# The formation of gradient-driven singular structures of codimension one and two in two-dimensions: The case study of ferronematics
# Part I: Energy estimates and compactness results

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**Abstract**

We study a two-dimensional variational model for ferronematics — composite materials formed by dispersing magnetic nanoparticles into a liquid crystal matrix. The model features two coupled order parameters: a Landau-de Gennes $\mathbf{Q}$-tensor for the liquid crystal component and a magnetisation vector field $\mathbf{M}$, both of them governed by a Ginzburg-Landau-type energy. The energy, the largest part of which is carried by the $\mathbf{Q}$-component, includes a singular coupling term favouring alignment between $\mathbf{Q}$ and $\mathbf{M}$. In this article and in the companion paper [18], we analyse the asymptotic behaviour of (not necessarily minimizing) critical points as a small parameter $\varepsilon$ tends to zero. In this paper, we prove that the (rescaled) energy density for the $\mathbf{Q}$-component, concentrates, to leading order, on a finite number of singular points. Moreover, we prove energy estimates and compactness results that will be crucially used in [18] to determine the structure of the energy concentration set for the $\mathbf{M}$-component as well as the relationship between the two singular sets.

**Keywords:** Ginzburg-Landau functional, Allen-Cahn equation, vectorial problems, topological singularities, rectifiable sets.

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The formation of gradient-driven singular structures of codimension one and two has been the object of intensive investigation over the past few decades. Singular structures of codimension one, i.e. interfaces, arise in both scalar and vectorial problems, with notable examples in the literature on diffusion-reaction equations, such as the Allen-Cahn equations for instance. While the mathematical theory on the scalar Allen-Cahn equation is well-advanced, much less is known in the vectorial case. The available results on Allen-Cahn systems focus on the asymptotic analysis, via Γ-convergence, of minimising solutions [3, 22] — with the exception of a recent paper [11], which obtains convergence results for non-minimising solutions of Allen-Cahn systems in dimension two. On the other hand, singular objects of codimension two arise only in vectorial problems. A prototypical example is the Ginzburg-Landau functional for superconductivity, which has been extensively studied since the seminal monograph [6]. There is now quite a well-established theory on Ginzburg-Landau solutions, encompassing both minimising and non-minimising ones as well as solutions to evolution problems and, to a lesser degree, generalisations to different functionals with a similar structure.

In this work, we investigate a two-dimensional problem where singularities of both codimension one and two emerge, due to the coupling between two different components of the problem. The physical motivation for this model is the study of *ferronematics*, a class of composite materials obtained as suspensions of magnetic nanoparticles in a nematic liquid crystal host [14, 26]. According to the approach proposed in [12], ferronematics are described by two order parameters. The orientation of the liquid crystal molecules is described by the Landau-de Gennes $\mathbf{Q}$-tensor, which is a map from the physical domain $\Omega \subseteq \mathbb{R}^2$ to the space $\mathscr{S}_0^{2\times 2}$ of $2 \times 2$, symmetric, real matrices with trace equal to zero. Nonzero values of $\mathbf{Q}$ correspond to liquid crystal configurations with a well-defined direction of molecular alignment, represented by the eigenspace of $\mathbf{Q}$ associated with the positive eigenvalue, while $\mathbf{Q} = 0$ indicates an isotropic state, where all the directions of molecular alignment are equally likely. The distribution of magnetic nanoparticles is described by the average magnetisation vector, $\mathbf{M}\colon \Omega \to \mathbb{R}^2$. The system is governed by a free energy functional which depends on both $\mathbf{Q}$ and $\mathbf{M}$:

$$\mathscr{F}_\varepsilon(\mathbf{Q}, \mathbf{M}) := \int_G \left\{ \frac{1}{2} \left|\nabla \mathbf{Q}\right|^2 + \frac{\varepsilon}{2} \left|\nabla \mathbf{M}\right|^2 + \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}, \mathbf{M}) \right\} \mathrm{d}x, \tag{1}$$

where $\varepsilon$ is a non-dimensional parameter. The interaction between the liquid crystal host and the magnetic inclusions is mediated by the potential $f_\varepsilon$, which takes the form

$$f_\varepsilon(\mathbf{Q}, \mathbf{M}) := \frac{1}{4}\left(1 - |\mathbf{Q}|^2\right)^2 + \frac{\varepsilon}{4}\left(1 - |\mathbf{M}|^2\right)^2 - \varepsilon\beta\, \mathbf{Q}\mathbf{M}\cdot\mathbf{M} + \kappa_\varepsilon. \tag{2}$$

Here $\beta > 0$ is given and $\kappa_\varepsilon$ is an additive constant, depending on $\varepsilon$ and $\beta$ only, uniquely determined by imposing that $\inf f_\varepsilon = 0$. Potential energies similar to (2) have been derived via suitable homogenisation limits [17]. For positive values of $\beta$, the potential $f_\varepsilon$ promotes alignment between the liquid crystal molecules and the magnetisation vector. Indeed, the potential $f_\varepsilon(\mathbf{Q}, \mathbf{M})$ is minimised by pairs $(\mathbf{Q}_\varepsilon^{\mathrm{pot}}, \mathbf{M}_\varepsilon^{\mathrm{pot}})$ that satisfy the conditions

$$\left|\mathbf{M}_\varepsilon^{\mathrm{pot}}\right| = \lambda_{\varepsilon,\beta}, \qquad \mathbf{Q}_\varepsilon^{\mathrm{pot}} = \sqrt{2} s_{\varepsilon,\beta}\left(\frac{\mathbf{M}_\varepsilon^{\mathrm{pot}} \otimes \mathbf{M}_\varepsilon^{\mathrm{pot}}}{\lambda_{\varepsilon,\beta}^2} - \frac{\mathbf{I}}{2}\right) \tag{3}$$

where $\mathbf{I}$ is the $2 \times 2$ identity matrix and $\lambda_{\varepsilon,\beta}$, $s_{\varepsilon,\beta}$ are positive constants, uniquely determined by $\varepsilon$ and $\beta$, such that

$$\lambda_{\varepsilon,\beta} \to \left(\sqrt{2}\beta + 1\right)^{1/2}, \qquad s_{\varepsilon,\beta} \to 1$$

as $\varepsilon \to 0$ (see [20, Lemma B.2]). We are interested in the limit as $\varepsilon \to 0$, which is physically motivated as the "large domain limit" (i.e., the size of the domain is much larger than the typical correlation length for the liquid crystal molecules). The asymptotic analysis of the Ginzburg-Landau functional shows that the energy for the $\mathbf{Q}$-component concentrates around a finite number of points, which correspond to *non-orientable* singularities of the limiting liquid crystal configuration. In particular, the $\mathbf{Q}$-component of minimisers converges to a matrix-valued map that admits no continuous orthonormal frame of eigenvectors. However, as the coupling promotes alignment between $\mathbf{M}$ and the eigenvectors of $\mathbf{Q}$, the energy for the $\mathbf{M}$-component should concentrate on singular lines, corresponding to jumps in the eigenvector frame.

The asymptotic analysis of minimisers for $\mathscr{F}_\varepsilon$, subject to Dirichlet boundary conditions, has been carried out in the paper [20]. In the present paper, instead, we consider general *critical points* $(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)$ of the functional $\mathscr{F}_\varepsilon$, that is, finite-energy solutions of the Euler-Lagrange system of equations

$$-\Delta\mathbf{Q}_\varepsilon + \frac{1}{\varepsilon^2}(|\mathbf{Q}_\varepsilon|^2 - 1)\mathbf{Q}_\varepsilon - \frac{\beta}{\varepsilon}\left(\mathbf{M}_\varepsilon \otimes \mathbf{M}_\varepsilon - \frac{|\mathbf{M}_\varepsilon|^2}{2}\mathbf{I}\right) = 0 \tag{4}$$

$$-\Delta\mathbf{M}_\varepsilon + \frac{1}{\varepsilon^2}(|\mathbf{M}_\varepsilon|^2 - 1)\mathbf{M}_\varepsilon - \frac{2\beta}{\varepsilon^2}\mathbf{Q}_\varepsilon\mathbf{M}_\varepsilon = 0. \tag{5}$$

At first sight, both (4) and (5) look like perturbed Ginzburg-Landau systems. However, (5) can be regarded as the Euler-Lagrange equation of a 'perturbed' vectorial Allen-Cahn-type energy functional with wells depending on $\mathbf{Q}_\varepsilon$. Indeed, the coupling term favours $\mathbf{M}_\varepsilon$ to align with the eigenvector of $\mathbf{Q}_\varepsilon$ corresponding to the positive eigenvalue, so that for each nozero value of $\mathbf{Q}_\varepsilon$, there are exactly two values of $\mathbf{M}_\varepsilon$ that minimise the potential. We will consider Dirichlet boundary conditions for $\mathbf{Q}_\varepsilon$ and either Dirichlet or Neumann boundary conditions for $\mathbf{M}_\varepsilon$ (see (7), (9) below for details).

There are physical motivations behind the analysis of general critical points, including both stable and unstable ones. In setting up variational models for physical systems, there is some consensus that observable configurations (often also referred to as *stable configurations*, according to Gibbs) should correspond to minimisers or local minimisers of the energy, subject to boundary conditions as appropriate. However, there are experimentally and numerically observed cases [15, 16, 24] in which, before decaying into a state of minimal or locally minimal energy an unperturbed system remains in a state of *unstable equilibrium* (also called a *metastable phase*) for quite a long time, with respect to the typical time-scale of the overall dynamics. It is been argued in [24] that such unstable equilibria should correspond to general critical points, so that their mathematical analysis is physically justified. These general remarks are particularly relevant to the functional (1) we are considering, since numerical results [12, 20] show an abundance of critical points, stable and unstable.

In the study of the system (4), (5), a major difficulty is provided by the coupling term. Roughly speaking, while we expect the integral of $\frac{\beta}{\varepsilon}\mathbf{Q}_\varepsilon\mathbf{M}_\varepsilon \cdot \mathbf{M}_\varepsilon$ in the free energy functional to be relatively small because of the alignment between $\mathbf{M}_\varepsilon$ and $\mathbf{Q}_\varepsilon$, the coupling terms in the equations (4), (5) have large prefactors, of order $\frac{1}{\varepsilon}$ or $\frac{1}{\varepsilon^2}$. To deal with these issues, here and in the companion paper [18], we adapt and combine methods from the literature on the Ginzburg-Landau functional [6, 7] with the approach introduced in the recent paper [11] for non-minimising solutions of the vectorial Allen-Cahn problem.

### Assumptions and setting of the problem

We consider two alternative sets of boundary conditions for $\mathbf{Q}_\varepsilon$ and $\mathbf{M}_\varepsilon$. One option is to impose Dirichlet boundary conditions for both $\mathbf{Q}_\varepsilon$ and $\mathbf{M}_\varepsilon$:

$$\mathbf{Q}_\varepsilon = \mathbf{Q}_{\mathrm{bd}}, \quad \mathbf{M}_\varepsilon = \mathbf{M}_{\mathrm{bd}} \qquad \text{on } \partial\Omega. \tag{6}$$

We assume the boundary data $\mathbf{Q}_{\rm bd} \in C^1(\partial\Omega,\, \mathscr{S}_0^{2\times 2})$, $\mathbf{M}_{\rm bd} \in C^1(\partial\Omega,\, \mathbb{R}^2)$ are $\varepsilon$-independent maps that satisfy

$$
|\mathbf{M}_{\rm bd}| = \left(\sqrt{2}\beta + 1\right)^{1/2}, \qquad \mathbf{Q}_{\rm bd} = \sqrt{2}\left(\frac{\mathbf{M}_{\rm bd}\otimes\mathbf{M}_{\rm bd}}{\sqrt{2}\beta+1} - \frac{\mathbf{I}}{2}\right) \tag{7}
$$

at any point of $\partial\Omega$. This particular form of the boundary datum approximates the set of minimisers (3) for the potential and ensures that the potential energy is small, i.e. $f_\varepsilon(\mathbf{Q}_{\rm bd},\, \mathbf{M}_{\rm bd}) \leq C\varepsilon^2$ for some $\varepsilon$-independent constant $C$ (see Lemma 1.1 below). Alternatively to (6), we consider 'mixed' boundary conditions, i.e. Dirichlet boundary conditions for $\mathbf{Q}_\varepsilon$ and homogeneous Neumann boundary conditions for $\mathbf{M}_\varepsilon$:

$$
\mathbf{Q}_\varepsilon = \mathbf{Q}_{\rm bd}, \quad \partial_{\boldsymbol{\nu}}\mathbf{M}_\varepsilon = 0 \qquad \text{on } \partial\Omega, \tag{8}
$$

where $\boldsymbol{\nu}$ is the exterior unit normal to $\partial\Omega$. We then assume the boundary datum $\mathbf{Q}_{\rm bd}$ (does not depend on $\varepsilon$ and) takes the form

$$
\mathbf{Q}_{\rm bd} = \sqrt{2}\left(\mathbf{n}_{\rm bd}\otimes\mathbf{n}_{\rm bd} - \frac{\mathbf{I}}{2}\right) \tag{9}
$$

on $\partial\Omega$, for some map $\mathbf{n}_{\rm bd} \in C^1(\partial\Omega,\, \mathbb{R}^2)$ which, a priori, is completely independent of the values of $\mathbf{M}_\varepsilon$ on the boundary.

Regardless of our choice of boundary conditions, we will always *assume* that there exists a constant $\mathrm{C}_{\rm pot} > 0$, independent of $\varepsilon$, such that

$$
\frac{1}{\varepsilon^2}\int_\Omega f_\varepsilon(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon) \leq \mathrm{C}_{\rm pot} \tag{10}
$$

for any $\varepsilon$ small enough.

*Remark* 1. If $\{(\mathbf{Q}_\varepsilon^\star, \mathbf{M}_\varepsilon^\star)\}$ is a sequence of *minimisers* of $\mathscr{F}_\varepsilon$, subject to either Dirichlet (6)–(7) or mixed boundary conditions (8)–(9), then the assumption (10) is automatically satisfied (for the pure Dirichlet boundary conditions (6)–(7), this follows from [20, Lemma 4.3 and Lemma 4.10]; for the 'mixed' boundary conditions (8)–(9), the arguments are completely analogous, see [19]).

*Remark* 2. If the domain $\Omega$ is star-shaped, then solutions to (4)–(5) subject to Dirichlet boundary conditions as in (6)–(7) necessarily satisfy the condition (10). This is a consequence of a Pohozaev identity (see Lemma 1.8 below), exactly as in [6]. By contrast, solutions to (4)–(5) subject to *mixed* boundary conditions need not satisfy (10). Indeed, if $\mathbf{Q}_\varepsilon$ is a critical point of the (uncoupled) Ginzburg-Landau functional subject to appropriate Dirichlet conditions, then $(\mathbf{Q}_\varepsilon,\, 0)$ is a solution to (4)–(5) with boundary conditions (8), yet

$$
\frac{1}{\varepsilon^2}\int_\Omega f_\varepsilon(\mathbf{Q}_\varepsilon,\, 0) = \left(\frac{1}{4\varepsilon} + \frac{\kappa_\varepsilon}{\varepsilon^2}\right)|\Omega| = \mathrm{O}\left(\frac{1}{\varepsilon}\right),
$$

for Lemma 1.1 below implies that $\kappa_\varepsilon = \mathrm{O}(\varepsilon)$ as $\varepsilon \to 0$.

*Remark* 3. Under the assumption (10) (and for boundary data as in (6)–(7) or (8)–(9)), we will later *prove* that

$$\int_\Omega |\nabla \mathbf{Q}_\varepsilon|^2 \leq C\, |\log \varepsilon|\,, \qquad \varepsilon \int_\Omega |\nabla \mathbf{M}_\varepsilon|^2 \leq C \tag{11}$$

for some constant $C$ that does not depend on $\varepsilon$ (see Proposition 2.1 and 2.9, respectively) depending only on $\beta$, $\mathrm{C_{pot}}$, and the boundary data.

We consider the functions

$$\mu_\varepsilon := \frac{1}{|\log \varepsilon|}\left(\frac{1}{2}\,|\nabla \mathbf{Q}_\varepsilon|^2 + \frac{\varepsilon}{2}\,|\nabla \mathbf{M}_\varepsilon|^2 + \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)\right) \tag{12}$$

$$\nu_\varepsilon := \frac{\varepsilon}{2}\,|\nabla \mathbf{M}_\varepsilon|^2 + \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon) \tag{13}$$

Under the assumption (10) and for boundary data as in (6)–(7) or (8)–(9), by Remark 3 it follows that that the families $(\mu_\varepsilon)_{\varepsilon>0}$, $(\nu_\varepsilon)_{\varepsilon>0}$ are bounded in $L^1(\Omega)$. Given a map $\mathbf{Q} \in W^{1,2}(\Omega,\, \mathscr{S}_0^{2\times 2})$, we define the pre-Jacobian of $\mathbf{Q}$ as a vector-field $j(\mathbf{Q})\colon \Omega \to \mathbb{R}^2$, given component-wise as

$$j(\mathbf{Q}) := (Q_{11}\partial_1 Q_{12} - Q_{12}\partial_1 Q_{11},\, Q_{11}\partial_2 Q_{12} - Q_{12}\partial_2 Q_{11})$$

(see (1.6) below for more details).

**Convergence results: Theorem A**

The main result of this paper is the following theorem.

**Theorem A.** *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, simply connected domain of class $C^2$. Let $\{(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)\} \subset W^{1,2}(\Omega,\, \mathscr{S}_0^{2\times 2}) \times W^{1,2}(\Omega,\, \mathbb{R}^2)$ be a sequence of critical points of $\mathscr{F}_\varepsilon$ subject to either* (6)–(7) *or to* (8)–(9)*. Assume that the condition* (10) *is satisfied. Then, as $\varepsilon \to 0$,*

*(i)* $\mathbf{Q}_\varepsilon \to \mathbf{Q}_\star$ *strongly in $W^{1,p}(\Omega)$ for any $p < 2$ and* $\mathbf{M}_\varepsilon \to \mathbf{M}_\star$ *strongly in $L^p(\Omega)$ for any $p < +\infty$.*

*(ii) The limiting maps $\mathbf{Q}_\star$, $\mathbf{M}_\star$ satisfy*

$$|\mathbf{Q}_\star(x)| = 1, \quad |\mathbf{M}_\star(x)| = \left(\sqrt{2}\beta + 1\right)^{1/2}, \quad \mathbf{Q}_\star(x) = \sqrt{2}\left(\frac{\mathbf{M}_\star(x) \otimes \mathbf{M}_\star(x)}{\sqrt{2}\beta + 1} - \frac{\mathbf{I}}{2}\right)$$

*for a.e. $x \in \Omega$ and*

$$\operatorname{div} j(\mathbf{Q}_\star) = 0 \tag{14}$$

*in the sense of distributions in $\Omega$.*

*(iii)* $\mu_\varepsilon \rightharpoonup \mu_\star$ *and* $\nu_\varepsilon \rightharpoonup \nu_\star$ *weakly* as measures in $\Omega$.*

*(iv)* $\operatorname{spt} \mu_\star$ *is finite.*

*(v)* $\mathbf{Q}_\varepsilon \to \mathbf{Q}_\star$ *strongly in* $W^{1,p}_{\mathrm{loc}}(\Omega \setminus \operatorname{spt}\mu_\star)$ *for any* $p < +\infty$.

Theorem A shows, in particular, that the energy of a sequence of critical points concentrates on a finite set $\operatorname{spt}\mu_\star$ of singular points, arising from the Ginzburg-Landau terms in $\mathbf{Q}_\varepsilon$. We also emphasise that, by (14) and item (v), $\mathbf{Q}_\star$ is a smooth harmonic map in $\Omega \setminus \operatorname{spt}\mu_\star$.

As we are going to discuss in more detail later on, the proof of Theorem A is fundamentally rooted in the classical Ginzburg-Landau theory, as developed in [6, 7], although the presence of the coupling terms in the Euler-Lagrange equations (4), (5) calls for special care and non-trivial refinements of the arguments in [6, 7]. The situation changes completely when coming to the analysis of the energy densities $\nu_\varepsilon$, the reason being that their behaviour is basically controlled by the system (5), which, as mentioned above, should be properly considered as a perturbed Allen-Cahn system. Correspondingly, the arguments should be completely different from those for the measures $\mu_\varepsilon$, and follow instead the lines of the asymptotic analysis of critical points of the vectorial Allen-Cahn functional, as developed in the recent breakthrough paper [11]. Nonetheless, since the arguments in [11] are ultimately based on refined energy estimates and since the systems (4), (5) are coupled, the estimates and compactness properties obtained in the present paper will be crucial to cope with the consequences of the presence of the coupling term in (5). However, because of the significantly different framework, we deferred the study of the asymptotic behaviour of energy densities $\nu_\varepsilon$ to the companion paper [18].

## Structure of the proof

The proof of Theorem A is substantially more complex than the analogous results for minimising solutions. Many of the arguments of [20, 19] do not carry over to this context, because they are based on energy minimality. Instead, our proofs combine different elements. In order to obtain compactness for the $\mathbf{Q}_\varepsilon$-component, we adapt arguments from the Ginzburg-Landau literature, mostly from [6] and [7], with some modifications due to the presence of coupling terms in equations (4) and (5). The same arguments also show that $\mu_\varepsilon$ concentrates on a finite number of singular points. To prove compactness for the $\mathbf{M}_\varepsilon$-component, we perform the same change of variables that is used in [20] (see Section 3.1), which enables us to apply compactness results for the vectorial Modica-Mortola problem [22].

The proof of the strong convergence $\mathbf{Q}_\varepsilon \to \mathbf{Q}_\star$ in $W^{1,2}_{\mathrm{loc}}(\Omega \setminus \operatorname{spt}\mu_\star)$ is more delicate and, once again, the main source of difficulty is the coupling term. The heart of the matter consists in proving that, for any ball $B \subset\subset \Omega \setminus \operatorname{spt}\mu_\star$, there holds

$$\int_B \left\{ \left|\nabla \left|\mathbf{Q}_\varepsilon\right|\right|^2 + \left(\frac{|\mathbf{Q}_\varepsilon| - 1}{\varepsilon} - \kappa_\star\right)^2 \right\} \mathrm{d}x \to 0, \qquad \text{as } \varepsilon \to 0, \tag{15}$$

where $\kappa_\star = \frac{\beta}{2\sqrt{2}}\left(\sqrt{2}\beta + 1\right)$ is a constant representing the ground level of the 'modified' Ginzburg-Landau potential arising because of the interaction between the $\mathbf{Q}$ and $\mathbf{M}$ (i.e., the term in round brackets in the left-hand side of (15)). The analogous statement in the classical Ginzburg-Landau theory, formally corresponding to $\beta = 0$, is well-known, see [6, Theorem X.2]. However, the argument of [6] does not work directly in our case, exactly because of the coupling term. The proof of (15), detailed in Lemma 3.7 below, is therefore trickier than its counterpart [6, Equation (127)]. Once (15) is proved, the rest of the argument in [6], based on a suitable Hodge decomposition of the pre-Jacobian $j(\mathbf{Q}_\varepsilon)$ of $\mathbf{Q}_\varepsilon$, carries on with small modifications (see Proposition 3.5 for full details).

The importance of (15) is not limited to the proof of the locally strong convergence away from $\operatorname{spt}\mu_\star$. As detailed in the companion article [18], it is also a key ingredient in the asymptotic analysis of the energy measures $\nu_\varepsilon$ as well as in the description of the support of the corresponding limiting measure and of its relationship with the support of $\mu_\star$.

### Plan of the paper

The paper is organised as follows. Section 1 contains some notation and preliminary results on the space of $\mathbf{Q}$-tensors, on the potential $f_\varepsilon$, and some first consequences of the equations (4)–(5) (such as the maximum principle, the Pohozaev identity, and a "clearing-out" property [6]). Section 2 is devoted to uniform estimates on both $\mathbf{Q}_\varepsilon$ and $\mathbf{M}_\varepsilon$. For instance, we prove uniform $W^{1,p}(\Omega)$-estimates on $\mathbf{Q}_\varepsilon$, for $p < 2$, as well as uniform $L^1(\Omega)$-estimates for $\mu_\varepsilon$ and $\nu_\varepsilon$. The main estimates obtained through the section are all summarised in Theorem 2.11. In Section 3, we will apply these estimates to prove compactness for both $\mathbf{Q}_\varepsilon$ and $\mathbf{M}_\varepsilon$. As a key tool, we use a change of coordinates, already introduced in [20], allowing us to write the functional as an almost decoupled one: a Ginzburg-Landau type term, a vectorial Allen-Cahn term, and a small error (vanishing with $\varepsilon$). The proof of Theorem A is presented in Section 4. Finally, Appendix A completes the paper, with the proof of some technical results.

**Notation.** We use the symbol $\{X_\varepsilon\}$ to denote a family of objects indexed by the parameter $\varepsilon > 0$, almost always used as a shorthand to denote the sequence $\{X_{\varepsilon_k}\}_k$, where $\varepsilon_k \to 0$ as $k \to +\infty$. Usually, for the sake of a lighter notation, we do not relabel subsequences.

In inequalities like $A \lesssim B$, the symbol $\lesssim$ means that there exists a constant $C$, independent of $A$ and $B$, such that $A \leq CB$. In particular, dealing with sequences indexed by $\varepsilon$, we use $\lesssim$ to denote inequality up to a constant independent of $\varepsilon$. Whenever it is relevant, we keep track of the dependences of the implicit constants. Whenever possible without inducing ambiguities, for the sake of a lighter notation, we avoid writing explicitly the measure of integration in the integrals.

## Contents

# 1 Preliminary estimates

## 1.1 The space of Q-tensors

In the Introduction, we defined the space of $\mathbf{Q}$-tensors $\mathscr{S}_0^{2\times 2}$ as

$$\mathscr{S}_0^{2\times 2} := \{\mathbf{Q} \in \mathbb{M}_{2\times 2}(\mathbb{R}) \,:\, \mathbf{Q} = \mathbf{Q}^{\mathrm{t}},\ \mathrm{tr}(\mathbf{Q}) = 0\}.$$

It follows immediately that $\mathscr{S}_0^{2\times 2}$ is a linear space of dimension 2 whose elements are $2\times 2$, real matrices of the form

$$\mathbf{Q} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{pmatrix}.$$

We endow $\mathscr{S}_0^{2\times 2}$ with the usual Frobenius norm of matrices, i.e.,

$$|\mathbf{Q}| := \sqrt{\mathrm{tr}(\mathbf{Q}^{\mathrm{t}}\mathbf{Q})}, \qquad \text{for any } \mathbf{Q} \in \mathscr{S}_0^{2\times 2}.$$

The map

$$\mathscr{S}_0^{2\times 2} \ni \mathbf{Q} \mapsto \mathbf{q} := \sqrt{2}\,(Q_{11},\, Q_{12}) \in \mathbb{R}^2 \tag{1.1}$$

establishes an isometric isomorphism between $\mathscr{S}_0^{2\times 2}$ and $\mathbb{R}^2$ (the latter provided with the standard scalar product). Of course, we can also identify any $\mathbf{Q} \in \mathscr{S}_0^{2\times 2}$ with the complex number $Q_{11} + iQ_{12}$, and thus we have the further identification $\mathscr{S}_0^{2\times 2} \simeq \mathbb{C}$.

We denote as $\mathscr{N}$ the unit sphere in $\mathscr{S}_0^{2\times 2}$, i.e.,

$$\mathscr{N} := \left\{\mathbf{Q} \in \mathscr{S}_0^{2\times 2} \,:\, |\mathbf{Q}| = 1\right\},$$

which can be identified with the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$ (or, equivalently, in $\mathbb{C}$). Moreover, in view of (1.1), we may also identify, equivalently,

$$\mathscr{N} = \left\{\sqrt{2}\left(\mathbf{n}\otimes\mathbf{n} - \frac{\mathbf{I}}{2}\right) \,:\, \mathbf{n} \in \mathbb{S}^1\right\}.$$

**Representing Q-tensors in "polar coordinates".** Let $G \subseteq \Omega$ be a simply connected, smooth subdomain, and let $\mathbf{Q}\colon G \to \mathscr{S}_0^{2\times 2}$ be a smooth map such that $|\mathbf{Q}| > 0$ on $G$. The spectral theorem implies that $\mathbf{Q}$ can be written in the form $\sqrt{2}\mathbf{Q} = |\mathbf{Q}|\,(\mathbf{n}\otimes\mathbf{n} - \mathbf{m}\otimes\mathbf{m})$, where $(\mathbf{n},\,\mathbf{m})$ is an orthonormal basis of eigenvectors of $\mathbf{Q}$. Both $\mathbf{n}$ and $\mathbf{m}$ are uniquely determined by $\mathbf{Q}$, up to their sign. (In other words, either $\mathbf{m} = \mathbf{n}^\perp$ or $\mathbf{m} = -\mathbf{n}^\perp$.) Similarly, $\mathbf{n}$ can be written as $\mathbf{n} = (\cos\varphi,\, \sin\varphi)$ for some scalar function $\varphi\colon G \to \mathbb{R}$. As a consequence, setting $\rho := |\mathbf{Q}|$, we can write

$$\mathbf{Q} = \frac{\rho}{\sqrt{2}}\begin{pmatrix} \cos(2\varphi) & \sin(2\varphi) \\ \sin(2\varphi) & -\cos(2\varphi)\end{pmatrix} \tag{1.2}$$

The function $\varphi$ is uniquely determined by $\mathbf{Q}$, up to a constant multiple of $\pi$. An explicit computation shows that

$$|\nabla\mathbf{Q}|^2 = |\nabla\rho|^2 + 4\rho^2\,|\nabla\varphi|^2 \tag{1.3}$$

Moreover, if we only assume that $\mathbf{Q}$ is bounded away from zero on $\partial G$, but not necessarily in the interior of $G$, we can still write $\mathbf{Q}$ as in (1.2), except that the function $\varphi$ can be discontinuous. However, we can always choose $\varphi$ to be continuous on $\partial G$, except for one jump discontinuity at most (see, e.g., [13, Proof of Lemma 1.3 and the remarks thereafter]). If $x_0 \in \partial G$ is the unique jump point of $G$ and $\varphi(x_0^+)$, $\varphi(x_0^-)$ are the traces of $\varphi$ on either side of $x_0$ (oriented in the anticlockwise direction), then we define

$$\deg(\mathbf{Q},\,\partial G) := \frac{1}{2\pi}\left(\varphi(x_0^+) - \varphi(x_0^-)\right) \tag{1.4}$$

The number $\deg(\mathbf{Q},\,\partial G)$ is the topological degree of $\mathbf{Q}$ on $\partial G$, and is a half-integer. If $\mathbf{Q}$ admits a continuous lifting on $\partial G$, then $\deg(\mathbf{Q},\,\partial G) = 0$ (and vice-versa). Similar considerations apply when $\mathbf{Q} \in W^{1,2}(\Omega,\,\mathscr{S}_0^{2\times 2})$, so long as $\rho$ is bounded away from zero (by lifting results for $\mathbf{Q}$-tensors — see, e.g., [4, 8] — and degree theory in $W^{1/2,2}$ — see, e.g., [13, Proposition 12.1]).

