A singularity theorem in terms of asymptotic expansion

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Abstract

We prove a sharp singularity theorem in which the classical focusing hypothesis of Hawking–Penrose theory is replaced by a condition on the asymptotic volume growth of timelike geodesics. More precisely, let (M, g) be a smooth, globally hyperbolic Lorentzian manifold satisfying the strong energy condition, and let $V \subset M$ be a smooth, compact Cauchy hypersurface. We associate to the future-directed timelike geodesics orthogonal to V asymptotic volume-expansion invariants θ_α, and show that a uniform lower bound $\theta_\alpha \geq c > 0$ forces a sharp upper bound on the time separation between V and its chronological past. In particular, (M, g) is past timelike geodesically incomplete.

The estimate is sharp and equality is attained by Lorentzian cones. Moreover, in the smooth setting we prove a rigidity theorem showing that equality characterizes these cone models.

The argument extends to globally hyperbolic Lorentzian length spaces satisfying the synthetic strong energy condition in the form of the Timelike Measure Contraction Property, $\mathsf{TMCP}^e(0, N)$. In this non-smooth setting we also obtain corresponding raywise and measure-rigidity statements.

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1 Introduction

The singularity theorems of Penrose [36] and Hawking [23] are among the foundational achievements of mathematical relativity. They demonstrate that spacetime singularities — understood as causal geodesic incompleteness — are not merely artifacts of highly symmetric solutions of Einstein's equations, but arise under broad and physically natural assumptions. Penrose's theorem predicts singularity formation in gravitational collapse, whereas Hawking's theorem applies to cosmological spacetimes undergoing expansion. Together with their many subsequent extensions and refinements, these results have profoundly shaped our understanding of black holes, cosmology, and the global structure of spacetime.

The underlying mechanism in both theorems is the combination of an energy condition with the focusing of causal geodesics. Concretely, one assumes that the spacetime is globally hyperbolic and that the Ricci curvature is nonnegative along timelike directions in Hawking's theorem and along null directions in Penrose's theorem. This is supplemented by a local geometric condition inducing the focusing of causal geodesics, typically formulated in terms of a mean-curvature bound on a spacelike hypersurface or the existence of trapped surfaces. Under these hypotheses, one concludes that the spacetime contains an incomplete causal geodesic. In the classical framework of general relativity, such geodesic incompleteness is regarded as the hallmark of a spacetime singularity.

In this paper we establish a singularity theorem of a different nature. Rather than imposing a local focusing condition on a Cauchy hypersurface, we assume a lower bound on the *asymptotic volume expansion* of the timelike geodesic congruence generated by it. Thus the mechanism leading to incompleteness is encoded in the large-time behaviour of the congruence, rather than in its infinitesimal behaviour at the initial hypersurface.

A second feature of the result is that its natural framework is the non-smooth theory of Lorentzian length spaces [25]. More precisely, the main theorem is proved under the synthetic strong energy condition $\mathsf{TMCP}^e(0,N)$ introduced by the authors in [16], which is rooted in the optimal transport characterizations of timelike Ricci curvature lower bounds established in the smooth setting by McCann [26] and by Suhr and the second author [32]. Recall that $\mathsf{TMCP}^e(0,N)$ is strictly weaker than requiring a timelike curvature-dimension condition $\mathsf{TCD}^e_p(0,N)$ and is the Lorentzian counterpart of the classical $\mathsf{MCP}(0,N)$ condition in metric geometry [34, 39]. The relevance of this weakening is twofold. On the one hand, the proof of the sharp singularity estimate ultimately relies only on a one-dimensional $\mathsf{MCP}(0,N)$ comparison. On the other hand, $\mathsf{TMCP}^e(0,N)$ is satisfied by important singular Lorentzian models, including generalized Lorentzian cones [12].

For the sake of exposition, let us first describe the result in the smooth setting. Let (M,g) be an $(n+1)$-dimensional, globally hyperbolic Lorentzian manifold satisfying the Hawking–Penrose strong energy condition

$$\mathrm{Ric}_g(v,v) \geq 0, \qquad \text{for every timelike vector } v \in TM. \tag{1.1}$$

Let $V \subset M$ be a smooth, compact Cauchy hypersurface and denote by $I^{\pm}(V)$ its chronological future and past. The signed time-separation from V is defined by

$$\tau_V(x) := \begin{cases} \sup_{y\in V} \tau(y,x), & x \in I^+(V), \\ -\sup_{y\in V} \tau(x,y), & x \in I^-(V), \\ 0, & \text{otherwise}. \end{cases} \tag{1.2}$$

Away from the cut locus, the integral curves of $-\nabla\tau_V$ are unit-speed timelike geodesics orthogonal to V. More generally, the localization theory for the time-separation function provides, up to a set of vanishing volume measure, a decomposition of $I^+(V) \cup I^-(V)$ into timelike transport rays X_α. Parametrizing each ray by $t = \tau_V$ and imposing $X_\alpha(0) \in V$, the spacetime volume disintegrates in the smooth case as

$$\mathrm{Vol}_{g \llcorner I^\pm(V)} = \int_V \mathfrak{m}_\alpha \, \mathrm{Vol}_{g_V}(\mathrm{d}\alpha), \tag{1.3}$$

where

$$\mathfrak{m}_\alpha = h_\alpha(t)\,\mathcal{L}^1, \qquad h_\alpha(0) = 1 \quad \text{for } \mathrm{Vol}_{g_V}\text{-a.e. } \alpha \in V. \tag{1.4}$$

The density h_α measures the infinitesimal transverse volume distortion along the ray X_α.

The strong energy condition implies a Bishop–Gromov-type monotonicity along each ray. In particular,

$$t \longmapsto \frac{h_\alpha(t)}{(n+1)t^n}$$

is non-increasing for $t > 0$. This motivates the definition of the one-dimensional asymptotic area ratio

$$\theta_\alpha := \lim_{t\to+\infty} \frac{h_\alpha(t)}{(n+1)t^n}. \tag{1.5}$$

The quantity θ_α records the asymptotic transverse volume expansion along the future-directed ray X_α. Our main observation is that positive asymptotic expansion in the future imposes a sharp upper bound on how far the same ray can extend into the past.

The main result

The one-dimensional core of the argument is particularly simple. In Lemma 3.2, we prove that if $(I, |\cdot|, h\,\mathcal{L}^1)$ satisfies $\mathsf{MCP}(0,N)$, if $\sup I = +\infty$, and if

$$\theta := \lim_{R\to+\infty} \frac{h(R)}{NR^{N-1}} > 0,$$

then

$$I \subset \left(-\left(\frac{h(0)}{N\theta}\right)^{\frac{1}{N-1}}, +\infty \right).$$

Combining this estimate with the localization of $\mathsf{TMCP}^e(0,N)$ along the transport rays of τ_V gives the following theorem (stated here for simplicity in the smooth setting, for the more general synthetic statement see Lemma 4.1).

Theorem 1.1 (Lemma 4.2)**.** *Let (M,g) be an $(n+1)$-dimensional, globally hyperbolic, smooth Lorentzian manifold satisfying Penrose-Hawking's strong energy condition* (1.1)*. Let $V \subset M$ be a smooth, compact, Cauchy hypersurface. Assume moreover that there exists a constant $c > 0$ such that*

$$\theta_\alpha \geq c > 0 \qquad \textit{for } \mathrm{Vol}_V\textit{-a.e. } \alpha \in V. \tag{1.6}$$

Then

$$\sup_{x\in I^-(V)} |\tau_V(x)| \leq \left(\frac{1}{(n+1)c}\right)^{\frac{1}{n}}.$$

In particular, (M,g) is not past timelike geodesically complete.

Moreover, for any (possibly non-smooth) past extension $(X,\mathsf{d},\mathfrak{m},\ll,\leq,\tau)$ of (M,g) which is a timelike non-branching, globally hyperbolic, Lorentzian geodesic space, satisfying the $\mathsf{TMCP}^e(0,n+1)$ condition, together with its causally-reversed structure, it holds that

$$\sup_{x\in I^-_X(V)} |\tau_V(x)| \leq \left(\frac{1}{(n+1)c}\right)^{\frac{1}{n}}, \tag{1.7}$$

where $I^-_X(V)\subset X$ is the chronological past of V in X.
In particular, $(X,\mathsf{d},\mathfrak{m},\ll,\leq,\tau)$ is not past timelike geodesically complete.

The second part of the theorem may be viewed as an inextendibility statement. If a smooth spacetime satisfying the assumptions above is embedded into a possibly non-smooth past extension which is timelike non-branching, globally hyperbolic and satisfies $\mathsf{TMCP}^e(0,n+1)$ in both time orientations, then the same upper bound (1.7) holds in the extension. Hence the singularity cannot be removed by passing to a lower-regularity extension within this synthetic curvature class.

Sharpness and rigidity

The estimate (1.7) is sharp. For $N>2$, let $(Z,\mathsf{d}_Z,\mathfrak{m}_Z)$ be a compact $\mathsf{CD}(-(N-2),N-1)$ metric measure space and consider the Minkowski $(N-1)$-cone

$$\mathcal{M}^{N-1}(Z) = {}^-[0,+\infty)\times^{N-1}_{\mathrm{id}} Z, \qquad \mathrm{d}\mathfrak{m}_{\mathcal{M}} = r^{N-1}\,\mathrm{d}r\,\mathrm{d}\mathfrak{m}_Z.$$

The results of [12] imply that this space satisfies $\mathsf{TMCP}^e(0,N)$. For the slice

$$V=\{r_0\}\times Z,$$

the transport rays are radial and

$$h_\alpha(s) = \left(1+\frac{s}{r_0}\right)^{N-1}, \qquad \theta_\alpha = \frac{1}{N r_0^{N-1}}.$$

Therefore

$$\sup_{I^-(V)} |\tau_V| = r_0 = \left(\frac{1}{N\theta_\alpha}\right)^{\frac{1}{N-1}},$$

so equality is attained.

