SEGREGATED SOLUTIONS OF A DEGENERATE CROSS-DIFFUSION SYSTEM WITH DRIFTS

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Abstract. We prove the global existence of segregated weak solutions of a one-dimensional degenerate cross-diffusion system with independent drifts, which is endowed with a Wasserstein gradient flow structure. We argue by a Lagrangian formulation written in terms of the (pseudo-)inverse for the cumulative mass function of the sum of the species, which we solve by a Minimising Movement Scheme in the setting of $L^2 \cap BV_{\text{loc}}$. This Lagrangian problem gives rise to a parabolic PDE similar to a p-Laplace equation, with typical range $p \in (-\infty, 1)$. We employ monotonicity methods *à la* Minty–Browder to obtain strong convergence and pass to the limit $\tau \to 0$ in the time-step of the discrete scheme. Our contribution simultaneously treats all porous medium degeneracies, the log-entropy, and fast diffusions of index $\alpha \in (\frac{1}{3}, 1)$, thereby complementing the recent results [31, 46, 47, 51], and the prior work [42].

Contents

1. Introduction

We study the following cross-diffusion system in one spatial dimension $(t,x) \in (0,\infty) \times \mathbb{R}$:

$$\begin{cases} \partial_t \varrho = \partial_x\big(\varrho \partial_x (f'(\varrho + \mu) + V)\big), \\ \partial_t \mu = \partial_x\big(\mu \partial_x (f'(\varrho + \mu) + W)\big), \end{cases} \tag{1.1}$$

where the terms V, W are given Lipschitz drifts depending only on x, and $f : (0,\infty) \to \mathbb{R}$ is a given function of the sum. In the sequel, we prove the existence of *segregated* solutions ϱ, μ belonging to the space of probability measures *i.e.* ϱ, μ are mutually singular; more precisely, they will be absolutely continuous measures such that $\varrho(x)\mu(x) = 0$ $\mathscr{L}$-a.e. x, *cf.* (1.8). Our strategy relies on a Lagrangian formulation of the problem via the pseudo-inverse of the cumulative mass function for the sum $S = \varrho + \mu$, which itself satisfies

$$\partial_t S = \partial_x\big(S \partial_x (f'(S))\big) + \partial_x\big(\varrho \partial_x V + \mu \partial_x W\big). \tag{1.2}$$

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1.1. **Context and novelty.** In this portion of the paper, we give a brief overview of cross-diffusion systems and outline the "state-of-the-art" for system (1.1).

1.1.1. *Background on cross-diffusion systems.* Systems of advection-diffusion equations arise naturally when studying collective behaviour in the biological and social sciences. Instances of such systems can be found in the modelling of multiple chemotactic populations in competition for nutrient [24, 32], tumour growth [40, 13], population biology [26, 35], neural networks [39], and semiconductors [23, 45].

One of the most pertinent features of cross-diffusion systems is their relevance in describing cell sorting. This is a reorganisation process in which cells of different categories regroup into subregions with clearly defined boundaries [41, 50]; thereby giving rise to segregation effects. This biological phenomenon is related to the inhibition/activation of growth whenever two populations occupy the same environment [9]. These considerations make the study of segregated solutions of cross-diffusion systems a central theme, both for theory and applications.

A complete well-posedness theory for cross-diffusion systems involving transport terms is currently out of reach. This is partly due to the lack of maximum principles, which means that the classical parabolic theory [3, 43] no longer applies. Furthermore, the aforementioned segregation effects between species naturally lead to the formation of boundaries and sharp interfaces, which make the analysis of such equations delicate. This is exacerbated by the presence of degenerate diffusions, which cause a loss of strict parabolicity and can give rise to singularities [48]. In turn, one often tailors the analysis to low-regularity settings which can accommodate for the presence of jumps and discontinuities. And indeed, when the drift terms in (1.1) are replaced with reaction terms, an existence theory was obtained in a one-dimensional BV setting in [20]; this setting was also highlighted in [14, 15], while for the Sobolev setting we refer to *e.g.* [2, 8, 22, 21, 37]. However, it is fair to say that, while the functional framework of BV seems suitably general for analysis, it does not exploit any special structure of the cross-diffusion system at hand.

Some cross-diffusion systems are intrinsically endowed with a gradient flow structure, meaning that they can be recast as the dynamic time-evolution of a minimisation problem associated to a particular energy functional. For instance, the system (1.1) can be rewritten as

$$\partial_t \varrho = \partial_x \left(\varrho \partial_x \frac{\delta \mathrm{E}}{\delta \varrho} \right), \quad \partial_t \mu = \partial_x \left(\mu \partial_x \frac{\delta \mathrm{E}}{\delta \mu} \right), \tag{1.3}$$

where $\frac{\delta \mathrm{E}}{\delta \varrho}$ and $\frac{\delta \mathrm{E}}{\delta \mu}$ denote the first variation of E with respect to its two variables ϱ and μ, respectively, where the energy E is given by

$$\mathrm{E}[\varrho, \mu] := \int_{\mathbb{R}} f(\varrho + \mu)\, \mathrm{d}x + \int_{\mathbb{R}} \big(\varrho V + \mu W\big)\, \mathrm{d}x. \tag{1.4}$$

While there exists ample literature concerning the well-posedness of Wasserstein gradient flows [5, 49] in the case of single population densities, these results do not generally apply to the case of systems such as (1.3), due to the lack of λ-geodesic convexity (unless the system is actually diagonal, thanks to a clever construction in [6] which shows necessary conditions for λ-geodesic convexity in $W_2 \times W_2$). Nevertheless, this gradient flow approach to cross-diffusion systems was successfully employed in [29, 28] for systems of non-local interaction equations with two species and non-symmetric cross-interactions, and was first used in a system with cross-diffusion terms but no transport terms in [44]. Other results focus on diagonal diffusion and bounded domains [18, 19], small cross-diffusion [1, 2], convergence to equilibrium [6], or triangular structures [25, 27]. In some cases, one may interpret the system of equations at hand as a perturbation of a gradient flow associated with a convex functional [12, 30], which sometimes makes them amenable to the boundedness-by-entropy method [16, 38]; see also [17] for a non-local version involving infinitely many species. All things considered, despite the simple gradient flow structure of (1.1), there is still much to discover concerning the behaviour of this system of equations.

1.1.2. *State-of-the-art for system* (1.1). The simplest case of (1.1) arises when $\partial_x V = \partial_x W = 0$, which makes the system of equations purely diffusive. We can then compute, for

$$\mathrm{F}[S] = \int_{\mathbb{R}} f(S)\,\mathrm{d}x, \tag{1.5}$$

the first variation as $\frac{\delta \mathrm{F}}{\delta S}[S] = f'(S)$. In this instance of the problem, the evolution of the sum S can be recast as the closed-form gradient flow:

$$\partial_t S = \partial_x\Big(S\partial_x \frac{\delta \mathrm{F}}{\delta S}[S]\Big), \tag{1.6}$$

and existence for (1.6) follows directly from the well-established Wasserstein gradient flow theory [5]. Once existence of S is known, one can go back to the individual equations for ϱ, μ and treat them as independent linear scalar conservation laws: $\partial_t \varrho = \partial_x(\varrho \partial_x f'(S))$ and similarly for μ. Since the density S that is obtained as a solution to (1.6) is sufficiently smooth, this linear equation is well-posed, which allows to obtain existence of ϱ and μ and also to verify that the ϱ, μ obtained in this manner satisfy $\varrho + \mu = S$. This purely diffusive problem and its segregating effects had already been studied in [7]; *cf.* system (1.6) therein. When the drifts are not null, and not identical, the situation is more intricate.

A first general existence result for system (1.1) was given in [42], which involved very restrictive assumptions on the ordering of the potentials V, W and on the initial configuration so as to obtain segregated solutions for all times. The problem then stayed with no progress for some years, until the next breakthrough, obtained in [47], where global existence to (1.1) with periodic boundary conditions, general Lipschitz potentials V, W, and a broad class of initial data was shown for the specific case $f(s) = s\log s - s$. The recent contribution [31] then improved the result of [47], and proved existence of periodic solutions for a relatively broad class of ("totally mixed") initial data with $f(s) \sim s^\alpha$ for $0 < \alpha < 1$; the case $\alpha = 1$ coincides precisely with the choice of f in [47]. Both of these recent contributions employ a formulation of the problem where the main quantities that are studied are the sum $\varrho + \mu$ and the quotient ϱ/μ. Therein, the authors obtain BV estimates for the logarithm of the quotient ϱ/μ and deduce suitable bounds on ϱ and μ from this and H^1 bounds on functions of S. We highlight that in one dimension, BV functions are L^∞, and that the boundedness from above and below of this logarithm corresponds to the case of *totally mixed solutions*, with overlapping supports. This structure condition was then relaxed in [46] (relying on the ratio of each density with respect to the sum, instead of using the logarithm). A much more general existence result was subsequently obtained in [51] using a compensated compactness approach; this contribution can accommodate for all choices of fast diffusion, porous medium degeneracy, and the log-entropy.

1.1.3. *Novelty.* The focus of the present manuscript is not to obtain as general an existence result as possible, but rather to study the segregation dynamics encoded in (1.1) using variational techniques. In particular, we are concerned with the *preservation of segregation* from segregated initial data (which was not studied in the aforementioned works [31, 46, 47, 51]).

This is a natural case to consider from a thermodynamic/energy-minimising perspective, since mixing is often energetically less favourable than staying segregated. Our contribution is closer in spirit to that of [42], which is the only other work to specifically study segregated solutions, but the result contained in the present paper is much stronger than that of [42] as we allow for arbitrary segregated initial data and general drifts. That is to say, we do not require all the mass of ϱ to be supported to the left of μ for example, and we allow part of the support of ϱ to be squeezed between two components of the support of μ. In terms of the the drifts, we do not require anymore their orientation to be such that it favours segregation. As a byproduct of our approach, which entails a Lagrangian formulation (see §1.2.2), we end up solving a parabolic p-Laplace problem with typical range of exponent $p \in (-\infty, 1)$ by considering a gradient flow in the setting $L^2 \cap BV_{\text{loc}}$, which is novel in and of itself.

1.2. **Set-up.** We describe our approach and the novel results contained herein in more detail. Firstly, we outline which segregation property will be preserved in the solution we build.

1.2.1. *Segregation.* This work is concerned with (1.1) where one assumes a particular segregated structure on the solution. To explain this notion, we introduce the cumulative distribution function for the sum $S_t = S(t, \cdot) = \varrho(t, \cdot) + \mu(t, \cdot)$, denoted by F_t:

$$F_t(x) := \int_{-\infty}^{x} S_t(x')\,\mathrm{d}x'. \tag{1.7}$$

We study solutions which are such that, for $A \subseteq [0, 2]$ a given measurable subset satisfying $\mathscr{L}(A) = 1 = \mathscr{L}(A^c)$, it holds

$$\varrho_t(x) = S_t(x)\mathbb{1}_A(F_t(x)), \qquad \mu_t(x) = S_t(x)\mathbb{1}_{A^c}(F_t(x)); \tag{1.8}$$

where $\varrho_t = \varrho(t, \cdot), \mu_t = \mu(t, \cdot) \in \text{Prob}(\mathbb{R})$. It follows that F_t takes values in $[0, 2]$, and furthermore, by writing the formal change of variables $y = F_t(x)$, *i.e.* $\mathrm{d}y = S_t(x)\,\mathrm{d}x$, it holds

$$1 = \int_{\mathbb{R}} \varrho_t(x)\,\mathrm{d}x = \int_{\mathbb{R}} \mathbb{1}_A(F_t(x))S_t(x)\,\mathrm{d}x = \int_{[0,2]} \mathbb{1}_A(y)\,\mathrm{d}y = \mathscr{L}(A),$$

and similarly for A^c; thus the requirement $\mathscr{L}(A) = 1 = \mathscr{L}(A^c)$ is necessary.

The condition (1.8) imposes that the solutions are *segregated* in the sense:

$$\{x \in \mathbb{R} : \varrho_t(x) > 0\} \cap \{x \in \mathbb{R} : \mu_t(x) > 0\} = \varnothing \qquad \forall t \geq 0.$$

Morally speaking, the structure condition (1.8) means that the supports of ϱ_t, μ_t have a fixed *ordering*, which does not change in time; however, the supports themselves do evolve in time. Note that the case studied in [42] corresponds to the case $A = [0, 1]$ (with additional assumptions on V and W).

1.2.2. *Strategy.* With the segregated formulation of §1.2.1, the final term of (1.2) equals

$$\partial_x\big(\varrho_t(x)\partial_x V(x) + \mu_t(x)\partial_x W(x)\big) = \partial_x\big(S_t(x)b(F_t(x), x)\big), \tag{1.9}$$

where we used (1.8), and where

$$b(y, s) := \mathbb{1}_A(y)\partial_x V(s) + \mathbb{1}_{A^c}(y)\partial_x W(s); \tag{1.10}$$

note that, for V, W Lipschitz continuous, it holds $b \in L^\infty([0, 2] \times \mathbb{R})$. It then follows that the entire system (1.1) effectively collapses into the single equation

$$\partial_t(\partial_x F_t) = \partial_x\big(\partial_x F_t \cdot \partial_x(f'(\partial_x F_t))\big) + \partial_x\big(\partial_x F_t \cdot b(F_t, x)\big), \tag{1.11}$$

and we have that ϱ_t, μ_t given by (1.8) are solutions of the original system (1.1).

The equation (1.11) is not in a particularly nice form, which motivates us to consider the equation satisfied by its (pseudo)-inverse function $u_t = F_t^{-1}$. As shown in §4.1, it turns out that the equation for this inverse—which we call the *Lagrangian reformulation*—has a familiar structure; in formal non-divergence form, it may be written

$$\partial_t u_t = f''(\frac{1}{\partial_y u_t})\frac{\partial_{yy}u_t}{(\partial_y u_t)^3} - b(y, u_t), \qquad y \in [0,2], \tag{1.12}$$

with b given by (1.10). For $f(s) \sim s^\alpha$ with $\alpha > 0$, (with a negative sign if $\alpha < 1$ so as to have a convex function f) one can see that the leading-order terms in (1.12) formally correspond to a *parabolic p-Laplace equation*, but with $p = 1 - \alpha \in (-\infty, 1)$. This formulation enables us to harness powerful monotonicity methods *à la* Minty–Browder. Furthermore, one can show that (1.12) is endowed with an L^2-gradient flow structure (*cf.* §4.2). Our approach is therefore to solve (1.12) by means of Minimising Movement Scheme (*cf.* §5–§6), and to then deduce existence to (1.11)—and hence (1.1)—by translating back into the original coordinates.

1.3. **Plan of the paper and notations.**

1.3.1. *Organisation of the paper.* In §2 we introduce the necessary functional framework for our analysis and to state our main results. §3 contains the statements of our main theorems. §4 explains the Lagrangian reformulation of the problem in (1.12) and its L^2-gradient flow structure. In §5, we write the discrete-time Minimising Movement Scheme associated to the gradient flow formulation of (1.12) with fixed time-step $\tau > 0$. §6 is concerned with passing to the limit $\tau \to 0$ in the time-step of the discrete-time scheme and proving existence to the Lagrangian reformulation. In §7, we translate the result for the Lagrangian reformulation into the original coordinates and prove existence to (1.1).

1.3.2. *Notations.* Throughout the paper, we denote the Lebesgue measure by $\mathscr{L}$ and, to avoid heavy notation, we also denote the two-dimensional Lebesgue on $(0,\infty)\times\mathbb{R}$ and the restriction of Lebesgue measure to a particular set by $\mathscr{L}$, where no confusion arises. For a set A, we denote its complement by A^c. The family of probability measures on $\mathbb{R}$ is denoted $\mathrm{Prob}(\mathbb{R})$, while $\mathrm{Prob}_2(\mathbb{R})$ denotes the set of probabilities with finite second moment, *i.e.* $\varrho \in \mathrm{Prob}(\mathbb{R})$ belongs to $\mathrm{Prob}_2(\mathbb{R})$ if $\int |x|^2 \,\mathrm{d}\varrho(x) < \infty$. For a given open set $U \subseteq \mathbb{R}$, we denote by $\mathcal{M}_{\mathrm{loc}}(U)$ (and $L^p_{\mathrm{loc}}(U)$, resp.) the family of locally finite measures (and locally finite L^p functions, resp.) on U, and $BV_{\mathrm{loc}}(U)$ for functions belonging to $L^1_{\mathrm{loc}}(U)$ with derivative belonging to $\mathcal{M}_{\mathrm{loc}}(U)$. Given $v \in BV_{\mathrm{loc}}((0,2))$ we denote by $\partial_y v$ its distributional derivative (w.r.t. the variable y, which is the standard choice of notation that we use in the paper for elements of $(0,2)$), while the notation v' denotes the absolutely continuous part of $\partial_y v$. W_2 denotes the 2-Wasserstein distance on $\mathbb{R}$. Throughout, τ denotes the time-step for the variational scheme in §5.

2. Functional Framework

In this section, we state some functional analytic results on the space $\mathcal{M}_{\mathrm{loc}}((0,2))$ of locally finite measures; in the sequel, we study variational problems over the space $BV_{\mathrm{loc}}((0,2))$.

Definition 2.1 (The space $C_c((0,2))$ and its dual)**.** We recall the space of continuous functions of compact support $C_c((0,2)) = \{\varphi \in C([0,2]) : \operatorname{supp}\varphi \subset (0,2)\}$, equipped with the norm $\|\cdot\|_{L^\infty([0,2])}$. The dual $(C_c((0,2)))' = \mathcal{M}((0,2))$ is the space of signed measures on $(0,2)$, equipped with the total variation norm $\|\nu\|_{\mathcal{M}((0,2))} = |\nu|((0,2))$, where $|\nu| = \nu^+ + \nu^-$ and $\nu^\pm$ are nonnegative measures in $\mathcal{M}((0,2))^2$, singular to each other (i.e. concentrated on two disjoint sets), such that $\nu = \nu^+ - \nu^-$.

In the sequel, we will often work with measures which belong to $\mathcal{M}_{\rm loc}((0,2))$, *i.e.* locally finite measures on $(0,2)$: when we write $\nu \in \mathcal{M}_{\rm loc}((0,2))$ we mean $\nu \in \mathcal{M}(K) = (C(K))'$ for every compact subset $K \subset (0,2)$. Often these measures will be non-negative: we write $\nu \geq 0$ if, for all compact subsets $K \subset (0,2)$, it holds $\nu(K) \geq 0$. Equivalently, $\nu \geq 0$ if, for all non-negative $\varphi \in C_c((0,2))$, it holds $\int_0^2 \varphi(y)\,\mathrm{d}\nu(y) \geq 0$; note that the integral is well-defined for $\varphi \in C_c((0,2))$ and $\nu \in \mathcal{M}_{\rm loc}((0,2))$.

