_i = d(x_i, \tilde{x}_i)$.

After passing to a subsequence such that Proposition 4.7 holds for some $R_i \rightarrow \infty$ and $\varepsilon_i \leq e^{-R_i}$, it is easy to verify, using Lemma 4.11 and the fact that $\sup_{\lambda>0} \lambda^2 U(\delta_\lambda(x)) \leq \frac{C}{|x|^{2n}}$, that eventually $\tilde{r}_i \geq R_i M_i^{-\frac{p_i-1}{2}}$.

Define $\tilde{u}_i = \tilde{r}_i^{\frac{2}{p_i-1}} u_i \circ \delta_{\tilde{r}_i} \circ L_{x_i}$ in B_2 . \tilde{u}_i satisfies

$$L_\theta \tilde{u}_i = \tilde{R} \tilde{u}_i^{p_i}$$

and satisfies the hypotheses of Lemma 4.12, therefore $\max_{\partial B_1} \tilde{u}_i(0) \tilde{u}_i < \infty$. Using the definition of \tilde{u}_i and Lemma 4.11, this contradicts formula (4.9).

Hence $M_i u_i$ is locally bounded in $B_{\frac{\rho}{2}}(\bar{x}) \setminus \{\bar{x}\}$, and satisfies

$$L_\theta(M_i u_i) = M_i^{1-p_i} (M_i u_i)^{p_i},$$

therefore, applying regularity theory,

$$M_i u_i \rightarrow v \quad \text{in} \quad C_{loc}^2(B_{\frac{\rho}{2}}(\bar{x}) \setminus \{\bar{x}\})$$

with v satisfying $L_\theta v = 0$. Known results about singular solutions (see for example Proposition 9.1 in [36], which can be adapted without difficulty to pseudohermitian geometry) imply the rest of the claim, except for the fact that $a > 0$. This can be proved by showing that v must be singular by the same proof as Lemma 4.12. \square

In order to proceed we need a generalization of Theorem 2.1.

Lemma 4.14. *If u is a function on \mathbb{H}^n satisfying*

$$-\Delta_b u = U^{\frac{Q+2}{Q-2}} u$$

and $\lim_{x \rightarrow \infty} u = 0$, then u is one of the Malchiodi-Uguzzoni solutions of Theorem 2.1.

Proof. The Cayley transform v of u satisfies

$$-\Delta_b v = U^{\frac{Q+2}{Q-2}} v$$

and $\lim_{x \rightarrow 0} |x|^{Q-2} v = 0$. In particular $v \in L_{loc}^1(\mathbb{H}^n)$, and therefore it is a distribution. Thus $-\Delta_b v - U^{\frac{Q+2}{Q-2}} v$ is a distribution on \mathbb{H}^n whose restriction to $\mathbb{H}^n \setminus \{0\}$ is zero, and therefore it is equal to $\sum_\alpha a^\alpha \partial_\alpha \delta$. Hence if ϕ is a smooth compactly supported function which is one in a neighborhood of the origin,

$$-\Delta_b(\phi v) - U^{\frac{Q+2}{Q-2}} \phi v = \psi + \sum_\alpha a^\alpha \partial_\alpha \delta$$

for a smooth compactly supported function ψ , therefore

$$\phi v = G * (U^{\frac{Q+2}{Q-2}} \phi v) + G * \psi + \sum_{\alpha} a^{\alpha} (-1)^{|\alpha|} \partial_{\alpha} G.$$

Since $|U^{\frac{Q+2}{Q-2}} \phi v| \lesssim \frac{1}{|x|^{Q-2}}$, $|G * (U^{\frac{Q+2}{Q-2}} \phi v)| \lesssim \frac{1}{|x|^{Q-4}}$ for $n \geq 2$, whereas $|G * (U^{\frac{Q+2}{Q-2}} \phi v)| \lesssim |\log |x||$ for $n = 1$. This and the fact that $\lim_{x \rightarrow 0} |x|^{Q-2} v = 0$ imply that $a^{\alpha} = 0$. Therefore $-\Delta_b v = U^{\frac{Q+2}{Q-2}} v$ on \mathbb{H}^1 and by regularity theory it is smooth; therefore, the result by Malchiodi and Uguzzoni applies. \square

From now on $2n + 1 = 5$.

Lemma 4.15. *If \bar{x} is an isolated simple blow-up point then there exists $\gamma > 0$ such that*

$$|v_i - U| \lesssim \max\{M_1^{-2} \log M_i, \tau_i\}$$

for $|x| \leq \gamma M_i^{\frac{p_i-1}{2}}$ (where $\tau_i = \frac{n+2}{n} - p_i$).