In the smooth category, these models are rigid. More precisely, assume that (M,g) is connected, V is connected, and equality holds in (1.7). The asymptotic lower bound first implies the mean-curvature estimate

$$H_V \geq \frac{n}{T}, \qquad T := \left(\frac{1}{(n+1)c}\right)^{1/n}.$$

Equality in the time-separation bound produces a maximal past-directed V-ray of length T. A Lorentzian maximal-distance rigidity theorem (see [20]), combined with the Raychaudhuri–Riccati equation, then propagates equality to the whole spacetime and yields

$$(M,g)\simeq \left((0,+\infty)\times V,\ -\mathrm{d}r^2 + \frac{r^2}{T^2}g_V\right).$$

Thus equality in the smooth theorem characterizes Lorentzian cones; in particular, if $g_Z := T^{-2}g_V$, then

$$\mathrm{Ric}_{g_Z} \geq -(n-1)g_Z.$$

In the non-smooth setting of $\mathsf{TMCP}^e(0, N)$ spaces, a meaningful rigidity theory survives at the level of the conditional measures, see Proposition 4.6. We prove that a maximal transport ray has the exact model density on its future half,

$$h_\alpha(t) = \left(1 + \frac{t}{T}\right)^{N-1}, \qquad t \geq 0,$$

and obtain quantitative bounds on its past density. If maximality holds for almost every ray and the corresponding sharp past-volume equality is satisfied, then the full conditional densities are conical. Under an additional uniqueness/common-tip assumption, inspired by Ohta's maximal-diameter rigidity theorem for $\mathsf{MCP}(N-1, N)$ spaces [35, Theorem 5.5], this yields a topological measure-cone structure.

Related literature

The present work belongs to the rapidly developing theory of synthetic Lorentzian geometry. The Timelike Curvature–Dimension condition and its measure-contraction counterpart were introduced in [16], building on the optimal-transport characterizations of timelike Ricci curvature obtained in the smooth setting by McCann [26] and by Mondino–Suhr [32]; see also the survey [15]. Among the applications of this theory are synthetic versions of Hawking's singularity theorem, timelike Bishop–Gromov and Bonnet–Myers inequalities, and Lorentzian isoperimetric-type inequalities [16, 17].

The use of TMCP^e rather than the stronger TCD^e_p condition is particularly natural here. Localization of $\mathsf{TMCP}^e(K, N)$ yields one-dimensional $\mathsf{MCP}(K, N)$ densities, and the sharp lifetime estimate proved in this paper is already encoded in this weaker one-dimensional comparison. This also makes the theorem applicable to generalized Lorentzian cones constructed in [12], which provide the equality models for our result.

Related developments include inextendibility theorems for Lorentzian pre-length spaces satisfying synthetic lower bounds on timelike sectional curvature [21, 1, 4], singularity theorems in the framework of closed cone structures [27], and further comparison and rigidity results in synthetic Lorentzian geometry; see, among others, [5, 3, 8, 6, 9].

A notable feature of the synthetic framework is its ability to accommodate singular spacetimes lying beyond the scope of classical Lorentzian geometry. Examples include Penrose's impulsive gravitational waves [31] and spacetimes endowed with Lipschitz Lorentzian metrics whose timelike Ricci curvature is bounded below in the sense of distributions [7]. The present paper contributes to this program by identifying a new, genuinely asymptotic mechanism for timelike geodesic incompleteness and by determining its sharp models and rigidity properties.

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2 Preliminaries

2.1 Lorentzian geodesic spaces and synthetic Ricci lower bounds

2.1.1 Some basics on Lorentzian pre-length spaces

The general framework for the paper is the one of Lorenzian pre-length spaces introduced by Kunzinger-Sämann in [25], inspired by earlier works of Kronheimer-Penrose [24] on causal spaces and of Busemann [10] on timelike spaces. An alternative approach put forward by Minguzzi-Suhr [30] derived many topological properties out of basic assumptions on the time-separation. Variants have been studied by several authors with different motivations, let us mention [26, 8, 11, 3, 33, 14].

Recall that $(X, \mathsf{d}, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* provided: $(X, \ll, \leq)$ is a *causal space* (i.e., $\leq$ is a preorder and $\ll$ a transitive relation contained in $\leq$) additionally equipped with a proper metric d (i.e., closed and bounded subsets are compact) and a lower semicontinuous function $\tau : X \times X \to [0, \infty]$, called *time-separation function*, satisfying

$$\begin{aligned} \tau(x,y) + \tau(y,z) \leq \tau(x,z) \quad \forall x \leq y \leq z \quad &\text{reverse triangle inequality} \\ \tau(x,y) = 0, \text{ if } x \not\leq y, \quad \tau(x,y) > 0 \Leftrightarrow x \ll y. & \end{aligned}$$

We write $x < y$ whenever $x \leq y$ and $x \neq y$. We say that x and y are *timelike* (resp. *causally*) related if $x \ll y$ (resp. $x \leq y$).

Let $A \subset X$ be an arbitrary subset. The *chronological* (resp. *causal*) future of A is defined as

$$\begin{aligned} I^+(A) &:= \{y \in X : \exists x \in A \text{ such that } x \ll y\}, \\ J^+(A) &:= \{y \in X : \exists x \in A \text{ such that } x \leq y\}. \end{aligned}$$

Analogously, one defines the *chronological* (resp. *causal*) past of A.

If $A = \{x\}$ is a singleton, we will slightly abuse notation and write $I^\pm(x)$ (resp. $J^\pm(x)$) instead of $I^\pm(\{x\})$ (resp. $J^\pm(\{x\})$).

We say that $(X, \mathsf{d}, \ll_r, \leq_r, \tau_r)$ is the *causally-reversed structure* of $(X, \mathsf{d}, \ll, \leq, \tau)$ if

$$x \ll_r y \iff y \ll x, \quad x \leq_r y \iff y \leq x, \quad \tau_r(x,y) = \tau(y,x).$$

The set of (resp. *timelike*) *geodesics* is defined as:

$$\begin{aligned} \mathrm{Geo}(X) &:= \{\gamma \in C([0,1], X) \,:\, \tau(\gamma_s, \gamma_t) = (t-s)\, \tau(\gamma_0, \gamma_1), \ \forall s < t\}, \\ \mathrm{TGeo}(X) &:= \{\gamma \in \mathrm{Geo}(X) \,:\, \tau(\gamma_0, \gamma_1) > 0\}. \end{aligned}$$

Definition 2.1 (Timelike non-branching)**.** A Lorentzian pre-length space $(X, \mathsf{d}, \ll, \leq, \tau)$ is said to be *forward timelike non-branching* if and only if for any $\gamma^1, \gamma^2 \in \mathrm{TGeo}(X)$, it holds:

$$\exists\, \bar{t} \in (0,1) \text{ such that } \ \forall t \in [0, \bar{t}\,] \quad \gamma^1_t = \gamma^2_t \quad \Longrightarrow \quad \gamma^1_s = \gamma^2_s, \quad \forall s \in [0,1].$$

X is said to be *backward timelike non-branching* if the causally-reversed structure is forward timelike non-branching. In case X is both forward and backward timelike non-branching it is said *timelike non-branching.*

Concerning the causal ladder (see [25, 29] for more details), we will only consider *Lorentzian geodesic spaces,* i.e. Lorentzian pre-length spaces $(X, \mathsf{d}, \ll, \leq, \tau)$ that additionally are:

- d-*Compatible:* every $x \in X$ admits a neighbourhood U and a constant C such that $L_{\mathsf{d}}(\gamma) \leq C$ for every causal curve γ contained in U;

- *Geodesic:* for all $x, y \in X$ with $x < y$ there is a future-directed causal curve γ from x to y with $\tau(x,y) = L_\tau(\gamma)$.

We consider the following version of global hyperbolicity that fits with the previous literature (see [29, Cor. 3.8]): a Lorentzian geodesic space $(X, \mathsf{d}, \ll, \leq, \tau)$ is called

- *Causal*: if $\leq$ is also antisymmetric, i.e. $\leq$ is an order;
- *Globally hyperbolic*: if it is causal and for every $x, y \in X$ the causal diamond $J^+(x) \cap J^-(y)$ is compact in X.

2.1.2 Timelike optimal transport in a Lorentzian pre-length space

Let $\mathcal{P}(X)$ be the set of Borel probability measures over X. Denote by $P_i : X \times X \to X$, $i = 1, 2$ the projection map and let $(P_i)_\sharp : \mathcal{P}(X \times X) \to \mathcal{P}(X)$ be the corresponding push-forward map.
Given $\mu, \nu \in \mathcal{P}(X)$, consider

$$\begin{aligned}
\Pi(\mu,\nu) &:= \{\pi \in \mathcal{P}(X \times X) \, : \, (P_1)_\sharp \pi = \mu, \, (P_2)_\sharp \pi = \nu\}, \\
\Pi_\leq(\mu,\nu) &:= \{\pi \in \Pi(\mu,\nu) \, : \, \pi(X^2_\leq) = 1\}, \\
\Pi_\ll(\mu,\nu) &:= \{\pi \in \Pi(\mu,\nu) \, : \, \pi(X^2_\ll) = 1\},
\end{aligned}$$

where $X^2_\leq := \{(x,y) \in X^2 \, : \, x \leq y\}$ and $X^2_\ll := \{(x,y) \in X^2 \, : \, x \ll y\}$.

We next recall the notion of p-Lorentz-Wasserstein distance, following the convention in [16]. The definition below extends to Lorentzian pre-length spaces the corresponding notion given in the smooth Lorentzian setting in [18] (see also [26, 32], and [40] for $p = 1$).

Definition 2.2. Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $p \in (0, 1]$. Given $\mu, \nu \in \mathcal{P}(X)$, the *p-Lorentz-Wasserstein distance* is defined by

$$\ell_p(\mu,\nu) := \sup_{\pi \in \Pi_\leq(\mu,\nu)} \left(\int_{X \times X} \tau(x,y)^p \, \pi(\mathrm{d}x\mathrm{d}y) \right)^{1/p}. \tag{2.1}$$

When $\Pi_\leq(\mu,\nu) = \emptyset$ we set $\ell_p(\mu,\nu) := -\infty$.

A coupling $\pi \in \Pi_\leq(\mu,\nu)$ maximising in (2.1) is said ℓ_p*-optimal*. The set of ℓ_p-*optimal* couplings from μ to ν is denoted by $\Pi^{p\text{-opt}}_\leq(\mu,\nu)$. We also set

$$\Pi^{p\text{-opt}}_\ll(\mu,\nu) := \Pi^{p\text{-opt}}_\leq(\mu,\nu) \cap \Pi_\ll(\mu,\nu)$$

the family of *timelike* ℓ_p-optimal couplings.

By gluing of couplings, one can prove that the p-Lorentz-Wasserstein distance ℓ_p satisfies the reverse triangle inequality on causally related measures (see [16, Proposition 2.5]), thus lifting the Lorentzian metric structure to the space of probability measures.