We recall the following weak-* compactness result for locally finite measures; its proof is standard, and follows by compact exhaustion of $(0,2)$ and a diagonal argument.

Theorem 2.2 (Alaoglu's Theorem for $\mathcal{M}_{\rm loc}((0,2))$)**.** *Let* $\{\nu_n\}_n$ *be a sequence belonging to* $\mathcal{M}_{\rm loc}((0,2))$ *for which it holds* $\sup_{n\in\mathbb{N}} \|\nu_n\|_{\mathcal{M}([y_0,y_1])} \leq C_{y_0,y_1}$ *for all* $[y_0,y_1] \subset (0,2)$*. Then, there exists a subsequence* $\{\nu_{\sigma(n)}\}_n$ *and* $\nu \in \mathcal{M}_{\rm loc}((0,2))$ *such that* $\int_0^2 \varphi\,\mathrm{d}\nu_{\sigma(n)} \to \int_0^2 \varphi\,\mathrm{d}\nu$ *for all* $\varphi \in C_c((0,2))$*; we write* $\nu_{\sigma(n)} \overset{*}{\rightharpoonup} \nu$ weakly-* in $\mathcal{M}_{\rm loc}((0,2))$*.*

Exactly as it happens for finite signed measures, each $\nu \in \mathcal{M}_{\rm loc}((0,2))$ can be uniquely decomposed (*Jordan decomposition*) as $\nu = \nu^+ - \nu^-$ where $\nu^+, \nu^- \geq 0$ are singular to each other. Moreover, every $\nu \in \mathcal{M}_{\rm loc}((0,2))$ is also decomposed uniquely into its *absolutely continuous and singular parts*, denoted $(\nu_a, \nu_s) \in L^1_{\rm loc}((0,2)) \times \mathcal{M}_{\rm loc}((0,2))$,

$$\nu^+ = \nu_a \cdot \mathscr{L} + \nu_s. \tag{2.1}$$

Note that, in the above, when $\nu \geq 0$ then $\nu_a \geq 0$ $\mathscr{L}$-a.e. and $\nu_s \geq 0$.

Definition 2.3 (The space $BV_{\rm loc}((0,2))$)**.** We define the space $BV_{\rm loc}((0,2))$ to be the subset of $L^1_{\rm loc}((0,2))$ composed of those functions whose distributional derivative is a locally finite measure on $(0,2)$, *i.e.*,

$$BV_{\rm loc}((0,2)) := \Big\{ v \in L^1_{\rm loc}((0,2)) : \partial_y v \in \mathcal{M}_{\rm loc}((0,2)) \Big\}.$$

For $v \in BV_{\rm loc}((0,2))$ we write $v' = (\partial_y v)_a$ in the decomposition (2.1), *i.e.*,

$$\partial_y v = v' \cdot \mathscr{L} + (\partial_y v)_s. \tag{2.2}$$

As a consequence of the compactness results for locally bounded measures, we note that a compactness result for sequences of functions in $BV_{\rm loc}$ is also available: if $\{v_n\}_n$ is a sequence belonging to $BV_{\rm loc}((0,2))$ for which it holds $\sup_{n\in\mathbb{N}} \|v_n\|_{TV([y_0,y_1])} \leq C_{y_0,y_1}$ for all $[y_0,y_1] \subset (0,2)$, there exists a subsequence $\{v_{\sigma(n)}\}_n$ and $v \in BV_{\rm loc}((0,2))$ such that $\partial_y v_{\sigma(n)} \overset{*}{\rightharpoonup} \partial_y v$ weakly-* in $\mathcal{M}_{\rm loc}((0,2))$, and $\lim_{n\to\infty} \|v_{\sigma(n)} - v\|_{L^1([y_0,y_1])} = 0$ for all $[y_0,y_1] \subset (0,2)$; we write $v_{\sigma(n)} \to v$ *strongly in* $L^1_{\rm loc}((0,2))$.

Remark 2.4 (Precise representative in $BV_{\rm loc}$). Recall from [4, Theorem 3.28 and p.139] that, for all $u \in BV_{\rm loc}((0,2))$, there exists a unique right-continuous function u^r defined everywhere in $(0,2)$ which satisfies $u^r = u$ $\mathscr{L}$-a.e.; we call u^r the (right-continuous) *precise representative* for u. For this representative, it holds

$$u^r(y_1) - u^r(y_0) = \partial_y u([y_0,y_1)) \qquad \forall [y_0,y_1] \subset (0,2).$$

Throughout the paper, we always identify $BV_{\rm loc}$ functions with a precise representative which, by arbitrary convention, can be chosen to be the right-continuous one (but all results would also work if choosing the left-continuous one, or the average of the right-continuous and of the left-continuous, or any other reasonable precise representative). Moreover, we use the shorthand notation $\int_{y_0}^{y_1} \partial_y u$ to denote $\partial_y u([y_0, y_1))$, whence the following version of the Fundamental Theorem of Calculus in BV_{loc} is satisfied:

$$u(y_1) - u(y_0) = \int_{y_0}^{y_1} \partial_y u \qquad \forall [y_0, y_1] \subset (0,2). \tag{2.3}$$

This clarification removes all ambiguity concerning the evaluation of the integral on the right-hand side when either of the endpoints are atoms for the measure $\partial_y u$.

Finally, we introduce the function space in which we will conduct the bulk of our analysis.

Definition 2.5 (The space X)**.** We define the function space $X := L^2([0,2]) \cap BV_{\mathrm{loc}}((0,2))$. For a sequence $\{v_n\}_n \subset X$, we say it is *bounded in* X if $\sup_n \|v_n\|_{L^2([0,2])} < \infty$ and if, for all $[y_0, y_1] \subset (0,2)$, it holds $\sup_{n\in\mathbb{N}} \|v_n\|_{TV([y_0,y_1])} \le C_{y_0,y_1} < \infty$. For a sequence $\{v_n\}_n \subset X$, we say that v_n converges to v in X if $||v_n - v||_{L^1([0,2])} \to 0$ and v_n is bounded in X.

We observe that bounded sets in X are precompact for this convergence. Indeed, the BV_{loc} bound provides strong L^1 compactness on each interval $[y_0, y_1]$ by Helly's Theorem [4, Theorem 3.23] and hence, up to a subsequence, $\mathscr{L}$-a.e. convergence, and the L^2 bound on the whole interval $[0,2]$ provides equi-integrability of v_n, which transforms (via an easy application of Egoroff's theorem) the $\mathscr{L}$-a.e. convergence into strong L^1 convergence. We also observe that when a sequence v_n converges to v in X, then we also have $v_n \to v$ in $L^2_{\mathrm{loc}}((0,2))$ and $v_n \to v$ in $L^p([0,2])$ for every $p < 2$.

3. Main Results

We give our notion of solution, list the assumptions on f in (1.1), and state our main results.

Definition 3.1 (Segregated weak solution)**.** We say that a pair of curves (ϱ_t, μ_t) valued in $L^1(\mathbb{R}) \cap \mathrm{Prob}(\mathbb{R})$ is a *segregated weak solution* of (1.1) if $\varrho_t, \mu_t \in L^\infty(\mathbb{R})$ for $\mathscr{L}$-a.e. t, there exists a measurable subset $A \subseteq [0,2]$ such that (1.8) holds, $\sqrt{\varrho_t + \mu_t}\,\partial_x(f'(\varrho_t + \mu_t)) \in L^2_{\mathrm{loc}}(0,\infty; L^2(\mathbb{R}))$, and for all $\varphi \in C^1_c(\mathbb{R})$, it holds for $\mathscr{L}$-a.e. $t > 0$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \varrho_t \varphi \,\mathrm{d}x &= - \int_{\mathbb{R}} \varrho_t \partial_x\big(f'(\varrho_t + \mu_t) + V\big) \partial_x \varphi \,\mathrm{d}x, \\ \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \mu_t \varphi \,\mathrm{d}x &= - \int_{\mathbb{R}} \mu_t \partial_x\big(f'(\varrho_t + \mu_t) + W\big) \partial_x \varphi \,\mathrm{d}x. \end{aligned} \tag{3.1}$$

In what follows, we will assume that $f \in C([0,\infty)) \cap C^2((0,\infty))$ is strictly convex on $(0,\infty)$ and $f(0) = 0$. Associated with f, we define the function $\tilde f$ through

$$\tilde f(s) := s f(1/s) \qquad \text{for all } s > 0. \tag{3.2}$$

Remark 3.2 (Properties of $\tilde f$). We compute the derivatives of $\tilde f$ and obtain the explicit formulas

$$\tilde f'(s) = f(1/s) - \frac{1}{s} f'(1/s), \qquad \tilde f''(s) = \frac{1}{s^3} f''(1/s). \tag{3.3}$$

Since (for a C^2 function) strict convexity is equivalent to the second derivative not vanishing on any interval, it follows from the previous expression for $\tilde f''$ that $\tilde f$ is also strictly convex. From $f(0) = 0$ and the strict convexity of f we deduce $f'(s)s > f(s) - f(0) = f(s)$, whence

$$\tilde f' < 0, \tag{3.4}$$

and hence $\tilde{f}$ is a strictly decreasing function. Moreover, we observe that we have

$$\lim_{s\to\infty} \tilde{f}'(s) = \lim_{s\to 0} \big(f(s) - sf'(s)\big) = 0. \tag{3.5}$$

In computing this limit, we use $f(0) = 0$ and $\lim_{s\to 0} sf'(s) \le 0$. The latter is justified by the fact that, if f' is unbounded close to 0^+, then it can only be negative (as a consequence of convexity). This implies $\liminf_{s\to\infty} \tilde{f}'(s) \ge 0$ which, together with $\tilde{f}'(s) < 0$, proves (3.5).

The function $\tilde{f}$ could be bounded or not in a neighbourhood of $s = 0$ but, in any case, it admits a (possibly infinite) limit for $s \to 0^+$ due to its monotonicity. We then extend $\tilde{f}$ to $[0, +\infty)$ by continuity, accepting the possible value $\tilde{f}(0) = +\infty$.

Our assumptions on f are listed below.

Assumption 3.3. We assume $f \in C([0,\infty)) \cap C^2((0,\infty))$ is strictly convex on $(0,\infty)$ and $f(0) = 0$, and we assume that f and $\tilde{f}$ satisfy the following two conditions:

$$\lim_{s\to 0^+} \tilde{f}'(s) = -\infty, \tag{3.6}$$

and, for some positive constant C and some $\alpha \in (\frac{1}{3}, 1)$, the lower bound

$$f(s) \ge -Cs^\alpha \qquad \forall s \ge 0. \tag{3.7}$$

We remark that we *do not impose* a quantitative convexity assumption such as $f'' > 0$ a.e. (even for $f \in C^2$ strictly convex, its second derivative f'' is allowed to vanish on a set of positive Lebesgue measure, just not on an interval). Furthermore, note that the condition (3.6) is weaker than superlinearity, *i.e.*

$$\lim_{s\to\infty} \frac{f(s)}{s} = +\infty, \tag{3.8}$$

which follows from the fact that the superlinearity of f is equivalent to $\lim_{s\to 0^+} \tilde{f}(s) = +\infty$.

Remark 3.4 (Admissible f). Assumption 3.3 allows for any f of porous medium type $f(s) = s^\alpha$ for $\alpha > 1$, the log-entropy $f(s) = s\log s - s$, and a large class of fast diffusions $f(s) = -s^\alpha$ for $\frac{1}{3} < \alpha < 1$. The reason for imposing $\alpha > \frac{1}{3}$ is connected with the choice of the L^2 space that we will use later and is standard whenever considering gradient flows in W_2: equations of fast diffusion type in $W_2(\mathbb{R}^d)$ can be considered as far as the exponent of the diffusion α satisfies $\alpha > \frac{d}{d+2}$, which is required for finiteness of the second moment of the Barenblatt solutions.

Our first main theorem is concerned with the global existence of weak segregated solution of the cross-diffusion system (1.1).

Theorem 3.5 (Global existence of segregated solutions). *Let f satisfy Assumption 3.3, $V, W \in W^{1,\infty}(\mathbb{R})$, let $\varrho_0, \mu_0 \in L^1(\mathbb{R}) \cap \mathrm{Prob}_2(\mathbb{R})$ be given initial data, and assume the finiteness of the energy $\mathrm{F}[S_0]$, where $S_0 := \varrho_0 + \mu_0$. Then, there exists (ϱ_t, μ_t) a segregated weak solution of (1.1) in the sense of Definition 3.1, and the initial data is achieved in the sense $\varrho_t \to \varrho_0$ and $\mu_t \to \mu_0$ in $W_2(\mathbb{R})$.*

As mentioned, the conclusion of Theorem 3.5 follows from a corresponding result for a Lagrangian reformulation (*cf.* (1.12) and §4), which is encapsulated in our second main theorem.

Theorem 3.6 (Lagrangian problem). *Let $u_0 \in X$ satisfy $\partial_y u_0 \ge 0$ in $\mathcal{M}_{\mathrm{loc}}((0,2))$ and $\int_0^2 \tilde{f}(u_0')\,\mathrm{d}y < \infty$. Let b be given by (1.10) with $V, W \in W^{1,\infty}(\mathbb{R})$ and $\mathscr{L}(A) = 1$. Let f satisfy*

Assumption 3.3. Then, there exists $u \in L^\infty_{\mathrm{loc}}([0,\infty); X) \cap H^1_{\mathrm{loc}}([0,\infty); L^2([0,2]))$ *satisfying* $0 \le -\tilde{f}'(u') \in L^2_{\mathrm{loc}}([0,\infty); H^1_0([0,2]))$*, and*

$$\begin{cases} \partial_t u = \partial_y(\tilde{f}'(u')) - b(y,u) & \mathscr{L}\text{-a.e. in } (0,\infty)\times[0,2], \\ u|_{t=0} = u_0, \end{cases} \tag{3.9}$$

where $\lim_{t\to 0^+} \|u(t,\cdot) - u_0\|_{L^2([0,2])} = 0$*,* u' *as per (2.2), and* $\operatorname{ess\,inf}_{[0,2]} u'(t,\cdot) > 0$ *for* $\mathscr{L}$*-a.e.* t*. Moreover, for* $\mathscr{L}$*-a.e.* t *the the measure* $(\partial_y u)_s$ *is concentrated on the set where the function* $\tilde{f}'(u')$ *vanishes.*

We remark that only the absolutely continuous part u' of the derivative $\partial_y u$ appears in the equation (3.9). Uniqueness for (3.9) can be obtained under certain semiconvexity conditions on b; this lies outside the scope of this paper, as we do not require it to solve (1.1).

Remark 3.7 (Comments on the main results). In this paper we study the cross-diffusion system (1.1) on the whole real line $\mathbb{R}$, but it would have been possible to instead consider a bounded interval $[a,b]$, with no-flux boundary conditions. This would have simplified some parts of the analysis, in particular since we would consider (in the Lagrangian reformulation) non-decreasing functions valued into $[a,b]$, which are automatically globally BV. This would have avoided the use of locally finite measures and BV_{loc} functions, and our functional setting would have been endowed with strong compactness in L^2 (and not only L^2_{loc}), *cf.* the topology on the space X of Definition 2.5. The restriction $\alpha > \frac{1}{3}$ in the assumption (3.7) is also related to bounding the second moment of the mass distribution $S = \varrho + \mu$, or equivalently the L^2 norm of u, and could be removed when working on $[a,b]$. On the other hand, the extra difficulty in the bounded domain case lies in the boundary conditions in the Lagrangian formulation, where it would not be possible to prove the Dirichlet boundary values $\tilde{f}'(u')|_{y=0} = \tilde{f}'(u')|_{y=2} = 0$.

4. Lagrangian Reformulation

In this section we rewrite the system (1.1) in Lagrangian form, by arguing via u_t the inverse function of the cumulative density function F_t introduced in (1.7). Section §4.1 contains heuristic arguments to give context to the reader before presenting the rigorous definitions (§4.2–§4.3) needed for the analysis of the Lagrangian problem in §5–§6.

4.1. **Formal Lagrangian reformulation.** Recall the cumulative mass function F_t for the sum S_t as defined in (1.7). Let us assume for the time being that F_t is strictly increasing, such that it admits a well-defined inverse function $u_t : [0,2] \to \mathbb{R}$ which is also strictly increasing. Our objective is to rewrite the equation (1.2) as an equation for u_t *i.e.* (1.12); this is what we mean by *Lagrangian reformulation.* All manipulations in this portion of the manuscript are formal, and for the purposes of exposition; in what follows, the Eulerian coordinate x and the Lagrangian coordinate y are linked via $y = F_t(x)$ and $x = u_t(y)$.

Notice that (1.7) formally implies $\partial_x F_t(x) = S_t(x)$, which, using also

$$F_t(u_t(y)) = y, \tag{4.1}$$

implies, by differentiating with respect to y and using the formula (1.7),

$$S_t(u_t(y)) = \partial_x F_t(u_t(y)) = \frac{1}{\partial_y u_t(y)}, \tag{4.2}$$

and taking a further derivative with respect to y yields $\partial_x S_t(u_t(y)) = -\frac{\partial_{yy}u_t(y)}{\partial_y u_t(y)^3}$. Using (1.7), we take the primitive of (1.2) and, using (1.9), we get $\partial_t F_t = S_t \partial_x f'(S_t) + S_t b$ *i.e.*,

$$\partial_t F_t(u_t(y)) = S_t(u_t(y))\Big[- f''(\frac{1}{\partial_y u_t(y)})\frac{\partial_{yy}u_t(y)}{(\partial_y u_t(y))^3} + b(y, u_t(y))\Big]. \tag{4.3}$$

By (4.1) and differentiating in t, we get $\partial_t F_t(u_t(y)) + \partial_x F_t(u_t(y))\partial_t u_t(y) = 0$. We deduce $\partial_t F_t(u_t(y)) = -S_t(u_t(y))\partial_t u_t(y)$. Substituting into (4.3) and cancelling by $S_t(u_t(y))$ yields the formal non-divergence equation (1.12), *i.e.*, $\partial_t u_t = f''(\frac{1}{\partial_y u_t})\frac{\partial_{yy}u_t}{(\partial_y u_t)^3} - b(y, u_t)$.

