A singularity theorem in terms of asymptotic expansion

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Abstract

We prove a singularity theorem in which the classical focusing hypothesis of Hawking–Penrose theory is replaced by a condition on asymptotic volume growth. Under the strong energy condition, we introduce asymptotic volume-expansion invariants associated with a compact Cauchy hypersurface and show that a uniform positive lower bound on these invariants implies past timelike geodesic incompleteness. More precisely, we obtain an explicit upper bound on the time-separation from the hypersurface to its chronological past. The theorem extends to globally hyperbolic Lorentzian length spaces satisfying the synthetic strong energy condition $\mathsf{TCD}^e_p(0,N)$, yielding an inextendibility result valid without any smoothness or differentiability assumption. We also prove an area comparison theorem for equidistant hypersurfaces and a volume singularity theorem based on related asymptotic expansion invariants.

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1 Introduction

The singularity theorems of Penrose [34] and Hawking [22] are among the foundational achievements of mathematical relativity. They demonstrate that spacetime singularities — understood as causal geodesic incompleteness — are not merely artifacts of highly symmetric solutions of Einstein's equations, but arise under broad and physically natural assumptions. Penrose's theorem predicts singularity formation in gravitational collapse, whereas Hawking's theorem applies to cosmological spacetimes undergoing expansion. Together with their many subsequent extensions and refinements, these results have profoundly shaped our understanding of black holes, cosmology, and the global structure of spacetime.

The underlying mechanism in both theorems is the combination of an energy condition with the focusing of causal geodesics. Concretely, one assumes that the spacetime is globally hyperbolic and that the Ricci curvature is nonnegative along timelike directions in Hawking's theorem and along null directions in Penrose's theorem. This is supplemented by a local geometric condition that induces the focusing of causal geodesics,

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typically formulated in terms of positive mean curvature of a compact Cauchy hypersurface or the existence of trapped surfaces. Under these hypotheses, one concludes that the spacetime contains an incomplete causal geodesic. In the classical framework of general relativity, such geodesic incompleteness is regarded as the hallmark of a spacetime singularity.

In this note we establish a singularity theorem of a different nature. Rather than assuming a local focusing condition on a compact Cauchy hypersurface, we impose a global condition on the asymptotic growth of volume. We show that, when combined with an appropriate lower bound on the timelike Ricci curvature, this large-scale geometric assumption alone suffices to force the existence of an incomplete timelike geodesic.

For the sake of exposition, we formulate the discussion in the introduction for $(n+1)$-dimensional, smooth, globally hyperbolic Lorentzian manifolds (M, g) satisfying the Hawking–Penrose strong energy condition:

$$\mathrm{Ric}_g(v, v) \geq 0, \quad \text{for every timelike vector } v \in TM. \tag{1.1}$$

The main results, however, hold in the substantially broader setting of Lorentzian length spaces [25]. In fact, they are proved under the synthetic strong energy condition $\mathsf{TCD}^e_p(0, N)$ introduced by the authors in [15], which is rooted in the optimal transport characterizations of timelike Ricci curvature lower bounds established in the smooth setting by McCann [26] and by Suhr and the second author [33].

The main result

We begin by fixing notation. Let $V \subset M$ be a smooth, compact, n-dimensional Cauchy hypersurface. In fact, the results of this paper are established under the weaker assumption that V is a compact achronal set with empty global edge (see Definition 2.8). Denote by $I^{\pm}(V)$ the chronological future and past of V, respectively. The *signed time-separation function* from V is defined by

$$\tau_V(x) := \begin{cases} \sup_{y \in V} \tau(y, x), & x \in I^+(V), \\ -\sup_{y \in V} \tau(x, y), & x \in I^-(V), \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

For the sake of exposition, let us first assume that τ_V is smooth on $I^{\pm}(V)$. Then the integral curves of the vector field $-\nabla \tau_V$ induce a foliation of $I^{\pm}(V)$. For each $\alpha \in V$, let X_α denote the unique integral curve through α that we assume to be future complete. We parametrize it on its maximal interval of existence $(\mathfrak{a}_\alpha, +\infty)$, with $\mathfrak{a}_\alpha < 0$, and normalized by the condition $X_\alpha(0) = \alpha$. When τ_V fails to be smooth, an analogous decomposition still holds up to a set of vanishing volume measure; see the body of the paper for the precise statements.

The Fubini–Tonelli theorem, or more generally the disintegration theorem, yields a decomposition of the spacetime volume measure along this family of timelike geodesics:

$$\mathrm{Vol}_{g \llcorner I^{\pm}(V)} = \int_V \mathfrak{m}_\alpha \, \mathrm{Vol}_{g_V}(\mathrm{d}\alpha), \tag{1.3}$$

where

$$\mathfrak{m}_\alpha = h_\alpha(\cdot) \, \mathcal{L}^1, \qquad h_\alpha(0) = 1 \quad \text{for } \mathrm{Vol}_{g_V}\text{-a.e. } \alpha \in V. \tag{1.4}$$

Here Vol_g denotes the $(n+1)$-dimensional volume measure of (M, g), while Vol_{g_V} is the induced n-dimensional volume measure on V. The measures $\mathfrak{m}_\alpha$ are supported on the geodesics X_α and describe the distribution of spacetime volume along the foliation.

The strong energy condition (1.1) implies, via a Lorentzian analogue of the Bishop–Gromov inequality (which, in this one-dimensional setting, reduces to an elementary computation), that for each α the function

$$(\mathfrak{a}_\alpha, +\infty) \ni t \longmapsto \frac{h_\alpha(t)}{(n+1)t^n}$$

is monotone non-increasing. We therefore define

$$\theta_\alpha := \lim_{t\to+\infty} \frac{h_\alpha(t)}{(n+1)t^n} = \inf_{t>\mathfrak{a}_\alpha} \frac{h_\alpha(t)}{(n+1)t^n}. \tag{1.5}$$

Geometrically, θ_α represents the asymptotic volume expansion along the timelike ray X_α. The central observation of this note is that a uniform positive asymptotic expansion is incompatible with past timelike geodesic completeness. Indeed, if θ_α is bounded below by a positive constant, then the past of V has uniformly bounded time-separation from V, yielding the existence of an incomplete past-directed timelike geodesic.

Theorem 1.1 (Corollary 5.8)**.** *Let (M, g) be an $(n+1)$-dimensional, globally hyperbolic, smooth Lorentzian manifold satisfying Penrose-Hawking's strong energy condition (1.1). Let $V \subset M$ be a smooth, compact, Cauchy hypersurface. Assume moreover that there exists a constant $c > 0$ such that*

$$\theta_\alpha \geq c > 0 \qquad \text{for } \mathrm{Vol}_V\text{-a.e. } \alpha \in V. \tag{1.6}$$

Then

$$\sup_{x\in I^-(V)} |\tau_V(x)| \leq \left(\frac{1}{c}\right)^{\frac{1}{n}}.$$

In particular, (M, g) is not past timelike geodesically complete.

Moreover, for any (possibly non-smooth) past extension $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ of (M, g) which is a timelike non-branching, globally hyperbolic, Lorentzian geodesic space, satisfying the $\mathsf{TCD}^e_p(0, n+1)$ condition, together with its causally-reversed structure, it holds that

$$\sup_{x\in I^-_X(V)} |\tau_V(x)| \leq \left(\frac{1}{c}\right)^{\frac{1}{n}},$$

where $I^-_X(V) \subset X$ is the chronological past of V in X.
In particular, $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ is not past timelike geodesically complete.

The second part of the theorem may be viewed as an inextendibility statement. It shows that the combination of the strong energy condition and the asymptotic volume growth assumption (1.6) enforces past timelike geodesic incompleteness not only in the smooth spacetime (M, g) itself, but in every globally hyperbolic extension satisfying the same synthetic energy condition, regardless of its regularity. In this sense, the singularity predicted by the theorem cannot be removed by passing to a lower-regularity extension.

Beyond the singularity theorem stated above, the techniques introduced in this paper yield several additional results of independent interest. These include an area comparison estimate for equidistant hypersurfaces from V (see Section 4) and a volume singularity theorem (see Section 5.1) in the sense of [19], both formulated in terms of asymptotic invariants capturing the large-time expansion of the geodesic congruence generated by V.

Related literature

The present work is part of the emerging theory of synthetic Lorentzian geometry. Among the achievements of the theory of $\mathsf{TCD}_p^e(K,N)$ Lorentzian length spaces there is the extension of Hawking's singularity theorem to a fully synthetic setting, obtained in [15]; see also the survey [14].

Related developments include inextendibility theorems for Lorentzian pre-length spaces satisfying synthetic lower bounds on timelike sectional curvature [20, 1, 4], as well as singularity theorems in the broader framework of closed cone structures [27].

More generally, the synthetic approach has led to analogues of a number of foundational comparison, rigidity, and singularity theorems from Lorentzian geometry highlighting the flexibility of synthetic curvature-dimension conditions, see for instance [5, 3, 8, 6, 9, 16].

A notable feature of the synthetic framework is its ability to accommodate singular spacetimes lying beyond the scope of classical Lorentzian geometry. In particular, Lorentzian pre-length spaces satisfying synthetic lower bounds on timelike Ricci curvature include physically relevant examples such as Penrose's impulsive gravitational waves [31] and spacetimes endowed with Lipschitz Lorentzian metrics whose timelike Ricci curvature is bounded below in the sense of distributions [7].

The present paper should be viewed as a contribution to this program: it identifies a new synthetic mechanism for geodesic incompleteness based on asymptotic volume expansion rather than local focusing.

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2 Preliminaries

2.1 Lorentzian geodesic spaces and synthetic Ricci lower bounds

2.1.1 Some basics on Lorentzian pre-length spaces

The general framework for the paper is the one of Lorenzian pre-length spaces introduced by Kunzinger-Sämann in [25], inspired by earlier works of Kronheimer-Penrose [24] on causal spaces and of Busemann [10] on timelike spaces. An alternative approach put forward by Minguzzi-Suhr [30] derived many topological properties out of basic assumptions on the time-separation. Variants have been studied by several authors with different motivations, let us mention [26, 8, 11, 3, 32, 13].