We conclude this subsection by recalling the notion of ℓ_p-geodesic $(\mu_s)_{s \in [0,1]} \subset \mathcal{P}(X)$. The evaluation map is defined by

$$\mathrm{e}_t : C([0,1], X) \to X, \qquad \gamma \mapsto \mathrm{e}_t(\gamma) := \gamma_t, \qquad \forall t \in [0,1]. \tag{2.2}$$

Definition 2.3 (ℓ_p-optimal dynamical plans and ℓ_p-geodesics)**.** Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $p \in (0, 1]$. We say that $\eta \in \mathcal{P}(\mathrm{Geo}(X))$ is an ℓ_p*-optimal dynamical plan* from $\mu_0 \in \mathcal{P}(X)$ to $\mu_1 \in \mathcal{P}(X)$ if

$$(\mathrm{e}_0, \mathrm{e}_1)_\sharp \eta \in \Pi^{p\text{-opt}}_\leq\big((\mathrm{e}_0)_\sharp \eta, (\mathrm{e}_1)_\sharp \eta\big). \tag{2.3}$$

The collection of all ℓ_p-optimal dynamical plans from μ_0 to μ_1 is denoted by $\mathrm{OptGeo}_{\ell_p}(\mu_0,\mu_1)$.
We say that a curve $[0,1] \ni t \mapsto \mu_t \in \mathcal{P}(X)$ is an ℓ_p*-geodesic* if there exists $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0,\mu_1)$ such that

$$\mu_t = (\mathrm{e}_t)_\sharp \eta, \qquad \forall t \in [0,1].$$

Observe that if $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$, then the associated curve

$$\mu_t := (\mathrm{e}_t)_\sharp \eta, \qquad \forall t \in [0,1],$$

is continuous with respect to the narrow topology and satisfies

$$\ell_p(\mu_s, \mu_t) = (t-s)\,\ell_p(\mu_0, \mu_1), \qquad \forall\, 0 \le s \le t \le 1.$$

Finally, recall that if X is a globally hyperbolic Lorentzian geodesic space, $\mu_0, \mu_1 \in \mathcal{P}(X)$ have compact support and $\Pi_\le(\mu_0, \mu_1) \neq \emptyset$, then there always exists an ℓ_p-optimal dynamical plan $\eta \in \mathrm{OptGeo}_{\ell_p}(\mu_0, \mu_1)$ (and hence an ℓ_p-geodesic) connecting μ_0 to μ_1; see [16, Proposition 2.33] for the proof and further properties of ℓ_p-geodesics.

2.1.3 Synthetic timelike Ricci lower bounds

A *measured Lorentzian pre-length space* $(X, \mathsf{d}, \mathfrak{m}, \ll, \le, \tau)$ is a Lorentzian pre-length space endowed with a Radon non-negative measure $\mathfrak{m}$. We say that $(X, \mathsf{d}, \mathfrak{m}, \ll, \le, \tau)$ is globally hyperbolic (resp. geodesic) if $(X, \mathsf{d}, \ll, \le, \tau)$ is so.

We next recall the definition of the Timelike Measure Contraction Property, denoted by $\mathsf{TMCP}^e(0,N)$, as introduced in [16] (after [26] and [32]). Here $K \in \mathbb{R}$ stands for a synthetic lower bound on the timelike Ricci curvature, and $N \in (0,\infty)$ stands for a synthetic upper bound on the dimension. Since in this paper we will only consider the case on non-negative timelike Ricci curvature, we will focus on the case $K = 0$. We refer the reader to [16] for more details and to [15] for a concise overview.

The definition is based on a suitable concavity property of the entropy functional, which we now briefly recall. Given $\mu \in \mathcal{P}(X)$, its Boltzmann-Shannon entropy w.r.t. $\mathfrak{m}$ is given by

$$\mathrm{Ent}(\mu|\mathfrak{m}) = \int_M \rho \log \rho \ \mathfrak{m},$$

if $\mu = \rho\,\mathfrak{m}$ is absolutely continuous with respect to $\mathfrak{m}$ and $(\rho \log \rho)_+$ is $\mathfrak{m}$-integrable. Otherwise we set $\mathrm{Ent}(\mu|\mathfrak{m}) = +\infty$. We set

$$\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})) := \{\mu \in \mathcal{P}(X) \,:\, \mathrm{Ent}(\mu|\mathfrak{m}) \in \mathbb{R}\},$$

to be the finiteness domain of the entropy. Finally, recall the definition of the Shannon entropy power:

$$U_N(\mu|\mathfrak{m}) := \exp\left(-\frac{\mathrm{Ent}(\mu|\mathfrak{m})}{N}\right), \quad \text{for all } \mu \in \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})). \tag{2.4}$$

Definition 2.4 ($\mathsf{TMCP}^e(0,N)$ condition, [16])**.** Fix $p \in (0,1)$, $K \in \mathbb{R}$, $N \in (0,\infty)$. The measured Lorentzian pre-lengh space $(X, \mathsf{d}, \mathfrak{m}, \ll, \le, \tau)$ satisfies $\mathsf{TMCP}^e(0,N)$ if and only if for any $\mu_0 \in \mathcal{P}_c(X) \cap \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))$ and for any $x_1 \in X$ such that $x \ll x_1$ for μ_0-a.e. $x \in X$, there exists an ℓ_p-geodesic $(\mu_t)_{t\in[0,1]}$ from μ_0 to $\mu_1 = \delta_{x_1}$ such that

$$U_N(\mu_t|\mathfrak{m}) \ge (1-t)\,U_N(\mu_0|\mathfrak{m}), \quad \forall t \in [0,1). \tag{2.5}$$

Remark 2.5. The validity of the $\mathsf{TMCP}^e(K,N)$ condition is independent of the choice of $p \in (0,1)$ in Definition 2.4. Indeed, for any $p, q \in (0,1)$, a curve $(\mu_t)_{t\in[0,1]}$ with $\mu_1 = \delta_{\bar{x}}$ is an ℓ_p-geodesic if and only if it is an ℓ_q-geodesic. This follows directly from the definition of ℓ_p-geodesics (see Definition 2.3) and from the observation that the unique coupling between μ_0 and $\mu_1 = \delta_{\bar{x}}$ is

$$\pi = \mu_0 \otimes \delta_{\bar{x}},$$

which is therefore automatically optimal for every $p \in (0,1)$. Hence the class of admissible geodesics entering Definition 2.4 does not depend on p, and neither does the $\mathsf{TMCP}^e(K,N)$ condition.

2.2 Time separation function from achronal sets

Another key object in our analysis is the synthetic analogue of the gradient flow lines associated with the time-separation function from a given set V. We now recall some basic constructions.

Standing assumptions. From now on we will always assume that $(X, \mathsf{d}, \ll, \leq, \tau)$ is a globally hyperbolic Lorentzian geodesic space (see subsection 2.1.1 for the definitions).

Under such standing assumptions, the time separation $\tau : X \times X \to \mathbb{R}$ is continuous (see [25, Theorem 3.28]). A subset $V \subset X$ is called *achronal* if $x \not\ll y$ for every $x, y \in V$. In particular, if V is achronal, then $I^+(V) \cap I^-(V) = \emptyset$, so we can define the *signed time-separation* to V, $\tau_V : X \to [-\infty, +\infty]$, by

$$\tau_V(x) := \begin{cases} \sup_{y \in V} \tau(y, x), & \text{for } x \in I^+(V) \\ -\sup_{y \in V} \tau(x, y), & \text{for } x \in I^-(V) \\ 0 & \text{otherwise} \end{cases}. \tag{2.6}$$

Note that τ_V is lower semi-continuous on $I^+(V)$ as supremum of continuous functions, and is upper semi-continuous on $I^-(V)$.

To introduce a synthetic analogue of the integral curves of $-\nabla\tau_V$, some preparatory tools are needed. We briefly recall the construction and refer to [16, Sec. 4.1] for a more detailed treatment.

Definition 2.6 (Future timelike complete (FTC) subsets, [19])**.** A subset $V \subset X$ is *future timelike complete* (FTC), if for each point $x \in I^+(V)$, the intersection $J^-(x) \cap V \subset V$ has compact closure (w.r.t. d) in V. Analogously, one defines *past timelike completeness* (PTC). A subset that is both FTC and PTC is called *timelike complete.*

Lemma 2.7 (Lemma 4.1, [16])**.** *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a globally hyperbolic Lorentzian geodesic space and let $V \subset X$ be an achronal FTC (resp. PTC) subset. Then for each $x \in I^+(V)$ (resp. $x \in I^-(V)$) there exists a point $y_x \in V$ with $\tau_V(x) = \tau(y_x, x) > 0$ (resp. $\tau_V(x) = -\tau(x, y_x) < 0$).*

Moreover for all $x, z \in I^+(V) \cup V$,

$$\tau_V(z) - \tau_V(x) \geq \tau(y_x, z) - \tau(y_x, x) \geq \tau(x, z), \tag{2.7}$$

provided $(x, z) \in X^2_{\leq}$. An analogous statement is valid for $x \in I^-(V)$.

The inequality (2.7) can be restated by saying that τ_V is timelike reverse 1-Lipschitz on $I^+(V)$ and, separately, on $I^-(V)$. As shown in [17], this permits to study the integral lines of maximal steep of $-\tau_V$ on $I^+(V)$ and, separately, on $I^-(V)$ giving a disintegration formula for the reference measure $\mathfrak{m}$ whose conditional measures localize the eventual Ricci curvature lower bound to the integral lines of $-\tau_V$.

For the scope of this note, the interest will be in performing this analysis jointly on $I^+(V)$ and $I^-(V)$. To start, for ease of writing, we will use the following notation

$$I^\pm(V) := (I^+(V) \cup I^-(V) \cup V).$$

In general τ_V fails to be reverse 1-Lipschitz on $I^\pm(V)$: consider for instance in Minkowski spacetime $V = \{0\}$, the origin. Let z be any point in the future light cone of 0 and x be any point in the past light cone of 0 additionally being in the timelike past of z. Then

$$\tau_V(z) - \tau_V(x) = \tau(0, z) + \tau(x, 0) = 0 < \tau(x, z),$$

contradicting the reverse 1-Lipschitz property.

Nevertheless, as discussed below, the reverse 1-Lipschitzianity holds in some special cases of interest.

Definition 2.8 (Empty global edge)**.** Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a globally hyperbolic Lorentzian geodesic space. Let $V \subset X$ be an achronal set. We say that V has *empty global edge* if for any $x \in I^-(V)$ and $z \in I^+(V)$, any $\gamma \in \mathrm{TGeo}(X)$ connecting x to z intersects V.

Lemma 2.8 may be viewed as a global counterpart of the condition that an achronal set has empty edge. Recall that the *edge* of an achronal set $S \subset X$ consists of those points $p \in S$ such that every neighbourhood $U \subset X$ of p contains points $p^+ \in I^+(p)$ and $p^- \in I^-(p)$, together with a timelike curve connecting p^- to p^+ that does not intersect S. Clearly, if V has empty global edge then its edge is empty as well, however the converse implication may fail.