Recalling the definition (3.2) for $\tilde{f}$ and the computations (3.3), this final equation can be recast in the divergence form $\partial_t u_t = \partial_y(\tilde{f}'(\partial_y u_t)) - b(y, u_t)$; as one can see in equation (3.9), a a distinction between $\partial_y u$ and u' has to be made, and will be made in the next section, where we also rewrite this divergence-form equation as an L^2-gradient flow.

4.2. **Gradient flow formulation of the Lagrangian problem.** In §4.2.1, we define the functionals over the spaces $\mathcal{M}_{\text{loc}}((0,2))$ and $BV_{\text{loc}}((0,2))$ needed for our analysis. §4.2.2 links this with the underlying L^2-gradient flow structure of the equation obtained in §4.1.

4.2.1. *Defining the functionals.* For what follows, we recall the function $\tilde{f}$ in (3.2). This function is now used to define an entropy-like functional.

Definition 4.1 (Functionals Ent and $\mathscr{F}$)**.** For all $\nu \in \mathcal{M}_{\text{loc}}((0,2))$, with ν_a as per (2.1), we define

$$\text{Ent}[\nu] := \begin{cases} +\infty & \text{if } \nu^- \neq 0, \\ \displaystyle\int_0^2 \tilde{f}(\nu_a(y))\,\mathrm{d}y & \text{if } \nu^- = 0. \end{cases} \tag{4.4}$$

We then define the functional $\mathscr{F}[v] := \text{Ent}[\partial_y v]$ for all $v \in BV_{\text{loc}}((0,2))$.

Note that the functional Ent is a local lower semicontinuous functional on measures (in the sense of the theory of lower semicontinuous functionals in the space of measures, see [10, 11]), and the fact that the singular part of ν does not appear in the functional is due to the condition $\lim_{s\to\infty} \tilde{f}(s)/s = \lim_{s\to\infty} f(1/s) = f(0) = 0$ (see, *e.g.*, [49, Chapter 7]).

Next, we introduce the functional $\mathscr{E}$ corresponding to E in (1.4).

Definition 4.2 (Functional $\mathscr{E}$)**.** For all $v \in X$ and $b \in L^\infty([0,2]\times\mathbb{R})$, we define

$$\mathscr{E}[v] := \mathscr{F}[v] + \int_0^2 B(y, v(y))\,\mathrm{d}y, \tag{4.5}$$

where $B(y,s) := \int_0^s b(y,s')\,\mathrm{d}s'$ is a primitive for b in the second variable.

Remark 4.3 (Control of B). Note that the assumption $b \in L^\infty([0,2]\times\mathbb{R})$—which follows from (1.10) when $\partial_x V, \partial_x W \in L^\infty(\mathbb{R})$—implies

$$|B(y,s_1) - B(y,s_2)| \le \|b\|_{L^\infty}|s_1 - s_2| \qquad \forall s_1, s_2 \in \mathbb{R}. \tag{4.6}$$

4.2.2. *Formal L^2-gradient flow structure.* By performing the usual formal computation $0 = \frac{\mathrm{d}}{\mathrm{d}\delta}|_{\delta=0}\mathscr{E}[v+\delta\varphi]$ for φ a smooth test function (assuming formally that $\partial_y v+\delta\partial_y\varphi \geq 0$, *cf.* Lemma 5.2 and Proposition 5.3), we obtain that the first variation of $\mathscr{E}$ is

$$\mathscr{E}'[v] = b(y,v) - \partial_y(\tilde{f}'(v')), \tag{4.7}$$

with the notation v' of Definition 2.3. Note that in this Hilbert setting we prefer to use the notation $\mathscr{E}'$ rather than the notation with $\frac{\delta}{\delta\varrho}$ that we used in the space of measures, but the meaning is still the first variation of the functional.

Using (3.3), equation (1.12) is formally equivalent to

$$\partial_t u = -\mathscr{E}'[u], \tag{4.8}$$

which is (up to identifying singular parts of the derivative) precisely the divergence-form equation written at the end of §4.1, *cf.* (3.9). We shall therefore prove the existence of a solution of this Lagrangian reformulation (*cf.* Theorem 3.6) by means of a Minimising Movement Scheme with respect to the functional $\mathscr{E}$ (see §5–§6).

The next section contain the properties required on the functionals $\mathscr{E}$, $\mathscr{F}$, Ent for our later analysis; namely lower semicontinuity, convexity, and boundedness from below.

4.3. **Properties of the functional $\mathscr{E}$.** We begin with the following lemma concerned with the convexity of $\mathscr{F}$ and its boundedness from below.

Lemma 4.4 (Convexity of $\mathscr{F}$ and lower bound)**.** *The functional $\mathscr{F}$ is convex and there exist positive constants C_1, C_2 depending only on α in (3.7) such that*

$$\mathscr{F}[u] \geq -C_1\int_0^2 u'(y)^{1-\alpha}\,\mathrm{d}y \geq -C_2\Big(1+\|u\|_{L^2([0,2])}\Big) \qquad \forall u\in X. \tag{4.9}$$

Moreover, $\tilde{f}(u')_- \in L^1([0,2])$ for all $u\in X$, where $\tilde{f}(u')_-$ is the negative part of $\tilde{f}(u')$.

In order to prove the above, we first need two technical lemmas (Lemmas 4.5 and 4.6).

Lemma 4.5 (BV_{loc} and L^∞_{loc} estimate for L^2 monotonic functions)**.** *Let $v\in X$ be such that $\partial_y v\geq 0$ in $\mathcal{M}_{\mathrm{loc}}((0,2))$. Then, for all $[y_0,y_1]\subseteq[0,2]$ we have the pointwise bounds*

$$-\frac{\|v\|_{L^2([0,y_0])}}{\sqrt{y_0}} \leq v(y) \leq \frac{\|v\|_{L^2([y_1,2])}}{\sqrt{2-y_1}} \qquad \mathscr{L}\text{-a.e. } y\in(y_0,y_1). \tag{4.10}$$

Consequently, given any compact subset $[y_0,y_1]\subset(0,2)$, we have the BV_{loc}-estimate:

$$\|v\|_{TV([y_0,y_1])} \leq \|v\|_{L^2([0,2])}\Big(\frac{1}{\sqrt{2-y_1}}+\frac{1}{\sqrt{y_0}}\Big). \tag{4.11}$$

We remark that the assumption $\partial_y v\geq 0$ in $\mathcal{M}_{\mathrm{loc}}((0,2))$ above implies immediately that the derivative is a locally finite measure, and hence $v\in BV_{\mathrm{loc}}$; the real content of the result are the estimates (4.10)–(4.11). This result is needed to obtain uniform estimates on the minimising sequences in the proof of Proposition 5.1 (when applying the direct method), since the functional $\mathscr{F}$ is *a priori* not coercive.

Proof. We begin with the upper bound. Note that v is an increasing function on $(0,2)$. We then distinguish two cases: if $v(y_1)<0$ the desired inequality is automatically satisfied since

$v(y) \le v(y_1) < 0$ for all $y < y_1$. If instead $v(y_1) \ge 0$, we use the fact that for $\mathscr{L}$-a.e. $y \in (y_1, 2)$ we have $v(y) \ge v(y_1)$. Squaring and integrating in y over $(y_1, 2)$ yields

$$(2-y_1)v(y_1)^2 \le \int_{y_1}^2 v(y)^2 \, \mathrm{d}y = \|v\|^2_{L^2([y_1,2])}.$$

This provides the upper bound

$$v(y_1) \le \frac{\|v\|_{L^2([y_1,2])}}{\sqrt{2-y_1}}$$

and, again by monotonicity, the claim. The proof of the lower bound is similar. For the total variation estimate, we just need to use the equality $||v||_{TV([y_0,y_1])} = v(y_1) - v(y_0)$, which is satisfied whenever v is non-decreasing; recall that when evaluating pointwise values of $v \in BV_{\mathrm{loc}}((0,2))$ we identify it with its precise representative given in Remark 2.4. ☐

The next result provides a useful lower bound; its proof relies on Fenchel's inequality.

Lemma 4.6. *Let α be as in (3.7), $0 < \beta < (1-\alpha)^{-1} - \frac{3}{2}$, and set*

$$h(y) := \big(y(2-y)\big)^{\frac{1}{2}+\beta}, \tag{4.12}$$

which satisfies $\int_0^2 h(y)^{-\frac{1-\alpha}{\alpha}} \, \mathrm{d}y < \infty$ and $h \in H^1_0([0,2])$. Then, there exists a positive constant C depending only on α, β, such that for all $[y_0, y_1] \subset (0,2)$ it holds for all $u \in X$

$$\int_{y_0}^{y_1} \tilde{f}(u')_- \, \mathrm{d}y \le C\Big(\|\partial_y h\|_{L^2([y_0,y_1])} \|u\|_{L^2([y_0,y_1])} + \int_{y_0}^{y_1} h(y)^{-\frac{1-\alpha}{\alpha}} \, \mathrm{d}y + \big[h(y)u(y)\big]_{y_0}^{y_1} \Big). \tag{4.13}$$

Proof. Using the bound (3.7) and $\tilde{f}(s) = sf(1/s)$, there exists $C > 0$ such that

$$\tilde{f}(u'(y))_- \le Cu'(y)^{1-\alpha} \qquad \mathscr{L}\text{-a.e. } (0,2). \tag{4.14}$$

We begin by showing that, for all $[y_0, y_1] \subset (0,2)$,

$$-\frac{1}{1-\alpha} \int_{y_0}^{y_1} u'(y)^{1-\alpha} \, \mathrm{d}y + \frac{\alpha}{1-\alpha} \int_{y_0}^{y_1} h(y)^{-\frac{1-\alpha}{\alpha}} \, \mathrm{d}y \ge -\int_{y_0}^{y_1} u'(y)h(y) \, \mathrm{d}y. \tag{4.15}$$

To see this, recall from Fenchel's inequality that, for all $x \in (0, \infty)$ and $z \in (-\infty, 0)$, it holds

$$-\frac{1}{1-\alpha} x^{1-\alpha} + \frac{\alpha}{1-\alpha}(-z)^{-\frac{1-\alpha}{\alpha}} \ge xz,$$

where we recognise $\frac{\alpha}{1-\alpha}(-z)^{-\frac{1-\alpha}{\alpha}} = \sup\{xz + \frac{1}{1-\alpha}x^{1-\alpha} : x \in (0,\infty)\}$ as the convex conjugate of the function $x \mapsto -\frac{1}{1-\alpha}x^{1-\alpha}$; the inequality (4.15) is then obtained directly from the above by setting $z = -h(y)$ for $h(y) \ge 0$ and $x = u'(y)$, and integrating over the interval $[y_0, y_1]$. Then, since $h \ge 0$ and $\partial_y u \ge 0$ in $\mathcal{M}_{\mathrm{loc}}((0,2))$, the final term in (4.15) is estimated as

$$0 \le \int_{y_0}^{y_1} u'(y)h(y) \, \mathrm{d}y \le \int_{y_0}^{y_1} h(y) \, \partial_y u(y) = -\int_{y_0}^{y_1} u(y)\partial_y h(y) \, \mathrm{d}y + \big[h(y_1)u(y_1) - h(y_0)u(y_0)\big],$$

whence, by returning to (4.15), it holds

$$\begin{aligned} -\frac{1}{1-\alpha} \int_{y_0}^{y_1} u'(y)^{1-\alpha} \, \mathrm{d}y &+ \frac{\alpha}{1-\alpha} \int_{y_0}^{y_1} h(y)^{-\frac{1-\alpha}{\alpha}} \, \mathrm{d}y \\ &\ge -\|u\|_{L^2([y_0,y_1])} \|\partial_y h\|_{L^2([y_0,y_1])} + \big[h(y_0)u(y_0) - h(y_1)u(y_1)\big]. \end{aligned} \tag{4.16}$$

Note that we have $0 \le \int_0^2 h(y)^{-\frac{1-\alpha}{\alpha}}\,\mathrm{d}y \le C\int_0^1 y^{-(\frac{1}{2}+\beta)\frac{1-\alpha}{\alpha}}\,\mathrm{d}y < \infty$, as a consequence of the condition on β, which implies $-(\frac{1}{2}+\beta)\frac{1-\alpha}{\alpha} > -1$. Moreover, we have $\partial_y h \in L^2([0,2])$ since $|\partial_y h|^2$ behaves as $(y(2-y))^{2\beta-1}$, which is integrable for $\beta > 0$. □

We are ready to prove Lemma 4.4.

Proof of Lemma 4.4. Since the convexity of the functional is straightforward, we are only concerned with the lower bound (4.9). Note that if the condition $\partial_y u \ge 0$ in $\mathcal{M}_{\mathrm{loc}}((0,2))$ is not satisfied, then $\mathscr{F}[u] = +\infty$ by Definition 4.1, and the inequality is trivially satisfied. Henceforth, we assume $\partial_y u \ge 0$ in $\mathcal{M}_{\mathrm{loc}}((0,2))$, and we use (4.13) with $y_0 = \varepsilon$ and $y_1 = 2-\varepsilon$ for an arbitrary $\varepsilon > 0$. We obtain

$$\int_\varepsilon^{2-\varepsilon} \tilde f'(u')_-\,\mathrm{d}y \le C\Big(\|u\|_{L^2([0,2])} + \int_\varepsilon^{2-\varepsilon} h(y)^{-\frac{1-\alpha}{\alpha}}\,\mathrm{d}y + h(2-\varepsilon)u(2-\varepsilon) - h(\varepsilon)u(\varepsilon)\Big). \tag{4.17}$$

Since $\partial_y u \ge 0$ in $\mathcal{M}_{\mathrm{loc}}((0,2))$, the assumptions of Lemma 4.5 hold, thus $|u(y)| \le C(\frac{1}{\sqrt{y}} + \frac{1}{\sqrt{2-y}})$ for some $C > 0$ depending only on $\|u\|_{L^2([0,2])}$. This implies $h(\varepsilon)u(\varepsilon) \to 0$ as $\varepsilon \to 0$ and, then

$$\int_0^2 \tilde f'(u')_-\,\mathrm{d}y \le C\Big(1 + \|u\|_{L^2([0,2])}\Big) < +\infty. \tag{4.18}$$

This allows to prove at the same time (4.9) and $\tilde f(u')_- \in L^1([0,2])$. □

Lemma 4.7 (Properties of $\mathscr{E}$)**.** *The functional $\mathscr{E} : X \to \mathbb{R}$ of Definition 4.2 is proper and*

$$\mathscr{E}[v] \ge -C\Big(1 + \|v\|_{L^2([0,2])}\Big) \qquad \text{for all } v \in X, \tag{4.19}$$

where the positive constant C depends only on $\|b\|_{L^\infty}$ and the exponent α in (3.7). Furthermore, if $\{v_n\}_n$ is a sequence converging to v in X, then $\mathscr{E}[v] \le \liminf_{n\to\infty} \mathscr{E}[v_n]$ and

$$\int_{y_0}^{y_1} \tilde f(v')\,\mathrm{d}y \le \liminf_{n\to\infty} \int_{y_0}^{y_1} \tilde f(v_n')\,\mathrm{d}y \qquad \forall [y_0, y_1] \subseteq [0,2]. \tag{4.20}$$

Proof. 1. *Estimate and proper functional.* The estimate (4.19) follows immediately from the lower bound (4.9) and the estimate (4.6) on the drift term along with an application of Hölder's inequality. It is clear that $\mathscr{E}$ is proper since, for instance, the linear function $v_*(y) = y$ for all $y \in [0,2]$ belongs to the space X, and it holds

$$\mathscr{E}[v_*] = \int_0^2 f(1)\,\mathrm{d}y + \int_0^2 B(y,y)\,\mathrm{d}y \le 2\big(f(1) + \|b\|_{L^\infty}\big) < \infty.$$

2. *Lower semicontinuity.* Note that B is continuous in its second argument and has at most linear growth. From the strong convergence $v_n \to v$ in $L^1([0,2])$ we deduce

$$\int_0^2 B(y,v)\,\mathrm{d}y = \lim_{n\to\infty} \int_0^2 B(y,v_n)\,\mathrm{d}y. \tag{4.21}$$

Denoting $v', v_n' \in L^1_{\mathrm{loc}}((0,2))$ the absolutely continuous parts of the derivatives as in (2.2), we obtain the lower semicontinuity:

$$\int_{y_0}^{y_1} \tilde f(v')\,\mathrm{d}y \le \liminf_{n\to\infty} \int_{y_0}^{y_1} \tilde f(v_n')\,\mathrm{d}y, \tag{4.22}$$

on any compact subset $[y_0, y_1] \subset (0,2)$ since the BV_{loc} bound on v_n implies weak-* convergence of $\partial_y v_n$ to $\partial_y v$ on $[y_0, y_1]$ and the functional Ent is lower semicontinuous for this convergence.