Recall that $(X, \mathsf{d}, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* provided: $(X, \ll, \leq)$ is a *causal space* (i.e., $\leq$ is a preorder and $\ll$ a transitive relation contained in $\leq$)

additionally equipped with a proper metric d (i.e., closed and bounded subsets are compact) and a lower semicontinuous function $\tau : X \times X \to [0, \infty]$, called *time-separation function*, satisfying

$$\tau(x,y) + \tau(y,z) \le \tau(x,z) \quad \forall x \le y \le z \quad \text{reverse triangle inequality}$$
$$\tau(x,y) = 0, \text{ if } x \not\le y, \quad \tau(x,y) > 0 \Leftrightarrow x \ll y.$$

We write $x < y$ whenever $x \le y$ and $x \ne y$. We say that x and y are *timelike* (resp. *causally*) related if $x \ll y$ (resp. $x \le y$).

Let $A \subset X$ be an arbitrary subset. The *chronological* (resp. *causal*) future of A is defined as

$$I^+(A) := \{y \in X : \exists\, x \in A \text{ such that } x \ll y\},$$
$$J^+(A) := \{y \in X : \exists\, x \in A \text{ such that } x \le y\}.$$

Analogously, one defines the *chronological* (resp. *causal*) past of A.

If $A = \{x\}$ is a singleton, we will slightly abuse notation and write $I^\pm(x)$ (resp. $J^\pm(x)$) instead of $I^\pm(\{x\})$ (resp. $J^\pm(\{x\})$).

We say that $(X, \mathsf{d}, \ll_r, \le_r, \tau_r)$ is the *causally-reversed structure* of $(X, \mathsf{d}, \ll, \le, \tau)$ if

$$x \ll_r y \iff y \ll x, \quad x \le_r y \iff y \le x, \quad \tau_r(x,y) = \tau(y,x).$$

The set of (resp. *timelike*) *geodesics* is defined as:

$$\mathrm{Geo}(X) := \{\gamma \in C([0,1], X) : \tau(\gamma_s, \gamma_t) = (t-s)\,\tau(\gamma_0, \gamma_1),\ \forall s < t\},$$
$$\mathrm{TGeo}(X) := \{\gamma \in \mathrm{Geo}(X) : \tau(\gamma_0, \gamma_1) > 0\}.$$

Definition 2.1 (Timelike non-branching)**.** A Lorentzian pre-length space $(X, \mathsf{d}, \ll, \le, \tau)$ is said to be *forward timelike non-branching* if and only if for any $\gamma^1, \gamma^2 \in \mathrm{TGeo}(X)$, it holds:

$$\exists\, \bar t \in (0,1) \text{ such that } \forall t \in [0, \bar t\,] \quad \gamma^1_t = \gamma^2_t \quad \Longrightarrow \quad \gamma^1_s = \gamma^2_s, \quad \forall s \in [0,1].$$

X is said to be *backward timelike non-branching* if the causally-reversed structure is forward timelike non-branching. In case X is both forward and backward timelike non-branching it is said *timelike non-branching*.

Concerning the causal ladder (see [25, 29] for more details), we will only consider *Lorentzian geodesic spaces,* i.e. Lorentzian pre-length spaces $(X, \mathsf{d}, \ll, \le, \tau)$ that additionally are:

- d-*Compatible:* every $x \in X$ admits a neighbourhood U and a constant C such that $L_{\mathsf{d}}(\gamma) \le C$ for every causal curve γ contained in U;
- *Geodesic:* for all $x, y \in X$ with $x < y$ there is a future-directed causal curve γ from x to y with $\tau(x,y) = L_\tau(\gamma)$.

We consider the following version of global hyperbolicity that fits with the previous literature (see [29, Cor. 3.8]): a Lorentzian geodesic space $(X, \mathsf{d}, \ll, \le, \tau)$ is called

- *Causal*: if $\le$ is also antisymmetric, i.e. $\le$ is an order;
- *Globally hyperbolic*: if it is causal and for every $x, y \in X$ the causal diamond $J^+(x) \cap J^-(y)$ is compact in X.

2.1.2 Timelike optimal transport in a Lorentzian pre-length space

Let $\mathcal{P}(X)$ be the set of Borel probability measures over X. Denote by $P_i : X \times X \to X$, $i = 1, 2$ the projection map and let $(P_i)_\sharp : \mathcal{P}(X \times X) \to \mathcal{P}(X)$ be the corresponding push-forward map.
Given $\mu, \nu \in \mathcal{P}(X)$, consider

$$\begin{aligned}\Pi(\mu,\nu) &:= \{\pi \in \mathcal{P}(X \times X) \,:\, (P_1)_\sharp\pi = \mu,\ (P_2)_\sharp\pi = \nu\},\\ \Pi_\leq(\mu,\nu) &:= \{\pi \in \Pi(\mu,\nu) \,:\, \pi(X^2_\leq) = 1\},\\ \Pi_\ll(\mu,\nu) &:= \{\pi \in \Pi(\mu,\nu) \,:\, \pi(X^2_\ll) = 1\},\end{aligned}$$

where $X^2_\leq := \{(x,y) \in X^2 \,:\, x \leq y\}$ and $X^2_\ll := \{(x,y) \in X^2 \,:\, x \ll y\}$.

We next recall the notion of p-Lorentz-Wasserstein distance, following the convention in [15]. The definition below extends to Lorentzian pre-length spaces the corresponding notion given in the smooth Lorentzian setting in [17] (see also [26, 33], and [37] for $p = 1$).

Definition 2.2. Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $p \in (0,1]$. Given $\mu, \nu \in \mathcal{P}(X)$, the *p-Lorentz-Wasserstein distance* is defined by

$$\ell_p(\mu,\nu) := \sup_{\pi \in \Pi_\leq(\mu,\nu)} \left(\int_{X \times X} \tau(x,y)^p \, \pi(\mathrm{d}x\mathrm{d}y) \right)^{1/p}. \tag{2.1}$$

When $\Pi_\leq(\mu,\nu) = \emptyset$ we set $\ell_p(\mu,\nu) := -\infty$.

A coupling $\pi \in \Pi_\leq(\mu,\nu)$ maximising in (2.1) is said *ℓ_p-optimal*. The set of *ℓ_p-optimal* couplings from μ to ν is denoted by $\Pi^{p\text{-opt}}_\leq(\mu,\nu)$. We also set

$$\Pi^{p\text{-opt}}_\ll(\mu,\nu) := \Pi^{p\text{-opt}}_\leq(\mu,\nu) \cap \Pi_\ll(\mu,\nu)$$

the family of *timelike* ℓ_p-optimal couplings.

By gluing of couplings, one can prove that the p-Lorentz-Wasserstein distance ℓ_p satisfies the reverse triangle inequality on causally related measures (see [15, Proposition 2.5]), thus lifting the Lorentzian metric structure to the space of probability measures.

In order to transfer the causal constraint in the optimal transport problem to the cost function (see [15, Remark 2.2] for more details), it is also useful to consider

$$\ell(x,y)^p := \begin{cases} \tau(x,y)^p & \text{if } x \leq y \\ -\infty & \text{otherwise} \end{cases}. \tag{2.2}$$

We next recall the notion of timelike p-dualisable pairs of measures from [15], relaxing the notion of q-separated measures introduced by McCann [26, Definition 4.1] in the smooth Lorentzian setting.

Definition 2.3 (Timelike p-dualisable)**.** Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $p \in (0,1]$. We say that $(\mu,\nu) \in \mathcal{P}(X)^2$ is *timelike p-dualisable (by $\pi \in \Pi_\ll(\mu,\nu)$)* if

1. $\ell_p(\mu,\nu) \in (0,\infty)$;
2. $\pi \in \Pi^{p\text{-opt}}_\ll(\mu,\nu)$;
3. there exist measurable functions $a, b : X \to \mathbb{R}$, with $a \oplus b \in L^1(\mu \otimes \nu)$ such that $\ell^p \leq a \oplus b$ on $\operatorname{supp}\mu \times \operatorname{supp}\nu$.

The motivation for considering p-dualisable pairs of measures is three-fold: firstly the p-optimal coupling $\pi(\mathrm{d}x\mathrm{d}y)$ matches events described by $\mu(\mathrm{d}x)$ with events described by $\nu(\mathrm{d}y)$ so that $x \ll y$, secondly Kantorovich duality holds (see [15, Proposition 2.19]; see also the earlier works [37, 23, 26] in the smooth Lorentzian setting), thirdly the class of p-dualisable pairs of measures will provide a key building block to define the synthetic timelike Ricci lower bounds. We conclude this subsection by recalling the notion of ℓ_p-geodesic $(\mu_s)_{s\in[0,1]} \subset \mathcal{P}(X)$.
The evaluation map is defined by

$$\mathrm{e}_t : C([0,1], X) \to X, \qquad \gamma \mapsto \mathrm{e}_t(\gamma) := \gamma_t, \qquad \forall t \in [0,1]. \tag{2.3}$$

Definition 2.4 (ℓ_p-optimal dynamical plans and ℓ_p-geodesics)**.** Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $p \in (0,1]$. We say that $\eta \in \mathcal{P}(\mathrm{Geo}(X))$ is an *ℓ_p-optimal dynamical plan* from $\mu_0 \in \mathcal{P}(X)$ to $\mu_1 \in \mathcal{P}(X)$ if

$$(\mathrm{e}_0, \mathrm{e}_1)_\sharp \eta \in \Pi_{\leq}^{p\text{-opt}}\big((\mathrm{e}_0)_\sharp \eta, (\mathrm{e}_1)_\sharp \eta\big). \tag{2.4}$$

The collection of all ℓ_p-optimal dynamical plans from μ_0 to μ_1 is denoted by $\mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$. We say that a curve $[0,1] \ni t \mapsto \mu_t \in \mathcal{P}(X)$ is an *ℓ_p-geodesic* if there exists $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$ such that

$$\mu_t = (\mathrm{e}_t)_\sharp \eta, \qquad \forall t \in [0,1].$$

Observe that if $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$, then the associated curve

$$\mu_t := (\mathrm{e}_t)_\sharp \eta, \qquad \forall t \in [0,1],$$

is continuous with respect to the narrow topology and satisfies

$$\ell_p(\mu_s, \mu_t) = (t-s)\,\ell_p(\mu_0, \mu_1), \qquad \forall\, 0 \leq s \leq t \leq 1.$$

Finally, recall that if X is a globally hyperbolic Lorentzian geodesic space, $\mu_0, \mu_1 \in \mathcal{P}(X)$ have compact support and $\Pi_{\leq}(\mu_0, \mu_1) \neq \emptyset$, then there always exists an ℓ_p-optimal dynamical plan $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$ (and hence an ℓ_p-geodesic) connecting μ_0 to μ_1; see [15, Proposition 2.33] for the proof and further properties of ℓ_p-geodesics.