The notion of the edge of an achronal set is classical in Lorentzian geometry and causality theory; see, for instance, the foundational references [37, 22] or the more recent survey [28].

Remark 2.9 (Sufficient conditions for $V \subset X$ to have empty global edge)**.**

- Let (M, g) be a spacetime with a continuous Lorentzian metric. If V is a Cauchy hypersurface, then V has empty global edge. For properties of Cauchy hypersurfaces in C^0-spacetimes, see [38].
- Recall the definition of future Cauchy development $D^+(V)$ of V:

  $$D^+(V) := \{p \in X \colon \text{every past inext. timelike curve through } p \text{ intersects } V\}.$$

  If $I^+(V) \supset D^+(V)$, then V has empty global edge.

Proposition 2.10. *Let $(X, \mathsf{d}, \ll, \leq, \tau)$ be a globally hyperbolic Lorentzian geodesic space and let $V \subset X$ be an achronal, timelike complete subset.*

If V has empty global edge, then τ_V is reverse 1-Lipschitz over $I^\pm(V)$.

Proof. It suffices to prove that, for all $x \leq z$ with $x \in I^-(V)$ and $z \in I^+(V)$:

$$\tau_V(z) - \tau_V(x) \geq \tau(x, z).$$

If $\tau(x, z) = 0$, the claim is verified as $\tau_V(x) \leq 0$ and $\tau_V(z) \geq 0$. If $\tau(x, z) > 0$, then there exists $\gamma \in \mathrm{TGeo}(X)$ with $\gamma_0 = x$ and $\gamma_1 = z$. Since V is achronal with empty geodesic boundary, there exists a unique $s \in (0, 1)$ such that $\gamma_s \in V$. Then $\tau(x, z) = \tau(x, \gamma_s) + \tau(\gamma_s, z) \leq \tau_V(z) - \tau_V(x)$; by definition indeed $-\tau_V(x) = \sup_{y \in V} \tau(x, y)$. The claim is therefore proved. $\square$

In [17, Section 6], the localization of a general reverse 1-Lipschitz function is discussed in detail. Hence we now confine ourselves to reporting the main definitions and results.

Given $V \subset X$, an achronal set with empty global edge, define:

$$\begin{aligned} \Gamma_V := \{&(x, z) \in I^\pm(V)^2 \cap X^2_\leq : \tau_V(z) - \tau_V(x) = \tau(x, z) > 0\} \\ &\cup \{(x, x) : x \in I^\pm(V)\}. \end{aligned} \tag{2.8}$$

The monotonicity of Γ_V ensures that pairs of points belonging to Γ_V are aligned along geodesics: if $(x, z) \in \Gamma_V$ with $x \neq z$ and $x \in I^+(V)$, then there exist $y \in V, \gamma \in \mathrm{TGeo}(X)$ and $t \in (0, 1)$ such that $y = \gamma_0,\ x = \gamma_t,\ z = \gamma_1$, and

$$\tau(y, \gamma_s) = \tau_V(\gamma_s) \quad \forall s \in [0, 1], \qquad (\gamma_s, \gamma_t) \in \Gamma_V, \quad \forall s \in [0, t].$$

An analogous property holds true if $z \in I^-(V)$. Next, define

$$\Gamma_V^{-1} := \{(x, y) : (y, x) \in \Gamma_V\}$$

and consider the *transport relation* R_V and the *transport set with endpoints* $\mathcal{T}_V^{end}$

$$R_V := \Gamma_V \cup \Gamma_V^{-1}, \qquad \mathcal{T}_V^{end} := P_1(R_V \setminus \{x = y\}). \tag{2.9}$$

The transport relation R_V will be an equivalence relation on a suitable subset of $\mathcal{T}_V^{end}$, once initial and final points of the integral lines are removed. The sets

$$\begin{aligned} \mathfrak{a} = \mathfrak{a}(\mathcal{T}_V^{end}) &:= \{x \in \mathcal{T}_V^{end} \colon \nexists y \in \mathcal{T}_V^{end} \ s.t. \ (y,x) \in \Gamma_V, y \neq x\} \\ \mathfrak{b} = \mathfrak{b}(\mathcal{T}_V^{end}) &:= \{x \in \mathcal{T}_V^{end} \colon \nexists y \in \mathcal{T}_V^{end} \ s.t. \ (x,y) \in \Gamma_V, y \neq x\}, \end{aligned} \tag{2.10}$$

are the *initial* and *final points*, respectively. The *transport set without endpoints* is defined by:

$$\mathcal{T}_V := \mathcal{T}_V^{end} \setminus (\mathfrak{a}(\mathcal{T}_V^{end}) \cup \mathfrak{b}(\mathcal{T}_V^{end})). \tag{2.11}$$

Remark 2.11. The set $\mathfrak{a}(\mathcal{T}_V^{end})$ shall be understood as the past cut-locus of V while $\mathfrak{b}(\mathcal{T}_V^{end})$ the future one, being indeed those points, in the smooth scenario, where the geodesics stops to be maximizing.

If additionally $V \subset X$ is timelike complete, repeating the argument of [16, Lemma 4.4] one can prove that

$$I^+(V) \cup I^-(V) = \mathcal{T}_V^{end} \setminus V.$$

If X is timelike (both backward and forward) non-branching, then the transport relation R_V defines an equivalence relation on $\mathcal{T}_V$. The corresponding equivalence classes are timelike geodesics that locally maximize the function τ_V; we denote them by X_α. Moreover, there exists a measurable quotient map

$$\mathfrak{Q} : \mathcal{T}_V \to Q,$$

associated with the equivalence relation R_V on $\mathcal{T}_V$.

The corresponding disintegration of the reference measure $\mathfrak{m}$ and the localization of the Ricci curvature bounds were established in [16] (see also [17, 8]). We denote by $\mathcal{M}_+(X)$ the space of non-negative Radon measures over X.

Theorem 2.12. *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a globally hyperbolic, timelike non-branching, Lorentzian geodesic space satisfying $\mathsf{TMCP}^e(0, N)$, assume that the causally-reversed structure satisfies the same conditions and let $V \subset X$ be a Borel achronal timelike complete subset with empty global edge.*
Let $\mathcal{T}_V^{end}, \mathfrak{a}(\mathcal{T}_V^{end}), \mathfrak{b}(\mathcal{T}_V^{end})$ and $\mathcal{T}_V$ defined in (2.9), (2.10) *and* (2.11).

Then $\mathfrak{m}(\mathfrak{a}(\mathcal{T}_V^{end})) = \mathfrak{m}(\mathfrak{b}(\mathcal{T}_V^{end}) = 0$ and the following disintegration formula holds:

$$\mathfrak{m}\llcorner_{\mathcal{T}_V^{end}} = \mathfrak{m}\llcorner_{\mathcal{T}_V} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(\mathrm{d}\alpha), \tag{2.12}$$

where $\mathfrak{q}$ is a Borel probability measure over a quotient set $Q \subset \mathcal{T}_V$ such that $\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_V}) \ll \mathfrak{q}$ and the map $Q \ni \alpha \mapsto \mathfrak{m}_\alpha \in \mathcal{M}_+(X)$ satisfies the following properties:

(1) for any $\mathfrak{m}$-measurable set $B \subset X$, the map $\alpha \mapsto \mathfrak{m}_\alpha(B)$ is $\mathfrak{q}$-measurable;

(2) for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha$ is concentrated on $\mathfrak{Q}^{-1}(\alpha) = X_\alpha$ (strong consistency);

(3) for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha \ll \mathcal{L}^1\llcorner_{X_\alpha}$;

(4) for $\mathfrak{q}$-a.e. $\alpha \in Q$, the one-dimensional metric measure space $(X_\alpha, |\cdot|, \mathfrak{m}_\alpha)$ satisfies the classical $\mathsf{MCP}(0, N)$; namely, writing $\mathfrak{m}_\alpha = h(\alpha, \cdot)\mathcal{L}^1\llcorner_{X_\alpha}$, then $h(\alpha, \cdot)$ has an almost everywhere representative that is locally Lipschitz and strictly positive in the interior of X_α, continuous on its closure, and satisfying

$$\left(\frac{b - \tau_V(x_1)}{b - \tau_V(x_0)}\right)^{N-1} \leq \frac{h(\alpha, x_1)}{h(\alpha, x_0)} \leq \left(\frac{\tau_V(x_1) - a}{\tau_V(x_0) - a)}\right)^{N-1}, \tag{2.13}$$

for all $x_0, x_1 \in X_\alpha$, with $a < \tau_V(x_0) < \tau_V(x_1) < b$, where

$$a := \inf_{x \in X_\alpha} \tau_V(x), \quad b = \sup_{x \in X_\alpha} \tau_V(x).$$

Moreover, fixed any $\mathfrak{q}$ *as above such that* $\mathfrak{Q}_\sharp(\mathfrak{m} \llcorner_{\mathcal{T}_V}) \ll \mathfrak{q}$*, the disintegration is* $\mathfrak{q}$*-essentially unique in the following sense: if any other map* $Q \ni \alpha \mapsto \bar{\mathfrak{m}}_\alpha \in \mathcal{P}(X)$ *satisfies points (1)-(2), then* $\bar{\mathfrak{m}}_\alpha = \mathfrak{m}_\alpha$ *for* $\mathfrak{q}$*-a.e.* $\alpha \in Q$*.*

2.3 Future Minkowski content

Building on Lemma 2.12, in the previous work [17], the authors established results concerning the area of achronal subsets of X. We begin by recalling the definition of the area of an achronal set.

Definition 2.13 (Timelike Minkowski content)**.** Let $A \subset X$ be a Borel achronal set and consider the signed time-separation function τ_A from A, see (2.6). We define the *future Minkowski content* of A by

$$\begin{gathered}\mathfrak{m}^+(A) := \inf_{U \in \mathcal{U}} \limsup_{\varepsilon \to 0} \frac{\mathfrak{m}(\tau_A^{-1}((0,\varepsilon)) \cap U)}{\varepsilon}, \\ \mathcal{U} := \{U \subset X : U \text{ open}, \ A \subset U\}.\end{gathered} \tag{2.14}$$

Analogously, the *past Minkowski content* of A is defined by

$$\mathfrak{m}^-(A) := \inf_{U \in \mathcal{U}} \limsup_{\varepsilon \to 0} \frac{\mathfrak{m}(\tau_A^{-1}((-\varepsilon,0)) \cap U)}{\varepsilon}. \tag{2.15}$$

Remark 2.14. If X is a smooth, globally hyperbolic, Lorentzian manifold and $A \subset X$ is a smooth, achronal, future causally complete hypersurface, then $\mathfrak{m}^+(A)$ coincides with the standard area of A computed with respect to the restriction of the ambient Lorentzian metric; see [17, Remark 4.2].