We have therefore verified (4.20) for all $[y_0, y_1] \subset (0,2)$, and it remains to verify it for $[y_0, y_1] = [0,2]$. Using (4.13) with lower endpoint 0 and upper endpoint $y_0 \in (0,1)$, noting that $v(0)h(0) = 0$ with h as prescribed by (4.12), we get

$$\int_0^{y_0} \tilde{f}(v_n')\,\mathrm{d}y \geq -C\bigg(\|\partial_y h\|_{L^2([0,y_0])} + \int_0^{y_0} h(y)^{-\frac{1-\alpha}{\alpha}}\,\mathrm{d}y + \frac{1}{\sqrt{y_0}} h(y_0)\bigg), \tag{4.23}$$

where we also used that $v_n(y_0) \geq -\frac{\sup_n \|v_n\|_{L^2([0,2])}}{\sqrt{y_0}}$ for $y_0 \in (0,1)$ from Lemma 4.5. Analogously, for $1 < y_1 < 2$, since $u(2)h(2) = 0$, we have

$$\int_{y_1}^{2} \tilde{f}(v_n')\,\mathrm{d}y \geq -C\bigg(\|\partial_y h\|_{L^2([y_1,2])} + \int_{y_1}^{2} h(y)^{-\frac{1-\alpha}{\alpha}}\,\mathrm{d}y + \frac{1}{\sqrt{2-y_1}} h(y_1)\bigg), \tag{4.24}$$

where we also used that $v_n(y_1) \leq \frac{\sup_n \|v_n\|_{L^2([0,2])}}{\sqrt{2-y_1}}$ for $y_1 \in (1,2)$ from Lemma 4.5. As such,

$$\begin{aligned}
\int_0^2 \tilde{f}(v_n')\,\mathrm{d}y &= \int_0^{y_0} \tilde{f}(v_n')\,\mathrm{d}y + \int_{y_0}^{y_1} \tilde{f}(v_n')\,\mathrm{d}y + \int_{y_1}^{2} \tilde{f}(v_n')\,\mathrm{d}y \\
&\geq \int_{y_0}^{y_1} \tilde{f}(v_n')\,\mathrm{d}y - C\bigg(\|\partial_y h\|_{L^2([0,y_0])} + \int_0^{y_0} h(y)^{-\frac{1-\alpha}{\alpha}}\,\mathrm{d}y + \frac{1}{\sqrt{y_0}} h(y_0)\bigg) \\
&\quad - C\bigg(\|\partial_y h\|_{L^2([y_1,2])} + \int_{y_1}^{2} h(y)^{-\frac{1-\alpha}{\alpha}}\,\mathrm{d}y + \frac{1}{\sqrt{2-y_1}} h(y_1)\bigg),
\end{aligned}$$

for all $0 < y_0 < 1 < y_1 < 2$. By taking the $\liminf_n$ on both sides and using (4.22), we get

$$\begin{aligned}
\liminf_{n\to\infty} \int_0^2 \tilde{f}(v_n')\,\mathrm{d}y \geq \int_{y_0}^{y_1} \tilde{f}(v')\,\mathrm{d}y - C\bigg(\|\partial_y h\|_{L^2([0,y_0])} + \int_0^{y_0} h(y)^{-\frac{1-\alpha}{\alpha}}\,\mathrm{d}y + \frac{1}{\sqrt{y_0}} h(y_0)\bigg) \\
- C\bigg(\|\partial_y h\|_{L^2([y_1,2])} + \int_{y_1}^{2} h(y)^{-\frac{1-\alpha}{\alpha}}\,\mathrm{d}y + \frac{1}{\sqrt{2-y_1}} h(y_1)\bigg).
\end{aligned} \tag{4.25}$$

We let $y_0 \to 0^+$ and $y_1 \to 2^-$ in (4.25). Recall from the final part of Lemma 4.4 that the negative part $\tilde{f}(v')_-$ is integrable, whence $\int_{y_0}^{y_1} \tilde{f}(v')\,\mathrm{d}y$ tends to $\int_0^2 \tilde{f}(v')\,\mathrm{d}y$ as $y_0 \to 0$ and $y_1 \to 1$; note that this integral may assume the value $+\infty$, but it is bounded from below by Lemma 4.4. Hence, using the integrability of h and $\partial_y h$ from Lemma 4.6, we get $\liminf_n \int_0^2 \tilde{f}(v_n')\,\mathrm{d}y \geq \int_0^2 \tilde{f}(v')\,\mathrm{d}y$. Using (4.21), the proof of (4.20) is complete. □

We now record the monotonicity properties of $\tilde{f}'$ which are key to passing to the limit in the time-step $\tau \to 0$ in §6 to prove existence to the Lagrangian reformulation.

Lemma 4.8 (Quantified strict convexity of $\tilde{f}$)**.** *Recall the function $\tilde{f}$ defined in (3.2). Then there exists a non-negative continuous function $\omega : [0,\infty)\times(0,\infty) \to [0,\infty]$ satisfying $\omega(s_1, s_2) = 0$ if and only if $s_1 = s_2$, and*

$$\tilde{f}(s_1) - \tilde{f}(s_2) - (s_1 - s_2)\tilde{f}'(s_2) \geq \omega(s_1, s_2) \qquad \forall s_1, s_2 \in (0,\infty). \tag{4.26}$$

Furthermore, for all fixed $s_1 \in (0,\infty)$, ω admits the limits

$$0 < \lim_{s_2\to 0} \omega(s_1, s_2) \leq \infty, \qquad 0 < \lim_{s_2\to\infty} \omega(s_1, s_2) \leq \infty. \tag{4.27}$$

Proof. Let $s_1, s_2 \in (0,\infty)$. We define the quantity ω by

$$\tilde{f}(s_1) - \tilde{f}(s_2) - (s_1 - s_2)\tilde{f}'(s_2) = \int_{s_2}^{s_1}\int_{s_2}^{r} \tilde{f}''(s)\,\mathrm{d}s\,\mathrm{d}r =: \omega(s_1, s_2). \tag{4.28}$$

This formula guarantees the continuity of ω on $[0,\infty)\times(0,\infty)$ using the integrability of $\tilde f''$ on all compact subsets of $(0,\infty)$, and shows that ω vanishes on the diagonal. Also, this expression clearly provides $\omega(s_1,s_2)>0$ for all $s_1\neq s_2$ because of the strict convexity of $\tilde f$, which implies that $\tilde f''$ does not vanish on any interval; hence there exists a point in the interval (s_2,r) where $\tilde f''$ is strictly positive, and by continuity of $\tilde f''$ a small neighbourhood on which this holds, giving rise to a strictly positive integral. Similarly, we obtain for the same reason $\lim_{s_2\to 0}\omega(s_1,s_2)>0$ and $\lim_{s_2\to\infty}\omega(s_1,s_2)>0$. □

5. Minimising Movement Scheme for the Lagrangian Problem

In order to show existence to (3.9), we proceed by means of an implicit Euler (Minimising Movement) scheme discretised with respect to the time variable. Our result is the following.

Proposition 5.1 (Implicit Euler time-stepping scheme)**.** *Let $u_0\in X$ be such that $\mathscr{F}[u_0]$ is finite, $T>0$ be arbitrary, $\tau>0$, and $N:=\lceil T/\tau\rceil$. For $k\in\{1,\dots,N\}$, define the functionals*

$$\mathscr{E}^k[v]:=\frac{1}{2\tau}\|v-v_{k-1}\|^2_{L^2([0,2])}+\mathscr{E}[v], \tag{5.1}$$

with $v_0:=u_0$, and $v_k\in\arg\min\left\{\mathscr{E}^k[v]:v\in X\right\}$. Then, the minimisers $\{v_k\}_{k=1}^N\subset X$ are well-defined, and satisfy $\partial_y v_k\geq 0$ in $\mathcal{M}_{\mathrm{loc}}((0,2))$ and $\int_0^2\tilde f(v_k')\,\mathrm{d}y<+\infty$.

Proof. We emphasise at the start of the proof that τ and k are fixed throughout. By the estimate (4.9), the functional $\mathscr{E}^k$ satisfies

$$\mathscr{E}^k[v]\geq\frac{1}{2\tau}\|v-v_{k-1}\|^2_{L^2([0,2])}-2C\big(1+\|v\|_{L^2([0,2])}\big).$$

It follows that $\mathscr{E}^k$ is bounded from below and coercive in terms of the L^2 norm.

We now argue by the direct method of the calculus of variations. By definition of the infimum, there exists a minimising sequence $\{v_n\}_{n\in\mathbb{N}}\subset X$:

$$\mathscr{E}^k[v_n]<+\infty,\qquad\lim_{n\to\infty}\mathscr{E}^k[v_n]=\inf\left\{\mathscr{E}^k[w]:w\in X\right\}=:M^k>-\infty.$$

The aforementioned coercivity implies that the sequence v_n is bounded in $L^2([0,2])$. Recall also $\partial_y v_n\geq 0$ for all n, and these two facts together imply (by (4.11)) that v_n is bounded in X, and hence compact for the convergence in X. The lower semicontinuity result of Lemma 4.7 and the weak lower semicontinuity of the L^2 norm imply the existence of a minimiser $v\in X$ for the functional $\mathscr{E}^k$, which is such that $\int_0^2\tilde f(v')\,\mathrm{d}y$ is finite and satisfies $(\partial_y v)_s\geq 0$ in the sense $\mathcal{M}_{\mathrm{loc}}((0,2))$ and $v'\geq 0$ $\mathscr{L}$-a.e. in [0,2]. We set $v_k=v$. □

In fact, a stronger lower bound is available than simply $\partial_y v_k\geq 0$ in $\mathcal{M}_{\mathrm{loc}}((0,2))$ for minimisers v_k of $\mathscr{E}^k$. This is encapsulated in the next lemma, and is fundamental in obtaining the Euler–Lagrange equation associated to $\mathscr{E}^k$, *cf.* Proposition 5.3.

Lemma 5.2 (Strictly positive lower bound on v_k')**.** *Assume the conditions of Proposition 5.1 hold, and let $\mathscr{E}^k$ be as per (5.1) and $v_k\in\arg\min\{\mathscr{E}^k[v]:v\in X\}$. Then, it holds*

$$\operatorname{ess\,inf}_{[0,2]}v_k'>0, \tag{5.2}$$

where we use the notation v_k' of Definition 2.3.

Proof. For ease of notation we omit the subscript k from the minimiser v_k of $\mathscr{E}^k$ in this proof. Suppose for contradiction that (5.2) does not hold, *i.e.*, we suppose

$$\operatorname{ess\,inf}_{[0,2]} v' = 0. \tag{5.3}$$

Our strategy is now to construct a competitor $\tilde{v} \in X$ such that $\mathscr{E}^k[\tilde{v}] < \mathscr{E}^k[v]$, which contradicts the minimality of v in the class X; we explain the underlying idea in the lines that follow. By (3.6), $\tilde{f}'(v')$ explodes as v' approaches the value zero and, by the contradiction hypothesis (5.3), v' takes arbitrarily small values on sets of positive measure. We therefore build a competitor $\tilde{v}$, modifiying some small values of v' into larger values, so that $\int_0^2 \tilde{f}(\tilde{v}')\,\mathrm{d}y$ is much less than $\int_0^2 \tilde{f}(v')\,\mathrm{d}y$ and such that the gain in this term overcomes the possible loss in the other terms of the functional $\mathscr{E}^k$.

1. *Construction of a competitor.* For all $\varepsilon > 0$, we define the competitor

$$\tilde{v}(y) := v(y) + \int_0^y (2\varepsilon - v'(z))\mathbb{1}_{\{0 \le v'(z) < \varepsilon\}}\,\mathrm{d}z \qquad \mathscr{L}\text{-a.e. } y \in [0,2]; \tag{5.4}$$

we omit the explicit dependence on ε to avoid heavy notation. Since the integrand is bounded by 2ε, the integral is well-defined.

Moreover, $\tilde{v} \in X$ for all choices of $\varepsilon > 0$ and, by differentiating (5.4) distributionally,

$$\partial_y \tilde{v} = \partial_y v + (2\varepsilon - v')\mathbb{1}_{\{0 \le v' < \varepsilon\}} \ge 0,$$

where the final inequality holds in the sense $\mathcal{M}_{\mathrm{loc}}((0,2))$, and follows directly from $\partial_y v \ge 0$. By taking the absolutely continous part of the measure above, which is a linear operation,

$$\tilde{v}' = v' + (2\varepsilon - v')\mathbb{1}_{\{0 < v' < \varepsilon\}} = v'\mathbb{1}_{\{v' \ge \varepsilon\}} + 2\varepsilon\mathbb{1}_{\{0 \le v' < \varepsilon\}} \qquad \mathscr{L}\text{-a.e. in } [0,2],$$

from which it follows from Definition 4.1 of the functional $\mathscr{F}$ that

$$\mathscr{F}[\tilde{v}] = \int_0^2 \tilde{f}(v')\mathbb{1}_{\{v' \ge \varepsilon\}}\,\mathrm{d}y + \tilde{f}(2\varepsilon)m(\varepsilon), \tag{5.5}$$

where we have used the shorthand notation $m(\varepsilon) := \mathscr{L}(\{0 \le v' < \varepsilon\})$. Note that the contradiction hypothesis (5.3) implies $m(\varepsilon) > 0$ for all $\varepsilon > 0$.

2. *Estimates on competitor.* We compute the functional value for the competitor $\tilde{v}$,

$$\begin{aligned}\mathscr{E}^k[\tilde{v}] - \mathscr{E}^k[v] =& \frac{1}{2\tau}\|\tilde{v} - v_{k-1}\|^2_{L^2([0,2])} - \frac{1}{2\tau}\|v - v_{k-1}\|^2_{L^2([0,2])} \\ &+ \int_0^2 \Big(B(y,\tilde{v}(y)) - B(y,v(y))\Big)\,\mathrm{d}y + \mathscr{F}[\tilde{v}] - \mathscr{F}[v].\end{aligned}$$

By expanding the L^2-inner product, denoted $\langle\cdot,\cdot\rangle$, and using the Lipschitz bound (4.6) for B,

$$\begin{aligned}&\Big|\mathscr{E}^k[\tilde{v}] - \mathscr{E}^k[v] - (\mathscr{F}[\tilde{v}] - \mathscr{F}[v])\Big| \\ &\qquad \le \frac{1}{2\tau}\Big|\|\tilde{v}\|^2_{L^2([0,2])} - \|v\|^2_{L^2([0,2])} - 2\langle \tilde{v} - v, v_{k-1}\rangle\Big| + \|b\|_{L^\infty}\|\tilde{v} - v\|_{L^1([0,2])}.\end{aligned}$$

Returning to (5.4), we note the pointwise estimate $|\tilde{v}(y) - v(y)| \le 2\varepsilon m(\varepsilon)$ for $\mathscr{L}$-a.e. y, and

$$\|\tilde{v}\|^2_{L^2} = \|v\|^2_{L^2} + 2\langle v, \tilde{v} - v\rangle + \|\tilde{v} - v\|^2_{L^2([0,2])},$$

from which we get $|\|\tilde v\|^2_{L^2([0,2])} - \|v\|^2_{L^2([0,2])}| \le 4\sqrt{2}\|v\|_{L^2([0,2])}\varepsilon m(\varepsilon) + 8\varepsilon^2 m(\varepsilon)^2$. In turn,

$$\left|\mathscr{E}^k[\tilde v] - \mathscr{E}^k[v] - (\mathscr{F}[\tilde v] - \mathscr{F}[v])\right| \le C\varepsilon m(\varepsilon)\big(1+\varepsilon m(\varepsilon)\big),$$

where the positive constant C depends on $\|v\|_{L^2}, \|v_{k-1}\|_{L^2}, \|b\|_{L^\infty}$, τ, but is independent of ε. Meanwhile, by (5.5), we also have $\mathscr{F}[\tilde v] - \mathscr{F}[v] = -\int_0^2 \tilde f(v')\mathbb{1}_{\{0\le v'<\varepsilon\}}\,\mathrm{d}y + \tilde f(2\varepsilon)m(\varepsilon)$ so that the previous inequality implies the upper bound

$$\mathscr{E}^k[\tilde v] \le \mathscr{E}^k[v] - \int_0^2 \tilde f(v')\mathbb{1}_{\{0\le v'<\varepsilon\}}\,\mathrm{d}y + \tilde f(2\varepsilon)m(\varepsilon) + C\varepsilon m(\varepsilon)\big(1+\varepsilon m(\varepsilon)\big); \tag{5.6}$$

the above holds for all choices of $\varepsilon > 0$, with C independent of ε.

3. *Competitor beats minimiser.* Our aim is to show that the contribution from the negative integral term in (5.6) dominates, so that we have the strict inequality $\mathscr{E}^k[\tilde v] < \mathscr{E}^k[v]$.

As $\tilde f$ is strictly decreasing (Remark 3.2), it holds $\int_0^2 \tilde f(v')\mathbb{1}_{\{0\le v'<\varepsilon\}}\,\mathrm{d}y \ge \tilde f(\varepsilon)m(\varepsilon)$, whence

$$\mathscr{E}^k[\tilde v] \le \mathscr{E}^k[v] + m(\varepsilon)\Big[-\tilde f(\varepsilon) + \tilde f(2\varepsilon) + C\varepsilon\big(1+\varepsilon m(\varepsilon)\big)\Big] \qquad \text{for all } \varepsilon > 0, \tag{5.7}$$

and because of the hypothesis (5.3), it suffices to show that the term in the square brackets is strictly negative for some choice of ε; indeed, recall that the quantity $m(\varepsilon)$ is strictly positive for all choices of $\varepsilon > 0$. This quantity is also bounded above by $m(\varepsilon) \le \mathscr{L}([0,2]) = 2$. It follows that, for all $\varepsilon \in (0,1)$ we have $1+\varepsilon m(\varepsilon) \le 3$, and (5.7) yields

$$\mathscr{E}^k[\tilde v] < \mathscr{E}^k[v] + \varepsilon m(\varepsilon)\Big[\frac{\tilde f(2\varepsilon) - \tilde f(\varepsilon)}{\varepsilon} + 3C\Big] \qquad \forall \varepsilon \in (0,1). \tag{5.8}$$

Meanwhile, the convexity of $\tilde f$ gives $\frac{\tilde f(2\varepsilon)-\tilde f(\varepsilon)}{\varepsilon} \le \tilde f'(2\varepsilon) \to -\infty$ as $\varepsilon \to 0^+$ by (3.6). Thus, by choosing ε sufficiently small, we contradict the minimality of v in X. □

The strictly positive lower bound on v_k' from the previous lemma allows us to compute the Euler–Lagrange equation satisfied by the minimiser v_k; this is the content of the next result, which relies crucially on the aforementioned strictly positive lower bound on the derivative.

Proposition 5.3 (Discrete-time Euler–Lagrange)**.** *The minimisers $\{v_k\}_{k=1}^N \subset X$ of Proposition 5.1 satisfy* $\operatorname{ess\,inf}_{[0,2]} v_k' > 0$, $(\partial_y v_k)_s \ge 0$ *in* $\mathcal{M}_{\mathrm{loc}}((0,2))$, $\tilde f'(v_k') \in H^1([0,2])$, *and*

$$\frac{v_k - v_{k-1}}{\tau} = \partial_y(\tilde f'(v_k')) - b(y, v_k), \tag{5.9}$$

with v_k' as per (2.2)*. For some positive universal constant C we also have*

$$\|\tilde f'(v_k')\|_{H^1([0,2])} \le C\Big(\Big\|\frac{v_k - v_{k-1}}{\tau}\Big\|_{L^2([0,2])} + \|b\|_{L^\infty}\Big). \tag{5.10}$$

Proof. The proof is divided into three steps.