**Pre-Jacobians.** We recall some notation from [20]. We define the vector product of two matrices $\mathbf{Q} \in \mathscr{S}_0^{2\times 2}$, $\mathbf{P} \in \mathscr{S}_0^{2\times 2}$ as

$$\mathbf{Q}\times\mathbf{P} := Q_{11}P_{12} - Q_{12}P_{11} + Q_{21}P_{22} - Q_{22}P_{21} = 2\,(Q_{11}P_{12} - Q_{12}P_{11}) \tag{1.5}$$

For any $\mathbf{Q} \in (L^\infty \cap W^{1,1})\left(\Omega,\,\mathscr{S}_0^{2\times 2}\right)$, we define the vector field $j(\mathbf{Q})\colon \Omega \to \mathbb{R}^2$ as

$$j(\mathbf{Q}) := \frac{1}{2}\left(\mathbf{Q}\times\partial_1\mathbf{Q},\,\mathbf{Q}\times\partial_2\mathbf{Q}\right). \tag{1.6}$$

For notational convenience, we shall often write (1.6) in the form

$$j(\mathbf{Q}) = \frac{1}{2}\mathbf{Q}\times\nabla\mathbf{Q}. \tag{1.7}$$

By analogy with the Ginzburg-Landau literature, $j(\mathbf{Q})$ could be called the “pre-Jacobian” of $\mathbf{Q}$. This terminology is justified because, for smooth maps $\mathbf{Q}$, we have

$$\frac{1}{2}\operatorname{curl} j(\mathbf{Q}) = \partial_1 Q_{11}\, \partial_2 Q_{12} - \partial_2 Q_{11}\, \partial_1 Q_{12} \tag{1.8}$$

(and the same remains true if $\mathbf{Q} \in W^{1,2}(\Omega,\, \mathscr{S}_0^{2\times 2})$, by a density argument). In particular, for any $\mathbf{Q}$-tensor field $\mathbf{Q} \in (L^\infty \cap W^{1,1})\left(\Omega,\, \mathscr{S}_0^{2\times 2}\right)$ it makes sense to define

$$J(\mathbf{Q}) := \operatorname{curl} j(\mathbf{Q}), \tag{1.9}$$

if we agree that the derivatives are taken in the sense of distributions. The distribution $J(\mathbf{Q})$ is called the *distributional Jacobian* of $\mathbf{Q}$.

If $\mathbf{Q}$ can be represented in the form (1.2), then

$$j(\mathbf{Q}) = \rho^2\, \nabla\varphi. \tag{1.10}$$

As a consequence, for any smooth $\mathbf{Q}\colon \Omega \to \mathscr{S}_0^{2\times 2}$, there holds

$$|\mathbf{Q}|^2\, |\nabla\mathbf{Q}|^2 = |\mathbf{Q}|^2\, |\nabla(|\mathbf{Q}|)|^2 + 4\, |j(\mathbf{Q})|^2 \tag{1.11}$$

The equality (1.11) holds at any point of $\{\mathbf{Q} \neq 0\}$ because of (1.3), (1.10), and is satisfied trivially at almost any point of $\{\mathbf{Q} = 0\}$. Another consequence of (1.10) is the equality

$$\operatorname{curl}\left(\frac{j(\mathbf{Q})}{|\mathbf{Q}|^2}\right) = 0, \tag{1.12}$$

which holds at any point $x \in \Omega$ such that $\mathbf{Q}(x) \neq 0$. Moreover, if $G \subseteq \Omega$ is a simply connected, smooth subdomain such that $|\mathbf{Q}| > 0$ on $\partial G$ and $\boldsymbol{\tau}$ is the unit tangent vector to $\partial G$ oriented anti-clockwise direction, then

$$\int_{\partial G} \frac{j(\mathbf{Q})\cdot\boldsymbol{\tau}}{|\mathbf{Q}|^2}\, \mathrm{d}s = \pi \deg(\mathbf{Q},\, \partial G) \tag{1.13}$$

because of (1.4) and (1.10).

*Remark* 1.4. The distributional pre-Jacobian and Jacobian are continuous with respect to the weak $W^{1,p}$-convergence, for any $p > 1$. In other words, given a sequence of uniformly bounded $\mathbf{Q}$-tensor fields $\mathbf{Q}_\varepsilon \in W^{1,p}\left(\Omega,\, \mathscr{S}_0^{2\times 2}\right)$ which converges weakly in $W^{1,p}\left(\Omega,\, \mathscr{S}_0^{2\times 2}\right)$ to some $\mathbf{Q}$-tensor field $\mathbf{Q}_\star$ as $\varepsilon \to 0$, then $j(\mathbf{Q}_\varepsilon) \to j(\mathbf{Q}_\star)$ and $J(\mathbf{Q}_\varepsilon) \to J(\mathbf{Q}_\star)$ as $\varepsilon \to 0$ in the sense of distributions in $\Omega$. (Cf., e.g., [1, Remark 3.6(iii)].)

**Notation.** Besides the pre-Jacobian of $\mathbf{Q}$-tensor fields, occasionally we shall also consider the pre-Jacobian of a vector field $\mathbf{u} \in (L^\infty \cap W^{1,1})\,(\Omega,\, \mathbb{R}^2)$. Denoted $j(\mathbf{u})$, such a pre-Jacobian can be identified with the vector field with components

$$(j(\mathbf{u}))_k = (u_1\partial_k u_2 - u_2\partial_k u_1)_k\,, \qquad \text{for } k \in \{1,\, 2\}, \tag{1.14}$$

i.e.,

$$j(\mathbf{u}) = \mathbf{u} \times \nabla\mathbf{u},$$

exactly as in the standard Ginzburg-Landau theory.

## 1.2 Properties of the potential $f_\varepsilon$.

We summarise below some properties of the potential $f_\varepsilon$. As in [20, Equation (3.1)], we define

$$\kappa_\star := \frac{1}{2\sqrt{2}}\beta\left(\sqrt{2}\beta + 1\right). \tag{1.15}$$

**Lemma 1.1** ([20, Lemma B.3])**.** *The potential $f_\varepsilon$ satisfies the following properties.*

*(i) The constant $\kappa_\varepsilon$ in* (2)*, uniquely defined by imposing the condition* $\inf f_\varepsilon = 0$*, satisfies*

$$\kappa_\varepsilon = \frac{1}{2}\left(\beta^2 + \sqrt{2}\beta\right)\varepsilon + \kappa_\star^2\varepsilon^2 + \mathrm{o}(\varepsilon^2)$$

*as $\varepsilon \to 0$.*

*(ii) If $(\mathbf{Q}, \mathbf{M}) \in \mathscr{S}_0^{2\times 2} \times \mathbb{R}^2$ is such that*

$$|\mathbf{M}| = \left(\sqrt{2}\beta + 1\right)^{1/2}, \qquad \mathbf{Q} = \sqrt{2}\left(\frac{\mathbf{M}\otimes\mathbf{M}}{\sqrt{2}\beta + 1} - \frac{\mathbf{I}}{2}\right)$$

*then $f_\varepsilon(\mathbf{Q}, \mathbf{M}) = \kappa_\star^2\varepsilon^2 + \mathrm{o}(\varepsilon^2)$ as $\varepsilon \to 0$.*

*(iii) If $\varepsilon$ is sufficiently small, then for any $(\mathbf{Q}, \mathbf{M}) \in \mathscr{S}_0^{2\times 2} \times \mathbb{R}^2$ we have*

$$\frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}, \mathbf{M}) \geq \frac{1}{8\varepsilon^2}\left(|\mathbf{Q}|^2 - 1\right)^2 - \beta^2|\mathbf{M}|^4. \tag{1.16}$$

For any $(\mathbf{Q}, \mathbf{M}) \in \mathscr{S}_0^{2\times 2} \times \mathbb{R}^2$, we can write

$$\frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}, \mathbf{M}) = \frac{1}{4\varepsilon^2}\left(|\mathbf{Q}|^2 - 1\right)^2 + \frac{1}{\varepsilon}\ell(\mathbf{Q}, \mathbf{M}) + \chi_\varepsilon, \tag{1.17}$$

where

$$\ell(\mathbf{Q}, \mathbf{M}) := \frac{1}{4}\left(|\mathbf{M}|^2 - 1\right)^2 - \beta\,\mathbf{Q}\mathbf{M}\cdot\mathbf{M} + \frac{1}{2}\left(\beta^2 + \sqrt{2}\beta\right) \tag{1.18}$$

and

$$\chi_\varepsilon := \frac{\kappa_\varepsilon}{\varepsilon^2} - \frac{1}{2\varepsilon}\left(\beta^2 + \sqrt{2}\beta\right). \tag{1.19}$$

Note that

$$\chi_\varepsilon \to \kappa_\star^2 \qquad \text{as } \varepsilon \to 0. \tag{1.20}$$

In the next lemma, we consider minimisers of the function $\ell(\mathbf{Q}, \cdot)$, for a given $\mathbf{Q}$. We recall that any $\mathbf{Q} \in \mathscr{S}_0^{2\times 2} \setminus \{0\}$ has exactly one positive and one negative eigenvalues, because of the constraint $\operatorname{tr}\mathbf{Q} = 0$.

**Lemma 1.2.** *For any* $\mathbf{Q} \in \mathscr{S}_0^{2\times 2} \setminus \{0\}$*, the function* $\ell(\mathbf{Q},\,\cdot\,)$ *has exactly two minimisers, given by*

$$\mathbf{M}_\pm := \pm\left(\sqrt{2}\beta\rho + 1\right)^{1/2}\mathbf{n} \tag{1.21}$$

*where* $\rho := |\mathbf{Q}|$ *and* $\mathbf{n}$ *is a unit eigenvector of* $\mathbf{Q}$ *corresponding to its positive eigenvalue. Moreover, there holds*

$$\min \ell(\mathbf{Q},\,\cdot\,) = \ell(\mathbf{Q},\,\mathbf{M}_\pm) = \frac{\beta}{2}(1-\rho)\left(\sqrt{2}+\beta+\beta\rho\right). \tag{1.22}$$

*Proof.* The gradient and the Hessian of $\ell$ with respect to the $\mathbf{M}$-variable are given by

$$\begin{aligned}
\nabla_{\mathbf{M}}\ell(\mathbf{Q},\,\mathbf{M}) &= \left(|\mathbf{M}|^2 - 1\right)\mathbf{M} - 2\beta\mathbf{Q}\mathbf{M} \\
\mathrm{D}^2_{\mathbf{M}}\ell(\mathbf{Q},\,\mathbf{M}) &= \left(|\mathbf{M}|^2 - 1\right)\mathbf{I} + 2\mathbf{M}\otimes\mathbf{M} - 2\beta\mathbf{Q}.
\end{aligned}$$

We immediately see that any critical point of $\ell(\mathbf{Q},\,\cdot\,)$ must be an eigenvector of $\mathbf{Q}$. Since $\operatorname{tr}\mathbf{Q} = 0$ and $\operatorname{tr}(\mathbf{Q}^2) = \rho^2$, the eigenvalues of $\mathbf{Q}$ must be equal to $\pm\rho/\sqrt{2}$. Let $(\mathbf{n},\,\mathbf{m})$ be an orthonormal basis of eigenvectors of $\mathbf{Q}$, such that $\mathbf{n}$ is associated with the positive eigenvalue of $\mathbf{Q}$ and $\mathbf{m}$ is associated with the negative one. Then, substituting $\mathbf{M} = \lambda\,\mathbf{n}$, $\mathbf{M} = \mu\,\mathbf{m}$ in the condition $\nabla_{\mathbf{M}}\ell = 0$, we find that critical points (with respect to the $\mathbf{M}$-variable only) occur for

$$\mathbf{M} = 0, \qquad \mathbf{M} = \mathbf{M}_\pm := \pm\left(1+\sqrt{2}\beta\rho\right)^{1/2}\mathbf{n}, \qquad \mathbf{M} = \widetilde{\mathbf{M}}_\pm := \pm\left(1-\sqrt{2}\beta\rho\right)^{1/2}\mathbf{m}$$

— the latter pair, $\widetilde{\mathbf{M}}_\pm$, being defined only when $\sqrt{2}\beta\rho \leq 1$. However, $\mathrm{D}^2_{\mathbf{M}}\ell(\mathbf{Q},\,0) = -\mathbf{I} - 2\beta\mathbf{Q}$ cannot be positive semi-definite, because its trace is equal to $-2$. Moreover, we have $\ell(\mathbf{Q},\,\widetilde{\mathbf{M}}_+) = \ell(\mathbf{Q},\,\widetilde{\mathbf{M}}_-)$, $\ell(\mathbf{Q},\,\mathbf{M}_+) = \ell(\mathbf{Q},\,\mathbf{M}_-)$ and

$$\ell(\mathbf{Q},\,\widetilde{\mathbf{M}}_\pm) - \ell(\mathbf{Q},\,\mathbf{M}_\pm) = \sqrt{2}\beta\rho > 0.$$

Therefore, the minimisers of $\ell(\mathbf{Q},\,\cdot\,)$ are exactly those given in (1.21), and (1.22) follows by an explicit computation. ☐

**Corollary 1.3.** *For any* $\mathbf{Q} \in \mathscr{S}_0^{2\times 2} \setminus \{0\}$ *and any* $\mathbf{M} \in \mathbb{R}^2$*, there holds*

$$\ell(\mathbf{Q},\,\mathbf{M}) - \ell(\mathbf{Q},\,\mathbf{M}_\pm) = \frac{1}{4}\left|\mathbf{M}-\mathbf{M}_+\right|^2\left|\mathbf{M}-\mathbf{M}_-\right|^2 + \sqrt{2}\beta\rho(\mathbf{M}\cdot\mathbf{m})^2, \tag{1.23}$$

*where* $(\mathbf{n},\,\mathbf{m})$ *is an eigenframe for* $\mathbf{Q}$ *and* $\mathbf{n}$ *is a unit eigenvector of* $\mathbf{Q}$ *corresponding to its positive eigenvalue.*

*Proof.* Since $\rho = |\mathbf{Q}| > 0$, we can write $\mathbf{Q} = \frac{\rho}{\sqrt{2}}(\mathbf{n}\otimes\mathbf{n} - \mathbf{m}\otimes\mathbf{m})$, where $\mathbf{n}$ is an eigenvector of $\mathbf{Q}$ corresponding to its positive eigenvalue and $\mathbf{m}$ is orthogonal to $\mathbf{n}$. Furthermore, we can decompose $\mathbf{M}$ along $(\mathbf{n},\,\mathbf{m})$, i.e, we can write

$$\mathbf{M} = (\mathbf{M}\cdot\mathbf{n})\mathbf{n} + (\mathbf{M}\cdot\mathbf{m})\mathbf{m},$$

and it follows that

$$\mathbf{QM} \cdot \mathbf{M} = \frac{\rho}{\sqrt{2}} \left( (\mathbf{M} \cdot \mathbf{n})^2 - (\mathbf{M} \cdot \mathbf{m})^2 \right)$$

On the other hand, we may obviously rephrase

$$\left( |\mathbf{M}|^2 - 1 \right)^2 = \left( |\mathbf{M}|^2 - \left( 1 + \sqrt{2}\beta\rho \right) \right)^2 + 2\sqrt{2}\beta\rho \left( |\mathbf{M}|^2 - 1 - \sqrt{2}\beta\rho \right) + 2\beta^2\rho^2.$$

Combining the above formulas with the definition (1.18) of $\ell$ and with (1.22), we obtain

$$\begin{aligned} \ell(\mathbf{Q},\, \mathbf{M}) &= \frac{1}{4} \left( |\mathbf{M}|^2 - \left( 1 + \sqrt{2}\beta\rho \right) \right)^2 + \frac{\sqrt{2}}{2}\beta\rho \left( 2(\mathbf{M} \cdot \mathbf{m})^2 - \left( 1 + \sqrt{2}\beta\rho \right) \right) \\ &\quad + \frac{1}{2}\beta^2\rho^2 + \frac{1}{2} \left( \beta^2 + \sqrt{2}\beta \right) \\ &= \ell(\mathbf{Q},\, \mathbf{M}_\pm) + \frac{1}{4} \left( |\mathbf{M}|^2 - \left( 1 + \sqrt{2}\beta\rho \right) \right)^2 + \sqrt{2}\beta\rho(\mathbf{M} \cdot \mathbf{m})^2. \end{aligned}$$

After rearrangement, the conclusion follows by noticing that, in view of (1.21),

$$\left( |\mathbf{M}|^2 - \left( 1 + \sqrt{2}\beta\rho \right) \right)^2 = |\mathbf{M} - \mathbf{M}_+|^2 \, |\mathbf{M} - \mathbf{M}_-|^2 .$$

□

For $\mathbf{Q} \in \mathscr{S}_0^{2\times 2} \setminus \{0\}$, let $\Sigma(\mathbf{Q}) := \{\mathbf{M}_+,\, \mathbf{M}_-\} \subseteq \mathbb{R}^2$ be the set of minimisers of $\ell(\mathbf{Q},\, \cdot)$, as given by (1.21). If $\mathbf{Q} \neq 0$ and $\mathrm{dist}(\mathbf{M},\, \Sigma(\mathbf{Q})) \leq 1$, we define

$$\pi(\mathbf{Q},\, \mathbf{M}) := \begin{cases} \mathbf{M}_+ & \text{if } |\mathbf{M} - \mathbf{M}_+| \leq 1 \\ \mathbf{M}_- & \text{if } |\mathbf{M} - \mathbf{M}_-| \leq 1. \end{cases} \tag{1.24}$$

$\pi(\mathbf{Q},\, \mathbf{M})$ is the closest projection of $\mathbf{M}$ to $\Sigma(\mathbf{Q})$. It is defined in a non-ambiguous way, because $|\mathbf{M}_+ - \mathbf{M}_-| = 2(\sqrt{2}\beta\rho + 1)^{1/2} > 2$.

*Remark* 1.5. Let $\mathbf{Q} \in \mathscr{S}_0^{2\times 2} \setminus \{0\}$ be arbitrary. As an obvious consequence of (1.23), and since $|\mathbf{M}_+ - \mathbf{M}_-| > 2$, letting $\mathbf{N} = \pi(\mathbf{Q},\, \mathbf{M})$, for any $\mathbf{M} \in \mathbb{R}^2$ such that $\mathrm{dist}(\mathbf{M},\, \Sigma(\mathbf{Q})) \leq 1$, there holds

$$\ell(\mathbf{Q},\, \mathbf{M}) - \ell(\mathbf{Q},\, \mathbf{N}) \geq \frac{1}{4} |\mathbf{M} - \mathbf{N}|^2 . \tag{1.25}$$

**Lemma 1.4.** *For any $\Lambda > 1$, there exist positive constants $\delta_0(\Lambda)$ and $c_0(\Lambda)$ (depending on $\Lambda$ and $\beta$ only) such that the following statement holds: for any $(\mathbf{Q},\, \mathbf{M}) \in \mathscr{S}_0^{2\times 2} \times \mathbb{R}^2$ such that $\Lambda^{-1} \leq |\mathbf{Q}| \leq \Lambda$ and $\mathrm{dist}(\mathbf{M},\, \Sigma(\mathbf{Q})) \leq \delta_0(\Lambda)$, there holds*

$$\nabla_{\mathbf{M}}\ell(\mathbf{Q},\, \mathbf{M}) \cdot (\mathbf{M} - \pi(\mathbf{Q},\, \mathbf{M})) \geq c_0(\Lambda)\Big( \ell(\mathbf{Q},\, \mathbf{M}) - \ell(\mathbf{Q},\, \pi(\mathbf{Q},\, \mathbf{M})) \Big) \geq 0.$$

*Proof.* Let $X(\Lambda) \subseteq \mathscr{S}_0^{2\times 2} \times \mathbb{R}^2$ be the set of pairs $(\mathbf{Q},\, \mathbf{M})$ such that $\Lambda^{-1} \leq |\mathbf{Q}| \leq \Lambda$ and $\mathrm{dist}(\mathbf{M},\, \Sigma(\mathbf{Q})) \leq 1$. Let $(\mathbf{Q},\, \mathbf{M}) \in X(\Lambda)$ and $\mathbf{N} := \pi(\mathbf{Q},\, \mathbf{M})$. Since $|\mathbf{Q}| > 0$, we may write $\sqrt{2}\mathbf{Q} = \rho(\mathbf{n} \otimes \mathbf{n} - \mathbf{m} \otimes \mathbf{m})$, where $\rho := |\mathbf{Q}|$ and $(\mathbf{n},\, \mathbf{m})$ is an orthonormal set of eigenvectors, with $\mathbf{n}$ corresponding to the positive eigenvalue of $\mathbf{Q}$. In view of Corollary 1.3, of the immediate inequality $|\mathbf{M}-\mathbf{N}|^2 \geq (\mathbf{M}\cdot\mathbf{m})^2$, and of the assumption $\mathrm{dist}(\mathbf{M},\,\Sigma(\mathbf{Q})) \leq 1$, we obtain

$$0 \leq \ell(\mathbf{Q},\,\mathbf{M}) - \ell(\mathbf{Q},\,\mathbf{N}) \leq L(\Lambda)\,|\mathbf{M}-\mathbf{N}|^2\,, \tag{1.26}$$

where the first inequality holds because $\mathbf{N}$ is a minimiser of $\ell(\mathbf{Q},\,\cdot\,)$, and $L(\Lambda)$ can be taken of the form

$$L(\Lambda) = \sqrt{2}\beta\Lambda + C, \tag{1.27}$$

form some $C$ that does not depend on $\Lambda$, nor on $\beta$.

On the other hand, by Taylor-expanding $\nabla_{\mathbf{M}}\ell$ around the point $(\mathbf{Q},\,\mathbf{N})$, we obtain

$$\nabla_{\mathbf{M}}\ell(\mathbf{Q},\,\mathbf{M})\cdot(\mathbf{M}-\mathbf{N}) \geq \mathrm{D}^2_{\mathbf{M}}\ell(\mathbf{Q},\,\mathbf{N})(\mathbf{M}-\mathbf{N})\cdot(\mathbf{M}-\mathbf{N}) - L'\,|\mathbf{M}-\mathbf{N}|^3\,,$$

where $L'$ is an upper bound for the third derivatives of $\ell(\mathbf{Q},\,\cdot\,)$ on $X(\Lambda)$. (The third derivatives of $\ell(\mathbf{Q},\,\cdot)$ do not depend on $\mathbf{Q}$, so we can choose $L'$ to be independent of $\Lambda$.) We evaluate explicitly the second derivatives of $\ell$. Recalling (1.21), we obtain

$$\mathrm{D}^2_{\mathbf{M}}\ell(\mathbf{Q},\,\mathbf{N}) = \sqrt{2}\beta\rho\,\mathbf{I} + (\sqrt{2}\beta\rho+2)\mathbf{n}\otimes\mathbf{n} + \sqrt{2}\beta\rho\,\mathbf{m}\otimes\mathbf{m} = 2\sqrt{2}\beta\rho\,\mathbf{I} + 2\mathbf{n}\otimes\mathbf{n} \tag{1.28}$$

and hence,

$$\mathrm{D}^2_{\mathbf{M}}\ell(\mathbf{Q},\,\mathbf{N})(\mathbf{M}-\mathbf{N})\cdot(\mathbf{M}-\mathbf{N}) \geq 2\sqrt{2}\,\beta\rho\,|\mathbf{M}-\mathbf{N}|^2 \geq \frac{2\sqrt{2}\,\beta}{\Lambda}\,|\mathbf{M}-\mathbf{N}|^2 \tag{1.29}$$

for any $(\mathbf{Q},\,\mathbf{M}) \in X(\Lambda)$. Combining (1.26), (1.27), and (1.29), and choosing, for instance, $\delta_0(\Lambda) := \frac{\sqrt{2}\beta}{\Lambda L'}$, the lemma follows. □

**Lemma 1.5.** *For any $\delta > 0$ there exist positive constants $\varepsilon_1(\delta) > 0$ and $c_1(\delta) > 0$ (depending on $\beta$ and $\delta$, but not on $\varepsilon$) such that the following statement holds: for any $\varepsilon \in (0,\,\varepsilon_1(\delta)]$ and any $(\mathbf{Q},\,\mathbf{M}) \in \mathscr{S}_0^{2\times 2}\times\mathbb{R}^2$ such that $|\mathbf{Q}| \leq 3/4$ or $\mathrm{dist}(\mathbf{M},\,\Sigma(\mathbf{Q})) \geq \delta$, there holds*

$$f_\varepsilon(\mathbf{Q},\,\mathbf{M}) \geq c_1(\delta)\,\varepsilon. \tag{1.30}$$

*Proof.* Let $\delta > 0$ be fixed. Should the lemma be false, there would exist sequences $(\varepsilon_k)_{k\in\mathbb{N}}$, $(\mathbf{Q}_k)_{k\in\mathbb{N}}$, $(\mathbf{M}_k)_{k\in\mathbb{N}}$ such that $\varepsilon_k \to 0$ as $k \to +\infty$,

$$|\mathbf{Q}_k| \leq \frac{3}{4} \quad \text{or} \quad \mathrm{dist}(\mathbf{M}_k,\,\Sigma(\mathbf{Q}_k)) \geq \delta \qquad \text{for any } k\in\mathbb{N}, \tag{1.31}$$

and

$$\frac{f_{\varepsilon_k}(\mathbf{Q}_k,\,\mathbf{M}_k)}{\varepsilon_k} = \frac{1}{4\varepsilon_k}\left(|\mathbf{Q}_k|^2-1\right)^2 + \ell(\mathbf{Q}_k,\,\mathbf{M}_k) + \varepsilon_k\,\chi_{\varepsilon_k} \to 0 \qquad \text{as } k\to+\infty. \tag{1.32}$$

We can give an estimate for $|\mathbf{Q}_k|$ by combining (1.32) with Equation (1.22), which provides a lower bound for $\ell_{\varepsilon_k}(\mathbf{Q}_k, \mathbf{M}_k)$, and Lemma 1.1, which shows that the sequence $(\chi_{\varepsilon_k})_{k\in\mathbb{N}}$ is bounded. We obtain

$$
\begin{aligned}
\frac{1}{4\varepsilon_k}\left(|\mathbf{Q}_k|^2 - 1\right)^2 &\leq -\min \ell(\mathbf{Q}_k, \cdot) + \mathrm{o}_{k\to+\infty}(1) \\
&\leq \frac{\beta}{2}\left(|\mathbf{Q}_k| - 1\right)\left(\sqrt{2} + \beta + \beta\,|\mathbf{Q}_k|\right) + \mathrm{o}_{k\to+\infty}(1).
\end{aligned} \tag{1.33}
$$

This inequality implies that the sequence $(\mathbf{Q}_k)_{k\in\mathbb{N}}$ is bounded. Then, from (1.32) we deduce

$$
\limsup_{k\to+\infty}\left(\frac{1}{4}\left(|\mathbf{M}_k|^2 - 1\right)^2 - \beta\, C\,|\mathbf{M}_k|^2\right) \leq \limsup_{k\to+\infty} \ell(\mathbf{Q}_k, \mathbf{M}_k) \leq 0,
$$

where $C$ is an upper bound for $|\mathbf{Q}_k|$. Therefore, $(\mathbf{M}_k)_{k\in\mathbb{N}}$, too, is bounded. Up to extraction of a subsequence, we can assume that $\mathbf{Q}_k \to \mathbf{Q}$, $\mathbf{M}_k \to \mathbf{M}$. By passing to the limit in (1.33) and (1.32), we see that $|\mathbf{Q}| = 1$ and $\ell(\mathbf{Q}, \mathbf{M}) \leq 0$. On the other hand, (1.22) implies that $\ell(\mathbf{P}, \mathbf{N}) \geq 0$ for any $(\mathbf{P}, \mathbf{N})$ such that $|\mathbf{P}| = 1$, with strict inequality unless $\mathbf{N} \in \Sigma(\mathbf{P})$. Therefore, we must have $|\mathbf{Q}| = 1$ and $\mathbf{M} \in \Sigma(\mathbf{Q})$, but this contradicts (1.31). The lemma follows. □

## 1.3 First consequences of the Euler-Lagrange equations

**The maximum principle.** From now on, we denote by $(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)$ a solution to (4)–(5), subject to either Dirichlet or mixed boundary conditions, as in (6)–(7) or (8)–(9) respectively. Either way, we assume that (10) is satisfied. We will denote the energy density as

$$
e_\varepsilon(\mathbf{Q}, \mathbf{M}) := \frac{1}{2}\,|\nabla\mathbf{Q}|^2 + \frac{\varepsilon}{2}\,|\nabla\mathbf{M}|^2 + \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}, \mathbf{M}) \tag{1.34}
$$

for $(\mathbf{Q}, \mathbf{M}) \in W^{1,2}(\Omega, \mathscr{S}_0^{2\times 2}) \times W^{1,2}(\Omega, \mathbb{R}^2)$.