In particular, as shown in [41], in this smooth setting the time separation function τ_A is smooth on $I^+(A) \backslash C_+(A)$, where $C_+(A)$ is the future cut-locus of A that is a closed set of zero measure. This implies that each level set $\tau_A^{-1}(s)$ is a smooth submanifold of $I^+(A) \backslash C_+(A)$ having therefore a well-defined volume measure induced by the ambient metric. Finally the coarea formula [41, Proposition 3] gives the claim.

3 One-dimensional inequalities

Leaving the Lorentzian setting for this short section, we will prove some one-dimensional inequalities for $\mathsf{MCP}(0,N)$ spaces, i.e. for $(I, |\cdot|, h\,dx)$, where $I \subset \mathbb{R}$ is an open interval, $h : I \to (0,\infty)$ is continuous and satisfies (2.13).

The Bishop-Gromov inequality (which, in this one-dimensional setting, reduces to an elementary computation) yields that the function

$$(0,\infty) \ni R \mapsto \frac{h(R)}{NR^{N-1}} \searrow$$

is monotone non-increasing. Assuming that $\sup I = \infty$, we denote

$$\theta = \theta_{h,N} := \lim_{R \to \infty} \frac{h(R)}{NR^{N-1}} = \inf_{R>0} \frac{h(R)}{NR^{N-1}}. \tag{3.1}$$

In the next lemma, we relate the asymptotic “area ratio” (3.1) to the more familiar asymptotic volume ratio,

$$\mathrm{AVR}_N(h) := \lim_{R \to \infty} \frac{1}{R^N} \int_0^R h(t)\,dt$$

which appeared in the recent metric geometry literature (see for instance [2, 13]).

Lemma 3.1. *Let $N \geq 1$ and let $h : [0, \infty) \to [0, \infty)$ be a continuous function such that*

$$t \mapsto \frac{h(t)}{Nt^{N-1}}$$

is decreasing on $(0, \infty)$ and

$$\lim_{t \to \infty} \frac{h(t)}{Nt^{N-1}} = \theta$$

for some $\theta \geq 0$. Then

$$\lim_{R \to \infty} \frac{1}{R^N} \int_0^R h(t)\, dt = \theta.$$

Proof. Set

$$f(t) := \frac{h(t)}{Nt^{N-1}}.$$

By assumption, f is decreasing on $(0, \infty)$ and $f(t) \to \theta$ as $t \to \infty$. Fix $\varepsilon > 0$. Then there exists $T > 0$ such that

$$\theta \leq f(t) \leq \theta + \varepsilon, \qquad \text{for all } t \geq T.$$

For $R > T$, we write

$$\frac{1}{R^N} \int_0^R h(t)\, dt = \frac{1}{R^N} \int_0^T h(t)\, dt + \frac{1}{R^N} \int_T^R Nt^{N-1} f(t)\, dt.$$

The first term in the right hand side tends to 0 as $R \to \infty$. For the second term, the bounds on f give

$$\theta \cdot \frac{1}{R^N} \int_T^R t^{N-1}\, dt \leq \frac{1}{R^N} \int_T^R t^{N-1} f(t)\, dt \leq (\theta + \varepsilon) \cdot \frac{1}{R^N} \int_T^R t^{N-1}\, dt.$$

Since

$$\frac{1}{R^N} \int_T^R Nt^{N-1}\, dt = \frac{R^N - T^N}{R^N} \longrightarrow 1 \qquad \text{as } R \to \infty,$$

it follows that

$$\lim_{R \to \infty} \frac{1}{R^N} \int_T^R t^{N-1} f(t)\, dt = \theta,$$

completing the proof. ☐

Lemma 3.2 (The key one-dimensional estimates)**.** *Let $N \in (1, \infty)$ and let*

$$X = (I, |\cdot|, h\,\mathcal{L}^1)$$

be a one-dimensional metric measure space satisfying $\mathsf{MCP}(0, N)$, where $I \subset \mathbb{R}$ is an open interval, $0 \in I$, $\sup I = +\infty$, and $h > 0$ in I. Then the function

$$(0, \infty) \ni R \longmapsto \frac{h(R)}{NR^{N-1}}$$

is non-increasing. In particular, the limit

$$\theta = \theta_{h,N} := \lim_{R \to +\infty} \frac{h(R)}{NR^{N-1}}$$

exists.

Moreover,

$$I \subset \left(-\left(\frac{h(0)}{N\theta} \right)^{\frac{1}{N-1}}, +\infty \right), \tag{3.2}$$

with the convention that the left endpoint equals $-\infty$ if $\theta = 0$.

Proof. We use the one-dimensional formulation of the $\mathsf{MCP}(0, N)$ condition. Namely, for every $x_0, x_1 \in I$ and every $t \in [0, 1]$, one has

$$h\big((1-t)x_0 + tx_1\big) \geq (1-t)^{N-1} h(x_0), \tag{3.3}$$

whenever the segment joining x_0 and x_1 is contained in I.

We first prove the monotonicity of the asymptotic area ratio. Fix $0 < R_1 < R_2$ and apply (3.3) with

$$x_0 = R_2, \qquad x_1 = 0, \qquad t = 1 - \frac{R_1}{R_2}.$$

Since

$$(1-t)R_2 + t\,0 = R_1,$$

we obtain

$$h(R_1) \geq \left(\frac{R_1}{R_2}\right)^{N-1} h(R_2).$$

Equivalently,

$$\frac{h(R_1)}{R_1^{N-1}} \geq \frac{h(R_2)}{R_2^{N-1}}.$$

It follows that

$$R \longmapsto \frac{h(R)}{NR^{N-1}}$$

is non-increasing, and hence the limit defining θ exists.

Let now $s > 0$ be such that $-s \in I$. For any $R > 0$, apply (3.3) to

$$x_0 = R, \qquad x_1 = -s, \qquad t = \frac{R}{R+s}.$$

Since

$$(1-t)R + t(-s) = 0, \qquad 1 - t = \frac{s}{R+s},$$

we infer that

$$h(0) \geq \left(\frac{s}{R+s}\right)^{N-1} h(R). \tag{3.4}$$

Rewriting the right-hand side gives

$$h(0) \geq N s^{N-1} \left(\frac{R}{R+s}\right)^{N-1} \frac{h(R)}{NR^{N-1}}.$$

Passing to the limit as $R \to +\infty$ yields

$$h(0) \geq N\theta\, s^{N-1}. \tag{3.5}$$

If $\theta > 0$, this implies

$$s \leq \left(\frac{h(0)}{N\theta}\right)^{\frac{1}{N-1}}.$$

In fact, the inequality is strict for every $-s \in I$. Indeed, since I is open, there exists $\varepsilon > 0$ such that $-(s+\varepsilon) \in I$. Applying (3.5) with $s + \varepsilon$ in place of s gives

$$s + \varepsilon \leq \left(\frac{h(0)}{N\theta}\right)^{\frac{1}{N-1}},$$

and therefore

$$s < \left(\frac{h(0)}{N\theta}\right)^{\frac{1}{N-1}}.$$

This proves (3.2). If $\theta = 0$, the claim is void by convention. $\square$

4 A timelike incompleteness theorem in terms of asymptotic expansion

Before stating and proving the result, we fix some notation. Provided $\mathfrak{m}^+(V) < \infty$, it will be convenient to slightly change the normalizations of the conditional probabilities $h(\alpha, \cdot)$ in the following manner.

First, since Q can be identified with a measurable subset of V, we may, without loss of generality, parametrize each ray X_α so that

$$X_\alpha(0) = X_\alpha \cap V.$$

It then follows that

$$\int_Q h(\alpha, 0)\, \mathfrak{q}(\mathrm{d}\alpha) \leq \mathfrak{m}^+(V) < \infty,$$

and hence $h(\cdot, 0) \in L^1(\mathfrak{q})$.

For our purposes, it will suffice to consider only those rays X_α for which $X_\alpha \cap V$ is a relative interior point of the ray. We denote by Q^- the set of indices with this property. Hence, since 0 is an interior point of the chosen parametrization, it follows that for $\mathfrak{q}$-a.e. $\alpha \in Q^-$,

$$h(\alpha, 0) > 0.$$

We may therefore normalize the densities so that

$$h(\alpha, 0) = 1 \qquad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q^-, \tag{4.1}$$

by correspondingly modifying the quotient measure according to

$$\bar{\mathfrak{q}} := h(\alpha, 0)\, \mathfrak{q}\llcorner_{Q^-} + \mathfrak{q}\llcorner_{Q \setminus Q^-}. \tag{4.2}$$

Note that $\bar{\mathfrak{q}}$ is no longer a probability measure. Nevertheless, it satisfies

$$\bar{\mathfrak{q}}(Q) \leq \mathfrak{m}^+(V). \tag{4.3}$$

Finally, we assume that the map

$$\alpha \mapsto \theta_\alpha^{-1}$$

belongs to $L^1(\bar{\mathfrak{q}}, Q^-)$, where $\theta_\alpha := \theta_{h_\alpha}$ is defined as in (3.1).

Observe that the normalization $h(\alpha, 0) = 1$ is also needed in order to fix, once and for all, the scaling of the one-dimensional asymptotic area ratio θ_α.

Theorem 4.1 (A synthetic singularity theorem, based on asymptotic volume growth)**.** *Let $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ be a timelike non-branching, globally hyperbolic, Lorentzian geodesic space. Let $V \subset X$ be an achronal, timelike complete set having finite area, i.e. $\mathfrak{m}^+(V) < \infty$, and empty global edge. Assume moreover that:*

- *Both $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ and the causally-reverse structure satisfy the $\mathsf{TMCP}^e(0, N)$ condition, for some $N \in (1, \infty)$;*
- *there exists $c > 0$ such that, adopting the normalization (4.1), it holds*

$$\theta_\alpha \geq c > 0 \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q^-. \tag{4.4}$$

Then

$$\sup_{x \in I^-(V)} |\tau_V(x)| \leq \left(\frac{1}{Nc}\right)^{\frac{1}{N-1}}. \tag{4.5}$$

In particular, $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ is not past timelike geodesically complete.

Proof. For $\mathfrak{q}$-a.e. $\alpha \in Q^-$, the positivity of the one-dimensional asymptotic area ratio $\theta_\alpha > 0$ implies that the corresponding ray X_α has no future endpoint.