1. *Euler–Lagrange equation in duality with $C^1([0,2])$.* Let $\varphi \in C^1([0,2])$. Recall from Lemma 5.2 that it holds $\ell := \operatorname{ess\,inf}_{[0,2]} v_k' > 0$. In turn, for all $\delta \in \mathbb{R}$ satisfying

$$|\delta| < \delta_* := \frac{\ell}{(1+\|\partial_y\varphi\|_{L^\infty})}, \tag{5.11}$$

it holds, using the notation (2.2), $\partial_y(v_k + \delta\varphi) = \partial_y v_k + \delta\partial_y\varphi \ge \ell_\varphi \cdot \mathscr{L} > 0$, where $\ell_\varphi := \ell(1 - \frac{\|\partial_y\varphi\|_{L^\infty}}{1+\|\partial_y\varphi\|_{L^\infty}})$. Thus, for all $\varphi \in C^1([0,2])$ and δ satisfying (5.11), it holds

$$\mathscr{E}^k[v_k + \delta\varphi] = \frac{1}{2\tau}\|v_k + \delta\varphi - v_{k-1}\|^2_{L^2([0,2])} + \int_0^2 \tilde{f}(v_k' + \delta\partial_y\varphi)\,\mathrm{d}y + \int_0^2 B(y, v_k + \delta\varphi)\,\mathrm{d}y.$$

The minimality of v_k in the class $X \ni v_k + \delta\varphi$ implies $\mathscr{E}^k[v_k] \le \mathscr{E}^k[v_k + \delta\varphi]$, whence

$$\begin{aligned} 0 \le \int_0^2 \Big(\tilde{f}(v_k' + \delta\partial_y\varphi) - \tilde{f}(v_k')\Big)\,\mathrm{d}y + \int_0^2 \Big(B(y, v_k + \delta\varphi) - B(y, v_k)\Big)\,\mathrm{d}y \\ + \frac{1}{2\tau}\int_0^2 \Big(|v_k + \delta\varphi - v_{k-1}|^2 - |v_k - v_{k-1}|^2\Big)\,\mathrm{d}y. \end{aligned} \tag{5.12}$$

Choosing $\delta \in (0, \delta_*)$ and dividing by δ, and then letting $\delta \to 0^+$, we get

$$0 \le \lim_{\delta\to 0^+} \int_0^2 \frac{\tilde{f}(v_k' + \delta\partial_y\varphi) - \tilde{f}(v_k')}{\delta}\,\mathrm{d}y + \int_0^2 \Big(b(y, v_k) + \frac{v_k - v_{k-1}}{\tau}\Big)\varphi\,\mathrm{d}y. \tag{5.13}$$

To evaluate the first term on the right-hand side, we use

$$\Big|\frac{\tilde{f}(v_k' + \delta\partial_y\varphi) - \tilde{f}(v_k')}{\delta}\Big| \le \sup_{s\ge\ell_\varphi} |\tilde{f}'(s)| \cdot \|\partial_y\varphi\|_{L^\infty} = -\tilde{f}'(\ell_\varphi)\|\partial_y\varphi\|_{L^\infty} \qquad \mathscr{L}\text{-a.e.},$$

where we used that $\tilde{f}' \le 0$ and $\tilde{f}'$ is increasing (*cf.* Remark 3.2) to obtain the final inequality. In turn, by returning to (5.13) and applying the Dominated Convergence Theorem, we get

$$0 \le \int_0^2 \tilde{f}'(v_k')\partial_y\varphi\,\mathrm{d}y + \int_0^2 \Big(b(y, v_k) + \frac{v_k - v_{k-1}}{\tau}\Big)\varphi\,\mathrm{d}y \qquad \forall\varphi \in C^1([0,2]).$$

Replacing φ with $-\varphi$ we finally obtain equality for all $\varphi \in C^1([0,2])$, which proves (5.9) in duality with any $\varphi \in C^1([0,2])$.

2. *L^2-estimate on $\partial_y(\tilde{f}'(v_k'))$.* Directly from (5.9), we obtain in the sense of distributions $\partial_y(\tilde{f}'(v_k')) = \frac{v_k - v_{k-1}}{\tau} - b$, which shows that $\partial_y(\tilde{f}'(v_k'))$ belongs to $L^2([0,2])$ with

$$\|\partial_y(\tilde{f}'(v_k'))\|_{L^2([0,2])} \le \Big(\Big\|\frac{v_k - v_{k-1}}{\tau}\Big\|_{L^2([0,2])} + \sqrt{2}\|b\|_{L^\infty}\Big). \tag{5.14}$$

3. *L^1-estimate on $\tilde{f}'(v_k')$.* In order to obtain a full bound in H^1 on $\tilde{f}'(v_k')$ we will also derive an L^1-estimate on it. To this end, we test the equation with $\varphi(y) = y$, for which $\partial_y\varphi = 1$ (note $\varphi \in C^1([0,2])$), and obtain (using $\tilde{f}' \le 0$)

$$\|\tilde{f}'(v_k')\|_{L^1} = \int_0^2 -\tilde{f}'(v_k')\,\mathrm{d}y = \int_0^2 y\Big(\frac{v_k - v_{k-1}}{\tau} + b(y, v_k)\Big)\,\mathrm{d}y.$$

Then, by the Poincaré–Wirtinger inequality, we also deduce a bound on the whole H^1 norm of $\tilde{f}'(v_k')$. Now that we know that we have $\tilde{f}'(v_k') \in H^1$, the equation holds in weak form in duality with less smooth functions φ, and also in strong form as an aquality a.e. with the (L^2) distributional derivative of $\tilde{f}'(v_k')$. □

Remark 5.4 (Improved weak formulation). Choosing to test (5.9) with $\varphi \in C_c^1((0,2))$, using the H^1-bound on $\tilde{f}'(v_k')$ obtained in (5.10), and integrating by parts, we deduce

$$\int_0^2 \varphi\Big(\frac{v_k - v_{k-1}}{\tau} + b(y, v_k) - \partial_y(\tilde{f}'(v_k'))\Big)\,\mathrm{d}y = 0. \tag{5.15}$$

Moreover, due to the density of $C_c^1((0,2))$ in $L^2([0,2])$, the weak formulation (5.15) holds for all $\varphi \in L^2([0,2])$, and in fact the Fundamental Lemma of the calculus of variations implies

$$\frac{v_k - v_{k-1}}{\tau} = \partial_y(\tilde{f}'(v_k')) - b(y, v_k) \qquad \mathscr{L}\text{-a.e. in } [0,2]. \tag{5.16}$$

Corollary 5.5 ($\tilde{f}'(v_k')$ vanishes at the endpoints)**.** *It holds* $\tilde{f}'(v_k') \in H_0^1([0,2])$, *i.e.*

$$\tilde{f}'(v_k')|_{y=0} = \tilde{f}'(v_k')|_{y=2} = 0. \tag{5.17}$$

Proof. Recall from Proposition 5.3 that the Euler–Lagrange equation (5.9) holds in duality with $H^1([0,2])$. In particular, for all $r > 0$, we define the truncation

$$\varphi_r(y) = \min\{\frac{y}{r}, 1\} - 1, \qquad y \in [0,2].$$

Note that $\partial_y \varphi_r = \frac{1}{r}\mathbb{1}_{\{y<r\}}$ in the sense of distributions, whence $\varphi_r \in H^1([0,2])$ for all $r > 0$. Moreover we have $|\varphi_r| \le 1$. By inserting φ_r into the weak formulation of (5.9), we get

$$\int_0^r \Big(\frac{y}{r} - 1\Big)\Big(\frac{v_k - v_{k-1}}{\tau} + b(y, v_k)\Big)\,\mathrm{d}y = -\frac{1}{r}\int_0^r \tilde{f}'(v_k')\,\mathrm{d}y. \tag{5.18}$$

The left-hand side in (5.18) is controlled as follows:

$$\Big|\int_0^r \underbrace{\Big(\frac{y}{r} - 1\Big)}_{\in[-1,0]}\Big(\frac{v_k - v_{k-1}}{\tau} + b(y, v_k)\Big)\,\mathrm{d}y\Big| \le \int_0^r \Big|\frac{v_k - v_{k-1}}{\tau} + b(y, v_k)\Big|\,\mathrm{d}y \le C\sqrt{r}.$$

Returning to (5.18) and letting $r \to 0$ (keeping k, τ fixed), an application of the Dominated Convergence Theorem and Lebesgue's Differentiation Theorem imply

$$\tilde{f}'(v_k')|_{y=0} = \lim_{r\to 0}\frac{1}{r}\int_0^r \tilde{f}'(v_k')\,\mathrm{d}y = 0.$$

An analogous procedure yields $\tilde{f}'(v_k')|_{y=2} = 0$ by considering instead $\varphi_r(2-y)$. □

We conclude this section with the proof of a particular optimality condition which is needed in the sequel. Morally speaking, this says that $(\partial_y v_k)_s$ and $\tilde{f}'(v_k')$ have disjoint supports.

Lemma 5.6 (Optimality condition on the singular part of the derivative)**.** *Let* $v_k \in X$ *be a minimiser of* $\mathscr{E}^k$ *provided by Proposition 5.1. Then, we have* $\tilde{f}'(v_k') = 0 \quad (\partial_y v_k)_s$*-a.e.*

Proof. For clarity of presentation, in this proof, we omit the k subscript in our notation for v_k the minimiser of $\mathscr{E}^k$.

1. *Defining w such that $\partial_y w = (\partial_y v)_s$.* Fix any Lebesgue point $y_0 \in (0,2)$ of v, and define

$$w := v - \int_{y_0}^y v'(s)\,\mathrm{d}s \in X. \tag{5.19}$$

It follows directly directly from the above that

$$\partial_y w = \partial_y v - v' \cdot \mathscr{L} = (\partial_y v)_s \ge 0, \qquad w' = 0 \quad \mathscr{L}\text{-a.e.}, \tag{5.20}$$

where the inequalities hold in $\mathcal{M}_{\mathrm{loc}}((0,2))$. In the lines that follow, we verify $w \in X$.

For $\mathscr{L}$-a.e. $y \in (0,2)$, the non-negativity of the measure $(\partial_y v)_s$ in $\mathcal{M}_{\mathrm{loc}}((0,2))$ and $v' \ge 0$ $\mathscr{L}$-a.e. (*cf.* Proposition 5.3) imply

$$0 \le \int_{y_0}^y v'(s)\,\mathrm{d}s \le \int_{y_0}^y v'(s)\,\mathrm{d}s + \int_{y_0}^y (\partial_y v)_s(s) = \int_{y_0}^y \partial_y v(s) = v(y) - v(y_0),$$

where we used the identification with the precise representative from Remark 2.4 and the formula (2.3). Hence, returning to (5.19), we have $0 \leq |w(y)| \leq 2|v(y)| + |v(y_0)|$ $\mathscr{L}$-a.e. $y \in (0,2)$. As such, by integrating the previous inequality in y we get $\int_0^2 |w(y)|^2 \,\mathrm{d}y \leq 4|v(y_0)|^2 + 8\int_0^2 |v(y)|^2 \,\mathrm{d}y$. Hence $w \in L^2([0,2])$, and it follows that $w \in X$, as required.

2. *Variations along w.* Our next objective is to show

$$\int_0^2 \Big(b(y, v_k) + \frac{v_k - v_{k-1}}{\tau}\Big) w \,\mathrm{d}y = 0. \tag{5.21}$$

We do this by computing variations along w. Using (5.20), for all $|\delta| < 1$, we have

$$\partial_y(v + \delta w) \geq 0 \,\text{in } \mathcal{M}_{\mathrm{loc}}((0,2)), \qquad (v + \delta w)' = v'. \tag{5.22}$$

The minimality of v in $X \ni (v+\delta w)$ implies $\mathscr{E}^k[v] \leq \mathscr{E}^k[v+\delta w]$, from which we deduce (5.21).

3. *Conclusion.* The strong form (5.16) of the Euler–Lagrange equation allows to test with w without needing to approximate by mollification or cut-off. Multiplying (5.16) with w and integrating, we get

$$0 = -\int_0^2 w\Big(\frac{v_k - v_{k-1}}{\tau} + b(y, v_k)\Big) \,\mathrm{d}y = \int_0^2 w\partial_y(-\tilde{f}'(v_k')) \,\mathrm{d}y. \tag{5.23}$$

We would like to deduce (by integration by parts) from (5.23) that we have $\int \tilde{f}'(v_k')\partial_y w = 0$, which would give the claim. Yet, the difficulty comes from the fact that $\partial_y w$ is not necessarily a finite measure (the boundary term should formally disappear because $\tilde{f}'(v_k') \in H_0^1$). Actually, using $\partial_y w \geq 0$ and $\tilde{f}'(v_k') \leq 0$, it would be enough to obtain $\int \tilde{f}'(v_k')\partial_y w \geq 0$

The conclusion is obtained using the following facts: w is an L^2 function, and $\partial_y w$ is a positive measure; $\tilde{f}'(v_k')$ is a non-positive H_0^1 function, which can be approximated strongly in H_0^1 (and hence uniformly) by a sequence of functions $\varphi_m \in C_c^1$ with $\varphi_m \leq 0$. We then write

$$\int_0^2 \tilde{f}'(v_k') \,\partial_y w \geq \lim_{m\to\infty} \int_0^2 \varphi_m \,\partial_y w = -\lim_{m\to\infty} \int_0^2 w\partial_y\varphi_m \,\mathrm{d}y = -\int_0^2 w\partial_y(\tilde{f}'(v_k')) \,\mathrm{d}y = 0,$$

where the first inequality above comes from Fatou's lemma, the first equality is an integration by part on a compact subset of $(0,2)$, the next one comes from $w \in L^2$ and the strong L^2 convergence of $\partial_y\varphi_m$ to $\partial_y(\tilde{f}'(v_k'))$ and the last is (5.23). □

6. Solution of the Lagrangian Problem

In this section, we prove the main well-posedness result for the Lagrangian problem (*cf.* Theorem 3.6). The main result of this section is the following.

Proposition 6.1. *Let $u_0 \in X$ be such that $\mathscr{F}[u_0]$ is finite, and $T > 0$. Then, there exists $u \in L^\infty(0,T;X) \cap H^1([0,T];L^2([0,2]))$ satisfying $\partial_y u(t,\cdot) \geq 0$ in $\mathcal{M}_{\mathrm{loc}}((0,2))$ and $0 \leq -\tilde{f}'(u') \in L^2(0,T;H_0^1([0,2]))$, and the equation*

$$\begin{cases} \partial_t u = \partial_y(\tilde{f}'(u')) - b(y,u), \\ u|_{t=0} = u_0, \end{cases} \tag{6.1}$$

where the initial data is achieved in the sense $\lim_{t\to 0^+} \|u(t,\cdot) - u_0\|_{L^2([0,2])} = 0$, and with u' as per (2.2). Moreover, for $\mathscr{L}$-a.e. $t \in [0,T]$, there exists c_t such that

$$\operatorname{ess\,inf}_{[0,2]} u'(t,\cdot) \geq c_t > 0. \tag{6.2}$$

We will prove Proposition 6.1 via a sequence of lemmas. Firstly, we prove uniform estimates for the sequence of piecewise constant time-interpolations $\{u_\tau\}_\tau$. The solution of (6.1) provided by Proposition 6.1 will be obtained as the limit as $\tau \to 0^+$ of the sequence $\{u_\tau\}_\tau$.

Lemma 6.2 (Properties of interpolations). *Let $u_0 \in X$ be such that $\mathscr{F}[u_0]$ is finite, $0 < \tau < 1$, $T > 0$, and $N := \lceil T/\tau \rceil$. For $k \in \{1, \dots, N\}$, let the functionals $\mathscr{E}^k$ and their minimisers v_k be as in Proposition 5.1. For all $t \in [0, T]$, define the piecewise constant interpolation*

$$u_\tau(t, \cdot) := v_k, \qquad \text{for all } t \in ((k-1)\tau, k\tau], \quad k \in \{0, 1, \dots, N\}. \tag{6.3}$$

Then, $\partial_y u_\tau \geq 0$, $u_\tau(0, \cdot) = u_0$, and for some positive $C = C(\|u_0\|_{L^2}, \mathscr{E}[u_0], \|b\|_{L^\infty}, T)$ independent of τ, it holds the equicontinuity estimate:

$$\|u_\tau(t, \cdot) - u_\tau(s, \cdot)\|_{L^2([0,2])} \leq C\big(\sqrt{|t-s|} + \sqrt{\tau}\big) \qquad \text{a.e. } t, s \in [0, T], \tag{6.4}$$

(in particular, the functions $u_\tau(t, \cdot)$ are bounded in $L^2([0, 2])$) and the uniform estimates:

$$\int_0^2 \tilde{f}(u'_\tau)\,\mathrm{d}y \leq C,\ \|u_\tau\|_{L^\infty(0,T;L^2([0,2])} \leq C,\ \|\tilde{f}'(u'_\tau)\|_{L^2(0,T;H^1_0([0,2])} \leq C, \tag{6.5}$$

with u'_τ as per (2.2). Moreover, for all compact subsets $[y_0, y_1] \subset (0, 2)$, we have

$$\|u_\tau(t, \cdot)\|_{TV([y_0,y_1])} \leq C_{y_0,y_1}, \tag{6.6}$$

where $C_{y_0,y_1} = C_{y_0,y_1}(T, y_0, y_1, \|u_0\|_{L^2}, \mathscr{E}[u_0], \|b\|_{L^\infty})$ is independent of τ. Also, we have

$$\tilde{f}'(u'_\tau) = 0 \quad (\partial_y u_\tau)_s\text{-a.e.}, \quad \mathscr{L}\text{-a.e. } t. \tag{6.7}$$

Proof. Since $\partial_y v_k \geq 0$ in the sense $\mathcal{M}_{\mathrm{loc}}((0, 2))$ for all k, direct computation yields:

$$0 \leq \partial_y u_\tau(t, \cdot) \quad \text{in } \mathcal{M}_{\mathrm{loc}}((0, 2)), \qquad \mathscr{L}\text{-a.e. } t \in [0, T]. \tag{6.8}$$

By construction, $u_\tau(0, \cdot) = v_0 = u_0 \in X$ and $\mathscr{E}[u_0] \in \mathbb{R}$. Equality (6.7) follows from definition (6.3) and Lemma 5.6 for v'_k. The rest of the proof deals with the uniform-in-τ estimates.

