A CLASSICAL ELLIPTIC REGULARITY APPROACH TO ALMOST HARMONIC MAPS AND RELATED SYSTEMS

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Abstract. We establish interior regularity results for a broad class of two-dimensional nonlinear elliptic systems. Our approach isolates the core integrability mechanism within a unified abstract framework built around a Campanato-type discrete iteration scheme coupled with a Caccioppoli-type estimate. Specifically, we show that within any class of admissible pairs $(\boldsymbol{u}, \boldsymbol{f})$ that is stable under rescaling and satisfies a discrete oscillation-decay axiom, the map $\boldsymbol{u}$ is automatically locally Hölder continuous. Furthermore, the resulting Hölder exponent is explicit and optimally attains the classical Morrey–Campanato threshold dictated by the Lebesgue integrability of the source term $\boldsymbol{f}$. This purely analytic framework systematically avoids the $\mathcal{H}^1$–BMO duality, Wente's inequality, moving frames, and conformal uniformization techniques that underpin existing regularity theories.

We apply this principle to derive regularity results in regimes lying strictly beyond the reach of existing gauge-theoretic methods. As a foundational example, we provide a new direct proof of local Hölder continuity for almost harmonic maps $-\Delta\boldsymbol{u} = |\nabla\boldsymbol{u}|^2\boldsymbol{u} + \boldsymbol{f}$ into $\mathbb{S}^n$ with L^q-integrable tension fields. We then extend the analysis to systems of the form $-\Delta\boldsymbol{u} = \boldsymbol{\Omega}\cdot\nabla\boldsymbol{u} + \boldsymbol{f}$, replacing Rivière's geometric antisymmetry assumption on the connection form $\boldsymbol{\Omega} \in L^2$ with the purely analytic condition $\mathbf{div}\,\boldsymbol{\Omega} \in L^q$ for some $q > 1$. This demonstrates that, although antisymmetry provides the fundamental algebraic mechanism underlying gauge theoretic reformulations, most notably through the construction of a *divergence free* connection frame ($\mathbf{div}\,\boldsymbol{\Omega} = \mathbf{0}$), the considerably weaker and purely analytic condition $\mathbf{div}\,\boldsymbol{\Omega} \in L^q$ is already sufficient to guarantee regularity, entirely independently of any underlying geometric gauge structure.

Furthermore, for anisotropic systems of the form $-\,\mathbf{div}(A\nabla\boldsymbol{u}) = \boldsymbol{\Omega}\cdot\nabla\boldsymbol{u} + \boldsymbol{f}$ with a Hölder-continuous, non-unimodular coefficient matrix A, we establish local Hölder continuity in a setting where conformal uniformization is unavailable. Finally, we develop a dimension-independent, stationarity-free bootstrap procedure that upgrades initial continuity to full C^∞ regularity. As a primary physical application, we apply this framework to establish the interior regularity of magnetic skyrmions. More precisely, we prove that finite energy critical points of the full Brown micromagnetic energy in dimensions ($m \in \{2,3\}$), incorporating the Dzyaloshinskii–Moriya interaction, magnetocrystalline anisotropy, and the nonlocal demagnetizing field, inherit the optimal regularity ($C^{k+1,\beta}$) dictated by the regularity of the underlying magnetocrystalline anisotropy density.

Contents

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1. Introduction and Statement of Results

1.1. **Motivation and Problem Setting.** Harmonic maps and their generalizations occupy a central position in geometric analysis, the calculus of variations, and a wide array of models arising in mathematical physics. Emerging as critical points of the Dirichlet energy, they provide a unifying framework that connects nonlinear elliptic systems, minimal surface theory, and geometric flows. The rich analytical structure of these systems has catalyzed the development of foundational techniques across the discipline, driving major advances in regularity theory, singularity analysis, and the mathematics of pattern formation.

In this paper, we investigate the interior regularity for a broad class of strongly coupled, nonlinear elliptic systems exhibiting critical quadratic growth in the gradient. The prototypical example driving this investigation is the regularity theory of *almost harmonic maps* from an open subset of $\mathbb{R}^m$ into the sphere $\mathbb{S}^n$. More precisely, these correspond to critical points of perturbed Dirichlet energies whose Euler–Lagrange equation takes the form

$$-\Delta \boldsymbol{u} = |\nabla \boldsymbol{u}|^2 \boldsymbol{u} + \boldsymbol{f}, \tag{1.1}$$

where the lower-order term $\boldsymbol{f}$ represents an inhomogeneous perturbation of the classical harmonic map equation. Such systems arise naturally in physically relevant models, including the Oseen–Frank theory of liquid crystals and variational models in micromagnetics, where external fields, anisotropies, or defects in the crystal lattice introduce nontrivial forcing terms.

The core analytical difficulty of (1.1), and of the broader hierarchy of systems we consider, lies in the critical growth of the nonlinearity: under standard dilational rescaling, the quadratic gradient term scales exactly like the Laplacian, placing the equation at the borderline of classical Calderón–Zygmund theory. In dimension two, it is well known that weak solutions in the energy space H^1 are continuous. However, the established proofs of this fact rely on a substantial analytical apparatus developed over several decades, including compensated compactness, the Hardy space $\mathcal{H}^1$, its duality with BMO, Wente's inequality, and gauge-theoretic decompositions. Although these approaches are remarkably powerful, they rely fundamentally on the presence of a rigid divergence curl structure. As a consequence, they are highly effective within the classical geometric setting, yet extremely difficult to extend beyond it. Even comparatively mild perturbations of the governing equations, such as the loss of antisymmetry or the introduction of anisotropic coefficients, typically destroy the algebraic structures on which the theory depends.

The present work is motivated by the observation that the regularity mechanism underlying (1.1) is substantially more robust than the analytical frameworks traditionally used to capture it. By isolating this mechanism and reformulating it as an abstract integrability principle, we develop a regularity theory that applies in regimes lying far beyond the scope of existing methods. This viewpoint naturally leads to the following four questions, which remain unresolved, or only partially understood, in the current literature:

- **($\boldsymbol{Q}_1$)** Is $\mathcal{H}^1$–BMO duality genuinely necessary to establish the interior regularity of two-dimensional almost harmonic maps with L^q tension fields? Alternatively, can these results be recovered using classical linear elliptic theory alone, yielding explicit quantitative estimates in terms of q?
- **($\boldsymbol{Q}_2$)** Modern regularity arguments rely heavily on the geometric antisymmetry of the connection form $\boldsymbol{\Omega} = \boldsymbol{u} \wedge \nabla \boldsymbol{u}$. Is this algebraic structure essential, or can it be substituted by a purely analytic integrability condition on the divergence $\operatorname{\mathbf{div}} \boldsymbol{\Omega}$?
- **($\boldsymbol{Q}_3$)** Can the regularity theory successfully accommodate variable-coefficient principal parts of the form $\operatorname{\mathbf{div}}(A\nabla \boldsymbol{u})$? This seems to pose analytical challenges in the non-unimodular regime ($\det A \neq \text{const}$), a setting where conformal uniformization is unavailable and standard gauge-fixing techniques are rendered ineffective.
- **($\boldsymbol{Q}_4$)** Once continuity is achieved, the passage to higher Schauder regularity is frequently presented in a manner that obscures the precise handling of critical nonlinearities. Can this bootstrap procedure be formulated as a direct, explicit mechanism that avoids cumbersome intermediate machinery and transitions seamlessly into classical subcritical estimates?

1.2. **Background and State of the Art.** We begin by recalling the classical results on harmonic maps and then highlight the distinct analytical challenges introduced by almost harmonic maps and their broader structural generalizations, thereby clarifying the precise scope and context of the present work.

For harmonic maps from two-dimensional domains ($m = 2$), the theory is well established, ultimately grounded in the conformal invariance of the Dirichlet energy at the critical dimension. The regularity of energy-minimizing harmonic maps from surfaces rests on Morrey's foundational analysis of multiple integrals in the calculus of variations [Mor66]. This theory was subsequently advanced by Schoen [Sch84] who proved the interior smoothness of *stationary* harmonic maps, that is, of weak solutions subject to the additional variational assumption of stability under domain deformations.

The definitive regularity result for generic weak solutions is due to Hélein. He first proved that every weakly harmonic map (i.e., every $W^{1,2}$ solution of the Euler–Lagrange equation) from a surface into the sphere $\mathbb{S}^n$ is smooth [Hél90], and subsequently generalized this optimal regularity result to maps taking values in closed Riemannian manifolds [Hél91, Hé91, Hé02]. For the specific case of $\mathbb{S}^n$, Hélein's argument famously relies on rewriting the critical nonlinearity using a divergence-free antisymmetric structural potential and applying compensated-compactness techniques. Obtaining continuity in this framework relies on Wente's inequality and the duality between the Hardy space $\mathcal{H}^1$ and BMO.

In dimensions $m \geqslant 3$ the picture changes markedly, precisely because the two-dimensional conformal invariance is lost. As documented in Hardt's survey [Har97], regularity may fail even for minimizers, with singularities forced by topology or by energy considerations. The canonical instance is the radial projection $\boldsymbol{u}(x) = x/|x|$ from the unit ball of $\mathbb{R}^3$ into $\mathbb{S}^2$: it minimizes the energy among maps sharing its boundary data, yet its singular set is the single point at the origin, of Hausdorff dimension zero. The earliest positive results imposed geometric restrictions precluding such defects; thus Hildebrandt, Kaul, and Widman [HKW77] obtained everywhere regularity for minimizers whose image lies within a sufficiently small geodesic ball. The decisive advance, removing all restrictions on the image, was achieved by Schoen and Uhlenbeck [SU82] (see also the historical account in [SBM$^+$18]): through the monotonicity formula and a systematic analysis of tangent maps obtained by blow-up, they proved that the singular set of any energy minimizer is closed and has Hausdorff dimension at most $m-3$. In particular, minimizers from a surface are everywhere continuous, and in three dimensions the singularities reduce to isolated points.

For the broader class of *stationary* weak solutions Evans [Eva91] established partial regularity for maps into spheres, demonstrating that the singular set has $(m-2)$-dimensional Hausdorff measure zero. Evans exploited the specific coordinate representation of the spherical harmonic map equation to reveal the antisymmetric structure of the quadratic nonlinearity, deploying $\mathcal{H}^1$–BMO duality via the theorem of Coifman, Lions, Meyer, and Semmes [CLMS93]. Bethuel [Bet93] subsequently extended this bound to stationary maps into arbitrary compact target manifolds by utilizing moving frames to uncover a hidden antisymmetric structure.

When the classical harmonic map equation is perturbed by a lower-order inhomogeneous source term, the corresponding analytic theory demands a strict distinction between two fundamentally different mathematical paradigms.

The first concerns the asymptotic analysis of defect formation. In this framework, one studies sequences of almost (or approximate) harmonic maps whose tension fields vanish or remain uniformly bounded in a suitable Lebesgue space. In this direction, Topping [Top04] utilized sequences with parametrically small L^2 tension fields to establish sharp quantization and repulsion phenomena for emerging singularities within a bubbling regime, a perspective later extended to L^p-bounded tension fields ($p > 1$) by Wang, Wei, and Zhang [WWZ17].

The second paradigm—and the one relevant to the present work—concerns the interior regularity of an individual map subject to a fixed, prescribed tension field. In this setting, the governing system exhibits severe analytical obstructions that require a dedicated geometric and analytic treatment, as discussed for instance in Moser's monograph [Mos05b].

For generic weak solutions subject to a fixed (not necessarily small) source term, positive regularity results in dimension two were initially achieved only in highly specialized contexts, such as Carbou's analysis of nonlocal micromagnetic energies [Car97]. Beyond specific physical models, systematic treatments (e.g., Moser [Mos05a, Mos05b]) frequently focus on higher-dimensional settings ($m \geqslant 3$), where stationarity is necessary to derive monotonicity formulas underlying partial regularity.

For conformally invariant systems without stationarity, a significant generalization of Hélein's method was achieved by Rivière [Riv06], who systematically studied generic equations of the form $-\Delta \boldsymbol{u} = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u}$, where $\boldsymbol{\Omega}$ is an arbitrary antisymmetric matrix in $L^2\left(B_1, \mathfrak{so}(n+1) \otimes \mathbb{R}^m\right)$, $B_1 \subseteq \mathbb{R}^m$. His decisive observation was that the Euler–Lagrange equation of *any* two-dimensional conformally invariant variational problem can be brought into this antisymmetric form. Combining this with Uhlenbeck's gauge decomposition, he proved that all such systems admit continuous weak solutions; the framework was carried up to the boundary, and to systems carrying an inhomogeneous term, by Müller and Schikorra [MS09]. We stress that, even with the optimal gauge in hand, the passage to continuity continues to rely on Wente's inequality and $\mathcal{H}^1$–BMO duality.

The present work develops a different route to the regularity of almost harmonic maps, combining ideas of Chang [CWY99b, CWY99a] and Carbou [Car97] with the structural insight of Hélein [Hé91]. Working entirely within the classical linear theory of divergence-form elliptic equations, we dispense with compensated-compactness machinery and with $\mathcal{H}^1$–BMO duality. In their place we establish a Campanato-type iterative decay estimate; a new Caccioppoli-type inequality then makes explicit how the fractional integrability of the source $\boldsymbol{f}$ governs the local Hölder exponent of the solution. The same scheme yields a transparent, self-contained bootstrap from initial continuity to full C^∞ regularity. Finally, the estimates apply directly to Brown's static micromagnetic equations and establish the full regularity of topological spin textures [DFMRS20, DFS23].

In sum, the stationarity-free method developed here shows that the regularity of two-dimensional almost harmonic maps can be recovered in its entirety from classical linear elliptic theory, with quantitative control on the perturbation that is well suited to physical applications.

1.3. Contributions of the present work.

To clarify the scope of our approach, we first highlight a model corollary that emerges from our framework:

Theorem 1.1. *Any weakly almost harmonic map $\boldsymbol{u} \in H^1\left(B_1, \mathbb{S}^n\right)$ from the unit two-dimensional disk $B_1 \subseteq \mathbb{R}^2$ to a sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, solving the equation*

$$-\Delta \boldsymbol{u} = |\nabla \boldsymbol{u}|^2 \boldsymbol{u} + \boldsymbol{f}, \tag{1.2}$$

with a source term $\boldsymbol{f} \in L^q(B_1, \mathbb{R}^{n+1})$ for some $q > 1$, is locally Hölder continuous. Specifically, there exists an exponent $\eta \in (0,1)$ such that $\boldsymbol{u} \in C^{0,\eta}_{\mathrm{loc}}(B_1, \mathbb{R}^{n+1})$.

The proof of Theorem 1.1 rests on absorbing the critical geometric nonlinearity into an antisymmetric connection matrix. While continuity for L^q sources was previously obtained by Müller and Schikorra [MS09] via gauge transformations, the novelty of our result lies in the *method*. Specifically, we answer ($\boldsymbol{Q_1}$) by establishing regularity entirely through a discrete iteration scheme, bypassing both the construction of a gauge and the $\mathcal{H}^1$–BMO duality. This provides a direct, quantitative route to local Hölder continuity that is governed strictly by the fractional integrability of the source term, independent of classical gauge-theoretic machinery.

To illustrate the flexibility of our linear approach, we establish a broad generalization in which the geometric assumption of antisymmetry is replaced by a purely analytic divergence condition. A precedent for this perspective is found in the work of Bethuel [Bet92], who, in the context of the prescribed mean curvature equation, obtained regularity by imposing integrability assumptions that force the divergence of the corresponding connection matrix to belong to a Lorentz space. Indeed, our framework naturally reveals that the antisymmetry of $\boldsymbol{\Omega}$ is not required for regularity, provided that $\boldsymbol{u}$ is globally bounded and $\mathbf{div}\,\boldsymbol{\Omega} \in L^q$ for some $q > 1$.

Theorem 1.2 (Regularity of generalized systems)**.** *Let* $B_1 \subset \mathbb{R}^2$*. Let* $\boldsymbol{u} \in H^1 \cap L^\infty(B_1, \mathbb{R}^{n+1})$ *be a weak solution to the generalized system*

$$-\Delta \boldsymbol{u} = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} + \boldsymbol{f} \quad \text{in } B_1. \tag{1.3}$$

Assume that the structural matrix and the source terms satisfy the purely analytic hypotheses for some $q > 1$*:*

- $\boldsymbol{\Omega} \in L^2(B_1, \mathbb{R}^{(n+1)\times(n+1)} \otimes \mathbb{R}^2)$,
- $\boldsymbol{f} \in L^q(B_1, \mathbb{R}^{n+1})$,
- $\mathbf{div}\,\boldsymbol{\Omega} \in L^q(B_1, \mathbb{R}^{(n+1)\times(n+1)})$.

Then, $\boldsymbol{u}$ *is locally Hölder continuous in* B_1*. Specifically, there exists an exponent* $\eta \in (0,1)$ *such that* $\boldsymbol{u} \in C^{0,\eta}_{\text{loc}}(B_1, \mathbb{R}^{n+1})$*.*

To place the condition $\mathbf{div}\,\boldsymbol{\Omega} \in L^q$ in context, it is useful to compare our setting with the classical gauge-theoretic framework. In Hélein's original proof for $\mathbb{S}^n$-valued harmonic maps [Hél90], the antisymmetry of the connection is used to recast the system in a gauge in which the resulting connection matrix is divergence-free, namely $\mathbf{div}\,\boldsymbol{\Omega} = \mathbf{0}$. Only after this divergence-free structure has been obtained can one invoke the analytical $\mathcal{H}^1$–BMO regularity theory. A similar strategy appears in the general gauge-theoretic constructions of Rivière [Riv06] and Müller–Schikorra [MS09], where antisymmetry acts as the algebraic mechanism that produces a zero-divergence gauge, after which purely analytic compensation estimates apply.

By contrast, our framework shows that this geometric step is not necessary for regularity. The enforcement of $\mathbf{div}\,\boldsymbol{\Omega} = \mathbf{0}$ through a gauge transformation can be entirely avoided: we prove that an a priori L^q bound on $\mathbf{div}\,\boldsymbol{\Omega}$ already suffices to initiate the Campanato iteration and derive regularity, independently of any underlying geometric gauge structure. This underscores that, in the classical theory, antisymmetry functions primarily as a means of producing a divergence-free gauge, rather than as an intrinsic analytic requirement of the regularity mechanism itself.

Our third result extends the theory to non-unimodular anisotropic media. Through a localized coefficient-freezing argument, we generalize the isotropic principal part $-\Delta \boldsymbol{u}$ to a fully variable-coefficient divergence-form operator $-\mathbf{div}(A\nabla \boldsymbol{u})$. This extension successfully establishes regularity in a regime that cannot be treated by uniformization.

Theorem 1.3 (Regularity of anisotropic systems)**.** *Let $B_1 \subset \mathbb{R}^2$. Let $\boldsymbol{u} \in H^1 \cap L^\infty(B_1, \mathbb{R}^{n+1})$ be a weak solution of*

$$-\operatorname{\mathbf{div}}(A\nabla \boldsymbol{u}) = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} + \boldsymbol{f} \qquad \text{in } B_1. \tag{1.4}$$

Assume the structural data satisfy, for some $q > 1$ and some $\gamma_0 \in (0, 1]$:

(A1) *$A : B_1 \to \mathbb{R}^{2\times 2}$ is symmetric, uniformly elliptic, and Hölder continuous:*

$$\lambda|\xi|^2 \leqslant A(x)\xi \cdot \xi \leqslant \Lambda|\xi|^2 \qquad \forall x \in B_1, \xi \in \mathbb{R}^2, \quad 0 < \lambda \leqslant \Lambda < +\infty, \tag{1.5}$$

and $A \in C^{0,\gamma_0}(\overline{B_1})$;

(A2) *$\boldsymbol{\Omega} \in L^2(B_1, \mathbb{R}^{(n+1)\times(n+1)} \otimes \mathbb{R}^2)$;*

(A3) *$\boldsymbol{f} \in L^q(B_1, \mathbb{R}^{n+1})$;*

(A4) *$\operatorname{\mathbf{div}} \boldsymbol{\Omega} \in L^q(B_1, \mathbb{R}^{(n+1)\times(n+1)})$.*

Then $\boldsymbol{u}$ is locally Hölder continuous in B_1. Specifically, there exists an exponent $\eta \in (0, 1)$ such that $\boldsymbol{u} \in C^{0,\eta}_{\mathrm{loc}}(B_1, \mathbb{R}^{n+1})$.

When $\det A$ is constant, a conformal change of variables reduces the system to the isotropic setting. Theorem 1.3 answers ($\boldsymbol{Q_3}$) by resolving the *non-unimodular* regime ($\det A \neq \text{const}$), which is inaccessible to uniformization.

While initial Hölder continuity addresses the critical geometric barrier, physical applications demand full classical smoothness. To complete the regularity paradigm, our final main theorem provides a universal bootstrap mechanism. Valid in arbitrary dimensions, it demonstrates that initial continuity unconditionally forces higher-order smoothness, dictated solely by the integrability of the source.

Theorem 1.4 (Bootstrap Regularity for Almost Harmonic Maps)**.** *Let $U \subset \mathbb{R}^m$ be an open domain, and let $\boldsymbol{u} \in W^{1,2}_{\mathrm{loc}}(U, \mathbb{S}^n) \cap C^0(U, \mathbb{S}^n)$ be a weakly almost harmonic map solving the Euler–Lagrange equation:*

$$-\Delta \boldsymbol{u} = |\nabla \boldsymbol{u}|^2 \boldsymbol{u} + \boldsymbol{f} \quad \text{in } \mathcal{D}'(U), \tag{1.6}$$

where the source term initially satisfies $\boldsymbol{f} \in L^q_{\mathrm{loc}}(U, \mathbb{R}^{n+1})$.
If the source integrability q satisfies $2 \leqslant q \leqslant m/2$ (which can only occur in dimensions $m \geqslant 4$), we additionally assume the map is already locally Hölder continuous: $\boldsymbol{u} \in C^{0,\gamma}_{\mathrm{loc}}(U, \mathbb{S}^n)$ for some $\gamma \in (0, 1]$.
Then, the regularity of $\boldsymbol{u}$ depends on the regularity of $\boldsymbol{f}$:

(1) *If $\boldsymbol{f} \in L^q_{\mathrm{loc}}(U, \mathbb{R}^{n+1})$ and $2 \leqslant q \leqslant m$ then the regularity is bounded by the critical Sobolev embedding threshold:*

$$\boldsymbol{u} \in W^{1,q^*}_{\mathrm{loc}}(U, \mathbb{S}^n) \quad \text{with } q^* = \frac{mq}{m-q}, \tag{1.7}$$

with the usual understanding that q^* (necessarily greater than 1) can be any finite real exponent if $m = q$. In particular, if $m = q$ then $\boldsymbol{u} \in C^{0,\alpha}_{\mathrm{loc}}(U, \mathbb{S}^n)$ for all $0 < \alpha < 1$, while if $q = 2$ and $m = 3$ then $\boldsymbol{u} \in C^{0,1/2}_{\mathrm{loc}}(U, \mathbb{S}^n)$.

(2) *If $\boldsymbol{f} \in L^q_{\mathrm{loc}}(U, \mathbb{R}^{n+1})$ with $q > m$, then*

$$\boldsymbol{u} \in C^{1,\eta}_{\mathrm{loc}}(U, \mathbb{S}^n), \quad \text{where } \eta := 1 - m/q. \tag{1.8}$$

Note that η does not depend on any intermediate fractional exponent. In particular, if $\boldsymbol{f} \in L^\infty_{\mathrm{loc}}(U, \mathbb{R}^{n+1})$ then $\boldsymbol{u} \in C^{1,\alpha}_{\mathrm{loc}}(U, \mathbb{S}^n)$ for all $0 < \alpha < 1$.

(3) *For any integer $k \geqslant 1$ and exponent $\beta \in (0,1)$, if $\boldsymbol{f} \in C^{k-1,\beta}_{\mathrm{loc}}(U, \mathbb{R}^{n+1})$ then $\boldsymbol{u} \in C^{k+1,\beta}_{\mathrm{loc}}(U, \mathbb{S}^n)$. In particular, if $\boldsymbol{f} \in C^\infty(U, \mathbb{R}^{n+1})$, then $\boldsymbol{u} \in C^\infty(U, \mathbb{S}^n)$. Again, note that η does not play any role in the higher regularity class we end up in.*

Theorem 1.4 answers ($\boldsymbol{Q_4}$). The algorithmic chain 1.→2.→3. is dimension-free, requires no stationarity assumption, and relies on Calderón–Zygmund and Schauder theory. In $m = 2$, the continuity hypothesis is supplied by Theorem 1.1, making the bootstrap unconditional.

As a direct application of the preceding structural theorems, we turn to micromagnetics and determine the maximal interior regularity of topological spin textures such as magnetic skyrmions. These configurations arise as weak critical points of a non-convex variational energy $\mathcal{G}$ that governs the magnetization field $\boldsymbol{u} : U \to \mathbb{S}^2$:

$$\mathcal{G}(\boldsymbol{u}) := \int_U \left[\frac{1}{2}|\nabla \boldsymbol{u}|^2 + \kappa \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{u} + \varphi(\boldsymbol{u})\right] \mathrm{d}x - \frac{1}{2}\int_U \boldsymbol{h}[\boldsymbol{u}] \cdot \boldsymbol{u}\, \mathrm{d}x. \tag{1.9}$$

This functional combines the Dirichlet exchange energy (favoring uniform spin alignment), the antisymmetric Dzyaloshinskii–Moriya interaction (inducing chiral twisting), the magnetocrystalline anisotropy density φ (favoring alignment along preferred crystal axes), and the nonlocal demagnetizing stray field $\boldsymbol{h}[\boldsymbol{u}]$. The constrained Euler–Lagrange equations attached to (1.9) are of almost harmonic map type. Deferring the rigorous physical derivation to Section 7, we state the regularity result for these critical points autonomously.

Theorem 1.5 (Interior Regularity of Micromagnetic Maps)**.** *Let $U \subseteq \mathbb{R}^m$ ($m = 2, 3$) be a bounded open set and let $\boldsymbol{u} \in H^1 \cap C^0(U, \mathbb{S}^2)$ be a continuous critical point of the micromagnetic energy $\mathcal{G}$. Assume that $\boldsymbol{h} : L^2(U, \mathbb{R}^3) \to L^2(U, \mathbb{R}^3)$ satisfies the regularity hypotheses* (7.2) *and* (7.3)*.*

If the anisotropy density φ is of class $C^{k,\beta}$ for $k \geqslant 1$ and exponent $\beta \in (0,1)$, then $\boldsymbol{u} \in C^{k+1,\beta}_{\mathrm{loc}}(U, \mathbb{S}^2)$. In particular, if $\varphi \in C^\infty$, then $\boldsymbol{u} \in C^\infty(U, \mathbb{S}^2)$.

For $m = 2$, the initial continuity assumption is automatically satisfied for finite-energy solutions.

1.4. **Strategy of Proof.** To prove the theorems stated above without relying on standard conformal uniformization or $\mathcal{H}^1$–BMO duality, we develop a purely analytic architecture. Synthesizing insights for classical harmonic maps originating in Chang et al. [CWY99b, CWY99a], the structural observations of Hélein [Hé91], and the direct methods of Carbou [Car97], we establish regularity through a flexible Campanato-type iteration scheme.

Framing the problem within classical linear elliptic theory yields a decisive analytical advantage when addressing the anisotropic generalizations of Theorem 1.3. When the coefficient matrix A is uniformly elliptic and merely Hölder continuous, and its determinant $\det A$ is non-constant, the system cannot be conformally reduced to an isotropic equation on a two-dimensional Riemannian manifold. Such rough, non-unimodular systems represent a severe theoretical roadblock for established gauge-theoretic methods.

The architecture of the paper is built upon three structural mechanisms, contrasting sharply with traditional gauge-theoretic methods:

Algebraic Reformulation. For sphere-valued maps, the geometric constraint $|\boldsymbol{u}| = 1$ and orthogonality $\boldsymbol{f} \cdot \boldsymbol{u} = 0$ yield $|\nabla \boldsymbol{u}|^2 \boldsymbol{u} = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u}$, which allows a Leibniz expansion against an arbitrary constant $\boldsymbol{c} \in \mathbb{R}^{n+1}$:

$$-\Delta \boldsymbol{u} = \mathbf{div}\,(\boldsymbol{\Omega}(\boldsymbol{u} - \boldsymbol{c})) + (\boldsymbol{u} \wedge \boldsymbol{f})(\boldsymbol{u} - \boldsymbol{c}) + \boldsymbol{f}. \tag{1.10}$$

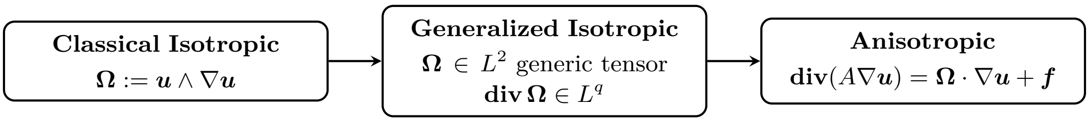
The Coupled Caccioppoli Estimate. Comparing $\boldsymbol{u}$ to its harmonic extension $\boldsymbol{h}_r$ on B_r, we establish a coupled Caccioppoli-type bound:

$$\|\nabla(\boldsymbol{u} - \boldsymbol{h}_r)\|_{L^q(B_r)} \leqslant C\,\|\nabla \boldsymbol{u}\|_{L^2(B_r)}\,\left(r^m\,\Psi_p(\boldsymbol{u}, B_r)\right)^{1/p} + Cr\,\|\boldsymbol{f}\|_{L^q(B_r)}. \tag{1.11}$$

The algebraic coincidence occurs here: the exponents are coupled via $1/q = 1/2 + 1/p$. In dimension $m = 2$, this forces the oscillation exponent p to be the Sobolev conjugate $q^* = 2q/(2-q)$. This serendipitous conjugacy is what permits the Campanato iteration to close in 2D without Wente's inequality.
An abstract regularity principle. We isolate the mechanism behind Hölder regularity as a property of an abstract class $\boldsymbol{\mathcal{S}}$ satisfying closure under rescaling and a discrete Campanato decay axiom:

$$\Psi_p(\boldsymbol{u}, B_\theta) \leqslant \gamma\,\Psi_p(\boldsymbol{u}, B_1) + \kappa\,\|\boldsymbol{f}\|^p_{L^q(B_1)}. \tag{1.12}$$

The paper's progression can be mapped as a structural hierarchy:



By formally decomposing the theory into these sequential stages, we provide a transparent framework for identifying what is mathematically lost, and what is gained, when passing from purely geometric analysis to models of heterogeneous physical media.

1.5. **Structure of the Paper.** The manuscript is structured to reflect a progressive relaxation of structural hypotheses: from the classical isotropic setting to generalized non-antisymmetric systems, and ultimately to fully anisotropic operators. Throughout this hierarchy, the core analytic engine—relying on Campanato-type iterations and coupled Caccioppoli estimates—remains unified.

Outline. In Section 2, we establish the functional setting and record the foundational linear PDE estimates that drive the subsequent nonlinear iteration schemes. Section 3 introduces the baseline abstract regularity principle (Theorem 3.3), establishing Hölder continuity via an algebraic iteration lemma. These axioms are verified for weakly almost harmonic maps into $\mathbb{S}^n$, where the dimensional restriction $m = 2$ enters exclusively through the Sobolev conjugacy coincidence.

This framework is extended in Section 4 to Rivière-type systems lacking geometric antisymmetry, demonstrating that the algebraic antisymmetry of the connection form can be securely replaced by the condition $\mathbf{div}\,\boldsymbol{\Omega} \in L^q$. Subsequently, Section 5 addresses the non-unimodular anisotropic regime $-\,\mathbf{div}(A\nabla \boldsymbol{u})$. We formulate an anisotropic regularity principle (Theorem 5.6) and verify its axioms via localized coefficient-freezing, which imposes the strict dimensional ceiling $m < 4$.

Transitioning away from the L^2-critical regime, Section 6 establishes a dimension-free bootstrap mechanism (Theorem 1.4) to unconditionally upgrade continuous weak solutions to C^∞. Section 7 applies this theory to physical models in micromagnetics. Finally, standard interior gradient estimates for weakly harmonic functions are collected in Appendix 8.

Logical Dependence. Section 2 is prerequisite to the entire paper. Sections 3, 4, and 5 are parallel: each develops an abstract principle and verifies it for a specific PDE structure. Section 6 operates independently, requiring only initial continuity as a black box. Section 7 synthesizes the 2D initial continuity and the higher-dimensional bootstrap to address the physical models.

Before proving the main results, it is essential to establish the analytic framework and introduce the notation employed throughout the manuscript. In the next section, we formulate the variational setting for almost harmonic maps, define the localized oscillation functionals underlying the Campanato iteration scheme, and record the precise scaling laws together with the linear elliptic estimates that form the foundation of our regularity theory. To streamline the exposition and remain consistent with the conventions of classical geometric analysis, we omit the proofs of standard functional inequalities and elementary scaling identities, focusing instead on the structural results required for the subsequent analysis.

2. Analytic Setting and Preliminary Tools

Our primary focus is on the interior regularity of vector-valued fields defined on an open subset of flat Euclidean space—specifically the unit ball $B_1 \subset \mathbb{R}^m$—taking values in the standard unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Throughout, $B_r(x_0) \subset \mathbb{R}^m$ denotes the open ball of radius $r > 0$ centered at x_0; when the center is the origin or is clear from context, we simply write B_r. For a bounded measurable set $U \subset \mathbb{R}^m$, we denote the integral average of a measurable map $\boldsymbol{v} : U \to \mathbb{R}^{n+1}$ by $\langle \boldsymbol{v} \rangle_U := \fint_U \boldsymbol{v}$.

2.1. Geometric Setup and Governing Equations.

For a Sobolev map $\boldsymbol{u} \in H^1(B_1, \mathbb{S}^n)$, the *Dirichlet energy* on a sub-ball $B_r(x_0) \subseteq B_1$ is

$$\mathcal{E}(\boldsymbol{u}, B_r(x_0)) := \frac{1}{2} \int_{B_r(x_0)} |\nabla \boldsymbol{u}|^2 \mathrm{d}x. \tag{2.1}$$

Definition 2.1 (Harmonic and Almost Harmonic Maps)**.** A map $\boldsymbol{u} \in H^1(B_1, \mathbb{S}^n)$ is a *weakly harmonic map* if it is a critical point of the Dirichlet energy subject to the pointwise constraint $|\boldsymbol{u}| = 1$, satisfying the Euler–Lagrange equation $-\Delta \boldsymbol{u} = |\nabla \boldsymbol{u}|^2 \boldsymbol{u}$ in $\mathcal{D}'(B_1)$.

We extend this structure to systems driven by lower-order source terms. A map $\boldsymbol{u} \in H^1(B_1, \mathbb{S}^n)$ is a *weakly almost harmonic map* if it solves the perturbed equation:

$$-\Delta \boldsymbol{u} = |\nabla \boldsymbol{u}|^2 \boldsymbol{u} + \boldsymbol{f} \quad \text{in } \mathcal{D}'(B_1), \tag{2.2}$$

for a source field $\boldsymbol{f} \in L^q(B_1, \mathbb{R}^{n+1})$, $q > 1$, satisfying the orthogonality condition $\boldsymbol{f} \cdot \boldsymbol{u} = 0$ almost everywhere in B_1.