Since

$$\mathfrak{Q}_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_V}) \ll \mathfrak{q},$$

it follows that, up to a set of $\mathfrak{m}$-measure zero, the set $I^-(V)$ is foliated by integral curves of τ_V having no future endpoint and indexed by elements of Q^-.

Hence, combining the assumption (4.4) with the bound (3.2), we obtain that for $\mathfrak{q}$-a.e. α,

$$\sup_{x \in X_\alpha \cap I^-(V)} \big(-\tau_V(x)\big) \leq \left(\frac{1}{Nc}\right)^{\frac{1}{N-1}}.$$

The disintegration formula (2.12) implies that

$$-\tau_V(x) \leq \left(\frac{1}{Nc}\right)^{\frac{1}{N-1}}, \quad \text{for } \mathfrak{m}\text{-a.e. } x \in I^-(V).$$

Since $-\tau_V$ is lower semicontinuous on $I^-(V)$ and $\operatorname{supp}\mathfrak{m} = X$ by standing assumption, the above almost-everywhere estimate extends to all of $I^-(V)$. This concludes the proof of (4.5).

We finally show that X is not past timelike geodesically complete. Consider any timelike geodesic γ parametrized by arclength and defined on a maximal (on the left) interval $(-a, 0] \subset (-\infty, 0]$ such that $\gamma_0 \in V$. We claim that

$$a \leq \left(\frac{1}{Nc}\right)^{\frac{1}{N-1}}.$$

Indeed, if by contradiction for some $s_0 \in [0, a)$

$$\tau(\gamma_{s_0}, \gamma_0) = s_0 > \left(\frac{1}{Nc}\right)^{\frac{1}{N-1}},$$

the very definition (2.6) of τ_V would imply

$$-\tau_V(\gamma_{s_0}) > \left(\frac{1}{Nc}\right)^{\frac{1}{N-1}},$$

contradicting (4.5). $\square$

In the case where (M, g) is an $(n+1)$-dimensional globally hyperbolic smooth Lorentzian manifold, and $V \subset M$ is a smooth, compact, achronal hypersurface with empty global edge, every ray associated with τ_V intersects V. This allows one to take V itself as the quotient set and the induced hypersurface measure Vol_V as the quotient measure. In this setting, $X_\alpha \cap V$ is an interior point of the ray X_α for every α. Thus, we can choose

$$Q^- = V. \tag{4.6}$$

Moreover, the normalization introduced above appears entirely natural and yields the following disintegration formula (which, in this case, corresponds to Fubini-Tonelli's theorem):

$$\mathrm{Vol}_{g\llcorner_{I^\pm(V)}} = \int_V h(\alpha, \cdot)\, \mathrm{Vol}_{g_V}(\mathrm{d}\alpha), \qquad h(\alpha, 0) = 1,\ \mathrm{Vol}_V - \text{a.e. } \alpha, \tag{4.7}$$

where we have denoted by Vol_g (resp. Vol_{g_V}) the $(n+1)$-dimensional volume measure of (M, g) (resp. the n-dimensional volume measure of $(V, g|_{TV})$). Below we report the smooth version of Lemma 4.1.

Theorem 4.2 (A singularity result, based on asymptotic volume growth – smooth setting)**.** *Let (M,g) be an $(n+1)$-dimensional, globally hyperbolic, smooth Lorentzian manifold satisfying Penrose-Hawking's strong energy condition, i.e.* $\mathrm{Ric}_g(v,v) \geq 0$ *for all $v \in TM$ timelike. Let $V \subset M$ be a smooth, compact, Cauchy hypersurface. Assume moreover that there exists a constant $c > 0$ such that*

$$\theta_\alpha \geq c > 0 \qquad \textit{for } \mathrm{Vol}_V\textit{-a.e. } \alpha \in V.$$

Then

$$\sup_{x \in I^-(V)} |\tau_V(x)| \leq \left(\frac{1}{(n+1)c}\right)^{\frac{1}{n}}.$$

In particular, (M,g) is not past timelike geodesically complete.

Moreover, for any (possibly non-smooth) past extension $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ of (M,g) which is a timelike non-branching, globally hyperbolic, Lorentzian geodesic space, satisfying the $\mathsf{TMCP}^e(0,n+1)$ condition, together with its causally-reversed structure, it holds that

$$\sup_{x \in I^-_X(V)} |\tau_V(x)| \leq \left(\frac{1}{(n+1)c}\right)^{\frac{1}{n}},$$

where $I^-_X(V) \subset X$ is the chronological past of V in X.
In particular, $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ is not past timelike geodesically complete.

Proof. Theorem 4.2 follows directly from Theorem 4.1, once noticing that

- (M,g) is, in particular, a timelike non-branching, globally hyperbolic, Lorentzian geodesic space satisfying the $\mathsf{TMCP}^e(0,n+1)$ condition, together with its causally-reverse structure (see [16, Theorem A.1], after [26, 32]);
- since V is a Cauchy hypersurface, it has empty global edge;
- the smoothness of V implies that all the rays X_α, $\alpha \in V$, maximizing the time separation τ_V from V can be extended across V, i.e. so that $X_\alpha(0) = X_\alpha \cap V$ is an interior point of X_α;
- the disintegration formula

$$(\mathrm{Vol}_g)|_{I^+(V)} = \int_V h_\alpha \, \mathrm{Vol}_V(\mathrm{d}\alpha)$$

  corresponds to choosing $Q = V$ and $\mathfrak{q} = \mathrm{Vol}_{g_V}$ in the disintegration theorem, which imply that $h_\alpha(0) = 1$ for Vol_{g_V}-a.e. $\alpha \in V$;
- The compactness of V implies that it is timelike complete and has finite n-dimensional volume. Moreover,

$$\mathrm{Vol}^+_g(V) = \mathrm{Vol}_{g_V}(V),$$

  see Remark 2.14.

The final claim on the non-smooth extension is a consequence of the following observations: if X is a past extension of (M,g), then (M,g) is isomorphically embedded into X; moreover, denoting with I^+_M and I^+_X the future sets in M and X respectively, then $I^+_M(V) = I^+_X(V)$ and $I^-_M(V) \subset I^-_X(V)$. One can then repeat verbatim the proof of Theorem 4.1 with the aforementioned simplifications. $\square$

4.1 Sharpness of Theorem 4.1

We briefly recall the notation for Lorentzian generalized cones, following [12]. Let $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ be a metric measure space and let $I \subset \mathbb{R}$ be an interval. Given a continuous warping function $f : I \to [0, \infty)$, the Lorentzian generalized cone

$$^{-}I \times_f Z$$

is the Lorentzian warped product with one-dimensional base I endowed with the negative metric $-dr^2$ and fibre (Z, d_Z). More precisely, if $\gamma = (r, \beta)$ is an absolutely continuous causal curve, its Lorentzian length is given by

$$L(\gamma) = \int \sqrt{\dot{r}^{\,2} - f(r)^2 |\dot{\beta}|^2}\, dt,$$

where $|\dot{\beta}|$ denotes the metric speed of β. A causal curve is future-directed when $\dot{r} > 0$ a.e., and the corresponding time-separation is obtained by taking the supremum of the Lorentzian lengths of future-directed causal curves joining two given points.

For $N > 2$, the *Minkowski* $(N-1)$*-cone* over $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ is the particular Lorentzian generalized cone

$$\mathcal{M}^{N-1}(Z) := {}^{-}[0, \infty) \times_{\mathrm{id}}^{N-1} Z, \tag{4.8}$$

corresponding to the warping function $f(r) = r$. Here the superscript $N-1$ records the dimensional parameter of the fibre measure. The canonical reference measure is

$$\mathfrak{m}_{\mathcal{M}} = r^{N-1}\, dr \otimes \mathfrak{m}_Z. \tag{4.9}$$

Since the warping function vanishes at $r = 0$, all points of $\{0\} \times Z$ are identified with a single point o, called the *tip* of the cone. Thus, as a set,

$$\mathcal{M}^{N-1}(Z) = \big((0, \infty) \times Z\big) \cup \{\mathsf{o}\}.$$

On the regular part $(0, \infty) \times Z$, the synthetic construction is the nonsmooth analogue of the Lorentzian warped metric

$$-dr^2 + r^2\, \mathsf{d}_Z^2.$$

In particular, the radial curves

$$s \longmapsto (r + s, z)$$

are unit-speed timelike geodesics, as long as $r + s > 0$.

Proposition 4.3 (Sharpness of Theorem 4.1)**.** *Let $N \in (2, \infty)$ and let $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ be a compact $\mathsf{CD}(-(N-2), N-1)$ metric measure space with $\operatorname{supp} \mathfrak{m}_Z = Z$. Consider the Minkowski $(N-1)$-cone $\mathcal{M}^{N-1}(Z)$ endowed with the measure $\mathfrak{m}_{\mathcal{M}}$, defined above. For $r_0 > 0$, set*

$$V := \{r_0\} \times Z.$$

Then all the assumptions of Theorem 4.1 are satisfied and, adopting the normalization (4.1),

$$\theta_\alpha = \frac{1}{N r_0^{N-1}} \qquad \text{for } \bar{\mathfrak{q}}\text{-a.e. } \alpha \in Q^-. \tag{4.10}$$

Consequently, choosing

$$c := \frac{1}{N r_0^{N-1}},$$

equality holds in (4.5)*, namely*

$$\sup_{x \in I^-(V)} |\tau_V(x)| = r_0 = \left(\frac{1}{Nc}\right)^{\frac{1}{N-1}}. \tag{4.11}$$

In particular, the estimate in Theorem 4.1 is sharp.

Proof. The fact that the Minkowski cone satisfies $\mathsf{TMCP}^e(0,N)$ follows from [12, Theorem 5.2] with dimension $N-1$, warping function

$$f(r) = r,$$

and parameters

$$\kappa = 0, \qquad \eta = -1.$$

Indeed, since Z is non-branching, the cone X is timelike non-branching by [12, Remark 6.2]. In particular, it is timelike p-essentially non-branching, and hence [12, Remark 2.16] yields the entropic condition $\mathsf{TMCP}^e(0,N)$. The same conclusion holds for the causally-reversed structure. Finally, X is a globally hyperbolic Lorentzian geodesic space; see, for instance, [12, Theorem 3.17].

We next discuss the slice $V = \{r_0\} \times Z$. It is compact and achronal, and therefore it is timelike complete. Moreover, it has empty global edge. Indeed, the radial component of every future-directed timelike curve is strictly increasing, and hence every timelike curve joining a point in the chronological past of V to a point in its chronological future must intersect the level set $\{r = r_0\}$.