Proof. Let $\ell_i = \gamma M_i^{\frac{p_i-1}{2}}$ and

$$\Lambda_i = \max_{|y| \leq \ell_i} |v_i(y) - U(y)|$$

realized by some y_i . If $|y_i| \geq c \ell_i$ for some c then, since $w_i \lesssim U$ and $U(y) \lesssim \frac{1}{|y|^{-2n}}$ on the considered domain, then

$$|v_i(y_i) - U(y_i)| \lesssim \frac{1}{|y_i|^{-2n}} \lesssim \ell_i^{-2n} \lesssim M_i^{-2}$$

(using the lemma...) and therefore by definition of y_i we obtain the conclusion. Therefore we can suppose that $|y_i| \leq \frac{\ell_i}{2}$. Define

$$w_i(x) = \frac{1}{\Lambda_i} (v_i(x) - U(x)).$$

Then w_i satisfies

$$\begin{aligned} L_{\tilde{\theta}_i} w_i(x) &= \frac{1}{\Lambda_i} v_i^{p_i}(x) - \frac{1}{\Lambda_i} U(x)^{\frac{n+2}{n-2}} + \frac{1}{\Lambda_i} (L_{\tilde{\theta}_i} - \Delta_{\theta_{\mathbb{H}^2}}) U \\ &= \frac{1}{\Lambda_i} (v_i^{p_i} - U^{p_i}) + \frac{1}{\Lambda_i} \left(U^{p_i} - U^{\frac{n+2}{n-2}} + (L_{\tilde{\theta}_i} - \Delta_{\theta_{\mathbb{H}^2}}) U \right) \\ &= \frac{v_i^{p_i} - U^{p_i}}{v_i - U} w_i + \\ &+ \frac{1}{\Lambda_i} \left(U^{p_i} - U^{\frac{n+2}{n-2}} + M_i^{-(p_i-1)} R \circ \delta_{M_i^{-\frac{p_i-1}{2}}} U + O \left(M_i^{-(p_i-1)} (1 + |x|)^{-2} \right) \right) \end{aligned}$$

(where, we recall, $\theta_i = M_i^{-(p_i-1)} \left(\delta_{M_i^{-\frac{p_i-1}{2}}} \circ L_{x_i} \right)^* \theta$).

Setting $b_i = \frac{v_i^{p_i} - U^{p_i}}{v_i - U}$ by the estimates of the preceding lemmas

$$|b_i(x)| \lesssim (1 + |x|)^{-4} \quad |x| \leq \ell_i.$$

Call also $Q_i = L_{\tilde{\theta}_i} w_i - b_i w_i$.

Then

$$w_i(y) = \int_{B_{\ell_i}} G_i(y, \xi) (b_i(\xi) w_i(\xi) - Q_i(\xi)) d\xi - \int_{\partial B_{\ell_i}} \nu_{\xi} G_i$$

Let $t_i = M_i^{-2} \log M_i$. Suppose, for contradiction, that $\Lambda_i^{-1} \max\{t_i, \tau_i\} \rightarrow 0$. In CR normal coordinates $R = O(|x|^2)$ (see [32]), then

$$|Q_i(y)| \lesssim \Lambda_i^{-1} \left(\tau_i \log U (1 + |x|)^{-8} + M_i^{-\frac{2}{3}} |x|^2 (1 + |x|)^{-6} + O\left(M_i^{-4/3} (1 + |x|)^{-2}\right) \right)$$

The estimates above show $|w_i| \lesssim \Lambda_i^{-1} M_i^{-2}$, and it is standard that $|G_i(y - \xi)| \lesssim |y - \xi|^{-4}$ for $|y - \xi| \leq \frac{\ell_i}{2}$, therefore

$$(4.10) \quad |w_i(y)| \lesssim \left((1 + |y|)^{-2} + \Lambda_i \log M_i M_i^{-2} \right).$$

By Schauder estimates, after passing to a subsequence, $w_i \rightarrow w$ in the subriemannian Hölder space satisfying

$$L_{\theta_{\mathbb{H}^2}} w = U w$$

and $|w(y)| \lesssim (1 + |y|)^{-2}$. By Lemma 4.14, w is one of the Malchiodi-Uguzzoni solutions. The definition of v implies that $w(0) = dw(0) = 0$, and therefore $w = 0$, which implies (since $w_i(y_i) = 1$) that $|y_i| \rightarrow \infty$, but this contradicts equation (4.10) since we had supposed by contradiction that $\Lambda_i \log M_i M_i^{-2} \rightarrow 0$. \square

Lemma 4.16.

$$\tau_i \lesssim \log M_i M_i^{-2}.$$

Proof. If this were not true, then by the previous lemma $|v_i - U_0| \lesssim \tau_i$. Setting

$$w_i(x) = \frac{1}{\tau_i} (v_i(x) - U(x)),$$

w_i satisfies

$$L_{\tilde{\theta}_i} w_i(x) = \frac{v_i^{p_i} - U^{p_i}}{v_i - U} w_i + \tilde{Q}_i$$

where

$$\tilde{Q}_i = \frac{1}{\tau_i} O\left(\tau_i \log U (1 + |x|)^{-8} + M_i^{-\frac{2}{3}} |x|^2 (1 + |x|)^{-6} + M_i^{-(p_i-1)} (1 + |x|)^{-2}\right).$$