**Lemma 1.6.** *The maps $\mathbf{Q}^*_\varepsilon$, $\mathbf{M}^*_\varepsilon$ are smooth inside $\Omega$ and of class $C^1$ up to the boundary of $\Omega$. Moreover, there exist a constant $C_\beta$, depending only on $\beta$, and a constant $C_{\beta,\Omega}$, depending only on $\beta$ and $\Omega$, such that*

$$
\|\mathbf{Q}^*_\varepsilon\|_{L^\infty(\Omega)} + \|\mathbf{M}^*_\varepsilon\|_{L^\infty(\Omega)} \leq C_\beta \tag{1.35}
$$

$$
\|\nabla\mathbf{Q}^*_\varepsilon\|_{L^\infty(\Omega)} + \|\nabla\mathbf{M}^*_\varepsilon\|_{L^\infty(\Omega)} \leq \frac{C_{\beta,\Omega}}{\varepsilon}. \tag{1.36}
$$

*Proof of Lemma 1.6.* Elliptic regularity theory (both for the Dirichlet and the Neumann problems) implies that $\mathbf{Q}_\varepsilon$ and $\mathbf{M}_\varepsilon$ are smooth in $\Omega$ and of class $C^1$ up to $\partial\Omega$. We focus on the proof of (1.35); once (1.35) is proved, the gradient bounds (1.36) will follow from elliptic estimates, by reasoning along the lines of [5, Lemma A.2]. The bounds (1.35) are a consequence of the maximum principle. For the problem with Dirichlet boundary conditions, the details can be found in [20, Lemma 4.1]. Here, we focus on the problem with mixed boundary conditions.

As an intermediate step towards (1.35), we show that

$$\max_{\overline{\Omega}} |\mathbf{M}_\varepsilon|^2 \leq m_\varepsilon := 1 + \sqrt{2}\beta \max_{\overline{\Omega}} |\mathbf{Q}_\varepsilon| \tag{1.37}$$

Indeed, by taking the scalar product of (5) against $\mathbf{M}_\varepsilon$, we obtain

$$-\frac{1}{2}\Delta\left(|\mathbf{M}_\varepsilon|^2\right) + |\nabla \mathbf{M}_\varepsilon|^2 + \frac{1}{\varepsilon^2}(|\mathbf{M}_\varepsilon|^2 - 1)\,|\mathbf{M}_\varepsilon|^2 - \frac{2\beta}{\varepsilon^2}\mathbf{Q}_\varepsilon\mathbf{M}_\varepsilon\cdot\mathbf{M}_\varepsilon = 0.$$

and hence (observing that $\mathbf{Q}\mathbf{M}\cdot\mathbf{M} \leq \frac{1}{\sqrt{2}}|\mathbf{Q}|\,|\mathbf{M}|^2$ for any $\mathbf{Q}\in\mathscr{S}_0^{2\times 2}$ and $\mathbf{M}\in\mathbb{R}^2$)

$$-\frac{1}{2}\Delta\left(|\mathbf{M}_\varepsilon|^2\right) \leq -\frac{|\mathbf{M}_\varepsilon|^2}{\varepsilon^2}\left(|\mathbf{M}_\varepsilon|^2 - 1 - \sqrt{2}\beta\,|\mathbf{Q}_\varepsilon|\right) \tag{1.38}$$

at every point of $\Omega$. Now, let $x_0 \in \overline{\Omega}$ be a maximum point for $|\mathbf{M}_\varepsilon|^2$ and suppose, towards a contradiction, that $|\mathbf{M}_\varepsilon(x_0)| > m_\varepsilon$. Then, the right-hand side of (1.38) would attain a strictly negative value at the point $x_0$ and, by continuity of $(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)$, it would be strictly negative in a (smooth) neighbourhood $V \subseteq \overline{\Omega}$ of $x_0$. If $x_0$ lies in the interior of $\Omega$, we obtain a contradiction because $-\Delta(|\mathbf{M}_\varepsilon|^2)(x_0) \geq 0$. On the other hand, if $x_0 \in \partial\Omega$, then Hopf's lemma (applied on $V$) implies that $\partial_{\boldsymbol{\nu}}(|\mathbf{M}_\varepsilon|^2)(x_0) > 0$, which contradicts the boundary conditions (6). Therefore, (1.37) is proved.

Now, (1.35) follows by (1.37) by the maximum principle. Indeed, by taking the scalar product of (4) against $\mathbf{Q}_\varepsilon$, we deduce

$$-\frac{1}{2}\Delta\left(|\mathbf{Q}_\varepsilon|^2\right) \leq -\frac{1}{\varepsilon^2}\left(|\mathbf{Q}_\varepsilon|^4 - |\mathbf{Q}_\varepsilon|^2 - \frac{\beta}{\sqrt{2}}\,\varepsilon\,|\mathbf{Q}_\varepsilon|\,|\mathbf{M}_\varepsilon|^2\right) \tag{1.39}$$

inside $\Omega$. Let $x_1 \in \overline{\Omega}$ be a maximum point for $|\mathbf{Q}_\varepsilon|^2$. If $x_1 \in \partial\Omega$, then $|\mathbf{Q}_\varepsilon(x_1)|^2 = 1$ because of the assumption (7) on the boundary datum. If $x_1 \in \Omega$, then (1.39) and (1.37) imply that

$$|\mathbf{Q}_\varepsilon(x_1)|^3 \leq |\mathbf{Q}_\varepsilon(x_1)| + \frac{\beta}{\sqrt{2}}\,\varepsilon\,|\mathbf{M}_\varepsilon(x_1)|^2 \leq \left(1 + \beta^2\,\varepsilon\right)|\mathbf{Q}_\varepsilon(x_1)| + \frac{\beta}{\sqrt{2}}\,\varepsilon \tag{1.40}$$

and hence, $|\mathbf{Q}_\varepsilon|$ is uniformly bounded, in terms of $\beta$ only. □

*Remark* 1.6. The arguments above apply, with minor modifications, to the problem with Dirichlet boundary conditions. In both cases, from (1.37) and (1.40) we obtain

$$|\mathbf{Q}_\varepsilon| \leq s_*(\beta,\, \varepsilon), \quad |\mathbf{M}_\varepsilon|^2 \leq 1 + \sqrt{2}\beta\, s_*(\beta,\, \varepsilon) \qquad \text{in } \Omega, \tag{1.41}$$

where $s_*(\beta,\, \varepsilon) > 1$ is the largest root of the polynomial $P(X) = X^3 - (1+\beta^2\varepsilon)X - \frac{\beta}{\sqrt{2}}\varepsilon$. Elementary calculus shows that $s_*(\beta,\, \varepsilon) \to 1$ as $\varepsilon \to 0$, for any given value of $\beta$. Then, by differentiating the constraint $P(s_*(\beta,\, \varepsilon)) = 0$ with respect to $\varepsilon$, we deduce

$$s_*(\beta,\, \varepsilon) = 1 + \varepsilon\kappa_\star + \mathrm{O}(\varepsilon^2) \tag{1.42}$$

as $\varepsilon \to 0$.

*Remark* 1.7. Lemma 1.6 does not depend on the assumption (10). However, combining (10) with the $L^\infty$-estimate (1.35) and the inequality (1.16), we obtain the bound

$$\frac{1}{\varepsilon^2}\int_\Omega (|\mathbf{Q}_\varepsilon|^2 - 1)^2 \lesssim \int_\Omega \left(\frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon) + \beta^2\,|\mathbf{M}_\varepsilon|^4\right) \leq C, \tag{1.43}$$

for some constant $C$ depending only on $\beta$ and $\mathrm{C}_{\mathrm{pot}}$. This estimate will be useful in the sequel.

**The Pohozaev identity.** We define the *stress-energy tensor* associated to $(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)$ as

$$T^\varepsilon_{jk} := \partial_j \mathbf{Q}_\varepsilon \cdot \partial_k \mathbf{Q}_\varepsilon + \varepsilon\, \partial_j \mathbf{M}_\varepsilon \cdot \partial_k \mathbf{M}_\varepsilon - e_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\, \delta_{jk} \tag{1.44}$$

for $(j, k) \in \{1, 2\}^2$. Here $e_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)$ is the energy density, defined by (1.34).

**Lemma 1.7.** *The stress-energy tensor satisfies $\partial_j T^\varepsilon_{jk} = 0$ in $\Omega$.*

*Proof.* For ease of notation, we will write $\mathbf{Q}$, $\mathbf{M}$ instead of $\mathbf{Q}_\varepsilon$, $\mathbf{M}_\varepsilon$. Let $G$ be a smooth subdomain of $\Omega$ and let $\mathbf{X} \in C^\infty_c(\mathbb{R}^2, \mathbb{R}^2)$ be a test vector field, which may or may not be supported in $\overline{\Omega}$. We test the equation (4) against $X_k\, \partial_k \mathbf{Q}$, test (5) against $\varepsilon\, X_k\, \partial_k \mathbf{M}$, then integrate over $G$. We obtain

$$-\int_G X_k \left(\partial_j\partial_j \mathbf{Q} \cdot \partial_k \mathbf{Q} + \varepsilon\, \partial_j\partial_j \mathbf{M} \cdot \partial_k \mathbf{M}\right) + \frac{1}{\varepsilon^2}\int_G X_k \left(\nabla_{\mathbf{Q}} f_\varepsilon \cdot \partial_k \mathbf{Q} + \nabla_{\mathbf{M}} f_\varepsilon \cdot \partial_k \mathbf{M}\right) = 0,$$

where $\nabla_{\mathbf{Q}} f_\varepsilon$, $\nabla_{\mathbf{M}} f_\varepsilon$ denote the (partial) gradients of $f_\varepsilon$ with respect to the variables $\mathbf{Q}$ and $\mathbf{M}$, evaluated at $(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)$. By the chain rule, the previous equality can also be written as

$$-\int_G X_k \left(\partial_j\partial_j \mathbf{Q} \cdot \partial_k \mathbf{Q} + \varepsilon\, \partial_j\partial_j \mathbf{M} \cdot \partial_k \mathbf{M}\right) + \frac{1}{\varepsilon^2}\int_G X_k\, \partial_k \left(f_\varepsilon(\mathbf{Q}, \mathbf{M})\right) = 0, \tag{1.45}$$

Now, we integrate by parts repeatedly. The potential term can be written as

$$\int_G X_k\, \partial_k \left(f_\varepsilon(\mathbf{Q}, \mathbf{M})\right) = -\int_G (\operatorname{div} \mathbf{X})\, f_\varepsilon(\mathbf{Q}, \mathbf{M}) + \int_{\partial G} (\mathbf{X} \cdot \boldsymbol{\nu})\, f_\varepsilon(\mathbf{Q}, \mathbf{M})\, \mathrm{d}s, \tag{1.46}$$

where $\boldsymbol{\nu}$ is the outward unit normal to $G$. We rewrite the first term by integrating by parts twice:

$$\begin{aligned} -\int_G X_k\, \partial_j\partial_j \mathbf{Q} \cdot \partial_k \mathbf{Q} &= \int_G \partial_j X_k\, \partial_j \mathbf{Q} \cdot \partial_k \mathbf{Q} + \int_G X_k\, \partial_j \mathbf{Q} \cdot \partial_j\partial_k \mathbf{Q} - \int_{\partial G} (\boldsymbol{\nu} \cdot \nabla \mathbf{Q}) \cdot (\mathbf{X} \cdot \nabla \mathbf{Q})\, \mathrm{d}s \\ &= \int_G \partial_j X_k\, \partial_j \mathbf{Q} \cdot \partial_k \mathbf{Q} - \frac{1}{2}\int_G (\operatorname{div} \mathbf{X})\, \lvert\nabla \mathbf{Q}\rvert^2 \\ &\qquad - \int_{\partial G} (\boldsymbol{\nu} \cdot \nabla \mathbf{Q}) \cdot (\mathbf{X} \cdot \nabla \mathbf{Q})\, \mathrm{d}s + \frac{1}{2}\int_{\partial G} (\boldsymbol{\nu} \cdot \mathbf{X})\, \lvert\nabla \mathbf{Q}\rvert^2\, \mathrm{d}s \end{aligned} \tag{1.47}$$

A similar computation applies to the term containing the derivatives of $\mathbf{M}$. By combining (1.45), (1.46) and (1.47), we obtain

$$\begin{aligned} \int_G T^\varepsilon_{jk}\, \partial_j X_k &= \int_{\partial G} \left((\boldsymbol{\nu} \cdot \nabla \mathbf{Q}) \cdot (\mathbf{X} \cdot \nabla \mathbf{Q}) + \varepsilon\, (\boldsymbol{\nu} \cdot \nabla \mathbf{M}) \cdot (\mathbf{X} \cdot \nabla \mathbf{M})\right) \mathrm{d}s \\ &\qquad - \int_{\partial G} (\mathbf{X} \cdot \boldsymbol{\nu})\, e_\varepsilon(\mathbf{Q}, \mathbf{M})\, \mathrm{d}s \end{aligned} \tag{1.48}$$

By choosing $G = \Omega$ and taking $\mathbf{X}$ compactly supported in $\Omega$, the boundary terms vanish, and the lemma follows. □

*Remark* 1.8. Completely analogous computations to those above return the following fact, which will be used later on. Suppose that $\mathbf{Q}_\star\colon G \to \mathbb{S}^1$ is a smooth harmonic map, i.e., a smooth solution to the equation

$$\Delta \mathbf{Q}_\star + |\nabla \mathbf{Q}_\star|^2 \, \mathbf{Q}_\star = 0 \qquad \text{in } G,$$

and let $\mathbf{X}$ be any smooth vector field with compact support in $G$. Then,

$$\int_G \left\{ \partial_j \mathbf{Q}_\star \cdot \partial_k \mathbf{Q}_\star \partial_j X_k - \frac{1}{2} (\operatorname{div} \mathbf{X}) \, |\nabla \mathbf{Q}_\star|^2 \right\} \mathrm{d}x = 0. \tag{1.49}$$

**Lemma 1.8.** *For each ball $B = B(x_0, R) \subseteq \Omega$, there holds*

$$\begin{aligned} &\frac{2}{\varepsilon^2} \int_B f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \, \mathrm{d}x + \frac{R}{2} \int_{\partial B} \left( |\partial_{\boldsymbol{\nu}} \mathbf{Q}_\varepsilon|^2 + \varepsilon \, |\partial_{\boldsymbol{\nu}} \mathbf{M}_\varepsilon|^2 \right) \mathrm{d}s \\ &\qquad = \frac{R}{2} \int_{\partial B} \left( |\partial_{\boldsymbol{\tau}} \mathbf{Q}_\varepsilon|^2 + \varepsilon \, |\partial_{\boldsymbol{\tau}} \mathbf{M}_\varepsilon|^2 + \frac{2}{\varepsilon^2} f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \right) \mathrm{d}s \end{aligned} \tag{1.50}$$

*where $\boldsymbol{\nu}$ is the outward unit normal and $\boldsymbol{\tau}$ is the unit tangent field on $\partial B$, oriented in such a way that $(\boldsymbol{\nu}, \boldsymbol{\tau})$ is a positive basis.*

*Proof.* The lemma follows by taking $G = B(x_0, R)$ and $\mathbf{X}(x) := \varphi(x)(x - x_0)$ in (1.48), where $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth cut-off function such that $\varphi \equiv 1$ in a neighbourhood of $\Omega$. ☐

**The clearing-out lemma.** This version of the "clearing-out lemma" is inspired by [7, Theorem 2], but its proof is much simpler, because we are working in a two-dimensional domain. Roughly stated, it implies that, if the energy of $(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)$ on a ball of radius $R$ is sufficiently small with respect to $\log(R/\varepsilon)$, then $\mathbf{Q}_\varepsilon$ is nonzero at the centre of the ball.

**Lemma 1.9** (Clearing-out)**.** *Let $B = B(x_0, R)$ be a ball contained in $\Omega$ and let $\varepsilon$ be such that $0 < \varepsilon < R$. Then,*

$$||\mathbf{Q}_\varepsilon(x_0)| - 1| \lesssim \left( \frac{\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; B)}{\log(R/\varepsilon)} \right)^{1/4} + \varepsilon^{1/2}$$

*The implicit constant on the right-hand side depends only on $\beta$ and $\Omega$.*

*Proof.* First of all, we claim that there exists $\rho \in (\varepsilon, R)$ such that

$$\int_{\partial B(x_0, \rho)} e_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \, \mathrm{d}s \le \frac{\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; B)}{\rho \, \log(R/\varepsilon)} \tag{1.51}$$

For otherwise, we would have

$$\begin{aligned} \mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; B) \ge \int_\varepsilon^R \left( \int_{\partial B(x_0, \rho)} e_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \, \mathrm{d}s \right) \mathrm{d}\rho > \int_\varepsilon^R \frac{\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; B)}{\rho \, \log(R/\varepsilon)} \mathrm{d}\rho \\ = \mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; B), \end{aligned}$$

which is a contradiction. For such a value of $\rho$, Lemma 1.8 gives

$$\begin{aligned} \frac{2}{\varepsilon^2}\int_{B(x_0,\,\varepsilon)} f_\varepsilon(\mathbf{Q}_\varepsilon,\,\mathbf{M}_\varepsilon) \leq \frac{2}{\varepsilon^2}\int_{B(x_0,\,\rho)} f_\varepsilon(\mathbf{Q}_\varepsilon,\,\mathbf{M}_\varepsilon) \leq \rho\int_{\partial B(x_0,\,\rho)} e_\varepsilon(\mathbf{Q}_\varepsilon,\,\mathbf{M}_\varepsilon)\,\mathrm{d}s \\ \leq \frac{\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon,\,\mathbf{M}_\varepsilon;\,B)}{\log(R/\varepsilon)} \end{aligned} \tag{1.52}$$

(see [20, Lemma B.1]). Therefore, taking (1.16) and Lemma 1.6 into account, from (1.52) we deduce

$$\begin{aligned} \frac{1}{\varepsilon^2}\int_{B(x_0,\,\varepsilon)} \left(|\mathbf{Q}_\varepsilon|^2 - 1\right)^2 \lesssim \int_{B(x_0,\,\varepsilon)} \left(\frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}_\varepsilon,\,\mathbf{M}_\varepsilon) + \beta^2\,|\mathbf{M}_\varepsilon|^4\right) \\ \lesssim \frac{\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon,\,\mathbf{M}_\varepsilon;\,B)}{\log(R/\varepsilon)} + \beta^2\varepsilon^2 \end{aligned} \tag{1.53}$$

Finally, the gradient estimate (1.36) implies that

$$||\mathbf{Q}_\varepsilon(x_0)| - 1| \lesssim \left(\frac{1}{\varepsilon^2}\int_{B(x_0,\,\varepsilon)} \left(|\mathbf{Q}_\varepsilon|^2 - 1\right)^2\right)^{1/4} \tag{1.54}$$

— see [7, Lemma III.3] for the details. (The statement in [7] only provides a one-sided bound for $1 - |\mathbf{Q}_\varepsilon|$, but the very same argument can be used to prove the opposite inequality.) The lemma follows by combining (1.53) with (1.54). ☐

# 2 Estimates for $\mathbf{Q}_\varepsilon$ and $\mathbf{M}_\varepsilon$

In this section, we derive several $\varepsilon$-independent estimates on $\mathbf{Q}_\varepsilon$ and $\mathbf{M}_\varepsilon$. For the reader's convenience, such estimates are all collected in Theorem 2.11 at the end of the section.

Our arguments follow a classical path and rely on testing the Euler-Lagrange equations (4), (5) against suitable maps. In particular, starting from a basic clearing-out lemma for the $\mathbf{Q}_\varepsilon$-component (reminiscent of classical results in Ginzburg-Landau theory), we show that, for boundary conditions as in (6)–(7) or as in (8)–(9) and under assumption (10), the energy of a sequence of critical points blows-up (at most) logarithmically in $\varepsilon$. Moreover, the largest part of the energy is carried $\mathbf{Q}_\varepsilon$-component; in fact, all terms in $\mathscr{F}_\varepsilon$ different from $\int_\Omega |\nabla\mathbf{Q}_\varepsilon|^2$ are uniformly bounded; furthermore, even the elastic energy of $\mathbf{Q}_\varepsilon$ remains finite around points at which $|\mathbf{Q}_\varepsilon| \geq 1/2$ for any $\varepsilon$ small enough.

The results in this section will be crucially exploited all along the rest of the paper. In particular, in Section 3, we use them to prove compactness properties for both the $\mathbf{Q}_\varepsilon$-component and $\mathbf{M}_\varepsilon$-component of a sequence $\{(\mathbf{Q}_\varepsilon,\,\mathbf{M}_\varepsilon)\}$ of critical points of $\mathscr{F}_\varepsilon$.

## 2.1 Bounds in $W^{1,p}(\Omega)$ for $\mathbf{Q}_\varepsilon$

The goal of this section is to prove the following result.

**Proposition 2.1.** *Let* $(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)$ *be a sequence of solutions to* (4)–(5)*, subject to either* (6)–(7) *or* (8)–(9)*. Assume that the condition* (10) *is satisfied. Then, there exists a constant* $C$*, depending only on* $\Omega$*,* $\beta$*,* $\mathrm{C}_{\mathrm{pot}}$*, and the boundary data, such that, for any* $\varepsilon$ *small enough, there holds*

$$\int_\Omega |\nabla \mathbf{Q}_\varepsilon|^2 \leq C\,|\log \varepsilon|. \tag{2.1}$$

*Moreover, for any* $p \in [1, 2)$ *there exists a constant* $C_p > 0$*, depending only on* $\Omega$*,* $p$*,* $\beta$*,* $\mathrm{C}_{\mathrm{pot}}$*, and the boundary data such that, for any* $\varepsilon$ *small enough, there holds*

$$\int_\Omega |\nabla \mathbf{Q}_\varepsilon|^p \leq C_p. \tag{2.2}$$

*Remark* 2.1. More precisely, the constants on the right-hand side of (2.1) and of (2.2) depend on the boundary data only through the $L^1(\partial\Omega)$- and the $L^2(\partial\Omega)$-norm of $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}} \mathbf{Q}_{\mathrm{bd}}$.

As in [6, 7], the main step in the proof of Proposition 2.1 is to obtain bounds on $j(\mathbf{Q}_\varepsilon)$. To this end, we write the equation satisfied by $j(\mathbf{Q}_\varepsilon)$ by taking the vector product of Equations (4)–(5) against $\mathbf{Q}_\varepsilon$ and $\mathbf{M}_\varepsilon$ respectively:

$$-\operatorname{div}\left(j(\mathbf{Q}_\varepsilon)\right) = \frac{\varepsilon}{2}\,\partial_k\left(\mathbf{M}_\varepsilon \times \partial_k \mathbf{M}_\varepsilon\right) \qquad \text{in } \Omega \tag{2.3}$$

(see [20, Lemma 4.8] for details). Lemma 1.6 implies that the right-hand side of Equation (2.3) is bounded in $W^{-1,\infty}(\Omega)$: indeed,

$$\varepsilon \left\|\mathbf{M}_\varepsilon \times \partial_k \mathbf{M}_\varepsilon\right\|_{L^\infty(\Omega)} \leq \varepsilon \left\|\mathbf{M}_\varepsilon\right\|_{L^\infty(\Omega)} \left\|\partial_k \mathbf{M}_\varepsilon\right\|_{L^\infty(\Omega)} \leq C \tag{2.4}$$

for $k \in \{1, 2\}$ and some constant $C$ that does not depend on $\varepsilon$ but only on $\beta$. This allows us to obtain $\varepsilon$-independent bounds on $j(\mathbf{Q}_\varepsilon)$.

**Lemma 2.2.** *For any* $p \in [1, 2)$*, there exists an* $\varepsilon$*-independent constant* $C_p > 0$ *such that*

$$\left\|j(\mathbf{Q}_\varepsilon)\right\|_{L^p(\Omega)} \leq C_p$$

*for any* $\varepsilon$ *small enough. The constant* $C_p$ *depends only on* $p$*,* $\beta$*,* $\mathrm{C}_{\mathrm{pot}}$*, and the* $L^1(\partial\Omega)$*-norm of* $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}} \mathbf{Q}_{\mathrm{bd}}$*.*

*Proof.* As in [6, 7], the strategy is to consider (a suitable variant of) the Helmoltz, or Hodge, decomposition of $j(\mathbf{Q}_\varepsilon)$. As a preliminary step, we will decompose $j(\mathbf{Q}_\varepsilon)$ in a suitable way. We use the notation $\rho_\varepsilon := |\mathbf{Q}_\varepsilon|$, $\mathbf{j}_\varepsilon := j(\mathbf{Q}_\varepsilon)$.

*Step* 1 (Splitting $\operatorname{curl}\mathbf{j}_\varepsilon$)*.* Let $A\colon [0, +\infty) \to [0, 2]$ be a smooth function such that

$$A(0) = 1, \qquad A(t) = \frac{1}{t} \quad \text{if } t \geq \frac{1}{2}, \qquad |A'(t)| \leq 4 \qquad \text{for any } t \geq 0, \tag{2.5}$$

and let $\alpha_\varepsilon := A(\rho_\varepsilon)$. We write

$$\operatorname{curl}\mathbf{j}_\varepsilon = \operatorname{curl}(\alpha_\varepsilon^2\,\mathbf{j}_\varepsilon) + \operatorname{curl}((1 - \alpha_\varepsilon^2)\,\mathbf{j}_\varepsilon).$$

The first term, $\mathrm{curl}(\alpha_\varepsilon^2\,\mathbf{j}_\varepsilon)$, is supported in $\{\rho_\varepsilon \leq 1/2\}$, because of (1.12) and (2.5). Lemma 1.6 implies the pointwise inequality

$$\left|\mathrm{curl}(\alpha_\varepsilon^2\,\mathbf{j}_\varepsilon)\right| \lesssim \frac{\mathbb{1}_{\{\rho_\varepsilon\leq 1/2\}}}{\varepsilon^2} \lesssim \frac{1}{\varepsilon^2}\left(\rho_\varepsilon^2 - 1\right)^2, \tag{2.6}$$

where $\mathbb{1}_{\{\rho_\varepsilon\leq 1/2\}}$ is the indicator function of the set $\{\rho_\varepsilon \leq 1/2\}$. Then, recalling (1.43), we obtain the $\varepsilon$-independent bound

$$\left\|\mathrm{curl}(\alpha_\varepsilon^2\,\mathbf{j}_\varepsilon)\right\|_{L^1(\Omega)} \leq C. \tag{2.7}$$

In fact, the constant $C$ depends only on $\beta$ and $\mathrm{C}_{\mathrm{pot}}$.

As for the second term, $\mathrm{curl}((1-\alpha_\varepsilon^2)\,\mathbf{j}_\varepsilon)$, Lemma 1.6 and (2.5) imply the pointwise bound

$$\left|(1-\alpha_\varepsilon^2)\,\mathbf{j}_\varepsilon\right|^2 \lesssim \frac{(\rho_\varepsilon^2-1)^2}{\varepsilon^2\,\rho_\varepsilon^4}\mathbb{1}_{\{\rho_\varepsilon\geq 1/2\}} + \frac{\mathbb{1}_{\{\rho_\varepsilon\leq 1/2\}}}{\varepsilon^2} \lesssim \frac{1}{\varepsilon^2}(\rho_\varepsilon^2-1)^2.$$

Therefore, (1.43) gives

$$\left\|(1-\alpha_\varepsilon^2)\mathbf{j}_\varepsilon\right\|_{L^2(\Omega)} \leq C, \tag{2.8}$$

where $C$ does not depend on $\varepsilon$ but only on $\beta$ and $\mathrm{C}_{\mathrm{pot}}$.