2.1.3 Synthetic timelike Ricci lower bounds

A *measured Lorentzian pre-length space* $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ is a Lorentzian pre-length space endowed with a Radon non-negative measure $\mathfrak{m}$. We say that $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ is globally hyperbolic (resp. geodesic) if $(X, \mathsf{d}, \ll, \leq, \tau)$ is so.

We next recall the definition of the Timelike Curvature–Dimension condition, denoted by $\mathsf{TCD}^e_p(K,N)$ (and by $\mathsf{wTCD}^e_p(K,N)$ in its weak form), as introduced in [15] (after [26] and [33]). Here $K \in \mathbb{R}$ stands for a synthetic lower bound on the timelike Ricci curvature, and $N \in (0,\infty)$ stands for a synthetic upper bound on the dimension. Since in this paper we will only consider the case on non-negative timelike Ricci curvature, we will focus on the case $K = 0$. We refer the reader to [15] for more details and to [14] for a concise overview.

The definition is based on a suitable concavity property of the entropy functional, which we now briefly recall. Given $\mu \in \mathcal{P}(X)$, its Boltzmann-Shannon entropy w.r.t. $\mathfrak{m}$ is given by

$$\mathrm{Ent}(\mu|\mathfrak{m}) = \int_M \rho \log \rho\, \mathfrak{m},$$

if $\mu = \rho\,\mathfrak{m}$ is absolutely continuous with respect to $\mathfrak{m}$ and $(\rho \log \rho)_+$ is $\mathfrak{m}$-integrable. Otherwise we set $\mathrm{Ent}(\mu|\mathfrak{m}) = +\infty$. We set

$$\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})) := \{\mu \in \mathcal{P}(X) \,:\, \mathrm{Ent}(\mu|\mathfrak{m}) \in \mathbb{R}\},$$

to be the finiteness domain of the entropy.

Definition 2.5 ($\mathsf{TCD}_p^e(0,N)$ condition, [15])**.** Fix $p \in (0,1)$ and $N \in (0,\infty)$. We say that a measured Lorentzian pre-length space $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ satisfies $\mathsf{TCD}_p^e(0,N)$ if the following holds. For any pair $(\mu_0, \mu_1) \in (\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})))^2$ which is timelike p-dualisable by some $\pi \in \Pi_{\ll}^{p\text{-opt}}(\mu_0, \mu_1)$, there exists an ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ such that the function $[0,1] \ni t \mapsto e(t) := \mathrm{Ent}(\mu_t|\mathrm{Vol}_g)$ is convex (and thus in particular it is locally Lipschitz in $(0,1)$) and it satisfies

$$e''(t) - \frac{1}{N} e'(t)^2 \geq 0. \tag{2.5}$$

in the distributional sense on $[0,1]$.

A variant on the TCD_p^e condition, denoted by TCD_p^* was later introduced by Braun [5]. The same paper also proved the equivalence between the TCD_p^e and the TCD_p^* conditions in timelike non-branching Lorentzian pre-length spaces.

2.2 Time separation function from achronal sets

Another key object in our analysis is the synthetic analogue of the gradient flow lines associated with the time-separation function from a given set V. We now recall some basic constructions.

Standing assumptions. From now on we will always assume that $(X, \mathsf{d}, \ll, \leq, \tau)$ is a globally hyperbolic Lorentzian geodesic space (see subsection 2.1.1 for the definitions).

Under such standing assumptions, the time separation $\tau : X \times X \to \mathbb{R}$ is continuous (see [25, Theorem 3.28]). A subset $V \subset X$ is called *achronal* if $x \not\ll y$ for every $x, y \in V$. In particular, if V is achronal, then $I^+(V) \cap I^-(V) = \emptyset$, so we can define the *signed time-separation* to V, $\tau_V : X \to [-\infty, +\infty]$, by

$$\tau_V(x) := \begin{cases} \sup_{y\in V} \tau(y,x), & \text{for } x \in I^+(V) \\ -\sup_{y\in V} \tau(x,y), & \text{for } x \in I^-(V) \\ 0 & \text{otherwise} \end{cases}. \tag{2.6}$$

Note that τ_V is lower semi-continuous on $I^+(V)$ as supremum of continuous functions, and is upper semi-continuous on $I^-(V)$.

To introduce a synthetic analogue of the integral curves of $-\nabla\tau_V$, some preparatory tools are needed. We briefly recall the construction and refer to [15, Sec. 4.1] for a more detailed treatment.

Definition 2.6 (Future timelike complete (FTC) subsets, [18])**.** A subset $V \subset X$ is *future timelike complete* (FTC), if for each point $x \in I^+(V)$, the intersection $J^-(x) \cap V \subset V$ has compact closure (w.r.t. d) in V. Analogously, one defines *past timelike completeness* (PTC). A subset that is both FTC and PTC is called *timelike complete.*

Lemma 2.7 (Lemma 4.1, [15])**.** *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a globally hyperbolic Lorentzian geodesic space and let $V \subset X$ be an achronal FTC (resp. PTC) subset. Then for each $x \in I^+(V)$ (resp. $x \in I^-(V)$) there exists a point $y_x \in V$ with $\tau_V(x) = \tau(y_x, x) > 0$ (resp. $\tau_V(x) = -\tau(x, y_x) < 0$).*

Moreover for all $x, z \in I^+(V) \cup V$,

$$\tau_V(z) - \tau_V(x) \geq \tau(y_x, z) - \tau(y_x, x) \geq \tau(x, z), \tag{2.7}$$

provided $(x,z) \in X^2_{\leq}$. An analogous statement is valid for $x \in I^-(V)$.

The inequality (2.7) can be restated by saying that τ_V is timelike reverse 1-Lipschitz on $I^+(V)$ and, separately, on $I^-(V)$. As shown in [16], this permits to study the integral lines of maximal steep of $-\tau_V$ on $I^+(V)$ and, separately, on $I^-(V)$ giving a disintegration formula for the reference measure $\mathfrak{m}$ whose conditional measures localize the eventual Ricci curvature lower bound to the integral lines of $-\tau_V$.

For the scope of this note, the interest will be in performing this analysis jointly on $I^+(V)$ and $I^-(V)$. To start, for ease of writing, we will use the following notation

$$I^{\pm}(V) := (I^+(V) \cup I^-(V) \cup V).$$

In general τ_V fails to be reverse 1-Lipschitz on $I^{\pm}(V)$: consider for instance in Minkowski spacetime $V = \{0\}$, the origin. Let z be any point in the future light cone of 0 and x be any point in the past light cone of 0 additionally being in the timelike past of z. Then

$$\tau_V(z) - \tau_V(x) = \tau(0, z) + \tau(x, 0) = 0 < \tau(x, z),$$

contradicting the reverse 1-Lipschitz property.

Nevertheless, as discussed below, the reverse 1-Lipschitzianity holds in some special cases of interest.

Definition 2.8 (Empty global edge)**.** Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a globally hyperbolic Lorentzian geodesic space. Let $V \subset X$ be an achronal set. We say that V has *empty global edge* if for any $x \in I^-(V)$ and $z \in I^+(V)$, any $\gamma \in \mathrm{TGeo}(X)$ connecting x to z intersects V.

Definition 2.8 may be viewed as a global counterpart of the condition that an achronal set has empty edge. Recall that the *edge* of an achronal set $S \subset X$ consists of those points $p \in S$ such that every neighbourhood $U \subset X$ of p contains points $p^+ \in I^+(p)$ and $p^- \in I^-(p)$, together with a timelike curve connecting p^- to p^+ that does not intersect S. Clearly, if V has empty global edge then its edge is empty as well, however the converse implication may fail.

The notion of the edge of an achronal set is classical in Lorentzian geometry and causality theory; see, for instance, the foundational references [35, 21] or the more recent survey [28].

Remark 2.9 (Sufficient conditions for $V \subset X$ to have empty global edge)**.**

- Let (M, g) be a spacetime with a continuous Lorentzian metric. If V is a Cauchy hypersurface, then V has empty global edge. For properties of Cauchy hypersurfaces in C^0-spacetimes, see [36].
- Recall the definition of future Cauchy development $D^+(V)$ of V:

  $$D^+(V) := \{p \in X\colon \text{every past inext. timelike curve through } p \text{ intersects } V\}.$$

  If $I^+(V) \supset D^+(V)$, then V has empty global edge.

Proposition 2.10. *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a globally hyperbolic Lorentzian geodesic space and let $V \subset X$ be an achronal, timelike complete subset.*

If V has empty global edge, then τ_V is reverse 1-Lipschitz over $I^{\pm}(V)$.