Suppose by contradiction that $\log M_i M_i^{-2} \tau_i^{-1} \rightarrow 0$. By Schauder estimates $w_i \rightarrow w$ on compact sets. Let $\psi = \frac{Q-2}{2} U + \Xi U$. Then we have

$$\int_{|y| \leq \ell_i/2} \psi \frac{1}{\tau_i} O\left(M_i^{-\frac{2}{3}} |x|^2 (1 + |x|)^{-6} + M_i^{-4/3} (1 + |x|)^{-2}\right) \rightarrow 0$$

therefore

$$\lim_{i \rightarrow \infty} \int_{|y| \leq \ell_i/2} \psi \tilde{Q}_i = \int_{\mathbb{H}^2} \psi \log U U^2.$$

At the same time

$$\begin{aligned} \int_{|y| \leq \ell_i/2} \psi \tilde{Q}_i &= \int_{|y| \leq \ell_i/2} \psi (L_{\tilde{\theta}_i} w_i + b_i w_i) = \\ &= \int_{|y| \leq \ell_i/2} (L_{\tilde{\theta}_i} \psi + b_i \psi) w_i + \int_{|y| = \ell_i/2} (\psi \nu w_i - w_i \nu \psi) \end{aligned}$$

therefore passing to the limit

$$\lim_{i \rightarrow \infty} \int_{|y| \leq \ell_i/2} \psi \tilde{Q}_i = \int_{\mathbb{H}^2} (L_{\mathbb{H}^2} \psi + U^2 \psi) w = 0$$

which is a contradiction. \square

Corollary 4.17. *If \bar{x} is an isolated simple blow-up point then there exists $\gamma > 0$ such that*

$$|v_i - U| \lesssim M_i^{-2} \log M_i$$

for $|x| \leq \gamma M_i^{\frac{p_i-1}{2}}$ (where $\tau_i = \frac{n+2}{n} - p_i$).

Arguing in a similar way as in the previous lemmas, following the proof of Proposition 5.5 and the following Remark 1 in [38], we also obtain the following estimates.

Lemma 4.18. *Under the previous hypotheses,*

$$\begin{aligned} |v_i - U|(y) &\lesssim M_i^{-3/2} (1 + |y|)^{-1} \\ |\nabla_b(v_i - U)|(y) &\lesssim M_i^{-3/2} (1 + |y|)^{-2} \\ |\nabla_b^2(v_i - U)|(y) &\lesssim M_i^{-3/2} (1 + |y|)^{-3} \end{aligned}$$

Lemma 4.19. *If \bar{x} is an isolated simple blow-up point then $S(\bar{x}) = 0$ (where S is the Chern tensor defined in Section 2).*

Proof. We want to apply the Pohozaev identity from Proposition 4.1. Now

$$\begin{aligned} A_i(r) &:= M_i^2 \int_{B_r} \left(-\frac{1}{3} \left(R + \frac{1}{2} \Xi(R) \right) u_i^2 - (\Xi u_i + 2u_i) (\Delta_b u_i(x) - \mathring{\Delta}_b u_i) \right) d\mathring{V} = \\ &= M_i^2 M_i^{2-2(p_i-1)} \int_{B_{rM_i^{\frac{p_i-1}{2}}}} \left(-\frac{1}{3} M_i^{-(p_i-1)} \left(R + \frac{1}{2} \Xi(R) \right) \circ \delta_{-M_i^{\frac{p_i-1}{2}}} v_i^2 + \right. \\ &\quad \left. - (\Xi v_i + 2v_i) (\Delta'_b v_i - \mathring{\Delta}_b v_i) \right) d\mathring{V} \end{aligned}$$

Defining

$$\begin{aligned} \hat{A}_i(r) &= M_i^2 M_i^{2-2(p_i-1)} \int_{B_{rM_i^{\frac{p_i-1}{2}}}} \left(-\frac{1}{3} M_i^{-(p_i-1)} \left(R + \frac{1}{2} \Xi(R) \right) \circ \delta_{-M_i^{\frac{p_i-1}{2}}} U^2 + \right. \\ &\quad \left. - (\Xi U + 2U) (\Delta'_b U - \mathring{\Delta}_b U) \right) d\mathring{V} \end{aligned}$$

then

$$\begin{aligned} |A_i(r) - \hat{A}_i(r)| &\lesssim M_i \int_{B_{rM_i^{\frac{p_i-1}{2}}}} (|v_i - U|(y) (1 + |y|)^{-4} + \\ &+ |\nabla_b(v_i - U)|(y) (1 + |y|)^{-3} + |\nabla_b^2(v_i - U)|(y) (1 + |y|)^{-2}) \lesssim \\ &\lesssim M_i^{-1/2} \int_{B_{rM_i^{\frac{p_i-1}{2}}}} (1 + |y|)^{-5} \lesssim 1. \end{aligned}$$