*Step* 2 (Hodge decomposition of $\mathbf{j}_\varepsilon$). We consider the following elliptic problems:

$$\begin{cases} \Delta\phi_\varepsilon = \mathrm{curl}(\alpha_\varepsilon^2\,\mathbf{j}_\varepsilon) & \text{in } \Omega \\ \partial_{\boldsymbol{\nu}}\phi_\varepsilon = \dfrac{1}{2}\mathbf{Q}_{\mathrm{bd}}\times\partial_{\boldsymbol{\tau}}\mathbf{Q}_{\mathrm{bd}} & \text{on } \partial\Omega \end{cases} \tag{2.9}$$

and

$$\begin{cases} \Delta\xi_\varepsilon = \mathrm{curl}((1-\alpha_\varepsilon^2)\,\mathbf{j}_\varepsilon) & \text{in } \Omega \\ \partial_{\boldsymbol{\nu}}\xi_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

The right-hand sides of (2.9) and (2.10) satisfy the solvability conditions for Neumann problems. Indeed, if $\boldsymbol{\tau}$ denotes the unit tangent vector to $\partial\Omega$ in the anti-clockwise direction, we have

$$\alpha_\varepsilon^2\,\mathbf{j}_\varepsilon\cdot\boldsymbol{\tau} = \frac{1}{2}\mathbf{Q}_{\mathrm{bd}}\times\partial_{\boldsymbol{\tau}}\mathbf{Q}_{\mathrm{bd}}, \quad (1-\alpha_\varepsilon^2)\,\mathbf{j}_\varepsilon = 0 \qquad \text{on } \partial\Omega \tag{2.11}$$

because our assumptions on the boundary conditions (either (7) or (9)) imply $|\mathbf{Q}_\varepsilon| = |\mathbf{Q}_{\mathrm{bd}}| = 1$ on $\partial\Omega$. As a consequence, we obtain

$$\begin{aligned} \int_\Omega \mathrm{curl}\left(\alpha_\varepsilon^2\,\mathbf{j}_\varepsilon\right) &= \int_{\partial\Omega}\alpha_\varepsilon^2\,\mathbf{j}_\varepsilon\cdot\boldsymbol{\tau} = \frac{1}{2}\int_{\partial\Omega}\mathbf{Q}_{\mathrm{bd}}\times\partial_{\boldsymbol{\tau}}\mathbf{Q}_{\mathrm{bd}} \\ \int_\Omega \mathrm{curl}\left((1-\alpha_\varepsilon^2)\,\mathbf{j}_\varepsilon\right) &= \int_{\partial\Omega}(1-\alpha_\varepsilon^2)\,\mathbf{j}_\varepsilon\cdot\boldsymbol{\tau} = 0, \end{aligned}$$

and the solutions $\phi_\varepsilon$, $\xi_\varepsilon$ exist and are unique up to additive constants. Moreover, writing $\nabla^\perp := (-\partial_2, \partial_1)$, we have

$$\operatorname{curl}\left(\mathbf{j}_\varepsilon - \nabla^\perp\phi_\varepsilon - \nabla^\perp\xi_\varepsilon\right) = \operatorname{curl}\mathbf{j}_\varepsilon - \Delta\phi_\varepsilon - \Delta\xi_\varepsilon = 0 \qquad \text{in } \Omega.$$

As $\Omega$ is assumed to be simply connected, there exists a function $H_\varepsilon\colon \Omega \to \mathbb{R}$ such that

$$\mathbf{j}_\varepsilon = \nabla H_\varepsilon + \nabla^\perp\phi_\varepsilon + \nabla^\perp\xi_\varepsilon \qquad \text{in } \Omega. \tag{2.12}$$

By taking the divergence of both sides of (2.12), and keeping (2.3) into account, we obtain

$$-\Delta H_\varepsilon = \frac{\varepsilon}{2}\,\partial_k\left(\mathbf{M}_\varepsilon \times \partial_k\mathbf{M}_\varepsilon\right) \qquad \text{in } \Omega. \tag{2.13}$$

In a similar way, by taking the *tangential component* of the trace at the boundary in both sides of (2.12), and using (2.9), (2.10), (2.11), we deduce

$$\partial_{\boldsymbol{\tau}} H_\varepsilon = 0 \qquad \text{on } \partial\Omega.$$

The boundary of $\Omega$ is connected, because $\Omega$ is simply connected. Therefore, up to an additive constant, we can assume that

$$H_\varepsilon = 0 \qquad \text{on } \partial\Omega. \tag{2.14}$$

*Step* 3 (Elliptic estimates on $\phi_\varepsilon$, $\xi_\varepsilon$, $H_\varepsilon$). We estimate $\phi_\varepsilon$ by applying estimates for the Poisson problem with integrable right-hand side (see e.g. [7, Proposition A.2]). Keeping (2.7) into account, we obtain

$$\|\nabla\phi_\varepsilon\|_{L^p(\Omega)} \leq C_p\left(\left\|\operatorname{curl}(\alpha_\varepsilon^2\,\mathbf{j}_\varepsilon)\right\|_{L^1(\Omega)} + \|\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}}\mathbf{Q}_{\mathrm{bd}}\|_{L^1(\partial\Omega)}\right) \leq C_p, \tag{2.15}$$

for any $p \in [1, 2)$ and some constant $C_p$ that does not depend on $\varepsilon$ but only on $p$, $\beta$, $\mathrm{C}_{\mathrm{pot}}$, and the boundary data. As for $\xi_\varepsilon$, we test Equation (2.10) against $\xi_\varepsilon$ and integrate by parts, recalling that $(1-\alpha_\varepsilon^2)\,\mathbf{j}_\varepsilon = 0$ on $\partial\Omega$:

$$\int_\Omega |\nabla\xi_\varepsilon|^2 = -\int_\Omega (1-\alpha_\varepsilon^2)\,\mathbf{j}_\varepsilon \times \nabla\xi_\varepsilon.$$

Then, Young's inequality and (2.8) give

$$\|\nabla\xi_\varepsilon\|_{L^2(\Omega)} \leq \left\|(1-\alpha_\varepsilon^2)\,\mathbf{j}_\varepsilon\right\|_{L^2(\Omega)} \leq C, \tag{2.16}$$

for the same constant $C$ as in (2.8). In a similar way, testing (2.13)–(2.14) against $H_\varepsilon$, integrating by parts, and applying (2.4), we obtain

$$\|\nabla H_\varepsilon\|_{L^2(\Omega)} \leq \varepsilon\sum_{k=1}^{2}\|\mathbf{M}_\varepsilon \times \partial_k\mathbf{M}_\varepsilon\|_{L^2(\Omega)} \leq C. \tag{2.17}$$

Combining (2.15), (2.16) and (2.17), the lemma follows. □

**Lemma 2.3.** *Assume* (10) *holds. For any $\varepsilon$ small enough, we have*

$$\|j(\mathbf{Q}_\varepsilon)\|_{L^2(\Omega)} \lesssim |\log \varepsilon|^{1/2} \,,$$

*where the implicit constant on the right-hand side depends only on $|\Omega|$, $\beta$, $\mathrm{C}_{\mathrm{pot}}$, and the $L^2(\partial\Omega)$-norm of $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}} \mathbf{Q}_{\mathrm{bd}}$.*

*Proof.* We consider again the decomposition of $j(\mathbf{Q}_\varepsilon)$ given in (2.12). In view of (2.16) and (2.17), the lemma will follow if we prove that the solution $\phi_\varepsilon$ to (2.9) satisfies

$$\|\nabla \phi_\varepsilon\|_{L^2(\Omega)} \lesssim |\log \varepsilon|^{1/2} \,. \tag{2.18}$$

We follow the same strategy as in [6, Lemma X.5]. To simplify the notation, we set $h_\varepsilon := \mathrm{curl}(\alpha_\varepsilon^2 \, j(\mathbf{Q}_\varepsilon))$ and $2g := \mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}} \mathbf{Q}_{\mathrm{bd}}$. By testing (2.9) against $\phi_\varepsilon$, we obtain

$$\int_\Omega |\nabla \phi_\varepsilon|^2 = \int_{\partial\Omega} g \, \phi_\varepsilon \, \mathrm{d}s - \int_\Omega h_\varepsilon \, \phi_\varepsilon \tag{2.19}$$

We estimate the boundary term first. Since the solution $\phi_\varepsilon$ is only identified up to an additive constant, we can assume without loss of generality that $\phi_\varepsilon$ has zero average in $\Omega$. Then, the trace inequality and the Poincaré inequality give

$$\left| \int_{\partial\Omega} g \, \phi_\varepsilon \, \mathrm{d}s \right| \leq \|g\|_{L^2(\partial\Omega)} \|\phi_\varepsilon\|_{L^2(\partial\Omega)} \lesssim \|\phi_\varepsilon\|_{H^1(\Omega)} \lesssim \|\nabla \phi_\varepsilon\|_{L^2(\Omega)} \tag{2.20}$$

(the boundary datum $g$ depends only on $\mathbf{Q}_{\mathrm{bd}}$, not on $\varepsilon$, and is continuous). Now, we estimate the term that depends on $h_\varepsilon$. Let $Z_\varepsilon := \{|\mathbf{Q}_\varepsilon| \leq 1/2\}$. We recall (see (2.6)) that $|h_\varepsilon| \lesssim \varepsilon^{-2}$ and $h_\varepsilon$ is supported in $Z_\varepsilon$. Moreover, the estimate (1.43) (which holds thanks to the assumption (10)) for the Ginzburg-Landau potential implies

$$|Z_\varepsilon| \lesssim \int_\Omega \left( |\mathbf{Q}_\varepsilon|^2 - 1 \right)^2 \lesssim \varepsilon^2. \tag{2.21}$$

As a consequence, we obtain

$$\left| \int_\Omega h_\varepsilon \, \phi_\varepsilon \right| \lesssim \frac{1}{\varepsilon^2} \int_{Z_\varepsilon} |\phi_\varepsilon| \lesssim \|\phi_\varepsilon\|_{L^\infty(Z_\varepsilon)} \tag{2.22}$$

We can further estimate the $L^\infty(Z_\varepsilon)$-norm of $\phi_\varepsilon$ by applying Trudinger's inequality (see e.g [23, Theorem 7.15]), which gives

$$\int_\Omega \exp\left( \frac{|\phi_\varepsilon|^2}{\sigma_1^2 \, \|\nabla \phi_\varepsilon\|^2_{L^2(\Omega)}} \right) \leq \sigma_2 \, |\Omega|$$

for some universal constants $\sigma_1$, $\sigma_2$. It follows that

$$\exp\left( \frac{\|\phi_\varepsilon\|^2_{L^\infty(Z_\varepsilon)}}{\sigma_1^2 \, \|\nabla \phi_\varepsilon\|^2_{L^2(\Omega)}} \right) |Z_\varepsilon| \leq \sigma_2 \, |\Omega|$$

and, hence,

$$\|\phi_\varepsilon\|_{L^\infty(Z_\varepsilon)} \leq \sigma_1 \|\nabla\phi_\varepsilon\|_{L^2(\Omega)} \left(\log\left(\frac{\sigma_2\,|\Omega|}{|Z_\varepsilon|}\right)\right)^{1/2} \lesssim \|\nabla\phi_\varepsilon\|_{L^2(\Omega)} \left(|\log\varepsilon|^{1/2} + 1\right) \tag{2.23}$$

for any $i \in \{1, \ldots, N_\varepsilon\}$. Combining (2.20) with (2.22) and (2.23), we obtain

$$\|\nabla\phi_\varepsilon\|^2_{L^2(\Omega)} \lesssim \left(|\log\varepsilon|^{1/2} + 1\right) \|\nabla\phi_\varepsilon\|_{L^2(\Omega)}$$

and (2.19) follows. □

Next, we need to estimate $\rho_\varepsilon := |\mathbf{Q}_\varepsilon|$. To this end, we write an equation for $\rho_\varepsilon$ by taking the scalar product of both sides of (4) against $\mathbf{Q}_\varepsilon$. We obtain

$$-\frac{1}{2}\Delta(\rho_\varepsilon^2) + |\nabla\mathbf{Q}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left(\rho_\varepsilon^2 - 1\right)\rho_\varepsilon^2 - \frac{\beta}{\varepsilon}\mathbf{Q}_\varepsilon\mathbf{M}_\varepsilon\cdot\mathbf{M}_\varepsilon = 0 \qquad \text{in } \Omega. \tag{2.24}$$

From this equation, we deduce the following bound:

**Lemma 2.4.** *For any $\delta \in (0, 1/2)$, any $p \in [1, 2]$ and any $\varepsilon > 0$ small enough, there holds*

$$\int_\Omega |\nabla\rho_\varepsilon|^p \lesssim \left(\delta\int_\Omega |\nabla\mathbf{Q}_\varepsilon|^2\right)^{p/2} + \frac{\varepsilon^{2-p}}{\delta^2} + 1,$$

*where the implicit constant in front of the right-hand side is independent of $\delta$, $p$, $\varepsilon$ and depends only on $\Omega$, $\beta$, $\mathrm{C}_{\mathrm{pot}}$, and the $L^1(\partial\Omega)$- and the $L^2(\partial\Omega)$-norm of $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}}\mathbf{Q}_{\mathrm{bd}}$.*

*Proof.* For a given value of $\delta \in (0, 1/2)$, let

$$E_\varepsilon := \{x \in \Omega\colon\ |\rho_\varepsilon(x) - 1| \leq \delta\}.$$

The bound (1.43) implies

$$|\Omega \setminus E_\varepsilon| \lesssim \frac{1}{\delta^2}\int_\Omega (\rho_\varepsilon - 1)^2 \lesssim \frac{\varepsilon^2}{\delta^2}, \tag{2.25}$$

where the implicit constant on the right-hand side depends only on $\beta$ and $\mathrm{C}_{\mathrm{pot}}$, and hence, recalling (1.36),

$$\int_{\Omega\setminus E_\varepsilon} |\nabla\rho_\varepsilon|^2 \lesssim \frac{1}{\delta^2}, \tag{2.26}$$

where, again, the implicit constant on the right-hand side depends only on $\beta$ and $\mathrm{C}_{\mathrm{pot}}$.

It remains to estimate $\nabla\rho_\varepsilon$ on $E_\varepsilon$. To this purpose, we define

$$\overline{\rho}_\varepsilon := \begin{cases} 1+\delta & \text{if } \rho_\varepsilon \geq 1+\delta \\ \rho_\varepsilon & \text{if } 1-\delta < \rho_\varepsilon < 1+\delta \\ 1-\delta & \text{if } \rho_\varepsilon \leq 1-\delta. \end{cases}$$

We have $\overline{\rho}_\varepsilon = 1$ on $\partial\Omega$, because $|\mathbf{Q}_\varepsilon| = |\mathbf{Q}_{\mathrm{bd}}| = 1$ on $\partial\Omega$ by (7), (9). To estimate $\nabla\rho_\varepsilon$ on $E_\varepsilon$, we test Equation (2.24) against $1 - \overline{\rho}_\varepsilon$ and integrate by parts. We obtain

$$
\begin{aligned}
\int_{E_\varepsilon} \rho_\varepsilon \left|\nabla\rho_\varepsilon\right|^2 &= \int_\Omega \left( |\nabla\mathbf{Q}_\varepsilon|^2 + \frac{1}{\varepsilon^2}(\rho_\varepsilon^2 - 1)\rho_\varepsilon^2 - \frac{\beta}{\varepsilon}\mathbf{Q}_\varepsilon\mathbf{M}_\varepsilon \cdot \mathbf{M}_\varepsilon \right)(1 - \overline{\rho}_\varepsilon) \\
&\leq \delta \int_\Omega |\nabla\mathbf{Q}_\varepsilon|^2 + \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon,
\end{aligned}
\tag{2.27}
$$

where

$$
\mathrm{I}_\varepsilon := \frac{1}{\varepsilon^2}\int_\Omega \left|\rho_\varepsilon^2 - 1\right| |1 - \overline{\rho}_\varepsilon| \,, \qquad \mathrm{II}_\varepsilon := \frac{\beta}{\varepsilon}\int_\Omega |1 - \overline{\rho}_\varepsilon| \; |\mathbf{Q}_\varepsilon\mathbf{M}_\varepsilon \cdot \mathbf{M}_\varepsilon|
$$

To estimate $\mathrm{I}_\varepsilon$, we observe that $|1 - \overline{\rho}_\varepsilon| \leq |1 - \rho_\varepsilon|$ and hence,

$$
|\mathrm{I}_\varepsilon| \leq \frac{1}{\varepsilon^2}\int_\Omega \frac{(\rho_\varepsilon^2 - 1)^2}{1 + \rho_\varepsilon} \leq \frac{1}{\varepsilon^2}\int_\Omega (\rho_\varepsilon^2 - 1)^2 \leq C,
\tag{2.28}
$$

because of (1.43). To estimate $\mathrm{II}_\varepsilon$, we apply the $L^\infty$-bound (1.35), the Hölder inequality, and (1.43) again:

$$
|\mathrm{II}_\varepsilon| \lesssim \frac{1}{\varepsilon}\int_\Omega |1 - \rho_\varepsilon| \lesssim \left(\frac{1}{\varepsilon^2}\int_\Omega (\rho_\varepsilon^2 - 1)^2\right)^{1/2} \leq C,
\tag{2.29}
$$

where the constant $C$ depends only on $\Omega$, $\beta$, and $\mathrm{C}_{\mathrm{pot}}$. From (2.27), (2.28) and (2.29), we deduce

$$
\int_{E_\varepsilon} |\nabla\rho_\varepsilon|^p \lesssim \left(\int_{E_\varepsilon} \rho_\varepsilon \left|\nabla\rho_\varepsilon\right|^2\right)^{p/2} \lesssim \left(\delta\int_\Omega |\nabla\mathbf{Q}_\varepsilon|^2\right)^{p/2} + C
\tag{2.30}
$$

(we have used that $\rho_\varepsilon \geq 1 - \delta \geq 1/2$ on $E_\varepsilon$). Taking (2.26) into account, the lemma follows. □

We can now complete the proof of Proposition 2.1.

*Proof of Proposition 2.1.* Let $\rho_\varepsilon := |\mathbf{Q}_\varepsilon|$, and let

$$
F_\varepsilon := \left\{ x \in \Omega \colon \rho_\varepsilon(x) \geq \frac{1}{2} \right\}
$$

The inequality (1.43) implies that $|\Omega \setminus F_\varepsilon| \lesssim \varepsilon^2$, so the $L^\infty$-bound (1.36) on $\nabla\mathbf{Q}_\varepsilon$ gives

$$
\int_{\Omega \setminus F_\varepsilon} |\nabla\mathbf{Q}_\varepsilon|^2 \leq C
\tag{2.31}
$$

for some $\varepsilon$-independent constant $C$. To estimate the $L^2$-norm of $\nabla\mathbf{Q}_\varepsilon$ in $F_\varepsilon$, we use the identity (1.11), which gives

$$
\int_{F_\varepsilon} |\nabla\mathbf{Q}_\varepsilon|^2 = \int_{F_\varepsilon} \left( |\nabla\rho_\varepsilon|^2 + \frac{|j(\mathbf{Q}_\varepsilon)|^2}{\rho_\varepsilon^2} \right) \lesssim \int_{F_\varepsilon} \left( |\nabla\rho_\varepsilon|^2 + |j(\mathbf{Q}_\varepsilon)|^2 \right)
$$

Let $\delta \in (0, 1/2)$ be a small parameter. Lemma 2.3 and Lemma 2.4 imply

$$\int_{F_\varepsilon} |\nabla \mathbf{Q}_\varepsilon|^2 \lesssim \delta \int_\Omega |\nabla \mathbf{Q}_\varepsilon|^2 + \delta^{-2} + |\log \varepsilon| \tag{2.32}$$

Combining (2.31) with (2.32), and choosing $\delta$ small enough (but still independent of $\varepsilon$), we obtain (2.1). Now, let $p \in [1, 2)$. The identity (1.11), Lemma 2.2, and Lemma 2.4 (applied with the choice $\delta = |\log \varepsilon|^{-2}$) imply

$$\begin{aligned} \int_{F_\varepsilon} |\nabla \mathbf{Q}_\varepsilon|^p &\lesssim \int_{F_\varepsilon} \left(|\nabla \rho_\varepsilon|^p + |j(\mathbf{Q}_\varepsilon)|^p\right) \\ &\lesssim \left(\frac{1}{|\log \varepsilon|^2} \int_\Omega |\nabla \mathbf{Q}_\varepsilon|^2\right)^{p/2} + \varepsilon^{2-p} |\log \varepsilon|^4 + C_p \end{aligned}$$

where $C_p$ is a constant that depends on $p$, but not on $\varepsilon$. Therefore, keeping (2.1) into account, we obtain

$$\int_{F_\varepsilon} |\nabla \mathbf{Q}_\varepsilon|^p \lesssim \frac{1}{|\log \varepsilon|^{p/2}} + \varepsilon^{2-p} |\log \varepsilon|^4 + C_p \leq 1 + C_p \tag{2.33}$$

for $\varepsilon$ small enough. The estimate (2.2) follows from (2.31) and (2.33). $\square$

### 2.2 $\eta$-ellipticity for $\mathbf{Q}_\varepsilon$

The goal of this section is to prove the following result.

**Proposition 2.5.** *There exists positive numbers $\varepsilon_*$, $\eta_*$ and, for any $x_0 \in \Omega$, $R > 0$, a constant $C_*(x_0, R) > 0$ such that the following statement holds. Let $(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)$ be a solution to* (4)–(5)*, subject to boundary conditions either as in* (6)–(7) *or as in* (8)–(9)*. Assume that the condition* (10) *is satisfied, that $B(x_0, R) \subset\subset \Omega$ and that $0 < \varepsilon \leq \varepsilon_* R$. If*

$$\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon;\, B(x_0, R)) \leq \eta_* \log \frac{R}{\varepsilon}, \tag{2.34}$$

*then*

$$\textit{for any } x \textit{ in } B(x_0, 3R/4), \quad |\mathbf{Q}_\varepsilon(x)| \geq \frac{1}{2}. \tag{2.35}$$

*Moreover, whenever* (2.35) *holds (irrespectively of* (2.34)*), there holds*

$$\int_{B(x_0, R/2)} |\nabla \mathbf{Q}_\varepsilon|^2 \leq C_*(x_0, R). \tag{2.36}$$

*Proof.* Assume that $B(x_0, R) \subset\subset \Omega$ and $\varepsilon \in (0, \varepsilon_* R]$ are such that (2.34) is satisfied, for some positive constants $\eta_*$, $\varepsilon_*$ to be determined later.

*Step* 1 (Proof of (2.35)). Assume that $\varepsilon_* \leq 1/8$, so that $\varepsilon \leq R/8$. Then, for any $x \in B(x_0,\, 3R/4)$, Lemma 1.9 implies

$$
\begin{aligned}
||\mathbf{Q}_\varepsilon(x)| - 1| &\lesssim \left(\frac{\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon;\, B(x,\, R/4))}{\log\left(\dfrac{R}{4\varepsilon}\right)}\right)^{1/4} + \varepsilon^{1/2} \\
&\overset{(2.34)}{\lesssim} \left(\frac{\eta_* \log(R/\varepsilon)}{\log(R/\varepsilon) - \log 4}\right)^{1/4} + \varepsilon^{1/2} \lesssim 3^{1/4}\,\eta_*^{1/4} + \varepsilon_*^{1/2}\, R^{1/2}
\end{aligned}
$$

If $\eta_*$ and $\varepsilon_*$ are chosen small enough, then (2.35) follows.

*Step* 2 (Bounds on $j(\mathbf{Q}_\varepsilon)$). Let $\rho_\varepsilon := |\mathbf{Q}_\varepsilon|$. Thanks to (2.35), we can apply topological lifting results and write $\mathbf{Q}_\varepsilon$ in "polar coordinates", as in (1.2), for some (smooth) scalar function $\varphi_\varepsilon \colon B(x_0,\, 3R/4) \to \mathbb{R}$. Up to an additive constant, we can assume without loss of generality that

$$
0 \leq \fint_{B(x_0,\, 3R/4)} \varphi_\varepsilon < 2\pi. \tag{2.37}
$$

The pre-Jacobian of $\mathbf{Q}_\varepsilon$ can the be written as $j(\mathbf{Q}_\varepsilon) = \rho_\varepsilon^2\, \nabla\varphi_\varepsilon$. Then, Equation (2.3) reduces to

$$
-\operatorname{div}\left(\rho_\varepsilon^2\, \nabla\varphi_\varepsilon\right) = \frac{\varepsilon}{2}\, \partial_k\left(\mathbf{M}_\varepsilon \times \partial_k \mathbf{M}_\varepsilon\right) \qquad \text{in } B(x_0,\, 3R/4). \tag{2.38}
$$

The left-hand side of (2.38) is uniformly elliptic, because of (2.35) and the $L^\infty$-bound (1.35), while the right-hand side is uniformly bounded in $W^{-1,\infty}(\Omega)$, because of (1.35) and (1.36). Moreover, Proposition 2.1 and (1.3) imply that

$$
\|\nabla\varphi_\varepsilon\|_{L^p(B(x_0,\, 3R/4))} \leq C_p, \qquad \text{for any } p \in [1,\, 2)
$$

and for some constant $C_p$ that depends on $p$, but not on $\varepsilon$. By applying Calderon-Zygmund estimates to Equation (2.38), we obtain the interior estimate

$$
\|\nabla\varphi_\varepsilon\|_{L^p(B(x_0,\, 5R/8))} \leq C_p, \qquad \text{for any } p \in [1,\, +\infty), \tag{2.39}
$$

for some constant $C_p$ that depends on $p$, $x_0$, $R$ but not on $\varepsilon$. (More precisely, the constant $C_p$ depends only on $\beta$, $\Omega$, $p$, $x_0$, $R$, $\mathrm{C}_{\mathrm{pot}}$, and $\mathbf{Q}_{\mathrm{bd}}$ but just through the $L^1(\partial\Omega)$- and the $L^2(\partial\Omega)$-norm of $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}}\mathbf{Q}_{\mathrm{bd}}$.)

*Step* 3 (Bounds on the modulus). Let us consider the equation (2.24) for $\rho_\varepsilon$. Thanks to (2.35), we can divide both sides by $\rho_\varepsilon$. Then, applying (1.3), we obtain

$$
-\Delta\rho_\varepsilon + 4\rho_\varepsilon\, |\nabla\varphi_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left(\rho_\varepsilon^2 - 1\right)\rho_\varepsilon - \frac{1}{\varepsilon}\sigma_\varepsilon = 0 \qquad \text{in } B(x_0,\, 3R/4), \tag{2.40}
$$

where $\sigma_\varepsilon := \beta\rho_\varepsilon^{-1}\, \mathbf{Q}_\varepsilon\mathbf{M}_\varepsilon \cdot \mathbf{M}_\varepsilon$. Let $B := B(x_0, 5R/8)$. Let $\zeta \in C^\infty_{\mathrm{c}}(B)$ be a cut-off function such that $\zeta = 1$ in $B(x_0,\, R/2)$ and $0 \leq \zeta \leq 1$ in $B$. By testing Equation (2.40) against $(1 - \rho_\varepsilon)\,\zeta^2$, integrating by parts, and applying Young's inequality, we obtain

$$\begin{aligned}\frac{1}{2}\int_B \zeta^2 \left|\nabla\rho_\varepsilon\right|^2 \leq 2\int_B (1-\rho_\varepsilon)^2 \left|\nabla\zeta\right|^2 + 4\int_B \rho_\varepsilon(1-\rho_\varepsilon)\left|\nabla\varphi_\varepsilon\right|^2\zeta^2 \\ + \frac{1}{\varepsilon^2}\int_B \rho_\varepsilon\left(\rho_\varepsilon^2-1\right)(1-\rho_\varepsilon)\,\zeta^2 - \frac{1}{\varepsilon}\int_B \sigma_\varepsilon(1-\rho_\varepsilon)\,\zeta^2 \\ =: \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon + \mathrm{IV}_\varepsilon.\end{aligned}$$

We claim that all the terms at the right-hand side are bounded uniformly with respect to $\varepsilon$. For $\mathrm{I}_\varepsilon$, this is an immediate consequence of the $L^\infty$-bound (1.35). Boundedness of $\mathrm{II}_\varepsilon$ follows from (1.35) and (2.39). For $\mathrm{III}_\varepsilon$, we observe that

$$\left|\mathrm{III}_\varepsilon\right| \leq \frac{1}{\varepsilon^2}\int_B \frac{\rho_\varepsilon\left(\rho_\varepsilon^2-1\right)^2}{1+\rho_\varepsilon},$$

then apply (1.35) and (1.43) to conclude that $|\mathrm{III}_\varepsilon| \leq C$ for some $\varepsilon$-independent $C$. Finally, we estimate $\mathrm{IV}_\varepsilon$ by applying the Hölder inequality:

$$\left|\mathrm{IV}_\varepsilon\right| \leq 4\left(\int_B \sigma_\varepsilon^2\right)^{1/2}\left(\frac{1}{\varepsilon^2}\int_B \left(\rho_\varepsilon^2-1\right)^2\right)^{1/2}$$

Since $\sigma_\varepsilon$ is bounded (because of (1.35) and (2.35)), the estimate (1.43) implies that $|\mathrm{IV}_\varepsilon| \leq C$. Therefore, we have proved that

$$\int_B \zeta^2\left|\nabla\rho_\varepsilon\right|^2 \leq C, \tag{2.41}$$

where the constant $C$ does not depend on $\varepsilon$ (but does depend on $x_0$, $R$, in general). Together with (2.39), this inequality implies (2.36). □

*Remark* 2.2. As it can be easily checked from the proof above, the constant $C$ on the right-hand side of (2.41) depends only on $x_0$, $R$, $\beta$, $\Omega$, $\mathrm{C}_{\mathrm{pot}}$, and $\mathbf{Q}_{\mathrm{bd}}$, but just through the $L^1(\partial\Omega)$- and the $L^2(\partial\Omega)$-norm of $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}}\mathbf{Q}_{\mathrm{bd}}$. Keeping (2.39) into account, it follows that the constant $C_*(x_0, R)$ on the right-hand side of (2.36) depends only on the quantities listed above.

We now improve on Proposition 2.5 so to obtain the following refined bounds.

**Proposition 2.6.** *Let* $(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)$ *be a solution to* (4)–(5), *subject to boundary conditions either as in* (6)–(7) *or as in* (8)–(9). *Assume that the condition* (10) *is satisfied. Let* $B(x_0, R) \subset\subset \Omega$ *be a ball such that* $|\mathbf{Q}_\varepsilon| \geq 1/2$ *on* $B(x_0, R)$. *Then,*

$$\left\|\nabla\mathbf{Q}_\varepsilon\right\|_{L^p(B(x_0,R/2))} + \varepsilon^{-1}\left\|\left|1-\left|\mathbf{Q}_\varepsilon\right|\right|\right\|_{L^p(B(x_0,R/2))} \leq C_p(x_0,R). \tag{2.42}$$

Before proceeding to the proof of Proposition 2.6, we point out the following observation.

*Remark* 2.3. If $K$ is a compact set contained in $\{|\mathbf{Q}_\varepsilon| \geq 1/2\}$, then (2.42) holds on $K$ (by a covering argument), with a constant $C_p(K)$ depending on $p$ and $K$ (but not on $\varepsilon$). In particular, for any ball $B_R = B(x_0, R) \subseteq K$, applying the Hölder inequality, we get

$$\int_{B_R}\left(\frac{1}{2}\left|\nabla\mathbf{Q}_\varepsilon\right|^2 + \frac{1}{4\varepsilon^2}\left(\left|\mathbf{Q}_\varepsilon\right|^2-1\right)^2\right)\mathrm{d}x \leq C_\alpha(K)R^\alpha \tag{2.43}$$

for any $\alpha \in (0,2)$.

Moreover, we state the following lemma, which replaces [5, Lemma 2].