Proof. It suffices to prove that, for all $x \leq z$ with $x \in I^-(V)$ and $z \in I^+(V)$:

$$\tau_V(z) - \tau_V(x) \geq \tau(x, z).$$

If $\tau(x, z) = 0$, the claim is verified as $\tau_V(x) \leq 0$ and $\tau_V(z) \geq 0$. If $\tau(x, z) > 0$, then there exists $\gamma \in \mathrm{TGeo}(X)$ with $\gamma_0 = x$ and $\gamma_1 = z$. Since V is achronal with empty geodesic boundary, there exists a unique $s \in (0, 1)$ such that $\gamma_s \in V$. Then $\tau(x, z) = \tau(x, \gamma_s) + \tau(\gamma_s, z) \leq \tau_V(z) - \tau_V(x)$; by definition indeed $-\tau_V(x) = \sup_{y \in V} \tau(x, y)$. The claim is therefore proved. $\square$

In [16, Section 6], the localization of a general reverse 1-Lipschitz function is discussed in detail. Hence we now confine ourselves to reporting the main definitions and results.

Given $V \subset X$, an achronal set with empty global edge, define:

$$\begin{aligned}\Gamma_V := \{(x,z) \in I^{\pm}(V)^2 \cap X^2_{\leq} : \tau_V(z) - \tau_V(x) = \tau(x,z) > 0\} \\ \cup \{(x,x) : x \in I^{\pm}(V)\}.\end{aligned} \tag{2.8}$$

The monotonicity of Γ_V ensures that pairs of points belonging to Γ_V are aligned along geodesics: if $(x,z) \in \Gamma_V$ with $x \neq z$ and $x \in I^+(V)$, then there exist $y \in V, \gamma \in \mathrm{TGeo}(X)$ and $t \in (0,1)$ such that $y = \gamma_0,\ x = \gamma_t,\ z = \gamma_1$, and

$$\tau(y,\gamma_s) = \tau_V(\gamma_s) \quad \forall s \in [0,1], \qquad (\gamma_s,\gamma_t) \in \Gamma_V, \quad \forall s \in [0,t].$$

An analogous property holds true if $z \in I^-(V)$. Next, define

$$\Gamma_V^{-1} := \{(x,y) : (y,x) \in \Gamma_V\}$$

and consider the *transport relation* R_V and the *transport set with endpoints* $\mathcal{T}_V^{end}$

$$R_V := \Gamma_V \cup \Gamma_V^{-1}, \qquad \mathcal{T}_V^{end} := P_1(R_V \setminus \{x = y\}). \tag{2.9}$$

The transport relation R_V will be an equivalence relation on a suitable subset of $\mathcal{T}_V^{end}$, once initial and final points of the integral lines are removed. The sets

$$\begin{aligned}\mathfrak{a} = \mathfrak{a}(\mathcal{T}_V^{end}) := \{x \in \mathcal{T}_V^{end} \colon \nexists y \in \mathcal{T}_V^{end} \ s.t.\ (y,x) \in \Gamma_V, y \neq x\} \\ \mathfrak{b} = \mathfrak{b}(\mathcal{T}_V^{end}) := \{x \in \mathcal{T}_V^{end} \colon \nexists y \in \mathcal{T}_V^{end} \ s.t.\ (x,y) \in \Gamma_V, y \neq x\},\end{aligned} \tag{2.10}$$

are the *initial* and *final points*, respectively. The *transport set without endpoints* is defined by:

$$\mathcal{T}_V := \mathcal{T}_V^{end} \setminus (\mathfrak{a}(\mathcal{T}_V^{end}) \cup \mathfrak{b}(\mathcal{T}_V^{end})). \tag{2.11}$$

Remark 2.11. The set $\mathfrak{a}(\mathcal{T}_V^{end})$ shall be understood as the past cut-locus of V while $\mathfrak{b}(\mathcal{T}_V^{end})$ the future one, being indeed those points, in the smooth scenario, where the geodesics stops to be maximizing.

If additionally $V \subset X$ is timelike complete, repeating the argument of [15, Lemma 4.4] one can prove that

$$I^+(V) \cup I^-(V) = \mathcal{T}_V^{end} \setminus V.$$

If X is timelike (both backward and forward) non-branching, then the transport relation R_V defines an equivalence relation on $\mathcal{T}_V$. The corresponding equivalence classes are timelike geodesics that locally maximize the function τ_V; we denote them by X_α. Moreover, there exists a measurable quotient map

$$\mathfrak{Q} : \mathcal{T}_V \to Q,$$

associated with the equivalence relation R_V on $\mathcal{T}_V$.

The corresponding disintegration of the reference measure $\mathfrak{m}$ and the localization of the Ricci curvature bounds were established in [15, 16] (see also [8] for the extension to variable timelike Ricci lower bounds). We denote by $\mathcal{M}_+(X)$ the space of non-negative Radon measures over X.

Theorem 2.12. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a globally hyperbolic, timelike non-branching, Lorentzian geodesic space satisfying $\mathsf{TCD}_p^e(0,N)$, assume that the causally-reversed structure satisfies the same conditions and let $V \subset X$ be a Borel achronal timelike complete subset with empty global edge.*
Let $\mathcal{T}_V^{end}, \mathfrak{a}(\mathcal{T}_V^{end}), \mathfrak{b}(\mathcal{T}_V^{end})$ and $\mathcal{T}_V$ defined in (2.9)*,* (2.10) *and* (2.11)*.*

Then $\mathfrak{m}(\mathfrak{a}(\mathcal{T}_V^{end})) = \mathfrak{m}(\mathfrak{b}(\mathcal{T}_V^{end}) = 0$ *and the following disintegration formula holds:*

$$\mathfrak{m}\llcorner_{\mathcal{T}_V^{end}} = \mathfrak{m}\llcorner_{\mathcal{T}_V} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(\mathrm{d}\alpha), \tag{2.12}$$

where $\mathfrak{q}$ *is a Borel probability measure over a quotient set* $Q \subset \mathcal{T}_V$ *such that* $\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_V}) \ll \mathfrak{q}$ *and the map* $Q \ni \alpha \mapsto \mathfrak{m}_\alpha \in \mathcal{M}_+(X)$ *satisfies the following properties:*

(1) for any $\mathfrak{m}$*-measurable set* $B \subset X$*, the map* $\alpha \mapsto \mathfrak{m}_\alpha(B)$ *is* $\mathfrak{q}$*-measurable;*

(2) for $\mathfrak{q}$*-a.e.* $\alpha \in Q$*,* $\mathfrak{m}_\alpha$ *is concentrated on* $\mathfrak{Q}^{-1}(\alpha) = X_\alpha$ *(strong consistency);*

(3) for $\mathfrak{q}$*-a.e.* $\alpha \in Q$*,* $\mathfrak{m}_\alpha \ll \mathcal{L}^1\llcorner_{X_\alpha}$*;*

(4) for $\mathfrak{q}$*-a.e.* $\alpha \in Q$*, the one-dimensional metric measure space* $(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ *satisfies the classical* $\mathsf{CD}(0, N)$*; namely, writing* $\mathfrak{m}_\alpha = h(\alpha, \cdot)\mathcal{L}^1\llcorner_{X_\alpha}$*, then* $h(\alpha, \cdot)$ *is semi-concave (and thus twice differentiable* $\mathcal{L}^1$*-a.e. on* X_α*) and it satisfies the differential inequality*

$$\frac{\partial^2}{\partial x^2} \log h(\alpha, x) + \frac{1}{N-1} \left(\frac{\partial}{\partial x} \log h(\alpha, x) \right)^2 \le 0, \tag{2.13}$$

at any point x *in the interior of* X_α *where* $h(\alpha, \cdot)$ *is twice differentiable.*

Moreover, fixed any $\mathfrak{q}$ *as above such that* $\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_V}) \ll \mathfrak{q}$*, the disintegration is* $\mathfrak{q}$*-essentially unique in the following sense: if any other map* $Q \ni \alpha \mapsto \bar{\mathfrak{m}}_\alpha \in \mathcal{P}(X)$ *satisfies points (1)-(2), then* $\bar{\mathfrak{m}}_\alpha = \mathfrak{m}_\alpha$ *for* $\mathfrak{q}$*-a.e.* $\alpha \in Q$*.*

2.3 Future Minkowski content

Building on Theorem 2.12, in the previous work [16], the authors established results concerning the area of achronal subsets of X. We begin by recalling the definition of the area of an achronal set.

Definition 2.13 (Timelike Minkowski content)**.** Let $A \subset X$ be a Borel achronal set and consider the signed time-separation function τ_A from A, see (2.6). We define the *future Minkowski content* of A by

$$\begin{gathered} \mathfrak{m}^+(A) := \inf_{U \in \mathcal{U}} \limsup_{\varepsilon \to 0} \frac{\mathfrak{m}(\tau_A^{-1}((0, \varepsilon)) \cap U)}{\varepsilon}, \\ \mathcal{U} := \{U \subset X \colon U \text{ open}, \ A \subset U\}. \end{gathered} \tag{2.14}$$

Analogously, the *past Minkowski content* of A is defined by

$$\mathfrak{m}^-(A) := \inf_{U \in \mathcal{U}} \limsup_{\varepsilon \to 0} \frac{\mathfrak{m}(\tau_A^{-1}((-\varepsilon, 0)) \cap U)}{\varepsilon}. \tag{2.15}$$

Remark 2.14. If X is a smooth, globally hyperbolic, Lorentzian manifold and $A \subset X$ is a smooth, achronal, future causally complete hypersurface, then $\mathfrak{m}^+(A)$ coincides with the standard area of A computed with respect to the restriction of the ambient Lorentzian metric; see [16, Remark 4.2].

In particular, as shown in [38], in this smooth setting the time separation function τ_A is smooth on $I^+(A) \setminus C_+(A)$, where $C_+(A)$ is the future cut-locus of A that is a closed set of zero measure. This implies that each level set $\tau_A^{-1}(s)$ is a smooth submanifold of $I^+(A) \setminus C_+(A)$ having therefore a well-defined volume measure induced by the ambient metric. Finally the coarea formula [38, Proposition 3] gives the claim.