Finally

$$\int_{B_r} \left(\frac{Q}{p+1} - \frac{Q-2}{2} \right) \tilde{R} u^{p+1} \geq 0$$

and therefore, since $|M_i^2 \mathcal{B}(x, u, \mathring{\nabla}_b u_i)| \lesssim 1$ by the fact that $M_i u_i$ has limit in $\Gamma^{2,\alpha}$ on compact subsets of $M \setminus \{x\}$, we get that $\hat{A}_i(r) \lesssim 1$. Now

$$M_i^2 M_i^{2-2(p_i-1)} \int_{B_{rM_i^{\frac{p_i-1}{2}}}} (\Xi U + 2U) (\Delta'_b U - \mathring{\Delta}_b U) d\mathring{V} \lesssim$$

$$\lesssim M_i^2 M_i^{2-2(p_i-1)} M_i^{-(p_i-1)} \int_{B_{rM_i \frac{p_i-1}{2}}} (1+|y|)^{-4} (1+|y|)^{-2} \lesssim 1$$

By following the proof of Proposition 4.2 in [32], it can be shown that

$$M_i^2 \int_{B_{rM_i \frac{p_i-1}{2}}} \left(-\frac{1}{3} M_i^{-(p_i-1)} \left(R + \frac{1}{2} \Xi(R) \right) \circ \delta_{-M_i \frac{p_i-1}{2}} U^2 \right) d\mathring{V} \gtrsim |S(x)|^2 \log M_i$$

therefore the claim follows. \square

Lemma 4.20. *If \bar{x} is an isolated simple blow-up point and $M_i u_i \rightarrow h$ in $M \setminus \{\bar{x}\}$ then*

$$\liminf_{r \rightarrow 0} \mathcal{B}(x, h, \mathring{\nabla}_b h) \geq 0.$$

Proof. Using the Pohozaev identity it can be proved like in Lemma 4.14 in [1], in a manner similar to the proof of the previous lemma, that

$$\liminf_{r \rightarrow 0} \mathcal{B}(x, h, \mathring{\nabla}_b h) \gtrsim \liminf_{r \rightarrow 0} \int_{B_{rM_i \frac{p_i-1}{2}}} (\Xi U + 2U) R U$$

which is nonnegative, by reasoning as in the proof of Proposition 4.2 in [32] \square

Lemma 4.21. *Isolated blow-up points are isolated simple.*

Proof. The proof is the same as that of Proposition 4.15 in [1] or Proposition 4.1 in [36], using Lemma 4.20. \square

Lemma 4.22. *In the hypotheses of Theorem 1.3, the set of blow-up points is finite and it consists of isolated simple blow-up points.*

Proof. The proof is the same as [36] or [1]. \square

Proposition 4.23. *Let M be a five-dimensional pseudoconvex CR manifold of positive CR Yamabe class, and let $p \in M$ such that $S(p) = 0$. Then the Green function G_p of the conformal sub-Laplacian at p in CR normal coordinates centered at p satisfies*

$$G_p(x) = \frac{a}{|x|^4} + A_p + O(|x|)$$

where $A_p = bm_x$ for some constant $b > 0$.

Proof. Since in CR normal coordinates $R_{\alpha\bar{\beta}}(p) = 0$ by Proposition 3.12 in [32], $R_{\beta\bar{\alpha}}(p) = 0$. Thanks to this and to Proposition 2.5 in [32], by repeating the proof of Lemma A.5, it can be checked that

$$(\Delta_b - \mathring{\Delta}_b) \frac{1}{|x|^4}$$

is locally bounded around the origin, and therefore

$$G_p(x) = \frac{a}{|x|^4} + A_p + O(|x|).$$

The last assertion of the proposition is proved by following the proof of Proposition 3.7 in [12]. \square

Proof of Theorem 1.3. By standard arguments through elliptic theory, it suffices to show that the set is bounded in C^0 .

If a sequence u_i violates this, by Lemma 4.22, after passing to a subsequence, it has a blow-up set $S = \{\bar{x}^1, \dots, \bar{x}^N\}$ formed by isolated simple blow-up points. After passing to a subsequence, we can suppose that $u_i(x_i^1) \leq u_i(x_i^k)$ for every k , and setting $w_i = u_i(x_i^1)u_i$, by the previous results we can deduce that

$$w_i(x) \rightarrow h(x) = \sum_{k=1}^N a_k G_{\bar{x}^k}(x).$$

Thanks to the hypothesis and to Lemma 4.19 the Chern tensor satisfies $S(\bar{x}^1) = 0$, and therefore thanks to Proposition 4.23 we obtain that in CR normal coordinates

$$h(x) = \frac{c}{|x|^4} + A'$$

where $A' > 0$, but from this and Lemma 4.20 we get a contradiction. □

APPENDIX A

Let M be a $2n+1$ -dimensional pseudoconvex pseudohermitian manifold and $x \in M$.