The area of V is finite. More precisely, directly from the definition of the future Minkowski content,

$$\begin{aligned}
\mathfrak{m}^+_{\mathcal{M}}(V) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathfrak{m}_{\mathcal{M}}\big(\{r_0 < r < r_0 + \varepsilon\}\big) \\
&= \lim_{\varepsilon \downarrow 0} \frac{\mathfrak{m}_Z(Z)}{\varepsilon} \int_{r_0}^{r_0+\varepsilon} r^{N-1}\, dr \\
&= r_0^{N-1} \mathfrak{m}_Z(Z) < \infty.
\end{aligned}$$

We now compute the signed time-separation from V. If $\gamma = (r, \beta)$ is any future-directed causal curve, then

$$L(\gamma) = \int \sqrt{\dot r^{\,2} - r^2 |\dot\beta|^2}\, dt \le \int \dot r\, dt.$$

On the other hand, equality is attained by radial curves, namely curves with constant Z-component. It follows that

$$\tau_V((r,z)) = r - r_0. \tag{4.12}$$

In particular,

$$I^-(V) = \{(r,z) : 0 \le r < r_0\},$$

up to the usual identification at the tip, and therefore

$$\sup_{x \in I^-(V)} |\tau_V(x)| = \sup_{0 \le r < r_0} (r_0 - r) = r_0. \tag{4.13}$$

It remains to compute the one-dimensional asymptotic expansion. The transport rays associated with τ_V are precisely the radial curves

$$s \longmapsto (r_0 + s, z), \qquad s \in (-r_0, \infty),$$

indexed by $z \in Z$. In these coordinates, the reference measure disintegrates as

$$\mathfrak{m}_{\mathcal{M}} = (r_0 + s)^{N-1}\, ds \otimes \mathfrak{m}_Z.$$

After imposing the normalization (4.1), we may write

$$d\bar{\mathfrak{q}}(z) := r_0^{N-1}\, d\mathfrak{m}_Z(z), \qquad h_z(s) := \left(1 + \frac{s}{r_0}\right)^{N-1}, \qquad s \in (-r_0, \infty).$$

In particular,

$$h_z(0) = 1$$

for every $z \in Z$. Recalling the definition of the one-dimensional asymptotic area ratio, we obtain

$$\begin{aligned}\theta_z &= \lim_{R\to+\infty} \frac{h_z(R)}{NR^{N-1}} \\ &= \lim_{R\to+\infty} \frac{(r_0+R)^{N-1}}{Nr_0^{N-1}R^{N-1}} = \frac{1}{Nr_0^{N-1}}.\end{aligned}$$

Thus (4.4) holds with equality for

$$c = \frac{1}{Nr_0^{N-1}}.$$

Consequently,

$$\left(\frac{1}{Nc}\right)^{\frac{1}{N-1}} = \left(r_0^{N-1}\right)^{\frac{1}{N-1}} = r_0.$$

Combining this identity with (4.13) gives (4.11), and proves the sharpness of Theorem 4.1. □

Remark 4.4. The restriction $N > 2$ above comes from the range of [12, Theorem 5.2]. Indeed, in order to construct a cone satisfying $\mathsf{TMCP}^e(0, N)$, that theorem is applied with fibre dimension parameter $N-1$, which is required to belong to $(1, \infty)$. For the Minkowski warping function $f(r) = r$, the corresponding sharp curvature condition on the fibre is $\mathsf{CD}(-(N-2), N-1)$.

4.2 Rigidity properties

Theorem 4.5 (Geometric rigidity in the smooth setting)**.** *Let (M, g) be a connected, $(n+1)$-dimensional, globally hyperbolic, smooth Lorentzian manifold and let $V \subset M$ be a smooth, compact, connected Cauchy hypersurface. Assume that*

$$\mathrm{Ric}_g(v, v) \geq 0 \qquad \text{for every timelike vector } v \in TM. \tag{4.14}$$

Let h_α be the densities in the disintegration (4.7), normalized by $h_\alpha(0) = 1$, and let θ_α be the corresponding one-dimensional asymptotic area . Suppose that, for some $c > 0$,

$$\theta_\alpha \geq c \qquad \text{for } \mathrm{Vol}_V\text{-a.e. } \alpha \in V. \tag{4.15}$$

Set

$$T := \left(\frac{1}{(n+1)c}\right)^{\frac{1}{n}}. \tag{4.16}$$

If equality holds in the estimate of Lemma 4.2, namely

$$\sup_{x\in I^-(V)} |\tau_V(x)| = T, \tag{4.17}$$

then (M, g) is isometric to the Lorentzian cone

$$\left((0, +\infty) \times V, -\mathrm{d}r^2 + \frac{r^2}{T^2} g_V\right), \tag{4.18}$$

where $g_V = g|_{TV}$ and V is identified with the slice $\{T\} \times V$. In particular, for

$$g_Z := T^{-2} g_V,$$

one has

$$\mathrm{Ric}_{g_Z} \geq -(n-1) g_Z. \tag{4.19}$$

Proof. We divide the proof into three steps.

Step 1: the asymptotic expansion gives a mean-curvature bound. Let ν denote the future-pointing unit normal to V, and let H_V be the mean curvature of V with respect to ν, with the sign convention that the slices of the cone in (4.18) have positive mean curvature. On every future normal ray on which the disintegration is defined, put

$$u_\alpha(s) := h_\alpha(s)^{1/n}.$$

The traced Riccati equation and (4.14) imply that u_α is concave. Moreover,

$$u_\alpha(0) = 1, \qquad u'_\alpha(0) = \frac{H_V(\alpha)}{n}. \tag{4.20}$$

Since

$$\lim_{R\to+\infty} \frac{u_\alpha(R)}{R} = \big((n+1)\theta_\alpha\big)^{1/n}, \tag{4.21}$$

concavity and (4.15) yield

$$\frac{H_V(\alpha)}{n} \geq \big((n+1)\theta_\alpha\big)^{1/n} \geq \frac{1}{T} \tag{4.22}$$

for Vol_V-a.e. $\alpha \in V$. The smoothness of V and the continuity of H_V extend the last inequality to every point of V. Hence

$$H_V \geq \frac{n}{T} \qquad \text{on } V. \tag{4.23}$$

Step 2: rigidity of the chronological past. We first observe that the equality in (4.17) produces a past-directed V-ray of length T. Indeed, choose $x_j \in I^-(V)$ such that $-\tau_V(x_j) \to T$. Global hyperbolicity and the compactness of V give a future-directed maximizing geodesic from x_j to V. Read backwards from V, it is a past-directed unit-speed geodesic

$$\gamma_j : [0, s_j] \longrightarrow M, \qquad s_j = -\tau_V(x_j) \longrightarrow T,$$

which maximizes the time separation from V on each of its initial segments. Writing $\gamma_j(0) = \alpha_j \in V$, compactness gives, after passing to a subsequence, $\alpha_j \to \alpha \in V$. Continuous dependence of the geodesic flow then yields a past-directed unit-speed geodesic

$$\gamma : [0, T) \longrightarrow M, \qquad -\tau_V(\gamma(t)) = t,$$

where the maximizing property follows by continuity of the time separation. Thus γ is a V-ray of maximal length T.

Reverse now the time orientation. The timelike Ricci lower bound is unchanged, while (4.23) becomes the upper bound $H_V \leq -n/T$. Therefore [20, Theorem 5.12] applies with

$$\kappa = 0, \qquad \beta = -\frac{n}{T}, \qquad b_{\kappa,\beta} = T.$$

The existence of the maximal V-ray obtained above implies

$$(I^-(V), g) \simeq \left((0,T) \times V, -\mathrm{d}t^2 + \left(1 - \frac{t}{T}\right)^2 g_V\right). \tag{4.24}$$

In particular, the future shape operator of V is

$$B_0 = \frac{1}{T}\mathbb{I}, \qquad H_V = \frac{n}{T}. \tag{4.25}$$

Step 3: rigidity of the chronological future. Combining (4.25) with (4.20), (4.21) and the concavity of u_α, we infer that for Vol_V-a.e. α and every $s > 0$ in the corresponding future ray,

$$1 + \big((n+1)\theta_\alpha\big)^{1/n} s \leq u_\alpha(s) \leq 1 + \frac{s}{T}. \tag{4.26}$$

Here the first inequality follows by (4.21) and letting the second endpoint tend to $+\infty$ in the secant-slope inequality for the concave function u_α, while the second one is the tangent-line inequality at $s = 0$ combined with (4.20) and (4.25). In view of (4.15) and (4.16), both inequalities in (4.26) are equalities. Consequently,

$$\theta_\alpha = c, \qquad h_\alpha(s) = \Big(1 + \frac{s}{T}\Big)^n \tag{4.27}$$

for Vol_V-a.e. $\alpha \in V$.

Equality in the traced Riccati inequality forces equality both in the Cauchy–Schwarz estimate for the second fundamental form and in the timelike Ricci bound. Hence the shear vanishes and

$$B_s = \frac{1}{T+s}\mathbb{I}, \qquad \mathrm{Ric}_g(\dot\gamma_\alpha, \dot\gamma_\alpha) = 0. \tag{4.28}$$

It follows that the induced metrics g_s on the regular level sets of τ_V satisfy

$$\partial_s g_s = 2B_s^\flat, \qquad g_s = \left(\frac{T+s}{T}\right)^2 g_V.$$

Finally, (4.27) gives the model volume over every compact subset of V. The volume-rigidity statement [20, Proposition 6.3] therefore yields

$$(I^+(V), g) \simeq \left((0,+\infty) \times V, -\mathrm{d}s^2 + \Big(1 + \frac{s}{T}\Big)^2 g_V\right). \tag{4.29}$$

Combining (4.24) and (4.29), and setting $r = T + \tau_V$, proves (4.18).

The warped-product Ricci identities, together with (4.14), imply $\mathrm{Ric}_{g_Z} \geq -(n-1)g_Z$ for $g_Z = T^{-2}g_V$, proving (4.19). $\square$

In the next proposition, we establish several rigidity statements in the non-smooth setting. The third statement is inspired by Ohta's rigidity result for the Bonnet–Myers theorem in $\mathsf{MCP}(N-1, N)$ metric measure spaces; see [35, Theorem 5.5].