As shown in [16], a convenient framework for studying the area of a given achronal set A is obtained by assuming that $A \subset I^+(V)$, where $V \subset X$ is a Borel, achronal, and timelike complete set. One may then exploit the disintegration formula associated with the function τ_V, recalled above, to derive an upper bound for the area of A in terms of its traces along the integral curves of τ_V.

Moreover, when A is given by a level set of τ_V, namely

$$V_t := \{\tau_V = t\},$$

the corresponding inequality becomes an identity. We recall below [16, Proposition 4.6, Proposition 4.8].

Proposition 2.15. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TCD}^e_p(0, N)$ and assume that the causally-reversed structure satisfies the same conditions. Let $V \subset X$ be a Borel, achronal, timelike complete subset and consider the disintegration given by Theorem 2.12.*

Then

$$\mathfrak{m}^+(V) \geq \int_Q \mathfrak{m}^+_\alpha(V \cap \overline{X_\alpha})\, \mathfrak{q}(\mathrm{d}\alpha). \tag{2.16}$$

Moreover for every $t > 0$:

$$\begin{aligned} \mathfrak{m}^+(V_t) &= \int_{Q_t} \mathfrak{m}^+_\alpha(V_t \cap \overline{X_\alpha})\, \mathfrak{q}(\mathrm{d}\alpha), \quad \textit{for all } t > 0, \\ \mathfrak{m}^-(V_t) &= \int_{Q_t} \mathfrak{m}^-_\alpha(V_t \cap \overline{X_\alpha})\, \mathfrak{q}(\mathrm{d}\alpha), \quad \textit{for all } t < 0, \end{aligned} \tag{2.17}$$

where

$$Q_t := \begin{cases} \{\alpha \in Q \colon \sup_{x \in X_\alpha} \tau_V(x) > t\}, & \textit{for } t > 0, \\ \{\alpha \in Q \colon \inf_{x \in X_\alpha} \tau_V(x) < t\}, & \textit{for } t < 0. \end{cases} \tag{2.18}$$

Finally a Bishop-Gromov type theorem is valid for $\mathfrak{m}^+(V_t)$, and analogously for $\mathfrak{m}^-(V_t)$ in case $t < 0$; for the proof see [16, Theorem 5.1].

Theorem 2.16 (Monotonicity formula for the area)**.** *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TCD}^e_p(0, N)$ and assume that the causally-reversed structure satisfies the same conditions. Let $V \subset X$ be a Borel, achronal, timelike complete subset. Then*

$$(0, \infty) \ni t \longmapsto \frac{\mathfrak{m}^+(V_t)}{t^{N-1}},$$

is monotone non-increasing.

Thanks to Proposition 2.10 and Proposition 2.15, a comparison of areas in the future of V with the one in its past will be possible.

3 One-dimensional inequalities

Leaving the Lorentzian setting for this short section, we will prove some one-dimensional inequalities for $\mathsf{CD}(0, N)$ spaces, i.e. for $(I, |\cdot|, h\, dx)$, where $I \subset \mathbb{R}$ is an open interval, $h : I \to (0, \infty)$ is semi-concave and satisfies the differential inequality (2.13) in distributional sense.

The Bishop-Gromov inequality (which, in this one-dimensional setting, reduces to an elementary computation) yields that the function

$$(0, \infty) \ni R \mapsto \frac{h(R)}{NR^{N-1}} \searrow$$

is monotone non-increasing. Assuming that $\sup I = \infty$, we denote

$$\theta = \theta_{h,N} := \lim_{R\to\infty} \frac{h(R)}{NR^{N-1}} = \inf_{R>0} \frac{h(R)}{NR^{N-1}}. \tag{3.1}$$

In the next lemma, we relate the asymptotic "area ratio" (3.1) to the more familiar asymptotic volume ratio,

$$\mathrm{AVR}_N(h) := \lim_{R\to\infty} \frac{1}{R^N} \int_0^R h(t)\, dt$$

which appeared in the recent metric geometry literature (see for instance [2, 12]).

Lemma 3.1. *Let $N \geq 1$ and let $h : [0,\infty) \to [0,\infty)$ be a continuous function such that*

$$t \mapsto \frac{h(t)}{Nt^{N-1}}$$

is decreasing on $(0,\infty)$ and

$$\lim_{t\to\infty} \frac{h(t)}{Nt^{N-1}} = \theta$$

for some $\theta \geq 0$. Then

$$\lim_{R\to\infty} \frac{1}{R^N} \int_0^R h(t)\, dt = \theta.$$

Proof. Set

$$f(t) := \frac{h(t)}{Nt^{N-1}}.$$

By assumption, f is decreasing on $(0,\infty)$ and $f(t) \to \theta$ as $t \to \infty$. Fix $\varepsilon > 0$. Then there exists $T > 0$ such that

$$\theta \leq f(t) \leq \theta + \varepsilon, \qquad \text{for all } t \geq T.$$

For $R > T$, we write

$$\frac{1}{R^N} \int_0^R h(t)\, dt = \frac{1}{R^N} \int_0^T h(t)\, dt + \frac{1}{R^N} \int_T^R Nt^{N-1} f(t)\, dt.$$

The first term in the right hand side tends to 0 as $R \to \infty$. For the second term, the bounds on f give

$$\theta \cdot \frac{1}{R^N} \int_T^R t^{N-1}\, dt \leq \frac{1}{R^N} \int_T^R t^{N-1} f(t)\, dt \leq (\theta + \varepsilon) \cdot \frac{1}{R^N} \int_T^R t^{N-1}\, dt.$$

Since

$$\frac{1}{R^N} \int_T^R Nt^{N-1}\, dt = \frac{R^N - T^N}{R^N} \longrightarrow 1 \qquad \text{as } R \to \infty,$$

it follows that

$$\lim_{R\to\infty} \frac{1}{R^N} \int_T^R t^{N-1} f(t)\, dt = \theta,$$

completing the proof. □

Lemma 3.2. *Let $X = (I, |\cdot|, h\, dx)$ be a one-dimensional m.m.s. verifying the $\mathsf{CD}(0,N)$ condition for some $N \in (1,\infty)$, where $I \subset \mathbb{R}$ is an open interval such that $h(t) > 0$ for all $t \in I$. Then:*

$$h^{\frac{1}{N-1}}(-s) + \frac{s}{R} h^{\frac{1}{N-1}}(R) \leq \left(1 + \frac{s}{R}\right) h^{\frac{1}{N-1}}(0), \tag{3.2}$$

provided $-s, 0, R$ lie in the interior of I and $R, s > 0$.

In particular:

- *If $N \in [2, \infty)$, then, for all $R, s > 0$:*

$$h(-s) \le \left(1 + \frac{s}{R}\right)^{N-1} h(0) - \left(\frac{s}{R}\right)^{N-1} h(R); \tag{3.3}$$

- *If $\sup I = \infty$, then*

$$I \subset \left[-\left(\frac{h(0)}{N\theta}\right)^{\frac{1}{N-1}}, +\infty\right). \tag{3.4}$$

Moreover, assuming that $0 \in I$ and denoting $-a := \inf I$, $\mathfrak{m} = h \cdot \mathcal{L}^1$, then

$$\mathfrak{m}([-a, 0]) \le h(0) \left(\frac{h(0)}{N\theta}\right)^{\frac{1}{N-1}}. \tag{3.5}$$

Proof. Let $s, R > 0$. The fact that $h : I \to (0, \infty)$ is semi-concave and satisfies the differential inequality (2.13) in distributional sense is equivalent to the concavity of the function $t \mapsto h^{1/N-1}(t)$. Such a concavity property implies:

$$\frac{h^{\frac{1}{N-1}}(0) - h^{\frac{1}{N-1}}(-s)}{s} \ge \frac{h^{\frac{1}{N-1}}(R) - h^{\frac{1}{N-1}}(0)}{R}.$$

Hence

$$h^{\frac{1}{N-1}}(-s) + \frac{s}{R}\left(h^{\frac{1}{N-1}}(R) - h^{\frac{1}{N-1}}(0)\right) \le h^{\frac{1}{N-1}}(0),$$

or

$$h^{\frac{1}{N-1}}(-s) + \frac{s}{R} h^{\frac{1}{N-1}}(R) \le h^{\frac{1}{N-1}}(0)\left(1 + \frac{s}{R}\right),$$

giving (3.2). The inequality (3.3) follows directly from (3.2) once recalling that the convexity of the function $t \mapsto t^{N-1}$, for $N \in [2, \infty)$, ensures that

$$a^{N-1} + b^{N-1} \le (a + b)^{N-1}, \quad \text{for all } a, b \ge 0. \tag{3.6}$$

Passing to the limit as $R \to \infty$ in (3.2) yields:

$$h^{\frac{1}{N-1}}(-s) + s(N\theta)^{\frac{1}{N-1}} \le h^{\frac{1}{N-1}}(0). \tag{3.7}$$

We deduce two consequences from (3.7). Firstly, that $s \le (h(0)/N\theta)^{\frac{1}{N-1}}$ and therefore

$$I \subset \{h > 0\} \subset \left[-\left(\frac{h(0)}{N\theta}\right)^{\frac{1}{N-1}}, +\infty\right).$$

Secondly, (3.7) implies trivially that:

$$h(-s) \le h(0).$$

Integrating from $-a$ to 0 yields:

$$\mathfrak{m}([-a, 0]) \le a h(0),$$

which, combined with (3.4), implies that

$$\mathfrak{m}([-a, 0]) \le h(0) \left(\frac{h(0)}{N\theta}\right)^{\frac{1}{N-1}}. \tag{3.8}$$

□

4 An area bound on equidistant sets in terms of asymptotic geometry

Definition 4.1. Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, globally hyperbolic, Lorentzian geodesic space satisfying $\mathsf{TCD}^e_p(0, N)$ and assume that the causally-reversed structure satisfies the same conditions. Let $V \subset X$ be an achronal, timelike complete set having finite area, i.e. $\mathfrak{m}^+(V) < \infty$, and empty global edge.