Remark 2.2 (Algebraic Compatibility). The orthogonality condition $\boldsymbol{f} \cdot \boldsymbol{u} = 0$ is not an artificial assumption, but a rigid algebraic necessity. For any sphere-valued map $|\boldsymbol{u}| = 1$, testing the Laplacian against the map itself yields $-\boldsymbol{u} \cdot \Delta \boldsymbol{u} = |\nabla \boldsymbol{u}|^2$. Taking the inner product of (2.2) with $\boldsymbol{u}$ thus immediately forces $\boldsymbol{f} \cdot \boldsymbol{u} = 0$ almost everywhere.

Terminology Note. We reserve the term *harmonic maps* strictly for $\mathbb{S}^n$-valued functions solving the nonlinear constraint equation. Classical $\mathbb{R}^{n+1}$-valued solutions to the homogeneous linear equation $-\Delta \boldsymbol{u} = \mathbf{0}$ will be referred to as *harmonic functions*.

2.2. The Oscillation Functional and Scaling Properties.

The engine of our regularity theory relies on quantifying the local deviation of maps from constant states via a variational oscillation functional.

Definition 2.3 (L^p-Oscillation)**.** Let $U \subseteq \mathbb{R}^m$ be an open subset, $\boldsymbol{v} \in L^p(U)$ for $p \geqslant 1$, and $B \subseteq U$ a ball. The variational L^p-oscillation of $\boldsymbol{v}$ on B is:

$$\Psi_p(\boldsymbol{v}, B) := \inf_{\boldsymbol{c} \in \mathbb{R}^{n+1}} \fint_B |\boldsymbol{v}(y) - \boldsymbol{c}|^p \mathrm{d}y. \tag{2.3}$$

It is immediate that $\Psi_p(\boldsymbol{v}, B)$ is equivalent to the standard mean oscillation up to a dimensional constant: $\Psi_p(\boldsymbol{v}, B) \leqslant \fint_B |\boldsymbol{v} - \langle \boldsymbol{v} \rangle_B|^p \leqslant 2^p \Psi_p(\boldsymbol{v}, B)$. The functional Ψ_p obeys the following exact scaling and monotonicity rules, which drive our subsequent Campanato iterations.

Lemma 2.4 (Scaling and Monotonicity). *Let $\boldsymbol{v} : U \to \mathbb{R}^{n+1}$ be measurable, $p \geqslant 1$, and $B \subseteq U$. Then:*

i. The map $\boldsymbol{v} \mapsto \Psi_p^{1/p}(\boldsymbol{v}, B)$ is a seminorm on $L^p(B)$.
ii. For $B_r(x_0) \subseteq U$, the rescaled map $\boldsymbol{v}_{x_0,\lambda}(x) := \boldsymbol{v}(x_0 + \lambda x)$ defined on $B_{r/\lambda}$ satisfies:

$$\Psi_p\left(\boldsymbol{v}, B_r(x_0)\right) = \Psi_p\left(\boldsymbol{v}_{x_0,\lambda}, B_{r/\lambda}\right). \tag{2.4}$$

iii. For any radii $0 < r_1 \leqslant r_2$, the volume normalization imposes the geometric decay bound:

$$\Psi_p\left(\boldsymbol{v}, B_{r_1}\right) \leqslant \left(\frac{r_2}{r_1}\right)^m \Psi_p\left(\boldsymbol{v}, B_{r_2}\right). \tag{2.5}$$

When evaluating the oscillation of a uniformly bounded map (such as a map into a compact manifold), the minimization problem characterizing Ψ_p can be strictly localized, a fact that will prove technically useful.

Lemma 2.5 (Compactification of the Oscillation Infimum). *Let $E \subset \mathbb{R}^m$ be a domain of finite measure, and let $\boldsymbol{u} \in L^\infty(E, \mathbb{R}^{n+1})$. For any exponent $p \geqslant 1$, the infimum defining the L^p-oscillation can be restricted to the closed ball of radius $M := \|\boldsymbol{u}\|_{L^\infty(E)}$ without altering its value:*

$$\inf_{\boldsymbol{c} \in \mathbb{R}^{n+1}} \fint_E |\boldsymbol{u}(y) - \boldsymbol{c}|^p \mathrm{d}y = \inf_{\substack{\boldsymbol{c} \in \mathbb{R}^{n+1} \\ |\boldsymbol{c}| \leqslant M}} \fint_E |\boldsymbol{u}(y) - \boldsymbol{c}|^p \mathrm{d}y. \tag{2.6}$$

Proof. By scaling, it suffices to prove the case $M = 1$. The inequality "$\leqslant$" is trivial. For the reverse, let $\boldsymbol{c} \in \mathbb{R}^{n+1}$ with $|\boldsymbol{c}| > 1$, and define its radial projection $\boldsymbol{c}_* := \boldsymbol{c}/|\boldsymbol{c}| \in \mathbb{S}^n$. Because $|\boldsymbol{u}| \leqslant 1$ a.e., Cauchy–Schwarz ensures $\boldsymbol{u}(y) \cdot \boldsymbol{c} \leqslant |\boldsymbol{c}|$. We can then estimate the difference in squared distances:

$$\begin{aligned} |\boldsymbol{u}(y) - \boldsymbol{c}|^2 - |\boldsymbol{u}(y) - \boldsymbol{c}_*|^2 &= |\boldsymbol{c}|^2 - 1 - 2\boldsymbol{u}(y) \cdot \boldsymbol{c} \left(1 - \frac{1}{|\boldsymbol{c}|}\right) \\ &\geqslant |\boldsymbol{c}|^2 - 1 - 2|\boldsymbol{c}| \left(1 - \frac{1}{|\boldsymbol{c}|}\right) = (|\boldsymbol{c}| - 1)^2 > 0. \end{aligned}$$

Consequently, $|\boldsymbol{u}(y) - \boldsymbol{c}_*| < |\boldsymbol{u}(y) - \boldsymbol{c}|$ strictly pointwise. Integrating this strict inequality against the exponent p demonstrates that any constant vector outside the unit ball is suboptimal compared to its projection, concluding the proof. □

Standard PDE Estimates. Throughout the remainder of the paper, we will freely utilize classical interior linear estimates (specifically the global and local Calderón–Zygmund bounds for divergence-form operators, and standard interior Schauder estimates). We refer the reader to Giaquinta [Gia83, Chapters 5 and 7] for the precise statements and structural dependencies of these constants. Similarly, Campanato's classical characterization of $C^{0,\alpha}$ spaces will be invoked directly [Gia83, Theorem 1.3].

3. The first abstract regularity principle and the proof of Theorem 1.1

The main goal of this section is to establish the result already stated in the introduction, namely:

Theorem 1.1. *Any weakly almost harmonic map $\boldsymbol{u} \in H^1(B_1, \mathbb{S}^n)$ from the unit two-dimensional disk $B_1 \subseteq \mathbb{R}^2$ to a sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, solving the equation*

$$-\Delta \boldsymbol{u} = |\nabla \boldsymbol{u}|^2 \boldsymbol{u} + \boldsymbol{f}, \tag{3.1}$$

with a source term $\boldsymbol{f} \in L^q(B_1, \mathbb{R}^{n+1})$ for some $q > 1$, is locally Hölder continuous. Specifically, there exists an exponent $\eta \in (0,1)$ such that $\boldsymbol{u} \in C^{0,\eta}_{\mathrm{loc}}(B_1, \mathbb{R}^{n+1})$.

This result was first established by Hélein in the source-free case and later extended by Müller and Schikorra [MS09]. Our proof follows the classical elliptic regularity strategy in the spirit of Campanato.

The proof of Theorem 1.1 will be presented in Subsection 4.4. To this end, we first introduce an abstract regularity principle (stated in Theorem 3.3) by defining an appropriate abstract class of configurations. We then formulate and establish a second (generalized) abstract regularity theorem, namely Theorem 4.5, in general base dimension $m \geqslant 2$, although the framework will ultimately be applied to the regularity theory of generalized systems in the two-dimensional setting to conclude the proof of Theorem 1.1.

3.1. The first abstract regularity principle.

We present an abstract regularity theory for maps in the Sobolev space $H^1(B_1, \mathbb{R}^N)$, where B_1 is the unit disk in $\mathbb{R}^2$. The framework cleanly separates the structural features of the underlying equation (such as scaling invariance) from the analytic ingredients, notably oscillation decay estimates, thereby yielding a unified approach to partial regularity in geometric analysis.

Let $\boldsymbol{\mathcal{S}} \subseteq H^1(B_1, \mathbb{R}^N) \times L^q(B_1, \mathbb{R}^N)$ be a class of *admissible pairs* $(\boldsymbol{u}, \boldsymbol{f})$ defined on the unit disk $B_1 \subset \mathbb{R}^2$, which obey the following two axioms:

Axiom 3.1 (Locality and Closure)**.** The class $\boldsymbol{\mathcal{S}}$ is closed under rescaling. Specifically, if $(\boldsymbol{u}, \boldsymbol{f}) \in \boldsymbol{\mathcal{S}}$, then for any ball $B_r(x_0) \subset B_1$, the rescaled triplet $(\boldsymbol{u}_{x_0,r}, \tilde{\boldsymbol{f}}_{x_0,r})$ defined by

$$\boldsymbol{u}_{x_0,r}(y) := \boldsymbol{u}(x_0 + ry), \qquad \tilde{\boldsymbol{f}}_{x_0,r}(y) := r^2 \boldsymbol{f}(x_0 + ry), \tag{3.2}$$

is still in $\boldsymbol{\mathcal{S}}$ over the unit ball $B_1 \subseteq \mathbb{R}^m$.

Axiom 3.2 (The Decay Property)**.** There exist structural constants $\theta, \gamma \in (0,1)$, $p \geqslant 1$, and $\kappa > 0$, such that for any pair $(\boldsymbol{u}, \boldsymbol{f}) \in \boldsymbol{\mathcal{S}}$, the oscillation strictly contracts at the interior scale:

$$\Psi_p(\boldsymbol{u}, B_\theta) \leqslant \gamma \cdot \Psi_p(\boldsymbol{u}, B_1) + \kappa \|\boldsymbol{f}\|^p_{L^q(B_1)}. \tag{3.3}$$

Throughout the sequel, we repeatedly exploit several standard scaling identities specific to the two-dimensional setting. By a direct change of variables, the Dirichlet energy is invariant under this rescaling:

$$\mathcal{E}(\boldsymbol{u}_{x_0,r}, B_1) = \mathcal{E}(\boldsymbol{u}, B_r(x_0)). \tag{3.4}$$

By contrast, the source term measured in L^q acquires the scaling factor $\delta := 2 - \frac{2}{q}$, which is strictly positive precisely when $q > 1$. More precisely,

$$\|\tilde{\boldsymbol{f}}_{x_0,r}\|_{L^q(B_1)} = r^\delta \|\boldsymbol{f}\|_{L^q(B_r(x_0))} \leqslant r^\delta \|\boldsymbol{f}\|_{L^q(B_1)}. \tag{3.5}$$

Motivating Example. Later in this work, we will apply the abstract framework by defining $\boldsymbol{\mathcal{S}}$ to be the class of almost harmonic maps on the unit disk $B_1 \subseteq \mathbb{R}^2$, whose

Dirichlet energy lies below a fixed critical threshold ε_*:

$$\boldsymbol{\mathcal{S}} := \Big\{(\boldsymbol{u},\boldsymbol{f}) \in H^1(B_1,\mathbb{S}^n)\times L^q(B_1,\mathbb{R}^{n+1}) \,:\, -\Delta\boldsymbol{u} = |\nabla\boldsymbol{u}|^2\boldsymbol{u}+\boldsymbol{f} \quad \text{and} \quad \mathcal{E}(\boldsymbol{u},B_1)<\varepsilon_*\Big\}. \tag{3.6}$$

Note that, we will formally set $N := n+1$.

Our first objective is to show that every map belonging to this two-dimensional admissible class is locally Hölder continuous. The key point is that, in dimension $m=2$, the initial mean oscillation on B_1 is naturally controlled by the scale-invariant H^1 Dirichlet energy through the Poincaré–Sobolev inequality. Consequently, no additional L^∞ is required in this setting.

Theorem 3.3 (Abstract Regularity Principle)**.** *Let* $\boldsymbol{\mathcal{S}} \subseteq H^1(B_1,\mathbb{R}^N)\times L^q(B_1,\mathbb{R}^N)$ *defined on* $B_1\subset\mathbb{R}^2$ *be a class of admissible pairs satisfying* **Axioms 3.1, 3.2***.*

If $(\boldsymbol{u},\boldsymbol{f})\in\boldsymbol{\mathcal{S}}$*, then* $\boldsymbol{u}$ *is locally Hölder continuous in* B_1*. Specifically,* $\boldsymbol{u}\in C^{0,\eta}_{\mathrm{loc}}(B_1)$ *for every Hölder exponent* η *satisfying :*

$$\eta < \min\left(\frac{\ln\gamma}{p\ln\theta}, 2-\frac{2}{q}\right). \tag{3.7}$$

For any compact subset $K \Subset B_1$*, the corresponding Hölder seminorm depends only on the chosen exponent* η*, the structural parameters* θ,γ,κ*, the exponent* p*, the initial energy* $\mathcal{E}(\boldsymbol{u},B_1)$*, the source norm* $\|\boldsymbol{f}\|_{L^q(B_1)}$*, and the distance* $\operatorname{dist}(K,\partial B_1)$*.*

The proof of Theorem 3.3 is presented in the next subsection. However, the same structural argument extends naturally to domains of arbitrary dimension $m\geqslant 2$, provided one imposes an additional analytic assumption. Indeed, when $m\geqslant 3$, the H^1-Dirichlet energy alone no longer controls the L^p-oscillation for arbitrarily large exponents p. This dimensional obstruction can be circumvented by restricting the admissible class to maps that are uniformly bounded in L^∞. Under this assumption, the initial mean oscillation is trivially controlled by the amplitude $\|\boldsymbol{u}\|_{L^\infty}$. More precisely, if $\boldsymbol{u}\in H^1(B_1,E)$, where $E\subset\mathbb{R}^N$ is compact, then

$$\Psi_p(\boldsymbol{u},B_1)\leqslant \operatorname{diam}(E)^p. \tag{3.8}$$

Consequently, one may avoid the dimension-dependent Sobolev embeddings used in the final part of the proof of Theorem 3.3 (see (3.25)). We omit the proof, as it is essentially identical to that of Theorem 3.3, modulo the modifications discussed above; see also the proof of the abstract regularity principle in arbitrary dimension, Theorem 4.5.

Theorem 3.4 (Abstract Regularity Principle in general dimension)**.** *Let* $B_1\subset\mathbb{R}^m$ *with* $m\geqslant 2$*. Let* $\boldsymbol{\mathcal{S}}\subseteq (H^1\cap L^\infty)(B_1,\mathbb{R}^N)\times L^q(B_1,\mathbb{R}^N)$ *be a class of admissible pairs satisfying* **Axioms 3.1, 3.2***, with the source integrability condition* $q>m/2$*.*

If $(\boldsymbol{u},\boldsymbol{f})\in\boldsymbol{\mathcal{S}}$*, then* $\boldsymbol{u}$ *is locally Hölder continuous in* B_1*. Specifically,* $\boldsymbol{u}\in C^{0,\eta}_{\mathrm{loc}}(B_1)$ *for every Hölder exponent* η *satisfying:*

$$\eta < \min\left(\frac{\ln\gamma}{p\ln\theta}, 2-\frac{m}{q}\right). \tag{3.9}$$

For any compact subset $K\Subset B_1$*, the corresponding Hölder seminorm depends only on the chosen exponent* η*, the structural parameters* θ,γ,κ*, the exponent* p*, the amplitude* $\|\boldsymbol{u}\|_{L^\infty(B_1)}$*, the initial energy* $\mathcal{E}(\boldsymbol{u},B_1)$*, the source norm* $\|\boldsymbol{f}\|_{L^q(B_1)}$*, and the distance* $\operatorname{dist}(K,\partial B_1)$*.*

Remark 3.5 (Optimality of the Hölder Exponent)*.* The two competing mechanisms defining this upper bound are the homogeneous decay rate $(\ln\gamma)/(p\ln\theta)$ from the decay axiom and the inhomogeneous contribution $(2 - m/q)$ coming from the L^q-integrability of the source via the m-dimensional scaling.

The upper bound for the Hölder exponent η in Theorem 3.4 is sharp. The inhomogeneous threshold $2 - m/q$ coincides exactly with the optimal Morrey–Campanato regularity for the linear Poisson equation driven by an L^q source. Furthermore, the strict inequality $\eta < \min\left(\frac{\ln\gamma}{p\ln\theta}, 2 - \frac{m}{q}\right)$ reflects an intrinsic analytic obstruction rather than a suboptimal estimate. In the resonant case, where the homogeneous decay rate matches the inhomogeneous source scaling, the iteration unavoidably accumulates logarithmic penalties of the form $|\ln r|$. Conceding an arbitrarily small ε-loss absorbs these factors, allowing the application of Campanato's theorem and yielding the supremum as a strict limit.

3.2. **Proof of Theorem 3.3.** For the proof of Theorem 3.3 we need to obtain a continuous decay rate from discrete scaling estimates. For that, a refined version of the standard algebraic iteration lemma is required. While similar results are common in the regularity literature (see, for example, Lemma 2.1 in Chapter III of Giaquinta [Gia83] or Lemma 7.3 in Giusti [Giu03]), they often absorb the iteration constant into a generic, unspecified parameter. For our purposes, it is essential to explicitly track how this constant depends on the interplay between the homogeneous and inhomogeneous decay rates. These rates are encoded in the parameters γ and κ, as well as by the scaling factor associated with the source norm $\|\boldsymbol{f}\|^p_{L^q(B_1)}$ introduced in Axiom **3.2**.

Lemma 3.6 (Algebraic Iteration Lemma). *Let $\psi : (0, r_0] \to [0, \infty)$ be a non-decreasing function. Suppose there exist constants $\theta \in (0, 1)$, $\gamma \in (0, 1)$, $b \geqslant 0$, and $\beta > 0$ such that*

$$\psi(\theta r) \leqslant \gamma\psi(r) + br^\beta \tag{3.10}$$

holds for all $r \in (0, r_0]$. Let $\alpha = (\ln\gamma)/(\ln\theta)$ (so that $\gamma = \theta^\alpha$) and assume $\alpha < \beta$. Then there exists a structural constant

$$c := \frac{1}{\theta^{2\alpha}(1 - \theta^{\beta-\alpha})} > 0, \tag{3.11}$$

depending only on θ, α, and β, such that

$$\psi(r) \leqslant c\left[\frac{\psi(r_0)}{r_0^\alpha} + br_0^{\beta-\alpha}\right] r^\alpha \tag{3.12}$$

for all $r \in (0, r_0]$.

The proof of the Algebraic Iteration Lemma 3.6 is postponed to the end of the section.

Proof of Theorem 3.3. For convenience, we divide the proof into five steps.

Step 1: The Master Iteration Inequality. Let $(\boldsymbol{u}, \boldsymbol{f}) \in \boldsymbol{\mathcal{S}}$. Our goal is to verify the Campanato condition for $\boldsymbol{u}$ uniformly on $B_{1/2}$. Fix an arbitrary center point $x_0 \in B_{1/2}$ and a macroscopic starting radius $r_0 := 1/2$ so that $B_r(x_0) \subset B_1$ for every $x_0 \in B_{1/2}$ and every $r \in (0, r_0]$. For any radius $r \in (0, r_0]$, we set $\psi(r) := r^2\Psi_p(\boldsymbol{u}, B_r(x_0))$. We wish to establish an algebraic iteration inequality for ψ.

By Axiom **3.1**, the rescaled pair $(\boldsymbol{u}_{x_0,r}, \tilde{\boldsymbol{f}}_{x_0,r})$ belongs to $\boldsymbol{\mathcal{S}}$. Applying the contractive decay property (Axiom **3.2**) to this rescaled pair yields:

$$\Psi_p(\boldsymbol{u}_{x_0,r}, B_\theta) \leqslant \gamma \cdot \Psi_p(\boldsymbol{u}_{x_0,r}, B_1) + \kappa\|\tilde{\boldsymbol{f}}_{x_0,r}\|^p_{L^q(B_1)}. \tag{3.13}$$

We translate the oscillations back to the physical coordinates on B_1. Using the scaling identity (2.4) for the oscillation (mean integral), we get:

$$\begin{aligned}\text{LHS:}\quad \Psi_p(\boldsymbol{u}_{x_0,r}, B_\theta) &= \Psi_p(\boldsymbol{u}, B_{\theta r}(x_0)) = (\theta r)^{-2}\,\psi(\theta r), &(3.14)\\ \text{RHS:}\quad \Psi_p(\boldsymbol{u}_{x_0,r_0}, B_1) &= \Psi_p(\boldsymbol{u}, B_r(x_0)) = r^{-2}\psi(r). &(3.15)\end{aligned}$$

Therefore, (3.13) can be rewritten as

$$\psi(\theta r) \leqslant \theta^2\gamma\cdot\psi(r) + \kappa\theta^2 r^2\|\tilde{\boldsymbol{f}}_{x_0,r}\|^p_{L^q(B_1)}. \tag{3.16}$$

For the source term, we observe that $\|\tilde{\boldsymbol{f}}_{x_0,r}\|^q_{L^q(B_1)} = r^{2q-2}\|\boldsymbol{f}\|^q_{L^q(B_r(x_0))} \leqslant r^{2q-2}\|\boldsymbol{f}\|^q_{L^q(B_1)}$. Hence,

$$\|\tilde{\boldsymbol{f}}_{x_0,r}\|^p_{L^q(B_1)} \leqslant r^{p(2-2/q)}\|\boldsymbol{f}\|^p_{L^q(B_1)}. \tag{3.17}$$

Substituting the previous bound into the decay property (3.16), we obtain the master iteration inequality:

$$\psi(\theta r) \leqslant \theta^2\gamma\cdot\psi(r) + br^\beta \quad \text{for all } r\in(0,r_0], \tag{3.18}$$

where $\beta := 2 + p\,(2-2/q)$ and $b := \kappa\theta^2\|\boldsymbol{f}\|^p_{L^q(B_1)}$.

Step 2: Continuous Algebraic Decay via the Iteration Lemma. To extract a continuous algebraic decay rate from this discrete bound, we wish to apply the algebraic iteration lemma (Lemma 3.6). This is possible because ψ is non-decreasing. The natural algebraic decay exponent associated with the homogeneous term is given by $\ln(\theta^2\gamma)/(\ln\theta)$. The iteration lemma requires this exponent to be strictly less than the source exponent $\beta = 2+p\,(2-2/q)$. To achieve the sharpest possible decay while satisfying this strict inequality, we fix an arbitrarily small $\varepsilon > 0$ and consider the slightly less sharp master iteration inequality

$$\psi(\theta r) \leqslant \theta^2\gamma_\varepsilon\cdot\psi(r) + br^\beta \quad \text{for all } r\in(0,r_0], \tag{3.19}$$

where we define $\gamma_\varepsilon := \max(\gamma, \theta^{\beta-2-\varepsilon})$. Indeed, with this choice of γ_ε we have $\gamma_\varepsilon\in(0,1)$ because $\gamma,\theta\in(0,1)$ and we can choose ε small enough such that $\beta-2-\varepsilon>0$. We now define our operational exponent $\alpha_\varepsilon := \ln(\theta^2\gamma_\varepsilon)/(\ln\theta)$ so that $\theta^2\gamma_\varepsilon = \theta^{\alpha_\varepsilon}$. By construction, we have

$$\frac{\ln\gamma_\varepsilon}{\ln\theta} = \frac{|\ln\gamma_\varepsilon|}{|\ln\theta|} = \frac{\min(|\ln\gamma_\varepsilon|, (\beta-2-\varepsilon)|\ln\theta|)}{|\ln\theta|} = \min\left(\frac{\ln\gamma_\varepsilon}{\ln\theta}, \beta-2-\varepsilon\right). \tag{3.20}$$

Therefore,

$$\alpha_\varepsilon := 2 + \frac{|\ln\gamma_\varepsilon|}{|\ln\theta|} = 2 + \min\left(\frac{\ln\gamma}{\ln\theta}, \beta-2-\varepsilon\right). \tag{3.21}$$

In particular, $\beta-\alpha_\varepsilon = \beta-2-\min\left(\frac{\ln\gamma}{\ln\theta},\beta-2-\varepsilon\right) \geqslant (\beta-2)-(\beta-2-\varepsilon) = \varepsilon > 0$.

Because $\alpha_\varepsilon < \beta$, the iteration lemma (Lemma 3.6) applies, guaranteeing the existence of a purely structural constant $c_\varepsilon > 0$ such that for all $r\in(0,r_0]$:

$$\psi\,(r) \leqslant c_\varepsilon\left[\frac{\psi\,(r_0)}{r_0^{\alpha_\varepsilon}} + br_0^{\beta-\alpha_\varepsilon}\right] r^{\alpha_\varepsilon}. \tag{3.22}$$

Dividing both sides by r^2, we recover the uniform algebraic decay for the mean oscillation:

$$\Psi_p(\boldsymbol{u}, B_r(x_0)) \leqslant c_\varepsilon\left[\frac{\Psi_p(\boldsymbol{u}, B_{r_0}(x_0))}{r_0^{\alpha_\varepsilon-2}} + br_0^{\beta-\alpha_\varepsilon}\right] r^{\alpha_\varepsilon-2}, \tag{3.23}$$

where, we recall, $b := \kappa\theta^2\|\boldsymbol{f}\|^p_{L^q(B_1)}$.

Step 3: Uniform Bound via the Poincaré-Sobolev Inequality. To conclude the proof, we must uniformly bound the right-hand side of (3.23) independent of x_0. We invoke the Poincaré-Sobolev inequality on $B_{r_0}(x_0)$. Proposition applied with $q = 2$ and the integrability exponent p (any finite real number, since $q = m$) yields the scale-invariant estimate

$$\left(\fint_{B_{r_0}(x_0)} |\boldsymbol{u} - \langle \boldsymbol{u} \rangle_{B_{r_0}(x_0)}|^p\right)^{1/p} \leqslant K_S \left(\int_{B_{r_0}(x_0)} |\nabla \boldsymbol{u}|^2\right)^{1/2}. \tag{3.24}$$

Note that the integral on the right is the total energy (not the mean), which is scale-invariant in 2D. That is, K_S does not depend on the radius r_0. Thus, applying (3.24) to our specific ball $B_{r_0}(x_0)$ (where $r_0 = 1/2$):

$$\Psi_p(\boldsymbol{u}, B_{r_0}(x_0)) \leqslant \fint_{B_{r_0}(x_0)} |\boldsymbol{u} - \langle \boldsymbol{u} \rangle|^p \leqslant K_S^p \left(\int_{B_{r_0}(x_0)} |\nabla \boldsymbol{u}|^2\right)^{p/2} \leqslant K_S^p \, 2^{p/2} \left(\mathcal{E}(\boldsymbol{u}, B_{r_0}(x_0))\right)^{p/2}. \tag{3.25}$$

Using the monotonicity of the energy ($\mathcal{E}(\boldsymbol{u}, B_{r_0}(x_0)) \leqslant \mathcal{E}(\boldsymbol{u}, B_1)$), we obtain the uniform bound:

$$\Psi_p(\boldsymbol{u}, B_{r_0}(x_0)) \leqslant C_S \mathcal{E}(\boldsymbol{u}, B_1)^{p/2}, \tag{3.26}$$

where $C_S := K_S^p \, 2^{p/2}$ is a structural constant independent of x_0. Substituting this back into (3.23), we obtain the estimate:

$$\Psi_p(\boldsymbol{u}, B_r(x_0)) \leqslant M r^{\alpha_\varepsilon - 2}, \tag{3.27}$$

where M is a universal constant (independent of $x_0 \in B_{1/2}$ and $r \in (0, r_0]$) and

$$\alpha_\varepsilon - 2 = \min\left(\frac{\ln \gamma}{\ln \theta}, \beta - 2 - \varepsilon\right) = \min\left(\frac{\ln \gamma}{\ln \theta}, p\,(2 - 2/q) - \varepsilon\right). \tag{3.28}$$

Finally, observe that since $q < 2$, we have $(2 - 2/q) < 1$. Consequently, for sufficiently small $\varepsilon > 0$, $\alpha_\varepsilon - 2 < p$.

Step 4: The Campanato Condition and Hölder Continuity on $B_{1/2}$. Since inequality (3.27) holds uniformly for every $x_0 \in B_{1/2}$, it satisfies precisely the Campanato condition with exponent $(\alpha_\varepsilon - 2)$. Therefore, $\boldsymbol{u} \in C^{0,\eta_\varepsilon}(B_{1/2})$ with $\eta_\varepsilon = (\alpha_\varepsilon - 2)/p$. Substituting our exact choice for $(\alpha_\varepsilon - 2)$ directly yields:

$$\eta_\varepsilon = \min\left(\frac{\ln \gamma}{p \ln \theta}, 2 - \frac{2}{q} - \frac{\varepsilon}{p}\right), \tag{3.29}$$

matching the formula in the theorem statement as soon as we pass to the limit for $\varepsilon \to 0$.

The Hölder seminorm $[\boldsymbol{u}]_{C^{0,\eta_\varepsilon}(B_{1/2})}$ depends only on $\theta, \gamma, \kappa, p, \mathcal{E}(\boldsymbol{u}, B_1)$, and $\|\boldsymbol{f}\|_{L^q(B_1)}$ through the constants appearing in the iteration lemma and the Poincaré-Sobolev constant K_S.

Step 5: Extension to arbitrary compact subsets of B_1. To conclude the proof, we extend this regularity from $B_{1/2}$ to the entirety of B_1 and explicitly track the seminorm dependency. Let $K \subset B_1$ be an arbitrary compact subset, and let $R := \operatorname{dist}(K, \partial B_1) > 0$ be its distance to the boundary. For any point $x_0 \in K$, the ball $B_R(x_0)$ is strictly contained within B_1. We define the rescaled pair $(\boldsymbol{v}, \boldsymbol{g}) := (\boldsymbol{u}_{x_0,R}, \tilde{\boldsymbol{f}}_{x_0,R})$ on the unit ball B_1. By Axiom **3.1**, this rescaled pair belongs to $\boldsymbol{\mathcal{S}}$.

Applying the exact uniform bound established in the previous steps to this rescaled pair, we know that $\boldsymbol{v} \in C^{0,\eta}(B_{1/2})$. Furthermore, its Hölder seminorm $[\boldsymbol{v}]_{C^{0,\eta}(B_{1/2})}$ is bounded uniformly by the structural constants, the energy $\mathcal{E}(\boldsymbol{v}, B_1)$, and the source norm $\|\boldsymbol{g}\|_{L^q(B_1)}$. By the monotonicity of the energy and the 2D source scaling, these are universally bounded by the initial global quantities: $\mathcal{E}(\boldsymbol{v}, B_1) \leqslant \mathcal{E}(\boldsymbol{u}, B_1)$ and $\|\boldsymbol{g}\|_{L^q(B_1)} \leqslant R^{2-2/q}\|\boldsymbol{f}\|_{L^q(B_1)} \leqslant \|\boldsymbol{f}\|_{L^q(B_1)}$.

Finally, we translate this back to the original map $\boldsymbol{u}$ using the equivalence of scaled Hölder seminorms: the restriction of $\boldsymbol{v}$ to $B_{1/2}$ corresponds directly to the restriction of $\boldsymbol{u}$ to $B_{R/2}(x_0)$. The seminorms obey the scaling identity

$$[\boldsymbol{u}]_{C^{0,\eta_\varepsilon}(B_{R/2}(x_0))} = R^{-\eta_\varepsilon}[\boldsymbol{v}]_{C^{0,\eta}(B_{1/2})}. \tag{3.30}$$

Because x_0 was an arbitrary point in K, a standard finite covering argument guarantees that $\boldsymbol{u} \in C^{0,\eta_\varepsilon}(K)$. The explicit appearance of the scaling factor $R^{-\eta_\varepsilon} = \mathrm{dist}(K, \partial B_1)^{-\eta_\varepsilon}$ dictates precisely why the final Hölder seminorm on K depends on the distance to the boundary.

Finally, since $\varepsilon > 0$ can be chosen arbitrarily small, this guarantees that $\boldsymbol{u} \in C^{0,\eta}(K)$ for every exponent η strictly less than the critical threshold, concluding the proof. $\square$

Proof of the Algebraic Iteration Lemma 3.6. We first derive the bound along the discrete sequence $r_j = \theta^j r_0$ for $j \in \mathbb{N}_0$. For $j = 1$ the given inequality yields $\psi(\theta r_0) \leqslant \gamma\psi(r_0) + br_0^\beta$. Iterating, using induction on $j \geqslant 1$, gives

$$\psi(\theta^j r_0) \leqslant \gamma^j \psi(r_0) + b\sum_{i=0}^{j-1} \gamma^i (\theta^{j-1-i} r_0)^\beta. \tag{3.31}$$

Substituting $\gamma = \theta^\alpha$ and rewriting the general term in the sum, $\gamma^i(\theta^{j-1-i})^\beta = \theta^{\alpha i} \cdot \theta^{\beta(j-1-i)} = \theta^{\alpha(j-1)} \cdot \theta^{(\beta-\alpha)(j-1-i)}$, we obtain

$$\begin{aligned}\psi(\theta^j r_0) &\leqslant \theta^{\alpha j}\psi(r_0) + br_0^\beta \theta^{\alpha(j-1)} \sum_{i=0}^{j-1} \theta^{(\beta-\alpha)(j-1-i)} \\ &= \theta^{\alpha j}\left[\psi(r_0) + \frac{br_0^\beta}{\theta^\alpha}\sum_{k=0}^{j-1}(\theta^{\beta-\alpha})^k\right],\end{aligned}$$

where for the last equality we re-indexed the sum by setting $k := j - 1 - i$. Since $\beta > \alpha$ and $\theta \in (0, 1)$, the ratio $\theta^{\beta-\alpha} < 1$. Thus the finite geometric sum is bounded by the sum of the geometric series:

$$\psi(\theta^j r_0) \leqslant \theta^{\alpha j}\left[\psi(r_0) + \frac{br_0^\beta}{\theta^\alpha(1-\theta^{\beta-\alpha})}\right]. \tag{3.32}$$

To extend the estimate to arbitrary $r \in (0, r_0]$, choose the unique integer $j \geqslant 0$ such that $\theta^{j+1} r_0 < r \leqslant \theta^j r_0$. Since ψ is non-decreasing and $r \leqslant \theta^j r_0$, we have $\psi(r) \leqslant \psi(\theta^j r_0)$. From the lower bound on r we deduce $\theta^j < \theta^{-1}(r/r_0)$. Raising to the power $\alpha > 0$ (which preserves the inequality) gives

$$\theta^{\alpha j} < \frac{1}{\theta^\alpha}\left(\frac{r}{r_0}\right)^\alpha. \tag{3.33}$$

Inserting this into the discrete bound produces

$$\psi(r) \leqslant \psi(\theta^j r_0) \overset{(3.32)}{\leqslant} \theta^{\alpha j}\left[\psi(r_0) + \frac{br_0^\beta}{\theta^\alpha(1-\theta^{\beta-\alpha})}\right] \overset{(3.33)}{\leqslant} \frac{1}{\theta^\alpha}\left(\frac{r}{r_0}\right)^\alpha\left[\psi(r_0) + \frac{br_0^\beta}{\theta^\alpha(1-\theta^{\beta-\alpha})}\right]. \tag{3.34}$$

Distributing terms,

$$\psi(r) \leqslant \frac{1}{\theta^\alpha}\frac{\psi(r_0)}{r_0^\alpha}r^\alpha + \frac{1}{\theta^{2\alpha}(1-\theta^{\beta-\alpha})}br_0^{\beta-\alpha}r^\alpha. \tag{3.35}$$

Because $\theta \in (0,1)$ and $\alpha > 0$, we have $1/\theta^{\alpha} \leqslant 1/\theta^{2\alpha}$ and $1/\theta^{2\alpha} < 1/[\theta^{2\alpha}(1-\theta^{\beta-\alpha})]$ (since $1-\theta^{\beta-\alpha} < 1$). Therefore both coefficients are bounded by the structural constant $c := 1/[\theta^{2\alpha}(1-\theta^{\beta-\alpha})]$, which completes the proof. □

3.3. **Hölder regularity of almost harmonic maps: proof of Theorem 1.1.** The proof of Theorem 1.1 hinges on the following proposition, which demonstrates that small Dirichlet energy on the unit ball, coupled with a sufficiently integrable source term, implies a contraction of the L^p-oscillation at a smaller scale. Specifically, the content of the next result is to prove that the class of almost harmonic maps satisfies the epsilon-decay property formulated in Axiom 3.2.