1. *Estimate on discrete time-derivative.* The minimality of each v_k implies $\mathscr{E}^k[v_k] \leq \mathscr{E}^k[v_{k-1}]$ for all $k \in \{1, \dots, N\}$, *i.e.*, it holds

$$\frac{1}{2\tau}\|v_k - v_{k-1}\|^2_{L^2([0,2])} + \mathscr{E}[v_k] \leq \mathscr{E}[v_{k-1}] \qquad \forall k \in \{1, \dots, N\}, \tag{6.9}$$

from which we deduce $\|v_k - v_{k-1}\|^2_{L^2([0,2])} \leq 2\tau(\mathscr{E}[v_{k-1}] - \mathscr{E}[v_k])$, and summing over k yields

$$\sum_{k=1}^N \|v_k - v_{k-1}\|^2_{L^2([0,2])} \leq 2\tau(\mathscr{E}[v_0] - \mathscr{E}[v_N]) \leq 2\tau\mathscr{E}[u_0] + \tau C\big(1 + \|v_N\|_{L^2([0,2])}\big), \tag{6.10}$$

where we used the lower bound (4.19). By the triangle inequality $\|v_N\|_{L^2([0,2])} \leq \|u_0\|_{L^2([0,2])} + \sum_{k=1}^N \|v_k - v_{k-1}\|_{L^2([0,2])}$, and thus by returning to (6.10), we deduce

$$\begin{aligned}\sum_{k=1}^N \|v_k - v_{k-1}\|^2_{L^2([0,2])} &\leq C\tau\Big(\mathscr{E}[u_0] + 1 + \|u_0\|_{L^2([0,2])}\Big) + C\tau\sum_{k=1}^N \|v_k - v_{k-1}\|_{L^2([0,2])} \\ &\leq C\tau\Big(\mathscr{E}[u_0] + 1 + \|u_0\|_{L^2([0,2])}\Big) + \frac{C^2}{2}N\tau^2 + \frac{1}{2}\sum_{k=1}^N \|v_k - v_{k-1}\|^2_{L^2([0,2])},\end{aligned}$$

where we used Young's inequality in the final sum. In turn, using also that $N\tau \le T+1$ since $N = \lceil T/\tau \rceil$, we absorb the final term into the left-hand side, and we find

$$\sum_{k=1}^{N} \|v_k - v_{k-1}\|^2_{L^2([0,2])} \le C_0 \tau, \tag{6.11}$$

for some positive $C_0 = C_0(T, \|u_0\|_{L^2}, \|b\|_{L^\infty}, \mathscr{E}[u_0])$ independent of the time-step τ.

2. *Equicontinuity in time.* We rewrite the previous estimate (6.11) in terms of the interpolation (6.3). Let $0 < t < s < T$ such that $t \in ((n-1)\tau, n\tau]$ and $s \in ((m-1)\tau, m\tau]$; *i.e.* $n < m$. Then, by the triangle inequality, it holds

$$\|u_\tau(t,\cdot) - u_\tau(s,\cdot)\|_{L^2([0,2])} \le \sum_{k=n+1}^{m} \|v_k - v_{k-1}\|_{L^2([0,2])} \le \Bigg(\sum_{k=n+1}^{m} \|v_k - v_{k-1}\|^2_{L^2([0,2])} \Bigg)^{\frac{1}{2}} |n-m|^{\frac{1}{2}},$$

where we applied the Cauchy–Schwarz inequality; note that the above also holds if $n = m$, since in this case the left-hand side is zero. We note that $|n-m| \le \frac{|t-s|}{\tau} + 1$ by our choice of t, s and thus, by (6.11), we deduce $\|u_\tau(t,\cdot) - u_\tau(s,\cdot)\|_{L^2([0,2])} \le C_0(\sqrt{|t-s|} + \sqrt{\tau})$, where we recall C_0 is independent of the time-step τ. We have proved (6.4).

3. *Uniform $L^\infty_t L^2_y$-bound on u_τ.* Using the triangle inequality as before,

$$\begin{aligned} \|v_k\|_{L^2([0,2])} &\le \|u_0\|_{L^2([0,2])} + \sum_{j=1}^{k} \|v_j - v_{j-1}\|_{L^2([0,2])} \\ &\le \|u_0\|_{L^2([0,2])} + \sqrt{N} \Bigg(\sum_{j=1}^{N} \|v_j - v_{j-1}\|^2_{L^2([0,2])} \Bigg)^{\frac{1}{2}} \le \|u_0\|_{L^2([0,2])} + \sqrt{C_0 N \tau}, \end{aligned}$$

where we used the Cauchy–Schwarz inequality and (6.11). Using again $N\tau \le T+1$, we get that there exists a positive $C_1 = C_1(T, \|u_0\|_{L^2}, \|b\|_{L^\infty}, \mathscr{E}[u_0])$ independent of τ such that

$$\|v_k\|_{L^2([0,2])} \le C_1 \quad \forall k \implies \|u_\tau\|_{L^\infty(0,T;L^2([0,2]))} \le C_1. \tag{6.12}$$

4. *Estimates on the functional.* By dropping the positive contribution from the L^2 bound in (6.9), we get $\mathscr{E}[v_k] \le \mathscr{E}[v_{k-1}]$ for all k, and thus $\mathscr{E}[v_k] \le \mathscr{E}[u_0]$. It then follows from the Lipschitz estimate (4.6) on B and Jensen's inequality that

$$\mathscr{F}[v_k] = \mathscr{E}[v_k] - \int_0^2 B(y, v_k)\,\mathrm{d}y \le \mathscr{E}[u_0] + \sqrt{2}\|b\|_{L^\infty} \|v_k\|_{L^2([0,2])} \le \mathscr{E}[u_0] + \sqrt{2}\|b\|_{L^\infty} C_1 =: C_2,$$

using the uniform L^2-estimate (6.12). Hence, $\mathscr{F}[v_k] \le C_2$, which implies $\int_0^2 \tilde{f}(u_\tau')\,\mathrm{d}y \le C_2$, where C_2 is independent of τ. This proves the first part of (6.5).

5. *H^1-bound on $\tilde{f}'(u_\tau')$.* Directly from estimate (5.10) of Proposition 5.3 and summing in k,

$$\int_0^T \|\tilde{f}'(u_\tau'(t,\cdot))\|^2_{H^1([0,2])}\,\mathrm{d}t \le C \sum_{k=1}^{N} \Big(\Big\| \frac{v_k - v_{k-1}}{\tau} \Big\|^2_{L^2([0,2])} + \|b\|^2_{L^\infty} \Big) \tau \le C\big(C_0 + \|b\|^2_{L^\infty}(1+T)\big),$$

where the constant C is independent of τ by (6.11). We deduce the final part of the estimate (6.5); moreover, $\tilde{f}'(u_\tau') \in L^2(0,T; H^1_0([0,2]))$ by Corollary 5.5.

6. *BV_{loc} bound.* Since $\partial_y u_\tau \geq 0$ from (6.8), Lemma 4.5 yields

$$\|u_\tau(t,\cdot)\|_{TV([y_0,y_1])} \leq \|u_\tau\|_{L^\infty(0,T;L^2([0,2]))}\Big(\frac{1}{\sqrt{2-y_1}}+\frac{1}{\sqrt{y_0}}\Big)$$

for any compact interval $[y_0,y_1]\subset(0,2)$. Using (6.12) to bound the right-hand side independently of τ, we get (6.6). □

The next lemma gives a first result concerning the convergence of the sequence $\{u_\tau\}_\tau$ to a limit u. It is not sufficient to pass to the limit in the non-linear terms of the Euler–Lagrange equation, but it will subsequently be used to obtain a stronger convergence (*cf.* Lemma 6.4).

Lemma 6.3. *Let τ and u_τ be as in Lemma 6.2. Then, there exists a subsequence of $\{u_\tau\}_\tau$, which we do not relabel, for which we have $u_\tau(t)\to u(t)$ in X for every t, for some $u\in L^\infty(0,T;X)\cap C^{0,\frac12}([0,T];L^2([0,2]))$ such that $\partial_y u(t,\cdot)\geq 0$. Furthermore, $\partial_y u_\tau \overset{*}{\rightharpoonup} \partial_y u$ in $L^\infty(0,T;\mathcal{M}_{\text{loc}}((0,2)))$, and for $\mathscr{L}$-a.e. t,*

$$\partial_y u_\tau(t,\cdot)\overset{*}{\rightharpoonup}\partial_y u(t,\cdot)\qquad \text{in } \mathcal{M}_{\text{loc}}((0,2)). \tag{6.13}$$

Also, for all $[y_0,y_1]\subseteq[0,2]$,

$$\int_0^T\int_{y_0}^{y_1}\tilde f(u')\,\mathrm{d}y\,\mathrm{d}t \leq \liminf_{\tau\to0^+}\int_0^T\int_{y_0}^{y_1}\tilde f(u'_\tau)\,\mathrm{d}y\,\mathrm{d}t, \tag{6.14}$$

where we recall the notations u'_τ, u' of Definition 2.3. Moreover, u_τ satisfies

$$\begin{aligned}\Big|\int_0^T\int_0^2\Big(-\Big(\frac{\varphi(t+\tau)-\varphi(t)}{\tau}\Big)u_\tau+b(y,u_\tau)\varphi+\tilde f'(u'_\tau)\partial_y\varphi\Big)\,\mathrm{d}y\,\mathrm{d}t\\ -\frac1\tau\int_0^\tau\int_0^2\varphi(t,y)u_0(y)\,\mathrm{d}y\,\mathrm{d}t+\frac1\tau\int_{T-\tau}^T\int_0^2\varphi(t+\tau)u_\tau(t,y)\,\mathrm{d}y\,\mathrm{d}t\Big|\leq C_\varphi\sqrt\tau,\end{aligned} \tag{6.15}$$

for all $\varphi\in C^1([0,T]\times[0,2])$, where the positive constant C_φ is independent of τ.

Proof. By (6.4), the sequence $\{u_\tau\}_\tau$ satisfies an approximate modulus of continuity in time (with an error of the order of $\sqrt\tau$), when valued in $L^2([0,2])$, and hence in $L^1([0,2])$. Moreover, all functions $u_\tau(t,\cdot)$ belong to a bounded subset of X, and this subset is compact for the convergence in X, *i.e.* the L^1 convergence. A standard variant of the Ascoli–Arzelà Theorem (see [5, Section 2.2]), allows to deduce that a subsequence converges uniformly (in X) towards some limit $u\in C^{0,\frac12}([0,T];L^2([0,2]))$ which is valued in the same bounded subset of X. Of course the functions $u(t,\cdot)$ also satisfy the condition $\partial_y u\geq0$ in the sense of $\mathcal{M}_{\text{loc}}((0,2))$.

The BV_{loc} and L^∞_{loc} estimate (6.6) allows to improve the convergence $u_\tau(t,\cdot)\to u(t,\cdot)$ which is *a priori* stated in $L^1([0,2])$ and have $u_\tau(t,\cdot)\to u(t,\cdot)$ in $L^2_{\text{loc}}((0,2))$ and $\partial_y u_\tau(t,\cdot)\overset{*}{\rightharpoonup}\partial_y u(t,\cdot)$ (these conditions hold for every t).

In order to prove (6.14) it is enough to observe that the lower semicontinuity for each t, *i.e.* $\int_{y_0}^{y_1}\tilde f(u')\,\mathrm{d}y\leq\liminf_{\tau\to0^+}\int_{y_0}^{y_1}\tilde f(u'_\tau)\,\mathrm{d}y$ is a consequence of (4.20). Moreover, the quantity $\int_{y_0}^{y_1}\tilde f(u'_\tau)\,\mathrm{d}y$ is bounded from below in terms of $\|u_\tau(t,\cdot)\|_{L^2([0,2])}$, which is itself uniformly bounded by (6.5). This allows to obtain a lower bound on the integrand and apply Fatou's lemma to obtain lower semicontinuity after integrating in time.

Multiplying equation (5.9) by τ and summing over k, using (6.3) the definition of u_τ,

$$\int_\tau^T\int_0^2\Big(\varphi\Big(\frac{u_\tau(t)-u_\tau(t-\tau)}{\tau}+b(y,u_\tau)\Big)+\tilde f'(u'_\tau)\partial_y\varphi\Big)\,\mathrm{d}y\,\mathrm{d}t=0, \tag{6.16}$$

for all $\varphi \in C^1([0,T]\times[0,2])$. Direct computation yields (6.15), with the final term given by

$$\int_0^\tau \Big(\varphi b(y,u_\tau) + \tilde{f}'(u_\tau')\partial_y\varphi\Big)\,\mathrm{d}y\,\mathrm{d}t \le \sqrt{\tau}\|\varphi\|_{C^1([0,T]\times[0,2])}(\|b\|_{L^\infty} + \|\tilde{f}'(u_\tau')\|_{L^2([0,T]\times[0,2])}),$$

whence the constant C_φ is given by $\|\varphi\|_{C^1([0,T]\times[0,2])}(\|b\|_{L^\infty} + \sup_\tau\|\tilde{f}'(u_\tau')\|_{L^2([0,T]\times[0,2])})$. □

The next result is fundamental in passing to the limit as $\tau \to 0^+$ in the equation (6.15).

Lemma 6.4 (Absolutely continuous parts converge a.e.)**.** *There exists a subsequence of* $\{u_\tau\}_\tau$ (*which we do not relabel*) *such that* $u_\tau' \to u'$ $\mathscr{L}$*-a.e.*

The proof of this result relies on the convexity of $\tilde{f}$ and an argument inspired by the Minty–Browder monotonicity method for compactness (*cf. e.g.* [33, §2 and §5]). We note in passing that this situation is specific to the gradient flow structure of the problem; *e.g.* for any approximation of the identity, the absolutely continuous part of the gradients does not converge $\mathscr{L}$-a.e. (since the gradient concentrates as δ_0' the derivative of the Dirac).

Proof. The convexity estimate (4.26) applied to $s_1 = u'$ and $s_2 = u_\tau'$ yields

$$\tilde{f}(u') - \tilde{f}(u_\tau') - \tilde{f}'(u_\tau')(u' - u_\tau') \ge \omega(u', u_\tau') \qquad \mathscr{L}\text{-a.e.}$$

By integrating only in the y variable on a compact interval $[y_0, y_1] \subset (0,2)$, we get

$$\int_{y_0}^{y_1} (\tilde{f}(u') - \tilde{f}(u_\tau'))\,\mathrm{d}y + \int_{y_0}^{y_1} -\tilde{f}'(u_\tau')(u' - u_\tau')\,\mathrm{d}y \ge \int_{y_0}^{y_1} \omega(u', u_\tau')\,\mathrm{d}y \qquad \mathscr{L}\text{-a.e. } t. \tag{6.17}$$

To avoid heavy notation, we omit the explicit dependence on t in what follows. We estimate the second integral in (6.17). Using the identification with the precise representative from Remark 2.4 and the formula (2.3), we have

$$\begin{aligned}\int_{y_0}^{y_1} -\tilde{f}'(u_\tau')(u' - u_\tau')\,\mathrm{d}y &= \int_{y_0}^{y_1} -\tilde{f}'(u_\tau')\,\partial_y(u - u_\tau) - \int_{y_0}^{y_1} -\tilde{f}'(u_\tau')\,(\partial_y u)_s + \int_{y_0}^{y_1} -\tilde{f}'(u_\tau')\,(\partial_y u_\tau)_s \\ &= \big[-\tilde{f}'(u_\tau')(u - u_\tau)\big]_{y_0}^{y_1} + \int_{y_0}^{y_1} (u - u_\tau)\partial_y(\tilde{f}'(u_\tau'))\,\mathrm{d}y \\ &\quad - \underbrace{\int_{y_0}^{y_1} -\tilde{f}'(u_\tau')\,(\partial_y u)_s}_{\ge 0} + \underbrace{\int_{y_0}^{y_1} -\tilde{f}'(u_\tau')\,(\partial_y u_\tau)_s}_{=0},\end{aligned} \tag{6.18}$$

where we integrated by parts to obtain the second line, used $(\partial_y u)_s \ge 0$ in $\mathcal{M}_{\mathrm{loc}}((0,2))$ by Lemma 6.3, and used Lemma 5.6 to make the final term vanish. We deduce

$$\int_{y_0}^{y_1} -\tilde{f}'(u_\tau')(u' - u_\tau')\,\mathrm{d}y \le \big[-\tilde{f}'(u_\tau')(u - u_\tau)\big]_{y_0}^{y_1} + \|u - u_\tau\|_{L^2([y_0,y_1])}\|\tilde{f}'(u_\tau')\|_{H_0^1([0,2])},$$

and we estimate the boundary terms on the right-hand side as follows: using the BV_{loc} estimate of Lemma 4.5, $u_\tau \in L^\infty(0,T;X)$, and $-\tilde{f}'(u_\tau') \ge 0$, we have the one-sided esimate

$$\begin{aligned}\big[-\tilde{f}'(u_\tau')(u - u_\tau)\big]_{y_0}^{y_1} &= -\tilde{f}'(u_\tau')u\big|_{y_1} + \tilde{f}'(u_\tau')u\big|_{y_0} + \tilde{f}'(u_\tau')u_\tau\big|_{y_1} - \tilde{f}'(u_\tau')u_\tau\big|_{y_0} \\ &\le -\tilde{f}'(u_\tau'(y_1))\frac{\|u\|_{L^2([y_1,2])}}{\sqrt{2 - y_1}} - \tilde{f}'(u_\tau'(y_0))\frac{\|u\|_{L^2([0,y_0])}}{\sqrt{y_0}} \\ &\quad - \tilde{f}'(u_\tau'(y_1))\frac{\|u_\tau\|_{L^2([0,2])}}{\sqrt{y_1}} - \tilde{f}'(u_\tau'(y_0))\frac{\|u_\tau\|_{L^2([0,2])}}{\sqrt{2 - y_0}}.\end{aligned} \tag{6.19}$$

As $\tilde{f}'(u'_\tau) \in H^1_0([0,2])$, we compute $-\tilde{f}'(u'_\tau(y_1)) = \int_{y_1}^2 \partial_y(\tilde{f}'(u'_\tau))\,\mathrm{d}y \le \sqrt{2-y_1}\|\tilde{f}'(u'_\tau)\|_{H^1([0,2])}$ and $-\tilde{f}'(u'_\tau(y_0)) = -\int_0^{y_0} \partial_y(\tilde{f}'(u'_\tau))\,\mathrm{d}y \le \sqrt{y_0}\|\tilde{f}'(u'_\tau)\|_{H^1([0,2])}$. By returning to (6.19), we get

$$\begin{aligned}\left[-\tilde{f}'(u'_\tau)(u-u_\tau)\right]_{y_0}^{y_1} &\le \|\tilde{f}'(u'_\tau)\|_{H^1_0([0,2])}\Big(\|u\|_{L^2([y_1,2])} + \|u\|_{L^2([0,y_0])}\Big)\\ &\quad + \|\tilde{f}'(u'_\tau)\|_{H^1_0([0,2])}\|u_\tau\|_{L^2([0,2])}\Big(\frac{\sqrt{2-y_1}}{\sqrt{y_1}} + \frac{\sqrt{y_0}}{\sqrt{2-y_0}}\Big),\end{aligned} \tag{6.20}$$