Proposition A.1. *In pseudohermitian normal coordinates*

$$\begin{aligned} \theta^\alpha &= (1 + O(|x|^2))\overset{\circ}{\theta}^\alpha + O(|x|^2)\overset{\circ}{\theta}^\beta + O(|x|^2)\overset{\circ}{\theta}^{\bar{\beta}} + O(|x|)\overset{\circ}{\theta} \\ \theta &= (1 + O(|x|^2))\overset{\circ}{\theta} + O(|x|^3)\theta^\beta + O(|x|^3)\theta^{\bar{\beta}}. \end{aligned}$$

$$(A.1) \quad \omega_1^1 = O(|x|)\overset{\circ}{\theta}^\beta + O(|x|)\overset{\circ}{\theta}^{\bar{\beta}} + O(|x|)\overset{\circ}{\theta}$$

Proof. It follows from Proposition 2.5 in [32]. □

Lemma A.2. *In pseudohermitian normal coordinates*

$$(A.2) \quad \begin{cases} Z_\alpha = \overset{\circ}{Z}_\alpha + O(|x|^2)\overset{\circ}{Z}_\beta + O(|x|^2)\overset{\circ}{Z}_{\bar{\beta}} + O(|x|^3)\overset{\circ}{T} \\ Z_{\bar{\alpha}} = \overset{\circ}{Z}_{\bar{\alpha}} + O(|x|^2)\overset{\circ}{Z}_\beta + O(|x|^2)\overset{\circ}{Z}_{\bar{\beta}} + O(|x|^3)\overset{\circ}{T} \\ T = O(|x|)\overset{\circ}{Z}_\beta + O(|x|)\overset{\circ}{Z}_{\bar{\beta}} + (1 + O(|x|^2))\overset{\circ}{T} \end{cases},$$

and

$$(A.3) \quad \begin{cases} \overset{\circ}{Z}_\alpha = Z_\alpha + O(|x|^2)Z_\beta + O(|x|^2)Z_{\bar{\beta}} + O(|x|^3)T \\ \overset{\circ}{Z}_{\bar{\alpha}} = Z_{\bar{\alpha}} + O(|x|^2)Z_\beta + (1 + O(|x|^2))Z_{\bar{\beta}} + O(|x|^3)T \\ \overset{\circ}{T} = O(|x|)Z_\beta + O(|x|)Z_{\bar{\beta}} + (1 + O(|x|^2))T \end{cases}$$

Proof. Letting $Z_\alpha = a^\beta \overset{\circ}{Z}_\beta + b^{\bar{\beta}} \overset{\circ}{Z}_{\bar{\beta}} + c \overset{\circ}{T}$ and applying θ^β , $\theta^{\bar{\beta}}$ and θ , we get

$$\begin{cases} \delta_\alpha^\beta = (\delta_\alpha^\beta + O(|x|^2))a^\beta + O(|x|^2)b^{\bar{\beta}} + O(|x|)c, \\ \delta_{\bar{\alpha}}^{\bar{\beta}} = (1 + O(|x|^2))b^{\bar{\beta}} + O(|x|^2)a^\beta + O(|x|)c, \\ 0 = (1 + O(|x|^2))c + O(|x|^3)a^{\bar{\beta}} + O(|x|^3)b^\beta \end{cases}$$

respectively. The third one implies that $c = O(|x|^3)$, and using this in the other two allows to deduce that $a^\beta = \delta_\alpha^\beta + O(|x|^2)$ and $b^{\bar{\beta}} = O(|x|^2)$. Therefore

$$Z_\alpha = \overset{\circ}{Z}_\alpha + O(|x|^2)\overset{\circ}{Z}_\beta + O(|x|^2)\overset{\circ}{Z}_{\bar{\beta}} + O(|x|^3)\overset{\circ}{T}.$$

Letting $T = d^\beta \overset{\circ}{Z}_\beta + e^{\bar{\beta}} \overset{\circ}{Z}_{\bar{\beta}} + f \overset{\circ}{T}$ and applying θ^β , $\theta^{\bar{\beta}}$ and θ , we get

$$\begin{cases} 0 = (1 + O(|x|^2))d^\beta + O(|x|^2)e^{\bar{\beta}} + O(|x|)f, \\ 0 = O(|x|^2)d^\beta + (1 + O(|x|^2))e^{\bar{\beta}} + O(|x|)f \\ 1 = (1 + O(|x|^2))f + O(|x|^3)d^\beta + O(|x|^3)e^{\bar{\beta}}. \end{cases}$$

Arguing as before, these imply that $d^\beta = O(|x|)$, $e^{\bar{\beta}} = O(|x|)$ and $f = 1 + O(|x|^2)$. Therefore

$$T = O(|x|)\overset{\circ}{Z}_\beta + O(|x|)\overset{\circ}{Z}_{\bar{\beta}} + (1 + O(|x|^2))\overset{\circ}{T}.$$

So we proved the first part of the Lemma.