Lemma 3.7 (Decay Property). *For any integrability exponent $q \in (1,2)$, we denote by $p := 2q/(2-q)$ the corresponding oscillation exponent. There exist structural constants $\theta \in (0,1)$, $\gamma \in (0,1)$, $\kappa > 0$, and an energy threshold $\varepsilon_* > 0$ such that for an arbitrary almost harmonic map $\boldsymbol{u} \in H^1(B_1, \mathbb{S}^n)$ defined on the unit ball $B_1 \subset \mathbb{R}^2$ with associated source term $\boldsymbol{f} \in L^q(B_1, \mathbb{R}^{n+1})$:*

$$\text{if} \quad \mathcal{E}(\boldsymbol{u}, B_1) < \varepsilon_*, \quad \text{then} \quad \Psi_p(\boldsymbol{u}, B_\theta) \leqslant \gamma \cdot \Psi_p(\boldsymbol{u}, B_1) + \kappa \|\boldsymbol{f}\|^p_{L^q(B_1)}. \tag{3.36}$$

In fact, one can always take $\gamma = 1/2$.

Remark 3.8 (The L^2 Integrability Ceiling). Let q_{src} denote the physical integrability of the source term $\boldsymbol{f}$. As B_1 is bounded, we may without loss of generality restrict our analysis to an operational exponent $1 < q < 2$, even when $q_{\mathrm{src}} \geqslant 2$. This reduction is not merely a matter of convenience: it is a structural necessity imposed by the critical quadratic nonlinearity $|\nabla \boldsymbol{u}|^2 \boldsymbol{u}$. Specifically, recasting this term as $\mathbf{div}(\boldsymbol{\Omega u})$ with $\boldsymbol{\Omega} \in L^2$ imposes a strict analytic threshold: estimating this flux in L^q via Hölder's inequality requires measuring the oscillation of $\boldsymbol{u}$ in L^p, which couples the exponents via $1/q = 1/2 + 1/p$. Simultaneously, closing the Campanato iteration via the Sobolev–Poincaré inequality dictates that p must equal the two-dimensional Sobolev conjugate $q^* = 2q/(2-q)$. This algebraic rigidity fundamentally limits the operational scheme to $q < 2$. Because the oscillation exponent $p \to \infty$ as $q \to 2^-$, optimal Hölder regularity is achieved by maximizing q. If $q_{\mathrm{src}} < 2$, the optimal choice is exactly $q = q_{\mathrm{src}}$. Conversely, if $q_{\mathrm{src}} \geqslant 2$, any source integrability beyond L^2 is analytically inaccessible to this iteration architecture; optimal regularity is instead captured by taking q arbitrarily close to 2. We therefore unify our subsequent analysis under the assumption that $\boldsymbol{f} \in L^q(B_1)$ for a fixed exponent $1 < q < 2$.

Remark 3.9 (Interdependence of Structural Constants). The parameters θ, γ, κ, and ε_* introduced in Lemma 3.7 are strictly interdependent. The constructive proof explicitly fixes $\gamma = 1/2$ and, via Hölder's inequality, rigidly locks the exponent to $p = 2q/(2-q)$. This algebraic coupling determines the scaling factor $\theta \equiv \theta(C_{m,q}, p)$, which ultimately dictates the critical energy threshold $\varepsilon_* = \varepsilon_*(\theta)$.

Remark 3.10 (The Structural Coincidence of Dimension Two). The success of the fundamental decay estimate (3.90) hinges on a structural coincidence specific to dimension $m = 2$. To close the Campanato iteration, the integrability gained via the Sobolev embedding must perfectly match the integrability demanded by Hölder's inequality. Specifically, bounding the error $\boldsymbol{w} := \boldsymbol{u} - \boldsymbol{h}$ via the Sobolev embedding $W^{1,q} \hookrightarrow L^{p_S}$ yields the dimensional relation $1/p_S = 1/q - 1/m$. Conversely, controlling the right-hand side $\mathbf{div}(\boldsymbol{\Omega}(\boldsymbol{u} - \boldsymbol{c}))$ in L^q given the fixed energy density $\boldsymbol{\Omega} \in L^2$ requires measuring the deviation in L^p such that $1/q = 1/2 + 1/p$. The factor of 2 in this latter relation originates entirely from the natural energy space and is strictly independent of the spatial dimension. For the iteration to close, the available Sobolev output must serve as the required Hölder input ($p_S = p$). Equating these relations yields $1/q - 1/m = 1/q - 1/2$, forcing $m = 2$. Thus, dimension two is the unique setting where the regularizing effect of the Laplacian exactly balances the integrability loss induced by the critical quadratic nonlinearity. We note that a similar observation regarding this critical algebraic balance was previously highlighted in [CWY99a] in their study of harmonic maps.

We postpone the proof of Lemma 3.7 to the next section. Here instead, we show how we can apply the abstract regularity theorem (Theorem 3.3) to infer the local Hölder regularity of weakly harmonic maps in dimension two stated in Theorem 1.1.

Proof of Theorem 1.1. We apply the abstract regularity framework. Let ε_* be the critical energy threshold provided by Lemma 3.7. We define the admissible class $\boldsymbol{\mathcal{S}}$ as the set of all $\mathbb{S}^n$-valued almost harmonic maps whose Dirichlet energy respects this threshold:

$$\boldsymbol{\mathcal{S}} = \Big\{ (\boldsymbol{u}, \boldsymbol{f}) \in H^1(B_1, \mathbb{S}^n) \times L^q(B_1, \mathbb{R}^{n+1}) \,:\, -\Delta \boldsymbol{u} = |\nabla \boldsymbol{u}|^2 \boldsymbol{u} + \boldsymbol{f} \quad \text{and} \quad \mathcal{E}(\boldsymbol{u}, B_1) < \varepsilon_* \Big\}. \tag{3.37}$$

We need to verify that this class satisfies Axiom 3.1 and Axiom 3.2.

Step 1: Verification of Axiom 3.1 (closure under rescaling). Let $(\boldsymbol{u}, \boldsymbol{f}) \in \boldsymbol{\mathcal{S}}$ and let $B_r(x_0) \subset B_1$. Consider the rescaled pair defined for $x \in B_1$ by $\boldsymbol{u}_{x_0,r}(x) := \boldsymbol{u}(x_0 + rx)$ and $\tilde{\boldsymbol{f}}_{x_0,r}(x) := r^2 \boldsymbol{f}(x_0 + rx)$.

- *Constraint:* Since $\boldsymbol{u}$ takes values in $\mathbb{S}^n$ almost everywhere, clearly $\boldsymbol{u}_{x_0,r}$ also takes values in $\mathbb{S}^n$ almost everywhere.
- *Equation:* In dimension $m = 2$, the Laplacian scales as $\Delta \boldsymbol{u}_{x_0,r}(y) = r^2 \Delta \boldsymbol{u}\,(x_0 + ry)$. Similarly, the quadratic term scales as $|\nabla \boldsymbol{u}_{x_0,r}(y)|^2 = r^2 |\nabla \boldsymbol{u}\,(x_0 + ry)\,|^2$. Thus, multiplying the original equation evaluated at $x_0 + rx$ by r^2, we obtain:
$$-\Delta \boldsymbol{u}_{x_0,r} = |\nabla \boldsymbol{u}_{x_0,r}|^2 \boldsymbol{u}_{x_0,r} + \tilde{\boldsymbol{f}}_{x_0,r}. \tag{3.38}$$
Thus, the structural equation is invariant.
- *Energy Threshold:* Because we are in dimension two, the Dirichlet energy is invariant under this rescaling. By the monotonicity of the energy integral over subdomains, we have:
$$\mathcal{E}(\boldsymbol{u}_{x_0,r}, B_1) = \mathcal{E}(\boldsymbol{u}, B_r(x_0)) \leqslant \mathcal{E}(\boldsymbol{u}, B_1) < \varepsilon_*. \tag{3.39}$$

Thus, the rescaled pair satisfies all structural conditions and preserves the strict energy bound, ensuring $(\boldsymbol{u}_{x_0,r}, \tilde{\boldsymbol{f}}_{x_0,r}) \in \boldsymbol{\mathcal{S}}$.

Step 2: Verification of Axiom 3.2 (the decay property). Let $(\boldsymbol{u}, \boldsymbol{f}) \in \boldsymbol{\mathcal{S}}$. By the explicit definition of our class $\boldsymbol{\mathcal{S}}$, the pair satisfies the global bound $\mathcal{E}(\boldsymbol{u}, B_1) < \varepsilon_*$. Therefore, the hypothesis of Lemma 3.7 is met, which immediately yields:

$$\Psi_p(\boldsymbol{u}, B_\theta) \leqslant \gamma \cdot \Psi_p(\boldsymbol{u}, B_1) + \kappa \|\boldsymbol{f}\|^p_{L^q(B_1)}. \tag{3.40}$$

This guarantees Axiom 3.2 is satisfied for all elements in $\boldsymbol{\mathcal{S}}$.

Step 3: Conclusion and Removal of the Small-Energy Hypothesis. Having verified the two axioms, the abstract regularity principle (Theorem 3.3) dictates that any map $\boldsymbol{u}$ belonging to a pair in $\boldsymbol{\mathcal{S}}$ is locally Hölder continuous in B_1.

To conclude the proof of the main theorem, we must explain why this regularity holds for *any* weakly almost harmonic map, not just those with small initial energy. Let $\boldsymbol{u} \in H^1(B_1, \mathbb{S}^n)$ be an arbitrary almost harmonic map with source $\boldsymbol{f} \in L^q(B_1, \mathbb{R}^{n+1})$. This map $\boldsymbol{u}$ might have a large total Dirichlet energy, meaning $(\boldsymbol{u}, \boldsymbol{f}) \notin \boldsymbol{\mathcal{S}}$.

However, let $x_0 \in B_1$ be an arbitrary point, and let $R := 1-|x_0| > 0$ denote its distance to the boundary. Because $\nabla \boldsymbol{u} \in L^2(B_1)$, by the absolute continuity of the Lebesgue integral we can always find a sufficiently small radius $r \in (0, R)$ such that $\mathcal{E}(\boldsymbol{u}, B_r(x_0)) < \varepsilon_*$. If we now construct the rescaled pair $(\boldsymbol{u}_{x_0,r}, \tilde{\boldsymbol{f}}_{x_0,r})$, it is defined on the unit ball B_1, solves the required equation, and satisfies the small-energy threshold $\mathcal{E}(\boldsymbol{u}_{x_0,r}, B_1) < \varepsilon_*$. Consequently, this rescaled pair belongs to our class $\boldsymbol{\mathcal{S}}$. By our abstract theorem, $\boldsymbol{u}_{x_0,r}$ is locally Hölder continuous in B_1. Scaling back to the original spatial coordinates, this implies that the original map $\boldsymbol{u}$ is Hölder continuous in the neighborhood $B_{r/2}(x_0)$. Since the point $x_0 \in B_1$ was chosen arbitrarily, the map $\boldsymbol{u}$ is locally Hölder continuous everywhere in B_1, completing the proof. □

3.4. **Proof of the Fundamental Lemma 3.7.** The rest of these subsection provide the complete proof of the fundamental decay estimate stated in Lemma 3.7. The core objective is to demonstrate that if the Dirichlet energy of an almost harmonic map is sufficiently small on a given ball, its local oscillation decays geometrically as the observation radius shrinks, thus triggering the Campanato regularity iteration. The architecture of the proof is built upon three main analytical milestones:

(1) *Algebraic Restructuring* (*Hélein's Trick*)*:* We introduce a generalized wedge product to define an antisymmetric potential. This allows us to recast the critical quadratic nonlinearity of the almost harmonic map equation into a divergence form.
(2) *Harmonic Approximation:* We compare the almost harmonic map to a carefully chosen harmonic extension, exploiting standard interior estimates for the Poisson integral.
(3) *Reverse Sobolev Estimates:* We derive a bound that controls the L^q norm of the gradient of the approximation error using the L^p oscillation of the map itself. Balancing these competing terms ultimately yields the required geometric decay.

Remark 3.11 (On the use of the symbol m)*.* Before detailing the functional setup, we establish a convention regarding our dimensional notation. Although the Hölder regularity results of this paper are inherently specific to spatial dimension equal to two, we will systematically retain the algebraic symbol m to denote the dimension of the base domain B_1. This notational choice is deliberate: it allows the reader to distinguish the occurrences of the number 2 that originate from the geometry of the spatial domain (e.g., the volume scaling r^m and the Sobolev embedding exponents) from the occurrences of the number 2 that natively arise from the L^2 Lebesgue integrability of the Dirichlet energy. As highlighted at the end of this section, it is precisely the algebraic collision of these two distinct sources that yields the exact exponent match required to close the regularity loop in two dimensions.

Notation and Algebraic Setup. We begin by establishing our notational conventions. Although the underlying linear algebra is elementary, the specific notation for operations such as the wedge product between vectors and matrices of vector fields is not standardized across the literature. To prevent any ambiguity and ensure that our subsequent algebraic manipulations remain completely transparent, we explicitly define these operations below.

Definition 3.12 (Convention on gradient and divergence)**.** Let $\boldsymbol{u} : B_1 \subset \mathbb{R}^m \to \mathbb{R}^{n+1}$. We adopt the convention that the gradient $\nabla \boldsymbol{u}$ is the transpose of the Jacobian matrix. Thus, $\nabla \boldsymbol{u}$ is an $m \times (n+1)$ matrix whose columns consist of the gradients of the scalar components of $\boldsymbol{u}$:

$$\nabla \boldsymbol{u} := \begin{pmatrix} | & & | \\ \nabla u^1 & \dots & \nabla u^{n+1} \\ | & & | \end{pmatrix}. \tag{3.41}$$

Consequently, the notation ∇u^β refers to the β-th column of this matrix, which is a vector in $\mathbb{R}^m$. The squared norm $|\nabla \boldsymbol{u}|^2$ is the Frobenius norm, given by $|\nabla \boldsymbol{u}|^2 = \sum_{\beta=1}^{n+1} |\nabla u^\beta|^2$.

For a standard vector field $\boldsymbol{v} : \mathbb{R}^m \to \mathbb{R}$, we denote its classical scalar divergence by $\operatorname{div} \boldsymbol{v}$. We extend this operator componentwise using the bold notation $\mathbf{div}$ for higher-order tensor fields. Specifically, if $\boldsymbol{\omega}$ is a vector of $n+1$ vector fields (i.e., an element of $\mathbb{R}^{n+1} \otimes \mathbb{R}^m$), $\mathbf{div}\, \boldsymbol{\omega}$ is the vector in $\mathbb{R}^{n+1}$ whose entries are the classical divergences of the individual vector fields. Similarly, if $\boldsymbol{\Omega}$ is an $(n+1) \times (n+1)$ matrix whose entries are vector fields in $\mathbb{R}^m$ (an element of $\mathbb{R}^{(n+1)\times(n+1)} \otimes \mathbb{R}^m$), then $\mathbf{div}\, \boldsymbol{\Omega}$ is the $(n+1) \times (n+1)$ matrix of their respective divergences.

Under these conventions, by identifying $\nabla \boldsymbol{u}$ as an element of $\mathbb{R}^{n+1} \otimes \mathbb{R}^m$, we naturally recover the classical componentwise relation $\mathbf{div}(\nabla \boldsymbol{u}) = \Delta \boldsymbol{u}$.

Definition 3.13 (The generalized wedge product)**.** For any two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^{n+1}$, their exterior product $\boldsymbol{v} \wedge \boldsymbol{w}$ is canonically identified with the skew-symmetric matrix (or bivector) in $\mathfrak{so}(n+1)$ whose entries are given by:

$$(\boldsymbol{v} \wedge \boldsymbol{w})^{\alpha\beta} := v^\alpha w^\beta - v^\beta w^\alpha, \quad \text{for } 1 \leqslant \alpha, \beta \leqslant n+1. \tag{3.42}$$

We extend this algebraic operation to objects that carry additional spatial structure over the domain $\mathbb{R}^m$. If $\boldsymbol{\Phi}$ is a standard vector in $\mathbb{R}^{n+1}$ and $\boldsymbol{\Psi} \in \mathbb{R}^{n+1} \otimes \mathbb{R}^m$ is a collection of spatial vector fields, their wedge product $\boldsymbol{\Phi} \wedge \boldsymbol{\Psi}$ acts on the target space components. The result is a skew-symmetric matrix of spatial vector fields taking values in $\mathfrak{so}(n+1) \otimes \mathbb{R}^m$, with entries defined identically:

$$(\boldsymbol{\Phi} \wedge \boldsymbol{\Psi})^{\alpha\beta} := \Phi^\alpha \Psi^\beta - \Phi^\beta \Psi^\alpha. \tag{3.43}$$

Definition 3.14 (The antisymmetric potential $\boldsymbol{\Omega}$)**.** Let $\boldsymbol{u} \in \mathbb{R}^{n+1}$ and let $\nabla \boldsymbol{u} \in \mathbb{R}^{m\times(n+1)}$ be defined as above. We define the *antisymmetric potential* $\boldsymbol{\Omega}$ as the matrix of vector fields of size $(n+1) \times (n+1)$:

$$\boldsymbol{\Omega} := \boldsymbol{u} \wedge \nabla \boldsymbol{u} \in \mathfrak{so}(n+1) \otimes \mathbb{R}^m. \tag{3.44}$$

Explicitly, the entry of $\boldsymbol{\Omega}$ at *row* α and *column* β, denoted by $\Omega^{\alpha\beta}$, is the $\mathbb{R}^m$-valued vector field given by:

$$\Omega^{\alpha\beta} := u^\alpha \nabla u^\beta - u^\beta \nabla u^\alpha. \tag{3.45}$$

Note that ∇u^β is the β-th column of the matrix $\nabla \boldsymbol{u}$ and that $\boldsymbol{\Omega}$ is a *skew–symmetric matrix* consisting of $\mathbb{R}^m$-valued vector fields: $\Omega^{\alpha\beta} = -\Omega^{\beta\alpha}$. In other words, $\boldsymbol{\Omega}$ is interpreted geometrically as a connection 1-form taking values in $\mathfrak{so}(n+1)$, i.e., $\boldsymbol{\Omega} \in \mathfrak{so}(n+1) \otimes \mathbb{R}^m \subset \mathbb{R}^{(n+1)\times(n+1)} \otimes \mathbb{R}^m$.

Remark 3.15 (Norm compatibility). Let $\boldsymbol{w} := \boldsymbol{\Omega} \boldsymbol{v}$. Here $\boldsymbol{v} \in \mathbb{R}^{n+1}$, while $\boldsymbol{\Omega}$ is an $(n+1) \times (n+1)$ matrix where each entry $\Omega^{\alpha\beta}$ is a vector in $\mathbb{R}^m$. The resulting $\boldsymbol{w}$ is a vector of size $n+1$, where each component $w^\alpha \in \mathbb{R}^m$ is an m-dimensional vector given by $w^\alpha = \sum_{\beta=1}^{n+1} \Omega^{\alpha\beta} v^\beta$. We use the standard Euclidean norm for $\boldsymbol{v}$ and the Frobenius norm for $\boldsymbol{\Omega}$:

$$|\boldsymbol{v}|^2 := \sum_{\beta=1}^{n+1} |v^\beta|^2, \qquad |\boldsymbol{\Omega}|^2 := \sum_{\alpha,\beta=1}^{n+1} |\Omega^{\alpha\beta}|^2_{\mathbb{R}^m}. \tag{3.46}$$

A straightforward application of the Cauchy-Schwarz inequality gives the norm compatibility relation

$$|\boldsymbol{\Omega v}| \leqslant |\boldsymbol{\Omega}| \cdot |\boldsymbol{v}|, \tag{3.47}$$

which holds for every tensor $\boldsymbol{\Omega} \in \mathbb{R}^{(n+1)\times(n+1)} \otimes \mathbb{R}^m$. Moreover, if $\boldsymbol{\Omega} = \boldsymbol{u} \wedge \nabla \boldsymbol{u}$ then

$$|\boldsymbol{\Omega}| = \sqrt{2}\, |\nabla \boldsymbol{u}|\,. \tag{3.48}$$

Indeed, $|\boldsymbol{\Omega}|^2 = 2|\nabla \boldsymbol{u}|^2 - 2|(\nabla \boldsymbol{u})\boldsymbol{u}|^2$, and $(\nabla \boldsymbol{u})\boldsymbol{u} = \mathbf{0}$ because of $|\boldsymbol{u}| = 1$.

Lemma 3.16. *A function $\boldsymbol{u} \in H^1(B_1, \mathbb{S}^n)$ is an almost harmonic map if, and only if, for every constant vector $\boldsymbol{c} \in \mathbb{R}^{n+1}$, the following identity holds in the sense of distributions:*

$$-\Delta \boldsymbol{u} = \mathbf{div}\,(\boldsymbol{\Omega}(\boldsymbol{u}-\boldsymbol{c})) + (\boldsymbol{u} \wedge \boldsymbol{f})\,(\boldsymbol{u}-\boldsymbol{c}) + \boldsymbol{f}. \tag{3.49}$$

In coordinates:

$$-\Delta u^\alpha = \operatorname{div}\left[\sum_{\beta=1}^{n+1} \Omega^{\alpha\beta}(u^\beta - c^\beta)\right] + \sum_{\beta=1}^{n+1} (\boldsymbol{u} \wedge \boldsymbol{f})^{\alpha\beta}(u^\beta - c^\beta) + f^\alpha, \tag{3.50}$$

where, we recall, $\Omega^{\alpha\beta} := u^\alpha \nabla u^\beta - u^\beta \nabla u^\alpha$ is an $\mathbb{R}^m$-valued vector field and $(\boldsymbol{u} \wedge \boldsymbol{f})^{\alpha\beta} := u^\alpha f^\beta - u^\beta f^\alpha$.

Remark 3.17 (Notation)*.* The expression $\boldsymbol{w} := \boldsymbol{\Omega}(\boldsymbol{u}-\boldsymbol{c})$ denotes a matrix-vector product. The result, $\boldsymbol{w}$, is a vector consisting of $(n+1)$ vector fields in $\mathbb{R}^m$. Its α-th component is:

$$w^\alpha = \sum_{\beta=1}^{n+1} \Omega^{\alpha\beta}(u^\beta - c^\beta) \in \mathbb{R}^m. \tag{3.51}$$

The operator **div** acts on this vector $\boldsymbol{w}$ component-wise. That is, the α-th component of the RHS of (3.49) is $\operatorname{div}(w^\alpha)$, where div is the standard divergence in $\mathbb{R}^m$. This gives the expression in (3.50). For any generic vector $\boldsymbol{v}$ consisting of $(n+1)$ vector fields in $\mathbb{R}^m$, the following Leibniz rule holds:

$$\mathbf{div}(\boldsymbol{\Omega v}) = (\mathbf{div}\,\boldsymbol{\Omega})\boldsymbol{v} + \boldsymbol{\Omega} \cdot \nabla \boldsymbol{v} \tag{3.52}$$

where $\mathbf{div}\,\boldsymbol{\Omega}$ is taken entrywise and the dot product denotes the contraction of the second index β, that is, $(\boldsymbol{\Omega} \cdot \nabla \boldsymbol{v})^\alpha := \sum_{\beta=1}^{n+1} \Omega^{\alpha\beta} \cdot \nabla v^\beta$. Recall that $\boldsymbol{\Omega}$ is an $(n+1) \times (n+1)$ matrix of $\mathbb{R}^m$-valued vector fields (entries in $\mathbb{R}^m$) and the divergence operator **div** gives an $(n+1) \times (n+1)$ matrix whose entries are scalars, the divergence of the $\mathbb{R}^m$-valued vector fields. Applying $\boldsymbol{\Omega}$ to a vector $\boldsymbol{u} \in \mathbb{R}^{n+1}$ yields $\mathbb{R}^m$-valued vector fields, in number of $(n+1)$. Again, $\mathbf{div}(\boldsymbol{\Omega v})$ is the vector obtained by applying the m-dimensional divergence operator to these $(n+1)$ vector fields. Therefore, the result will be an element of $\mathbb{R}^{(n+1)}$.

Proof. We break the proof into three easily verifiable steps.
Step 1: Rewriting the non-linearity. Starting from the almost harmonic map equation $-\Delta u^\alpha = u^\alpha |\nabla \boldsymbol{u}|^2 + f^\alpha$ we compute the nonlinear term using the orthogonality $(\nabla \boldsymbol{u})\boldsymbol{u} = \mathbf{0}$:

$$u^\alpha |\nabla \boldsymbol{u}|^2 = \sum_{\beta=1}^{n+1} \sum_{k=1}^{m} (u^\alpha \partial_k u^\beta - u^\beta \partial_k u^\alpha)\partial_k u^\beta = \sum_{\beta=1}^{n+1} \Omega^{\alpha\beta} \cdot \nabla u^\beta = (\boldsymbol{\Omega} \cdot \nabla \boldsymbol{u})^\alpha \tag{3.53}$$

Hence, the original almost-harmonic map can be rewritten as

$$-\Delta \boldsymbol{u} = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} + \boldsymbol{f} \tag{3.54}$$

Step 2: Applying the Leibniz rule. Applying the Leibniz rule (3.52) with $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{c}$ gives:

$$\boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} = \boldsymbol{\Omega} \cdot \nabla\,(\boldsymbol{u} - \boldsymbol{c}) = \mathbf{div}\,(\boldsymbol{\Omega}\,(\boldsymbol{u} - \boldsymbol{c})) - (\mathbf{div}\,\boldsymbol{\Omega})\,(\boldsymbol{u} - \boldsymbol{c}) \tag{3.55}$$

Substituting this back into (3.54) yields:

$$-\Delta \boldsymbol{u} = \mathbf{div}\,(\boldsymbol{\Omega}\,(\boldsymbol{u}-\boldsymbol{c})) - (\mathbf{div}\,\boldsymbol{\Omega})\,(\boldsymbol{u}-\boldsymbol{c}) + \boldsymbol{f}. \tag{3.56}$$

Step 3: Evaluating the divergence of $\boldsymbol{\Omega}$*.* For an *almost* harmonic map $\boldsymbol{u}$, which satisfies (3.54), i.e., $\Delta u^\alpha = -u^\alpha|\nabla \boldsymbol{u}|^2 - f^\alpha$, the divergence of $\Omega^{\alpha\beta}$ is

$$\begin{aligned}
\operatorname{div}(\Omega^{\alpha\beta}) &= (\nabla u^\alpha \cdot \nabla u^\beta - \nabla u^\beta \cdot \nabla u^\alpha) + (u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha)\\
&= u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha\\
&\overset{(3.54)}{=} -|\nabla \boldsymbol{u}|^2\, u^\alpha u^\beta + |\nabla \boldsymbol{u}|^2\, u^\alpha u^\beta - u^\alpha f^\beta + u^\beta f^\alpha\\
&= f^\alpha u^\beta - f^\beta u^\alpha\\
&= (\boldsymbol{f}\wedge \boldsymbol{u})^{\alpha\beta},
\end{aligned}$$

with $\boldsymbol{f}\wedge\boldsymbol{u} = \boldsymbol{f}\otimes\boldsymbol{u} - \boldsymbol{u}\otimes\boldsymbol{f}$. Thus $\mathbf{div}\,\boldsymbol{\Omega} = \boldsymbol{f}\wedge\boldsymbol{u}$, which is the claimed identity. □

3.5. **Harmonic Approximation and Caccioppoli-type estimates.** From (3.50) we know that if $\boldsymbol{u}$ is an *almost harmonic map* (i.e., satisfying $-\Delta\boldsymbol{u} = \boldsymbol{u}|\nabla\boldsymbol{u}|^2 + \boldsymbol{f}$), then for every constant vector $\boldsymbol{c}\in\mathbb{R}^{n+1}$, the following identity holds in the ball B_r:

$$-\Delta \boldsymbol{u} = \mathbf{div}(\boldsymbol{\Omega}(\boldsymbol{u}-\boldsymbol{c})) + (\boldsymbol{u}\wedge\boldsymbol{f})\,(\boldsymbol{u}-\boldsymbol{c}) + \boldsymbol{f} \tag{3.57}$$

The following result plays a central role in our argument. It establishes a control of the L^q-norm of the gradient deviation in terms of the L^p-norm of the map itself. This "reverse" nature (bounding derivatives by values) is typical of elliptic systems with critical growth.

Lemma 3.18 (Caccioppoli-type estimate)**.** *Let $q\in(1,2)$ be the fixed integrability exponent of the source term $\boldsymbol{f}$, and let $p\in(2,\infty)$ be the corresponding oscillation exponent determined by the relation*

$$\frac{1}{q} = \frac{1}{2} + \frac{1}{p}, \quad \textit{that is,} \quad p = \frac{2q}{2-q}. \tag{3.58}$$

Let $\boldsymbol{u}\in H^1(B_1,\mathbb{S}^n)$ be a solution to the almost harmonic map equation:

$$-\Delta \boldsymbol{u} = \mathbf{div}\,(\boldsymbol{\Omega}(\boldsymbol{u}-\boldsymbol{c})) + (\boldsymbol{u}\wedge\boldsymbol{f})\,(\boldsymbol{u}-\boldsymbol{c}) + \boldsymbol{f}. \tag{3.59}$$

Let $r\in(0,1)$ be a fixed radius. Let $\boldsymbol{h}_r\in H^1(B_r,\mathbb{R}^{n+1})$ denote the unique harmonic extension of $\boldsymbol{u}_{|\partial B_r}$ to the ball B_r:

$$-\Delta \boldsymbol{h}_r = \boldsymbol{0} \quad \textit{in } B_r, \qquad \boldsymbol{h}_r = \boldsymbol{u}_{|\partial B_r} \quad \textit{on } \partial B_r. \tag{3.60}$$

With these exponents, the following estimate holds:

$$\|\nabla(\boldsymbol{u}-\boldsymbol{h}_r)\|_{L^q(B_r)} \leqslant C_q\cdot\|\nabla\boldsymbol{u}\|_{L^2(B_r)}\,(r^m\,\Psi_p(\boldsymbol{u},B_r))^{1/p} + C_q r\|\boldsymbol{f}\|_{L^q(B_r)}, \tag{3.61}$$

where C_q depends only on q (and, in general, on the base dimension m; here $m=2$). In particular C_q is independent of r and $\boldsymbol{u}$.

Proof. Put $\boldsymbol{w}_r := \boldsymbol{u}-\boldsymbol{h}_r$. Since $-\Delta\boldsymbol{h}_r = 0$, the function $\boldsymbol{w}_r$ satisfies the same Poisson equation as $\boldsymbol{u}$ but with zero boundary data (cf. (3.57)):

$$\begin{cases} -\Delta\boldsymbol{w}_r = \mathbf{div}\,(\boldsymbol{\Omega}(\boldsymbol{u}-\boldsymbol{c})) + (\boldsymbol{u}\wedge\boldsymbol{f})\,(\boldsymbol{u}-\boldsymbol{c}) + \boldsymbol{f} & \text{in } B_r,\\ \boldsymbol{w}_r = \boldsymbol{0} & \text{on } \partial B_r, \end{cases} \tag{3.62}$$

for any fixed $\boldsymbol{c}\in\mathbb{R}^{n+1}$. Here, we recall, $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\boldsymbol{u}) := \boldsymbol{u}\wedge\nabla\boldsymbol{u}$ is the antisymmetric potential defined in (3.44).

If we set $\boldsymbol{g}_1 := \mathbf{div}\,(\boldsymbol{\Omega}(\boldsymbol{u}-\boldsymbol{c}))$, $\boldsymbol{g}_2 := (\boldsymbol{u}\wedge\boldsymbol{f})\,(\boldsymbol{u}-\boldsymbol{c})$, and $\boldsymbol{g}_3 := \boldsymbol{f}$, by the linearity of the Laplacian operator, we can decompose the error into three parts, $\boldsymbol{w} = \boldsymbol{w}_1+\boldsymbol{w}_2+\boldsymbol{w}_3$, with $-\Delta\boldsymbol{w}_i = \boldsymbol{g}_i$ in B_r and $\boldsymbol{w}_i = \boldsymbol{0}$ on ∂B_r.

Step 1: Elliptic Estimates. By standard elliptic estimates scaled to the ball B_r, for every $1 < q < 2$, we have

$$\|\nabla \boldsymbol{w}_2\|_{L^q(B_r)} \leqslant C_q r \|(\boldsymbol{u} \wedge \boldsymbol{f})(\boldsymbol{u} - \boldsymbol{c})\|_{L^q(B_r)}, \quad \|\nabla \boldsymbol{w}_3\|_{L^q(B_r)} \leqslant C_q r \|\boldsymbol{f}\|_{L^q(B_r)}. \tag{3.63}$$

Because $\boldsymbol{u}$ takes values in the unit sphere, $|\boldsymbol{u}| \equiv 1$, we can safely restrict our attention to constant vectors $\boldsymbol{c}$ belonging to the closed unit ball in $\mathbb{R}^{n+1}$ (see Lemma 2.5). Therefore, we may assume $|\boldsymbol{c}| \leqslant 1$. Thus, we have the universal pointwise bound $|\boldsymbol{u} - \boldsymbol{c}| \leqslant 2$. Furthermore, the operator norm of the wedge product gives $|(\boldsymbol{u} \wedge \boldsymbol{f})(\boldsymbol{u} - \boldsymbol{c})| \leqslant |\boldsymbol{f}||\boldsymbol{u} - \boldsymbol{c}| \leqslant 2|\boldsymbol{f}|$. Therefore, the source terms are trivially bounded directly in L^q:

$$\|\nabla \boldsymbol{w}_2\|_{L^q(B_r)} + \|\nabla \boldsymbol{w}_3\|_{L^q(B_r)} \leqslant 3 C_q r \|\boldsymbol{f}\|_{L^q(B_r)}. \tag{3.64}$$

It remains to bound $\|\nabla \boldsymbol{w}_1\|_{L^q(B_r)}$. For that, observe that the term $\boldsymbol{\Omega}(\boldsymbol{u} - \boldsymbol{c})$ belongs to $L^2\left(B_r, \mathbb{R}^{n+1} \otimes \mathbb{R}^m\right)$. In particular, for every $1 < q < 2$, applying Calderón–Zygmund estimates [GM12, Theorems 7.1 and 7.2] we get that

$$\|\nabla \boldsymbol{w}_1\|_{L^q(B_r)} \leqslant C_q \|\boldsymbol{\Omega}(\boldsymbol{u} - \boldsymbol{c})\|_{L^q(B_r)}, \tag{3.65}$$

valid for the stated range of q. The constant C_q depends only on the base dimension m, the exponent q, and on the shape of the domain; because the domain is a ball the estimate is scale invariant, so C_q is independent of $r, \boldsymbol{u}, \boldsymbol{c}$.