**Lemma 2.7.** *Let $B = B(x_0, R)$ be a ball. Let $B'$ be the ball concentric with $B$ and with radius $r < R$. Assume that $f$ is a continuous function that belongs to $L^p(B)$ for any $p \in [1, +\infty)$ and that $v \in C^2(B)$ is a non-negative function satisfying*

$$-\frac{1}{2}\Delta v + \frac{1}{4\varepsilon^2} v \leq f \qquad \text{in } B \tag{2.44}$$

*and*

$$\|v\|_{L^\infty(B)} < +\infty.$$

*Then,*

$$\|v\|_{L^p(B')} \lesssim \varepsilon^2, \qquad \text{for any } p \in [1, +\infty), \tag{2.45}$$

*where the implicit constant in front of the right-hand side depends only on $p$, $r$, $R$, $\|v\|_{L^\infty(B)}$, and $\|f\|_{L^p(B)}$.*

*Proof.* We fix $\zeta \in C_c^\infty(B)$, a smooth cut-off function satisfying

$$0 \leq \zeta \leq 1 \ \text{ in } B, \qquad \zeta \equiv 1 \ \text{ in } B', \qquad |\nabla \zeta| \lesssim \frac{1}{R - r} \ \text{ in } B. \tag{2.46}$$

We observe that, by Hölder's inequality, it is enough to prove (2.45) for any $p$ large enough. Therefore, we can assume that $p > 2$. We define

$$\psi := \zeta^{2p} v^{p-1}.$$

The function $\psi$ belongs to $C_c^1(B)$ and we can test the inequality (2.44) against $\psi$. Since $\psi \geq 0$, this yields

$$\int_B \frac{1}{2} \nabla \psi \cdot \nabla v \, \mathrm{d}x + \frac{1}{4\varepsilon^2} \int_B \zeta^{2p} v^p \, \mathrm{d}x \leq \int_B \zeta^{2p} v^{p-1} f \, \mathrm{d}x. \tag{2.47}$$

Since $0 \leq \zeta \leq 1$, we have $\zeta^{2p} \leq \zeta^{2(p-1)}$ pointwise and therefore, by Hölder's inequality,

$$\int_B \zeta^{2p} v^{p-1} f \, \mathrm{d}x \leq \int_B \zeta^{2(p-1)} v^{p-1} |f| \, \mathrm{d}x \leq \|f\|_{L^p(B)} \left( \int_B \zeta^{2p} v^p \, \mathrm{d}x \right)^{1-1/p}. \tag{2.48}$$

On the other hand, we have

$$\int_B \nabla \psi \cdot \nabla v \, \mathrm{d}x = \int_B \left( 2p \zeta^{2p-1} v^{p-1} \nabla \zeta \cdot \nabla v + (p-1) \zeta^{2p} v^{p-2} |\nabla v|^2 \right) \mathrm{d}x,$$

so that, by (2.47), we obtain

$$\begin{aligned} &\int_B \frac{1}{2}(p-1) \zeta^{2p} v^{p-2} |\nabla v|^2 \, \mathrm{d}x + \frac{1}{4\varepsilon^2} \int_B \zeta^{2p} v^p \, \mathrm{d}x \\ &\qquad \leq \|f\|_{L^p(B)} \left( \int_B \zeta^{2p} v^p \, \mathrm{d}x \right)^{1-1/p} + \int_B p \zeta^{2p-1} v^{p-1} |\nabla \zeta| \, |\nabla v| \, \mathrm{d}x. \end{aligned} \tag{2.49}$$

Since $p > 2$, we may rewrite the integrand in the last term on the right-hand side as

$$p\zeta^{2p-1}v^{p-1}\left|\nabla\zeta\right|\left|\nabla v\right| = p\left(\zeta^{p-1}v^{p/2}\left|\nabla\zeta\right|\right)\left(\zeta^{p}v^{p/2-1}\left|\nabla v\right|\right)$$

Thus, by applying Young's inequality and using (2.46), we obtain the bound

$$p\zeta^{2p-1}v^{p-1}\left|\nabla\zeta\right|\left|\nabla v\right| \leq \frac{C'_p}{(R-r)^2}\zeta^{2(p-1)}v^p + \frac{1}{2}(p-1)\zeta^{2p}v^{p-2}\left|\nabla v\right|^2, \tag{2.50}$$

where $C'_p$ is a positive constant that depends only on $p$. Plugging (2.50) into (2.49) and recalling that $v^p \leq \|v\|_{L^\infty(B)}\, v^{p-1}$ pointwise, we arrive at

$$\frac{1}{4\varepsilon^2}\int_B \zeta^{2p}v^p\,\mathrm{d}x \leq C_p\left(r,\, R,\, \|v\|_{L^\infty(B)},\, \|f\|_{L^p(B)}\right)\left[\left(\int_B \zeta^{2p}v^p\,\mathrm{d}x\right)^{1-1/p} + \int_B \zeta^{2(p-1)}v^{p-1}\,\mathrm{d}x\right],$$

whence, by a further application of Hölder's inequality on the last term on the right-hand side, we obtain

$$\frac{1}{4\varepsilon^2}\int_B \zeta^{2p}v^p\,\mathrm{d}x \leq C_p\left(r,\, R,\, \|v\|_{L^\infty(B)},\, \|f\|_{L^p(B)}\right)\left(\int_B \zeta^{2p}v^p\,\mathrm{d}x\right)^{1-1/p},$$

for a possibly larger constant $C_p\left(r,\, R,\, \|v\|_{L^\infty(B)},\, \|f\|_{L^p(B)}\right)$ (still depending only on $p$, $R$, and $\|f\|_{L^p(B)}$), so that

$$\left(\int_B \zeta^{2p}v^p\,\mathrm{d}x\right)^{1/p} \leq C_p\left(r,\, R,\, \|v\|_{L^\infty(B)},\, \|f\|_{L^p(B)}\right)\varepsilon^2,$$

and, recalling that $\zeta \equiv 1$ in $B'$, the desired inequality (2.45) follows. $\square$

*Remark* 2.4. If we happen to know that $f \in L^\infty(B)$, then, instead of the above lemma, we may apply [5, Lemma 2] and obtain the stronger conclusion

$$\|v\|_{L^\infty(B')} \lesssim \varepsilon^2.$$

*Proof of Proposition 2.6.* The proof is inspired by classical arguments in [10, 9] and, to make it more transparent, it is split into several steps.

*Step* 1. Set $\rho_\varepsilon := |\mathbf{Q}_\varepsilon|$ and let $\varphi_\varepsilon : B\left(x_0, 3R/4\right) \to \mathbb{R}$ be the angle function given by (1.2). By assumption, $\rho_\varepsilon \geq 1/2$ on $B(x_0,\, 3R/4)$ and then, by (2.39), we have $\|\nabla\varphi_\varepsilon\|_{L^p\left(B\left(x_0, \frac{5}{8}R\right)\right)} \leq C_p$ for any $p < +\infty$. From now on, we drop the subscript $\varepsilon$, for ease of notation.

*Step* 2. Writing the equation for $\rho$ in the form

$$-\Delta\rho + 4\rho\left|\nabla\varphi\right|^2 + \frac{1}{\varepsilon^2}\left(\rho^2 - 1\right)\rho = \frac{\sigma}{\varepsilon} \qquad \text{in } B\left(x_0, \frac{3R}{4}\right), \tag{2.51}$$

where $\sigma := \frac{\beta\mathbf{Q}\mathbf{M}\cdot\mathbf{M}}{\rho}$ is bounded, and multiplying it by $(\rho - 1)$, we obtain

$$(1-\rho)\Delta\rho + \frac{1}{\varepsilon^2}\left(\rho^2 - 1\right)(\rho - 1)\rho = 4(1-\rho)\rho\left|\nabla\varphi\right|^2 + \frac{\sigma}{\varepsilon}(\rho - 1). \tag{2.52}$$

We manipulate the terms at the left-hand side of the above equation as follows:

$$(1-\rho)\Delta\rho = \operatorname{div}((1-\rho)\nabla\rho) + |\nabla\rho|^2 = -\frac{1}{2}\Delta\left((\rho-1)^2\right) + |\nabla\rho|^2 \tag{2.53}$$

and

$$\frac{1}{\varepsilon^2}\left(\rho^2-1\right)(\rho-1) \geq \frac{1}{2\varepsilon^2}(\rho-1)^2,$$

while for the right-hand side we observe that, by Young's inequality,

$$\left|\frac{\sigma}{\varepsilon}(\rho-1)\right| \leq \frac{(\rho-1)^2}{4\varepsilon^2} + \sigma^2.$$

Thus, we obtain

$$-\frac{1}{2}\Delta\left((\rho-1)^2\right) + \frac{1}{4\varepsilon^2}(\rho-1)^2 \leq \widetilde{\sigma}, \tag{2.54}$$

where $\widetilde{\sigma} := \sigma^2 + 4(1-\rho)\rho\,|\nabla\varphi|^2$. Since $\sigma$ and $\rho$ are (uniformly) bounded in $L^\infty(\Omega)$ and $\varphi$ satisfies (2.39) in $B\left(x_0, \frac{5R}{8}\right)$, it follows that

$$\|\widetilde{\sigma}\|_{L^p\left(B\left(x_0,\frac{5R}{8}\right)\right)} \leq C_p, \qquad \text{for any } p < +\infty, \tag{2.55}$$

where, by (2.39), $C_p$ depends only on $p$, $x_0$, $R$, $\beta$, $\Omega$, $\mathrm{C}_{\mathrm{pot}}$, and the $L^1(\partial\Omega)$- and $L^2(\partial\Omega)$-norms of $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}}\mathbf{Q}_{\mathrm{bd}}$.

We now claim that

$$\|\rho-1\|_{L^p\left(B\left(x_0,\frac{9R}{16}\right)\right)} \leq C_p(x_0,R)\varepsilon, \qquad \text{for any } p < +\infty, \tag{2.56}$$

where $C_p(x_0,R)$ depends only on $p$, $x_0$, $R$, the coupling constant $\beta$, $\Omega$, $\mathrm{C}_{\mathrm{pot}}$, and $\mathbf{Q}_{\mathrm{bd}}$ (but just through the $L^1(\partial\Omega)$- and the $L^2(\partial\Omega)$-norm of $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}}\mathbf{Q}_{\mathrm{bd}}$).

*Proof of* (2.56). Let

$$B := B\left(x_0, \frac{5}{8}R\right), \qquad B' := B\left(x_0, \frac{9}{16}R\right).$$

Recasting (2.54) as an inequality for the non-negative function $v := (\rho-1)^2$ leads to

$$-\frac{1}{2}\Delta v + \frac{1}{4\varepsilon^2}v \leq \widetilde{\sigma} \qquad \text{in } B. \tag{2.57}$$

Note that $\|v\|_{L^\infty(\Omega)}$ is bounded only in terms of $\beta$, by (1.35). From Lemma 2.7, we have

$$\|v\|_{L^p(B')} \lesssim \varepsilon^2, \qquad \text{for any } p \in [1,+\infty),$$

where the implicit constant in front of the right-hand side depends only on $p$, $R$, $\beta$, and $\|\widetilde{\sigma}\|_{L^p(B)}$. Since the latter norm is controlled by a constant depending only on $x_0$, $R$, $\beta$, $\Omega$, $\mathrm{C}_{\mathrm{pot}}$, and $\mathbf{Q}_{\mathrm{bd}}$ (but just through the $L^1(\partial\Omega)$- and the $L^2(\partial\Omega)$-norm of $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}}\mathbf{Q}_{\mathrm{bd}}$.), going back from $v$ to $(\rho-1)^2$, we obtain (2.56), where the implicit constant in front of the right-hand side depends only on $p$, $x_0$, $R$, and $\beta$, as claimed.

*Step* 3 ($L^p$-estimates for $\nabla\rho$ and conclusion). Writing (4) in the form

$$-\Delta \mathbf{Q} = \frac{1}{\varepsilon}\left(\frac{(1-|\mathbf{Q}|^2)}{\varepsilon}\mathbf{Q} + \beta\left(\mathbf{M}\otimes\mathbf{M} - \frac{|\mathbf{M}|^2}{2}\,\mathrm{Id}\right)\right),$$

the global $L^\infty$-bounds on $\mathbf{Q}$ and $\mathbf{M}$ provided by Lemma 1.6 and (2.56) tell us that the term in round brackets above is bounded in $L^p\left(B\left(x_0, \frac{9R}{16}\right)\right)$. Thus, by standard elliptic regularity estimates,

$$\begin{aligned} \|\mathbf{Q}\|_{W^{2,p}(B(x_0,\frac{R}{2}))} &\lesssim \|\Delta\mathbf{Q}\|_{L^p(B(x_0,\frac{9R}{16}))} + \|\mathbf{Q}\|_{L^p(B(x_0,\frac{9R}{16}))} \\ &\lesssim C_p(x_0, R)\varepsilon^{-1}, \end{aligned} \tag{2.58}$$

where $C_p(x_0, R)$ depends only on $p$, $x_0$, $R$, and the quantities listed in Remark 2.2. By interpolation,

$$\begin{aligned} \|\nabla\rho\|_{L^p(B(x_0,\frac{R}{2}))} &\lesssim \|\mathbf{Q}\|^{1/2}_{W^{2,p}(B(x_0,\frac{R}{2}))}\,\|1-\rho\|^{1/2}_{L^p(B(x_0,\frac{R}{2}))} \\ &\lesssim C_p(x_0, R)\varepsilon^{-1/2}\varepsilon^{1/2} = C_p(x_0,\,\beta,\,R), \end{aligned} \tag{2.59}$$

whence

$$\|\nabla\mathbf{Q}\|_{L^p(B(x_0,\frac{R}{2}))} \lesssim \|\nabla\rho\|_{L^p(B(x_0,\frac{R}{2}))} + \|\nabla\varphi\|_{L^p(B(x_0,\frac{R}{2}))} \leq C_p(x_0,\,\beta,\,R),$$

where, once again, $C_p(x_0,\,\beta,\,R)$ depends only on $p$, $x_0$, $\beta$, $R$, and the quantities list in Remark 2.2. The conclusion follows. $\square$

**Corollary 2.8.** *If $K \subset \Omega$ is any compact set such that $|\mathbf{Q}_\varepsilon| \geq 1/2$ in $K$ for any $\varepsilon$ small enough, then*

$$\||\mathbf{Q}_\varepsilon| - 1\|_{L^\infty(K)} \leq C_\alpha \varepsilon^{1-\alpha} \tag{2.60}$$

$$\|\nabla\mathbf{Q}_\varepsilon\|_{L^\infty(K)} \leq C_\alpha \varepsilon^{-\alpha}, \tag{2.61}$$

*for any $\alpha \in (0, 1]$, where the number $C_\alpha$ depends only on $\alpha$, $\beta$, $K$, and the quantities listed in Remark 2.2.*

*Proof.* By a standard covering argument, it suffices to prove (2.60), (2.61) for a ball $B = B(x_0,\,R)$, with constants depending only on $\alpha$, $\beta$, $x_0$, $R$, and the quantities listed in Remark 2.2. Given $\alpha > 0$ and letting $p := \max\{2, 2/\alpha\}$, by the Gagliardo-Nirenberg interpolation inequality we have

$$\||\mathbf{Q}_\varepsilon| - 1\|_{L^\infty(B)} \lesssim \|\mathbf{Q}_\varepsilon\|^{2/p}_{W^{1,p}(B)}\,\||\mathbf{Q}_\varepsilon| - 1\|^{1-2/p}_{L^p(B)}.$$

Then, by (2.42) we obtain

$$\||\mathbf{Q}_\varepsilon| - 1\|_{L^\infty(B)} \leq C_\alpha \varepsilon^{1-\alpha},$$

where the constant $C_\alpha$ on the right-hand side depends only on $\alpha$, $\beta$, $x_0$, and $R$.

Similarly, again by the Gagliardo-Nirenberg interpolation inequality, we have

$$\|\nabla \mathbf{Q}_\varepsilon\|_{L^\infty(B)} \lesssim \|\mathbf{Q}_\varepsilon\|_{W^{2,p}(B)}^{2/p} \|\nabla \mathbf{Q}_\varepsilon\|_{L^p(B)}^{1-2/p}$$

and then, by (2.42) and (2.58), we obtain

$$\|\nabla \mathbf{Q}_\varepsilon\|_{L^\infty(B)} \leq C_p \varepsilon^{-2/p} \leq C_\alpha \varepsilon^{-\alpha},$$

where the constant $C_\alpha$ in the right-hand side depends only on $\alpha$, $\beta$, $x_0$, and $R$. □

## 2.3 An estimate for the energy of $\mathbf{M}_\varepsilon$

**Proposition 2.9.** *Assume that* (10) *holds. There exists a constant $C > 0$, depending only on $\beta$, $\mathrm{C_{pot}}$, and the boundary data, such that, for any $\varepsilon$ small enough, there holds*

$$\varepsilon \int_\Omega |\nabla \mathbf{M}_\varepsilon|^2 \leq C. \tag{2.62}$$

*Proof.* The strategy for the proof is to test Equation (5) against a suitable test function, then apply the assumption (10) on the potential and the estimate for $\nabla \mathbf{Q}_\varepsilon$ given by Proposition 2.1. Before going to the details, let us recall some notation. Let $\ell$ be defined as in (1.18). We recall from Lemma 1.2 that, for any $x \in \Omega$ such that $\mathbf{Q}_\varepsilon(x) \neq 0$, the function $\ell(\mathbf{Q}_\varepsilon(x), \cdot\,)$ has exactly two minimisers in $\mathbb{R}^2$, that is

$$\mathbf{M}_\pm(x) := \pm \left(\sqrt{2}\beta\rho_\varepsilon(x) + 1\right)^{1/2} \mathbf{n}_\varepsilon(x) \tag{2.63}$$

where $\rho_\varepsilon(x) := |\mathbf{Q}_\varepsilon(x)|$ and $\mathbf{n}_\varepsilon(x)$ is a unit eigenvector corresponding to the (unique) positive eigenvalue of $\mathbf{Q}_\varepsilon(x)$. Let $\Sigma_x := \Sigma(\mathbf{Q}_\varepsilon(x)) := \{\mathbf{M}_\pm(x)\} \subseteq \mathbb{R}^2$. For any $\mathbf{N} \in \mathbb{R}^2$ such that $\mathrm{dist}(\mathbf{N}, \Sigma_x) \leq 1$, let $\pi_x(\mathbf{N}) := \pi(\mathbf{Q}_\varepsilon(x), \mathbf{N})$ be the projection of $\mathbf{N}$ to $\Sigma_x$, as defined in (1.24).

*Step* 1 (Construction of $\mathbf{N}_\varepsilon$). Let $\delta \in (0, 1/2)$ be a fixed parameter, to be chosen later (independently of $\varepsilon$). Let $\xi, \zeta \colon [0, +\infty) \to [0, 1]$ be smooth functions such that

$$\xi(t) = 0 \quad \text{if } 0 \leq t \leq \frac{1}{2}, \qquad \xi(t) = 1 \quad \text{if } t \geq \frac{3}{4} \tag{2.64}$$

$$\zeta(t) = 1 \quad \text{if } 0 \leq t \leq \delta, \qquad \zeta(t) = 0 \quad \text{if } t \geq 2\delta \tag{2.65}$$

For $x \in \Omega$, we define

$$\eta_\varepsilon(x) := \xi\left(\rho_\varepsilon(x)\right) \zeta\left(\mathrm{dist}(\mathbf{M}_\varepsilon(x), \Sigma_x)\right)$$

and

$$\mathbf{N}_\varepsilon(x) := (1 - \eta_\varepsilon(x))\, \mathbf{M}_\varepsilon(x) + \eta_\varepsilon(x)\, \pi_x(\mathbf{M}_\varepsilon(x)) \tag{2.66}$$

The value $\eta_\varepsilon(x)$ is non-zero only if $\rho_\varepsilon(x) \geq 1/2$ and $\mathrm{dist}(\mathbf{M}_\varepsilon(x), \Sigma_x) \leq 2\delta < 1$, therefore $\pi_x(\mathbf{M}_\varepsilon(x))$ is well-defined if $\eta_\varepsilon(x) \neq 0$.

We use $\mathbf{N}_\varepsilon$ as a test function in Equation (5). Assume for a moment we are imposing mixed boundary conditions, as in (8), (9). Then, by testing Equation (5) against $\varepsilon\mathbf{M}_\varepsilon - \varepsilon\mathbf{N}_\varepsilon$, integrating by parts, and applying Young's inequality, we obtain

$$\frac{\varepsilon}{2}\int_\Omega |\nabla \mathbf{M}_\varepsilon|^2 \leq \int_\Omega \left(\frac{\varepsilon}{2}\left|\nabla \mathbf{N}_\varepsilon\right|^2 + \frac{1}{\varepsilon}\nabla_{\mathbf{M}}\ell(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)\cdot(\mathbf{N}_\varepsilon - \mathbf{M}_\varepsilon)\right), \tag{2.67}$$

where $\nabla_{\mathbf{M}}$ denotes the (partial) gradient with respect to the $\mathbf{M}$-variable. Equation (2.67) remains true if we are considering Dirichlet boundary conditions as in (6), (7). Indeed, the assumption (7) implies that the boundary data satisfy $|\mathbf{Q}_{\mathrm{bd}}(x)| = 1$, $\mathbf{M}_{\mathrm{bd}}(x) \in \Sigma_x$ and, hence, $\eta_\varepsilon(x) = 1$, $\mathbf{N}_\varepsilon(x) = \pi_x(\mathbf{M}_{\mathrm{bd}}(x)) = \mathbf{M}_{\mathrm{bd}}(x)$ for each $x \in \partial\Omega$. As a consequence, $\varepsilon\mathbf{M}_\varepsilon - \varepsilon\mathbf{N}_\varepsilon = 0$ on $\partial\Omega$ and (2.67) follows, irrespective of whether we are imposing Dirichlet or mixed boundary conditions.

*Step* 2 (Estimate on $\nabla\mathbf{N}_\varepsilon$). We claim that

$$\varepsilon\int_\Omega |\nabla \mathbf{N}_\varepsilon|^2 \lesssim \int_\Omega \left(\varepsilon\left|\nabla \mathbf{Q}_\varepsilon\right|^2 + \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)\right). \tag{2.68}$$

Here (and in the rest of this proof, wherever we use the notation $\lesssim$) the implicit constant in front of the right-hand side depends on $\delta$, but not on $\varepsilon$. In order to prove this claim, we consider the (open) set

$$E_\varepsilon := \left\{x \in \Omega\colon \rho_\varepsilon(x) > \frac{3}{4},\ \mathrm{dist}(\mathbf{M}_\varepsilon(x),\, \Sigma_x) < \delta\right\}. \tag{2.69}$$

By Lemma 1.5, there exists a constant $c_1(\delta) > 0$, depending on $\delta$ and $\beta$ but not on $\varepsilon$, such that $f_\varepsilon(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon) \geq c_1(\delta)\,\varepsilon$ on $\Omega \setminus E_\varepsilon$. As a consequence, we have

$$|\Omega \setminus E_\varepsilon| \leq \frac{1}{c_1(\delta)\,\varepsilon}\int_\Omega f_\varepsilon(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon). \tag{2.70}$$

On the other hand, the bound (1.36) implies the pointwise estimate $|\nabla\mathbf{N}_\varepsilon| \lesssim \varepsilon^{-1}$. Therefore, from (2.70) we deduce

$$\varepsilon\int_{\Omega\setminus E_\varepsilon} |\nabla \mathbf{N}_\varepsilon|^2 \lesssim \frac{|\Omega \setminus E_\varepsilon|}{\varepsilon} \lesssim \frac{1}{\varepsilon^2}\int_\Omega f_\varepsilon(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon). \tag{2.71}$$

It remains to estimate $|\nabla\mathbf{N}_\varepsilon|$ on $E_\varepsilon$. For any $x \in E_\varepsilon$, we have $\eta_\varepsilon(x) = 1$ (because of (2.64), (2.65)) and $\mathbf{N}_\varepsilon(x) = \pi_x(\mathbf{M}_\varepsilon(x))$. By definition of $\pi_x$ (see (1.24)), $\mathbf{N}_\varepsilon(x)$ is a nonzero eigenvector of $\mathbf{Q}_\varepsilon(x)$ corresponding to the (unique) positive eigenvalue. We claim that $\mathbf{N}_\varepsilon$ is smooth in the open set $E_\varepsilon$. Indeed, let $B \subseteq E_\varepsilon$ be an arbitrary open ball. The spectral theorem implies that $\mathbf{Q}_\varepsilon$ can be written in the form

$$\mathbf{Q}_\varepsilon = \frac{\rho_\varepsilon}{\sqrt{2}}\left(\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon - \mathbf{m}_\varepsilon \otimes \mathbf{m}_\varepsilon\right) \qquad \text{in } B,$$

where $(\mathbf{n}_\varepsilon,\, \mathbf{m}_\varepsilon)$ is a smooth orthonormal frame of eigenvectors for $\mathbf{Q}_\varepsilon$, and (see (2.63))

$$\mathbf{N}_\varepsilon(x) = \pi_x(\mathbf{M}_\varepsilon(x)) = \pm\left(\sqrt{2}\beta\rho_\varepsilon(x) + 1\right)^{1/2}\mathbf{n}_\varepsilon(x) \qquad \text{at any point } x \in B, \tag{2.72}$$

By definition of $\pi_x$, the sign of $\mathbf{N}_\varepsilon(x) \cdot \mathbf{n}_\varepsilon(x)$ is the same as the sign of $\mathbf{M}_\varepsilon(x) \cdot \mathbf{n}_\varepsilon(x)$ and is nonzero, because $\mathrm{dist}(\mathbf{M}_\varepsilon(x), \Sigma_x) \leq 1/2$. Therefore, the sets $B^+ := \{x \in B \colon \mathbf{N}_\varepsilon(x) \cdot \mathbf{n}_\varepsilon(x) > 0\}$ and $B^- := \{x \in B \colon \mathbf{N}_\varepsilon(x) \cdot \mathbf{n}_\varepsilon(x) < 0\}$ are open, disjoint, and their union contains $B$. As $B$ is connected, it follows that either $B = B^+$ or $B = B^-$ — that is, the sign in (2.72) is constant. Either way, $\mathbf{N}_\varepsilon$ is smooth in $B$ and there holds

$$|\nabla \mathbf{N}_\varepsilon| \lesssim |\nabla \rho_\varepsilon| + |\nabla \mathbf{n}_\varepsilon| \lesssim |\nabla \mathbf{Q}_\varepsilon|$$

pointwise in $B$. As the ball $B \subseteq E_\varepsilon$ is chosen arbitrarily, we have

$$\int_{E_\varepsilon} |\nabla \mathbf{N}_\varepsilon|^2 \lesssim \int_\Omega |\nabla \mathbf{Q}_\varepsilon|^2 . \tag{2.73}$$

Now, (2.68) follows from (2.71) and (2.73).

*Step* 3 (Estimate on the potential term). We claim that

$$\frac{1}{\varepsilon} \int_\Omega \nabla_{\mathbf{M}} \ell(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \cdot (\mathbf{N}_\varepsilon - \mathbf{M}_\varepsilon) \lesssim \frac{1}{\varepsilon^2} \int_\Omega f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon). \tag{2.74}$$

To this end, we consider again the set $E_\varepsilon$ defined by (2.69) above. The $L^\infty$-bound (1.35) and the estimate (2.71) imply

$$\frac{1}{\varepsilon} \int_{\Omega \setminus E_\varepsilon} \nabla_{\mathbf{M}} \ell(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \cdot (\mathbf{N}_\varepsilon - \mathbf{M}_\varepsilon) \lesssim \frac{|\Omega \setminus E_\varepsilon|}{\varepsilon} \lesssim \frac{1}{\varepsilon^2} \int_\Omega f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon). \tag{2.75}$$

To estimate the contribution from $E_\varepsilon$, we observe that $\mathbf{N}_\varepsilon(x) = \pi_x(\mathbf{M}_\varepsilon(x))$ at each point $x \in E_\varepsilon$. As a consequence, Lemma 1.4 and the $L^\infty$-bound (1.35) imply

$$\frac{1}{\varepsilon} \int_{E_\varepsilon} \nabla_{\mathbf{M}} \ell(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \cdot (\mathbf{N}_\varepsilon - \mathbf{M}_\varepsilon) \leq 0, \tag{2.76}$$

so long as we choose $\delta$ small enough. (We can choose a suitable value of $\delta$ that depends only on $\beta$ and the $L^\infty$-norm of $\mathbf{Q}_\varepsilon$; however, the latter is bounded uniformly with respect to $\varepsilon$, so $\delta$ can be chosen independently of $\varepsilon$.) Combining (2.75) with (2.76), we obtain (2.74).

*Step* 4 (Conclusion). From (2.67), (2.68) and (2.74), we deduce

$$\varepsilon \int_\Omega |\nabla \mathbf{M}_\varepsilon|^2 \lesssim \int_\Omega \left( \varepsilon \, |\nabla \mathbf{Q}_\varepsilon|^2 + \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \right), \tag{2.77}$$

The terms at the right-hand side are bounded uniformly with respect of $\varepsilon$, because of Proposition 2.1 and Assumption (10). The lemma follows. $\square$

*Remark* 2.5. We stress that the estimate (2.77) is completely independent of the assumption (10), although it relies on the assumptions (7), (9) on the boundary data and the maximum principle, Lemma 1.6 (which is itself independent of (10), see Remark 1.6).

*Remark* 2.6. More precisely, like in Proposition 2.1, the constant $C$ on the right-hand side of (2.62) depends on the boundary data just through the $L^1(\partial\Omega)$- and $L^2(\partial\Omega)$-norms of $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}} \mathbf{Q}_{\mathrm{bd}}$.

From Proposition 2.9, we immediately draw the following consequence.