We can write the formulas we proved as

$$\begin{pmatrix} Z_\alpha \\ Z_{\bar{\alpha}} \\ T \end{pmatrix} = \left(I + \begin{pmatrix} O(|x|^2) & O(|x|^2) & O(|x|^3) \\ O(|x|^2) & O(|x|^2) & O(|x|^3) \\ O(|x|) & O(|x|) & O(|x|^2) \end{pmatrix} \right) \begin{pmatrix} \overset{\circ}{Z}_\alpha \\ \overset{\circ}{Z}_{\bar{\alpha}} \\ \overset{\circ}{T} \end{pmatrix}.$$

Applying the Taylor expansion for the matrix inverse $(I+A)^{-1} = I - A + A^2 + O(\|A\|^3)$, we get

$$\begin{aligned} \begin{pmatrix} \overset{\circ}{Z}_\alpha \\ \overset{\circ}{Z}_{\bar{\alpha}} \\ \overset{\circ}{T} \end{pmatrix} &= \left(I + \begin{pmatrix} O(|x|^2) & O(|x|^2) & O(|x|^3) \\ O(|x|^2) & O(|x|^2) & O(|x|^3) \\ O(|x|) & O(|x|) & O(|x|^2) \end{pmatrix} \right) + \\ &+ \begin{pmatrix} O(|x|^4) & O(|x|^4) & O(|x|^5) \\ O(|x|^4) & O(|x|^4) & O(|x|^5) \\ O(|x|^3) & O(|x|^3) & O(|x|^4) \end{pmatrix} + O(|x|^3) \begin{pmatrix} Z_\alpha \\ Z_{\bar{\alpha}} \\ T \end{pmatrix} = \\ &= \left(I + \begin{pmatrix} O(|x|^2) & O(|x|^2) & O(|x|^3) \\ O(|x|^2) & O(|x|^2) & O(|x|^3) \\ O(|x|) & O(|x|) & O(|x|^2) \end{pmatrix} \right) \begin{pmatrix} Z_\alpha \\ Z_{\bar{\alpha}} \\ T \end{pmatrix}. \end{aligned}$$

This implies the second part of the claim. \square

Lemma A.3. *In pseudohermitian normal coordinates*

(A.4)