Proposition 4.6 (Raywise and measure rigidity)**.** *Let* $(X, \mathsf{d}, \mathfrak{m}, \ll, \leq, \tau)$ *and* $V \subset X$ *satisfy the assumptions of Lemma 4.1. Adopt the normalization* (4.1)*, and assume that*

$$\theta_\alpha \geq c > 0 \qquad \textit{for } \bar{\mathfrak{q}}\textit{-a.e. } \alpha \in Q^-.$$

Set

$$T := \left(\frac{1}{Nc}\right)^{\frac{1}{N-1}}. \tag{4.30}$$

For $\bar{\mathfrak{q}}$*-a.e.* $\alpha \in Q^-$*, write*

$$X_\alpha \simeq (a_\alpha, +\infty), \qquad \mathfrak{m}_\alpha = h_\alpha(t)\, \mathcal{L}^1(dt),$$

where $t = \tau_V$ *and* $h_\alpha(0) = 1$*.*

Then the following assertions hold.

(i) *Assume that, for some* $\alpha \in Q^-$*,*

$$a_\alpha = -T. \tag{4.31}$$

Then

$$\theta_\alpha = c \tag{4.32}$$

and

$$h_\alpha(t) = \left(1 + \frac{t}{T}\right)^{N-1}, \qquad t \in [0, +\infty). \tag{4.33}$$

Moreover, on the past portion of the ray one has

$$\left(1 + \frac{t}{T}\right)^{N-1} \leq h_\alpha(t) \leq 1, \qquad t \in (-T, 0]. \tag{4.34}$$

(ii) *Assume that* (4.31) *holds for* $\bar{\mathfrak{q}}$*-a.e.* $\alpha \in Q^-$ *and that*

$$\mathfrak{m}(I^-(V)) = \frac{T}{N}\,\bar{\mathfrak{q}}(Q^-). \tag{4.35}$$

Then

$$h_\alpha(t) = \left(1 + \frac{t}{T}\right)^{N-1}, \qquad t \in (-T, +\infty), \tag{4.36}$$

for $\bar{\mathfrak{q}}$*-a.e.* $\alpha \in Q^-$.
Consequently, after the change of variable

$$r = T + t,$$

the disintegration of $\mathfrak{m}$ *along the transport rays takes the form*

$$\mathfrak{m} = \int_{Q^-} r^{N-1}\,dr\,\nu(d\alpha), \qquad \nu := T^{-(N-1)}\bar{\mathfrak{q}}\llcorner_{Q^-}. \tag{4.37}$$

(iii) *Assume, in addition to the assumptions in* (ii)*, the following Ohta-type uniqueness condition. There exists a point* $\mathsf{o} \in X$ *such that:*

- *the closure of every ray* X_α*,* $\alpha \in Q^-$*, has past endpoint* o*;*
- *every point of* $X \setminus \{\mathsf{o}\}$ *belongs to a unique maximal transport ray associated with* τ_V*;*
- *the family of such rays is precompact, on every bounded* t*-interval, in the compact-open topology.*

Then there exists a topological space Y*, naturally identified with the family of transport rays, such that* $(X, \mathfrak{m})$ *is a measure cone over* (Y, ν)*. More precisely, setting*

$$\mathcal{C}(Y) := Y \times [0, +\infty)/\big(Y \times \{0\}\big),$$

there exists a homeomorphism

$$\Psi : \mathcal{C}(Y) \longrightarrow X$$

sending the tip of $\mathcal{C}(Y)$ *to* o *and satisfying*

$$\Psi_\sharp\big(r^{N-1}\,dr \otimes \nu\big) = \mathfrak{m}. \tag{4.38}$$

Thus $(X, \mathfrak{m})$ *is a cone as a topological measure space.*

Proof. We start with the raywise statement. By the localization of the $\mathsf{TMCP}^e(0, N)$ condition, for $\bar{\mathfrak{q}}$-a.e. $\alpha \in Q^-$ the one-dimensional metric measure space

$$\big((a_\alpha, +\infty), |\cdot|, h_\alpha\,\mathcal{L}^1\big)$$

satisfies $\mathsf{MCP}(0, N)$. In particular, the one-dimensional $\mathsf{MCP}(0, N)$ comparison yields

$$\left(\frac{b - t_1}{b - t_0}\right)^{N-1} \le \frac{h_\alpha(t_1)}{h_\alpha(t_0)} \le \left(\frac{t_1 - a}{t_0 - a}\right)^{N-1}, \tag{4.39}$$

whenever

$$a < t_0 < t_1 < b$$

belong to the ray.

Assume first that $a_\alpha = -T$. Letting $a \downarrow -T$ in the right-hand inequality of (4.39), we infer that

$$(-T, +\infty) \ni t \longmapsto \frac{h_\alpha(t)}{(T + t)^{N-1}} \tag{4.40}$$

is non-increasing. Since $h_\alpha(0) = 1$, it follows that

$$\frac{h_\alpha(0)}{T^{N-1}} = \frac{1}{T^{N-1}} = Nc. \tag{4.41}$$

On the other hand,

$$\begin{aligned}\lim_{t\to+\infty} \frac{h_\alpha(t)}{(T+t)^{N-1}} &= \lim_{t\to+\infty} \frac{h_\alpha(t)}{t^{N-1}} \\ &= N\theta_\alpha \geq Nc.\end{aligned}$$

Since the function in (4.40) is non-increasing, its limit at $+\infty$ cannot exceed its value at $t = 0$. Hence all the previous inequalities are equalities. In particular,

$$\theta_\alpha = c$$

and

$$\frac{h_\alpha(t)}{(T+t)^{N-1}} = \frac{1}{T^{N-1}}, \qquad t \geq 0,$$

which proves (4.33).

For $t \in (-T, 0]$, the monotonicity (4.40) gives

$$\frac{h_\alpha(t)}{(T+t)^{N-1}} \geq \frac{h_\alpha(0)}{T^{N-1}},$$

and therefore

$$h_\alpha(t) \geq \left(1 + \frac{t}{T}\right)^{N-1}.$$

Moreover, letting $b \to +\infty$ in the left-hand inequality of (4.39), we obtain that h_α is non-decreasing. Since $h_\alpha(0) = 1$, this gives

$$h_\alpha(t) \leq 1, \qquad t \in (-T, 0].$$

This proves (4.34) and completes the proof of (i).

We now prove (ii). By assumption, $a_\alpha = -T$ for $\bar{\mathfrak{q}}$-a.e. $\alpha \in Q^-$. Hence (4.34) gives

$$\begin{aligned}\int_{-T}^{0} h_\alpha(t)\,dt &\geq \int_{-T}^{0} \left(1 + \frac{t}{T}\right)^{N-1} dt \\ &= \frac{T}{N}\end{aligned} \tag{4.42}$$

for $\bar{\mathfrak{q}}$-a.e. $\alpha \in Q^-$. Integrating (4.42) with respect to $\bar{\mathfrak{q}}$ and using the disintegration formula yields

$$\mathfrak{m}(I^-(V)) \geq \frac{T}{N}\bar{\mathfrak{q}}(Q^-). \tag{4.43}$$

Under the equality assumption (4.35), equality must therefore hold in (4.42) for $\bar{\mathfrak{q}}$-a.e. α. Since

$$h_\alpha(t) - \left(1 + \frac{t}{T}\right)^{N-1} \geq 0$$

on $(-T, 0)$, we conclude that

$$h_\alpha(t) = \left(1 + \frac{t}{T}\right)^{N-1}$$

for $\mathcal{L}^1$-a.e. $t \in (-T, 0)$. Recalling that the canonical representative of an $\mathsf{MCP}(0, N)$ density is continuous in the interior of its domain, the identity holds for every $t \in (-T, 0]$. Together with (4.33), this proves (4.36).

Changing variables according to $r = T + t$, we then have

$$h_\alpha(t)\,dt\,\bar{\mathfrak{q}}(d\alpha) = \frac{r^{N-1}}{T^{N-1}}\,dr\,\bar{\mathfrak{q}}(d\alpha) = r^{N-1}\,dr\,\nu(d\alpha),$$

which proves (4.37).

We finally prove (iii). Let Y denote the family of maximal transport rays, endowed with the compact-open topology, and define

$$\Psi : Y \times (0, +\infty) \longrightarrow X \setminus \{\mathsf{o}\}$$

by

$$\Psi(\alpha, r) := X_\alpha(r - T).$$

The common-tip assumption permits to extend Ψ continuously to $r = 0$ by setting

$$\Psi(\alpha, 0) := \mathsf{o} \qquad \text{for every } \alpha \in Y.$$

Hence Ψ induces a map

$$\Psi : \mathcal{C}(Y) \longrightarrow X.$$

The uniqueness assumption implies that this map is bijective.

We claim that it is a homeomorphism. Continuity follows from the definition of the compact-open topology on the family of rays. To prove continuity of the inverse, argue as in the proof of [35, Lemma 5.4]. Assume by contradiction that $x_j \to x$ in $X \setminus \{\mathsf{o}\}$ while the corresponding rays α_j do not converge to the ray α containing x. By the assumed precompactness, after passing to a subsequence the α_j converge on every bounded parameter interval to a maximal transport ray $\tilde{\alpha}$. The continuity of τ and of the geodesics implies that $\tilde{\alpha}$ passes through x. By the uniqueness assumption, necessarily $\tilde{\alpha} = \alpha$, yielding a contradiction. The same argument at the tip, together with $\tau_V(\Psi(\alpha, r)) = r - T$, gives continuity of the inverse there. Thus Ψ is a homeomorphism.

It remains to identify the measure. Let $W \subset Y$ be Borel and $0 < a < b < +\infty$. By (4.37),

$$\begin{aligned} \mathfrak{m}\big(\Psi(W \times [a, b])\big) &= \int_W \int_a^b r^{N-1}\,dr\,\nu(d\alpha) \\ &= \nu(W) \int_a^b r^{N-1}\,dr. \end{aligned}$$

Since sets of this form generate the Borel σ-algebra of $\mathcal{C}(Y)$, a monotone-class argument gives

$$\Psi_\sharp\big(r^{N-1}\,dr \otimes \nu\big) = \mathfrak{m}.$$

This proves (4.38). □

Remark 4.7. The maximal-lifetime condition $a_\alpha = -T$ alone does not force the full cone density on the past portion of an $\mathsf{MCP}(0, N)$ ray. For instance,

$$h(t) := \begin{cases} 1, & -T < t \le 0, \\ \left(1 + \dfrac{t}{T}\right)^{N-1}, & t \ge 0, \end{cases}$$

satisfies the one-dimensional $\mathsf{MCP}(0, N)$ comparison inequalities, is normalized by $h(0) = 1$, has

$$\theta = \frac{1}{NT^{N-1}}, \qquad a = -T,$$

but it does not agree with the cone density on $(-T, 0)$. Thus an additional equality condition, such as (4.35), is necessary in order to obtain full measure-cone rigidity under $\mathsf{TMCP}^{\mathsf{e}}(0, N)$ alone.

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