**Corollary 2.10.** *Assume that* (10) *holds. Then, for any $\varepsilon > 0$ small enough,*

$$\|j(\mathbf{M}_\varepsilon)\|_{L^p(\Omega)} \lesssim \varepsilon^{-1/2} \tag{2.78}$$

*for any $p \in [1, 2]$. As a consequence,*

$$\varepsilon \left\|\operatorname{div} j(\mathbf{M}_\varepsilon)\right\|_{W^{-1,p}(\Omega)} \lesssim \varepsilon^{1/2} \tag{2.79}$$

*for any $p \in [1, 2]$ and any $\varepsilon$ small enough.*

*Proof.* Let $p \in [1, 2]$ be arbitrary and assume $\varepsilon$ is so small that (2.62) holds. Since $j(\mathbf{M}_\varepsilon) = \mathbf{M}_\varepsilon \times \nabla\mathbf{M}_\varepsilon$, by the uniform bound $\|\mathbf{M}_\varepsilon\|_{L^\infty(\Omega)} \lesssim 1$, the Hölder inequality, and (2.62) we have

$$\begin{aligned} \int_\Omega |j(\mathbf{M}_\varepsilon)|^p \,\mathrm{d}x &= \int_\Omega |\mathbf{M}_\varepsilon \times \nabla\mathbf{M}_\varepsilon|^p \,\mathrm{d}x \leq \int_\Omega |\mathbf{M}_\varepsilon|^p \,|\nabla\mathbf{M}_\varepsilon|^p \,\mathrm{d}x \\ &\lesssim \left(\int_\Omega |\nabla\mathbf{M}_\varepsilon|^2\right)^{p/2} \overset{(2.62)}{\lesssim} \left(\frac{1}{\varepsilon}\right)^{p/2}, \end{aligned}$$

which yield (2.78). Now, pick arbitrarily $\Psi \in W_0^{1,p'}(\Omega)$. Then, for any $\varepsilon$ as before,

$$\begin{aligned} \left|\langle \varepsilon \operatorname{div}(j(\mathbf{M}_\varepsilon)),\, \Psi\rangle_{W^{-1,p}(\Omega), W_0^{1,p'}(\Omega)}\right| &= \varepsilon \left|\int_\Omega j(\mathbf{M}_\varepsilon)\cdot\nabla\Psi \,\mathrm{d}x\right| \\ &\leq \varepsilon \,\|j(\mathbf{M}_\varepsilon)\|_{L^p(\Omega)} \,\|\nabla\Psi\|_{L^{p'}(\Omega)} \\ &\lesssim \varepsilon^{1/2} \,\|\Psi\|_{W^{1,p'}(\Omega)}, \end{aligned}$$

which implies (2.79). □

## 2.4 Summary of the energy estimates

Combining assumption (10) with Lemma 1.1, Proposition 2.1, Proposition 2.5, and Proposition 2.9, we obtain the following lemma, summarising the various energy estimates at our disposal.

**Theorem 2.11.** *Let $\{(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\}$ be a sequence of critical points of the functional $\mathscr{F}_\varepsilon$, subject to boundary conditions as in* (6)*,* (7) *or as in* (8)*,* (9)*. Assume that* (10) *holds. Then,*

$$\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \lesssim |\log\varepsilon|, \tag{2.80}$$

$$\int_\Omega |\nabla\mathbf{Q}_\varepsilon|^2 \lesssim |\log\varepsilon|, \tag{2.81}$$

$$\int_\Omega \varepsilon\,|\nabla\mathbf{M}_\varepsilon|^2 \lesssim 1, \tag{2.82}$$

$$\frac{1}{\varepsilon^2}\int_\Omega f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \lesssim 1, \tag{2.83}$$

$$\frac{1}{\varepsilon^2}\int_\Omega \left(1 - |\mathbf{Q}_\varepsilon|^2\right)^2 \lesssim 1, \tag{2.84}$$

*where the implicit constants on the right-hand side depend only on $\beta$, $\mathrm{C}_{\mathrm{pot}}$, and the $L^1(\partial\Omega)$- and the $L^2(\partial\Omega)$-norm of $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}}\mathbf{Q}_{\mathrm{bd}}$.*

*Moreover, on any ball $B = B(x_0, R) \subset\subset \Omega$ on which $|\mathbf{Q}_\varepsilon| \geq 1/2$, there hold*

$$\int_B |\nabla \mathbf{Q}_\varepsilon|^2 \, \mathrm{d}x \leq C(x_0, R, \beta, \mathrm{C}_{\mathrm{pot}}), \tag{2.85}$$

$$\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; B) \leq C(x_0, R, \beta, \mathrm{C}_{\mathrm{pot}}), \tag{2.86}$$

*where the constant $C(x_0, R, \beta, \mathrm{C}_{\mathrm{pot}}, \mathbf{Q}_{\mathrm{bd}})$ depends only on $x_0$, $R$, $\beta$, $\mathrm{C}_{\mathrm{pot}}$, and the $L^1(\partial\Omega)$- and the $L^2(\partial\Omega)$-norm of $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}}\mathbf{Q}_{\mathrm{bd}}$. Consequently, if $K \subset \Omega$ is any compact set such that $|\mathbf{Q}_\varepsilon| \geq 1/2$ on $K$ for any $\varepsilon$ small enough, then there holds*

$$\lim_{\varepsilon \to 0} \frac{\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; K)}{|\log \varepsilon|} = 0. \tag{2.87}$$

*Remark* 2.7. It follows from (3.1) below and classical results in Ginzburg-Landau theory ([25, 27] that, under both (6)–(7) and (9), as soon as $\deg(\mathbf{Q}_{\mathrm{bd}}, \partial\Omega) \neq 0$, the energy $\mathscr{F}_\varepsilon$ grows up *at least* logarithmically in $\varepsilon$. In particular, if (10) holds, then $\int_\Omega |\nabla \mathbf{Q}_\varepsilon|^2 \, \mathrm{d}x$ grows up at least logarithmically in $\varepsilon$.

# 3 Compactness

In this section, we prove compactness properties, in appropriate norms, for both the $\mathbf{Q}_\varepsilon$-component and the $\mathbf{M}_\varepsilon$-component of sequences $\{(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\}$ of critical points of $\mathscr{F}_\varepsilon$, subject to boundary condition either as in (6)–(7) or as in (8)–(9) and satisfying assumption (10). We first prove convergence (up to subsequences) for the $\mathbf{Q}_\varepsilon$-component to a limiting map $\mathbf{Q}_\star : \Omega \to \mathscr{N}$. For this task, we employ methods reminiscent of the standard Ginzburg-Landau theory but appropriately modified to take the coupling term into account. A key step consists in proving that the energy densities $\mu_\varepsilon$ defined in (12) converge to a limiting measure $\mu_\star$ whose support is a finite set of points. Then, as in [20], we observe that, away from the (finite) energy-concentration set and up to a suitable change of variable (presented in Section 3.1 below), we can rewrite the functional $\mathscr{F}_\varepsilon$ as the sum of a Ginzburg-Landau type functional (with a slightly modified potential) and a vectorial Allen-Cahn functional including a perturbation term which is small in energy. Taking advantage of this fact and of classical compactness results for sequences with bounded Allen-Cahn energy from [3, 22], we eventually show that a subsequence of $\{\mathbf{M}_\varepsilon\}$ converges in $L^p(\Omega)$, for any finite $p \geq 1$.

## 3.1 Changes of variables: from Q to q and from M to u

Following [20, Section 3], we introduce suitable types of “change of variables” for both the $\mathbf{Q}$-component and the $\mathbf{M}$-component of a pair $(\mathbf{Q}, \mathbf{M}) \in \mathscr{S}_0^{2\times 2} \times \mathbb{R}^2$.

As for the $\mathbf{Q}$-component, we recall from (1.1) that the map

$$\mathscr{S}_0^{2\times 2} \ni \mathbf{Q} \mapsto \mathbf{q} := \sqrt{2}\,(Q_{11}, Q_{12}) \in \mathbb{R}^2$$

provides an isometric isomorphism between $\mathscr{S}_0^{2\times 2}$ (endowed with the Frobenius scalar product) and $\mathbb{R}^2$ (with the standard scalar product). Of course, we can also view $\mathbf{q}$ as a complex number, in the obvious way. This correspondence obviously extends to an isometric correspondence between $\mathbf{Q}$-tensor fields and vector fields. With the help of Lemma 1.1, we deduce that

$$\int_\Omega \left(\frac{1}{2}\left|\nabla\mathbf{q}\right|^2 + \frac{1}{8\varepsilon^2}\left(\left|\mathbf{q}\right|^2 - 1\right)^2\right) \mathrm{d}x \leq \mathcal{F}_\varepsilon(\mathbf{Q},\, \mathbf{M}) + \beta^2 \int_\Omega \left|\mathbf{M}\right|^2 \mathrm{d}x. \tag{3.1}$$

Thus, in view of Theorem 2.11 and of the uniform bound (1.35), the Ginzburg-Landau energy of a sequence $\{\mathbf{q}_\varepsilon\}$ associated with a sequence of critical points $\{(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)\}$ is logarithmically bounded. This allows us to apply to $\mathbf{Q}$-tensor fields several results typical of the Ginzburg-Landau theory with essentially no changes. Usually, we will understood the usage of the above correspondence, whenever convenient, for the rest of this paper.

Let $\mathbf{Q} \in \mathscr{S}_0^{2\times 2} \setminus \{0\}$. Then, by the spectral theorem, we may write

$$\mathbf{Q} = \frac{|\mathbf{Q}|}{\sqrt{2}}(\mathbf{n}\otimes\mathbf{n} - \mathbf{m}\otimes\mathbf{m})$$

for orthogonal vectors $\mathbf{n}$, $\mathbf{m} \in \mathbb{S}^1$, where $\mathbf{n}$ denotes the eigenvector of $\mathbf{Q}$ relative to the positive eigenvalue while $\mathbf{m}$, because of the zero-trace constraint, is relative to the negative eigenvalue. In turn, we may decompose any $\mathbf{M} \in \mathbb{R}^2$ as

$$\mathbf{M} = (\mathbf{M}\cdot\mathbf{n})\,\mathbf{n} + (\mathbf{M}\cdot\mathbf{m})\,\mathbf{m}.$$

As in [20, Equation (3.6)], we define

$$u_1 := (\mathbf{M}\cdot\mathbf{n}), \qquad u_2 := (\mathbf{M}\cdot\mathbf{m}).$$

Upon setting $\mathbf{u} := (u_1,\, u_2)$, we clearly have

$$|\mathbf{M}| = |\mathbf{u}|\,.$$

The above discussion generalises to $\mathbf{Q}$-tensor field and vector fields as follows. Let $G \subseteq \Omega$ be a smooth, simply connected domain. Let

$$\{(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)\} \subset W^{1,2}\left(G,\, \mathscr{S}_0^{2\times 2}\right) \times W^{1,2}(G,\, \mathbb{R}^2)$$

be a sequence satisfying, for any $\varepsilon > 0$,

$$\int_G \left(\frac{1}{2}\left|\nabla\mathbf{Q}_\varepsilon\right|^2 + \frac{1}{4\varepsilon^2}\left(\left|\mathbf{Q}_\varepsilon\right|^2 - 1\right)^2\right) \mathrm{d}x \lesssim \left|\log\varepsilon\right|, \tag{3.2}$$

$$|\mathbf{Q}_\varepsilon(x)| \geq \frac{1}{2}, \quad |\mathbf{M}_\varepsilon(x)| \leq A \qquad \text{for any } x \in G, \tag{3.3}$$

where the implicit constant on the right-hand side of (3.2) does not depend on $\varepsilon$ and $A$ is some positive constant that does not depend on $\varepsilon$ as well. By (1.35) and Proposition 2.1, these assumptions are verified for critical points under either (6)-(7) or (8)-(9), provided that assumption (10) is in force. Then, since $G$ is simply connected and smooth, we may coherently decompose

$$\mathbf{Q}_\varepsilon = \frac{|\mathbf{Q}_\varepsilon|}{\sqrt{2}} \left(\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon - \mathbf{m}_\varepsilon \otimes \mathbf{m}_\varepsilon\right) \qquad \text{in } G, \tag{3.4}$$

where $(\mathbf{n}_\varepsilon,\, \mathbf{m}_\varepsilon)$ is an orthonormal frame of eigenvectors of $\mathbf{Q}_\varepsilon$ in $G$, with $\mathbf{n}_\varepsilon \in W^{1,2}(G,\, \mathbb{S}^1)$ and $\mathbf{m}_\varepsilon \in W^{1,2}(G,\, \mathbb{S}^1)$ — in fact, since $\mathbf{Q}_\varepsilon$ is smooth, $\mathbf{n}_\varepsilon$, $\mathbf{m}_\varepsilon$ are also smooth. We will always denote with $\mathbf{n}_\varepsilon(x)$ the eigenvector of $\mathbf{Q}_\varepsilon(x)$ relative to its (strictly) positive eigenvalue. As in the above, we then let

$$(u_\varepsilon)_1 := \mathbf{M}_\varepsilon \cdot \mathbf{n}_\varepsilon, \qquad (u_\varepsilon)_2 := \mathbf{M}_\varepsilon \cdot \mathbf{m}_\varepsilon, \tag{3.5}$$

so that

$$\mathbf{M}_\varepsilon = (u_\varepsilon)_1 \mathbf{n}_\varepsilon + (u_\varepsilon)_2 \mathbf{m}_\varepsilon.$$

Then, we define

$$\mathbf{u}_\varepsilon \colon G \to \mathbb{R}^2, \qquad \mathbf{u}_\varepsilon := ((u_\varepsilon)_1,\, (u_\varepsilon)_2).$$

Again, there holds

$$|\mathbf{u}_\varepsilon(x)| = |\mathbf{M}_\varepsilon(x)|\,, \qquad \text{for any } x \in G, \tag{3.6}$$

so that any sequence $\{\mathbf{u}_\varepsilon\}$ constructed in $G$ out of maps $\mathbf{M}_\varepsilon$ belonging to critical pairs is bounded in $L^\infty(G)$ by a constant that does not depend on $G$ (but only on the coupling parameter $\beta$). For later purposes, we also notice that

$$\frac{\mathbf{Q}}{|\mathbf{Q}|}\mathbf{M} \cdot \mathbf{M} = \frac{1}{\sqrt{2}} \left(u_1^2 - u_2^2\right) \qquad \text{in } G. \tag{3.7}$$

Next, we introduce the functions (see [20, (3.7), (3.8)])

$$g_\varepsilon(\mathbf{Q}) := \frac{1}{4\varepsilon^2}\left(|\mathbf{Q}|^2 - 1\right)^2 - \frac{2\kappa_\star}{\varepsilon}\left(|\mathbf{Q}| - 1\right) + \kappa_\star^2, \tag{3.8}$$

$$h(\mathbf{u}) := \frac{1}{4}\left(|\mathbf{u}|^2 - 1\right)^2 - \frac{\beta}{\sqrt{2}}\left(u_1^2 - u_2^2\right) + \frac{\beta^2 + \sqrt{2}\beta}{2}. \tag{3.9}$$

By [20, Lemma 3.3 and Lemma 3.4], $g_\varepsilon$ and $h$ are non-negative functions. They are related to $f_\varepsilon$ as follows (just recall (1.17) or cf. [20, Equation (3.14)]):

$$\begin{aligned} \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q},\, \mathbf{M}) = g_\varepsilon(\mathbf{Q}) + \frac{1}{\varepsilon}h(\mathbf{u}) + \frac{|\mathbf{Q}| - 1}{\varepsilon}\left(2\kappa_\star - \frac{\beta}{\sqrt{2}}\left(u_1^2 - u_2^2\right)\right) \\ + \frac{\kappa_\varepsilon}{\varepsilon^2} - \frac{1}{2\varepsilon}\left(\beta^2 + \sqrt{2}\beta\right) - \kappa_\star^2. \end{aligned} \tag{3.10}$$

By (1.20), the last line in (3.10) vanishes as $\varepsilon \to 0$. Then, it follows from (10), (2.84), (1.35), the nonnegativity of $g_\varepsilon$, and an application of Hölder's inequality on the third term at the right-hand side (recalling also (1.15)) that, whenever $\{(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\}$ is a sequence of critical points satisfying (6)–(7) or (8)–(9) and (10), there holds

$$\frac{1}{\varepsilon}\int_G h(\mathbf{u}_\varepsilon)\,\mathrm{d}x \leq C, \tag{3.11}$$

where $C$ depends only on $\beta$ and $\mathrm{C_{pot}}$.

For the reader's convenience and for later reference, we recall the following result from [20].

**Lemma 3.1** ([20, Lemma 3.4])**.** *The function $h : \mathbb{R}^2 \to \mathbb{R}$ defined in* (3.9) *has the following properties:*

*(i) For any $\mathbf{u} \in \mathbb{R}^2$, we have $h(\mathbf{u}) \geq 0$, with $h(\mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{u}_\pm$, where*

$$\mathbf{u}_\pm := \left(\pm\left(\sqrt{2}\beta + 1\right)^{1/2}, 0\right). \tag{3.12}$$

*(ii) The Hessian matrix of $h$ at both $\mathbf{u}_+$ and $\mathbf{u}_-$ is strictly positive definite. (In particular, $h$ behaves quadratically near the wells $\mathbf{u}_\pm$.)*

With the functions $g_\varepsilon$ and $h$ at hand, we may rewrite the functional $\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; G)$ as

$$\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; G) = \int_G \left(\frac{1}{2}\left|\nabla\mathbf{Q}_\varepsilon\right|^2 + g_\varepsilon(\mathbf{Q}_\varepsilon)\right)\mathrm{d}x + \int_G \left(\frac{\varepsilon}{2}\left|\nabla\mathbf{u}_\varepsilon\right|^2 + \frac{1}{\varepsilon}h(\mathbf{u}_\varepsilon)\right)\mathrm{d}x + R_\varepsilon, \tag{3.13}$$

where $R_\varepsilon$ is a remainder term (defined in [20, Equation (3.15)]), small in the sense that $R_\varepsilon = \mathrm{o}_{\varepsilon\to 0}(1)$ (cf. [20, (3.9) and Lemma 4.10]). The definition of the remainder term $R_\varepsilon$ involves *both* $\mathbf{Q}_\varepsilon$ and $\mathbf{u}_\varepsilon$, and therefore the change of variable in (3.5) doest not yield a complete decoupling of the functional $\mathscr{F}_\varepsilon$ into a term depending only on $\mathbf{Q}_\varepsilon$ and a term depending only on $\mathbf{u}_\varepsilon$. However, it allows us to consider the functional

$$\mathcal{AC}_\varepsilon(\mathbf{u}_\varepsilon; G) := \int_G \left(\frac{1}{2}\left|\nabla\mathbf{u}_\varepsilon\right|^2 + \frac{1}{\varepsilon}h(\mathbf{u}_\varepsilon)\right)\mathrm{d}x. \tag{3.14}$$

Since the function $\mathbf{u} \mapsto h(\mathbf{u})$ has superlinear growth as $|\mathbf{u}| \to +\infty$, Lemma 3.1 implies that $h$ can be regarded as an *Allen-Cahn potential*, of the type considered in [22] (see also [3, 28]). Thus, $\mathbf{u} \mapsto \mathcal{AC}_\varepsilon(\mathbf{u}; G)$ can be seen as an *Allen-Cahn functional with multiple wells of equal depth.*

In Lemma 3.2 below, we show that $\{\mathbf{u}_\varepsilon\}$ is a sequence with bounded energy (i.e., $\mathcal{AC}_\varepsilon(\mathbf{u}_\varepsilon; G)$ is bounded uniformly with respect to $\varepsilon$ and $G$). Thanks to well-known compactness statements for energy-bounded sequences in the (vectorial) Allen-Cahn theory (again, see [22] and also [3, 28]), in the next subsection we will obtain rather easily compactness in $L^p(\Omega)$, for any $p$ with $1 \leq p < +\infty$, for the sequence $\{\mathbf{u}_\varepsilon\}$ and, joining this with the compactness results for the $\mathbf{Q}_\varepsilon$-component, we obtain compactness in $L^p(\Omega)$ for $\{\mathbf{M}_\varepsilon\}$ as well.

**Lemma 3.2.** *Let $\{(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\}$ be a sequence of critical points of $\mathscr{F}_\varepsilon$ subject to either* (6)-(7) *or* (8)-(9) *and suppose that assumption* (10) *holds. Suppose that there exist a finite set $\Sigma_\star \subset \overline{\Omega}$ such that for any $K \subset \Omega \setminus \Sigma_\star$ there exist $\varepsilon_0(K) > 0$ such that $|\mathbf{Q}_\varepsilon| \geq 1/2$ in $\Omega \setminus \Sigma_\star$ for any*

$0 < \varepsilon \leq \varepsilon_0(K)$. *Then, for any simply connected open set* $G \subset K$ *with smooth boundary and any* $0 < \varepsilon \leq \varepsilon_0(K)$, *there holds*

$$\int_G \left( \frac{1}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} h(\mathbf{u}_\varepsilon) \right) \mathrm{d}x \leq C, \tag{3.15}$$

*where* $C$ *is a positive constant that depends* ***only*** *on* $\Omega$, $\beta$, *the constant* $C_{\mathrm{pot}}$ *on the right-hand side of* (10), *and the* $L^1(\partial\Omega)$- *and the* $L^2(\partial\Omega)$-*norm of* $\mathbf{Q}_{\mathrm{bd}}$.

*Remark* 3.1. We stress that the constant $C$ in (3.15) does **not** depend on the subdomain $G$.

*Remark* 3.2. We shall prove later on that the assumptions of Lemma 3.2 are in fact satisfied whenever $\{\mathbf{u}_\varepsilon\}$ is obtained by a sequence of critical points $\{(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\}$ (see Lemma 3.3 below).

*Proof of Lemma 3.2.* By Proposition 2.9 and assumption (10), we have

$$\int_\Omega \left( \frac{\varepsilon}{2} |\nabla \mathbf{M}_\varepsilon|^2 + \frac{1}{\varepsilon^2} f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon) \right) \mathrm{d}x \leq C, \tag{3.16}$$

for some constant $C$ depending only on the bound $\mathrm{C}_{\mathrm{pot}}$ in (10), the parameter $\beta$, and on the $L^1(\partial\Omega)$- and the $L^2(\partial\Omega)$-norm of $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}} \mathbf{Q}_{\mathrm{bd}}$.

Fix any compact set $K \subset \Omega \setminus \Sigma_\star$ and any simply connected open set $G \subset K$ with smooth boundary. Then, by assumption, there exists a threshold value $\varepsilon_0(K) > 0$ so that $|\mathbf{Q}_\varepsilon| \geq 1/2$ for any $0 < \varepsilon \leq \varepsilon_0(K)$, so that we can switch from $\mathbf{M}_\varepsilon$ to $\mathbf{u}_\varepsilon$ in $G$.

By the definition (3.5) of $\mathbf{u}_\varepsilon$, there holds

$$|\nabla \mathbf{M}_\varepsilon|^2 = |\nabla \mathbf{u}_\varepsilon|^2 - 2 j(\mathbf{u}_\varepsilon) \cdot j(\mathbf{n}_\varepsilon) + |\mathbf{u}_\varepsilon|^2 |\nabla \mathbf{n}_\varepsilon|^2 \tag{3.17}$$

pointwise in $G$. Here, $j(\mathbf{u}_\varepsilon)$ and $j(\mathbf{n}_\varepsilon)$ are defined as in (1.14). Since

$$j(\mathbf{u}_\varepsilon) \cdot j(\mathbf{n}_\varepsilon) \leq |\mathbf{u}_\varepsilon| \, |\nabla \mathbf{u}_\varepsilon| \, |\nabla \mathbf{n}_\varepsilon| \,,$$

it follows from Young's inequality, (3.6), and (1.35) that

$$|\nabla \mathbf{u}_\varepsilon|^2 \lesssim |\nabla \mathbf{M}_\varepsilon|^2 + |\nabla \mathbf{Q}_\varepsilon|^2 \tag{3.18}$$

where the implicit constant on the right-hand side depends only on the parameter $\beta$ and $\Omega$. Combining (3.17) and (3.18), we have

$$\varepsilon \int_G |\nabla \mathbf{u}_\varepsilon|^2 \, \mathrm{d}x \lesssim \varepsilon \int_G \left( |\nabla \mathbf{M}_\varepsilon|^2 + |\nabla \mathbf{Q}_\varepsilon|^2 \right) \mathrm{d}x, \tag{3.19}$$

where the implicit constant on the right-hand side depends only on the bound $\mathrm{C}_{\mathrm{pot}}$ in (10), the parameter $\beta$, $\Omega$, and the $L^1(\partial\Omega)$- and the $L^2(\partial\Omega)$-norm of $\mathbf{Q}_{\mathrm{bd}} \times \partial_{\boldsymbol{\tau}} \mathbf{Q}_{\mathrm{bd}}$.

Putting (3.11) and (3.19) together and recalling once again the bounds in Theorem 2.11, we obtain (3.15). □

The strategy proceeds now as follows. By the estimates in Theorem 2.11, the leading part of the energy is carried by the $\mathbf{Q}_\varepsilon$-component. Once divided by $|\log \varepsilon|$, all the terms in the energy apart from the elastic energy of $\mathbf{Q}_\varepsilon$ are negligible in the limit as $\varepsilon \to 0$. So, we will start by proving compactness the for $\mathbf{Q}_\varepsilon$-component of a sequence $\{(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\}$ of critical points. In particular, we will show that the energy concentrates around at most finitely many points (possibly located on the boundary). After possible extraction of a subsequence, we have $|\mathbf{Q}_\varepsilon| \geq 1/2$ locally away from the concentration points for any $\varepsilon$ small enough, and this allows us to employ the change of variables described at the beginning of this section. The second point of the strategy relies on the bound (3.15) and it consists in proving compactness for the $\mathbf{u}_\varepsilon$-components through classical compactness results for sequences with equibounded vectorial Allen-Cahn energy. Finally, gathering the compactness results for $\{\mathbf{Q}_\varepsilon\}$ and $\{\mathbf{u}_\varepsilon\}$, we obtain a compactness theorem for the $\mathbf{M}_\varepsilon$-component.

### 3.2 Compactness for $\mathbf{Q}_\varepsilon$

In this section, we prove that the $\mathbf{Q}_\varepsilon$-component of a sequence $\{(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\}$ of critical points of $\mathscr{F}_\varepsilon$ satisfying the boundary conditions (6)–(7) or (8)–(9) and assumption (10) converges, in various norms, to a limiting map $\mathbf{Q}_\star : \Omega \to \mathscr{N}$ as $\varepsilon \to 0$. Moreover, we show that $\mathbf{Q}_\star$ is a harmonic map with values into $\mathscr{N}$ having only a finite number of point singularities. This extend to critical points some results that are proven in [20, Section 4] for minimisers. However, our arguments are substantially different from those in [20], since the latter often take advantage of minimality. Instead, we rely on PDE methods, adapting the existing theory about *solutions* of the Ginzburg-Landau system. In particular, we modify methods from [6, Chapter X] and from [7] to keep the perturbation into account. This task involves some extra challenges, the chief example being, perhaps, Lemma 3.7 below.

To keep the size of this section at the minimum, we address the reader to the literature on the Ginzburg-Landau theory whenever possible, exploiting the change of variables (1.1) and the related energy bound (3.1).

*Remark* 3.3. (Extension to a larger domain) For technical reasons, it will be convenient to extend any critical pair $(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)$ to an ($\varepsilon$-independent) open neighbourhood of $\Omega$, i.e., to some open, simply connected set $\Omega' \supset \overline{\Omega}$, while keeping (2.1), (2.62), and (10).

In the case of pure Dirichlet boundary conditions (6)–(7), this can be achieved by taking any $C^1$-smooth extension $\mathbf{M}$ of $\mathbf{M}_{\mathrm{bd}}$ to $\Omega' \setminus \overline{\Omega}$ satisfying the constraint $|\mathbf{M}| = \left(\sqrt{2}\beta + 1\right)^{1/2}$ and then letting $\mathbf{Q} = \sqrt{2}\left(\frac{\mathbf{M}\otimes\mathbf{M}}{\sqrt{2}\beta+1} - \frac{\mathbf{I}}{2}\right)$ in $\Omega' \setminus \overline{\Omega}$. Up to shrinking $\Omega'$ a little bit, we may assume that $\mathbf{M}$ and $\mathbf{Q}$ have finite energy. Moreover, by Lemma 1.1, we have $\frac{1}{\varepsilon^2} f(\mathbf{Q}, \mathbf{M}) = \kappa_\star^2 + \mathrm{o}_{\varepsilon\to 0}(1)$ as $\varepsilon \to 0$.

In the case of the 'mixed' boundary conditions (8)–(9), we can take $\delta > 0$ so small that in the $\delta$-neighbourhood $\Omega' := \Omega_\delta = \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \partial\Omega) < \delta\}$ of $\Omega$ we can extend $\mathbf{Q}_\varepsilon$ and $\mathbf{M}_\varepsilon$ by reflection across the boundary, keeping the energy estimates. For vector fields, this can be done, for instance, by exploiting [2, Proposition 8.1 and Remark 8.2] while for $\mathbf{Q}$-tensor fields it can be done, e.g., as described in [21, Section 2.2]. (However, differently from [21], here we do not need the extended maps to be solutions of an equation in the whole extended domain but only to control their energy. This goal can be achieved by requiring merely $C^2$-regularity of the boundary and $C^1$-regularity of the Dirichlet boundary condition for the $\mathbf{Q}_\varepsilon$-component.) Note that these extensions are $\varepsilon$-dependent, but preserve the boundary condition (8)–(9).

**Notation.** In all the statements below, it is understood that $\{(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\}$ is a given sequence of critical points of $\mathscr{F}_\varepsilon$ satisfying the boundary conditions (6)–(7) or (8)–(9) and assumption (10). Whenever necessary, we also assume (without changing notation and even without explicit mention) that each pair $(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)$ has been extended to a larger domain $\Omega'$ as explained in Remark 3.3. Finally, we set $\rho_\varepsilon = |\mathbf{Q}_\varepsilon|$ and we denote by

$$\mu_\varepsilon := \frac{1}{|\log\varepsilon|}\left(\frac{1}{2}\left|\nabla\mathbf{Q}_\varepsilon\right|^2 + \varepsilon\left|\nabla\mathbf{M}_\varepsilon\right|^2 + \frac{1}{\varepsilon^2} f(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)\right) \tag{3.20}$$

the energy density of the pair $(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)$, rescaled by $|\log\varepsilon|$ and dually seen as a measure in $\Omega$ (or in $\Omega'$).

The first result of this section characterises the possible limiting energy measures $\mu_\star$.