Step 2: Hölder's Inequality. We estimate the terms in L^q using Hölder's inequality. By the definition of the matrix-vector action, $|\boldsymbol{\Omega}(\boldsymbol{u} - \boldsymbol{c})| \leqslant |\boldsymbol{\Omega}||\boldsymbol{u} - \boldsymbol{c}|$. Since $\boldsymbol{\Omega}, \boldsymbol{f} \in L^2(B_r)$, for $p > 1$ such that $1/q = 1/2 + 1/p$, Hölder inequality gives

$$\|\boldsymbol{\Omega}(\boldsymbol{u} - \boldsymbol{c})\|_{L^q(B_r)} \leqslant \|\boldsymbol{\Omega}\|_{L^2(B_r)} \|\boldsymbol{u} - \boldsymbol{c}\|_{L^p(B_r)}. \tag{3.66}$$

Step 3: Conclusion. Overall, by (3.64) and (3.66) we get that

$$\begin{aligned} \|\nabla(\boldsymbol{u} - \boldsymbol{h}_r)\|_{L^q(B_r)} &\leqslant \|\nabla \boldsymbol{w}_1\|_{L^q(B_r)} + \|\nabla \boldsymbol{w}_2\|_{L^q(B_r)} + \|\nabla \boldsymbol{w}_3\|_{L^q(B_r)} \\ &\leqslant C_q \|\boldsymbol{\Omega}\|_{L^2(B_r)} \cdot \|\boldsymbol{u} - \boldsymbol{c}\|_{L^p(B_r)} + 3 C_q r \|\boldsymbol{f}\|_{L^q(B_r)}. \end{aligned}$$

Recalling that $\boldsymbol{\Omega} = \boldsymbol{u} \wedge \nabla \boldsymbol{u}$ and $|\boldsymbol{u}| \equiv 1$, as shown in (3.48), we have the pointwise estimate $|\boldsymbol{\Omega}| \leqslant \sqrt{2}|\nabla \boldsymbol{u}|$. Hence

$$\|\nabla(\boldsymbol{u} - \boldsymbol{h}_r)\|_{L^q(B_r)} \leqslant C_q \sqrt{2} \|\nabla \boldsymbol{u}\|_{L^2(B_r)} \cdot \|\boldsymbol{u} - \boldsymbol{c}\|_{L^p(B_r)} + 3 C_q r \|\boldsymbol{f}\|_{L^q(B_r)}. \tag{3.67}$$

The previous estimate holds for every $|\boldsymbol{c}| \leqslant 1$. Taking the infimum over $|\boldsymbol{c}| \leqslant 1$ yields (see Lemma 2.5)

$$\|\nabla(\boldsymbol{u} - \boldsymbol{h}_r)\|_{L^q(B_r)} \leqslant C_q \cdot \|\nabla \boldsymbol{u}\|_{L^2(B_r)} \left(r^m \Psi_p(\boldsymbol{u}, B_r)\right)^{1/p} + C_q r \|\boldsymbol{f}\|_{L^q(B_r)}, \tag{3.68}$$

with the harmless abuse of notation of redefining $C_q := \max\left(\sqrt{2}\omega_m^{1/p}, 3C_q\right)$. This concludes the proof. □

The preceding PDE estimate and the following purely real-analytic bound for harmonic extensions will play a key role in our analysis. We now establish this harmonic approximation, which relies on the properties of the Poisson kernel and requires no PDE hypotheses.

Lemma 3.19 (Harmonic approximation)**.** *Let $\boldsymbol{u} \in H^1(B_1, \mathbb{R}^{n+1})$. For any $p > 1$, there exists a radius $r \in [1/2, 1]$ and a harmonic function $\boldsymbol{h} : B_r \to \mathbb{R}^{n+1}$ satisfying the boundary trace $\boldsymbol{h} = \boldsymbol{u}$ on ∂B_r, such that the interior gradient satisfies:*

$$\sup_{x \in B_{1/4}} |\nabla \boldsymbol{h}(x)| \leqslant C_m \cdot \Psi_p^{1/p}(\boldsymbol{u}, B_1), \tag{3.69}$$

where $C_m > 0$ is a dimensional constant.

Proof. Step 1: Selection of a good radius. By the coercivity and strict convexity of $\boldsymbol{c} \mapsto \int_B |\boldsymbol{u} - \boldsymbol{c}|^p$ for $p > 1$, the infimum in the definition of the oscillation is attained. Let $\boldsymbol{c} \in \mathbb{R}^{n+1}$ be a constant vector that realizes this infimum in $\Psi_p(\boldsymbol{u}, B_1)$

$$\fint_{B_1} |\boldsymbol{u} - \boldsymbol{c}|^p = \Psi_p(\boldsymbol{u}, B_1). \tag{3.70}$$

By a standard consequence of Fubini's theorem (or the mean value theorem applied to radial slices), there exists a radius $r \in [1/2, 1]$, which depends on the function $\boldsymbol{u} - \boldsymbol{c}$, such that the boundary integral is controlled by the bulk integral:

$$\int_{\zeta \in \partial B_r} |\boldsymbol{u}(\zeta) - \boldsymbol{c}| \,\mathrm{d}\zeta \leqslant 2 \int_{B_1} |\boldsymbol{u}(x) - \boldsymbol{c}| \,\mathrm{d}x. \tag{3.71}$$

Indeed, integrating in polar coordinates we have

$$\int_{1/2}^{1} \left(\int_{\partial B_r} |\boldsymbol{u}(\zeta) - \boldsymbol{c}| \,\mathrm{d}\zeta \right) \mathrm{d}r = \int_{B_1 \setminus B_{1/2}} |\boldsymbol{u} - \boldsymbol{c}| \,\mathrm{d}x \leqslant \int_{B_1} |\boldsymbol{u} - \boldsymbol{c}| \,\mathrm{d}x, \tag{3.72}$$

so that the mean value theorem yields a radius $r \in [1/2, 1]$ for which (3.71) holds.

Note that r depends on the optimal $\boldsymbol{c}$, and that the optimal $\boldsymbol{c}$ depends only on $\boldsymbol{u}$ and p.

Step 2: Harmonic extension and interior estimates. We fix the radius r found in Step 1. Let $\boldsymbol{h}$ be the unique harmonic function in B_r satisfying the boundary condition $\boldsymbol{h} = \boldsymbol{u}$ on ∂B_r. Since $\boldsymbol{c}$ is a constant, the function $\boldsymbol{h} - \boldsymbol{c}$ is also harmonic in B_r. We estimate $\boldsymbol{h} - \boldsymbol{c}$ inside the ball using its Poisson Integral Formula (see Proposition 8.1 in the Appendix in Section 8). By standard estimates on the gradient of the Poisson kernel $P_r(x, \zeta)$, for any $x \in B_{r/2}$, we have (since $\boldsymbol{h} = \boldsymbol{u}$ on the boundary ∂B_r):

$$|\nabla \boldsymbol{h}(x)| = |\nabla(\boldsymbol{h}(x) - \boldsymbol{c})| \leqslant \frac{\tilde{C}_m}{r^m} \int_{\zeta \in \partial B_r} |\boldsymbol{h}(\zeta) - \boldsymbol{c}| \,\mathrm{d}\zeta = \frac{\tilde{C}_m}{r^m} \int_{\zeta \in \partial B_r} |\boldsymbol{u}(\zeta) - \boldsymbol{c}| \,\mathrm{d}\zeta. \tag{3.73}$$

We now restrict the domain. Since $r \geqslant 1/2$, we have $B_{1/4} \subset B_{r/2}$ and $1/r^m \leqslant 2^m$. Combining this with the radius estimate (3.71) we get:

$$\begin{aligned} \sup_{x \in B_{1/4}} |\nabla \boldsymbol{h}(x)| \quad &\leqslant \quad \sup_{x \in B_{r/2}} |\nabla \boldsymbol{h}(x)| \overset{(3.73)}{\leqslant} \frac{\tilde{C}_m}{r^m} \int_{\zeta \in \partial B_r} |\boldsymbol{u}(\zeta) - \boldsymbol{c}| \,\mathrm{d}\zeta \\ &\overset{(3.71)}{\leqslant} \quad \tilde{C}_m 2^m \left(2 \int_{B_1} |\boldsymbol{u} - \boldsymbol{c}| \right) \leqslant 2^{m+1} \tilde{C}_m \omega_m \left(\fint_{B_1} |\boldsymbol{u} - \boldsymbol{c}|^p \right)^{1/p}, \end{aligned} \tag{3.74}$$

where ω_m is the volume of the unit ball in $\mathbb{R}^m$. Because $\boldsymbol{c}$ was specifically chosen to realize the minimal oscillation, the right-hand side is proportional to $\Psi_p^{1/p}(\boldsymbol{u}, B_1)$. Setting $C_m := 2^{m+1} \tilde{C}_m \omega_m$ yields the gradient estimate (3.69). □

Having established the existence of a Caccioppoli-type estimate (Lemma 3.18) and a controlled harmonic approximation (Lemma 3.19), we can now combine these results to obtain the following result which will be the main tool in the derivation of the decay estimate for almost harmonic maps.

Lemma 3.20 (Coupled Caccioppoli-type estimate)**.** *Let $\boldsymbol{u} \in H^1(B_1, \mathbb{S}^n)$ be a solution to the almost harmonic map equation:*

$$-\Delta \boldsymbol{u} = \mathbf{div}\left(\boldsymbol{\Omega} (\boldsymbol{u} - \boldsymbol{c}) \right) + (\boldsymbol{u} \wedge \boldsymbol{f})\,(\boldsymbol{u} - \boldsymbol{c}) + \boldsymbol{f}, \tag{3.75}$$

with source term $\boldsymbol{f} \in L^q(B_1, \mathbb{R}^{n+1})$ *having integrability exponent* $q \in (1,2)$*. Let* $p := 2q/(2-q) \in (2, \infty)$ *be the corresponding oscillation exponent.*

Then, there exists a harmonic function $\boldsymbol{h} : B_{1/4} \to \mathbb{R}^{n+1}$ *such that the following gradient estimate holds on* $B_{1/4}$*:*

$$\|\nabla(\boldsymbol{u}-\boldsymbol{h})\|_{L^q(B_{1/4})} \leqslant C_q \|\nabla \boldsymbol{u}\|_{L^2(B_1)} \Psi_p^{1/p}(\boldsymbol{u}, B_1) + C_q \|\boldsymbol{f}\|_{L^q(B_1)}, \tag{3.76}$$

where $C_q > 0$ *depends only on the dimension* $m = 2$ *and the exponent* q*. Moreover,*

$$\sup_{x \in B_{1/4}} |\nabla \boldsymbol{h}(x)| \leqslant C_m \cdot \Psi_p^{1/p}(\boldsymbol{u}, B_1), \tag{3.77}$$

where $C_m > 0$ *is a dimensional constant.*

Proof. By Lemma 3.19, there exists a good radius $r \in [1/2, 1]$ and a harmonic approximation $\boldsymbol{h}$ of the almost harmonic map $\boldsymbol{u}$ on B_r, such that (3.77) holds. Because $\boldsymbol{h}$ is a solution of the problem

$$-\Delta \boldsymbol{h} = \mathbf{0} \quad \text{in } B_r, \qquad \boldsymbol{h} = \boldsymbol{u} \quad \text{on } \partial B_r, \tag{3.78}$$

we can apply Lemma 3.18 over the ball B_r to get (recall that $B_{1/4} \subset B_r$) from (3.61)

$$\|\nabla(\boldsymbol{u}-\boldsymbol{h})\|_{L^q(B_{1/4})} \leqslant C_q \cdot \|\nabla \boldsymbol{u}\|_{L^2(B_r)} \cdot \left(r^m\, \Psi_p\left(\boldsymbol{u}, B_r\right)\right)^{1/p} + C_q r \|\boldsymbol{f}\|_{L^q(B_1)}. \tag{3.79}$$

Using the fact that $r \leqslant 1$, extending the domains of integration to B_1, and by the monotonicity of the oscillation functional $r^m \Psi_p\left(\boldsymbol{u}, B_r\right) \leqslant \Psi_p(\boldsymbol{u}, B_1)$, we obtain (3.76). Overall, the restriction of $\boldsymbol{h}$ to $B_{1/4}$ gives (3.76) and (3.77). □

3.6. **Conclusion of the Proof of the Fundamental Lemma 3.7: The Decay Estimate.** We are now in the position to show that for the specific exponent $p = 2q/(2-q)$, arising from the Hölder inequality, the decay estimate (Axiom 3.2) holds. Specifically, we want to prove that if the gradient energy is small enough, then for a fixed small radius $\theta \in (0, 1/4)$:

$$\Psi_p(\boldsymbol{u}, B_\theta) \leqslant \gamma \Psi_p(\boldsymbol{u}, B_1) + \kappa \|\boldsymbol{f}\|^p_{L^q(B_1)}, \quad \gamma := \frac{1}{2} \tag{3.80}$$

where κ is a constant depending on θ, p, and the dimensions, but independent of $\boldsymbol{u}$. Explicitly, we want to prove that if the energy is small enough, then for a fixed small radius $\theta \in (0, 1/4)$:

$$\inf_{\boldsymbol{c} \in \mathbb{R}^{n+1}} \left(\fint_{B_\theta} |\boldsymbol{u}-\boldsymbol{c}|^p \right) \leqslant \frac{1}{2} \inf_{\boldsymbol{c} \in \mathbb{R}^{n+1}} \left(\fint_{B_1} |\boldsymbol{u}-\boldsymbol{c}|^p \right) + \kappa \left(\int_{B_1} |\boldsymbol{f}|^q \right)^{p/q}. \tag{3.81}$$

Proof. By Lemma 3.20, there exists a harmonic function $\boldsymbol{h} : B_{1/4} \to \mathbb{R}^{n+1}$ satisfying the gradient estimates (3.76) and (3.77). Let $\theta \in (0, 1/4)$. We estimate the oscillation of $\boldsymbol{u}$ on B_θ by comparing $\boldsymbol{u}$ with $\boldsymbol{h}$. Using the triangle inequality for the L^p-oscillation seminorm, we split the error:

$$\Psi_p^{1/p}(\boldsymbol{u}, B_\theta) \leqslant \Psi_p^{1/p}(\boldsymbol{u}-\boldsymbol{h}, B_\theta) + \Psi_p^{1/p}(\boldsymbol{h}, B_\theta) =: I + II. \tag{3.82}$$

Step 1: Estimate of term I*.* Recall that $\boldsymbol{u}-\boldsymbol{h}$ was constructed to vanish on the larger boundary ∂B_r for some $r \geqslant 1/2$. Consequently, it does not necessarily vanish on the much smaller internal boundary ∂B_θ. Because $\boldsymbol{u}-\boldsymbol{h} \notin W_0^{1,q}(B_\theta)$, we cannot rely on the zero-trace Sobolev embedding. Instead, we apply the standard Poincaré-Sobolev inequality on the ball B_θ:

$$
\begin{aligned}
I := \Psi_p^{1/p}(\boldsymbol{u}-\boldsymbol{h}, B_\theta) &\leqslant \left(\fint_{B_\theta} |(\boldsymbol{u}-\boldsymbol{h}) - \langle \boldsymbol{u}-\boldsymbol{h}\rangle_{B_\theta}|^p \right)^{1/p} \qquad (3.83)\\
&= \frac{1}{(\omega_m \theta^m)^{1/p}} \|(\boldsymbol{u}-\boldsymbol{h}) - \langle \boldsymbol{u}-\boldsymbol{h}\rangle_{B_\theta}\|_{L^{p_S}(B_\theta)}\\
&\leqslant \frac{C_S}{(\omega_m \theta^m)^{1/p}} \|\nabla(\boldsymbol{u}-\boldsymbol{h})\|_{L^q(B_\theta)}. \qquad (3.84)
\end{aligned}
$$

Now we substitute the Caccioppoli-type estimate (3.76) in the previous expression: Since $B_\theta \subset B_{1/4}$, the L^q norm over B_θ is bounded by the L^q norm over $B_{1/4}$:

$$
\|\nabla(\boldsymbol{u}-\boldsymbol{h})\|_{L^q(B_\theta)} \leqslant \|\nabla(\boldsymbol{u}-\boldsymbol{h})\|_{L^q(B_{1/4})} \overset{(3.76)}{\leqslant} C_q \cdot \|\nabla \boldsymbol{u}\|_{L^2(B_1)} \Psi_p^{1/p}(\boldsymbol{u}, B_1) + C_q \|\boldsymbol{f}\|_{L^q(B_1)}. \tag{3.85}
$$

Substituting this back into I we get:

$$
I := \Psi_p^{1/p}(\boldsymbol{u}-\boldsymbol{h}, B_\theta) \leqslant \frac{C_q}{\theta^{m/p}} \cdot \|\nabla \boldsymbol{u}\|_{L^2(B_1)} \cdot \Psi_p^{1/p}(\boldsymbol{u}, B_1) + \frac{C_q}{\theta^{m/p}} \|\boldsymbol{f}\|_{L^q(B_1)}. \tag{3.86}
$$

Here, by an harmless abuse of notation, the constant C_q has been redefined to absorb the Poincaré-Sobolev constant C_S and geometric constants. It remains to estimate $II := \Psi_p^{1/p}(\boldsymbol{h}, B_\theta)$.

Step 2: Estimate of Term II. Since the oscillation is an infimum over all constants, we can bound it from above by choosing the specific constant $\boldsymbol{c} = \boldsymbol{h}(0)$. Also, since $\boldsymbol{h}$ is harmonic and smooth, we apply the mean value theorem inside $B_{1/4}$ to infer:

$$
\begin{aligned}
II := \Psi_p^{1/p}(\boldsymbol{h}, B_\theta) \quad &\leqslant \quad \left(\fint_{B_\theta} |\boldsymbol{h}(x) - \boldsymbol{h}(0)|^p \,\mathrm{d}x \right)^{1/p} \quad \leqslant \quad \left(\fint_{B_\theta} |x|^p \left(\sup_{y \in B_\theta} |\nabla \boldsymbol{h}(y)|^p \right) \mathrm{d}x \right)^{1/p}\\
&\leqslant \theta \sup_{x \in B_{1/4}} |\nabla \boldsymbol{h}(x)|. \qquad (3.87)
\end{aligned}
$$

Applying the gradient bound (3.77), we conclude that

$$
II \leqslant C_m \theta \Psi_p^{1/p}(\boldsymbol{u}, B_1). \tag{3.88}
$$

Step 3: Collecting the estimates of I and II. Combining the estimates (3.86) and (3.88) for I and II, and factoring out $\Psi_p^{1/p}(\boldsymbol{u}, B_1)$, with the assignment $C_{m,q} := \max(C_m, C_q)$, we obtain:

$$
\Psi_p^{1/p}(\boldsymbol{u}, B_\theta) \leqslant C_{m,q} \left(\frac{\|\nabla \boldsymbol{u}\|_{L^2(B_1)}}{\theta^{m/p}} + \theta \right) \Psi_p^{1/p}(\boldsymbol{u}, B_1) + \frac{C_{m,q}}{\theta^{m/p}} \|\boldsymbol{f}\|_{L^q(B_1)}. \tag{3.89}
$$

Raising both sides to the power p (using the standard algebraic inequality $(a+b)^p \leqslant 2^{p-1}(a^p + b^p)$), we rewrite this purely in terms of the Dirichlet energy $\mathcal{E}(\boldsymbol{u}) := \|\nabla \boldsymbol{u}\|_{L^2(B_1)}^2$:

$$
\Psi_p(\boldsymbol{u}, B_\theta) \leqslant C_{m,q}^p \left(\frac{\mathcal{E}(\boldsymbol{u})^{p/2}}{\theta^m} + \theta^p \right) \Psi_p(\boldsymbol{u}, B_1) + \frac{C_{m,q}^p}{\theta^m} \|\boldsymbol{f}\|_{L^q(B_1)}^p, \tag{3.90}
$$

where, as usual, the constant $C_{m,q}^p$ has been harmlessly redefined to absorb the 2^{p-1} factors that arise.

Step 4: Choice of the parameters. To achieve the decay factor $\gamma = 1/2$, we choose θ and ε_* sequentially. First, choose the radius $\theta \in (0, 1/4)$ small enough such that the harmonic term dominates:

$$C_{m,q}^p \theta^p \leqslant \frac{1}{4} \quad \Longleftrightarrow \quad \theta \leqslant \left(\frac{1}{4}\right)^{1/p} \frac{1}{C_{m,q}}. \tag{3.91}$$

Second, with θ fixed, we choose the Dirichlet energy threshold ε_* small enough such that if $\mathcal{E}(\boldsymbol{u}) < \varepsilon_*$, the nonlinear error term is controlled:

$$C_{m,q}^p \frac{\varepsilon_*^{p/2}}{\theta^m} \leqslant \frac{1}{4}. \tag{3.92}$$

Notice that this smallness condition depends *only* on the Dirichlet energy; no size restriction is placed on $\boldsymbol{f}$. With these choices, the scaling bracket in (3.90) sums to at most $1/4 + 1/4 = 1/2$. Defining $\kappa(\theta) := C_{m,q}^p/\theta^m$, we arrive at the final Campanato iteration step:

$$\Psi_p(\boldsymbol{u}, B_\theta) \leqslant \frac{1}{2}\Psi_p(\boldsymbol{u}, B_1) + \kappa \|\boldsymbol{f}\|_{L^q(B_1)}^p. \tag{3.93}$$

This establishes the exact decay estimate formulated in Axiom **3.2**, concluding the proof of Lemma 3.7. □

4. Generalization: Regularity of non-antisymmetric systems

In the classical regularity theory of harmonic maps, the assumption that the field $\boldsymbol{u}$ takes values in a closed target manifold $E \subset \mathbb{R}^{n+1}$ naturally yields a connection matrix $\boldsymbol{\Omega}$ that is geometrically antisymmetric. However, a fundamental consequence of the classical elliptic approach developed in this paper is that this strict geometric requirement can be entirely dispensed with.

By replacing the geometric hypothesis of antisymmetry with an analytic integrability condition on the divergence, we seamlessly extend our Campanato and Caccioppoli-type framework to generic, non-antisymmetric systems. While the landmark results of Rivière [Riv06] and Müller and Schikorra [MS09] fundamentally rely on the compensated compactness afforded by an antisymmetric $\boldsymbol{\Omega}$, our method demonstrates that classical linear elliptic estimates are robust enough to handle general systems, provided the divergence is analytically controlled.

The primary objective of this section is to rigorously establish this generalization. We emphasize throughout that $\boldsymbol{\Omega}$ is *not* assumed to be antisymmetric, distinguishing our setting from the frameworks of [Riv06] and [MS09]:

Theorem 1.2. (Regularity of generalized systems) *Let $B_1 \subset \mathbb{R}^2$. Let $\boldsymbol{u} \in H^1 \cap L^\infty(B_1, \mathbb{R}^{n+1})$ be a weak solution to the generalized system*

$$-\Delta \boldsymbol{u} = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} + \boldsymbol{f} \quad \textit{in } B_1. \tag{4.1}$$

Assume that the structural matrix and the source terms satisfy the purely analytic hypotheses for some $q > 1$:

- $\boldsymbol{\Omega} \in L^2(B_1, \mathbb{R}^{(n+1)\times(n+1)} \otimes \mathbb{R}^2)$,
- $\boldsymbol{f} \in L^q(B_1, \mathbb{R}^{n+1})$,
- $\operatorname{\mathbf{div}} \boldsymbol{\Omega} \in L^q(B_1, \mathbb{R}^{(n+1)\times(n+1)})$.

Then, $\boldsymbol{u}$ is locally Hölder continuous in B_1. Specifically, there exists an exponent $\eta \in (0, 1)$ such that $\boldsymbol{u} \in C_{\text{loc}}^{0,\eta}(B_1, \mathbb{R}^{n+1})$.

Remark 4.1. In what follows, we may assume without loss of generality that $1 < q < 2$. The reason for this has already been explained in Remark 3.8.

The proof of Theorem 1.2 will be given in subsection 4.4. To this end, we first introduce a slight generalization of the abstract regularity principle stated in Theorem 3.3 by defining an abstract class of configurations. We then formulate and prove the second abstract regularity principle (Theorem 4.5) in arbitrary base dimension $m \geqslant 2$, although the framework will ultimately be applied to the regularity theory of generalized systems in the two-dimensional case $m = 2$ to conclude the proof of Theorem 1.2.

4.1. **The second abstract regularity principle.** Let $\boldsymbol{\mathcal{S}}$ be a class of triplets $(\boldsymbol{u}, \boldsymbol{\Omega}, \boldsymbol{f})$ belonging to the space

$$H^1 \cap L^\infty(B_1, \mathbb{R}^{n+1}) \times L^2(B_1, \mathbb{R}^{(n+1)\times(n+1)} \otimes \mathbb{R}^m) \times L^q(B_1, \mathbb{R}^{n+1}), \tag{4.2}$$

which obey the following two axioms:

Axiom 4.2 (Locality and Closure)**.** The class $\boldsymbol{\mathcal{S}}$ is closed under rescaling. Specifically, if $(\boldsymbol{u}, \boldsymbol{\Omega}, \boldsymbol{f}) \in \boldsymbol{\mathcal{S}}$, then for any ball $B_r(x_0) \subset B_1$, the rescaled triplet $(\boldsymbol{u}_{x_0,r}, \overline{\boldsymbol{\Omega}}_{x_0,r}, \tilde{\boldsymbol{f}}_{x_0,r})$ defined by

$$\boldsymbol{u}_{x_0,r}(y) := \boldsymbol{u}(x_0 + ry), \quad \overline{\boldsymbol{\Omega}}_{x_0,r}(y) := r\boldsymbol{\Omega}(x_0 + ry), \quad \tilde{\boldsymbol{f}}_{x_0,r}(y) := r^2 \boldsymbol{f}(x_0 + ry), \tag{4.3}$$

is still in $\boldsymbol{\mathcal{S}}$ over the unit ball $B_1 \subseteq \mathbb{R}^m$.

Remark 4.3 (Dimensional Scaling and Closure). Unlike the $m = 2$ case, where the Dirichlet energy is conformally invariant, the energy in dimension $m > 2$ scales super-critically: $\mathcal{E}(\boldsymbol{u}_{x_0,r}, B_1) = r^{2-m}\mathcal{E}(\boldsymbol{u}, B_r(x_0))$. Because $r < 1$ and $2 - m < 0$, the energy of the rescaled map can arbitrarily magnify, meaning a "small energy" hypothesis cannot be preserved under inward scaling. The generalized framework bypasses this entirely by anchoring the closure to the amplitude bound. The uniform bound is scale-invariant: $\|\boldsymbol{u}_{x_0,r}\|_{L^\infty(B_1)} = \|\boldsymbol{u}\|_{L^\infty(B_r(x_0))} \leqslant M$. Furthermore, the forcing terms scale sub-critically: The L^q norm of the rescaled source yields:

$$\|\tilde{\boldsymbol{f}}_{x_0,r}\|_{L^q(B_1)} = \left(\int_{B_1} r^{2q} |\boldsymbol{f}(x_0 + ry)|^q \mathrm{d}y \right)^{1/q} = r^{2-m/q} \|\boldsymbol{f}\|_{L^q(B_r(x_0))}. \tag{4.4}$$

Because we assume $q > m/2$, the exponent $2 - m/q > 0$. This guarantees that the rescaled source data inherently decays as $r \to 0$. Thus, the zooming process intrinsically preserves the axioms without any reliance on the L^2 gradient energy.

Axiom 4.4 (The Decay Property)**.** There exist structural constants $\theta, \gamma \in (0, 1)$, $p \geqslant 1$, an amplitude bound $M > 0$, and $\kappa > 0$, such that for any triplet $(\boldsymbol{u}, \boldsymbol{\Omega}, \boldsymbol{f}) \in \boldsymbol{\mathcal{S}}$ satisfying $\|\boldsymbol{u}\|_{L^\infty(B_1)} \leqslant M$, the oscillation strictly contracts at the interior scale:

$$\Psi_p(\boldsymbol{u}, B_\theta) \leqslant \gamma \cdot \Psi_p(\boldsymbol{u}, B_1) + \kappa \left(\|\boldsymbol{f}\|_{L^q(B_1)} + M \, \|\mathbf{div}\, \boldsymbol{\Omega}\|_{L^q(B_1)} \right)^p. \tag{4.5}$$

Proceeding as we did for Theorem 3.3 one gets

Theorem 4.5 (Second Abstract Regularity Principle in general dimension)**.** *Let $\boldsymbol{\mathcal{S}}$ be a class of triplets satisfying* Axioms **4.2** *and* **4.4**, *on the unit ball $B_1 \subset \mathbb{R}^m$. If $(\boldsymbol{u}, \boldsymbol{\Omega}, \boldsymbol{f}) \in \boldsymbol{\mathcal{S}}$ with $\|\boldsymbol{u}\|_{L^\infty(B_1)} \leqslant M$, and the integrability exponent satisfies $q > m/2$, then $\boldsymbol{u}$ is locally Hölder continuous in B_1. Specifically, $\boldsymbol{u} \in C^{0,\eta}_{\mathrm{loc}}(B_1)$ for every Hölder exponent η satisfying:*

$$\eta < \min\left(\frac{\ln \gamma}{p \ln \theta}, 2 - \frac{m}{q} \right). \tag{4.6}$$

For any compact subset $K \subset B_1$, the corresponding Hölder seminorm depends only on the chosen exponent η, the structural parameters $\theta, \gamma, \kappa, p, M$, $\|\,\mathbf{div}\,\boldsymbol{\Omega}\|_{L^q(B_1)}$, $\|\boldsymbol{f}\|_{L^q(B_1)}$, and the distance $\mathrm{dist}(K, \partial B_1)$.

Proof of the Second Abstract Regularity Principle, Theorem 4.5. We divide the proof in five steps.

Step 1: The Master Iteration Inequality. Let $(\boldsymbol{u}, \boldsymbol{\Omega}, \boldsymbol{f}) \in \boldsymbol{\mathcal{S}}$. Our goal is to verify the Campanato condition for $\boldsymbol{u}$ uniformly on $B_{1/2}$. Fix an arbitrary center point $x_0 \in B_{1/2}$ and a macroscopic starting radius $r_0 := 1/2$ so that $B_r(x_0) \subset B_1$ for every $x_0 \in B_{1/2}$ and every $r \in (0, r_0]$. For any radius $r \in (0, r_0]$, we set $\psi(r) := r^m \Psi_p(\boldsymbol{u}, B_r(x_0))$. We wish to establish an algebraic iteration inequality for ψ.

By Axiom 4.2, the rescaled triplet $(\boldsymbol{u}_{x_0,r}, \overline{\boldsymbol{\Omega}}_{x_0,r}, \tilde{\boldsymbol{f}}_{x_0,r})$ belongs to $\boldsymbol{\mathcal{S}}$. We apply the contractive decay property (Axiom 4.4) to this rescaled triplet:

$$\Psi_p(\boldsymbol{u}_{x_0,r}, B_\theta) \leqslant \gamma \cdot \Psi_p(\boldsymbol{u}_{x_0,r}, B_1) + \kappa \left(\|\tilde{\boldsymbol{f}}_{x_0,r}\|_{L^q(B_1)} + M \| \operatorname{\mathbf{div}}_y \overline{\boldsymbol{\Omega}}_{x_0,r}\|_{L^q(B_1)} \right)^p . \tag{4.7}$$

We translate the oscillations back to the physical coordinates on B_1. Using the scaling identity (2.4) for the oscillation (mean integral), we get:

$$\text{LHS:} \quad \Psi_p(\boldsymbol{u}_{x_0,r}, B_\theta) \overset{(2.4)}{=} \Psi_p(\boldsymbol{u}, B_{\theta r}(x_0)) = (\theta r)^{-m}\, \psi(\theta r), \tag{4.8}$$

$$\text{RHS:} \quad \Psi_p(\boldsymbol{u}_{x_0,r_0}, B_1) \overset{(2.4)}{=} \Psi_p(\boldsymbol{u}, B_r(x_0)) = r^{-m} \psi(r). \tag{4.9}$$

Therefore, (4.7) can be rewritten as

$$\psi(\theta r) \leqslant \theta^m \gamma \cdot \psi(r) + \kappa \theta^m r^m \left(\|\tilde{\boldsymbol{f}}_{x_0,r}\|_{L^q(B_1)} + M \| \operatorname{\mathbf{div}}_y \overline{\boldsymbol{\Omega}}_{x_0,r}\|_{L^q(B_1)} \right)^p . \tag{4.10}$$

We calculate the L^q norms of the rescaled data. Since $\tilde{\boldsymbol{f}}_{x_0,r}(y) := r^2 \boldsymbol{f}(x_0 + ry)$, we have $\|\tilde{\boldsymbol{f}}_{x_0,r}\|^q_{L^q(B_1)} = r^{2q-m} \|\boldsymbol{f}\|^q_{L^q(B_r(x_0))} \leqslant r^{2q-m} \|\boldsymbol{f}\|^q_{L^q(B_1)}$. Hence

$$\|\tilde{\boldsymbol{f}}_{x_0,r}\|^p_{L^q(B_1)} \leqslant r^{p(2-m/q)} \|\boldsymbol{f}\|^p_{L^q(B_1)}. \tag{4.11}$$

Similarly, because $\operatorname{\mathbf{div}}_y \overline{\boldsymbol{\Omega}}_{x_0,r}(y) = r^2 (\operatorname{\mathbf{div}}_x \boldsymbol{\Omega})(x_0 + ry)$, we obtain:

$$\| \operatorname{\mathbf{div}}_y \overline{\boldsymbol{\Omega}}_{x_0,r}\|_{L^q(B_1)} = r^{2-m/q} \| \operatorname{\mathbf{div}}_x \boldsymbol{\Omega}\|_{L^q(B_r(x_0))} \leqslant r^{2-m/q} \| \operatorname{\mathbf{div}}_x \boldsymbol{\Omega}\|_{L^q(B_1)}. \tag{4.12}$$

Substituting the previous bound into the decay property (4.10), we obtain the master iteration inequality:

$$\psi(\theta r) \leqslant \theta^m \gamma \cdot \psi(r) + b r^\beta \quad \text{for all } r \in (0, r_0], \tag{4.13}$$

where $\beta := m + p\,(2 - m/q)$, and $b := \kappa \theta^m \left(\|\boldsymbol{f}\|_{L^q(B_1)} + M \| \operatorname{\mathbf{div}}_x \boldsymbol{\Omega}\|_{L^q(B_1)} \right)^p$ is a constant acting as a universal bound: Indeed, since b depends on the fixed global data over the full unit ball B_1, it remains completely independent of the moving center x_0 and the shrinking radius r.

Step 2: Continuous Algebraic Decay via the Iteration Lemma. To extract a continuous algebraic decay rate from this discrete bound, we wish to apply the algebraic iteration lemma (Lemma 3.6). This is possible because ψ is non-decreasing. Arguing as in the proof of Theorem 3.3, we fix an arbitrary small $\varepsilon > 0$ and consider the iteration inequality

$$\psi(\theta r) \leqslant \theta^m \gamma_\varepsilon \cdot \psi(r) + b r^\beta \quad \text{for all } r \in (0, r_0], \tag{4.14}$$

with $\gamma_\varepsilon := \max(\gamma, \theta^{\beta - m - \varepsilon})$, so that, by construction, for ε sufficiently small, we have (cf. (3.20)) $\gamma_\varepsilon \in (0,1)$, $\beta - m - \varepsilon > 0$, and (cf. (3.21))

$$\beta - \alpha_\varepsilon \geqslant \varepsilon > 0 \quad \text{with} \quad \alpha_\varepsilon := m + \min\left(\frac{\ln \gamma}{\ln \theta}, \beta - m - \varepsilon \right). \tag{4.15}$$

Because $\alpha_\varepsilon < \beta$, the iteration lemma (Lemma 3.6) gives the existence of a purely structural constant $c_\varepsilon > 0$ such that for all $r \in (0, r_0]$ there holds (cf. (3.23)):

$$\Psi_p(\boldsymbol{u}, B_r(x_0)) = r^{-m}\psi(r) \leqslant c_\varepsilon \left[\frac{\Psi_p(\boldsymbol{u}, B_{r_0}(x_0))}{r_0^{\alpha_\varepsilon - m}} + b r_0^{\beta-\alpha_\varepsilon}\right] r^{\alpha_\varepsilon - m}, \tag{4.16}$$

where, we recall, $b := \kappa\theta^m \left(\|\boldsymbol{f}\|_{L^q(B_1)} + M\|\operatorname{\mathbf{div}}_x \boldsymbol{\Omega}\|_{L^q(B_1)}\right)^p$.