By integrating (6.17) in time and using Hölder's inequality, we get for all $[y_0, y_1] \subset (0,2)$

$$\begin{aligned}\int_0^T\int_{y_0}^{y_1} \omega(u', u'_\tau)\,\mathrm{d}y\,\mathrm{d}t &\le \int_0^T\int_{y_0}^{y_1} \big(\tilde{f}(u') - \tilde{f}(u'_\tau)\big)\,\mathrm{d}y\,\mathrm{d}t\\ &\quad + \|\tilde{f}'(u'_\tau)\|_{L^2(0,T;H^1_0([0,2]))}\Big(\|u\|_{L^2(0,T;L^2([y_1,2]))} + \|u\|_{L^2(0,T;L^2([0,y_0]))}\Big)\\ &\quad + \|\tilde{f}'(u'_\tau)\|_{L^2(0,T;H^1_0([0,2]))}\|u_\tau\|_{L^2(0,T;L^2([0,2]))}\Big(\frac{\sqrt{2-y_1}}{\sqrt{y_1}} + \frac{\sqrt{y_0}}{\sqrt{2-y_0}}\Big)\\ &\quad + \|u - u_\tau\|_{L^2(0,T;L^2([y_0,y_1]))}\|\tilde{f}'(u'_\tau)\|_{L^2(0,T;H^1_0([0,2]))},\end{aligned}$$

By Lemma 6.2, $\|\tilde{f}'(u'_\tau)\|_{L^2(0,T;H^1_0([0,2]))}, \|u_\tau\|_{L^2(0,T;L^2([0,2]))} \le C$ independent of τ, whence

$$\begin{aligned}\int_0^T\int_{y_0}^{y_1} \omega(u', u'_\tau)\,\mathrm{d}y\,\mathrm{d}t &\le \int_0^T\int_{y_0}^{y_1} \big(\tilde{f}(u') - \tilde{f}(u'_\tau)\big)\,\mathrm{d}y\,\mathrm{d}t + C\|u-u_\tau\|_{L^2(0,T;L^2([y_0,y_1]))}\\ &\quad + C\Big(\|u\|_{L^2(0,T;L^2([y_1,2]))} + \|u\|_{L^2(0,T;L^2([0,y_0]))} + \frac{\sqrt{2-y_1}}{\sqrt{y_1}} + \frac{\sqrt{y_0}}{\sqrt{2-y_0}}\Big).\end{aligned}$$

Since $u_\tau \to u$ in $L^\infty(0,T;L^2([y_0,y_1]))$ for all $[y_0,y_1] \subset (0,2)$, using the lower semicontinuity result (6.14), we let $\tau \to 0$ (for the subsequence that achieves the lim inf in (6.14)) and get

$$\lim_{\tau\to 0}\int_0^T\int_{y_0}^{y_1} \omega(u', u'_\tau)\,\mathrm{d}y\,\mathrm{d}t \le C\Big(\|u\|_{L^2(0,T;L^2([y_1,2]))} + \|u\|_{L^2(0,T;L^2([0,y_0]))} + \frac{\sqrt{2-y_1}}{\sqrt{y_1}} + \frac{\sqrt{y_0}}{\sqrt{2-y_0}}\Big).$$

It is clear that the left-hand side in this inequality increases if we replace $[y_0, y_1]$ with a larger interval $[y'_0, y'_1]$, so that we have

$$\lim_{\tau\to 0}\int_0^T\int_{y_0}^{y_1} \omega(u', u'_\tau)\,\mathrm{d}y\,\mathrm{d}t \le C\Big(\|u\|_{L^2(0,T;L^2([y'_1,2]))} + \|u\|_{L^2(0,T;L^2([0,y'_0]))} + \frac{\sqrt{2-y'_1}}{\sqrt{y'_1}} + \frac{\sqrt{y'_0}}{\sqrt{2-y'_0}}\Big).$$

We then consider $y'_0 \to 0$ and $y'_1 \to 2$ so that the right-hand side tends to 0 and obtain

$$\lim_{\tau\to 0}\int_0^T\int_{y_0}^{y_1} \omega(u', u'_\tau)\,\mathrm{d}y\,\mathrm{d}t = 0 \tag{6.21}$$

for any $0 < y_0 < y_1 < 2$. From (6.21), we have (for a subsequence which we do not relabel)

$$\lim_{\tau\to 0} \omega(u', u'_\tau) = 0 \qquad \mathscr{L}\text{-a.e.} \tag{6.22}$$

The conditions on ω, which only vanishes on the diagonal together with conditions (4.27), imply $u'_\tau \to u'$ $\mathscr{L}$-a.e. □

We are ready to give the proof of the main result of this section.

Proof of Proposition 6.1. The proof is divided into four steps. We fix a test function $\varphi \in C_c^1((0,T)\times(0,2))$. We assume $\operatorname{supp}\varphi(t,\cdot) \subseteq [y_0, y_1]$ for all $t \in [0,T]$, for some $[y_0, y_1] \subset (0,2)$.

1. *Passing to the limit in $\tilde{f}'(u'_\tau)$.* The continuity of $\tilde{f}'$ and the $\mathscr{L}$-a.e. convergence $u'_\tau \to u'$ of Lemma 6.4 implies $\tilde{f}'(u'_\tau) \to \tilde{f}'(u')$ $\mathscr{L}$-a.e. Additionally, since $\{\tilde{f}'(u'_\tau)\}_\tau$ is a bounded sequence in $L^2(0,T;H_0^1([0,2]))$ from the estimates of Lemma 6.2, we deduce that

$$\tilde{f}'(u'_\tau) \rightharpoonup \tilde{f}'(u') \quad \text{weakly in } L^2(0,T;H_0^1([0,2])), \tag{6.23}$$

whence $\tilde{f}'(u') \in L^2(0,T;H_0^1([0,2]))$ and $\lim_\tau \int_0^T \int_0^2 \tilde{f}'(u'_\tau)\partial_y\varphi \,\mathrm{d}y\,\mathrm{d}t = \int_0^T \int_0^2 \tilde{f}'(u')\partial_y\varphi \,\mathrm{d}y\,\mathrm{d}t$.

2. *Derivative lower bound.* The previous step implies $\tilde{f}'(u') \in L^2(0,T;H_0^1([0,2]))$, whence

$$\int_0^T \|\tilde{f}'(u')\|^2_{H_0^1([0,2])}\,\mathrm{d}t < \infty \implies \|\tilde{f}'(u')\|_{H_0^1([0,2])} < \infty \quad \mathscr{L}\text{-a.e. } t. \tag{6.24}$$

By Morrey's embedding, we deduce from (6.24) that $\|\tilde{f}'(u')\|_{L^\infty([0,2])} < \infty$ for $\mathscr{L}$-a.e. t, and

$$0 \le -\tilde{f}'(u') \le \|\tilde{f}'(u'(t,\cdot))\|_{L^\infty([0,2])} \qquad \mathscr{L}\text{-a.e.}, \tag{6.25}$$

where we used $\tilde{f}' < 0$. Since $\tilde{f}'$ is strictly increasing by the strict convexity of $\tilde{f}$, it has a well-defined inverse, and we set

$$c_t := (-\tilde{f}')^{-1}\big(\|\tilde{f}'(u'(t,\cdot))\|_{L^\infty([0,2])}\big) > 0;$$

this quantity is strictly positive since $\lim_{s\to 0} \tilde{f}'(s) = -\infty$ by (3.6) and $\tilde{f}$ is strictly decreasing. Estimate (6.2) now follows from applying the inverse $(-\tilde{f}')^{-1}$ to the whole inequality (6.25).

3. *Passing to the limit in $b(y,u_\tau)$.* Recall that the test function φ has $\operatorname{supp}\varphi(t,\cdot) \subseteq [y_0,y_1] \subset (0,2)$ for all t. Using the explicit form (1.10) for b and writing $A^1 = A$, $A^2 = A^c$, we get

$$\int_0^2 b(y,u_\tau)\varphi\,\mathrm{d}y = \int_{[y_0,y_1]\cap A} \overbrace{\partial_x V}^{=:b^1}(u_\tau)\varphi\,\mathrm{d}y + \int_{[y_0,y_1]\cap A^c} \overbrace{\partial_x W}^{=:b^2}(u_\tau)\varphi\,\mathrm{d}y = \sum_{j=1}^2 \int_{[y_0,y_1]\cap A^j} b^j(u_\tau)\varphi\,\mathrm{d}y, \tag{6.26}$$

and the same decomposition holds for $\int_0^2 b(y,u)\varphi\,\mathrm{d}y$ with u_τ replaced by u on the right. We shall approximate $b^1 = \partial_x V$ and $b^2 = \partial_x W$, which are merely L^∞, by continuous functions using Lusin's Theorem; we then pass to the limit in τ without requiring continuity of b_1, b_2.

To this end, notice that the uniform bounds of Lemma 6.2 and Lemma 4.5 imply

$$\|u\|_{L^\infty([0,T]\times[y_0,y_1])} + \|u_\tau\|_{L^\infty([0,T]\times[y_0,y_1])} \le C\Big(\frac{1}{\sqrt{y_0}} + \frac{1}{\sqrt{2-y_1}}\Big) =: C_{y_0,y_1},$$

where the constant C_{y_0,y_1} is independent of τ. We set $I = [-C_{y_0,y_1}, C_{y_0,y_1}]$, and remark that it suffices to consider the restrictions of b^1, b^2 to the bounded subset I instead of all of $\mathbb{R}$. By Lusin's Theorem, for all $\varepsilon > 0$ there exists compact subsets $K_{j,\varepsilon} \subseteq I$ such that $\mathscr{L}(I \setminus K_{j,\varepsilon}) < \varepsilon$ and b^j coincides on $K_{j,\varepsilon}$ with a continuous function (denoted b^j_ε) bounded by the same constants which bound b^j $(j=1,2)$. Define the error terms $R^j_\varepsilon := b^j - b^j_\varepsilon$, and note that $R^j_\varepsilon = 0$ in $K_{j,\varepsilon}$ and $\|R^j_\varepsilon\|_{L^\infty} \le 2\|b^j\|_{L^\infty} < \infty$ since $\partial_x V, \partial_x W \in L^\infty$. Then, for $j = 1,2$

$$\int_{[y_0,y_1]\cap A^j} b^j(u_\tau)\varphi\,\mathrm{d}y = \int_{[y_0,y_1]\cap A^j} b^j_\varepsilon(u_\tau)\varphi\,\mathrm{d}y + \int_{B^j_{\varepsilon,\tau}} R^j_\varepsilon(u_\tau)\varphi\,\mathrm{d}y \qquad \mathscr{L}\text{-a.e. } t, \tag{6.27}$$

where $B^j_{\varepsilon,\tau} = \{y \in [y_0, y_1] \cap A^j : u_\tau(t,y) \in K^c_{j,\varepsilon}\}$, and we emphasise that we identify u_τ, u with their precise representatives of Remark 2.4. We estimate the final term as

$$\left| \int_{B^j_{\varepsilon,\tau}} R^j_\varepsilon(u_\tau)\varphi \,\mathrm{d}y \right| \le 2\|b^j\|_{L^\infty}\|\varphi\|_{L^\infty}\mathscr{L}(B^j_{\varepsilon,\tau}). \tag{6.28}$$

Let $B^j_\varepsilon = \{y \in [y_0, y_1] \cap A^j : u(t,y) \in K^c_\varepsilon\}$ and $b^j(u) - b^j_\varepsilon(u) = R^j_\varepsilon(u)$. Returning to (6.27),

$$\begin{aligned}\left| \int_{[y_0,y_1]\cap A^j} \big(b^j(u_\tau) - b^j(u)\big)\varphi \,\mathrm{d}y \right| \le{}& \left| \int_{[y_0,y_1]\cap A^j} \big(b^j_\varepsilon(u_\tau) - b^j_\varepsilon(u)\big)\varphi \,\mathrm{d}y \right| \\ &+ 2\|b^j\|_{L^\infty}\|\varphi\|_{L^\infty}\Big(\mathscr{L}(B^j_\varepsilon) + \mathscr{L}(B^j_{\varepsilon,\tau})\Big),\end{aligned} \tag{6.29}$$

where, as per (6.28), we also used the estimate $|\int_{B^j_\varepsilon} R^j_\varepsilon(u)\varphi \,\mathrm{d}y| \le 2\|b\|_{L^\infty}\|\varphi\|_{L^\infty}\mathscr{L}(B^j_\varepsilon)$.

It remains to estimate $\mathscr{L}(B^j_\varepsilon)$ and $\mathscr{L}(B^j_{\varepsilon,\tau})$. We rewrite $B^j_{\varepsilon,\tau}$ as a preimage set, which yields

$$\mathscr{L}(B^j_{\varepsilon,\tau}) = \mathscr{L}\big([y_0, y_1] \cap A^j \cap u_\tau(t,\cdot)^{-1}(K^c_{j,\varepsilon})\big); \tag{6.30}$$

analogously $\mathscr{L}(B^j_\varepsilon) = \mathscr{L}([y_0, y_1] \cap A^j \cap u(t,\cdot)^{-1}(K^c_{j,\varepsilon}))$. We estimate the right-hand side: recall that for every Borel measurable set $E \subset [0,2]$ and for strictly increasing $v \in BV_{\mathrm{loc}}$,

$$\int_E v' \,\mathrm{d}y \le \mathscr{L}(v(E)),$$

where $v(E)$ is the image set corresponding to the precise representative of Remark 2.4 and v' the absolutely continuous part of $\partial_y v$. To obtain this inequality it is enough to use the measure-valued version of the coarea formula for BV functions presented in section 5.5 of [34].

We then obtain

$$\mathscr{L}(B^j_{\varepsilon,\tau}) \cdot \underbrace{\operatorname{ess\,inf}_{[0,2]} u'_\tau}_{>0 \text{ by Lemma 5.2}} \le \int_{B^j_{\varepsilon,\tau}} u'_\tau \,\mathrm{d}y \le \mathscr{L}(K^c_{j,\varepsilon}) < \varepsilon.$$

Arguing analogously for $\mathscr{L}(B^j_\varepsilon)$, we obtain the estimates

$$\mathscr{L}(B^j_{\varepsilon,\tau}) \le \frac{\varepsilon}{\operatorname{ess\,inf}_{[0,2]} u'_\tau(t,\cdot)}, \qquad \mathscr{L}(B^j_\varepsilon) \le \frac{\varepsilon}{\operatorname{ess\,inf}_{[0,2]} u'(t,\cdot)}, \tag{6.31}$$

for $\mathscr{L}$-a.e. t, and by returning to (6.29), we get

$$\begin{aligned}\left| \int_{[y_0,y_1]\cap A^j} \big(b^j(u_\tau) - b^j(u)\big)\varphi \,\mathrm{d}y \right| \le{}& \left| \int_{[y_0,y_1]\cap A^j} \big(b^j_\varepsilon(u_\tau) - b^j_\varepsilon(u)\big)\varphi \,\mathrm{d}y \right| \\ &+ 2\varepsilon\|b^j\|_{L^\infty}\|\varphi\|_{L^\infty}\Big(\frac{1}{\operatorname{ess\,inf} u'_\tau(t,\cdot)} + \frac{1}{\operatorname{ess\,inf} u'(t,\cdot)}\Big).\end{aligned} \tag{6.32}$$

Next, we split our analysis into two parts: for times t which are such that $\operatorname{ess\,inf}_{[0,2]} u'_\tau(t,\cdot)$ is small, and times for which it is large. To this end, for each τ, define the set of times

$$\mathcal{T}_{\varepsilon,\tau} := \{t \in [0,T] : \operatorname{ess\,inf}_{[0,2]} u'_\tau(t,\cdot) \ge \sqrt{\varepsilon}\} \cap \{t \in [0,T] : \operatorname{ess\,inf}_{[0,2]} u'(t,\cdot) \ge \sqrt{\varepsilon}\}.$$

Note that the sign and monotonicity of $\tilde{f}'$ implies

$$\begin{aligned}\mathscr{L}(\mathcal{T}^c_{\varepsilon,\tau}) &\le \mathscr{L}(\{t : \operatorname{ess\,inf} u'_\tau(t,\cdot) < \sqrt{\varepsilon}\}) + \mathscr{L}(\{t : \operatorname{ess\,inf} u'(t,\cdot) < \sqrt{\varepsilon}\}) \\ &= \mathscr{L}(\{t : \|\tilde{f}'(u'_\tau(t,\cdot))\|^2_{L^\infty} > |\tilde{f}'(\sqrt{\varepsilon})|^2\}) + \mathscr{L}(\{t : \|\tilde{f}'(u'(t,\cdot))\|^2_{L^\infty} > |\tilde{f}'(\sqrt{\varepsilon})|^2\}) \\ &\le \frac{C}{|\tilde{f}'(\sqrt{\varepsilon})|^2}\|\tilde{f}'(u'_\tau)\|^2_{L^2(0,T;H^1([0,2]))} + \frac{C}{|\tilde{f}'(\sqrt{\varepsilon})|^2}\|\tilde{f}'(u')\|^2_{L^2(0,T;H^1([0,2]))} \le \frac{C}{|\tilde{f}'(\sqrt{\varepsilon})|^2},\end{aligned}$$

where we used Markov's inequality and Morrey's embedding to obtain the penultimate inequality, and the uniform bound of Lemma 6.2 in the final one. By returning to (6.32),

$$\int_{\mathcal{T}_{\varepsilon,\tau}}\left|\int_{[y_0,y_1]\cap A^j}\big(b^j(u_\tau)-b^j(u)\big)\varphi\,\mathrm{d}y\right|\mathrm{d}t\le\int_0^T\left|\int_{[y_0,y_1]\cap A^j}\big(b^j_\varepsilon(u_\tau)-b^j_\varepsilon(u)\big)\varphi\,\mathrm{d}y\right|\mathrm{d}t+4\sqrt{\varepsilon}T\|b^j\|_{L^\infty}\|\varphi\|_{L^\infty},$$

while, on the other hand,

$$\int_{\mathcal{T}^c_{\varepsilon,\tau}}\left|\int_{[y_0,y_1]\cap A^j}\big(b^j(u_\tau)-b^j(u)\big)\varphi\,\mathrm{d}y\right|\mathrm{d}t\le 2\|b^j\|_{L^\infty}\mathscr{L}(\mathcal{T}^c_{\varepsilon,\tau})\le\frac{C\|b^j\|_{L^\infty}}{|\tilde{f}'(\sqrt{\varepsilon})|^2}.$$

Putting the previous two estimates together and letting $\tau\to 0^+$, using the continuity of b^j_ε, the convergence $u_\tau\to u$ $\mathscr{L}$-a.e. in $[0,T]\times[0,2]$, and the Dominated Convergence Theorem,

$$\limsup_{\tau\to 0^+}\int_0^T\left|\int_{[y_0,y_1]\cap A^j}\big(b^j(u_\tau)-b^j(u)\big)\varphi\,\mathrm{d}y\right|\mathrm{d}t\le 4\sqrt{\varepsilon}T\|b^j\|_{L^\infty}\|\varphi\|_{L^\infty}+\frac{C\|b^j\|_{L^\infty}}{|\tilde{f}'(\sqrt{\varepsilon})|^2}.$$

The left side is independent of ε. By letting $\varepsilon\to 0$ and using $\tilde{f}'(\sqrt{\varepsilon})\to-\infty$ by (3.6), we get

$$\lim_{\tau\to 0^+}\left|\int_0^T\int_{[y_0,y_1]\cap A^j}\big(b^j(u_\tau)-b^j(u)\big)\varphi\,\mathrm{d}y\,\mathrm{d}t\right|=0,$$

and, by returning to (6.26) and using the triangle inequality, it follows that

$$\lim_{\tau\to 0^+}\int_0^T\int_0^2 b(y,u_\tau)\varphi\,\mathrm{d}y\,\mathrm{d}t=\int_0^T\int_0^2 b(y,u)\varphi\,\mathrm{d}y\,\mathrm{d}t.$$

The convergence $u_\tau\to u$ in $L^\infty(0,T;L^2([y_0,y_1]))$ for all $[y_0,y_1]\subset(0,2)$ from Lemma 6.3 is sufficient to pass to the limit in all the other terms of the weak formulation (6.15), and also to preserve the continuity valued in L^2 of the limit curve.