- We define the *asymptotic area growth of* V, denoted by Θ_V, as:
$$\Theta_V := \lim_{R\to+\infty} \frac{\mathfrak{m}^+(V_R)}{NR^{N-1}}, \tag{4.1}$$
where $V_R := \{\tau_V = R\}$, and the limit exists thanks to Theorem 2.16.
- Let $s > 0$. We define the *asymptotic area growth of* V *relative to* V_{-s}, denoted by $\Theta_V(s)$, the quantity:
$$\Theta_V(s) := \lim_{R\to+\infty} \frac{1}{NR^{N-1}} \int_{Q_{-s}} h_\alpha(R)\, \mathfrak{q}(\mathrm{d}\alpha). \tag{4.2}$$

The geometric interpretation of $\Theta_V(s)$ is to capture the asymptotic (in the future) area growth of V, restricting in the directions of the rays X_α that extend in the past up to $-s$.

Theorem 4.2 (An area bound in terms of the asymptotic area growth)**.** *Let* $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ *be a timelike non-branching, globally hyperbolic, Lorentzian geodesic space satisfying* $\mathsf{TCD}^e_p(0, N)$ *and assume that the causally-reversed structure satisfies the same conditions. Let* $V \subset X$ *be an achronal, timelike complete set having finite area, i.e.* $\mathfrak{m}^+(V) < \infty$*, and empty global edge.*
Assume moreover that the set V *has empty future cut-locus i.e.* $\mathfrak{b} = \emptyset$.
Parametrize each ray X_α *corresponding to* τ_V *so that* $X_\alpha(0) = X_\alpha \cap V$.

Then
$$\mathfrak{m}^-(V_{-s}) \leq \mathfrak{m}^+(V) - N\Theta_V(s)\, s^{N-1}, \quad \textit{for all } s > 0. \tag{4.3}$$

Proof. The assumption that $\mathfrak{b} = \emptyset$ is equivalent to require that the domain of definition of each ray X_α contains $(0, +\infty)$, which in turn implies that $Q_R = Q$, for all $R > 0$. The combination of Theorem 2.12 with (2.16),(2.17) and (3.3) implies that

$$\begin{aligned}
\mathfrak{m}^-(V_{-s}) &= \int_{Q_{-s}} \mathfrak{m}^-_\alpha(V_{-s} \cap \overline{X_\alpha})\, \mathfrak{q}(\mathrm{d}\alpha) \leq \int_{Q_{-s}} h_\alpha(-s)\, \mathfrak{q}(\mathrm{d}\alpha) \\
&\leq \Big(1 + \frac{s}{R}\Big)^{N-1} \int_{Q_{-s}} h_\alpha(0)\, \mathfrak{q}(\mathrm{d}\alpha) - \Big(\frac{s}{R}\Big)^{N-1} \int_{Q_{-s}} h_\alpha(R)\, \mathfrak{q}(\mathrm{d}\alpha) \\
&\leq \Big(1 + \frac{s}{R}\Big)^{N-1} \mathfrak{m}^+(V) - \Big(\frac{s}{R}\Big)^{N-1} \int_{Q_{-s}} h_\alpha(R)\, \mathfrak{q}(\mathrm{d}\alpha)
\end{aligned}$$

Sending $R \to +\infty$ and recalling (4.2) gives (4.3). □

We reinterpret the previous inequality in smooth setting.

Corollary 4.3 (An area estimate in terms of the asymptotic area growth – smooth setting)**.** *Let* (M, g) *be an* $(n + 1)$*-dimensional, globally hyperbolic, smooth Lorentzian manifold satisfying Penrose-Hawking's strong energy condition, i.e.* $\mathrm{Ric}_g(v, v) \geq 0$ *for all* $v \in TM$ *timelike. Let* $V \subset M$ *be a smooth, compact, Cauchy hypersurface.*

Assume that V *has empty future cut-locus, i.e. all the geodesics* X_α, $\alpha \in V$*, maximizing the time separation* τ_V *from* V *can be extended indefinitely in the future.*

Denote by Vol_{g_V} *the* n*-dimensional volume measure of* $(V, g|_{TV})$*. Then*
$$\mathrm{Vol}_{g_{V_{-s}}}(V_{-s}) \leq \mathrm{Vol}_{g_V}(V) - (n + 1)\Theta_V(s)\, s^n, \quad \textit{for a.e. } s > 0. \tag{4.4}$$

Proof. Corollary 4.3 follows directly from Theorem 4.2, once noticing that:

- (M, g, Vol_g) is, in particular, a timelike non-branching, globally hyperbolic, Lorentzian geodesic space satisfying the $\mathsf{TCD}^e_p(0, n+1)$ condition, together with its causally-reverse structure (see [15, Theorem 3.1], after [26, 33]);
- the smoothness of V implies that $\mathrm{Vol}_{g_V}(V)$ is finite and coincides with the future Minkowski content (see Remark 2.14); by Remark 2.11, if V has empty future cut-locus, then $\mathfrak{b} = \emptyset$ and therefore all the hypothesis of Theorem 4.2 are satisfied and (4.3) holds true. To conlude the validity of (4.4) we still need to justify its left-hand side. To this end, it is enough to refer to the discussion of Remark 2.14, obtaining (4.4) for a.e. $-s \in \tau_V(M)$. If $s \notin \tau_V(M)$, the inequality is trivially satisfied.

□

Proposition 4.4. *In the setting of Theorem 4.2, the area bound* (4.3) *is sharp for* $N = 2$. *More precisely, let* $\mathbb{M}^2$ *be the* 2*-dimensional Minkowski space with metric* $g = -dy^2 + dx^2$, *let* $\beta > 1$, *and consider the cone*

$$X = \{(x, y) \colon y \geq \beta \|x\|\},$$

endowed with the classical Minkowski metric and Lebesgue measure.
Then (4.3) *is an identity for* $V := \{(x, y) \in X \colon -y^2 + \|x\|^2 = -1\}$ *and* $N = 2$.

Proof. It is rather straightforward to compute the length of

$$S_t := \{(x, y) \in X \colon -y^2 + x^2 = -t^2\}.$$

Indeed, computing the intersection of $\{y = \beta x\}$ with $\{-y^2 + x^2 = -t^2\}$ and using the standard parametrization of S_t given by $x \mapsto (x, \sqrt{t^2 + x^2})$ gives

$$\ell(S_t) = \int_{-\frac{t}{\sqrt{\beta^2-1}}}^{\frac{t}{\sqrt{\beta^2-1}}} \frac{t}{\sqrt{x^2 + t^2}}\, dx = 2t \int_0^{\frac{1}{\beta^2-1}} \frac{1}{\sqrt{y^2 + 1}}\, dy = 2t \operatorname{arcsinh}\left(\frac{1}{\beta^2 - 1}\right).$$

Now, for any $s < 1$, $V_{-s} = S_{1-s}$ and $V_r = S_{1+r}$ so that

$$\Theta = \Theta(s) = \lim_{R\to\infty} \frac{2(R+1)}{2R} \operatorname{arcsinh}\left(\frac{1}{\beta^2 - 1}\right) = \operatorname{arcsinh}\left(\frac{1}{\beta^2 - 1}\right),$$

giving indeed that $\mathfrak{m}^-(V_{-s}) = \mathfrak{m}^+(V) - 2s\Theta$, for any $0 < s < 1$. □

Remark 4.5. In the higher-dimensional case $N = n + 1 \geq 3$, and under the same assumptions as in Proposition 4.4, the inequality (4.3) becomes strict. Consider the surface

$$S_t = \{(x, y) \in X \colon -y^2 + \|x\|^2 = -t^2\}. \tag{4.5}$$

If $y = \beta\|x\|$, and $(x, y) \in S_t$, then $\|x\| = t/\sqrt{\beta^2 - 1}$. For computing the area of S_t, we parametrize S_t via the graph of the function $u(x) = \sqrt{t^2 + \|x\|^2}$ over $x \in B_{t/\sqrt{\beta^2-1}}(0) \subset \mathbb{R}^n$, giving

$$\begin{aligned}
\mathrm{Area}(S_t) &= \int_{B_{t/\sqrt{\beta^2-1}}(0)} \sqrt{1 - |\nabla u|^2}\, dx_1 \cdots dx_n \\
&= t \int_{B_{t/\sqrt{\beta^2-1}}(0)} \frac{1}{\sqrt{t^2 + \|x\|^2}}\, dx_1 \cdots dx_n \\
&= t n \omega_n \int_0^{t/\sqrt{\beta^2-1}} \frac{r^{n-1}}{\sqrt{t^2 + r^2}}\, dr \\
&= t^n n \omega_n \int_0^{1/\sqrt{\beta^2-1}} \frac{s^{n-1}}{\sqrt{1 + s^2}}\, ds.
\end{aligned}$$

As before, calling $V := S_1$, it follows that $V_t = S_{1+t}$, $V_{-s} = S_{1-s}$ and

$$\Theta = \Theta(t) = \frac{n}{n+1}\omega_n \int_0^{1/\sqrt{\beta^2-1}} \frac{s^{n-1}}{\sqrt{1+s^2}} ds,$$

implying that the inequality in (4.3) is strict.

Notice that, in the proof of (4.3), we used inequality (3.6), which is an identity when $N = 2$, but becomes strict for $N > 2$ whenever $a, b \neq 0$.

Summarizing, Proposition 4.4 shows that (4.3) is sharp in the case $N = 2$. By contrast, in light of Remark 4.5, we expect the inequality (3.6) to be strict whenever $N > 2$.

5 Singularity theorems in terms of asymptotic expansion

In this section, we establish two singularity results in the non-smooth setting of globally hyperbolic Lorentzian geodesic spaces with non-negative timelike Ricci curvature. The first concerns volume singularities in the sense of [19], while the second addresses singularities in a more classical sense.