Step 3: Uniform bound via the condition $\|\boldsymbol{u}\|_{L^\infty(B_1)} \leqslant M$. Given that $\|\boldsymbol{u}\|_{L^\infty(B_1)} \leqslant M$, choosing the test vector $\boldsymbol{c} = \boldsymbol{0}$ in the infimum trivially yields $\Psi_p(\boldsymbol{u}, B_{r_0}(x_0)) \leqslant M^p$. Therefore, (4.16) gives the uniform bound

$$\Psi_p(\boldsymbol{u}, B_r(x_0)) \leqslant M_* r^{\alpha_\varepsilon - m}, \tag{4.17}$$

where M_* is a universal constant (independent of $x_0 \in B_{1/2}$ and $r \in (0, r_0]$) and (cf. (3.28))

$$\alpha_\varepsilon - m = \min\left(\frac{\ln\gamma}{\ln\theta}, p\,(2 - m/q) - \varepsilon\right). \tag{4.18}$$

Step 4: The Campanato Condition and Hölder Continuity on $B_{1/2}$. Since inequality (4.17) holds uniformly for every $x_0 \in B_{1/2}$, it satisfies precisely the Campanato condition with exponent $(\alpha_\varepsilon - m)$. Therefore, $\boldsymbol{u} \in C^{0,\eta_\varepsilon}(B_{1/2})$ with $\eta_\varepsilon = (\alpha_\varepsilon - m)/p$. Substituting our exact choice for $(\alpha_\varepsilon - m)$ directly yields:

$$\eta_\varepsilon = \min\left(\frac{\ln\gamma}{p\,\ln\theta}, 2 - \frac{m}{q} - \frac{\varepsilon}{p}\right), \tag{4.19}$$

matching the formula in the theorem statement for $\varepsilon \to 0$. Finally, note that the Hölder seminorm $[\boldsymbol{u}]_{C^{0,\eta_\varepsilon}(B_{1/2})}$ depends only on $\theta, \gamma, \kappa, p, M, \|\operatorname{\mathbf{div}}_x \boldsymbol{\Omega}\|_{L^q(B_1)}$, and $\|\boldsymbol{f}\|_{L^q(B_1)}$.

Step 5: Extension to arbitrary compact subsets of B_1. Step 5 follows the same scheme as in the proof of Theorem 3.3, with the energy bound $\mathcal{E}(\boldsymbol{v}, B_1) \leqslant \mathcal{E}(\boldsymbol{u}, B_1)$ replaced by the (scale-invariant) amplitude bound $\|\boldsymbol{v}\|_{L^\infty(B_1)} \leqslant M$ and the sub-critical scaling identities (4.11)–(4.12). □

4.2. **Caccioppoli-type estimate for generalized systems in dimension $m = 2$.** To apply the second abstract regularity principle, we must now demonstrate that our generalized system (1.3) inherently controls the oscillation of the solution at small scales. We achieve this by locally restructuring the equation in divergence form, allowing us to tame the critical gradient term and establish a rigorous Caccioppoli-type estimate specific to dimension $m = 2$.

Lemma 4.6 (Generalized Caccioppoli-type estimate in dimension $m = 2$)**.** *Let $q \in (1, 2)$ be the fixed integrability exponent of the source term $\boldsymbol{f}$, and let $p \in (2, \infty)$ be the corresponding oscillation exponent determined by the relation $p = 2q/(2 - q)$. Assume that $\boldsymbol{\Omega} \in L^2(B_1, \mathbb{R}^{(n+1)\times(n+1)} \otimes \mathbb{R}^2)$, $\operatorname{\mathbf{div}} \boldsymbol{\Omega} \in L^q(B_1, \mathbb{R}^{(n+1)\times(n+1)})$, and $\boldsymbol{f} \in L^q(B_1, \mathbb{R}^{n+1})$. Let $\boldsymbol{u} \in H^1 \cap L^\infty(B_1, \mathbb{R}^{n+1})$ be a bounded weak solution to*

$$-\Delta\boldsymbol{u} = \boldsymbol{\Omega} \cdot \nabla\boldsymbol{u} + \boldsymbol{f} \tag{4.20}$$

with global bound $\|\boldsymbol{u}\|_{L^\infty(B_1)} \leqslant M$.

Let $r \in (0, 1)$ be a fixed radius, and let $\boldsymbol{h}_r \in H^1(B_r, \mathbb{R}^{n+1})$ denote the unique harmonic extension of $\boldsymbol{u}_{|\partial B_r}$ to the ball B_r.

With these exponents, the following estimate holds:

$$\|\nabla(\boldsymbol{u} - \boldsymbol{h}_r)\|_{L^q(B_r)} \leqslant C_q \|\boldsymbol{\Omega}\|_{L^2(B_r)} \left(r^m \Psi_p(\boldsymbol{u}, B_r)\right)^{1/p} + C_q r \left(\|\boldsymbol{f}\|_{L^q(B_r)} + M\|\operatorname{\mathbf{div}} \boldsymbol{\Omega}\|_{L^q(B_r)}\right), \tag{4.21}$$

where C_q depends only on q (and, in general, on the base dimension m; here $m = 2$). In particular C_q is independent of r and $\boldsymbol{u}$.

Proof. Since $\boldsymbol{\Omega} \in L^2$, $\mathbf{div}\,\boldsymbol{\Omega} \in L^q$, and $\boldsymbol{u} \in L^\infty$, we invoke the standard distributional Leibniz rule. For any constant vector $\boldsymbol{c} \in \mathbb{R}^{n+1}$, the system (1.3) can be rewritten locally as:

$$-\Delta \boldsymbol{u} = \mathbf{div}(\boldsymbol{\Omega}(\boldsymbol{u} - \boldsymbol{c})) + \underbrace{\boldsymbol{f} - (\mathbf{div}\,\boldsymbol{\Omega})(\boldsymbol{u} - \boldsymbol{c})}_{:=\boldsymbol{F}}. \tag{4.22}$$

Due to the global bound $\|\boldsymbol{u}\|_{L^\infty} \leqslant M$, by Lemma 2.5 we can focus on constants $\boldsymbol{c} \in \mathbb{R}^{n+1}$ such that $|\boldsymbol{c}| \leqslant M$, i.e., to constants ensuring pointwise $|\boldsymbol{u} - \boldsymbol{c}| \leqslant 2M$. By the triangle inequality, the local effective source $\boldsymbol{F}$ is bounded by:

$$\|\boldsymbol{F}\|_{L^q(B_r)} \leqslant \|\boldsymbol{f}\|_{L^q(B_r)} + 2M\|\,\mathbf{div}\,\boldsymbol{\Omega}\|_{L^q(B_r)}. \tag{4.23}$$

We proceed like in Lemma 3.18. We decompose $\boldsymbol{w}_r := \boldsymbol{u} - \boldsymbol{h}_r = \boldsymbol{w}_1 + \boldsymbol{w}_2 \in H^1_0(B_r)$, where $-\Delta \boldsymbol{w}_1 = \mathbf{div}(\boldsymbol{\Omega}(\boldsymbol{u} - \boldsymbol{c}))$ and $-\Delta \boldsymbol{w}_2 = \boldsymbol{F}$. Applying standard elliptic estimates and Hölder's inequality $(1/q = 1/2 + 1/p)$ we get

$$\|\nabla \boldsymbol{w}_1\|_{L^q(B_r)} \leqslant C_q\|\boldsymbol{\Omega}\|_{L^2(B_r)}\|\boldsymbol{u} - \boldsymbol{c}\|_{L^p(B_r)}. \tag{4.24}$$

For the subcritical source term involving $\boldsymbol{F}$, classical estimates scaled to B_r yield:

$$\|\nabla \boldsymbol{w}_2\|_{L^q(B_r)} \leqslant C_q r\|\boldsymbol{F}\|_{L^q(B_r)}. \tag{4.25}$$

The conclusion follows by the triangle inequality, summing the bounds, and absorbing the involved absolute constants into C_q. □

Having established (in Lemma 4.6) the existence of a generalized Caccioppoli-type estimate and the controlled harmonic approximation (Lemma 3.19), we can now combine these results to obtain the following result which will be the main tool in the derivation of the decay estimate for almost harmonic maps.

Lemma 4.7 (Generalized coupled Caccioppoli-type estimate)**.** *Let $q \in (1, 2)$ be the fixed integrability exponent of the source term $\boldsymbol{f}$, and let $p \in (2, \infty)$ be the corresponding oscillation exponent determined by the relation $p = 2q/(2 - q)$. Assume that $\boldsymbol{\Omega} \in L^2(B_1, \mathbb{R}^{(n+1)\times(n+1)} \otimes \mathbb{R}^2)$, $\mathbf{div}\,\boldsymbol{\Omega} \in L^q(B_1, \mathbb{R}^{(n+1)\times(n+1)})$, and $\boldsymbol{f} \in L^q(B_1, \mathbb{R}^{n+1})$. Let $\boldsymbol{u} \in H^1 \cap L^\infty(B_1, \mathbb{R}^{n+1})$ be a bounded weak solution to*

$$-\Delta \boldsymbol{u} = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} + \boldsymbol{f}, \tag{4.26}$$

with global bound $\|\boldsymbol{u}\|_{L^\infty(B_1)} \leqslant M$.

Then, there exists a harmonic function $\boldsymbol{h} : B_{1/4} \to \mathbb{R}^{n+1}$ such that the following gradient estimate holds on $B_{1/4}$:

$$\|\nabla(\boldsymbol{u} - \boldsymbol{h})\|_{L^q(B_{1/4})} \leqslant C_q\|\boldsymbol{\Omega}\|_{L^2(B_1)}\Psi_p^{1/p}(\boldsymbol{u}, B_1) + C_q\left(\|\boldsymbol{f}\|_{L^q(B_1)} + M\|\,\mathbf{div}\,\boldsymbol{\Omega}\|_{L^q(B_1)}\right), \tag{4.27}$$

where $C_q > 0$ depends only on the dimension $m = 2$ and the exponent q. Moreover,

$$\sup_{x\in B_{1/4}} |\nabla \boldsymbol{h}(x)| \leqslant C_m \cdot \Psi_p^{1/p}(\boldsymbol{u}, B_1), \tag{4.28}$$

where $C_m > 0$ is a dimensional constant.

Proof. The argument is structurally identical to the proof of Lemma 3.20, with the following two substitutions. First, Lemma 4.6 (rather than Lemma 3.18) is invoked on the good radius $r \in [1/2, 1]$, providing

$$\|\nabla(\boldsymbol{u}-\boldsymbol{h}_r)\|_{L^q(B_r)} \leqslant C_q\|\boldsymbol{\Omega}\|_{L^2(B_r)}(r^m\Psi_p(\boldsymbol{u}, B_r))^{1/p}+C_q r\left(\|\boldsymbol{f}\|_{L^q(B_r)} + M\|\operatorname{\mathbf{div}}\boldsymbol{\Omega}\|_{L^q(B_r)}\right). \tag{4.29}$$

Second, the bound on the harmonic extension $\boldsymbol{h}$ (Lemma 3.19) is unchanged. Restricting the LHS to $B_{1/4} \subset B_r$ and using $r \leqslant 1$ together with the monotonicity of $r \mapsto r^m\Psi_p$ yields (4.27) and (4.28). □

4.3. **The Generalized Decay Estimate.** We now prove that the structural matrix $\boldsymbol{\Omega}$ dictates the decay of the oscillation provided its L^2 energy is sufficiently small. Specifically, the content of the next result is to prove that the class of almost harmonic maps satisfies the decay property formulated in Axiom 4.4.

Lemma 4.8 (Generalized Decay Property)**.** *For any integrability exponent $q \in (1, 2)$, let $p := 2q/(2-q)$ be its corresponding oscillation exponent. There exist structural constants $\theta, \gamma \in (0, 1)$, $\kappa > 0$, and an $\boldsymbol{\Omega}$-energy threshold $\varepsilon_* > 0$ such that the following holds. Let $\boldsymbol{u}$ be a bounded weak solution on the unit ball B_1 to the equation* (*see* (1.3))*:*

$$-\Delta\boldsymbol{u} = \boldsymbol{\Omega}\cdot\nabla\boldsymbol{u} + \boldsymbol{f} \quad \text{in } B_1, \tag{4.30}$$

satisfying the global bound $\|\boldsymbol{u}\|_{L^\infty(B_1)} \leqslant M$ for some $M > 0$. If the coefficient $\boldsymbol{\Omega}$ satisfies the smallness condition $\|\boldsymbol{\Omega}\|_{L^2(B_1)} < \varepsilon_*$, then the oscillation of $\boldsymbol{u}$ decays according to the estimate:

$$\Psi_p(\boldsymbol{u}, B_\theta) \leqslant \gamma\cdot\Psi_p(\boldsymbol{u}, B_1) + \kappa\left(\|\boldsymbol{f}\|_{L^q(B_1)} + M\,\|\operatorname{\mathbf{div}}\boldsymbol{\Omega}\|_{L^q(B_1)}\right)^p. \tag{4.31}$$

In fact, one can always take $\gamma = 1/2$.

Proof. To distinguish the exponent originating from L^2-integrability from the spatial dimension, we will temporarily denote the domain dimension by m (where $m = 2$).

By Lemma 4.7, there exists a harmonic function $\boldsymbol{h} : B_{1/4} \to \mathbb{R}^{n+1}$ satisfying the gradient estimates (4.27) and (4.28). For $\theta \in (0, 1/4)$, the triangle inequality gives:

$$\Psi_p^{1/p}(\boldsymbol{u}, B_\theta) \leqslant \Psi_p^{1/p}(\boldsymbol{u}-\boldsymbol{h}, B_\theta) + \Psi_p^{1/p}(\boldsymbol{h}, B_\theta) =: I + II. \tag{4.32}$$

For I, applying the Poincaré-Sobolev inequality (noting the exact matching of the Sobolev conjugate exponent p in dimension two), the coupled Caccioppoli-type estimate (see Lemma 4.7), and using that $r^2\Psi_p(\boldsymbol{u}, B_r) \leqslant \Psi_p(\boldsymbol{u}, B_1)$ because of the monotonicity of $r^2\Psi_p(\boldsymbol{u}, B_r)$, we deduce that:

$$\begin{aligned} I &\leqslant \frac{C_S}{(\omega_m\theta^m)^{1/p}}\|\nabla(\boldsymbol{u}-\boldsymbol{h})\|_{L^q(B_\theta)} && (4.33)\\ &\leqslant \frac{C_q}{\theta^{m/p}}\|\boldsymbol{\Omega}\|_{L^2(B_1)}\Psi_p^{1/p}(\boldsymbol{u}, B_1) + \frac{C_q}{\theta^{m/p}}\left(\|\boldsymbol{f}\|_{L^q(B_1)} + M\|\operatorname{\mathbf{div}}\boldsymbol{\Omega}\|_{L^q(B_1)}\right). && (4.34)\end{aligned}$$

Here, the constant C_q has been harmlessly redefined to absorb the Poincaré-Sobolev constant C_S and geometric constants.

It remains to estimate $II := \Psi_p^{1/p}(\boldsymbol{h}, B_\theta)$. For that, the mean value theorem and the gradient bound for $\boldsymbol{h}$ yield

$$II \leqslant \theta\sup_{B_{1/4}}|\nabla\boldsymbol{h}| \leqslant C_m\theta\Psi_p^{1/p}(\boldsymbol{u}, B_1) \tag{4.35}$$

Combining the estimates for I and II, and raising to the power p, we obtain the master decay inequality purely in terms of the original PDE data:

$$\Psi_p(\boldsymbol{u}, B_\theta) \leqslant C_{m,q}^p \left(\frac{\|\boldsymbol{\Omega}\|_{L^2(B_1)}^p}{\theta^m} + \theta^p \right) \Psi_p(\boldsymbol{u}, B_1) + \frac{C_{m,q}^p}{\theta^m} \left(\|\boldsymbol{f}\|_{L^q(B_1)} + M \|\operatorname{\mathbf{div}} \boldsymbol{\Omega}\|_{L^q(B_1)} \right)^p, \tag{4.36}$$

with $C_{m,q}$ a positive constant that depends only on m and q.

We fix the geometric decay factor $\gamma = 1/2$. First, select $\theta \in (0, 1/4)$ sufficiently small such that $C_{m,q}^p \theta^p \leqslant 1/4$. With θ fixed, we define the threshold parameter $\varepsilon_* > 0$ such that $C_{m,q}^p \varepsilon_*^p / \theta^m \leqslant 1/4$. Provided $\|\boldsymbol{\Omega}\|_{L^2(B_1)} < \varepsilon_*$, the scaling brackets in (4.36) sum to at most $1/2$. Setting $\kappa := C_{m,q}^p / \theta^m$ concludes the proof. □

4.4. **Proof of the Main Regularity Theorem (Theorem 1.2).** Let $\boldsymbol{u}$ be an arbitrary bounded weak solution to the generalized system. We fix its global amplitude bound $M := \|\boldsymbol{u}\|_{L^\infty(B_1)}$. Let ε_* be the critical threshold from Lemma 4.8. To invoke Theorem 4.5, we define the admissible class $\boldsymbol{\mathcal{S}}$ as the set of triplets $(\boldsymbol{u}, \boldsymbol{\Omega}, \boldsymbol{f}) \in \mathcal{X}$ satisfying the original system, the amplitude bound M, and the strict energy threshold:

$$\boldsymbol{\mathcal{S}} := \Big\{ (\boldsymbol{u}, \boldsymbol{\Omega}, \boldsymbol{f}) \in \mathcal{X} \,:\, -\Delta \boldsymbol{u} = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} + \boldsymbol{f}, \;\; \|\boldsymbol{u}\|_{L^\infty(B_1)} \leqslant M \;\text{ and }\; \|\boldsymbol{\Omega}\|_{L^2(B_1)} < \varepsilon_* \Big\}, \tag{4.37}$$

where the ambient space explicitly includes the divergence regularity:

$$\mathcal{X} := \Big(H^1 \cap L^\infty(B_1, \mathbb{R}^{n+1}) \Big) \times \Big\{ \boldsymbol{\Omega} \in L^2(B_1) \,:\, \operatorname{\mathbf{div}} \boldsymbol{\Omega} \in L^q(B_1) \Big\} \times L^q(B_1, \mathbb{R}^{n+1}). \tag{4.38}$$

We need to verify Axioms 4.2 and 4.4 required by Theorem 4.5.

Step 1: Verification of Axiom 4.2 (*closure under rescaling*). Let $(\boldsymbol{u}, \boldsymbol{\Omega}, \boldsymbol{f}) \in \boldsymbol{\mathcal{S}}$ and let $B_r(x_0) \subset B_1$. Consider the rescaled triple $(\boldsymbol{u}_{x_0,r}, \overline{\boldsymbol{\Omega}}_{x_0,r}, \tilde{\boldsymbol{f}}_{x_0,r})$. We need to show that this triple remains in $\boldsymbol{\mathcal{S}}$. Simple scaling shows that

$$-\Delta \boldsymbol{u}_{x_0,r} = \overline{\boldsymbol{\Omega}}_{x_0,r} \cdot \nabla \boldsymbol{u}_{x_0,r} + \tilde{\boldsymbol{f}}_{x_0,r}. \tag{4.39}$$

Furthermore, the global bound is trivially preserved ($\|\boldsymbol{u}_{x_0,r}\|_{L^\infty(B_1)} \leqslant M$), and the conformal invariance of the L^2 norm in dimension two implies $\|\overline{\boldsymbol{\Omega}}_{x_0,r}\|_{L^2(B_1)} = \|\boldsymbol{\Omega}\|_{L^2(B_r(x_0))} \leqslant \|\boldsymbol{\Omega}\|_{L^2(B_1)} < \varepsilon_*$. Thus, the class $\boldsymbol{\mathcal{S}}$ is closed under rescaling.

Step 2: Verification of Axiom 4.4 (*the decay property*). Axiom 4.4 is immediately satisfied because of Lemma 4.8.

Step 3: Conclusion and Removal of the small $\boldsymbol{\Omega}$*-energy Hypothesis.* Having verified the two axioms, the abstract regularity principle (Theorem 4.5) dictates that any map $\boldsymbol{u}$ belonging to a triple in $\boldsymbol{\mathcal{S}}$ is locally Hölder continuous in B_1. To conclude the proof of Theorem 1.2, it remains to explain why this regularity holds for *any* weak solution of the generalized system (1.3), not just those with small initial $\boldsymbol{\Omega}$-energy.

Let $\boldsymbol{u} \in H^1(B_1, \mathbb{R}^{n+1})$ be an arbitrary solution of (1.3). This map $\boldsymbol{u}$ might have a large total $\boldsymbol{\Omega}$-energy, meaning $(\boldsymbol{u}, \boldsymbol{\Omega}, \boldsymbol{f}) \notin \boldsymbol{\mathcal{S}}$. However, let $x_0 \in B_1$ be an arbitrary point, and let $R := 1 - |x_0| > 0$ denote its distance to the boundary. Because $\boldsymbol{\Omega} \in L^2(B_1)$, by the absolute continuity of the Lebesgue integral we can always find a sufficiently small radius $r \in (0, R)$ such that $\|\boldsymbol{\Omega}\|_{L^2(B_r(x_0))} < \varepsilon_*$.

If we now construct the rescaled pair $(\boldsymbol{u}_{x_0,r}, \overline{\boldsymbol{\Omega}}_{x_0,r}, \tilde{\boldsymbol{f}}_{x_0,r})$, it is defined on the unit ball B_1, solves the required equation, and satisfies the small-energy threshold $\left\|\overline{\boldsymbol{\Omega}}_{x_0,r}\right\|_{L^2(B_1)} < \varepsilon_*$. Consequently, this rescaled triple belongs to our class $\boldsymbol{\mathcal{S}}$. By Theorem 4.5, the rescaled map $\boldsymbol{u}_{x_0,r}$ is locally Hölder continuous in B_1. Scaling back to the original spatial coordinates, this implies that the original map $\boldsymbol{u}$ is Hölder continuous in the neighborhood $B_{r/2}(x_0)$.

Since the point $x_0 \in B_1$ was chosen arbitrarily, the map $\boldsymbol{u}$ is locally Hölder continuous everywhere in B_1, completing the proof.

5. Regularity of anisotropic systems with general connection forms

In the preceding section, we showed that the strict geometric requirement of antisymmetry on the connection matrix $\boldsymbol{\Omega}$ can be replaced by the purely analytic condition $\mathbf{div}\,\boldsymbol{\Omega} \in L^q$, without further modifying the regularity machinery. The principal differential operator, however, remained the standard Laplacian; equivalently, the coefficient matrix multiplying $\nabla \boldsymbol{u}$ was implicitly the identity. The natural next question is whether the isotropy of the principal part can be relaxed as well and, if so, what regularity its coefficients must possess for the Campanato decay mechanism to survive. The motivation is not merely formal. Anisotropic Dirichlet energies of the form

$$\mathcal{E}(\boldsymbol{u}, B_1) := \frac{1}{2}\int_{B_1} A\nabla \boldsymbol{u} : \nabla \boldsymbol{u}\,\mathrm{d}x, \tag{5.1}$$

possibly augmented by lower-order potential terms, model inhomogeneous physical media such as liquid crystals with spatially varying dielectric tensors, micromagnetic energies in non-uniform materials, and elastic membranes with prescribed anisotropy. Their critical points solve systems of the form

$$-\,\mathbf{div}(A\nabla \boldsymbol{u}) = \boldsymbol{\Omega}\cdot\nabla\boldsymbol{u} + \boldsymbol{f}, \tag{5.2}$$

where $A(x)$ is a symmetric, uniformly elliptic matrix encoding the anisotropy of the medium, and $\boldsymbol{\Omega}$ captures the geometric or constraint structure of the target.

When A is constant, (5.2) reduces, up to an invertible linear change of variables, to the system treated in the previous section. When A is variable but its determinant $\det A$ is constant, a two-dimensional uniformization argument recasts (5.2) as a harmonic-map-type equation on a Riemannian surface conformally equivalent to the disc, after which the previous theory applies with minor adjustments. Neither reduction is available, however, in the *non-unimodular* regime where $\det A$ varies in space. Such systems lie outside the reach of gauge-theoretic and uniformization-based methods, demanding a direct PDE treatment.

The primary objective of this section is to formally establish the following generalization (we emphasize that $\boldsymbol{\Omega}$ is *not* assumed to be antisymmetric, and the principal operator is *not* assumed to be the Laplacian):

Theorem 1.3. (Regularity of anisotropic systems) *Let $B_1 \subset \mathbb{R}^2$. Let $\boldsymbol{u} \in H^1 \cap L^\infty(B_1, \mathbb{R}^{n+1})$ be a weak solution of*

$$-\,\mathbf{div}(A\nabla \boldsymbol{u}) = \boldsymbol{\Omega}\cdot\nabla\boldsymbol{u} + \boldsymbol{f} \qquad \text{in } B_1. \tag{5.3}$$

Assume the structural data satisfy, for some $q > 1$ and some $\gamma_0 \in (0, 1]$:

(A1) $A : B_1 \to \mathbb{R}^{2\times 2}$ *is symmetric, uniformly elliptic, and Hölder continuous:*

$$\lambda|\xi|^2 \leqslant A(x)\xi\cdot\xi \leqslant \Lambda|\xi|^2 \qquad \forall x \in B_1, \xi \in \mathbb{R}^2, \quad 0 < \lambda \leqslant \Lambda < +\infty, \tag{5.4}$$

and $A \in C^{0,\gamma_0}(\overline{B_1})$;

(A2) $\boldsymbol{\Omega} \in L^2(B_1, \mathbb{R}^{(n+1)\times(n+1)} \otimes \mathbb{R}^2)$;

(A3) $\boldsymbol{f} \in L^q(B_1, \mathbb{R}^{n+1})$;

(A4) $\mathbf{div}\,\boldsymbol{\Omega} \in L^q(B_1, \mathbb{R}^{(n+1)\times(n+1)})$.

Then $\boldsymbol{u}$ is locally Hölder continuous in B_1. Specifically, there exists an exponent $\eta \in (0, 1)$ such that $\boldsymbol{u} \in C^{0,\eta}_{\mathrm{loc}}(B_1, \mathbb{R}^{n+1})$.

Remark 5.1. In what follows, we may assume without loss of generality that $1 < q < 2$. The reason is identical to that explained in Remark 3.8.

Remark 5.2 (Necessity of Hölder Continuous Coefficients). The Hölder continuity of the principal part A is critical for controlling the freezing flux $\tilde{A}\nabla \boldsymbol{u}$ in the master iteration inequality (5.58). To deduce local Hölder continuity for $\boldsymbol{u}$ via the algebraic iteration lemma (Lemma 3.6), the coefficient error $\|\tilde{A}\|_{L^\infty(B_r)}$ must decay polynomially as $r \to 0$. The C^{0,γ_0} regularity of A explicitly provides this rate:

$$\|\tilde{A}\|_{L^\infty(B_r)} \leqslant r^{\gamma_0}[A]_{C^{0,\gamma_0}(\overline{B_1})} \qquad \forall r \in (0,1]. \tag{5.5}$$

This guarantees that the freezing contribution shrinks as $r^{p\gamma_0}$, allowing it to be safely absorbed into the recursive iteration. While uniform continuity of A would suffice to prove the mere continuity of $\boldsymbol{u}$, the Hölder hypothesis is structurally required to lock in the algebraic decay rate.

The proof of Theorem 1.3 is given in Subsection 5.4. To this end, we first introduce a slight extension of the abstract regularity principle of Theorem 4.5 to accommodate the additional structural datum A.

5.1. **The anisotropic abstract regularity principle.** We now introduce a second, distinct abstract regularity framework designed specifically for anisotropic systems. It is important to emphasize that this result does not subsume the generalized abstract regularity principle of the previous sections as a special case; rather, it stands as a complementary result. The necessity for a separate framework stems from the fundamental analytical cost of handling variable principal coefficients.

In the isotropic setting, the principal operator is the Laplacian, allowing the theory to bypass energy scaling barriers and float freely into arbitrary dimensions m. However, when the principal part A is variable, the Campanato iteration must rely on a localized coefficient-freezing argument. This approximation inherently introduces a residual error driven by the L^2-norm of the gradient which restricts the validity of the perturbative freezing technique to dimensions $m < 4$ (see Remark 5.8). Thus, the two abstract principles offer a clear analytical dichotomy: the isotropic framework sacrifices variable coefficients to achieve regularity in arbitrary dimensions, whereas the anisotropic framework sacrifices dimensional freedom to successfully navigate rough, inhomogeneous media.

Let $\boldsymbol{\mathcal{S}}$ be a class of quadruples $(\boldsymbol{u}, \boldsymbol{\Omega}, \boldsymbol{f}, A)$ belonging to the space

$$\begin{aligned}\mathcal{Y} \;:=\; & \Big(H^1 \cap L^\infty(B_1, \mathbb{R}^{n+1})\Big) \times L^2(B_1, \mathbb{R}^{(n+1)\times(n+1)} \otimes \mathbb{R}^m) \\ & \times L^q(B_1, \mathbb{R}^{n+1}) \times C^{0,\gamma_0}(\overline{B_1}, \mathbb{R}^{m\times m}_{\text{sym}}),\end{aligned} \tag{5.6}$$

where A is uniformly elliptic with constants $0 < \lambda \leqslant \Lambda < \infty$ and $\operatorname{\mathbf{div}} \boldsymbol{\Omega} \in L^q(B_1)$. For any ball $B_r(x_0) \subset B_1$, we define the rescaled quadruple on B_1 by

$$\boldsymbol{u}_{x_0,r}(y) := \boldsymbol{u}(x_0 + ry), \qquad \overline{\boldsymbol{\Omega}}_{x_0,r}(y) := r\boldsymbol{\Omega}(x_0 + ry), \tag{5.7}$$

$$\tilde{\boldsymbol{f}}_{x_0,r}(y) := r^2 \boldsymbol{f}(x_0 + ry), \qquad A_{x_0,r}(y) := A(x_0 + ry). \tag{5.8}$$

Remark 5.3 (Scaling identities)*.* The rescaling (5.8) preserves the ellipticity bounds of A identically and obeys the algebraic identities:

$$\begin{aligned}
\|\boldsymbol{u}_{x_0,r}\|_{L^\infty(B_1)} &= \|\boldsymbol{u}\|_{L^\infty(B_r(x_0))}, && (5.9)\\
\|\nabla \boldsymbol{u}_{x_0,r}\|_{L^2(B_1)} &= r^{(2-m)/2}\|\nabla \boldsymbol{u}\|_{L^2(B_r(x_0))}, && (5.10)\\
\|\overline{\boldsymbol{\Omega}}_{x_0,r}\|_{L^2(B_1)} &= r^{(2-m)/2}\|\boldsymbol{\Omega}\|_{L^2(B_r(x_0))}, && (5.11)\\
\|\operatorname{\mathbf{div}}_y \overline{\boldsymbol{\Omega}}_{x_0,r}\|_{L^q(B_1)} &= r^{2-m/q}\|\operatorname{\mathbf{div}}_x \boldsymbol{\Omega}\|_{L^q(B_r(x_0))}, && (5.12)\\
\|\tilde{\boldsymbol{f}}_{x_0,r}\|_{L^q(B_1)} &= r^{2-m/q}\|\boldsymbol{f}\|_{L^q(B_r(x_0))}, && (5.13)\\
[A_{x_0,r}]_{C^{0,\gamma_0}(\overline{B_1})} &= r^{\gamma_0}[A]_{C^{0,\gamma_0}(\overline{B_r(x_0)})}. && (5.14)
\end{aligned}$$

In dimension $m = 2$, the gradient L^2-norm and the connection L^2-norm are scale-invariant ((5.10)–(5.11)); the source norms decay with exponent $2 - m/q > 0$ (whenever $q > m/2$); and the Hölder seminorm of A contracts with exponent γ_0. These three decay mechanisms are the engine driving the iteration in Theorem 5.6 below.

The class is required to obey the two axioms below.

Axiom 5.4 (Locality and closure)**.** The class $\boldsymbol{\mathcal{S}}$ is closed under the rescaling (5.8): for any $(\boldsymbol{u}, \boldsymbol{\Omega}, \boldsymbol{f}, A) \in \boldsymbol{\mathcal{S}}$ and any $B_r(x_0) \subset B_1$, the rescaled quadruple $(\boldsymbol{u}_{x_0,r}, \overline{\boldsymbol{\Omega}}_{x_0,r}, \tilde{\boldsymbol{f}}_{x_0,r}, A_{x_0,r})$ is again in $\boldsymbol{\mathcal{S}}$ over the unit ball $B_1 \subseteq \mathbb{R}^m$.

Axiom 5.5 (Decay property)**.** There exist structural constants $\theta \in (0, 1)$, $\gamma \in (0, 1)$, $p \geqslant 1$, an amplitude bound $M > 0$, and $\kappa > 0$, such that for any quadruple $(\boldsymbol{u}, \boldsymbol{\Omega}, \boldsymbol{f}, A) \in \boldsymbol{\mathcal{S}}$ satisfying $\|\boldsymbol{u}\|_{L^\infty(B_1)} \leqslant M$, the oscillation contracts at the interior scale:

$$\Psi_p(\boldsymbol{u}, B_\theta) \leqslant \gamma \cdot \Psi_p(\boldsymbol{u}, B_1) + \kappa\,(D^p_{\mathrm{src}} + D^p_{\mathrm{frz}})\,, \tag{5.15}$$

where the source and freezing data are

$$D_{\mathrm{src}} := \|\boldsymbol{f}\|_{L^q(B_1)} + M\|\operatorname{\mathbf{div}}\boldsymbol{\Omega}\|_{L^q(B_1)}, \qquad D_{\mathrm{frz}} := \|\tilde{A}\|_{L^\infty(B_1)}\|\nabla \boldsymbol{u}\|_{L^2(B_1)}, \tag{5.16}$$

with $\tilde{A} := A - A(0)$.

Proceeding as in the proof of Theorem 4.5, one obtains the following abstract result.