**Lemma 3.3.** *There exist a (not-relabelled) subsequence and a bounded, non-negative Radon measure $\mu_\star$ in $\overline{\Omega}$ such that, as $\varepsilon \to 0$, the energy measures $\{\mu_\varepsilon\}$ converge to $\mu_\star$ weakly* as measures in $\overline{\Omega}$. Moreover,*

$$\mu_\star = \sum_{j=1}^{N_\star} \theta_j \delta_{c_j}, \tag{3.21}$$

*where $c_1, \ldots, c_{N_\star}$ are distinct points in $\overline{\Omega}$, $\theta_1, \ldots, \theta_{N_\star}$ are real numbers so that $\theta_j \geq \eta_*$ for any $j \in \{1, \ldots, N_\star\}$, and $\eta_*$ is the constant provided by Proposition 2.5*

*Proof.* For convenience, we extend each $\mu_\varepsilon$ to a measure (still denoted $\mu_\varepsilon$) on the whole $\mathbb{R}^2$ by defining $\mu_\varepsilon$ as zero outside $\Omega'$. By the logarithmic energy bound in $\Omega'$, given by Theorem 2.11 and Remark 3.3, $\{\mu_\varepsilon\}$ is a sequence of bounded, non-negative Radon measures in $\mathbb{R}^2$, whence the existence of a (not relabelled) subsequence and of a bounded, non-negative Radon measure $\mu_\star$ on $\mathbb{R}^2$ such that $\mu_\varepsilon \rightharpoonup^* \mu_\star$ weakly* in the sense of measures in $\mathbb{R}^2$ and, in particular, on $\overline{\Omega}$, follow by standard arguments. Again by Theorem 2.11 and Remark 3.3, $\int_{\Omega'} \varepsilon\left|\nabla\mathbf{M}_\varepsilon\right|^2$ and $\varepsilon^{-2}\int_{\Omega'} f(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)$ are both negligible in front of $|\log\varepsilon|$ as $\varepsilon \to 0$. By Proposition 2.5, there exists positive constants $\eta_*$, $\varepsilon_*$ such that if $B(x_0,\, R)$ is a ball on which $\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon;\, B(x_0,\, R)) \leq \eta_* \log(R/\varepsilon)$ and $0 < \varepsilon \leq \varepsilon_* R$, then $\int_{B(x_0,\, R/2)} \left|\nabla\mathbf{Q}_\varepsilon\right|^2 \leq C(x_0,\, R)$ independently of $\varepsilon$, so that $B(x_0,\, R/2)$ is outside the support of $\mu_\star$. This implies that, for any ball $B \subset \mathbb{R}^2$,

$$\text{either} \qquad \mu_\star(B) = 0 \qquad \text{or} \qquad \mu_\star(B) \geq \eta_*.$$

Now, let

$$S := \{x \in \mathbb{R}^n :\ \text{for any } r > 0,\ \mu_\star(B(x,\, r)) > 0\}\,.$$

Then, $S$ is finite, with $|S| \leq \mu_\star(\mathbb{R}^n)/\eta_*$, and $\mu_\star$ is supported by $S$. □

The support of $\mu_\star$ represents exactly the energy-concentration set, on which therefore, in view of the logarithmic bound on the energy, weak convergence in $W^{1,2}$ and stronger types of convergence may fail. The next lemma, analogous to the last part of [20, Lemma 4.4] and reminiscent of the results in [6, Section X.5], shows global weak compactness for $\{\mathbf{Q}_\varepsilon\}$ in $W^{1,p}(\Omega)$ when $1 \leq p < 2$ and weak $W^{1,p}_{\mathrm{loc}}$-compactness outside $\operatorname{spt}\mu_\star$ for any finite $p \geq 1$.

**Notation.** All the arguments below involve extraction of subsequences; however, to keep the statement shorter, we do not emphasise this (very standard) fact in the forthcoming statements.

**Lemma 3.4.** *There exists a map* $\mathbf{Q}_\star \colon \Omega \to \mathscr{N}$ *such that*

$$\mathbf{Q}_\varepsilon \rightharpoonup \mathbf{Q}_\star \qquad \textit{weakly in } W^{1,p}(\Omega) \textit{ for any } p < 2 \tag{3.22}$$

$$\mathbf{Q}_\varepsilon \rightharpoonup \mathbf{Q}_\star \qquad \textit{weakly in } W^{1,p}_{\mathrm{loc}}(\Omega \setminus \operatorname{spt}\mu_\star) \ \textit{ for any } p \in [1, +\infty). \tag{3.23}$$

*Moreover, there exists* $\mathbf{n}_\star \in W^{1,p}(\Omega, \mathbb{S}^1)$ *for any* $p < 2$ *and belonging to* $W^{1,p}_{\mathrm{loc}}(\Omega \setminus \operatorname{spt}\mu_\star)$ *for any* $p \in [1, +\infty)$*, so that*

$$\mathbf{Q}_\star(x) = \sqrt{2}\left(\mathbf{n}_\star(x) \otimes \mathbf{n}_\star(x) - \frac{\mathbf{I}}{2}\right) \tag{3.24}$$

*for a.e.* $x \in \Omega$.

*Proof.* The existence of a map $\mathbf{Q}_\star \in W^{1,p}\left(\Omega, \mathscr{S}_0^{2\times 2}\right)$ for any $p \in [1, 2)$ such that $\mathbf{Q}_\varepsilon \rightharpoonup \mathbf{Q}_\star$ (on a subsequence) in $W^{1,p}(\Omega)$ for any $p \in [1, 2)$ follows from Proposition 2.1. Since $\partial\Omega$ is bounded and Lipschitz, by the Rellich-Kondrachov theorem it also follows that, up to extracting another subsequence, $\mathbf{Q}_\varepsilon(x) \to \mathbf{Q}_\star(x)$ for a.e. $x \in \Omega$, hence $|\mathbf{Q}_\star(x)| = 1$ for a.e. $x \in \Omega$, so that $\mathbf{Q}_\star \in W^{1,p}(\Omega, \mathscr{S}_0^{2\times 2})$ for any $p \in [1, 2)$, with $|\mathbf{Q}_\star(x)| = 1$ for a.e. $x \in \Omega$. Thus, (3.22) is proven.

Now, let $K \subset \Omega' \setminus \operatorname{spt}\mu_\star$ be any compact set. We can choose $R > 0$ and cover $K$ by finitely many balls $B(x_j, R/2) \subset\subset \Omega'$ which are out of the support of $\mu_\star$ and such that the balls $B(x_j, R)$ are still contained in $\Omega' \setminus \operatorname{spt}\mu_\star$. On each of the balls $B(x_j, R)$, by Proposition 2.5 we must have $\mathscr{F}_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon; B(x_j, R)) \leq \eta_* \log(R/\varepsilon)$ for any $0 < \varepsilon \leq \varepsilon_* R$, where $\varepsilon_*$ depends only on $K$, and hence $|\mathbf{Q}_\varepsilon| \geq 1/2$ on $B(x_j, 3R/4)$ for any such $\varepsilon$. Thus, by Proposition 2.6, up to extraction of a subsequence, interpolation, and a diagonal argument, we have $\mathbf{Q}_\varepsilon \rightharpoonup \mathbf{Q}_\star$ in $W^{1,p}(\cup_j(B(x_j, R/2) \cap \Omega))$ for any $p$ with $1 \leq p < +\infty$, whence (3.23) follows.

Finally, let $K \subset \Omega \setminus \operatorname{spt}\mu_\star$ be any compact set with non-empty interior and let $G \subset K$ be any simply connected open set with smooth boundary. As in the above, in $\overline{G}$ we have $|\mathbf{Q}_\varepsilon(x)| \geq 1/2$ for any $x \in \overline{G}$ and any $\varepsilon > 0$ small enough, depending only on $K$. Thus, in $G$ we can write $\mathbf{Q}_\varepsilon$ in the form (3.4), for vector fields $\mathbf{n}_\varepsilon$, $\mathbf{m}_\varepsilon \in W^{1,2}(G, \mathbb{S}^1)$ globally defined on $G$, where, as usual, $\mathbf{n}_\varepsilon$ is the eigenvector related to the positive eigenvalue of $\mathbf{Q}_\varepsilon$. In fact, we may write $\mathbf{I} = \mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon + \mathbf{m}_\varepsilon \otimes \mathbf{m}_\varepsilon$ in $G$ and therefore,

$$\mathbf{Q}_\varepsilon = |\mathbf{Q}_\varepsilon|\sqrt{2}\left(\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon - \frac{\mathbf{I}}{2}\right)$$

in $G$. Since $|\nabla\mathbf{n}| \lesssim |\nabla\mathbf{Q}|$ pointwise, the estimates on $\|\nabla\mathbf{Q}\|_{L^p(G)}$ obtained above for any $1 \leq p < +\infty$ yield estimates on $\|\nabla\mathbf{Q}\|_{L^p(G)}$, and thus the weak convergence $\mathbf{n}_\varepsilon \rightharpoonup \mathbf{n}_\star$ in $W^{1,p}(G)$ for any

$1 \leq p < +\infty$. In turn, by interpolation and possibly after extraction of subsequence, it follows that $\mathbf{n}_\varepsilon \to \mathbf{n}_\star$ in $L^p(G)$ for any $1 \leq p < +\infty$ as well as $\mathbf{n}_\varepsilon(x) \to \mathbf{n}_\star(x)$ for a.e. $x \in G$, whence $|\mathbf{n}_\star(x)| = 1$ for a.e. $x \in G$ and

$$\mathbf{Q}_\star(x) = \sqrt{2}\left(\mathbf{n}_\star(x) \otimes \mathbf{n}_\star(x) - \frac{\mathbf{I}}{2}\right)$$

for a.e. $x \in G$. By letting $K$ and $G$ vary in $\Omega \setminus \operatorname{spt}\mu_\star$, it follows that (3.24) holds at a.e. $x \in \Omega$. ☐

*Remark* 3.4. We shall see in Theorem 3.14 that, in fact, $\mathbf{Q}_\star$ is oriented in $\Omega \setminus \operatorname{spt}\mu_\star$ by (a multiple of) the vector field $\mathbf{M}_\star$, the $L^p(\Omega)$-limit of the maps $\mathbf{M}_\varepsilon$.

Next, we improve on the weak $W^{1,p}_{\mathrm{loc}}$-convergence to get strong $W^{1,p}_{\mathrm{loc}}$-convergence outside $\operatorname{spt}\mu_\star$, for any $p$ with $1 \leq p < +\infty$.

**Proposition 3.5.** *As $\varepsilon \to 0$, we have $\mathbf{Q}_\varepsilon \to \mathbf{Q}_\star$ strongly in $W^{1,p}(\Omega)$ for any $p$ with $1 \leq p < 2$ and in $W^{1,p}_{\mathrm{loc}}(\Omega \setminus \operatorname{spt}\mu_\star)$ for any $p$ with $1 \leq p < +\infty$.*

The proof of Proposition 3.5 rests on the pivotal lemma below.

**Lemma 3.6.** *As $\varepsilon \to 0$, we have $\mathbf{Q}_\varepsilon \to \mathbf{Q}_\star$ strongly in $W^{1,2}_{\mathrm{loc}}(\Omega \setminus \operatorname{spt}\mu_\star)$.*

The proof of Lemma 3.6 follows a similar path to that traced in [6, Step 1 and Step 2 of Theorem X.2]. However, we have to face the additional difficulties posed by the coupling term. The key step in the proof Lemma 3.6 is the following auxiliary result, which replaces [6, Equation (127) in Chapter X].

**Lemma 3.7.** *For any ball $B \subset\subset \Omega \setminus \operatorname{spt}\mu_\star$, we have*

$$\int_B \left\{ |\nabla\rho_\varepsilon|^2 + \left(\frac{\rho_\varepsilon - 1}{\varepsilon} - \kappa_\star\right)^2 \right\} \mathrm{d}x \to 0 \tag{3.25}$$

*as $\varepsilon \to 0$.*

*Proof.* For notational convenience, we systematically drop the subscript $\varepsilon$ within this proof. Also, we denote by $B'$ and $B''$ the balls concentric with $B$ whose radii are $3/4$ and $1/2$ of that of $B$, respectively. Since $B$ is out of $\operatorname{spt}\mu_\star$, we may assume that $\rho = |\mathbf{Q}| \geq 1/2$ in $B'$ for any $\varepsilon$ small enough. Since $B$ is arbitrary, it is enough to obtain (3.25) with $B''$ in place of $B$ (then, (3.25) will follow by a standard covering argument).

We consider (once again) the equation satisfied by $\rho$:

$$-\Delta\rho + 4\rho\,|\nabla\varphi|^2 + \frac{1}{\varepsilon^2}(\rho - 1)(\rho + 1)\rho - \frac{\sigma}{\varepsilon} = 0, \tag{3.26}$$

which holds pointwise in $B'$. We recall that, thanks to the change of variables $\mathbf{M} \mapsto \mathbf{u}$ given by (3.5) and to (3.7), the term $\sigma = \beta\frac{\mathbf{QM}\cdot\mathbf{M}}{\rho}$ can be written in $B'$ in terms of the components $u_1, u_2$ of $\mathbf{u}$ as follows:

$$-\sigma = \frac{\beta}{\sqrt{2}}\left(u_2^2 - u_1^2\right).$$

As a consequence of the definition (3.9) of the Allen-Cahn potential $h(\mathbf{u})$ and of the bound (3.11), by explicit computation we have that

$$\|2\kappa_\star - \sigma\|_{L^2(B)} \lesssim \sqrt{h(\mathbf{u})} = \mathrm{O}(\sqrt{\varepsilon}) \qquad \text{as } \varepsilon \to 0, \tag{3.27}$$

where the implicit constant depends only on $\beta$ (through the uniform bound on $\|\mathbf{M}\|_{L^\infty(\Omega)}$ given by (1.35)) and on the constant $\mathrm{C}_{\mathrm{pot}}$ on the right-hand side of (10). Let $\zeta \in C_c^\infty(B',\, [0,1])$ be any smooth cut-off function such that

$$\operatorname{spt}\zeta \subset B', \qquad \zeta \equiv 1 \quad \text{on } B'', \qquad |\nabla\zeta| \lesssim 1.$$

Multiplying (3.26) by $\zeta^2(\rho - 1 - \kappa_\star\varepsilon)$, where $\kappa_\star$ is the constant defined in (1.15), and then integrating over $B'$, we obtain

$$\begin{aligned}\int_{B'} \Big\{ \zeta^2 \left|\nabla\rho\right|^2 + \frac{1}{2}\zeta\nabla\zeta\cdot\nabla\left(\rho - 1 - \kappa_\star\varepsilon\right)^2 + 4\zeta^2\rho(\rho - 1 - \kappa_\star\varepsilon)\left|\nabla\varphi\right|^2 \\ + \zeta^2\left(\frac{\rho-1}{\varepsilon} - \kappa_\star\right)\left(\frac{\rho-1}{\varepsilon}(\rho+1)\rho - \sigma\right)\Big\}\,\mathrm{d}x = 0.\end{aligned}$$

We rearrange this equality in the form

$$\begin{aligned}&\int_{B'} \left\{ \zeta^2 \left|\nabla\rho\right|^2 + \zeta^2\left(\frac{\rho-1}{\varepsilon} - \kappa_\star\right)\left(\frac{\rho-1}{\varepsilon}(\rho+1)\rho - \sigma\right)\right\}\mathrm{d}x \\ &\qquad = -\int_{B'} \left\{\frac{1}{2}\zeta\nabla\zeta\cdot\nabla\left(\rho - 1 - \kappa_\star\varepsilon\right)^2 + 4\zeta^2\rho(\rho - 1 - \kappa_\star\varepsilon)\left|\nabla\varphi\right|^2\right\}\mathrm{d}x.\end{aligned} \tag{3.28}$$

By the $W^{1,p}$-estimates (2.42) for $\mathbf{Q}$ and the uniform convergence $\rho \to 1$ on $B'$ as $\varepsilon \to 0$ provided by Corollary 2.8, it follows that

$$\int_{B'} \zeta^2\rho(\rho - 1 - \kappa_\star\varepsilon)\left|\nabla\varphi\right|^2\,\mathrm{d}x = \mathrm{O}(\varepsilon) \qquad \text{as } \varepsilon \to 0. \tag{3.29}$$

Moreover, by using Young's inequality, we have

$$-\int_{B'} \frac{1}{2}\zeta\nabla\zeta\cdot\nabla\left(\rho - 1 - \kappa_\star\varepsilon\right)^2\,\mathrm{d}x \le \int_{B'} \frac{1}{2}\left(\rho - 1 - \kappa_\star\varepsilon\right)^2\left|\nabla\zeta\right|^2\,\mathrm{d}x + \int_{B'}\frac{1}{2}\zeta^2\left|\nabla\rho\right|^2\,\mathrm{d}x. \tag{3.30}$$

We further observe that the first integral on the right-hand side above is of order $\varepsilon$ as $\varepsilon \to 0$ so that plugging (3.29), (3.30) into (3.28) we obtain

$$\int_{B'} \left\{\frac{1}{2}\zeta^2\left|\nabla\rho\right|^2 + \zeta^2\left(\frac{\rho-1}{\varepsilon} - \kappa_\star\right)\left(\frac{\rho-1}{\varepsilon}(\rho+1)\rho - \sigma\right)\right\}\mathrm{d}x = \mathrm{O}(\varepsilon) \qquad \text{as } \varepsilon \to 0. \tag{3.31}$$

Recalling (3.27) and (2.42), we notice that

$$\int_{B'}\left(\frac{\rho-1}{\varepsilon} - \kappa_\star\right)(2\kappa_\star - \sigma)\,\mathrm{d}x = \mathrm{O}\left(\sqrt{\varepsilon}\right) \qquad \text{as } \varepsilon \to 0,$$

so that we can rewrite (3.31) as

$$\int_{B'} \left\{ \frac{1}{2}\zeta^2 \left|\nabla\rho\right|^2 + \zeta^2 \left(\frac{\rho-1}{\varepsilon} - \kappa_\star\right) \left(\frac{\rho-1}{\varepsilon}(\rho+1)\rho - 2\kappa_\star\right) \right\} \mathrm{d}x = \mathrm{O}\left(\sqrt{\varepsilon}\right) \qquad \text{as } \varepsilon \to 0.$$

Once again by the uniform convergence $\rho \to 1$ on $B'$, we get

$$\begin{aligned} &\int_{B'} \zeta^2 \left(\frac{\rho-1}{\varepsilon} - \kappa_\star\right) \left(\frac{\rho-1}{\varepsilon}(\rho+1)\rho - 2\kappa_\star\right) \mathrm{d}x \\ &\qquad = \int_{B'} \zeta^2 \left(\frac{\rho-1}{\varepsilon} - \kappa_\star\right) \left(\frac{\rho-1}{\varepsilon}(2+\mathrm{o}(1))(1+\mathrm{o}(1)) - 2\kappa_\star\right) \mathrm{d}x \qquad \text{as } \varepsilon \to 0 \end{aligned}$$

and therefore, using again (2.42) and recalling that $\zeta \equiv 1$ on $B''$, we eventually obtain

$$\int_{B''} \left\{ \frac{1}{2} \left|\nabla\rho\right|^2 + 2\left(\frac{\rho-1}{\varepsilon} - \kappa_\star\right)^2 \right\} \mathrm{d}x = \mathrm{o}(1) \qquad \text{as } \varepsilon \to 0,$$

which, thanks to a standard covering argument, implies (3.25). □

*Remark* 3.5. Let $B \subset\subset \Omega \setminus \operatorname{spt}\mu_\star$ be any ball. Then, by the uniform convergence $\rho_\varepsilon \to 1$ in $B$, the bound (2.84), (3.25), and the definition of $g_\varepsilon$, it also follows that

$$\int_B g_\varepsilon(\mathbf{Q}_\varepsilon)\,\mathrm{d}x \to 0 \tag{3.32}$$

as $\varepsilon \to 0$.

*Remark* 3.6. It follows from (3.25) and Proposition 2.6 that

$$\|\nabla\rho_\varepsilon\|_{L^p(B)} + \left\|\frac{\rho_\varepsilon - 1}{\varepsilon} - \kappa_\star\right\|_{L^p(B)} \to 0 \qquad \text{as } \varepsilon \to 0 \tag{3.33}$$

in any ball $B \subset\subset \Omega \setminus \operatorname{spt}\mu_\star$.

We are now ready for the proof of Lemma 3.6.

*Proof of Lemma 3.6.* We argue similarly to as in Step 1 and Step 2 in the proof of [6, Theorem X.2], with the technical modifications needed to handle the coupling term in the Euler-Lagrange equations.

Let $K$ be any compact subset of $\Omega \setminus \operatorname{spt}\mu_\star$. For our purposes, it is enough to consider the case in which $K$ has non-empty interior, therefore we may assume that there exist open sets $A$, $U$ such that

$$A \subset\subset K \subset U \subset\subset \Omega \setminus \operatorname{spt}\mu_\star.$$

Moreover, we may assume that $K$ and $U$ have smooth boundary.

*Step* 1 (Convergence of $j(\mathbf{Q}_\varepsilon)$). We consider again the Hodge-type decomposition of $\mathbf{j}_\varepsilon := j(\mathbf{Q}_\varepsilon)$ in Step 2 of Proposition 2.1, i.e., we write

$$\mathbf{j}_\varepsilon = \nabla H_\varepsilon + \nabla^\perp \phi_\varepsilon + \nabla^\perp \xi_\varepsilon \qquad \text{in } \Omega,$$

where $H_\varepsilon$, $\phi_\varepsilon$, and $\xi_\varepsilon$ solve, respectively, (2.13)-(2.14), (2.9), and (2.10). We are going to show that

$$H_\varepsilon \to H_\star \quad \text{in } W^{1,2}(K), \tag{3.34}$$

$$\xi_\varepsilon \to \xi_\star \quad \text{in } W^{1,2}(K), \tag{3.35}$$

$$\phi_\varepsilon \to \phi_\star \quad \text{in } W^{1,2}(K) \tag{3.36}$$

as $\varepsilon \to 0$. Of course, this implies that $\mathbf{j}_\varepsilon \to \mathbf{j}_\star$ strongly in $L^2(K)$, where

$$\mathbf{j}_\star = \nabla H_\star + \nabla^\perp \phi_\star + \nabla^\perp \xi_\star,$$

and, on the other hand, the strong $W^{1,2}$-convergence and Remark 1.4 imply that $\mathbf{j}_\star = j(\mathbf{Q}_\star) = \frac{1}{2}\mathbf{Q}_\star \times \nabla \mathbf{Q}_\star$.

We start by noticing that the weak convergences $H_\varepsilon \rightharpoonup H_\star$ and $\xi_\varepsilon \rightharpoonup \xi_\star$ in $W^{1,2}(\Omega)$ are consequences of (2.17) and (2.16), respectively. Moreover, by (2.13), (2.14), Corollary 2.10, and standard elliptic estimates, it follows that

$$\|H_\varepsilon\|_{W^{1,2}(\Omega)} \to 0$$

as $\varepsilon \to 0$, whence (3.34) follows, with $H_\star = 0$. Concerning $\xi_\varepsilon$, we multiply Equation (2.10) by $\zeta(\xi_\varepsilon - \xi_\star)$, where $\zeta$ is an arbitrary smooth cut-off function satisfying

$$0 \le \zeta \le 1, \qquad \zeta \equiv 1 \text{ on } K, \qquad K \subset \operatorname{spt}\zeta \subset U,$$

and we integrate over $\Omega$ to get

$$\begin{aligned} &\int_\Omega \zeta\, |\nabla \xi_\varepsilon|^2 - \int_\Omega \zeta \nabla \xi_\varepsilon \cdot \nabla \xi_\star + \int_\Omega (\xi_\varepsilon - \xi_\star) \nabla\zeta \cdot \nabla \xi_\varepsilon \\ &= \int_\Omega -\zeta \left(1 - \alpha_\varepsilon^2\right) \mathbf{j}_\varepsilon \times \nabla \xi_\varepsilon + \int_\Omega \zeta \left(1 - \alpha_\varepsilon^2\right) \mathbf{j}_\varepsilon \times \nabla \xi_\star - \int_\Omega (\xi_\varepsilon - \xi_\star) \left(1 - \alpha_\varepsilon^2\right) \mathbf{j}_\varepsilon \times \nabla \zeta \end{aligned}$$

By weak convergence,

$$\int_\Omega \zeta \nabla \xi_\varepsilon \cdot \nabla \xi_\star \to \int_\Omega \zeta\, |\nabla \xi_\star|^2 \tag{3.37}$$

as $\varepsilon \to 0$. Moreover, since weak convergence in $W^{1,2}(\Omega)$ implies strong convergence in $L^2(\Omega)$, from (2.16) and (2.8) it follows that

$$\int_\Omega (\xi_\varepsilon - \xi_\star) \nabla \zeta \cdot \nabla \xi_\varepsilon \to 0, \qquad \int_\Omega (\xi_\varepsilon - \xi_\star) \left(1 - \alpha_\varepsilon^2\right) \mathbf{j}_\varepsilon \times \nabla \zeta \to 0,$$

as $\varepsilon \to 0$. By the uniform convergence $\rho_\varepsilon \to 1$ (and hence, $\alpha_\varepsilon \to 1$) in $\overline{U}$ (see Corollary 2.8) and the weak convergence $\mathbf{Q}_\varepsilon \rightharpoonup \mathbf{Q}_\star$ in $W^{1,2}(U)$ (implying $\mathbf{j}_\varepsilon \rightharpoonup^* \mathbf{j}_\star$ in the sense of distributions on $U$), we have

$$\int_\Omega -\zeta\left(1-\alpha_\varepsilon^2\right)\mathbf{j}_\varepsilon \times \nabla\xi_\varepsilon \to 0, \qquad \int_\Omega \zeta\left(1-\alpha_\varepsilon^2\right)\mathbf{j}_\varepsilon \times \nabla\xi_\star \to 0$$

as $\varepsilon \to 0$. Thus, weak convergence and (3.37) imply $\xi_\varepsilon \to \xi_\star$ strongly in $W^{1,2}(K)$.

Finally, we focus on $\phi_\varepsilon$. First of all, we notice that, by (2.15) and Sobolev embedding, we have

$$\|\phi_\varepsilon\|_{L^2(\Omega)} \leq C, \tag{3.38}$$

where the constant $C$ does not depend on $\varepsilon$. Moreover, there exists $\phi_\star \in W^{1,p}(\Omega)$ for any $p$ with $1 \leq p < 2$, so that $\phi_\varepsilon \rightharpoonup \phi_\star$ weakly in $W^{1,p}(\Omega)$ and, up to a subsequence, strongly in $L^p(\Omega)$ and pointwise a.e. in $\Omega$, for any $p$ with $1 \leq p < 2$.

Letting $\zeta$ be any smooth cut-off function as above and multiplying (2.9) by $\zeta^2\phi_\varepsilon$, we obtain

$$\int_\Omega \zeta^2 |\nabla\phi_\varepsilon|^2 = \int_\Omega \phi_\varepsilon\alpha_\varepsilon^2\mathbf{j}_\varepsilon \times \nabla\zeta^2 + \int_\Omega \zeta^2\alpha_\varepsilon^2\mathbf{j}_\varepsilon \times \nabla\phi_\varepsilon - \int_\Omega 2\phi_\varepsilon\zeta\nabla\zeta\cdot\nabla\phi_\varepsilon.$$

Using Young's inequality on the last two terms on the right-hand side, we obtain

$$\int_\Omega \zeta^2 |\nabla\phi_\varepsilon|^2 \lesssim \int_\Omega \phi_\varepsilon\alpha_\varepsilon^2\mathbf{j}_\varepsilon \times \nabla\zeta^2 + \int_\Omega \zeta^2\alpha_\varepsilon^4 |\mathbf{j}_\varepsilon|^2 + \int_\Omega |\phi_\varepsilon|^2 |\nabla\zeta|^2$$

In virtue of (3.38), of Proposition 2.6 and of the uniform convergence $\alpha_\varepsilon \to 1$ on $\overline{U}$, the right-hand side above is bounded independently of $\varepsilon$, whence

$$\|\phi_\varepsilon\|_{W^{1,2}(U)} \leq C,$$

where $C > 0$ is a constant that does not depend on $\varepsilon$. Thus, $\phi_\varepsilon \rightharpoonup \phi_\star$ weakly in $W^{1,2}(U)$. In order to obtain strong convergence in $W^{1,2}(K)$, it is now enough to show that

$$\int_K |\nabla\phi_\varepsilon|^2 \to \int_K |\nabla\phi_\star|^2 \tag{3.39}$$

as $\varepsilon \to 0$. To this purpose, we multiply (2.9) by $\zeta^2(\phi_\varepsilon - \phi_\star)$ and integrate over $\Omega$. This yields

$$\int_\Omega \zeta^2 |\nabla\phi_\varepsilon|^2 - \int_\Omega \zeta^2\nabla\phi_\varepsilon\cdot\nabla\phi_\star = \int_\Omega (\phi_\star - \phi_\varepsilon)\nabla\zeta^2\cdot\nabla\phi_\varepsilon + \int_\Omega \zeta^2(\phi_\varepsilon - \phi_\star)\,\mathrm{curl}\left(\alpha_\varepsilon^2\mathbf{j}_\varepsilon\right).$$

By the weak convergence $\phi_\varepsilon \rightharpoonup \phi_\star$ in $W^{1,2}(U)$, it follows that, as $\varepsilon \to 0$,

$$\int_\Omega \zeta^2\nabla\phi_\varepsilon\cdot\nabla\phi_\star \to \int_\Omega \zeta^2 |\nabla\phi_\star|^2$$

and, moreover, recalling also that weak convergence in $W^{1,2}(U)$ implies strong convergence in $L^2(U)$,

$$\int_\Omega (\phi_\star - \phi_\varepsilon)\nabla\zeta^2\cdot\nabla\phi_\varepsilon \to 0,$$

as $\varepsilon \to 0$. In addition, by the uniform convergence $\alpha_\varepsilon \to 1$ in $\overline{U}$, we have $\alpha_\varepsilon > 1/2$ eventually in $U$, therefore, by (1.12) and (2.5), the term $\operatorname{curl}(\alpha_\varepsilon^2 \mathbf{j}_\varepsilon)$ is identically zero in $K$ for any $\varepsilon > 0$ small enough. Thus,

$$\int_\Omega \zeta^2(\phi_\varepsilon - \phi_\star) \operatorname{curl}\left(\alpha_\varepsilon^2 \mathbf{j}_\varepsilon\right) = 0$$

for any $\varepsilon > 0$ small enough, and we obtain (3.39).