4. *Convergence to the initial data.* Setting $s=0$ in (6.4), we get $\|u_\tau(t,\cdot)-u_0\|_{L^2([0,2])}\le C_0(\sqrt{t}+\sqrt{\tau})$.Meanwhile, the convergence $u_\tau(t)\to u(t)$ in X implies $\|u(t,\cdot)-u_0\|_{L^2([0,2])}\le C_0\sqrt{t}\to 0$ as $t\to 0^+$. The proof is complete. ☐

Finally, we record the continuous-time version of the optimality condition of Lemma 5.6.

Lemma 6.5 (Condition on the singular part of the derivative)**.** *The function $\tilde{f}'(u')$ vanishes $(\partial_y u)_s$-a.e. for $\mathscr{L}$-a.e. $t\in[0,T]$.*

Proof. For clarity of presentation, we select a discrete subsequence $\{u_{\tau_n}\}_n$ of the sequence $\{u_\tau\}_\tau$ with which we pass to the limit in Proposition 6.1. We write u_n in place of u_{τ_n}.

1. *Boundedness of subsequences on time-slices.* Recall from (6.23) and the argument that precedes it that (for a subsequence which we do not relabel here) we have $\tilde{f}'(u'_n)\to\tilde{f}'(u')$ $\mathscr{L}$-a.e. and weakly in $L^2(0,T;H^1_0([0,2]))$. The purpose of this step is to show, for $\mathscr{L}$-a.e. $t\in[0,T]$, that there exists a subsequence $\{\tilde{f}'(u'_{\sigma_t(n)}(t,\cdot))\}_n$ depending on t and $C_t>0$ such that

$$\sup_n\|\tilde{f}'(u'_{\sigma_t(n)}(t,\cdot))\|_{H^1_0([0,2])}\le C_t<\infty. \tag{6.33}$$

The proof of (6.33) follows directly from Fatou's Lemma. Indeed, we have

$$\int_0^T\liminf_{n\to\infty}\|\tilde{f}'(u'_n(t,\cdot))\|^2_{H^1_0([0,2])}\,\mathrm{d}t\le\liminf_{n\to\infty}\int_0^T\|\tilde{f}'(u'_n(t,\cdot))\|^2_{H^1_0([0,2])}\,\mathrm{d}t\le C,$$

where we used (6.5). Hence $\liminf_{n\to\infty} \|\tilde{f}'(u_n'(t,\cdot))\|^2_{H_0^1([0,2])}$ is finite for $\mathscr{L}$-a.e. t. For each t, select $\sigma_t(n)$ to be a subsequence such that $\|\tilde{f}'(u'_{\sigma_t(n)}(t,\cdot))\|_{H_0^1([0,2])}$ is bounded (*a priori*, this subsequence depends on the point t, and it satisfies (6.33)).

2. *Uniform convergence on time-slices.* Applying Morrey's embedding to the estimate (6.33), the sequence $\{\tilde{f}'(u'_{\sigma_t(n)}(t,\cdot))\}_n$ is uniformly bounded in $L^\infty([0,2])$ and equicontinuous.Thus, by the Ascoli–Arzelà Theorem, each such sequence admits a further uniformly convergent subsequence $\{\tilde{f}'(u'_{\sigma_t(n)}(t,\cdot))\}_n$; again, this subsequence depends on t. Since $\tilde{f}'(u'_\tau) \to \tilde{f}'(u')$ $\mathscr{L}$-a.e., it holds

$$\lim_{n\to\infty} \|\tilde{f}'(u'_{\sigma_t(n)}(t,\cdot)) - \tilde{f}'(u'(t,\cdot))\|_{L^\infty([0,2])} = 0. \tag{6.34}$$

3. *Convergence as $\tau \to 0$.* By returning to (6.18), we observe that for every $[y_0, y_1] \subset (0,2)$

$$\int_{y_0}^{y_1} -\tilde{f}'(u'_\tau)\,(\partial_y u)_s + \int_{y_0}^{y_1} -\tilde{f}'(u'_\tau)(u'-u'_\tau)\,\mathrm{d}y \le \big[-\tilde{f}'(u'_\tau)(u-u_\tau)\big]_{y_0}^{y_1} + \int_{y_0}^{y_1} (u-u_\tau)\partial_y(\tilde{f}'(u'_\tau))\,\mathrm{d}y,$$

which, by then re-inserting into (6.17) and using the non-negativity of $\omega(u',u'_\tau)$, yields

$$\int_{y_0}^{y_1} \big(\tilde{f}(u')-\tilde{f}(u'_\tau)\big)\,\mathrm{d}y + \big[-\tilde{f}'(u'_\tau)(u-u_\tau)\big]_{y_0}^{y_1} + \int_{y_0}^{y_1} (u-u_\tau)\partial_y(\tilde{f}'(u'_\tau))\,\mathrm{d}y \ge \int_{y_0}^{y_1} -\tilde{f}'(u'_\tau)\,(\partial_y u)_s.$$

We estimate the second term on the left-hand side as per (6.20) and get

$$\begin{aligned}\int_{y_0}^{y_1} -\tilde{f}'(u'_\tau)\,(\partial_y u)_s \le \int_{y_0}^{y_1} \big(\tilde{f}(u')-\tilde{f}(u'_\tau)\big)\,\mathrm{d}y &+ \|u-u_\tau\|_{L^2([y_0,y_1])}\|\tilde{f}'(u'_\tau)\|_{H_0^1([0,2])} \\ &+ \|\tilde{f}'(u'_\tau)\|_{H_0^1([0,2])}\Big(\|u\|_{L^2([y_1,2])} + \|u\|_{L^2([0,y_0])}\Big) \\ &+ \|\tilde{f}'(u'_\tau)\|_{H_0^1([0,2])}\|u_\tau\|_{L^2([0,2])}\Big(\frac{\sqrt{2-y_1}}{\sqrt{y_1}} + \frac{\sqrt{y_0}}{\sqrt{2-y_0}}\Big);\end{aligned}$$

unlike in Step 3 of the proof of Lemma 6.4, we have not integrated in time. Then, we pass to the limit along the subsequence $\{\sigma_t(n)\}_n$ of Step 2, for which (6.33) and (6.34) hold, and get

$$\begin{aligned}0 \le \int_{y_0}^{y_1} -\tilde{f}'(u')\,(\partial_y u)_s = \lim_{n\to\infty}\int_{y_0}^{y_1} -\tilde{f}'(u'_{\sigma_t(n)})\,(\partial_y u)_s &\le C_t\Big(\|u\|_{L^2([y_1,2])} + \|u\|_{L^2([0,y_0])}\Big) \\ &+ C_t\Big(\frac{\sqrt{2-y_1}}{\sqrt{y_1}} + \frac{\sqrt{y_0}}{\sqrt{2-y_0}}\Big).\end{aligned}$$

By letting $y_0 \to 0^+$ and $y_1 \to 2^-$, the right-hand side vanishes, which proves the claim. □

Theorem 3.6 now follows directly from Proposition 6.1 and Lemma 6.5.

7. Existence of Segregated Solutions of the Cross-Diffusion System

We define $u_t(y) := u(t,y)$ where u is the solution of (6.1) provided by Proposition 6.1, and fix a precise representative defined everywhere and equal to u_t $\mathscr{L}$-a.e., such that $\tilde{f}'(u'_t) \in C_0([0,2])$ by Morrey's embedding and the Fundamental Theorem of Calculus in BV_{loc} (2.3) is satisfied. Henceforth, we do not distinguish u_t from this representative.

Proof of Theorem 3.5. The proof is divided into four steps.

1. *Well-defined inverse function.* By Proposition 6.1, u_t is strictly increasing on $(0,2)$: for all $y_1 < y_2$, it holds

$$u_t(y_2) - u_t(y_1) = \int_{y_1}^{y_2} \partial_y u_t(y) \geq \int_{y_1}^{y_2} u_t'(y)\,\mathrm{d}y \geq c_t(y_2 - y_1) > 0, \tag{7.1}$$

using (6.2). Therefore, its inverse function is well-defined and strictly increasing on the image set $u_t((0,2))$. We denote $F_t(x) := u_t^{-1}(x)$, with $F_t : u_t((0,2)) \to (0,2)$.

2. *Defining ϱ_t, μ_t.* Define the set $G_t := \{y \in [0,2] : \tilde{f}'(u_t'(y)) \neq 0\}$, and note that, since $\tilde{f}'(s) < 0$ whenever s is finite, and u' is a.e. finite, then G_t is of full measure in $[0,2]$.

Since $\tilde{f}'(u_t')$ is continuous for $\mathscr{L}$-a.e. t, the set G_t is open in $[0,2]$. It follows that G_t can be written as a countable union of open intervals, and hence as a countable union of compact intervals. By Lemma 6.5, we deduce that the support of $(\partial_y u)_s$ is contained in the complement of G_t, so

$$\partial_y u_t = u_t' \cdot \mathscr{L} \quad \text{in } G_t \implies u_t \in AC_{\mathrm{loc}}(G_t) \qquad (\mathscr{L}\text{-a.e. } t), \tag{7.2}$$

where AC_{loc} on the set G_t, which is open but may not be connected, means AC on every compact interval contained within G_t. Hence u_t is differentiable $\mathscr{L}$-a.e. in G_t. Since u_t is continuous and strictly increasing on G_t, it is an open map, whence the image $u_t(G_t)$ is open in $\mathbb{R}$. Furthermore, for all $x_1, x_2 \in u_t(G_t)$, the condition (7.1) and the monotonicity of F_t implies

$$0 \leq \frac{F_t(x_1) - F_t(x_2)}{x_1 - x_2} \leq \frac{1}{c_t},$$

whence F_t is Lipschitz and thus differentiable $\mathscr{L}$-a.e. Differentiating $u_t(F_t(x)) = x$ at $\mathscr{L}$-a.e. point x in the open set $u_t(G_t)$, we get

$$\partial_x F_t(x) = \frac{1}{u_t'(F_t(x))} \qquad \mathscr{L}\text{-a.e. } x \in u_t(G_t); \tag{7.3}$$

this is a standard result for absolutely continuous functions. Consequently, *cf.* (1.7), we set

$$S_t(x) := \partial_x F_t(x) \mathbb{1}_{u_t(G_t)} = \frac{1}{u_t'(F_t(x))} \mathbb{1}_{u_t(G_t)} \qquad \forall x \in \mathbb{R}, \tag{7.4}$$

and we remark $S_t \in L^\infty(\mathbb{R})$ and $S_t(x) \cdot \mathscr{L} = (u_t)_\sharp \mathscr{L} \llcorner G_t$. In accordance with (1.8), define

$$\varrho_t(x) := \frac{1}{u_t'(F_t(x))} \mathbb{1}_{u_t(A \cap G_t)}, \qquad \mu_t(x) := \frac{1}{u_t'(F_t(x))} \mathbb{1}_{u_t(A^c \cap G_t)}; \tag{7.5}$$

hence $\varrho_t(x) \cdot \mathscr{L} = (u_t)_\sharp \mathscr{L} \llcorner (A \cap G_t)$, while $\mu_t(x) \cdot \mathscr{L} = (u_t)_\sharp \mathscr{L} \llcorner (A^c \cap G_t)$, and $\varrho_t + \mu_t = S_t$.

3. $\partial_y \tilde{f}'(u_t') \circ F_t = -\partial_x f'(S_t)$. It follows from the relation (3.3) that, for all $x \in u_t(G_t)$,

$$\tilde{f}'(u_t')|_{y=F_t(x)} = f(S_t(x)) - S_t(x) f'(S_t(x)).$$

By differentiating both sides with respect to x in the set $u_t(G_t)$ (since absolutely continuous functions are differentiable $\mathscr{L}$-a.e.), it follows that, for $\mathscr{L}$-a.e. $x \in u_t(G_t)$,

$$\partial_y (\tilde{f}'(u_t'))|_{y=F_t(x)} \underbrace{\partial_x F_t}_{=S_t}(x) = \partial_x \big(f(S_t(x)) - S_t(x) f'(S_t(x))\big) = -S_t(x) \partial_x (f'(S_t(x))).$$

When $x \in u_t(G_t)$, (7.4) and (6.2) imply $S_t(x) > 0$, whence we divide by $S_t(x)$ and get

$$\partial_y (\tilde{f}'(u_t'))|_{y=F_t(x)} = -\partial_x (f'(S_t(x))) \qquad \mathscr{L}\text{-a.e. } x \in u_t(G_t). \tag{7.6}$$

It follows from the above, the change of coordinates $y = F_t(x)$, and (7.3), that

$$\int_{u_t(G_t)} S_t(x)|\partial_x(f'(S_t(x)))|^2\,\mathrm{d}x = \int_{u_t(G_t)} \frac{|\partial_y(\tilde{f}'(u_t'))|^2(F_t(x))}{u_t'(F_t(x))}\,\mathrm{d}x = \int_{G_t} |\partial_y(\tilde{f}'(u_t'))|^2\,\mathrm{d}y,$$

whence, given that $S_t \equiv 0$ outside of $u_t(G_t)$ by definition (7.4), integrating in time gives

$$\|\sqrt{S_t}\partial_x(f'(S_t))\|_{L^2([0,T]\times\mathbb{R})} = \|\tilde{f}'(u_t')\|_{L^2(0,T;H^1([0,2]))},$$

where we used from Step 2 that G_t is of full measure to have equality.

4. *Equations for ϱ_t, μ_t.* Let $\varphi \in C_c^1(\mathbb{R})$. By definition (7.5) and the pushforward relation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \varphi(x)\varrho_t(x)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{A\cap G_t} \varphi(u_t(y))\,\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t}\int_{A} \varphi(u_t(y))\,\mathrm{d}y = \int_{A\cap G_t} \partial_x\varphi(u_t(y))\partial_t u_t(y)\,\mathrm{d}y.$$

where we use that G_t is of full measure from Step 2 to repeatedly change the domain of integration from $A\cap G_t$ to A and back. Recall that u satisfies (6.1) $\mathscr{L}$-a.e., whence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \varphi(x)\varrho_t(x)\,\mathrm{d}x &= \int_{A\cap G_t} \partial_x\varphi(u_t(y))\Big(\partial_y(\tilde{f}'(u_t'(y))) - b(y,u_t(y))\Big)\,\mathrm{d}y \\
&= \int_{A\cap G_t} \partial_x\varphi(u_t(y))\partial_y(\tilde{f}'(u_t'(y)))\,\mathrm{d}y - \int_{\mathbb{R}} \partial_x\varphi(x)\,\underbrace{b(F_t(x),x)\mathbb{1}_{\{F_t(x)\in A\}}}_{=\partial_x V(x)}\,\varrho_t(x)\,\mathrm{d}x \\
&= -\int_{u_t(A\cap G_t)} \partial_x\varphi(x)S_t(x)\partial_x(f'(S_t(x)))\,\mathrm{d}x - \int_{\mathbb{R}} \partial_x\varphi(x)\varrho_t(x)\partial_x V(x)\,\mathrm{d}x,
\end{aligned}$$

where we used (7.6) and the Jacobian factor $(u_t)_\sharp\,\mathrm{d}y \llcorner (A\cap G_t) = F_t'(x)\,\mathrm{d}x \llcorner u_t(A\cap G_t) = S_t(x)\,\mathrm{d}x \llcorner u_t(A\cap G_t)$ to obtain the final line. Rewriting in terms of ϱ_t, we find

$$-\int_{\mathbb{R}} \partial_x\varphi(x)S_t(x)\mathbb{1}_{\{F_t(x)\in A\cap G_t\}}\partial_x(f'(S_t(x)))\,\mathrm{d}x = -\int_{\mathbb{R}} \partial_x\varphi(x)\varrho_t(x)\partial_x(f'(S_t(x)))\,\mathrm{d}x,$$

and the equation (3.1) for ϱ_t follows. The computation for μ_t is analogous, replacing A with A^c and V with W in the relevant manipulations. The convergence to the initial data in $W_2(\mathbb{R})$ follows from the strong L^2 convergence $u_t \to u_0$. □

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