5.1 A volume singularity theorem

Complementing the classical Penrose–Hawking notion of singularity via causal geodesic incompleteness, García-Heveling [19] recently introduced the notion of a *volume singularity*. A spacetime (M, g) is said to be past volume singular if for every $\varepsilon > 0$ there exists a point $x \in M$ such that

$$\mathrm{Vol}_g\big(I^-(x)\big) < \varepsilon. \tag{5.1}$$

In a chronological spacetime, it suffices that there exists a point $x \in M$ whose chronological past has finite spacetime volume,

$$\mathrm{Vol}_g\big(I^-(x)\big) < \infty. \tag{5.2}$$

Indeed, finite past volume implies past volume singularity by a monotonicity argument along timelike past; see [19, Thm. 2.1]. The physical motivation for this notion is that if the past of an event has spacetime volume smaller than a Planck volume, then quantum-gravitational effects are expected to dominate, causing the classical spacetime description of general relativity to break down. From this perspective, volume singularities provide an alternative manifestation of a singular behavior, complementary to geodesic incompleteness, by signaling the transition from a classical to a quantum regime.

Before stating and proving the result, we fix some notation. Provided $\mathfrak{m}^+(V) < \infty$, it will be convenient to slightly change the normalizations of the conditional probabilities $h(\alpha, \cdot)$ in the following manner.

First, since Q can be identified with a measurable subset of V, we may, without loss of generality, parametrize each ray X_α so that

$$X_\alpha(0) = X_\alpha \cap V.$$

It then follows that

$$\int_Q h(\alpha, 0)\, \mathfrak{q}(\mathrm{d}\alpha) \le \mathfrak{m}^+(V) < \infty,$$

and hence $h(\cdot, 0) \in L^1(\mathfrak{q})$.

For our purposes, it will suffice to consider only those rays X_α for which $X_\alpha \cap V$ is a relative interior point of the ray. We denote by Q^- the set of indices with this property. Hence, since 0 is an interior point of the chosen parametrization, it follows that for $\mathfrak{q}$-a.e. $\alpha \in Q^-$,

$$h(\alpha, 0) > 0.$$

We may therefore normalize the densities so that

$$h(\alpha, 0) = 1 \qquad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q^-, \tag{5.3}$$

by correspondingly modifying the quotient measure according to

$$\bar{\mathfrak{q}} := h(\alpha, 0)\, \mathfrak{q}\llcorner_{Q^-} + \mathfrak{q}\llcorner_{Q\setminus Q^-}. \tag{5.4}$$

Note that $\bar{\mathfrak{q}}$ is no longer a probability measure. Nevertheless, it satisfies

$$\bar{\mathfrak{q}}(Q) \leq \mathfrak{m}^+(V). \tag{5.5}$$

Finally, we assume that the map

$$\alpha \mapsto \theta_\alpha^{-1}$$

belongs to $L^1(\bar{\mathfrak{q}}, Q^-)$, where $\theta_\alpha := \theta_{h_\alpha}$ is defined as in (3.1).

Observe that the normalization $h(\alpha, 0) = 1$ is also needed in order to fix, once and for all, the scaling of the one-dimensional asymptotic area ratio θ_α.

Theorem 5.1 (A synthetic volume singularity result)**.** *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, globally hyperbolic, Lorentzian geodesic space. Let $V \subset X$ be an achronal, timelike complete set having finite area, i.e. $\mathfrak{m}^+(V) < \infty$, and empty global edge. Assume moreover that:*

- *Both $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ and the causally-reverse structure satisfy the $\mathsf{TCD}_p^e(0, N)$ condition, for some $N \in (1, \infty)$;*
- *the set V has empty future cut-locus, i.e. $\mathfrak{b} = \emptyset$.*

Then:

$$\mathfrak{m}(I^-(V))^{N-1} \leq \mathfrak{m}^+(V)^{N-2} \left(\int_{Q^-} \frac{1}{N\theta_\alpha} \bar{\mathfrak{q}}(\mathrm{d}\alpha) \right). \tag{5.6}$$

In particular, if $\alpha \mapsto \theta_\alpha^{-1} \in L^1(\bar{\mathfrak{q}}, Q^-)$ then $\mathfrak{m}(I^-(V)) < \infty$.

Proof. First, consider only those rays X_α for which $X_\alpha \cap V$ is a relative interior point of the ray, and denote the corresponding set of indices by Q^-. Then the disintegration formula (2.12) yields

$$\mathfrak{m}(I^-(V)) = \int_{Q^-} \mathfrak{m}_\alpha\big((\mathfrak{a}_\alpha, 0]\big)\, \mathfrak{q}(\mathrm{d}\alpha).$$

Since $\mathfrak{b} = \emptyset$, it follows that all the transport rays X_α are defined indefinitely in the future. In particular, each ray X_α can be parametrized on an open interval containing $[0, \infty)$ so that $X_\alpha \cap V = X_\alpha(0)$. Since by assumption $X_\alpha(0)$ is an interior point of X_α, we infer that (3.5) holds true for $\mathfrak{q}$-a.e. $\alpha \in Q$, yielding:

$$\begin{aligned}
\mathfrak{m}(I^-(V)) &= \int_{Q^-} \mathfrak{m}_\alpha((\mathfrak{a}_\alpha, 0])\, \bar{\mathfrak{q}}(\mathrm{d}\alpha) \\
&\leq \int_{Q^-} h(\alpha, 0)^{1+\frac{1}{N-1}} (N\theta_\alpha)^{-\frac{1}{N-1}}\, \bar{\mathfrak{q}}(\mathrm{d}\alpha) \\
&= \int_{Q^-} (N\theta_\alpha)^{-\frac{1}{N-1}}\, \bar{\mathfrak{q}}(\mathrm{d}\alpha),
\end{aligned}$$

where, in the last identity, we used the normalization (5.3).
By Jensen's inequality, we bound the right hand side as follows:

$$\begin{aligned}\int_{Q^-} (N\theta_\alpha)^{-\frac{1}{N-1}}\,\bar{\mathfrak{q}}(\mathrm{d}\alpha) &= \bar{\mathfrak{q}}(Q^-)\int_{Q^-}\left(\frac{1}{N\theta_\alpha}\right)^{\frac{1}{N-1}}\left(\frac{\bar{\mathfrak{q}}}{\bar{\mathfrak{q}}(Q^-)}\right)(\mathrm{d}\alpha)\\ &\le \bar{\mathfrak{q}}(Q^-)^{\frac{N-2}{N-1}}\left(\int_{Q^-}\frac{1}{N\theta_\alpha}\bar{\mathfrak{q}}(\mathrm{d}\alpha)\right)^{\frac{1}{N-1}}\\ &\le \mathfrak{m}^+(V)^{\frac{N-2}{N-1}}\left(\int_{Q^-}\frac{1}{N\theta_\alpha}\bar{\mathfrak{q}}(\mathrm{d}\alpha)\right)^{\frac{1}{N-1}},\end{aligned}$$

where, the last inequality, we used (5.5). □

Remark 5.2. Finding lower bounds on the θ_α might be a challenging task. A possible approach would be to consider, for any $W \subset V$, the set of indices Q_W

$$Q_W = \{\alpha \in Q^- : X_\alpha \cap W \neq \emptyset\},$$

and

$$\Theta_W := \liminf_{R\to\infty} \frac{1}{NR^{N-1}} \int_{Q_W} h_\alpha(R)\,\bar{\mathfrak{q}}(\mathrm{d}\alpha).$$

If an estimate of the form

$$\Theta_W \ge c\,\mathfrak{m}^+(W)$$

holds, with $c > 0$ independent of W, then monotonicity implies that

$$\theta_\alpha \ge c.$$

In the case where (M, g) is an $(n + 1)$-dimensional globally hyperbolic smooth Lorentzian manifold, and $V \subset M$ is a smooth, compact, achronal hypersurface with empty global edge, every ray associated with τ_V intersects V. This allows one to take V itself as the quotient set and the induced hypersurface measure Vol_V as the quotient measure. In this setting, $X_\alpha \cap V$ is an interior point of the ray X_α for every α. Thus, we can choose

$$Q^- = V. \tag{5.7}$$

Moreover, the normalization introduced above appears entirely natural and yields the following disintegration formula (which, in this case, corresponds to Fubini-Tonelli's theorem):

$$\mathrm{Vol}_{g\llcorner I^\pm(V)} = \int_V h(\alpha,\cdot)\,\mathrm{Vol}_{g_V}(\mathrm{d}\alpha), \qquad h(\alpha,0) = 1,\ \mathrm{Vol}_V - \text{a.e.}\,\alpha, \tag{5.8}$$

where we have denoted by Vol_g (resp. Vol_{g_V}) the $(n+1)$-dimensional volume measure of (M, g) (resp. the n-dimensional volume measure of $(V, g|_{TV})$). Below we report the smooth version of Theorem 5.1.

Corollary 5.3 (A volume singularity result in the smooth setting)**.** *Let (M, g) be an $(n + 1)$-dimensional, globally hyperbolic, smooth Lorentzian manifold satisfying Penrose-Hawking's strong energy condition, i.e.* $\mathrm{Ric}_g(v, v) \ge 0$ *for all $v \in TM$ timelike. Let $V \subset M$ be a smooth, compact, Cauchy hypersurface. Moreover:*

- *Assume that V has empty future cut-locus.*
- *Assume that the map $\alpha \mapsto \theta_\alpha^{-1}$ belongs to $L^1(V, \mathrm{Vol}_{g_V})$, where $\theta_\alpha := \theta_{h_\alpha}$ is defined as in* (3.1)*.*

Then:

$$\mathrm{Vol}_g(I^-(V))^n \leq \mathrm{Vol}_{g_V}(V)^{n-1}\left(\int_Q \frac{1}{(n+1)\theta_\alpha}\mathrm{Vol}_{g_V}(\mathrm{d}\alpha)\right). \tag{5.9}$$

In particular, $\mathrm{Vol}_g(I^-(V)) < \infty$.