Theorem 5.6 (Anisotropic abstract regularity principle)**.** *Let $\boldsymbol{\mathcal{S}}$ be a class of quadruples satisfying* Axioms 5.4 *and* 5.5 *on the unit ball $B_1 \subset \mathbb{R}^m$. Let $(\boldsymbol{u}, \boldsymbol{\Omega}, \boldsymbol{f}, A) \in \boldsymbol{\mathcal{S}}$ with $\|\boldsymbol{u}\|_{L^\infty(B_1)} \leqslant M$. Assume that A is Hölder continuous of exponent $\gamma_0 \in (0, 1]$ satisfying $\gamma_0 > \frac{m-2}{2}$ and that $q > m/2$. Then $\boldsymbol{u}$ is locally Hölder continuous in B_1: $\boldsymbol{u} \in C^{0,\eta}_{\mathrm{loc}}(B_1)$ for every exponent η satisfying*

$$\eta < \min\left(\frac{\ln\gamma}{p\ln\theta}, 2 - \frac{m}{q}, \gamma_0 + \frac{2-m}{2}\right). \tag{5.17}$$

For any compact $K \subset B_1$, the corresponding Hölder seminorm depends only on η, the structural parameters $\theta, \gamma, \kappa, p, M$, the data norms $\|\operatorname{\mathbf{div}}\boldsymbol{\Omega}\|_{L^q(B_1)}$, $\|\boldsymbol{f}\|_{L^q(B_1)}$, $\|\nabla \boldsymbol{u}\|_{L^2(B_1)}$, $[A]_{C^{0,\gamma_0}(\overline{B_1})}$, and the distance $\mathrm{dist}(K, \partial B_1)$*.*

Remark 5.7. Notice that when $m = 2$, the term $\frac{2-m}{2}$ vanishes, and we get that $\boldsymbol{u} \in C^{0,\eta}_{\mathrm{loc}}(B_1)$ for every exponent η satisfying

$$\eta < \min\left(\frac{\ln\gamma}{p\ln\theta}, 2 - \frac{2}{q}, \gamma_0\right). \tag{5.18}$$

That is, the third entry reduces to γ_0, completely independent of the integrability exponent p.

Remark 5.8 (The Coefficient-Freezing Barrier and Dimensionality). The requirement $\gamma_0 > (m-2)/2$ highlights a fundamental structural distinction between the anisotropic and isotropic frameworks. When A is variable, local coefficient freezing introduces a residual error governed by the flux $\tilde{A}\nabla \boldsymbol{u}$, which contributes $\|\tilde{A}\|_{L^\infty(B_r)}\|\nabla \boldsymbol{u}\|_{L^2(B_r)}$ to the Campanato iteration. As $r \to 0$, Hölder continuity yields $\|\tilde{A}\|_{L^\infty(B_r)} \lesssim r^{\gamma_0}$, while the critical Dirichlet scaling dictates $\|\nabla \boldsymbol{u}\|_{L^2(B_r)} \sim r^{(2-m)/2}$. For this residual to act as an absorbable higher-order perturbation, the net exponent must be positive: $\gamma_0 + (2-m)/2 > 0$. Because Hölder continuity restricts $\gamma_0 \leqslant 1$, this condition is only satisfiable when $(m-2)/2 < 1$, strictly limiting this technique to dimensions $m < 4$. Extending this iteration to $m \geqslant 4$ formally requires $\gamma_0 > 1$ (implying $A \in C^1$), but such strong regularity renders the freezing machinery redundant, as classical elliptic estimates would apply directly. In contrast, for the isotropic setting ($A \equiv I$), the residual $\tilde{A}$ vanishes entirely, excising this dimensional barrier and allowing the regularity principle to hold in arbitrary dimensions.

Proof. We follow the structure of the proof of Theorem 4.5 in five steps, indicating only the modifications required by the additional freezing datum.

Step 1: Master iteration inequality. Fix $x_0 \in B_{1/2}$ and set $r_0 := 1/2$ so that $B_r(x_0) \subset B_1$ for every $r \in (0, r_0]$. Define, for $r \in (0, r_0]$, $\psi(r) := r^m \Psi_p(\boldsymbol{u}, B_r(x_0))$. By Axiom 5.4, the rescaled quadruple $(\boldsymbol{u}_{x_0,r}, \overline{\boldsymbol{\Omega}}_{x_0,r}, \tilde{\boldsymbol{f}}_{x_0,r}, A_{x_0,r})$ belongs to $\boldsymbol{\mathcal{S}}$. Note that $\|\boldsymbol{u}_{x_0,r}\|_{L^\infty(B_1)} = \|\boldsymbol{u}\|_{L^\infty(B_r(x_0))} \leqslant M$. Applying Axiom 5.5 to the rescaled quadruple,

$$\Psi_p(\boldsymbol{u}_{x_0,r}, B_\theta) \leqslant \gamma \Psi_p(\boldsymbol{u}_{x_0,r}, B_1) + \kappa \left(\bar{D}^p_{\mathrm{src},r} + \bar{D}^p_{\mathrm{frz},r}\right), \tag{5.19}$$

where $\bar{D}_{\mathrm{src},r}$ and $\bar{D}_{\mathrm{frz},r}$ denote the source and freezing data computed for the rescaled quadruple.

Translating the oscillation estimates back to the original coordinates via the standard scaling identity $\Psi_p(\boldsymbol{u}x_0, r, B\theta r) = \Psi_p(\boldsymbol{u}, B_{\theta r}(x_0))$, we obtain

$$\begin{aligned} \Psi_p(\boldsymbol{u}_{x_0,r}, B_\theta) &= (\theta r)^{-m}\psi(\theta r), \\ \Psi_p(\boldsymbol{u}_{x_0,r}, B_1) &= r^{-m}\psi(r). \end{aligned} \tag{5.20}$$

For the source term, by (5.13) and (5.12),

$$\bar{D}_{\mathrm{src},r} \leqslant r^{2-m/q}\left(\|\boldsymbol{f}\|_{L^q(B_1)} + M\|\operatorname{\mathbf{div}}\boldsymbol{\Omega}\|_{L^q(B_1)}\right) = r^{2-m/q}D_{\mathrm{src}}. \tag{5.21}$$

For the freezing term, by (5.10) and (5.5),

$$\bar{D}_{\mathrm{frz},r} = \|\tilde{A}_{x_0,r}\|_{L^\infty(B_1)}\|\nabla \boldsymbol{u}_{x_0,r}\|_{L^2(B_1)} \leqslant r^{\gamma_0+(2-m)/2}[A]_{C^{0,\gamma_0}}\|\nabla \boldsymbol{u}\|_{L^2(B_1)}, \tag{5.22}$$

where in the rescaled quadruple $\tilde{A}_{x_0,r} = A_{x_0,r} - A_{x_0,r}(0) = A(x_0 + r\cdot) - A(x_0)$. Substituting the previous estimates into (5.19) and multiplying by $(\theta r)^m$, we obtain

$$\psi(\theta r) \leqslant \theta^m \gamma \psi(r) + b_{\mathrm{src}} r^{\beta_{\mathrm{src}}} + b_{\mathrm{frz}} r^{\beta_{\mathrm{frz}}}, \tag{5.23}$$

with exponents

$$\beta_{\mathrm{src}} := m + p(2 - m/q), \quad \beta_{\mathrm{frz}} := m + p\left(\gamma_0 + \frac{2-m}{2}\right), \tag{5.24}$$

and constants

$$b_{\mathrm{src}} := \kappa\theta^m D^p_{\mathrm{src}}, \quad b_{\mathrm{frz}} := \kappa\theta^m [A]^p_{C^{0,\gamma_0}(\overline{B_1})}\|\nabla \boldsymbol{u}\|^p_{L^2(B_1)}, \tag{5.25}$$

depending only on the global data $D_{\mathrm{src}}, D_{\mathrm{frz}}$, κ, θ, M, $[A]_{C^{0,\gamma_0}}$. Since both exponents satisfy $\beta_{\mathrm{src}} > m$ (because $q > m/2$) and $\beta_{\mathrm{frz}} > m$ (because we assumed $\gamma_0 > (m-2)/2$; in particular for $m = 2$, this reduces to $\gamma_0 > 0$), setting $\beta := \min(\beta_{\mathrm{src}}, \beta_{\mathrm{frz}})$ and $b := b_{\mathrm{src}} + b_{\mathrm{frz}}$ we obtain the master iteration inequality

$$\psi(\theta r) \leqslant \theta^m \gamma \psi(r) + br^\beta \qquad \forall r \in (0, r_0]. \tag{5.26}$$

Step 2: Continuous algebraic decay via the iteration lemma. Identical to Step 2 of the proof of Theorem 4.5. Fix an arbitrarily small $\varepsilon > 0$ and set $\gamma_\varepsilon := \max(\gamma, \theta^{\beta-m-\varepsilon})$. With $\alpha_\varepsilon := m + \min(\ln\gamma/\ln\theta, \beta - m - \varepsilon)$, we have $\beta - \alpha_\varepsilon \geqslant \varepsilon > 0$, and by Lemma 3.6 there exists a structural constant $c_\varepsilon > 0$ such that for all $r \in (0, r_0]$,

$$\Psi_p(\boldsymbol{u}, B_r(x_0)) \leqslant c_\varepsilon \left[\frac{\Psi_p(\boldsymbol{u}, B_{r_0}(x_0))}{r_0^{\alpha_\varepsilon - m}} + br_0^{\beta-\alpha_\varepsilon} \right] r^{\alpha_\varepsilon - m}. \tag{5.27}$$

Step 3: Uniform bound via $\|\boldsymbol{u}\|_{L^\infty} \leqslant M$. Choosing $\boldsymbol{c} = \boldsymbol{0}$ in the infimum defining the oscillation $\Psi_p(\boldsymbol{u}, B_{r_0}(x_0))$ yields $\Psi_p(\boldsymbol{u}, B_{r_0}(x_0)) \leqslant M^p$. Substituting into (5.27) produces

$$\Psi_p(\boldsymbol{u}, B_r(x_0)) \leqslant M_* r^{\alpha_\varepsilon - m}, \tag{5.28}$$

for a constant M_* independent of $x_0 \in B_{1/2}$ and $r \in (0, r_0]$, with

$$\alpha_\varepsilon - m = \min\left(\frac{\ln\gamma}{\ln\theta}, p(2 - m/q) - \varepsilon, p(\gamma_0 + (2-m)/2) - \varepsilon\right). \tag{5.29}$$

Step 4: Campanato condition and Hölder continuity on $B_{1/2}$. By Campanato's theorem, $\boldsymbol{u} \in C^{0,\eta_\varepsilon}(B_{1/2})$ with $\eta_\varepsilon = (\alpha_\varepsilon - m)/p$. Using our formula for $(\alpha_\varepsilon - m)$ yields:

$$\eta_\varepsilon = \min\left(\frac{\ln\gamma}{p\ln\theta}, 2 - \frac{m}{q} - \frac{\varepsilon}{p}, \gamma_0 + \frac{2-m}{2} - \frac{\varepsilon}{p}\right). \tag{5.30}$$

Sending $\varepsilon \to 0^+$ recovers (5.17) with the third entry $\gamma_0 + \frac{2-m}{2}$ for the general-dimension statement.

Step 5: Extension to arbitrary compact subsets. Identical to Step 5 in the proof of Theorem 4.5. □

5.2. **Anisotropic Caccioppoli-type estimate in dimension $m = 2$.** To apply the abstract regularity principle (Theorem 5.6) we must show that solutions of (5.3) inherently control the oscillation at small scales. We achieve this by locally restructuring the equation in divergence form, with a coefficient-freezing step that converts $-\,\mathbf{div}(A\nabla\cdot)$ into the constant-coefficient operator $-\,\mathbf{div}(A_0\nabla\cdot)$ at the cost of an explicit freezing flux.

Fix $x_0 \in B_1$ (which we may translate to the origin). Set $A_0 := A(x_0)$ and $\tilde{A}(x) := A(x) - A_0$. Then A_0 is symmetric, $\lambda I \leqslant A_0 \leqslant \Lambda I$, and (5.5) holds. For any constant vector $\boldsymbol{c} \in \mathbb{R}^{n+1}$, the Leibniz rule gives

$$\boldsymbol{\Omega} \cdot \nabla\boldsymbol{u} = \boldsymbol{\Omega} \cdot \nabla(\boldsymbol{u} - \boldsymbol{c}) = \mathbf{div}\,(\boldsymbol{\Omega}(\boldsymbol{u} - \boldsymbol{c})) - (\mathbf{div}\,\boldsymbol{\Omega})(\boldsymbol{u} - \boldsymbol{c}). \tag{5.31}$$

Substituting into (5.3) and adding/subtracting $\mathbf{div}(A_0\nabla\boldsymbol{u})$, we rewrite the equation as

$$-\,\mathbf{div}(A_0\nabla\boldsymbol{u}) = \mathbf{div}\underbrace{\left(\tilde{A}\nabla\boldsymbol{u} + \boldsymbol{\Omega}(\boldsymbol{u} - \boldsymbol{c})\right)}_{=:\boldsymbol{J}} + \underbrace{\boldsymbol{f} - (\mathbf{div}\,\boldsymbol{\Omega})(\boldsymbol{u} - \boldsymbol{c})}_{=:\boldsymbol{F}}. \tag{5.32}$$

The flux $\boldsymbol{J}$ has two natural sub-contributions: the *convective* term $\boldsymbol{\Omega}(\boldsymbol{u} - \boldsymbol{c})$, already present in the isotropic case, and the new *freezing* term $\tilde{A}\nabla\boldsymbol{u}$, which vanishes algebraically as $r \to 0$ by (5.5).

Lemma 5.9 (Anisotropic Caccioppoli-type estimate, dimension $m = 2$)**.** *Let $q \in (1,2)$, and let $p := 2q/(2-q) \in (2,\infty)$ be the corresponding oscillation exponent. Assume* (A1)–(A4)*. Let $\boldsymbol{u} \in H^1 \cap L^\infty(B_1, \mathbb{R}^{n+1})$ be a weak solution of* (5.3) *with $\|\boldsymbol{u}\|_{L^\infty(B_1)} \leqslant M$. Fix $r \in (0,1]$, and let $\boldsymbol{h}_r \in H^1(B_r, \mathbb{R}^{n+1})$ be the unique A_0-harmonic extension of $\boldsymbol{u}_{|\partial B_r}$:*

$$-\operatorname{\mathbf{div}}(A_0 \nabla \boldsymbol{h}_r) = \mathbf{0} \quad \text{in } B_r, \qquad \boldsymbol{h}_r = \boldsymbol{u} \quad \text{on } \partial B_r. \tag{5.33}$$

Set $A_0 := A(0)$ and $\tilde{A} := A - A_0$. Then

$$\|\nabla(\boldsymbol{u} - \boldsymbol{h}_r)\|_{L^q(B_r)} \leqslant C_0 \|\boldsymbol{\Omega}\|_{L^2(B_r)} r^{2/p} \Psi_p^{1/p}(\boldsymbol{u}, B_r) + C_0 \|\tilde{A}\|_{L^\infty(B_r)} r^{2/p} \|\nabla \boldsymbol{u}\|_{L^2(B_r)} + C_0 r \left(\|\boldsymbol{f}\|_{L^q(B_r)} + 2M \|\operatorname{\mathbf{div}} \boldsymbol{\Omega}\|_{L^q(B_r)} \right), \tag{5.34}$$

where $C_0 = C_0(m, q, \lambda, \Lambda)$ depends only on the dimension, the integrability exponent, and the ellipticity bounds. In particular, C_0 does not depend on the modulus of continuity of A, on r, or on $\boldsymbol{u}$.

Proof. Since $\|\boldsymbol{u}\|_{L^\infty} \leqslant M$, by Lemma 2.5 we may restrict to constants $\boldsymbol{c} \in \mathbb{R}^{n+1}$ with $|\boldsymbol{c}| \leqslant M$. Fix such an optimal $\boldsymbol{c} = \boldsymbol{c}_*$ in $\Psi_p(\boldsymbol{u}, B_r)$; then $|\boldsymbol{u} - \boldsymbol{c}| \leqslant 2M$ pointwise, and $\|\boldsymbol{u} - \boldsymbol{c}\|_{L^p(B_r)} = \omega_m^{1/p} r^{2/p} \Psi_p^{1/p}(\boldsymbol{u}, B_r)$.

Set $\boldsymbol{w} := \boldsymbol{u} - \boldsymbol{h}_r \in H_0^1(B_r, \mathbb{R}^{n+1})$. Since $\operatorname{\mathbf{div}}(A_0 \nabla \boldsymbol{h}_r) = \mathbf{0}$ (A_0 constant, $\boldsymbol{h}_r$ is A_0-harmonic), subtracting from (5.32) yields the constant-coefficient Dirichlet problem

$$\begin{cases} -\operatorname{\mathbf{div}}(A_0 \nabla \boldsymbol{w}) = \operatorname{\mathbf{div}} \boldsymbol{J} + \boldsymbol{F} & \text{in } B_r, \\ \boldsymbol{w} = \mathbf{0} & \text{on } \partial B_r, \end{cases} \tag{5.35}$$

with $\boldsymbol{J}, \boldsymbol{F}$ as in (5.32). By linearity we can decompose $\boldsymbol{w} = \boldsymbol{w}_{\boldsymbol{J}} + \boldsymbol{w}_{\boldsymbol{F}}$, with $\boldsymbol{w}_{\boldsymbol{J}}$ and $\boldsymbol{w}_{\boldsymbol{F}}$ the unique solutions of the Poisson problems $-\operatorname{\mathbf{div}}(A_0 \nabla \boldsymbol{w}_{\boldsymbol{J}}) = \operatorname{\mathbf{div}} \boldsymbol{J}$ and $-\operatorname{\mathbf{div}}(A_0 \nabla \boldsymbol{w}_{\boldsymbol{F}}) = \boldsymbol{F}$, both with zero boundary data on ∂B_r.

Step 1: Calderón–Zygmund bound for the flux part. By the divergence-form Calderón–Zygmund estimate for the constant-coefficient operator $-\operatorname{\mathbf{div}}(A_0 \nabla \cdot)$ on the ball B_r (which is scale-invariant in the form of the source),

$$\|\nabla \boldsymbol{w}_{\boldsymbol{J}}\|_{L^q(B_r)} \leqslant C_q(m, q, \lambda, \Lambda) \|\boldsymbol{J}\|_{L^q(B_r)}. \tag{5.36}$$

Step 2: Source-to-divergence representation. We re-express $\boldsymbol{F} \in L^q$ as a divergence: solve $-\Delta \boldsymbol{\varphi} = \boldsymbol{F}$ in B_r with $\boldsymbol{\varphi} = \mathbf{0}$ on ∂B_r. Standard $W^{2,q}$-theory gives $\|D^2 \boldsymbol{\varphi}\|_{L^q(B_r)} \leqslant C \|\boldsymbol{F}\|_{L^q(B_r)}$. Because $\boldsymbol{\varphi}$ vanishes on the boundary, the divergence theorem guarantees that its gradient has a mean of zero over B_r. Applying the Poincaré–Wirtinger inequality to $\nabla \boldsymbol{\varphi}$ yields $\|\nabla \boldsymbol{\varphi}\|_{L^q(B_r)} \leqslant C r \|D^2 \boldsymbol{\varphi}\|_{L^q(B_r)}$. Setting $\boldsymbol{G} := \nabla \boldsymbol{\varphi}$, we have $\operatorname{\mathbf{div}} \boldsymbol{G} = -\boldsymbol{F}$ with

$$\|\boldsymbol{G}\|_{L^q(B_r)} \leqslant C r \|\boldsymbol{F}\|_{L^q(B_r)}. \tag{5.37}$$

Applying (5.36) to the divergence-form problem $-\operatorname{\mathbf{div}}(A_0 \nabla \boldsymbol{w}_{\boldsymbol{F}}) = \operatorname{\mathbf{div}}(-\boldsymbol{G})$,

$$\|\nabla \boldsymbol{w}_{\boldsymbol{F}}\|_{L^q(B_r)} \leqslant C_q \|-\boldsymbol{G}\|_{L^q(B_r)} \leqslant C r \|\boldsymbol{F}\|_{L^q(B_r)}. \tag{5.38}$$

Step 3: Bounds on the flux pieces. Combining (5.36) and (5.38),

$$\|\nabla \boldsymbol{w}\|_{L^q(B_r)} \leqslant C \left[\|\tilde{A} \nabla \boldsymbol{u}\|_{L^q(B_r)} + \|\boldsymbol{\Omega}(\boldsymbol{u} - \boldsymbol{c})\|_{L^q(B_r)} + r \|\boldsymbol{F}\|_{L^q(B_r)} \right]. \tag{5.39}$$

We bound each piece via Hölder's inequality. By Hölder with exponents $(2, p)$, $1/q = 1/2 + 1/p$:

$$\|\boldsymbol{\Omega}(\boldsymbol{u} - \boldsymbol{c})\|_{L^q(B_r)} \leqslant \|\boldsymbol{\Omega}\|_{L^2(B_r)} \|\boldsymbol{u} - \boldsymbol{c}\|_{L^p(B_r)} = \omega_m^{1/p} \|\boldsymbol{\Omega}\|_{L^2(B_r)} r^{2/p} \Psi_p^{1/p}(\boldsymbol{u}, B_r). \tag{5.40}$$

A second Hölder with the constant function 1 and exponents $(2/q, 2/(2-q))$ (since $q<2$), together with $|B_r|^{1/p} = \omega_m^{1/p} r^{2/p}$, gives

$$\|\nabla \boldsymbol{u}\|_{L^q(B_r)} \leqslant \|\nabla \boldsymbol{u}\|_{L^2(B_r)} |B_r|^{1/p} = \omega_m^{1/p} r^{2/p} \|\nabla \boldsymbol{u}\|_{L^2(B_r)}. \tag{5.41}$$

Hence

$$\|\tilde{A}\nabla \boldsymbol{u}\|_{L^q(B_r)} \leqslant \omega_m^{1/p} \|\tilde{A}\|_{L^\infty(B_r)} r^{2/p} \|\nabla \boldsymbol{u}\|_{L^2(B_r)}. \tag{5.42}$$

Pointwise $|\boldsymbol{F}| \leqslant |\boldsymbol{f}| + 2M|\operatorname{\mathbf{div}} \boldsymbol{\Omega}|$, hence

$$\|\boldsymbol{F}\|_{L^q(B_r)} \leqslant \|\boldsymbol{f}\|_{L^q(B_r)} + 2M\|\operatorname{\mathbf{div}} \boldsymbol{\Omega}\|_{L^q(B_r)}. \tag{5.43}$$

Substituting (5.40)–(5.43) into (5.39) and absorbing the universal factor $\omega_m^{1/p}$ into the constant $C_0 = C_0(m, q, \lambda, \Lambda)$ yields (5.34). □

We now extend the harmonic approximation lemma previously established in the antisymmetric setting (Lemma 3.19) to the class of A_0-harmonic functions. To this end, we rely on estimates for the anisotropic Poisson kernel $P_{A_0}(x,\zeta)$, established in Proposition 8.5 of the Appendix, Section 8.

Lemma 5.10 (Anisotropic harmonic approximation)**.** *Let $A_0 \in \mathbb{R}^{m\times m}$ be a constant symmetric matrix satisfying the uniform ellipticity condition $\lambda I \leqslant A_0 \leqslant \Lambda I$. Let $\boldsymbol{u} \in H^1(B_1, \mathbb{R}^{n+1})$.*

For any $p>1$, there exists a radius $r \in [1/2, 1]$ and an A_0-harmonic function $\boldsymbol{h} : B_r \to \mathbb{R}^{n+1}$ satisfying the boundary trace $\boldsymbol{h} = \boldsymbol{u}$ on ∂B_r, such that the interior gradient satisfies:

$$\sup_{x\in B_{1/4}} |\nabla \boldsymbol{h}(x)| \leqslant C_A \cdot \Psi_p^{1/p}(\boldsymbol{u}, B_1), \tag{5.44}$$

where $C_A > 0$ is a dimensional constant depending on m, λ, and Λ.

Proof. Step 1: Selection of a good radius. Let $\boldsymbol{c} \in \mathbb{R}^{n+1}$ be a constant vector that realizes this minimum for $\Psi_p(\boldsymbol{u}, B_1)$:

$$⨍_{B_1} |\boldsymbol{u} - \boldsymbol{c}|^p = \Psi_p(\boldsymbol{u}, B_1). \tag{5.45}$$

As explained in the proof of Lemma 3.19, Fubini's theorem in polar coordinates guarantees the existence of a radius $r \in [1/2, 1]$ such that the boundary integral is controlled by the bulk integral:

$$\int_{\partial B_r} |\boldsymbol{u}(\zeta) - \boldsymbol{c}|\, \mathrm{d}\zeta \leqslant 2\int_{B_1} |\boldsymbol{u}(x) - \boldsymbol{c}|\, \mathrm{d}x. \tag{5.46}$$

Note that r depends on the optimal $\boldsymbol{c}$, which in turn depends on $\boldsymbol{u}$ and p.

Step 2: Anisotropic harmonic extension and interior estimates. We fix the radius r found in Step 1. Let $\boldsymbol{h}$ be the unique weak solution to $\operatorname{\mathbf{div}}(A_0 \nabla \boldsymbol{h}) = 0$ in B_r satisfying the boundary condition $\boldsymbol{h} = \boldsymbol{u}$ on ∂B_r. Since $\boldsymbol{c}$ is a constant vector, the translated function $\boldsymbol{h} - \boldsymbol{c}$ is also A_0-harmonic in B_r. We estimate $\boldsymbol{h} - \boldsymbol{c}$ inside the ball using the Poisson integral representation for the Euclidean ball B_r associated with the anisotropic operator. By elliptic estimates for the anisotropic Poisson kernel $P_{A_0}(x,\zeta)$ evaluated at interior points $x \in B_{r/2}$ (see Proposition 8.5) we get

$$|\nabla \boldsymbol{h}(x)| = |\nabla(\boldsymbol{h}(x) - \boldsymbol{c})| \leqslant \frac{\tilde{C}_A}{r^m} \int_{\zeta\in\partial B_r} |\boldsymbol{u}(\zeta) - \boldsymbol{c}|\, \mathrm{d}\zeta, \tag{5.47}$$

with $\tilde{C}_A := C_{\theta,\lambda,\Lambda,m}$ being the constant in Proposition 8.5 (applied with $\theta = 1/2$).

We now restrict the domain to the fixed interior ball. Since $r \geqslant 1/2$, we have $B_{1/4} \subset B_{r/2}$ and that $1/r^m \leqslant 2^m$. Combining these geometric bounds with the radius estimate (5.46), we obtain:

$$\sup_{x \in B_{1/4}} |\nabla \boldsymbol{h}(x)| \leqslant \sup_{x \in B_{r/2}} |\nabla \boldsymbol{h}(x)| \overset{(5.47)}{\leqslant} \frac{\tilde{C}_A}{r^m} \int_{\zeta \in \partial B_r} |\boldsymbol{u}(\zeta) - \boldsymbol{c}| \, \mathrm{d}\zeta$$

$$\overset{(5.46)}{\leqslant} \tilde{C}_A 2^m \left(2 \int_{B_1} |\boldsymbol{u} - \boldsymbol{c}| \, \mathrm{d}x \right) \leqslant 2^{m+1} \tilde{C}_A \omega_m \left(\fint_{B_1} |\boldsymbol{u} - \boldsymbol{c}|^p \, \mathrm{d}x \right)^{1/p}, \quad (5.48)$$

where ω_m denotes the volume of the unit ball in $\mathbb{R}^m$.

Since the constant $\boldsymbol{c}$ was chosen precisely to realize the minimum in the definition of the oscillation, the right-hand side is exactly $\Psi_p^{1/p}(\boldsymbol{u}, B_1)$. Setting $C_A := 2^{m+1}\tilde{C}_A\omega_m$ formally yields the gradient estimate (5.44), concluding the proof. □

We now combine the anisotropic Caccioppoli estimate (established in Lemma 5.9) with the anisotropic harmonic approximation result (Lemma 5.10) to derive the following estimate.

Lemma 5.11 (Anisotropic coupled Caccioppoli-type estimate)**.** *Under the assumptions and notation of Lemma 5.9, there exists an A_0-harmonic function $\boldsymbol{h} : B_{1/4} \to \mathbb{R}^{n+1}$ such that*

$$\|\nabla(\boldsymbol{u} - \boldsymbol{h})\|_{L^q(B_{1/4})} \leqslant C_0 \|\boldsymbol{\Omega}\|_{L^2(B_1)} \Psi_p^{1/p}(\boldsymbol{u}, B_1) + C_0 \|\tilde{A}\|_{L^\infty(B_1)} \|\nabla \boldsymbol{u}\|_{L^2(B_1)}$$
$$+ C_0 \left(\|\boldsymbol{f}\|_{L^q(B_1)} + 2M \|\operatorname{\mathbf{div}} \boldsymbol{\Omega}\|_{L^q(B_1)} \right), \qquad (5.49)$$

with $C_0 = C_0(m, q, \lambda, \Lambda)$. Moreover,

$$\sup_{x \in B_{1/4}} |\nabla \boldsymbol{h}(x)| \leqslant C_A \Psi_p^{1/p}(\boldsymbol{u}, B_1), \qquad (5.50)$$

where $C_A > 0$ is a constant depending only on m, λ, and Λ.

Proof. By the Anisotropic Harmonic Approximation (Lemma 5.10) applied on the unit ball with exponent p, there exists a radius $r_* \in [1/2, 1]$ and an A_0-harmonic extension $\boldsymbol{h} : B_{r_*} \to \mathbb{R}^{n+1}$ such that $\boldsymbol{h} = \boldsymbol{u}$ on ∂B_{r_*}. This extension satisfies the interior gradient bound (5.50) since $B_{1/4} \subset B_{r_*/2}$. Also, we may apply the estimate from Lemma 5.9 directly on the ball B_{r_*}. This yields an intermediate constant C_0 such that:

$$\|\nabla(\boldsymbol{u} - \boldsymbol{h})\|_{L^q(B_{r_*})} \leqslant C_0 r_*^{2/p} \|\boldsymbol{\Omega}\|_{L^2(B_{r_*})} \Psi_p^{1/p}(\boldsymbol{u}, B_{r_*}) + C_0 \|\tilde{A}\|_{L^\infty(B_{r_*})} \|\nabla \boldsymbol{u}\|_{L^2(B_{r_*})}$$
$$+ C_0 \left(\|\boldsymbol{f}\|_{L^q(B_{r_*})} + 2M \|\operatorname{\mathbf{div}} \boldsymbol{\Omega}\|_{L^q(B_{r_*})} \right). \qquad (5.51)$$

To bridge (5.51) to our target estimate (5.49), we systematically bound the terms on the right-hand side using the monotonicity of the data norms (e.g., $\|\boldsymbol{\Omega}\|_{L^2(B_{r_*})} \leqslant \|\boldsymbol{\Omega}\|_{L^2(B_1)}$ and likewise for $\|\tilde{A}\|_{L^\infty}$, $\|\nabla \boldsymbol{u}\|_{L^2}$, $\|\boldsymbol{f}\|_{L^q}$, $\|\operatorname{\mathbf{div}} \boldsymbol{\Omega}\|_{L^q}$), and the monotonicity $r^m \Psi_p(\boldsymbol{u}, B_r) \leqslant \Psi_p(\boldsymbol{u}, B_1)$ of the rescaled oscillation. Restricting the LHS from $\|\nabla(\boldsymbol{u} - \boldsymbol{h})\|_{L^q(B_{r_*})}$ to $\|\nabla(\boldsymbol{u} - \boldsymbol{h})\|_{L^q(B_{1/4})}$ (since $B_{1/4} \subset B_{r_*}$) yields (5.49). □

5.3. **The anisotropic decay estimate.** We now show that the structural matrix $\boldsymbol{\Omega}$ dictates the decay of the oscillation, provided its L^2-energy is sufficiently small. The freezing flux enters as a fixed additive contribution and *does not* require a smallness threshold; its smallness on small balls is automatic from (5.5) and is exploited in the global iteration of Subsection 5.4.

Lemma 5.12 (Anisotropic decay property)**.** *For any integrability exponent $q \in (1,2)$, let $p := 2q/(2-q)$. There exist structural constants $\theta \in (0,1/4)$, $\gamma = 1/2$, $\kappa > 0$, and a threshold $\varepsilon_* > 0$ (depending only on m, q, λ, Λ, with κ also depending on M) such that the following holds. Let $\boldsymbol{u} \in H^1 \cap L^\infty(B_1, \mathbb{R}^{n+1})$ be a weak solution of (5.3) under (A1)–(A4) with $\|\boldsymbol{u}\|_{L^\infty(B_1)} \leqslant M$ for some $M > 0$. If*

$$\|\boldsymbol{\Omega}\|_{L^2(B_1)} < \varepsilon_*, \tag{5.52}$$

then

$$\Psi_p(\boldsymbol{u}, B_\theta) \leqslant \gamma \Psi_p(\boldsymbol{u}, B_1) + \kappa \left(D_{\mathrm{src}}^p + D_{\mathrm{frz}}^p\right), \tag{5.53}$$

where D_{src} and D_{frz} are defined in (5.16).

Remark 5.13 (Asymmetry between the two smallness mechanisms)*.* Only the smallness of the norm $\|\boldsymbol{\Omega}\|_{L^2(B_1)}$ is required. The freezing data $D_{\mathrm{frz}} = \|\tilde{A}\|_{L^\infty(B_1)} \|\nabla \boldsymbol{u}\|_{L^2(B_1)}$ enters as fixed additive data: no smallness threshold is imposed. Structurally, in the master inequality (5.58) below, $\|\boldsymbol{\Omega}\|_{L^2}^p$ multiplies $\Psi_p(\boldsymbol{u}, B_1)$ (so it must be small for the contraction to operate), whereas $\|\tilde{A}\|_{L^\infty}^p \|\nabla \boldsymbol{u}\|_{L^2}^p$ appears only as an additive contribution. The eventual smallness of $\|\tilde{A}\|_{L^\infty}$ on small balls, needed in the global iteration of Subsection 5.4, comes for free from (5.5) and the rescaling (5.14).