$$\left\{ \begin{aligned} \mathring{Z}_\alpha^2 &= Z_\alpha^2 + O(|x|)Z_\beta + O(|x|)Z_{\bar{\beta}} + O(|x|^2)T + O(|x|^2)Z_\alpha Z_\beta + O(|x|^2)Z_\alpha Z_{\bar{\beta}} + \\ &+ O(|x|^3)Z_\alpha T + O(|x|^4)Z_\beta Z_\gamma + O(|x|^4)Z_\beta Z_{\bar{\gamma}} + O(|x|^4)Z_{\bar{\beta}} Z_{\bar{\gamma}} + \\ &+ O(|x|^5)Z_\beta T + O(|x|^5)Z_{\bar{\beta}} T + O(|x|^6)T^2 \\ \mathring{Z}_\alpha \mathring{Z}_\beta &= Z_\alpha Z_\beta + O(|x|)Z_\gamma + O(|x|)Z_{\bar{\gamma}} + O(|x|^2)T + \\ &+ O(|x|^2)Z_\alpha Z_\gamma + O(|x|^2)Z_\alpha Z_{\bar{\gamma}} + O(|x|^3)Z_\alpha T + \\ &+ O(|x|^2)Z_\gamma Z_\beta + O(|x|^2)Z_{\bar{\gamma}} Z_\beta + O(|x|^3)T Z_\beta + \\ &+ O(|x|^4)Z_\gamma Z_\mu + O(|x|^4)Z_\gamma Z_{\bar{\mu}} + O(|x|^4)Z_{\bar{\gamma}} Z_{\bar{\mu}} + \\ &+ O(|x|^5)Z_\gamma T + O(|x|^5)Z_{\bar{\gamma}} T + O(|x|^6)T^2 \\ \mathring{Z}_\alpha \mathring{Z}_{\bar{\beta}} &= Z_\alpha Z_{\bar{\beta}} + O(|x|)Z_\gamma + O(|x|)Z_{\bar{\gamma}} + O(|x|^2)T + \\ &+ O(|x|^2)Z_\alpha Z_\gamma + O(|x|^2)Z_\alpha Z_{\bar{\gamma}} + O(|x|^3)Z_\alpha T + \\ &+ O(|x|^2)Z_\gamma Z_{\bar{\beta}} + O(|x|^2)Z_{\bar{\gamma}} Z_{\bar{\beta}} + O(|x|^3)T Z_{\bar{\beta}} + \\ &+ O(|x|^4)Z_\gamma Z_\mu + O(|x|^4)Z_\gamma Z_{\bar{\mu}} + O(|x|^4)Z_{\bar{\gamma}} Z_{\bar{\mu}} + \\ &+ O(|x|^5)Z_\gamma T + O(|x|^5)Z_{\bar{\gamma}} T + O(|x|^6)T^2 \\ \mathring{Z}_{\bar{\alpha}} \mathring{Z}_{\bar{\beta}} &= Z_{\bar{\alpha}} Z_{\bar{\beta}} + O(|x|)Z_\gamma + O(|x|)Z_{\bar{\gamma}} + O(|x|^2)T + \\ &+ O(|x|^2)Z_{\bar{\alpha}} Z_\gamma + O(|x|^2)Z_{\bar{\alpha}} Z_{\bar{\gamma}} + O(|x|^3)Z_{\bar{\alpha}} T + \\ &+ O(|x|^2)Z_\gamma Z_{\bar{\beta}} + O(|x|^2)Z_{\bar{\gamma}} Z_{\bar{\beta}} + O(|x|^3)T Z_{\bar{\beta}} + \\ &+ O(|x|^4)Z_\gamma Z_\mu + O(|x|^4)Z_\gamma Z_{\bar{\mu}} + O(|x|^4)Z_{\bar{\gamma}} Z_{\bar{\mu}} + \\ &+ O(|x|^5)Z_\gamma T + O(|x|^5)Z_{\bar{\gamma}} T + O(|x|^6)T^2 \\ \mathring{Z}_\alpha \mathring{T} &= Z_\alpha T + O(1)Z_\beta + O(1)Z_{\bar{\beta}} + O(|x|)T + \\ &+ O(|x|)Z_\alpha Z_\beta + O(|x|)Z_\alpha Z_{\bar{\beta}} + O(|x|^2)Z_\beta T + O(|x|^2)Z_{\bar{\beta}} T + \\ &+ O(|x|^3)Z_\beta Z_\gamma + O(|x|^3)Z_\beta Z_{\bar{\gamma}} + O(|x|^3)Z_{\bar{\beta}} Z_{\bar{\gamma}} + O(|x|^3)T^2 \\ \mathring{T}^2 &= (1 + O(|x|^2))T^2 + O(1)Z_\beta + O(1)Z_{\bar{\beta}} + O(|x|)T + O(|x|)Z_\beta T + \\ &+ O(|x|)Z_{\bar{\beta}} T + O(|x|^2)Z_\beta Z_\gamma + O(|x|^2)Z_\beta Z_{\bar{\gamma}} + O(|x|^2)Z_{\bar{\beta}} Z_{\bar{\gamma}} \end{aligned} \right.$$

Proof. It follows from Lemma A.2 and straightforward computations. \square

Lemma A.4. *For any function f in CR normal coordinates around \bar{x}*

$$|f_{,\alpha} - \mathring{Z}_\alpha f| \lesssim (|f_{,\beta}| + |f_{,\bar{\beta}}|)|x|^2 + |f_{,0}|x^3,$$

$$|f_{,0} - \mathring{T}f| \lesssim (|f_{,\beta}| + |f_{,\bar{\beta}}|)|x| + |f_{,0}|x^2,$$

$$|f_{,\alpha\beta} - \mathring{Z}_\alpha \mathring{Z}_\beta f| \lesssim (|f_{,\gamma}| + |f_{,\bar{\gamma}}|)|x| + (|f_{,0}| + |f_{,\alpha\gamma}| + |f_{,\alpha\bar{\gamma}}| + |f_{,\gamma\beta}| + |f_{,\bar{\gamma}\bar{\beta}}|)|x|^2 + (|f_{,\alpha 0}| + |f_{,0\beta}|)|x|^3 + \\ + (|f_{,\gamma\mu}| + |f_{,\gamma\bar{\mu}}| + |f_{,\bar{\gamma}\bar{\mu}}|)|x|^4 + (|f_{,0\gamma}| + |f_{,0\bar{\gamma}}|)|x|^5 + |f_{,00}|x^6,$$

$$|f_{,\alpha\bar{\beta}} - \mathring{Z}_\alpha \mathring{Z}_{\bar{\beta}} f| \lesssim (|f_{,\gamma}| + |f_{,\bar{\gamma}}|)|x| + (|f_{,0}| + |f_{,\alpha\gamma}| + |f_{,\alpha\bar{\gamma}}| + |f_{,\gamma\bar{\beta}}| + |f_{,\bar{\gamma}\bar{\beta}}|)|x|^2 + (|f_{,\alpha 0}| + |f_{,0\bar{\beta}}|)|x|^3 + \\ + (|f_{,\gamma\mu}| + |f_{,\gamma\bar{\mu}}| + |f_{,\bar{\gamma}\bar{\mu}}|)|x|^4 + (|f_{,0\gamma}| + |f_{,0\bar{\gamma}}|)|x|^5 + |f_{,00}|x^6,$$