Moreover, for any (possibly non-smooth) past extension $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ *of* (M, g) *which is a timelike non-branching, globally hyperbolic, Lorentzian geodesic space, satisfying the* $\mathsf{TCD}^e_p(0, n+1)$ *condition, together with its causally-reversed structure, it holds that*

$$\mathfrak{m}(I^-(V))^n \leq \mathrm{Vol}_{g_V}(V)^{n-1}\left(\int_Q \frac{1}{(n+1)\theta_\alpha}\mathrm{Vol}_{g_V}(\mathrm{d}\alpha)\right), \tag{5.10}$$

in particular, $\mathfrak{m}(I^-(V)) < \infty$.

Proof. Corollary 5.3 follows directly from Theorem 5.1, once noticing that

- (M, g) is, in particular, a timelike non-branching, globally hyperbolic, Lorentzian geodesic space satisfying the $\mathsf{TCD}^e_p(0, n+1)$ condition, together with its causally-reverse structure (see [15, Theorem 3.1], after [26, 33]);
- since V is a Cauchy hypersurface, it has empty global edge;
- the smoothness of V implies that all the rays X_α, $\alpha \in V$, maximizing the time separation τ_V from V can be extended across V, i.e. so that $X_\alpha(0) = X_\alpha \cap V$ is an interior point of X_α;
- the disintegration formula

  $$(\mathrm{Vol}_g)|_{I^+(V)} = \int_V h_\alpha \,\mathrm{Vol}_V(\mathrm{d}\alpha)$$

  corresponds to choosing $Q = V$ and $\mathfrak{q} = \mathrm{Vol}_{g_V}$ in the disintegration theorem, which imply that $h_\alpha(0) = 1$ for Vol_{g_V}-a.e. $\alpha \in V$;
- The compactness of V implies that it is timelike complete and has finite n-dimensional volume. Moreover,

  $$\mathrm{Vol}^+_g(V) = \mathrm{Vol}_{g_V}(V),$$

  see Remark 2.14.

The final claim on the non-smooth extension is a consequence of the following observations: if X is a past extension of (M, g), then (M, g) is isomorphically embedded into X; moreover, denoting with I^+_M and I^+_X the future sets in M and X respectively, then $I^+_M(V) = I^+_X(V)$ and $I^-_M(V) \subset I^-_X(V)$. One can then repeat verbatim the proof of Theorem 5.1 with the aforementioned simplifications. □

5.2 The timelike asymptotic volume ratio and a conjecture

We define a timelike version of the classical Riemannian asymptotic volume ratio.

Definition 5.4. Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a measured Lorentzian pre-length space and $V \subset X$ be a Borel achronal set. Then the N-timelike asymptotic volume ratio of $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ w.r.t. to V is defined by

$$\mathrm{TAVR}_{V,N}(X) := \limsup_{R\to\infty} \frac{\mathfrak{m}(\{x \in I^+(V) \colon \tau_V(x) \leq R\})}{R^N}. \tag{5.11}$$

Lemma 5.5. *Let* $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ *be a timelike non-branching, globally hyperbolic, Lorentzian geodesic space satisfying* $\mathsf{TCD}^e_p(0, N)$ *and assume that the causally-reversed structure satisfies the same conditions. Let* $V \subset X$ *be a Borel achronal timelike complete subset. Then the* $\limsup$ *defining* $\mathrm{TAVR}_{V,N}(X)$ *in* (5.11) *can be promoted to a limit.*

Proof. It will be enough to show a similar monotonicity property to Theorem 2.16. Up to a measurable shift, it is not restrictive to assume that the parametrization of the transport ray all start from V at $s = 0$.

From Theorem 2.12 and the Riemannian Bishop-Gromov inequality satisfied by the conditional measures $\mathfrak{m}_\alpha$, we obtain for all $0 < S < R$:

$$\begin{aligned}\mathfrak{m}(\{x \in I^+(V)\colon \tau_V(x) \le R\} &= \int_Q \mathfrak{m}_\alpha(\{x \in I^+(V)\colon \tau_V(x) \le R\}\, \mathfrak{q}(\mathrm{d}\alpha) \\ &\le \left(\frac{R}{S}\right)^N \int_Q \mathfrak{m}_\alpha(\{x \in I^+(V)\colon \tau_V(x) \le S\}\, \mathfrak{q}(\mathrm{d}\alpha) \\ &= \left(\frac{R}{S}\right)^N \mathfrak{m}(\{x \in I^+(V)\colon \tau_V(x) \le S\},\end{aligned}$$

proving the claim. □

We conjecture that (5.6) admits the following sharp form, valid in every dimension.

Conjecture 5.6. *For every $N \ge 2$ there exists a constant $C_N > 0$ such that, if X and $V \subset X$ are as in Theorem 5.1, then*

$$C_N\ \mathrm{TAVR}_{V,N}(X)\ \mathfrak{m}(I^-(V))^{N-1} \le \mathfrak{m}^+(V)^N.$$

5.3 A timelike incompleteness theorem

The volume singularity theorem can be upgraded to a classical singularity theorem, provided a "pointwise" lower bound is enforced on the θ_α.

Theorem 5.7 (A synthetic singularity theorem, based on asymptotic volume growth)**.** *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \le, \tau)$ be a timelike non-branching, globally hyperbolic, Lorentzian geodesic space. Let $V \subset X$ be an achronal, timelike complete set having finite area, i.e. $\mathfrak{m}^+(V) < \infty$, and empty global edge. Assume moreover that:*

- *Both $(X, \mathsf{d}, \mathfrak{m}, \ll, \le, \tau)$ and the causally-reverse structure satisfy the $\mathsf{TCD}^e_p(0, N)$ condition, for some $N \in (1, \infty)$;*
- *there exists $c > 0$ such that, adopting the normalization (5.3), it holds*

$$\theta_\alpha \ge c > 0 \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q^-. \tag{5.12}$$

Then

$$\sup_{x \in I^-(V)} |\tau_V(x)| \le \left(\frac{1}{c}\right)^{\frac{1}{N-1}}. \tag{5.13}$$

In particular, $(X, \mathsf{d}, \mathfrak{m}, \ll, \le, \tau)$ is not past timelike geodesically complete.

Proof. For $\mathfrak{q}$-a.e. $\alpha \in Q^-$, the positivity of the one-dimensional asymptotic area ratio $\theta_\alpha > 0$ implies that the corresponding ray X_α has no future endpoint.

Since

$$\mathfrak{Q}_\sharp(\mathfrak{m} \llcorner \tau_V) \ll \mathfrak{q},$$

it follows that, up to a set of $\mathfrak{m}$-measure zero, the set $I^-(V)$ is foliated by integral curves of τ_V having no future endpoint and indexed by elements of Q^-.

Hence, combining the assumption (5.12) with the bound (3.4), we obtain that for $\mathfrak{q}$-a.e. α,

$$\sup_{x \in X_\alpha \cap I^-(V)} \big(-\tau_V(x)\big) \le \left(\frac{1}{c}\right)^{\frac{1}{N-1}}.$$

The disintegration formula (2.12) implies that

$$-\tau_V(x) \le \left(\frac{1}{c}\right)^{\frac{1}{N-1}}, \quad \text{for } \mathfrak{m}\text{-a.e. } x \in I^-(V).$$

Since $-\tau_V$ is lower semicontinuous on $I^-(V)$ and $\operatorname{supp}\mathfrak{m} = X$ by standing assumption, the above almost-everywhere estimate extends to all of $I^-(V)$. This concludes the proof of (5.13).

We finally show that X is not past timelike geodesically complete. Consider any timelike geodesic γ parametrized by arclength and defined on a maximal (on the left) interval $(-a, 0] \subset (-\infty, 0]$ such that $\gamma_0 \in V$. We claim that

$$a \le \left(\frac{1}{c}\right)^{\frac{1}{N-1}}.$$

Indeed, if by contradiction for some $s_0 \in [0, a)$

$$\tau(\gamma_{s_0}, \gamma_0) = s_0 > \left(\frac{1}{c}\right)^{\frac{1}{N-1}},$$

the very definition (2.6) of τ_V would imply

$$-\tau_V(\gamma_{s_0}) > \left(\frac{1}{c}\right)^{\frac{1}{N-1}},$$

contradicting (5.13). □

Recalling (5.7), (5.8), and arguing as in the proof of Corollary 5.3, we obtain the following corollary in the smooth setting.

Corollary 5.8 (A singularity result, based on asymptotic volume growth – smooth setting)**.** *Let (M, g) be an $(n+1)$-dimensional, globally hyperbolic, smooth Lorentzian manifold satisfying Penrose-Hawking's strong energy condition, i.e.* $\mathrm{Ric}_g(v, v) \ge 0$ *for all $v \in TM$ timelike. Let $V \subset M$ be a smooth, compact, Cauchy hypersurface. Assume moreover that there exists a constant $c > 0$ such that*

$$\theta_\alpha \ge c > 0 \qquad \textit{for } \mathrm{Vol}_V\textit{-a.e. } \alpha \in V.$$

Then

$$\sup_{x \in I^-(V)} |\tau_V(x)| \le \left(\frac{1}{c}\right)^{\frac{1}{n}}.$$

In particular, (M, g) is not past timelike geodesically complete.

Moreover, for any (possibly non-smooth) past extension $(X, \mathsf{d}, \mathfrak{m}, \ll, \le, \tau)$ of (M, g) which is a timelike non-branching, globally hyperbolic, Lorentzian geodesic space, satisfying the $\mathsf{TCD}^e_p(0, n+1)$ condition, together with its causally-reversed structure, it holds that

$$\sup_{x \in I^-_X(V)} |\tau_V(x)| \le \left(\frac{1}{c}\right)^{\frac{1}{n}},$$

where $I^-_X(V) \subset X$ is the chronological past of V in X.
In particular, $(X, \mathsf{d}, \mathfrak{m}, \ll, \le, \tau)$ is not past timelike geodesically complete.

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