Proof. To distinguish the exponent originating from L^2-integrability from the spatial dimension, we temporarily denote the latter by m (here $m = 2$). By Lemma 5.11, there exists a harmonic $\boldsymbol{h} : B_{1/4} \to \mathbb{R}^{n+1}$ satisfying (5.49) and (5.50). For $\theta \in (0,1/4)$, the triangle inequality for the seminorm $\Psi_p^{1/p}$ yields

$$\Psi_p^{1/p}(\boldsymbol{u}, B_\theta) \leqslant \Psi_p^{1/p}(\boldsymbol{u} - \boldsymbol{h}, B_\theta) + \Psi_p^{1/p}(\boldsymbol{h}, B_\theta) =: I + II. \tag{5.54}$$

Bound on II. Choosing the constant $\boldsymbol{h}(0)$ in the infimum and using the fundamental theorem of calculus on the segment $[0, x] \subset B_\theta$,

$$II \leqslant \left(\fint_{B_\theta} |\boldsymbol{h}(x) - \boldsymbol{h}(0)|^p \mathrm{d}x \right)^{1/p} \leqslant \theta \sup_{B_{1/4}} |\nabla \boldsymbol{h}| \overset{(5.50)}{\leqslant} C_A \theta \Psi_p^{1/p}(\boldsymbol{u}, B_1). \tag{5.55}$$

Bound on I. Since $\boldsymbol{u} - \boldsymbol{h}$ does not vanish on ∂B_θ, we apply the (mean-subtracted) Sobolev–Poincaré inequality on B_θ at the critical exponents $1/p = 1/q - 1/2$, which is scale-invariant in dimension $m = 2$:

$$I \leqslant \frac{C_S}{(\omega_m \theta^m)^{1/p}} \|\nabla(\boldsymbol{u} - \boldsymbol{h})\|_{L^q(B_\theta)}. \tag{5.56}$$

Bounding $\|\nabla(\boldsymbol{u} - \boldsymbol{h})\|_{L^q(B_\theta)} \leqslant \|\nabla(\boldsymbol{u} - \boldsymbol{h})\|_{L^q(B_{1/4})}$ and invoking (5.49), we obtain (after harmlessly redefining the constant C_0 to absorb C_S and $\omega_m^{-1/p}$):

$$\begin{aligned} I \;\leqslant\; & \frac{C_0}{\theta^{m/p}} \|\boldsymbol{\Omega}\|_{L^2(B_1)} \Psi_p^{1/p}(\boldsymbol{u}, B_1) + \frac{C_0}{\theta^{m/p}} \|\tilde{A}\|_{L^\infty(B_1)} \|\nabla \boldsymbol{u}\|_{L^2(B_1)} \\ & + \frac{C_0}{\theta^{m/p}} \left(\|\boldsymbol{f}\|_{L^q(B_1)} + 2M \|\operatorname{\mathbf{div}} \boldsymbol{\Omega}\|_{L^q(B_1)} \right). \end{aligned} \tag{5.57}$$

Master inequality. Combining (5.54) with the bounds on I and II and raising to the p-th power and using that $(a_1+\ldots+a_j)^p \leqslant j^{p-1}\sum_{i=1}^j a_i^p$, we obtain the master decay inequality

$$\Psi_p(\boldsymbol{u},B_\theta) \leqslant K_1\theta^p\Psi_p(\boldsymbol{u},B_1) + \frac{K_2}{\theta^m}\|\boldsymbol{\Omega}\|^p_{L^2(B_1)}\Psi_p(\boldsymbol{u},B_1) + \frac{K_2}{\theta^m}\left[\|\tilde{A}\|^p_{L^\infty(B_1)}\|\nabla\boldsymbol{u}\|^p_{L^2(B_1)} + \Big(\|\boldsymbol{f}\|_{L^q(B_1)} + 2M\|\operatorname{\mathbf{div}}\boldsymbol{\Omega}\|_{L^q(B_1)}\Big)^p\right], \quad (5.58)$$

with $K_1 = 2^{p-1}C_A^p$ and $K_2 = 2^{p-1}\cdot 4^{p-1}C_0^p$ depending only on m,q,λ,Λ,p.
Choice of parameters. The contractive coefficient of $\Psi_p(\boldsymbol{u},B_1)$ in (5.58) is given by $K_1\theta^p + (K_2/\theta^m)\|\boldsymbol{\Omega}\|^p_{L^2(B_1)}$. We make it less than or equal to $1/2$ in two steps. First, choose $\theta\in(0,1/4)$ small enough that $K_1\theta^p\leqslant 1/4$. Second, with this θ fixed, choose $\varepsilon_*>0$ such that $K_2\varepsilon_*^p/\theta^m\leqslant 1/4$. Under (5.52), the contractive coefficient is at most $1/2$. The remaining additive terms are bounded by $K_2\theta^{-m}(D^p_{\mathrm{frz}} + (2D_{\mathrm{src}})^p)$, recalling $\|\boldsymbol{f}\|_{L^q(B_1)} + 2M\|\operatorname{\mathbf{div}}\boldsymbol{\Omega}\|_{L^q(B_1)}\leqslant 2D_{\mathrm{src}}$. Setting $\kappa := 2^pK_2\theta^{-m}$ yields (5.53). □

5.4. **Proof of the main regularity theorem (Theorem 1.3).** Let $\boldsymbol{u}$ be an arbitrary weak solution of (5.3) under (A1)–(A4) with $\boldsymbol{u}\in H^1\cap L^\infty(B_1,\mathbb{R}^{n+1})$. We fix the global amplitude bound $M := \max(1,\|\boldsymbol{u}\|_{L^\infty(B_1)})$.

Let ε_* be the critical threshold from Lemma 5.12. To invoke Theorem 5.6, we define the admissible class $\boldsymbol{\mathcal{S}}$ as the set of quadruples $(\boldsymbol{u},\boldsymbol{\Omega},\boldsymbol{f},A)\in\mathcal{Y}$, $\mathcal{Y}$ defined in (5.6), satisfying the original system, the amplitude bound M, and the smallness threshold:

$$\boldsymbol{\mathcal{S}} := \left\{(\boldsymbol{u},\boldsymbol{\Omega},\boldsymbol{f},A)\in\mathcal{Y} \;:\; \begin{array}{l} -\operatorname{\mathbf{div}}(A\nabla\boldsymbol{u}) = \boldsymbol{\Omega}\cdot\nabla\boldsymbol{u}+\boldsymbol{f}, \\ \|\boldsymbol{u}\|_{L^\infty(B_1)}\leqslant M, \|\boldsymbol{\Omega}\|_{L^2(B_1)}<\varepsilon_* \end{array}\right\}. \quad (5.59)$$

We need to verify Axioms 5.4 and 5.5 required by Theorem 5.6.
Step 1: Verification of Axiom 5.4 (*closure under rescaling*). Let $(\boldsymbol{u},\boldsymbol{\Omega},\boldsymbol{f},A)\in\boldsymbol{\mathcal{S}}$ and $B_r(x_0)\subset B_1$. Consider the rescaled tuple $(\boldsymbol{u}_{x_0,r},\overline{\boldsymbol{\Omega}}_{x_0,r},\tilde{\boldsymbol{f}}_{x_0,r},A_{x_0,r})$. We need to show that this triple remains in $\boldsymbol{\mathcal{S}}$. By the rescaling (5.8), we have

$$-\operatorname{\mathbf{div}}(A_{x_0,r}\nabla\boldsymbol{u}_{x_0,r}) = \overline{\boldsymbol{\Omega}}_{x_0,r}\cdot\nabla\boldsymbol{u}_{x_0,r} + \tilde{\boldsymbol{f}}_{x_0,r}, \quad (5.60)$$

so the equation is invariant. Furthermore, the amplitude bound is preserved by (5.9); the ellipticity constants of A are preserved identically; and the L^2-norm of $\boldsymbol{\Omega}$ is conformally invariant in dimension $m=2$ (cf. (5.11)), so

$$\|\overline{\boldsymbol{\Omega}}_{x_0,r}\|_{L^2(B_1)} = \|\boldsymbol{\Omega}\|_{L^2(B_r(x_0))}\leqslant\|\boldsymbol{\Omega}\|_{L^2(B_1)}<\varepsilon_*. \quad (5.61)$$

Finally, $\operatorname{\mathbf{div}}_y\overline{\boldsymbol{\Omega}}_{x_0,r}\in L^q$ and $A_{x_0,r}\in C^{0,\gamma_0}$ by (5.12) and (5.14). Thus, the class $\boldsymbol{\mathcal{S}}$ is closed under rescaling.
Step 2: Verification of Axiom 5.5. Immediate from Lemma 5.12: any quadruple in $\boldsymbol{\mathcal{S}}$ satisfies the smallness assumption (5.52), and the decay estimate (5.53) is exactly (5.15) with $\gamma=1/2$.
Step 3: Conclusion and Removal of the small $\boldsymbol{\Omega}$*-energy condition.* Having verified the two axioms, the abstract regularity principle (Theorem 5.6) dictates that any map $\boldsymbol{u}$ belonging to a quadruple in $\boldsymbol{\mathcal{S}}$ is locally Hölder continuous in B_1. To conclude the proof of Theorem 1.3, it remains to explain why this regularity holds for any weak solution of the generalized anisotropic system, not just those with small initial $\boldsymbol{\Omega}$-energy.

The arbitrary solution $\boldsymbol{u}$ may not lie in $\boldsymbol{\mathcal{S}}$ (the L^2-energy of $\boldsymbol{\Omega}$ may be too large). Fix $x_0\in B_1$ and let $R := 1-|x_0|>0$ denote its distance to the boundary. Since $|\boldsymbol{\Omega}|^2\in L^1(B_1)$, the absolute continuity of the Lebesgue integral yields a radius $r\in(0,R)$ with $\|\boldsymbol{\Omega}\|_{L^2(B_r(x_0))}<\varepsilon_*$. By the conformal invariance of the L^2-norm in 2D, the rescaled quadruple $(\boldsymbol{u}_{x_0,r}, \overline{\boldsymbol{\Omega}}_{x_0,r}, \tilde{\boldsymbol{f}}_{x_0,r}, A_{x_0,r})$ is defined on the unit ball B_1, solves the rescaled equation, and satisfies $\|\overline{\boldsymbol{\Omega}}_{x_0,r}\|_{L^2(B_1)} < \varepsilon_*$ together with the amplitude bound. Hence this rescaled quadruple lies in $\boldsymbol{\mathcal{S}}$. By Theorem 5.6, the rescaled map $\boldsymbol{u}_{x_0,r}$ is locally Hölder continuous on $B_{1/2}$, with exponent η satisfying (5.17). Scaling back, $\boldsymbol{u}$ is Hölder continuous on $B_{r/2}(x_0)$. Since $x_0 \in B_1$ was arbitrary, $\boldsymbol{u} \in C^{0,\eta}_{\text{loc}}(B_1, \mathbb{R}^{n+1})$, completing the proof of Theorem 1.3.

6. Smoothness of continuous Almost Harmonic Maps

Theorem 1.4. (Bootstrap Regularity for Almost Harmonic Maps) *Let $U \subset \mathbb{R}^m$ be an open domain, and let $\boldsymbol{u} \in W^{1,2}_{\text{loc}}(U, \mathbb{S}^n) \cap C^0(U, \mathbb{S}^n)$ be a weakly almost harmonic map solving the Euler–Lagrange equation:*

$$-\Delta \boldsymbol{u} = |\nabla \boldsymbol{u}|^2 \boldsymbol{u} + \boldsymbol{f} \quad \text{in } \mathcal{D}'(U), \tag{6.1}$$

where the source term initially satisfies $\boldsymbol{f} \in L^q_{\text{loc}}(U, \mathbb{R}^{n+1})$.
If the source integrability q satisfies $2 \leqslant q \leqslant m/2$ (which can only occur in dimensions $m \geqslant 4$), we additionally assume the map is already locally Hölder continuous: $\boldsymbol{u} \in C^{0,\gamma}_{\text{loc}}(U, \mathbb{S}^n)$ for some $\gamma \in (0, 1]$.
Then, the regularity of $\boldsymbol{u}$ depends on the regularity of $\boldsymbol{f}$:

(1) *If $\boldsymbol{f} \in L^q_{\text{loc}}(U, \mathbb{R}^{n+1})$ and $2 \leqslant q \leqslant m$ then the regularity is bounded by the critical Sobolev embedding threshold:*

$$\boldsymbol{u} \in W^{1,q^*}_{\text{loc}}(U, \mathbb{S}^n) \quad \text{with } q^* = \frac{mq}{m-q}, \tag{6.2}$$

with the usual understanding that q^* (necessarily greater than 1) can be any finite real exponent if $m = q$. In particular, if $m = q$ then $\boldsymbol{u} \in C^{0,\alpha}_{\text{loc}}(U, \mathbb{S}^n)$ for all $0 < \alpha < 1$, while if $q = 2$ and $m = 3$ then $\boldsymbol{u} \in C^{0,1/2}_{\text{loc}}(U, \mathbb{S}^n)$.

(2) *If $\boldsymbol{f} \in L^q_{\text{loc}}(U, \mathbb{R}^{n+1})$ with $q > m$, then*

$$\boldsymbol{u} \in C^{1,\eta}_{\text{loc}}(U, \mathbb{S}^n), \quad \text{where } \eta := 1 - m/q. \tag{6.3}$$

Note that η does not depend on any intermediate fractional exponent. In particular, if $\boldsymbol{f} \in L^\infty_{\text{loc}}(U, \mathbb{R}^{n+1})$ then $\boldsymbol{u} \in C^{1,\alpha}_{\text{loc}}(U, \mathbb{S}^n)$ for all $0 < \alpha < 1$.

(3) *For any integer $k \geqslant 1$ and exponent $\beta \in (0, 1)$, if $\boldsymbol{f} \in C^{k-1,\beta}_{\text{loc}}(U, \mathbb{R}^{n+1})$ then $\boldsymbol{u} \in C^{k+1,\beta}_{\text{loc}}(U, \mathbb{S}^n)$. In particular, if $\boldsymbol{f} \in C^\infty(U, \mathbb{R}^{n+1})$, then $\boldsymbol{u} \in C^\infty(U, \mathbb{S}^n)$. Again, note that η does not play any role in the higher regularity class we end up in.*

Remark 6.1. The true analytical power of case *i.* lies in the upgraded *Sobolev* regularity: specifically, the higher Lebesgue integrability of the gradient, $\boldsymbol{u} \in W^{1,q^*}_{\text{loc}}$. Securing these strong W^{1,q^*} bounds is paramount when transitioning to more advanced, non-linear settings where the source term depends on the solution itself (e.g., treated as a frozen quantity $\boldsymbol{f}(x) \equiv \boldsymbol{g}(x, \boldsymbol{u}(x))$). In such scenarios, one must feed the improved gradient integrability back into the PDE to re-evaluate the integrability of the frozen source, thereby closing the bootstrap loop and unlocking higher regularity. We will explicitly see this self-improving mechanism in action in the next section (see Section 7), where we treat the micromagnetic case.

Remark 6.2. One must be careful about the initial hypothesis $\boldsymbol{u} \in C^0$. In dimension $m = 2$, as we already proved, two-dimensional geometry yields sufficient critical cancellations to ensure that every weakly almost harmonic map $\boldsymbol{u} \in W^{1,2}(U, \mathbb{S}^n)$ is automatically continuous. However, in dimension $m \geqslant 3$ continuity fails in general. Because the Dirichlet energy scales differently in higher dimensions, topological singularities may have finite energy. A classical example is the radial map $\boldsymbol{u}(x) = x/|x|$ from the unit ball in $\mathbb{R}^3$ into $\mathbb{S}^2$, which is weakly harmonic but has a point singularity at the origin. More strikingly, Rivière [Riv95], in 1995, constructed weakly harmonic maps in dimension three into $\mathbb{S}^2$ that are *discontinuous* everywhere. Therefore, for $m \geqslant 3$ the algebraic bootstrap that upgrades a locally continuous solution to C^∞ remains valid, but it applies only at regular points, i.e., at points where the map is already locally continuous.

Proof of Theorem 1.4. We divide the proof into nine steps.
Step 1: Localization and the Broken Conservation Law. Let $x_0 \in U$ be an arbitrary point. Since $\boldsymbol{u}$ is continuous, we can choose a sufficiently small radius $R > 0$ such that the oscillation of $\boldsymbol{u}$ on the ball $B_R(x_0) \subset U$ is strictly less than $1/2$. By applying a constant rotation to the target sphere, we can assume without loss of generality that $\boldsymbol{u}(x_0) = \boldsymbol{e}_{n+1}$ is the north pole. By continuity, the image $\boldsymbol{u}(B_R(x_0))$ is strictly contained in the upper hemisphere; in particular, $u^{n+1}(x) := \boldsymbol{u}(x) \cdot \boldsymbol{e}_{n+1} \geqslant 1/2$ for all $x \in B_R(x_0)$.

For almost harmonic maps into $\mathbb{S}^n$, the antisymmetric matrix of 1-forms defined by $\Omega^{\alpha\beta} = u^\alpha \nabla u^\beta - u^\beta \nabla u^\alpha$ is no longer divergence-free. Computing the weak divergence and substituting the PDE yields a perfect cancellation of the critical gradient terms, leaving only the source term:

$$\operatorname{div}(u^\alpha \nabla u^\beta - u^\beta \nabla u^\alpha) = u^\beta f^\alpha - u^\alpha f^\beta \quad \text{in } \mathcal{D}'(B_R(x_0)). \tag{6.4}$$

Step 2: The Gnomonic Projection. Because $u^{n+1} \geqslant 1/2$, we can smoothly project the map from the center of the sphere onto the affine tangent plane at the north pole. We define the $\mathbb{R}^n$-valued function $\boldsymbol{w} = (w^1, \dots, w^n) \in W^{1,2}_{\text{loc}} \cap C^0$ by $w^\alpha := u^\alpha / u^{n+1}$ for $\alpha = 1, \dots, n$. Applying the quotient rule for weak derivatives gives the algebraic identity:

$$(u^{n+1})^2 \nabla w^\alpha = u^{n+1} \nabla u^\alpha - u^\alpha \nabla u^{n+1}. \tag{6.5}$$

Step 3: Linearizing the Equation. Taking the divergence of both sides, and applying the broken conservation law from Step 1 (setting $\beta = n + 1$), we obtain:

$$\operatorname{div}((u^{n+1})^2 \nabla w^\alpha) = u^\alpha f^{n+1} - u^{n+1} f^\alpha =: g^\alpha(x) \quad \text{for } \alpha = 1, \dots, n. \tag{6.6}$$

Note that the quadratic gradient terms $|\nabla \boldsymbol{u}|^2$ have been completely eliminated. The critical nonlinear system has decoupled into a system of n linear divergence-form elliptic equations:

$$\mathbf{div}(A \nabla \boldsymbol{w}) = \boldsymbol{g}, \tag{6.7}$$

where the scalar coefficient $A = (\boldsymbol{u} \cdot \boldsymbol{e}_{n+1})^2$ is strictly positive ($A(x) \geqslant 1/4$), and the new right-hand side $\boldsymbol{g}$ depends linearly on $\boldsymbol{f}$. Since $|\boldsymbol{u}| = 1$ and $\boldsymbol{f} \in L^2_{\text{loc}}$, it follows that $\boldsymbol{g} \in L^2_{\text{loc}}$.
Step 4: Upgrading C^0 to $C^{0,\gamma}$ via De Giorgi–Nash–Moser. Because we assumed only that $\boldsymbol{u} \in C^0(U)$, the coefficient A is merely continuous, which momentarily stalls classical Schauder theory. We cross this gap by invoking the De Giorgi–Nash–Moser theorem on our decoupled system (6.7).

Since $A \geqslant 1/4$, the coefficient is bounded and uniformly elliptic. The theorem (see [HL11, Theorem 4.13, p.83]) guarantees that weak solutions gain local Hölder continuity provided the right-hand side is integrable enough. Specifically, if $\boldsymbol{g} \in L^q_{\mathrm{loc}}$ for some $q > m/2$, then $\boldsymbol{w} \in C^{0,\gamma}_{\mathrm{loc}}$ for some $0 < \gamma < 1$.

Because $g^\alpha = u^\alpha f^{n+1} - u^{n+1} f^\alpha$, the source $\boldsymbol{g}$ inherits the L^q_{loc} integrability of $\boldsymbol{f}$. The strict requirement to initiate the bootstrap is therefore $q > m/2$. Because we assume a baseline integrability of $q \geqslant 2$, this condition is automatically satisfied for domains of dimension $m \leqslant 3$. If $q \leqslant m/2$, instead, our initial hypothesis explicitly assumed $\boldsymbol{u} \in C^{0,\gamma}_{\mathrm{loc}}$, bypassing this step entirely.

Algebraic inversion gives $\boldsymbol{u} \cdot \boldsymbol{e}_{n+1} = (1 + |\boldsymbol{w}|^2)^{-1/2}$. Since Hölder spaces are closed under smooth compositions, $\boldsymbol{w} \in C^{0,\gamma}_{\mathrm{loc}}$ forces the vertical component to be Hölder continuous. Consequently, the coefficient $A = (\boldsymbol{u} \cdot \boldsymbol{e}_{n+1})^2$ is officially upgraded to $C^{0,\gamma}_{\mathrm{loc}}$, clearing the temporary roadblock.

Step 5: Linear Theory, the Flux Representation, and the Integrability Ceiling. Because $\boldsymbol{u} \in C^{0,\gamma}_{\mathrm{loc}}$ (from Step 4), the uniformly elliptic coefficient $A \in C^{0,\gamma}_{\mathrm{loc}}$. By classical linear elliptic theory for divergence-form equations, the regularity of $\boldsymbol{w}$ now depends on the source term $\boldsymbol{g} \in L^2_{\mathrm{loc}}$. To precisely determine the Sobolev space of $\boldsymbol{w}$, we must express the source term in divergence form so that Calderón–Zygmund theory can be applied. On the ball $B_R(x_0)$ defined in Step 1, let $\boldsymbol{v} \in W^{1,2}_0(B_R(x_0))$ be the unique weak solution to the Dirichlet problem $\Delta \boldsymbol{v} = \boldsymbol{g}$ with zero boundary conditions. By standard L^p elliptic regularity up to the boundary, since $\boldsymbol{g} \in L^2(B_R(x_0))$, we have $\boldsymbol{v} \in W^{2,2}(B_R(x_0))$. Setting $\boldsymbol{G} := \nabla \boldsymbol{v}$, we rewrite the linearized equation on $B_R(x_0)$ as:

$$\mathbf{div}(A \nabla \boldsymbol{w}) = \mathbf{div}\, \boldsymbol{G}, \quad \text{where } \boldsymbol{G} \in W^{1,2}(B_R(x_0)). \tag{6.8}$$

Proof of i. *The L^q Bottleneck and Dimensional Dependency.* Provided the coefficient A is at least uniformly continuous, Calderón–Zygmund theory guarantees that the gradient of the solution $\nabla \boldsymbol{w}$ inherits the *Lebesgue* integrability of the flux $\boldsymbol{G}$. In general, if $\boldsymbol{f} \in L^q_{\mathrm{loc}}$, the flux satisfies $\boldsymbol{G} \in W^{1,q}_{\mathrm{loc}}$. This integrability is governed by the classical Sobolev embedding theorem, which introduces a hard dependency on the dimension m: if $\boldsymbol{G} \in W^{1,q}_{\mathrm{loc}}$ with $q \leqslant m$, then $\boldsymbol{G} \in L^{q^*}_{\mathrm{loc}}$, where $q^* = mq/(m-q)$. Therefore, by Calderón–Zygmund, $\nabla \boldsymbol{w} \in L^{q^*}_{\mathrm{loc}}$, meaning that $\boldsymbol{w} \in W^{1,q^*}_{\mathrm{loc}}$. (As usual, if $q = m$, q^* can be any finite real exponent). If $q^* > m$ (which is algebraically equivalent to $q > m/2$), Morrey's inequality yields $\boldsymbol{w} \in C^{0,\alpha}_{\mathrm{loc}}$ with $\alpha = 1 - m/q^*$. In particular:

- If $q = m$, then q^* is arbitrarily large, yielding $\boldsymbol{w} \in C^{0,\alpha}_{\mathrm{loc}} \cap W^{1,p}_{\mathrm{loc}}$ for all $\alpha < 1$ and all $1 \leqslant p < \infty$. The regularity becomes arbitrarily close to C^1, but stalls just short of it.
- If $q = 2$ and $m = 3$, then $q^* = 6$, yielding $\boldsymbol{w} \in C^{0,1/2}_{\mathrm{loc}} \cap W^{1,6}_{\mathrm{loc}}$.
- If $q \leqslant m/2$ (e.g., $q = 2, m \geqslant 4$), we cannot even guarantee continuity from the integrability of the source alone.

Finally, because the algebraic inversion from $\boldsymbol{w}$ back to $\boldsymbol{u}$ is smooth, $\boldsymbol{u}$ inherits these W^{1,q^*}_{loc} and $C^{0,\alpha}_{\mathrm{loc}}$ spaces. This proves assertion *i.*

Proof of ii. *Step 6: Pushing the Bootstrap forward via Morrey and Campanato.* To push the bootstrap forward, we must cross the boundary from Lebesgue integrability (L^p) into classical Hölder continuity ($C^{1,\eta_0}_{\mathrm{loc}}$). The mathematical bridge between these two realms requires the right-hand side of our PDE, $\boldsymbol{g}$, to cross the dimensional threshold $q > m$. Recalling the algebraic definition $g^\alpha = u^\alpha f^{n+1} - u^{n+1} f^\alpha$ and the geometric bound $|\boldsymbol{u}| = 1$, we observe that $\boldsymbol{g}$ inherits the Lebesgue integrability of the original source $\boldsymbol{f}$. Therefore, we assume higher integrability on our original source: $\boldsymbol{f} \in L^q_{\mathrm{loc}}$ for some $q > m$, which guarantees $\boldsymbol{g} \in L^q_{\mathrm{loc}}$.

Following the exact same flux representation as above, the potential $\boldsymbol{v}$ (solving $\Delta \boldsymbol{v} = \boldsymbol{g}$) now belongs to $W^{2,q}(B_R(x_0))$, which implies our flux $\boldsymbol{G} := \nabla \boldsymbol{v}$ belongs to $W^{1,q}(B_R(x_0))$. Because $q > m$, by Morrey's inequality, the flux $\boldsymbol{G}$ is no longer just an integrable function; it is continuous. Specifically, it embeds into a Hölder space:

$$\boldsymbol{G} \in C^{0,\alpha}(B_R(x_0)), \quad \text{where } \alpha := 1 - m/q. \tag{6.9}$$

This is the critical transition point in the argument. In the equation (6.8), both the coefficient $A \in C^{0,\gamma}_{\mathrm{loc}}$ and the flux $\boldsymbol{G} \in C^{0,\alpha}_{\mathrm{loc}}$ are now locally Hölder continuous. The Calderón–Zygmund L^q-theory is no longer sufficient, and we instead appeal to the classical Campanato regularity theory for divergence-form elliptic equations. These estimates (see, for instance, Giaquinta [Gia83, Chapter III] or Giusti [Giu03, Theorem 5.19]) imply that $\nabla \boldsymbol{w}$ inherits the Hölder continuity of the coefficients and forcing terms, with regularity determined by the smaller of the two exponents. More precisely,

$$\nabla \boldsymbol{w} \in C^{0,\eta_0}_{\mathrm{loc}}(B_R(x_0)), \qquad \eta_0 = \min(\gamma, \alpha), \tag{6.10}$$

with $\alpha = 1 - m/q$. Consequently, we obtain:

$$\boldsymbol{w} \in C^{1,\eta_0}_{\mathrm{loc}}(B_R(x_0)), \quad \text{where } \eta_0 = \min(\gamma, 1 - m/q). \tag{6.11}$$

Step 7: Bootstrapping back to $\boldsymbol{u}$ and the Self-Improving Loop. Assuming the source meets the threshold integrability $\boldsymbol{f} \in L^q_{\mathrm{loc}}$ $(q > m)$, Step 6 guarantees that $\boldsymbol{w} \in C^{1,\eta_0}_{\mathrm{loc}}$. We now algebraically invert the Gnomonic projection to recover the original map $\boldsymbol{u}$. Since $|\boldsymbol{w}|^2 = (\boldsymbol{u} \cdot \boldsymbol{e}_{n+1})^{-2} - 1$, solving for the vertical component $u^{n+1} = \boldsymbol{u} \cdot \boldsymbol{e}_{n+1}$ yields $\boldsymbol{u} \cdot \boldsymbol{e}_{n+1} = (1 + |\boldsymbol{w}|^2)^{-1/2}$. Because Hölder spaces are Banach algebras and are closed under composition with smooth functions, the composition of $\boldsymbol{w} \in C^{1,\eta_0}_{\mathrm{loc}}$ with the smooth function $y \mapsto (1 + |y|^2)^{-1/2}$ implies $\boldsymbol{u} \cdot \boldsymbol{e}_{n+1} \in C^{1,\eta_0}_{\mathrm{loc}}(B_R(x_0))$. Consequently, the remaining components are recovered via smooth products: $u^\alpha = w^\alpha u^{n+1} \in C^{1,\eta_0}_{\mathrm{loc}}(B_R(x_0))$. Since the base point x_0 was arbitrary, we have successfully lifted the entire map to $\boldsymbol{u} \in C^{1,\eta_0}_{\mathrm{loc}}(U, \mathbb{S}^n)$.
Step 8: Unconditional Hölder Exponents (Dropping γ). We can now completely eliminate the initial fractional dependency. Because $\boldsymbol{u} \in C^{1,\eta_0}_{\mathrm{loc}}$, its gradient is locally continuous and thus locally bounded (L^∞_{loc}). We return to the original harmonic map system, treating the right-hand side as a fixed source for the constant-coefficient Laplacian: $-\Delta \boldsymbol{u} = \boldsymbol{\varphi}(x) := |\nabla \boldsymbol{u}|^2 \boldsymbol{u} + \boldsymbol{f}$. Since $\nabla \boldsymbol{u} \in L^\infty_{\mathrm{loc}}$, the nonlinear term $|\nabla \boldsymbol{u}|^2 \boldsymbol{u} \in L^\infty_{\mathrm{loc}} \subset L^q_{\mathrm{loc}}$. Therefore, the entire right-hand side $\boldsymbol{\varphi}$ belongs to L^q_{loc}. Applying standard Calderón–Zygmund estimates for the constant-coefficient Laplacian yields $\boldsymbol{u} \in W^{2,q}_{\mathrm{loc}}$. By Morrey's inequality, this embeds into $C^{1,\eta}_{\mathrm{loc}}$ with the pure exponent $\eta = 1 - m/q$, completely forgetting the initial γ.

In particular, if $\boldsymbol{f} \in L^\infty_{\mathrm{loc}}$ (which corresponds to $q \to \infty$), the right-hand side is in L^p_{loc} for all $p < \infty$, yielding $\boldsymbol{u} \in C^{1,\alpha}_{\mathrm{loc}}$ for all $0 < \alpha < 1$. This completes the proof of *ii.*

Step 9: Proof of iii. *The Standard Schauder Bootstrap* ($C^{1,\beta} \to C^\infty$) Once the gradient $\nabla \boldsymbol{u}$ is Hölder continuous, the geometric criticality of the PDE no longer obstructs the iteration. We treat the system as a standard Poisson equation (PE):

$$-\Delta \boldsymbol{u} = \boldsymbol{\varphi} := |\nabla \boldsymbol{u}|^2 \boldsymbol{u} + \boldsymbol{f}(x). \tag{6.12}$$

The regularity of $\boldsymbol{u}$ now depends exclusively upon the smoothness of the external source $\boldsymbol{f}$. We proceed by induction using standard Schauder estimates (SE) for the Laplacian.

- *Base Step* ($k = 1$)*:* Assume $\boldsymbol{f} \in C^{0,\beta}_{\text{loc}}$. Since $\boldsymbol{f}$ is locally bounded ($\boldsymbol{f} \in L^\infty_{\text{loc}}$), the acquired point *ii.* of the theorem guarantees that $\boldsymbol{u} \in C^{1,\alpha}_{\text{loc}}$ for *all* $\alpha < 1$. By choosing $\alpha = \beta$, we have $\nabla \boldsymbol{u} \in C^{0,\beta}_{\text{loc}}$. Consequently, the nonlinear term satisfies $|\nabla \boldsymbol{u}|^2 \boldsymbol{u} \in C^{0,\beta}_{\text{loc}}$. The total source $\boldsymbol{\varphi}$ is therefore in $C^{0,\beta}_{\text{loc}}$. Applying Schauder estimates to the Poisson equation yields:
$$\Delta \boldsymbol{u} \in C^{0,\beta}_{\text{loc}} \quad \overset{\text{SE}}{\Longrightarrow} \quad \boldsymbol{u} \in C^{2,\beta}_{\text{loc}}. \tag{6.13}$$
- *Inductive Step:* Assume that for some integer $k \geqslant 2$, we have $\boldsymbol{u} \in C^{k,\beta}_{\text{loc}}$ and we are given a smoother source $\boldsymbol{f} \in C^{k-1,\beta}_{\text{loc}}$. Because $\boldsymbol{u} \in C^{k,\beta}_{\text{loc}}$, its gradient is $\nabla \boldsymbol{u} \in C^{k-1,\beta}_{\text{loc}}$. Since Hölder spaces $C^{k-1,\beta}_{\text{loc}}$ are Banach algebras, the product $|\nabla \boldsymbol{u}|^2 \boldsymbol{u}$ belongs to $C^{k-1,\beta}_{\text{loc}}$. Therefore, the total source is $\boldsymbol{\varphi} \in C^{k-1,\beta}_{\text{loc}}$. The bootstrap pushes the solution up by two derivatives relative to the source:
$$\Delta \boldsymbol{u} \in C^{k-1,\beta}_{\text{loc}} \quad \overset{\text{SE}}{\Longrightarrow} \quad \boldsymbol{u} \in C^{k+1,\beta}_{\text{loc}}. \tag{6.14}$$

Overall, if $\boldsymbol{f} \in C^{k-1,\beta}_{\text{loc}}$ and $\boldsymbol{u} \in C^{0,\eta}_{\text{loc}}$ then $\boldsymbol{u} \in C^{k+1,\beta}$. Again, note that η does not play any role in the higher regularity class we end up in.

By applying this inductive loop recursively, we see that if $\boldsymbol{f} \in C^\infty$, then $\boldsymbol{u} \in C^{k,\beta}_{\text{loc}}$ for all integers $k \geqslant 0$. Thus, we conclude that $\boldsymbol{u} \in C^\infty(U, \mathbb{S}^n)$. □

7. Applications to Micromagnetics: Smoothness of Magnetic Skyrmions

We now illustrate the power of the Bootstrap Lemma 1.4 by applying it to the variational theory of micromagnetics. This macroscopic continuum theory is of paramount importance in modern condensed matter physics and spintronics, particularly for the rigorous analysis of complex topological spin textures such as magnetic skyrmions. Skyrmions are localized, particle-like chiral spin configurations characterized by a nontrivial topological winding number, which grants them inherent stability against continuous deformations (topological protection). Because they can be nucleated, manipulated, and driven by ultra-low electrical currents, skyrmions are currently at the forefront of next-generation solid-state technologies, including high-density non-volatile magnetic storage (such as racetrack memory architectures), logic devices, and neuromorphic computing paradigms [FCS13]. A rigorous understanding of the analytical properties, such as the maximal regularity, of these magnetic configurations is therefore fundamentally necessary to validate both theoretical predictions and numerical simulations.

The state of a rigid ferromagnetic body occupying a domain $U \subset \mathbb{R}^m$ (with $m = 2$ or 3) is described by its magnetization vector field $\boldsymbol{u}$. Below the Curie temperature, the magnitude of the magnetization is assumed to be constant, leading to the non-convex pointwise saturation constraint $|\boldsymbol{u}(x)| = 1$, meaning $\boldsymbol{u} \in H^1(U, \mathbb{S}^2)$.

The stable magnetic configurations are the local minimizers of the micromagnetic energy functional [ADF15, Ber98, Bro63, DFMRS20]:
$$\mathcal{G}(\boldsymbol{u}) := \int_U \left[\frac{1}{2}|\nabla \boldsymbol{u}|^2 + \kappa \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{u} + \varphi(\boldsymbol{u})\right] \mathrm{d}x + \mathcal{E}_{\text{stray}}(\boldsymbol{u}), \quad \mathcal{E}_{\text{stray}}(\boldsymbol{u}) := -\frac{1}{2}\int_U \boldsymbol{h}[\boldsymbol{u}] \cdot \boldsymbol{u}. \tag{7.1}$$
Each term in this functional models a distinct quantum-mechanical or macroscopic physical interaction:

- *Exchange Energy* ($\frac{1}{2}|\nabla \boldsymbol{u}|^2$): This is the standard Dirichlet energy, which penalizes spatial variations and favors the uniform, parallel alignment of magnetic spins.