$$|f_{,0\alpha} - \mathring{Z}_\alpha \mathring{T}f| \lesssim |f_{,\beta}| + |f_{,\bar{\beta}}| + (|f_{,0}| + |f_{,\alpha\beta}| + |f_{,\alpha\bar{\beta}}|)|x| + (|f_{,0\beta}| + |f_{,0\bar{\beta}}|)|x|^2 + \\ + (|f_{,00}| + |f_{,\beta\gamma}| + |f_{,\beta\bar{\gamma}}| + |f_{,\bar{\beta}\bar{\gamma}}|)|x|^3,$$

$$|f_{,00} - \mathring{T}^2 f| \lesssim |f_{,\beta} + |f_{,\bar{\beta}}| + (|f_{,0} + |f_{,0\beta} + |f_{,0\bar{\beta}}|)|x| + (|f_{,\beta\gamma} + |f_{,\beta\bar{\gamma}}| + |f_{,\bar{\beta}\bar{\gamma}}| + |f_{,00}|)|x|^2.$$

Proof. The first two estimates follow from formulas (A.3) and (A.1). The other estimates follow from formulas (A.4) and (A.1). \square

Lemma A.5. *In pseudohermitian normal coordinates around a point x*

$$\begin{aligned} \Delta_b f &= \mathring{\Delta}_b f + O(|x|)(|\mathring{Z}_\beta f| + |\mathring{Z}_{\bar{\beta}} f|) + \\ &+ O(|x|^2)(|Tf| + |\mathring{Z}_\beta \mathring{Z}_\alpha f| + |\mathring{Z}_{\bar{\beta}} \mathring{Z}_\alpha f| + |\mathring{Z}_{,\bar{\alpha}} \mathring{Z}_{,\beta} f| + |\mathring{Z}_{\bar{\alpha}} \mathring{Z}_{\bar{\beta}} f|) + \\ &+ O(|x|^3)(|\mathring{Z}_\alpha Tf| + |\mathring{Z}_{\bar{\alpha}} Tf|) + O(|x|^4)(|\mathring{Z}_\gamma \mathring{Z}_\beta f| + |\mathring{Z}_{\bar{\gamma}} \mathring{Z}_\beta f| + |\mathring{Z}_{\bar{\gamma}} \mathring{Z}_{\bar{\beta}} f|) + \\ &+ O(|x|^5)(|\mathring{Z}_\beta Tf| + |\mathring{Z}_{\bar{\beta}} Tf|) + O(|x|^6)|T^2 f| \end{aligned}$$

Proof. Using formulas (A.2) and Lemma A.4 we get

$$\begin{aligned} |(\Delta_b - \mathring{\Delta}_b)f| &= |f_{\alpha\bar{\alpha}} + f_{\bar{\alpha}\alpha} - \mathring{Z}_\alpha \mathring{Z}_{\bar{\alpha}} f - \mathring{Z}_{\bar{\alpha}} \mathring{Z}_\alpha f| \lesssim \\ &\lesssim (|f_{,\beta} + |f_{,\bar{\beta}}|)|x| + (|f_{,0} + |f_{,\alpha\beta} + |f_{,\alpha\bar{\beta}}| + |f_{,\beta\bar{\alpha}} + |f_{,\bar{\beta}\bar{\alpha}}|)|x|^2 + (|f_{,\alpha 0} + |f_{,0\bar{\alpha}}|)|x|^3 + \\ &+ (|f_{,\beta\gamma} + |f_{,\beta\bar{\gamma}}| + |f_{,\bar{\beta}\bar{\gamma}}|)|x|^4 + (|f_{,0\beta} + |f_{,0\bar{\beta}}|)|x|^5 + |f_{,00}|x|^6 \lesssim \\ &\lesssim (|\mathring{Z}_\beta f| + |\mathring{Z}_{\bar{\beta}} f|)|x| + (|Tf| + |\mathring{Z}_\beta \mathring{Z}_\alpha f| + |\mathring{Z}_{\bar{\beta}} \mathring{Z}_\alpha f| + |\mathring{Z}_{,\bar{\alpha}} \mathring{Z}_{,\beta} f| + |\mathring{Z}_{\bar{\alpha}} \mathring{Z}_{\bar{\beta}} f|)|x|^2 + (|\mathring{Z}_\alpha Tf| + |\mathring{Z}_{\bar{\alpha}} Tf|)|x|^3 + \\ &+ (|\mathring{Z}_\gamma \mathring{Z}_\beta f| + |\mathring{Z}_{\bar{\gamma}} \mathring{Z}_\beta f| + |\mathring{Z}_{\bar{\gamma}} \mathring{Z}_{\bar{\beta}} f|)|x|^4 + (|\mathring{Z}_\beta Tf| + |\mathring{Z}_{\bar{\beta}} Tf|)|x|^5 + |T^2 f||x|^6 \end{aligned}$$

\square

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