- *Dzyaloshinskii-Moriya Interaction* (DMI) ($\kappa \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{u}$): This anti-symmetric exchange term arises from spin-orbit coupling in bulk materials or interfaces lacking inversion symmetry. Unlike the standard exchange that favors parallel alignment, the DMI favors a specific chirality or handedness, promoting the formation of complex topological spin textures such as spin helices and magnetic skyrmions (see [DFS23] for a rigorous mathematical treatment of DMI).
- *Magnetocrystalline Anisotropy* ($\varphi(\boldsymbol{u})$): This term models the tendency of the magnetization to align with specific crystallographic axes. The function $\varphi : \mathbb{S}^2 \to \mathbb{R}$ is typically a smooth (at least C^k) function that vanishes at a finite set of points (the "easy axes"). A standard example is the uniaxial anisotropy $\varphi(\boldsymbol{u}) = K(1-(\boldsymbol{u}\cdot e_3)^2)$ for some material constant $K > 0$.
- *Demagnetizing / Stray Field Energy* ($\mathcal{E}_{\text{stray}}(\boldsymbol{u})$): This accounts for the long-range dipole-dipole interactions. In 3D, it is represented via the demagnetizing field $\boldsymbol{h}[\boldsymbol{u}]$ generated by $\boldsymbol{u}$, which solves the magnetostatic Maxwell equations. Mathematically, $\boldsymbol{h} = -\nabla\Delta^{-1}\operatorname{div}(\boldsymbol{u}\chi_U)$ where $\boldsymbol{u}\chi_U$ denotes the extension by zero of $\boldsymbol{u}$ to the whole of $\mathbb{R}^3$. Because it is a zero-order Calderón-Zygmund singular integral operator, the map $\boldsymbol{h} : L^2 \to L^2$ is bounded on L^p spaces for $1 < p < \infty$ and satisfies the standard elliptic properties for any $k \in \mathbb{N}$, $\eta \in (0,1)$, and $p \in (1,\infty)$:
$$\boldsymbol{u} \in L^p \quad \Longrightarrow \quad \boldsymbol{h}[\boldsymbol{u}] \in L^p. \tag{7.2}$$
$$\boldsymbol{u} \in C^{k,\eta}_{\text{loc}} \quad \Longrightarrow \quad \boldsymbol{h}[\boldsymbol{u}] \in C^{k,\eta}_{\text{loc}}. \tag{7.3}$$
In 2D the nonlocal stray field operator collapses to a local term of the form $-(\boldsymbol{u}\cdot\boldsymbol{e}_3)\boldsymbol{e}_3$ and, more generally to $-(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}$ if U is a compact surface in $\mathbb{R}^3$ (we refer to [DF20, DFS23, DFMS24] for the rigorous derivation of these reduced thin-film models).

Regardless, for what follows, it is sufficient for us to assume that $\boldsymbol{h} : L^2 \to L^2$ is a linear, possibly non-local operator satisfying the regularity behavior in (7.2) and (7.3).

The Euler-Lagrange equations associated with the critical points of $\mathcal{G}$, constrained to the sphere $\mathbb{S}^2$, are known as Brown's static equations. They take the form:
$$-\Delta\boldsymbol{u} = |\nabla\boldsymbol{u}|^2\boldsymbol{u} + \boldsymbol{u}\times(\boldsymbol{u}\times(2\kappa\,\mathbf{curl}\,\boldsymbol{u} + \nabla_{\boldsymbol{u}}\varphi(\boldsymbol{u}) - \boldsymbol{h}[\boldsymbol{u}])). \tag{7.4}$$
From a PDE regularity perspective, we can absorb the lower-order magnetic interactions into a single effective source term, rewriting the system as an almost harmonic map equation:
$$-\Delta\boldsymbol{u} = |\nabla\boldsymbol{u}|^2\boldsymbol{u} + \boldsymbol{f}, \quad \text{with } \boldsymbol{f} := \boldsymbol{u}\times(\boldsymbol{u}\times(2\kappa\,\mathbf{curl}\,\boldsymbol{u} + \nabla_{\boldsymbol{u}}\varphi(\boldsymbol{u}) - \boldsymbol{h}[\boldsymbol{u}])). \tag{7.5}$$
We now state the main regularity result for micromagnetic configurations, which follows cleanly from our abstract framework.

Theorem 1.5. (Interior Regularity of Micromagnetic Maps) *Let $U \subseteq \mathbb{R}^m$ ($m = 2,3$) be a bounded open set and let $\boldsymbol{u} \in H^1 \cap C^0(U,\mathbb{S}^2)$ be a continuous critical point of the micromagnetic energy $\mathcal{G}$. Assume that $\boldsymbol{h} : L^2(U,\mathbb{R}^3) \to L^2(U,\mathbb{R}^3)$ satisfies the regularity hypotheses* (7.2) *and* (7.3)*.*

If the anisotropy density φ is of class $C^{k,\beta}$ for $k \geqslant 1$ and exponent $\beta \in (0,1)$, then $\boldsymbol{u} \in C^{k+1,\beta}_{\text{loc}}(U,\mathbb{S}^2)$. In particular, if $\varphi \in C^\infty$, then $\boldsymbol{u} \in C^\infty(U,\mathbb{S}^2)$.

For $m = 2$, the initial continuity assumption is automatically satisfied for finite-energy solutions.

Remark 7.1. From a geometric perspective, the physical dimensionality of the magnetic material dictates the structure of its singular set. In $m = 2$ dimensions, finite-energy solutions are a priori known to be continuous everywhere; therefore, 2D micromagnetic configurations are unconditionally smooth if $\varphi \in C^\infty$. In $m = 3$ dimensions, however, topological point defects (such as the hedgehog singularity $x/|x|$) may exhibit finite energy. Thus, 3D solutions are C^∞ smooth outside of their singular set (the specific discrete points where initial continuity fails).

Proof. We apply the Bootstrap Theorem 1.4 to establish the higher regularity of weak critical points. Because the physical dimension is restricted to $m \leqslant 3$, the baseline integrability of the effective source term ($q = 2$) satisfies the De Giorgi–Nash–Moser non-stalling condition $q > m/2$. Therefore, the abstract bootstrap is unconditionally triggered from mere continuity, and we are not required to assume any fractional Hölder regularity a priori.

Because $\boldsymbol{u} \in W^{1,2}$, the differential term $\mathbf{curl}\ \boldsymbol{u} \in L^2$. Furthermore, the geometric constraint $|\boldsymbol{u}| = 1$ ensures $\boldsymbol{u} \in L^\infty$, meaning the singular integral operator guarantees (via (7.2)) that $\boldsymbol{h}[\boldsymbol{u}] \in L^p_{\text{loc}}$ for all finite p. It follows that the composite effective source term initiates at $\boldsymbol{f} \in L^2_{\text{loc}}$.

In the following argument, we focus on the physically critical dimension $m = 3$, where the baseline integrability is weakest. The proof for $m = 2$ proceeds identically but crosses the classical regularity thresholds sooner.

First Bootstrap Step (*Sobolev Improvement*). By Theorem 1.4.*i*, since $\boldsymbol{f} \in L^2_{\text{loc}}$, we automatically deduce that $\boldsymbol{u} \in W^{1,2^*}_{\text{loc}}(\Omega, \mathbb{S}^2)$. In $m = 3$ dimensions, the critical Sobolev exponent is $2^* = 6$. Consequently, $\nabla \boldsymbol{u} \in L^6_{\text{loc}}$. It follows that $\mathbf{curl}\,\boldsymbol{u} \in L^6_{\text{loc}}$ and $\boldsymbol{h}[\boldsymbol{u}] \in L^p_{\text{loc}}$ for all $p < \infty$. Overall, our source term improves to $\boldsymbol{f} \in L^6_{\text{loc}}$.

Second Bootstrap Step (*Hölder Gradients*). We have now established $\boldsymbol{f} \in L^q_{\text{loc}}$ with $q = 6$. Because $q > m$ (i.e., $6 > 3$), we invoke Theorem 1.4.*ii* to cross the threshold into classical differentiability. We deduce that $\boldsymbol{u} \in C^{1,\eta}_{\text{loc}}$ with $\eta := 1 - m/q = 1/2$.

Higher Regularity. Because $\boldsymbol{u} \in C^{1,\eta}_{\text{loc}}$, its first-order derivatives (such as $\mathbf{curl}\,\boldsymbol{u}$) belong to $C^{0,\eta}_{\text{loc}}$. Concurrently, by (7.3), the zero-order stray field $\boldsymbol{h}[\boldsymbol{u}]$ is $C^{1,\eta}_{\text{loc}}$. The regularity of the gradient-dependent source term $\boldsymbol{f}$ is dictated by its least regular component, ensuring $\boldsymbol{f} \in C^{0,\eta}_{\text{loc}}$. Defining the full right-hand side as $\boldsymbol{\psi} := |\nabla \boldsymbol{u}|^2 \boldsymbol{u} + \boldsymbol{f}$, we see that $\boldsymbol{\psi} \in C^{0,\eta}_{\text{loc}}$. Standard linear elliptic regularity for the Poisson equation then yields $\boldsymbol{u} \in C^{2,\eta}_{\text{loc}}$. This elevated regularity forces $\boldsymbol{\psi} \in C^{1,\eta}_{\text{loc}}$, subsequently yielding $\boldsymbol{u} \in C^{3,\eta}_{\text{loc}}$, and so forth.

Iterating this Schauder-type bootstrap, we conclude that the configuration $\boldsymbol{u}$ precisely inherits the maximal regularity permitted by the anisotropy density: $\boldsymbol{u} \in C^{k+1,\beta}_{\text{loc}}(U, \mathbb{S}^2)$ if $\varphi \in C^{k,\beta}$. In particular, $\boldsymbol{u} \in C^\infty(U, \mathbb{S}^2)$ whenever the anisotropy φ is smooth. □

8. Appendix: Interior Gradient Estimates for Harmonic Functions

In this appendix, we collect classical interior estimates for maps with harmonic components. These results, which bound the gradient of a harmonic (or anisotropic harmonic) map in terms of its boundary values, are frequently invoked throughout the paper and are provided here for completeness.

The section is structured around three main results. First, Proposition 8.1 leverages the explicit Poisson kernel on Euclidean balls to establish a scale-invariant gradient bound controlled by the L^1-norm of the boundary trace, including a precise characterization of the constant's blow-up rate near the boundary. Second, to address non-spherical geometries, Lemma 8.3 recalls a potential theory bound by Widman [Wid67] for the Poisson kernel on arbitrary $C^{1,1}$ domains. Finally, Proposition 8.5 synthesizes these concepts: by applying an affine transformation and invoking the Widman bound on the resulting ellipsoid, it establishes the analogous interior gradient estimate for anisotropic harmonic functions, explicitly tracking the constant's dependence on the ellipticity ratio.

8.1. **Interior Gradient Estimate for Harmonic Functions.**

Proposition 8.1 (Interior Gradient Estimate for Harmonic Functions via Traces)**.** *Let $u \in W^{1,1}(B_r(x_0), \mathbb{R})$ be a weakly harmonic function in the open ball $B_r(x_0) \subset \mathbb{R}^m$. Let $u_{|\partial B_r(x_0)} \in L^1(\partial B_r(x_0))$ denote its boundary trace. Then, for every shrinking factor $0 < \lambda < 1$, u is smooth in the interior, and the following interior gradient estimate holds:*

$$\sup_{x \in B_{\lambda r}(x_0)} |\nabla u(x)| \leqslant \frac{C_{\lambda,m}}{r^m} \int_{\xi \in \partial B_r(x_0)} |u(\xi)| \, \mathrm{d}\xi, \tag{8.1}$$

where $C_{\lambda,m} > 0$ depends only on λ and m.

Furthermore, the constant $C_{\lambda,m}$ satisfies the explicit upper bound:

$$C_{\lambda,m} \leqslant \frac{2(m+\lambda)}{\dot{\omega}_m (1-\lambda)^m}, \tag{8.2}$$

where $\dot{\omega}_m$ is the surface area of the unit sphere in $\mathbb{R}^m$. Consequently, the optimal constant exhibits a blow-up of order $\mathcal{O}((1-\lambda)^{-m})$ as $\lambda \to 1^-$.

Remark 8.2. Because the difference of harmonic functions is harmonic, the proposition implies that for any constant $a \in \mathbb{R}$, we have:

$$\sup_{x \in B_{\lambda r}(x_0)} |\nabla u(x)| \leqslant \frac{C_{\lambda,m}}{r^m} \int_{\xi \in \partial B_r(x_0)} |u(\xi) - a| \, \mathrm{d}\xi. \tag{8.3}$$

This shifted estimate is the key for bounding gradients using the L^1-oscillation on the boundary, even when the boundary values are only understood in the sense of traces.

Proof. By translation and scaling, it suffices to prove the estimate on the unit ball centered at the origin ($x_0 = 0$, $r = 1$). We define the rescaled function $v(y) := u(x_0 + ry)$. Because $u \in W^{1,1}(B_r(x_0))$ is weakly harmonic, $v \in W^{1,1}(B_1)$ is weakly harmonic and its trace belongs to $L^1(\partial B_1)$. We aim to show:

$$\sup_{|y| \leqslant \lambda} |\nabla v(y)| \leqslant C_{\lambda,m} \int_{\zeta \in \partial B_1} |v(\zeta)| \, \mathrm{d}\zeta. \tag{8.4}$$

The Poisson integral formula via traces. Because v is weakly harmonic and its boundary values exist in $L^1(\partial B_1)$ in the trace sense, classical elliptic regularity theory guarantees that v is equivalent to a smooth harmonic function in the interior of B_1, and it can be recovered by the Poisson integral formula of its trace. Thus, for almost every $y \in B_1$ (and everywhere for the smooth representative):

$$v(y) = \int_{\zeta \in \partial B_1} P(y,\zeta) v(\zeta) \, \mathrm{d}\zeta, \qquad P(y,\zeta) = \frac{1 - |y|^2}{\dot{\omega}_m |y - \zeta|^m}. \tag{8.5}$$

Since $v \in L^1(\partial B_1)$ and the kernel $P(y,\zeta)$ alongside all its spatial derivatives with respect to y are smooth and uniformly bounded on any compact sub-domain $|y| \leqslant \lambda < 1$, we may differentiate under the integral sign using the Leibniz integral rule (justified by Lebesgue's Dominated Convergence Theorem):

$$\nabla v(y) = \int_{\zeta\in\partial B_1} \nabla_y P(y,\zeta)v(\zeta)\,\mathrm{d}\zeta. \tag{8.6}$$

The kernel map $(y,\zeta)\mapsto \nabla_y P(y,\zeta)$ is continuous (hence bounded) on the compact set

$$K := \{(y,\zeta)\in\mathbb{R}^m\times\mathbb{S}^{m-1} \,:\, |y|\leqslant\lambda,\ |\zeta|=1\}. \tag{8.7}$$

Let $M_{\lambda,m} = \sup_K |\nabla_y P(y,\zeta)|$ (finite, depending only on λ and m). Then

$$|\nabla v(y)| \leqslant M_{\lambda,m}\int_{\zeta\in\partial B_1} |v(\zeta)|\,\mathrm{d}\zeta. \tag{8.8}$$

This is the desired estimate for the unit ball with $C_{\lambda,m} = M_{\lambda,m}$. To recover the general estimate on $B_r(x_0)$, we reverse the scaling and translation. Recall that $v(y) = u(x_0+ry)$. By the chain rule, the gradient transforms as:

$$\nabla v(y) = r\nabla u(x_0+ry) \quad\Longrightarrow\quad \sup_{|y|\leqslant\lambda}|\nabla v(y)| = r\sup_{x\in B_{\lambda r}(x_0)}|\nabla u(x)|. \tag{8.9}$$

For the right-hand side, we apply the change of variables $\xi = x_0 + r\zeta$ to the boundary integral. As ζ parameterizes the unit sphere $\partial B_1(0)$, the variable ξ parameterizes the sphere $\partial B_r(x_0)$. The surface area element transforms as $\mathrm{d}\mathcal{H}^{m-1}(\xi) = r^{m-1}\mathrm{d}\mathcal{H}^{m-1}(\zeta)$. Therefore:

$$\int_{\zeta\in\partial B_1}|v(\zeta)|\,\mathrm{d}\zeta = \int_{\xi\in\partial B_r(x_0)}|u(\xi)|\frac{1}{r^{m-1}}\,\mathrm{d}\xi. \tag{8.10}$$

Substituting these two relations back into the unit ball estimate yields:

$$r\sup_{x\in B_{\lambda r}(x_0)}|\nabla u(x)| \leqslant M_{\lambda,m}\frac{1}{r^{m-1}}\int_{\xi\in\partial B_r(x_0)}|u(\xi)|\,\mathrm{d}\xi. \tag{8.11}$$

Dividing both sides by r produces the exact $1/r^m$ scaling factor, yielding the general result. *The blow-up of $C_{\lambda,m}$ as $\lambda\to1^-$.* Finally, the blow-up $C_{\lambda,m}\to+\infty$ as $\lambda\to1^-$ follows from the explicit form of P. Precisely

$$\nabla_y P(y,\zeta) = \frac{1}{\dot\omega_m}\nabla_y\left(\frac{1-|y|^2}{|y-\zeta|^m}\right) = \frac{1}{\dot\omega_m}\left(\frac{-2y\,|y-\zeta|^2 - m(1-|y|^2)(y-\zeta)}{|y-\zeta|^{m+2}}\right). \tag{8.12}$$

Taking the norm, we get that

$$|\nabla_y P(y,\zeta)| \leqslant \frac{1}{\dot\omega_m}\left(\frac{2|y|}{|y-\zeta|^m} + \frac{m(1-|y|^2)}{|y-\zeta|^{m+1}}\right). \tag{8.13}$$

We now restrict our attention to $y\in\overline{B_\lambda}$ (so $|y|\leqslant\lambda$) and $\zeta\in\partial B_1$ (so $|\zeta|=1$). We utilize the algebraic factorization $1-|y|^2 = (1-|y|)(1+|y|)$. Since $|y|<1$, we can bound $1+|y|<2$. Furthermore, by the reverse triangle inequality, $|y-\zeta|\geqslant 1-|y|$. Combining these gives the cancellation:

$$1-|y|^2 \leqslant 2(1-|y|) \leqslant 2|y-\zeta|. \tag{8.14}$$

Substituting this cancellation into the second term of our bound yields:

$$|\nabla_y P(y,\zeta)| \leqslant \frac{1}{\dot\omega_m}\left(\frac{2\lambda}{|y-\zeta|^m} + \frac{2m}{|y-\zeta|^m}\right) = \frac{2(\lambda+m)}{\dot\omega_m|y-\zeta|^m}. \tag{8.15}$$

Finally, applying the lower bound $|y-\zeta| \geqslant 1-\lambda$, we secure the uniform upper bound:

$$\sup_{|y|\leqslant\lambda} |\nabla_y P(y,\zeta)| \leqslant \frac{2(m+\lambda)}{\dot{\omega}_m(1-\lambda)^m} =: C_{\lambda,m}. \tag{8.16}$$

This establishes the estimate for the unit ball, explicitly demonstrating the $\mathcal{O}((1-\lambda)^{-m})$ blow-up. □

8.2. **Interior Gradient Estimate for Anisotropic Harmonic Functions.** Proposition 8.1 establishes an interior gradient estimate for classical harmonic functions on Euclidean balls, deriving its constants directly from the closed-form expression of the Poisson kernel. However, linearizing variable-coefficient elliptic problems naturally yields the constant-coefficient equation $\operatorname{div}(A_0\nabla h) = 0$, where A_0 is a symmetric, uniformly elliptic matrix. A standard affine transformation reduces this anisotropic equation to the classical Laplace equation, but it simultaneously maps Euclidean balls into ellipsoids whose geometry depends on the spectrum of A_0. Because no explicit Poisson kernel formula exists for arbitrary ellipsoids, the direct computational approach of Proposition 8.1 cannot be applied. To overcome this deficit, we invoke a general gradient estimate for Poisson kernels on bounded $C^{1,1}$ domains, originally established by Widman [Wid67].

We start with the following preliminary estimate.

Lemma 8.3. *Let $U \subset \mathbb{R}^m$ be a bounded $C^{1,1}$ domain, and let $P(y,\zeta)$ denote the Poisson kernel for the Dirichlet Laplacian on U. There exists a constant $C > 0$, depending only on m and ∂U, such that*

$$|\nabla_y P(y,\zeta)| \leqslant \frac{C}{|y-\zeta|^m} \qquad \text{for all } y \in U, \zeta \in \partial U. \tag{8.17}$$

Remark 8.4. More precisely, the constant C depends on the domain U through the dimension m and the radii of its uniform interior and exterior tangent balls. This precise geometric dependence is crucial for the proof of Proposition 8.5, as it allows us to deduce a uniform bound across a family of ellipsoids, yielding a final constant that depends solely on the ellipticity parameters of A_0 rather than the particular matrix itself.

Proof. We first recall the standard pointwise estimate

$$P(y,\zeta) \leqslant K\frac{\operatorname{dist}(y,\partial U)}{|y-\zeta|^m}, \qquad y \in U, \zeta \in \partial U, \tag{8.18}$$

where $K > 0$ depends only on m and on the $C^{1,1}$ geometry of ∂U. This follows from Widman's interior gradient estimate for the Dirichlet Green function G of $-\Delta$ on U (see Widman [Wid67, Theorem 2.3(iii)]). Namely, for all distinct interior points $x, y \in U$,

$$|\nabla_x G(y,x)| \leqslant K\,\frac{\operatorname{dist}(y,\partial U)}{|y-x|^m}. \tag{8.19}$$

where K depends only on m and ∂U. Since the Poisson kernel is given by the normal derivative of the Green function,

$$P(y,\zeta) := -\frac{\partial G(y,\zeta)}{\partial \boldsymbol{n}_\zeta} = -\lim_{x\to\zeta, x\in U} \nabla_x G(y,x)\cdot \boldsymbol{n}(\zeta), \tag{8.20}$$

where $\boldsymbol{n}(\zeta)$ is the outward unit normal at ζ, we obtain, for fixed $y \in U$ and $\zeta \in \partial U$,

$$P(y,\zeta) \leqslant \limsup_{x\to\zeta} |\nabla_x G(y,x)| \leqslant K\,\frac{\operatorname{dist}(y,\partial U)}{|y-\zeta|^m}. \tag{8.21}$$

Thus (8.18) holds.

Now fix $\zeta \in \partial U$. It is well known that the function $y \mapsto P(y, \zeta)$ is positive and harmonic in U. Positivity follows from the Hopf boundary point lemma, while harmonicity follows from the fact that $P(\cdot, \zeta)$ arises as the local uniform limit of harmonic functions.

Let $y \in U$ and set $\rho := \frac{1}{2}\operatorname{dist}(y, \partial U)$. Then $B_\rho(y) \subset U$, and for every $w \in B_\rho(y)$ we have

$$\operatorname{dist}(w, \partial U) \leqslant \operatorname{dist}(y, \partial U) + |w - y| \leqslant 2\rho + \rho = 3\rho. \tag{8.22}$$

Moreover, since $\zeta \in \partial U$,

$$|w - \zeta| \geqslant |y - \zeta| - |w - y| \geqslant |y - \zeta| - \rho \geqslant |y - \zeta| - \tfrac{1}{2}|y - \zeta| = \tfrac{1}{2}|y - \zeta|. \tag{8.23}$$

Applying (8.18) at the point w yields

$$P(w, \zeta) \leqslant C_1 \frac{\operatorname{dist}(w, \partial U)}{|w - \zeta|^m} \leqslant C_1 \frac{3\rho}{\left(\frac{1}{2}|y - \zeta|\right)^m} = 3 \cdot 2^m \, C_1 \frac{\rho}{|y - \zeta|^m}. \tag{8.24}$$

Since $P(\cdot, \zeta)$ is harmonic in $B_\rho(y)$, the standard interior gradient estimate gives

$$|\nabla_y P(y, \zeta)| \leqslant \frac{C_m}{\rho} \sup_{w \in B_\rho(y)} P(w, \zeta), \tag{8.25}$$

where $C_m > 0$ depends only on the dimension. Combining this with the previous bound, we find

$$|\nabla_y P(y, \zeta)| \leqslant \frac{C_m}{\rho} \cdot 3 \cdot 2^m \, C_1 \frac{\rho}{|y - \zeta|^m} = \frac{C}{|y - \zeta|^m}, \tag{8.26}$$

with $C := 3 \cdot 2^m \, C_1 C_m$. This proves the lemma. □

Proposition 8.5. *Let $A_0 \in \mathbb{R}^{m \times m}$ be a symmetric matrix satisfying the uniform ellipticity condition $\lambda I \leqslant A_0 \leqslant \Lambda I$ for some $0 < \lambda \leqslant \Lambda < \infty$. Let $h \in W^{1,1}(B_r(x_0))$ be a weakly A_0-harmonic function on $B_r(x_0) \subset \mathbb{R}^m$; that is, $\operatorname{div}(A_0 \nabla h) = 0$ in the weak sense in $B_r(x_0)$. Then h is smooth in the interior of $B_r(x_0)$, and for every $\theta \in (0, 1)$,*

$$\sup_{x \in B_{\theta r}(x_0)} |\nabla h(x)| \leqslant \frac{C_{\theta,\lambda,\Lambda,m}}{r^m} \int_{\zeta \in \partial B_r(x_0)} |h(\xi)| \, \mathrm{d}\xi, \tag{8.27}$$

where $C_{\theta,\lambda,\Lambda,m} > 0$ depends only on θ, m, λ, and Λ, and satisfies $C_{\theta,\lambda,\Lambda,m} = \mathcal{O}((1 - \theta)^{-m})$ as $\theta \to 1^-$.

Proof. By translation invariance, we may assume $x_0 = 0$. Standard elliptic regularity for constant-coefficient operators gives $h \in C^\infty(B_r(0))$.

Step 1: Affine change of variables. Define the linear map

$$\Phi(x) := r^{-1} A_0^{-1/2} x, \qquad x \in \mathbb{R}^m, \tag{8.28}$$

and set $\tilde{h}(y) := h(r A_0^{1/2} y) = h(\Phi^{-1}(y))$. Since $|A_0^{1/2} y| = |x|/r$ whenever $x = r A_0^{1/2} y$, the map Φ sends B_r bijectively onto the ellipsoid

$$E_1 := \left\{ y \in \mathbb{R}^m \, : \, |A_0^{1/2} y| < 1 \right\} = \{ y \in \mathbb{R}^m \, : \, \langle y, A_0 y \rangle < 1 \}, \tag{8.29}$$

whose semi-axes have lengths in $[1/\sqrt{\Lambda}, \, 1/\sqrt{\lambda}]$.

A direct chain-rule computation gives

$$\Delta \tilde{h}(y) = r^2 \, \operatorname{div}(A_0 \nabla h)(x) = 0, \qquad y = \Phi(x), \tag{8.30}$$

so $\tilde{h}$ is classically harmonic on E_1. Moreover, the identity $\nabla h(x) = r^{-1} A_0^{-1/2} \nabla \tilde{h}(y)$ together with the operator-norm bound $\|A_0^{-1/2}\| \leqslant 1/\sqrt{\lambda}$ yelds the pointwise estimate

$$|\nabla h(x)| \leqslant \frac{1}{r\sqrt{\lambda}} |\nabla \tilde{h}(y)|, \qquad y = \Phi(x). \tag{8.31}$$

Step 2: Distance from the inner ellipsoid to ∂E_1. The image of $B_{\theta r}$ under Φ is the inner ellipsoid

$$E_\theta := \left\{ y \in \mathbb{R}^m \,:\, |A_0^{1/2} y| < \theta \right\}. \tag{8.32}$$

For every $y \in E_\theta$ and every $z \in \partial E_1$, the bound $\|A_0^{1/2}\| \leqslant \sqrt{\Lambda}$ and the triangle inequality yield

$$1 - \theta \leqslant |A_0^{1/2} z| - |A_0^{1/2} y| \leqslant |A_0^{1/2}(z - y)| \leqslant \sqrt{\Lambda}\, |z - y|. \tag{8.33}$$

Taking the infimum over all admissible pairs (y, z) produces the uniform lower bound

$$d := \operatorname{dist}(E_\theta, \partial E_1) \geqslant \frac{1 - \theta}{\sqrt{\Lambda}} > 0. \tag{8.34}$$

Step 3: Gradient bound for $\tilde{h}$ on E_1. The function $\tilde{h}$ is harmonic on the smooth bounded domain E_1 and has an L^1 boundary trace: $\tilde{h}_{|\partial E_1} \in L^1(\partial E_1)$. The classical Poisson integral representation therefore applies:

$$\tilde{h}(y) = \int_{\zeta \in \partial E_1} P_{E_1}(y, \zeta)\, \tilde{h}(\zeta)\, \mathrm{d}\zeta, \qquad y \in E_1, \tag{8.35}$$

and since $P_{E_1}(\cdot, \zeta)$ is smooth in the interior of E_1, differentiation under the integral sign gives

$$\nabla \tilde{h}(y) = \int_{\zeta \in \partial E_1} \nabla_y P_{E_1}(y, \zeta)\, \tilde{h}(\zeta) \mathrm{d}\zeta, \qquad y \in E_1. \tag{8.36}$$

Now we can apply Lemma 8.3 to E_1, to get that

$$|\nabla_y \tilde{h}(y)| \leqslant C \int_{\zeta \in \partial E_1} \frac{|\tilde{h}(\zeta)|}{|y - \zeta|^m}\, \mathrm{d}\zeta, \qquad y \in E_1. \tag{8.37}$$

for a positive constant $C > 0$, depending only on m and ∂E_1. However, a priori, ∂E_1 depends on the specific form of A_0. We want to show that the constant provided by the lemma can be chosen depending only on m, λ, Λ, and not on the specific matrix A_0. The boundary ∂E_1 is the regular level set $\{f = 1\}$ of $f(\zeta) := \langle \zeta, A_0 \zeta \rangle$, whose gradient and Hessian are

$$\nabla f(\zeta) = 2 A_0 \zeta, \qquad \nabla^2 f = 2 A_0. \tag{8.38}$$

On ∂E_1, the relation $\langle \zeta, A_0 \zeta \rangle = 1$ combined with the operator inequality $\lambda A_0 \leqslant A_0^2 \leqslant \Lambda A_0$ (which follows by applying the spectral theorem to A_0, since each eigenvalue μ of A_0 satisfies $\lambda \mu \leqslant \mu^2 \leqslant \Lambda \mu$) yields

$$\sqrt{\lambda} \leqslant |A_0 \zeta| \leqslant \sqrt{\Lambda}. \tag{8.39}$$

The outward unit normal at $\zeta \in \partial E_1$ is $\boldsymbol{n}(\zeta) = A_0 \zeta / |A_0 \zeta|$, and the shape operator on the tangent space $T_\zeta \partial E_1$ takes the form

$$S = \frac{P_T\, \nabla^2 f P_T}{|\nabla f(\zeta)|} = \frac{P_T A_0 P_T}{|A_0 \zeta|}\,, \tag{8.40}$$

where P_T denotes orthogonal projection onto $T_\zeta \partial_1$. Therefore, given that $\lambda I \leqslant A_0 \leqslant \Lambda I$, the principal curvatures $\kappa_1, \ldots, \kappa_{m-1}$ of ∂E_1 are the eigenvalues of the shape operator S, we have

$$\frac{\lambda}{\sqrt{\Lambda}} \leqslant \kappa_i \leqslant \frac{\Lambda}{\sqrt{\lambda}}, \qquad i = 1, \ldots, m-1. \tag{8.41}$$

The upper bound on the curvatures yields a uniform interior tangent-ball condition on E_1 with radius at least $\sqrt{\lambda}/\Lambda$, while convexity of E_1 provides the exterior tangent-ball condition trivially. Combined with the diameter bound $\operatorname{diam} E_1 \leqslant 2/\sqrt{\lambda}$, this places E_1 in a family of $C^{1,1}$ domains whose geometric data are controlled by λ and Λ alone. Consequently, Lemma 8.3 yields a constant $K = K(m, \lambda, \Lambda)$, independent of the particular matrix A_0, such that

$$|\nabla_y P_{E_1}(y, \zeta)| \leqslant \frac{K}{|y - \zeta|^m}, \qquad y \in E_1, \zeta \in \partial E_1. \tag{8.42}$$

For $y \in E_\theta$ and $\zeta \in \partial E_1$, the lower bound (8.34) gives $|y - \zeta| \geqslant d \geqslant (1-\theta)/\sqrt{\Lambda}$, whence

$$\sup_{y \in E_\theta, \zeta \in \partial E_1} |\nabla_y P_{E_1}(y, \zeta)| \leqslant \frac{K}{d^m} \leqslant \frac{K\, \Lambda^{m/2}}{(1-\theta)^m}, \tag{8.43}$$

and integrating along ∂E_1 gives

$$|\nabla \tilde{h}(y)| \leqslant \frac{K\, \Lambda^{m/2}}{(1-\theta)^m} \int_{\zeta \in \partial E_1} |\tilde{h}(\zeta)| \, \mathrm{d}\zeta, \qquad y \in E_\theta. \tag{8.44}$$

Step 4: Pull-back of the boundary integral via the area formula. It remains to express the boundary integral over ∂E_1 in terms of an integral over ∂B_r. For a linear map $L : \mathbb{R}^m \to \mathbb{R}^m$ and a hypersurface Σ with unit normal $\boldsymbol{\nu}$, the change-of-variables formula which links the $(m-1)$-dimensional Hausdorff measure $\mathcal{H}^{m-1}_{L(\Sigma)}$ on $L(\Sigma)$ to the corresponding measure $\mathcal{H}^{m-1}_{\Sigma}$ on Σ reads as

$$\mathrm{d}\mathcal{H}^{m-1}_{L(\Sigma)}(L\xi) = |\det L| \cdot |L^{-\top} \boldsymbol{\nu}(\xi)| \, \mathrm{d}\mathcal{H}^{m-1}_{\Sigma}(\xi). \tag{8.45}$$

With $L = r^{-1} A_0^{-1/2}$, $\Sigma = \partial B_r(0)$, $\boldsymbol{\nu}(\xi) = \xi/r$, we have $\det L = r^{-m} \det(A_0)^{-1/2}$ and $L^{-\top} = r A_0^{1/2}$, whence $|L^{-\top} \boldsymbol{\nu}(\xi)| = |A_0^{1/2} \xi|$ and

$$\mathrm{d}\mathcal{H}^{m-1}_{\partial E_1}(\zeta) = r^{-m} \det(A_0)^{-1/2} |A_0^{1/2} \xi| \, \mathrm{d}\mathcal{H}^{m-1}_{\partial B_r}(\xi), \qquad \zeta = \Phi(\xi). \tag{8.46}$$

Moreover, on ∂B_r, $|A_0^{1/2} \xi| \leqslant \sqrt{\Lambda}\, |\xi| = r\sqrt{\Lambda}$, and $\det(A_0)^{-1/2} \leqslant \lambda^{-m/2}$, so

$$\mathrm{d}\mathcal{H}^{m-1}_{\partial E_1}(\zeta) \leqslant \frac{\sqrt{\Lambda}}{r^{m-1}\, \lambda^{m/2}} \mathrm{d}\mathcal{H}^{m-1}_{\partial B_r}(\xi). \tag{8.47}$$

Combining (8.31), (8.44), and (8.47), and using that $\tilde{h}(\zeta) = h(\xi)$ for $\zeta = \Phi(\xi)$, we obtain for every $x \in B_{\theta r}$,

$$\begin{aligned} |\nabla h(x)| \quad &\leqslant \quad \frac{1}{r\sqrt{\lambda}} \cdot \frac{K\, \Lambda^{m/2}}{(1-\theta)^m} \cdot \frac{\sqrt{\Lambda}}{r^{m-1}\, \lambda^{m/2}} \int_{\xi \in \partial B_r} |h(\xi)| \, \mathrm{d}\xi \\ &\leqslant \quad \frac{K}{(1-\theta)^m\, r^m} \left(\frac{\Lambda}{\lambda}\right)^{(m+1)/2} \int_{\xi \in \partial B_r} |h(\xi)| \, \mathrm{d}\xi. \end{aligned}$$

Setting

$$C_{\theta,\lambda,\Lambda,m} := K(m, \lambda, \Lambda) \left(\frac{\Lambda}{\lambda}\right)^{(m+1)/2} (1-\theta)^{-m} \tag{8.48}$$

establishes (8.27) and displays the explicit $\mathcal{O}((1-\theta)^{-m})$ blow-up of the constant as $\theta \to 1^-$. □

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