*Step* 2 (Conclusion). By (3.34), (3.35), (3.36), it follows that, as $\varepsilon \to 0$,

$$\mathbf{j}_\varepsilon \to \mathbf{j}_\star \qquad \text{strongly in } L^2(K).$$

On the other hand, we have $\rho_\varepsilon \to 1$ uniformly on $K$, and hence, by (1.11) and Lemma 3.7 (and an elementary covering argument), it follows that, as $\varepsilon \to 0$,

$$\mathbf{Q}_\varepsilon \to \mathbf{Q}_\star \qquad \text{strongly in } W^{1,2}(K).$$

The conclusion follows. □

The proof of Proposition 3.5 is now immediate.

*Proof of Proposition 3.5.* The claim follows immediately by Lemma 3.4, the $W^{1,p}$-estimates provided by Proposition 2.6, and Lemma 3.6. □

In view of Proposition 3.5, we also obtain the locally uniform convergence of $\mathbf{Q}_\varepsilon$ towards $\mathbf{Q}_\star$ away from $\operatorname{spt}\mu_\star$.

**Corollary 3.8.** *Let $K \subset \Omega \setminus \operatorname{spt}\mu_\star$ be any compact set. As $\varepsilon \to 0$, we have*

$$\mathbf{Q}_\varepsilon \to \mathbf{Q}_\star \qquad \textit{uniformly on } K. \tag{3.40}$$

*Proof.* By a standard covering argument, it suffices to prove the claim for a ball $B \subset\subset \Omega \setminus \operatorname{spt}\mu_\star$. By the Gagliardo-Nirenberg inequality, we have

$$\|\mathbf{Q}_\varepsilon - \mathbf{Q}_\star\|_{L^\infty(B)} \lesssim \|\mathbf{Q}_\varepsilon - \mathbf{Q}_\star\|_{W^{1,p}(B)}^{2/p} \|\mathbf{Q}_\varepsilon - \mathbf{Q}_\star\|_{L^p(B)}^{1-2/p}$$

for any $p > 2$ and, by Proposition 3.5 and the $L^\infty$-bound (1.35), the right-hand side above tends to zero as $\varepsilon \to 0$, so the conclusion follows. □

Now, we show that the pre-jacobian $\mathbf{j}_\star := j(\mathbf{Q}_\star) = \frac{1}{2}\mathbf{Q}_\star \times \nabla\mathbf{Q}_\star$ of $\mathbf{Q}_\star$ is divergence-free in $\Omega$ in the sense of distributions in $\Omega$, i.e., that there holds

$$\operatorname{div}(\mathbf{Q}_\star \times \nabla\mathbf{Q}_\star) = 0 \tag{3.41}$$

in the sense of distributions in $\Omega$, that is, in $\mathscr{D}'(\Omega)$. As in the usual Ginzburg-Landau theory, we shall draw several important consequences from this equation.

**Proposition 3.9.** *The map $\mathbf{Q}_\star\colon \Omega \to \mathscr{N}$ satisfies (3.41) in the sense of distributions in $\Omega$. Moreover, $\mathbf{Q}_\star$ is a smooth harmonic map in $\Omega \setminus \operatorname{spt}\mu_\star$ with values into $\mathscr{N}$, it is continuous in $\overline{\Omega} \setminus \operatorname{spt}\mu_\star$, and it satisfies the boundary condition $\mathbf{Q}_{\mathrm{bd}}$ in the sense of $W^{1-1/p,p}(\partial\Omega)$, for any $p \in (1,2)$.*

*Proof.* By taking the vector product of (4), (5) with $\mathbf{Q}_\varepsilon$ and $\mathbf{M}_\varepsilon$, respectively, we obtain that any critical pair $(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)$ satisfies

$$-\operatorname{div}(\mathbf{Q}_\varepsilon \times \nabla \mathbf{Q}_\varepsilon) = \frac{\varepsilon}{2} \operatorname{div}(\mathbf{M}_\varepsilon \times \nabla \mathbf{M}_\varepsilon) \qquad \text{in } \Omega.$$

Let $p \in (1,\, 2)$. By Lemma 3.4, we have $\mathbf{Q}_\varepsilon \rightharpoonup \mathbf{Q}_\star$ weakly in $W^{1,p}(\Omega)$ as $\varepsilon \to 0$. Thus, by the trace theorem, $\mathbf{Q}_\star$ has a well-defined trace in $W^{1-1/p,p}(\partial\Omega)$, and in fact, $\mathbf{Q}_\star = \mathbf{Q}_{\mathrm{bd}}$ on $\partial\Omega$ in the sense of traces, i.e., in $W^{1-1/p,p}(\partial\Omega)$. Moreover, by the uniform bound in $L^\infty(\Omega)$ for $\mathbf{Q}_\varepsilon$ and Lebesgue's dominated convergence theorem, we also have $\mathbf{Q}_\varepsilon \to \mathbf{Q}_\star$ strongly in $L^q(\Omega)$, for any $q < +\infty$. As a consequence,

$$\operatorname{div}(\mathbf{Q}_\varepsilon \times \nabla \mathbf{Q}_\varepsilon) \rightharpoonup^* \operatorname{div}(\mathbf{Q}_\star \times \nabla \mathbf{Q}_\star) \qquad \text{in } \mathscr{D}'(\Omega), \quad \text{as } \varepsilon \to 0. \tag{3.42}$$

On the other hand, by Proposition 2.9 and the uniform bound in $L^\infty(\Omega)$ for $\mathbf{M}_\varepsilon$ given by (1.35), it follows that

$$\varepsilon \left\| \mathbf{M}_\varepsilon \times \nabla \mathbf{M}_\varepsilon \right\|_{L^2(\Omega)} \lesssim \varepsilon \left\| \mathbf{M}_\varepsilon \right\|_{L^\infty(\Omega)} \left\| \nabla \mathbf{M}_\varepsilon \right\|_{L^2(\Omega)} \lesssim \varepsilon^{1/2} \to 0$$

as $\varepsilon \to 0$, whence

$$\varepsilon \operatorname{div}(\mathbf{M}_\varepsilon \times \nabla \mathbf{M}_\varepsilon) \to 0 \qquad \text{in } W^{-1,2}(\Omega) \quad \text{as } \varepsilon \to 0 \tag{3.43}$$

Combining (3.42) and (3.43), we deduce that $\mathbf{Q}_\star$ satisfies (3.41) in the sense of distributions in $\Omega$.

To check that $\mathbf{Q}_\star$ is smooth in $\Omega \setminus \operatorname{spt} \mu_\star$ and continuous in $\overline{\Omega} \setminus \operatorname{spt} \mu_\star$, take $G \subset\subset \overline{\Omega} \setminus \operatorname{spt} \mu_\star$, an arbitrary simply connected domain with smooth boundary. Then, $\mathbf{Q}_\star \in W^{1,2}(G,\, \mathscr{N})$, so that we can apply lifting results (see e.g. [8, Theorem 1]) and write

$$\mathbf{Q}_\star = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta_\star & \sin\theta_\star \\ \sin\theta_\star & -\cos\theta_\star \end{pmatrix},$$

where $\theta_\star \in W^{1,2}(G)$ is a scalar function satisfying

$$\Delta\theta_\star = 0 \qquad \text{as distributions in } G.$$

Therefore, $\theta_\star$ is smooth in $G$, and so is $\mathbf{Q}_\star$. Moreover, if $G$ touches $\overline{\Omega}$, $\theta_\star$ is continuous up to $\partial\Omega$ and hence $\mathbf{Q}_\star$ is. By letting $G$ vary among all simply connected domain with smooth boundary compactly contained in $\overline{\Omega} \setminus \operatorname{spt} \mu_\star$, the conclusion follows. □

Next, we state an additional convergence property for $\mathbf{Q}_\varepsilon$. To this purpose, we first recall that for any map $\mathbf{Q} \in W^{1,1}(\Omega,\, \mathscr{N})$ having only a finite number of singular points, the distributional Jacobian $J(\mathbf{Q})$ writes as a sum of Dirac deltas, whose multiplicities are the degrees of $\mathbf{Q}$ computed on small circles around the singular points. The singularities with degree different from zero are called the *topological singularities* of $\mathbf{Q}$; these are captured, in location and charge, by $J(\mathbf{Q})$, cf., e.g., [13, Section 2.0].

Also, we recall that, given a closed ball $\overline{B}_\rho(z) \subset \Omega'$ and a **Q**-tensor field $\mathbf{Q}$ on $\overline{B}_\rho(z)$ such that $|\mathbf{Q}| \geq 1/2$ on $\partial B_\rho(z)$, the map

$$\frac{\mathbf{Q}}{|\mathbf{Q}|} \colon \partial B_\rho(z) \simeq \mathbb{S}^1 \to \mathscr{N} \simeq \mathbb{RP}^1$$

is well-defined and continuous and hence its topological degree is well-defined as an element of $\frac{1}{2}\mathbb{Z}$ (i.e., it is either an integer or a half-integer). In particular, being $\mathbf{Q}_\star \in W^{1,1}(\Omega, \mathscr{N})$, it has a well-defined topological degree around each point in $\operatorname{spt}\mu_\star$, its Jacobian has the structure described above, and since its topological singular set is the support of $J(\mathbf{Q}_\star)$, it also follows that the sum of the degrees of its topological singularities equals the degree of $\mathbf{Q}_{\mathrm{bd}}$.

Without loss of generality, we may suppose that first $N_{\mathrm{top}} \in \mathbb{N} \cup \{0\}$ points in $\operatorname{spt}\mu_\star$ are the topological singularities of $\mathbf{Q}_\star$.

**Proposition 3.10.** *We have*

$$J(\mathbf{Q}_\varepsilon) \to J(\mathbf{Q}_\star) \qquad \text{in } W^{-1,1}(\Omega) \tag{3.44}$$

*as* $\varepsilon \to 0$. *Moreover,*

$$|J(\mathbf{Q}_\star)| \leq \pi\mu_\star \tag{3.45}$$

*and, in fact,*

$$J(\mathbf{Q}_\star) = \pi \sum_{j=1}^{N_{\mathrm{top}}} d_j \delta_{c_j} \qquad \text{in } \mathscr{D}'(\Omega), \tag{3.46}$$

*where* $N_{\mathrm{top}} \leq N$ *denotes the number of topological singularities of* $\mathbf{Q}_\star$ *and the numbers* $d_1, \ldots, d_{N_{\mathrm{top}}} \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$ *are the degrees of* $\mathbf{Q}_\star$ *around the points in* $\operatorname{spt} J(\mathbf{Q}_\star)$.

*Proof.* The convergence in $W^{-1,1}(\Omega)$ claimed by (3.44) follows exactly as in the proof of [20, Lemma 4.7], using Lemma 3.4 above and the uniform bound (1.35) for $\|\mathbf{Q}_\varepsilon\|_{L^\infty(\Omega)}$.

Clearly, (3.45) holds, because $\mathbf{Q}_\star$ is a smooth $\mathscr{N}$-valued map outside $\operatorname{spt}\mu_\star$, by Proposition 3.9. (Hence, $\mathbf{q}_\star$ is a smooth $\mathbb{S}^1$-valued map outside $\operatorname{spt}\mu_\star$.)

Finally, (3.46) follows since $\mathbf{Q}_\star \in W^{1,1}(\Omega, \mathscr{N})$ and it is smooth in $\Omega$ outside the finite set $\operatorname{spt}\mu_\star$, hence $\mathbf{q}_\star \in W^{1,1}(\Omega, \mathbb{S}^1)$ and it is smooth in $\Omega$ outside $\operatorname{spt}\mu_\star$, so that we can apply classical structure theorems for Jacobians in two-dimensions (see, e.g., [13, Section 2.1]), jointly with the further observation that $\mathbf{Q}_\star$ has degree zero around points in $(\operatorname{spt}\mu_\star \setminus J(\mathbf{Q}_\star)) \cap \Omega$ (because these are either continuity points for $\mathbf{Q}_\star$ or singularities of degree zero). ☐

*Remark* 3.7. In the case of minimisers, considered in [20], equality occurs in (3.45), because of matching lower and upper energy bounds, which prevent energy concentration on boundary points and on points which are not topological singularities of $\mathbf{Q}_\star$. However, this is unclear, for general critical points. In general, $\operatorname{spt}\mu_\star$ contains the topological singularities of $\mathbf{Q}_\star$ and may contain, in addition, boundary singular points of $\mathbf{Q}_\star$, singularities of degree zero of $\mathbf{Q}_\star$, and even points (possibly located on the boundary) which are *not* singularities of $\mathbf{Q}_\star$.

In the case of pure Dirichlet boundary conditions and assuming in addition (as in [6]) that $\Omega$ is star-shaped, we can add information to the conclusion of Proposition 3.9 and Proposition 3.10, obtaining a result much closer to [6, Theorem X.4] as well as to [20, Proposition 4.9].

**Corollary 3.11.** *Assume that $\Omega$ is star-shaped and that $\{(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\}$ is a sequence of critical points of $\mathscr{F}_\varepsilon$, subject to the Dirichlet boundary conditions* (6)*–*(7)*, and that the assumption* (10) *holds. Then, $\mathbf{Q}_\star$ is the canonical harmonic map with singularities $(c_1, \ldots, c_{N_{\mathrm{top}}})$, degrees $(d_1, \ldots, d_{N_{\mathrm{top}}})$, and boundary datum $\mathbf{Q}_{\mathrm{bd}}$.*

*Proof.* We already know from Proposition 3.9 that $\mathbf{Q}_\star \colon \Omega \to \mathscr{N}$ satisfies (3.41) in the sense of distributions in $\Omega$, it is smooth harmonic in $\Omega \setminus \operatorname{spt} \mu_\star$, and continuous up to $\overline{\Omega} \setminus \operatorname{spt} \mu_\star$. The set of the topological singularities of $\mathbf{Q}_\star$ coincides with the support of the limiting Jacobian $J(\mathbf{Q}_\star)$ and therefore such singularities carry degrees $d_1, \ldots, d_{N_{\mathrm{top}}} \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$. The points in $\operatorname{spt} \mu_\star \setminus \operatorname{spt} J(\mathbf{Q}_\star)$, if any, are continuity points for $\mathbf{Q}_\star$ or singular points located on the boundary or interior singularities of degree zero. We let $N' \in \mathbb{N} \cup \{0\}$ be the number of points in $\operatorname{spt} \mu_\star$ which are either topological singularities of $\mathbf{Q}_\star$ or interior singularities of degree zero.

We now show that, in fact, all the singularities of $\mathbf{Q}_\star$ are contained in the interior of $\Omega$. Once this is done, since $\mathbf{Q}_\star$ solves (3.41) in $\mathscr{D}'(\Omega)$, it follows that $\mathbf{Q}_\star$ is the canonical harmonic map associated with the boundary condition $\mathbf{Q}_{\mathrm{bd}}$, the singular points $c_1, \ldots, c_{N'}$, and the degrees $d_1, \ldots, d_{N'}$. However, since canonical harmonic maps are proper harmonic maps (see [6, Definition on p. 14]), it also follows the singularities of degree zero are removable, whence we may always assume that only the topological singularities are left.

As in [6, Theorem X.4], full boundary regularity follows immediately once the hypotheses of [6, Lemma X.14] are verified. (In the following, for brevity's sake, we understood the switching from $\mathbf{Q}_\star$ to $\mathbf{q}_\star$ (and back) where necessary to apply the results in [6] in the form they are written there.) To this purpose, in view of Proposition 3.9, it only remains to prove that

$$\partial_\nu \mathbf{Q}_\star \in L^2(\partial\Omega). \tag{3.47}$$

To obtain (3.47), we argue as in [6]. Combining the Pohozaev identity with the ($\varepsilon$-independent) boundary conditions (6)–(7), we deduce that

$$\|\partial_\nu \mathbf{Q}_\varepsilon\|_{L^2(\partial\Omega)} \lesssim 1 \tag{3.48}$$

independently of $\varepsilon$. On the other hand, we also infer in Proposition A.1 in Appendix A that (similarly to as in [6, Theorem X.3])

$$\mathbf{Q}_\varepsilon \to \mathbf{Q}_\star \qquad \text{in } W^{1,2}_{\mathrm{loc}}\left(\overline{\Omega} \setminus \operatorname{spt} \mu_\star\right) \text{ as } \varepsilon \to 0 \tag{3.49}$$

Combining (3.48) and (3.49), we obtain (3.47). Thus, by [6, Lemma X.4], $\mathbf{Q}_\star$ is smooth in a neighbourhood of $\partial\Omega$. But then $\mathbf{Q}_\star$ is the canonical harmonic map with singularities at $(c_1, \ldots, c_{N_{\mathrm{top}}})$, degrees $(d_1, \ldots, d_{N_{\mathrm{top}}})$ around them, and boundary datum $\mathbf{Q}_{\mathrm{bd}}$. □

*Remark* 3.8. Even under the hypotheses of Corollary 3.11, it is generally unclear whether equality holds in (3.45) or whether $\operatorname{spt} \mu_\star \subset \Omega$.

### 3.3 Compactness for $\mathbf{u}_\varepsilon$

Here, we prove that any sequence $\{\mathbf{u}_\varepsilon\}$ of functions $\mathbf{u}_\varepsilon : G \to \mathbb{R}^2$ satisfying

$$\sup_{\varepsilon>0} \mathcal{AC}_\varepsilon(\mathbf{u}_\varepsilon;\, G) \leq C < +\infty$$

is relatively compact in $L^p(G,\, \mathbb{R}^2)$, for any $p$ with $1 \leq p < +\infty$. The argument relies on the following specialisation to our case of a classical compactness and lower-semicontinuity theorem for sequences with bounded Allen-Cahn energy.

**Theorem 3.12** ([22, Theorem 4.1 and Theorem 3.4])**.** *Any family $\{\mathbf{u}_\varepsilon\}$ with $\mathcal{AC}_\varepsilon(\mathbf{u}_\varepsilon;\, G) \leq C < +\infty$ for all $\varepsilon > 0$ is relatively compact in $L^1(G,\, \mathbb{R}^2)$, i.e., there exists $\mathbf{u}_\star \in L^1(G,\, \mathbb{R}^2)$ and a (non-relabelled subsequence) such that $\mathbf{u}_\varepsilon \to \mathbf{u}_\star$ a.e. in $G$ and strongly in $L^1(G,\, \mathbb{R}^2)$. Moreover,*

$$c_\beta \mathscr{H}^1\left(\mathrm{S}_{\mathbf{u}_\star} \cap G\right) \leq \liminf_{\varepsilon\to 0} \mathcal{AC}_\varepsilon(\mathbf{u}_\varepsilon;\, G),$$

*where*

$$c_\beta := \frac{2\sqrt{2}}{3}\left(\sqrt{2}\beta + 1\right)^{3/2}. \tag{3.50}$$

In fact, [22, Theorem 3.4] is a full $\Gamma$-convergence statement in the $L^1$-topology. Similar results, allowing for more general energies of Allen-Cahn type having an arbitrary (finite) number of wells of equal depth, can be found in [3].

*Remark* 3.9. In particular, $h(\mathbf{u}_\star) = 0$. Therefore, the limiting map $\mathbf{u}_\star$ necessarily takes the form

$$\mathbf{u}_\star(x) = \left(\tau(x)\left(\sqrt{2}\beta + 1\right)^{1/2},\, 0\right) \qquad \text{for a.e. } x \in G \tag{3.51}$$

where $\tau(x) \in \{1, -1\}$ is a sign. Since $\mathbf{u}_\star$ takes values into a finite set, its distributional derivative $\mathrm{D}\mathbf{u}_\star$ must be concentrated on $\mathrm{S}_{\mathbf{u}_\star}$, implying that $\mathbf{u}_\star \in \mathrm{SBV}(G,\, \mathbb{R}^2)$.

Using Lemma 3.3 together with the bounds (3.15), (1.35), and the $L^1$-compactness in Theorem 3.12, we immediately get the following compactness statement in all $L^p$-spaces with $1 \leq p < +\infty$. The measure $\mu_\star$ appearing in the statement below is exactly the limiting energy measure given by (3.21).

**Lemma 3.13.** *Let $\{(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)\}$ be a sequence of critical points of $\mathscr{F}_\varepsilon$ subject to either* (6)–(7) *or* (8)–(9) *and suppose that assumption* (10) *holds. Let $G \subset\subset \Omega \setminus \operatorname{spt}\mu_\star$ be any simply connected open set. Then, the sequence $\{\mathbf{u}_\varepsilon\}$ associated with $\{\mathbf{M}_\varepsilon\}$ according to* (3.5) *is relatively compact in $L^p(G,\, \mathbb{R}^2)$, for any $p$ with $1 \leq p < +\infty$.*

*Proof.* Since the sequence $\{(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)\}$ is made up of critical points of $\mathscr{F}_\varepsilon$, it follows that $\{(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)\}$ satisfies the energy estimates in Theorem 2.11. Moreover, since $\overline{G}$ stays at strictly positive distance from $\operatorname{spt}\mu_\star$, by Corollary 2.8 and a covering argument it also follows that $|\mathbf{Q}_\varepsilon| \to 1$ uniformly on $\overline{G}$, so that we have, eventually, $|\mathbf{Q}_\varepsilon| \geq 1/2$ on $\overline{G}$. Let $\varepsilon_0 > 0$ be the corresponding threshold value for $\varepsilon$. Since $G$ is simply connected, we can lift $\mathbf{Q}_\varepsilon$ in $G$ as in (3.4), at least for any $0 < \varepsilon \leq \varepsilon_0$. As a consequence, the sequence $\{\mathbf{u}_\varepsilon\}$ in the statement is well-defined, at least for any $0 < \varepsilon \leq \varepsilon_0$. By (3.6) and (1.35), it follows that $\{\mathbf{u}_\varepsilon\}$ is bounded in $L^\infty(G, \mathbb{R}^2)$ (by a constant independent of $G$ and, in fact, depending only the parameter $\beta$). Therefore, Lemma 3.2 holds and, by Theorem 3.12, $\{\mathbf{u}_\varepsilon\}$ is relatively compact in $L^1(G, \mathbb{R}^2)$. Then, the conclusion follows from (3.6) and (1.35), by interpolation. □

### 3.4 Compactness for $\mathbf{M}_\varepsilon$

In this section, putting together the results of Section 3.2 and of Section 3.1, we prove a compactness result for the $\mathbf{M}_\varepsilon$-component of a sequence of critical points $\{(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\}$ of $\mathscr{F}_\varepsilon$ satisfying the boundary conditions (6)-(7) or (8)-(9) and the assumption (10).

**Theorem 3.14.** *Let $\{(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\}$ be a sequence of critical points of $\mathscr{F}_\varepsilon$, subject to either* (6)-(7) *or* (8)-(9)*, and assume that* (10) *holds. Then, there exists a map $\mathbf{M}_\star \in \mathrm{SBV}(\Omega, \mathbb{R}^2)$ and a (non-relabelled) subsequence such that $\mathbf{M}_\varepsilon \to \mathbf{M}_\star$ a.e. and strongly in $L^p(\Omega, \mathbb{R}^2)$ for any $p$ with $1 \leq p < +\infty$ as $\varepsilon \to 0$. Moreover, $\mathscr{H}^1(\mathrm{S}_{\mathbf{M}_\star}) < +\infty$ and $\mathbf{M}_\star$ satisfies*

$$|\mathbf{M}_\star| = \left(\sqrt{2}\beta + 1\right)^{1/2}, \tag{3.52}$$

$$\mathbf{Q}_\star = \sqrt{2}\left(\frac{\mathbf{M}_\star \otimes \mathbf{M}_\star}{\sqrt{2}\beta + 1} - \frac{\mathbf{I}}{2}\right), \tag{3.53}$$

*a.e. in $\Omega$. In particular, $f(\mathbf{Q}_\star, \mathbf{M}_\star) = 0$.*

*Proof.* The proof goes exactly as in [20]. Therefore, we only sketch the main points, addressing the reader to [20, Proposition 4.11] for full details.

Up to extraction of a subsequence (and keeping Remark 3.3 into account), by Lemma 3.3 the energy densities $\mu_\varepsilon$ converge weakly* in the sense of measures in $\overline{\Omega}$ to a limiting energy density $\mu_\star$, whose support $\operatorname{spt}\mu_\star$ is a finite set contained in $\overline{\Omega}$.

By Corollary 2.8, we know that, up to extraction of a (further and non-relabelled) subsequence and a covering argument, $|\mathbf{Q}_\varepsilon| \to 1$ uniformly in compact sets of $\Omega \setminus \operatorname{spt}\mu_\star$ as $\varepsilon \to 0$ so that, given any compact set $K \subset \Omega \setminus \operatorname{spt}\mu_\star$, there exists $\varepsilon_K > 0$ so that $|\mathbf{Q}_\varepsilon| \geq 1/2$ for any $0 < \varepsilon \leq \varepsilon_K$. Let $K \subset \Omega \setminus \operatorname{spt}\mu_\star$ be any compact set and let $G \subset K$ be any simply connected open set with smooth boundary.

By Lemma 3.13 applied in $G$, the sequence $\{\mathbf{u}_\varepsilon\}$ associated with $\mathbf{M}_\varepsilon$ as prescribed by (3.5) is relatively compact in $L^p(G, \mathbb{R}^2)$, for any $p$ with $1 \leq p < +\infty$. As $\varepsilon \to 0$, besides the $L^p(G)$-convergence of the maps $\{\mathbf{u}_\varepsilon\}$, by Lemma 3.4 we also have that $\mathbf{Q}_\varepsilon \rightharpoonup \mathbf{Q}_\star$ weakly in $W^{1,2}(G)$ as $\varepsilon \to 0$. In particular, we have $\mathbf{n}_\varepsilon \rightharpoonup \mathbf{n}_\star$ and $\mathbf{m}_\varepsilon \rightharpoonup \mathbf{m}_\star$ weakly in $W^{1,2}(G)$ and, up to extracting a subsequence and using the Rellich-Kondrachov theorem, almost everywhere in $G$, for some pair $(\mathbf{n}_\star, \mathbf{m}_\star)$ forming an orthonormal frame for $\mathbf{Q}_\star$ in $G$. Thus, letting

$$\mathbf{M}_\star := (\mathbf{u}_\star)_1\mathbf{n}_\star + (\mathbf{u}_\star)_2\mathbf{m}_\star = \tau\left(\sqrt{2}\beta + 1\right)^{1/2}\mathbf{n}_\star \quad \text{in } G, \tag{3.54}$$

it follows that $\mathbf{M}_\star$ is well-defined, it does not depend on the choice of the orientation of $\mathbf{n}_\varepsilon$, $\mathbf{m}_\varepsilon$ (so long as it is chosen consistently as $\varepsilon \to 0$, so that $\mathbf{n}_\varepsilon \to \mathbf{n}_\star$ and $\mathbf{m}_\varepsilon \to \mathbf{m}_\star$ a.e. in $G$). By letting $K$ and $G$ vary in $\Omega \setminus \operatorname{spt} \mu_\star$, we can define $\mathbf{M}_\star$ almost everywhere on $\Omega$. Then, (3.52) is obvious from (3.54), which also shows that

$$\mathrm{S}_{\mathbf{M}_\star} = \mathrm{S}_{\mathbf{u}_\star}. \tag{3.55}$$

Moreover, since the sequence $\{\mathbf{M}_\varepsilon\}$ is uniformly bounded in $L^\infty(\Omega)$, Lebesgue's dominated convergence theorem implies that, as $\varepsilon \to 0$, $\mathbf{M}_\varepsilon \to \mathbf{M}_\star$ in $L^p(\Omega)$, for any $p$ with $1 \le p < +\infty$.

From here on, continuing to argue as in the proof of [20, Proposition 4.11], one checks that $\mathscr{H}^1(\mathrm{S}_{\mathbf{M}_*}) < +\infty$ and that $\mathbf{M}_\star$ belongs to $\mathrm{SBV}(\Omega, \mathbb{R}^2)$. Finally, (3.53) is an obvious consequence of (3.24) and of (3.54). This concludes the proof. □

As in [20], a regularity property of $\mathbf{M}_\star$ outside of $\overline{\mathrm{S}_{\mathbf{M}_\star}}$ follows immediately.

**Proposition 3.15.** *The map $\mathbf{M}_\star$ is locally harmonic in $\Omega \setminus \overline{\mathrm{S}_{\mathbf{M}_\star}}$, with values in the unit circle of radius $\left(\sqrt{2}\beta + 1\right)^{1/2}$. In particular, $\mathbf{M}_\star$ is smooth in $\Omega \setminus \overline{\mathrm{S}_{\mathbf{M}_\star}}$.*

*Proof.* The proof follows exactly as in [20, Proposition 4.12]. □

*Remark* 3.10. At this stage, however, we do not know yet whether $\overline{\mathrm{S}_{\mathbf{M}_\star}}$ is a small set or not. We shall see in [18, Remark 3.4] that the set $\overline{\mathrm{S}_{\mathbf{M}_\star}}$ has locally finite $\mathscr{H}^1$-measure.

The question of the size of the set $\overline{\mathrm{S}_{\mathbf{M}_\star}}$ is connected with that of the uniform convergence of $\mathbf{M}_\varepsilon$ towards $\mathbf{M}_\star$ away from $\operatorname{spt} \mu_\star \cup \operatorname{spt} \nu_\star$. Both these questions will be addressed in the companion paper [18], as they need a clearing-out result for $\nu_\star$, which will be obtained in [18] at the end of a quite long route. With the purpose of keeping [18] at reasonable size, we prefer to avoid introducing again the change of variables $\mathbf{M} \mapsto \mathbf{u}$ in that paper, so we anticipate here a couple of lemmata that will be useful to prove the uniform convergence mentioned above.

The first lemma expresses $V(\mathbf{M})$ in terms of $h(\mathbf{u})$, $\mathbf{u}$, and $|\mathbf{Q}|$, and it follows immediately from the definitions.

**Lemma 3.16.** *Given any pair $(\mathbf{Q}, \mathbf{M}) \in \mathscr{S}_0^{2\times 2} \times \mathbb{R}^2$ with $\mathbf{Q} \neq 0$, there holds*

$$V(\mathbf{M}) = h(\mathbf{u}) + \frac{\beta}{\sqrt{2}}(1 - |\mathbf{Q}|)\left(u_1^2 - u_2^2 - 1 - \frac{\beta + \beta\,|\mathbf{Q}|}{\sqrt{2}}\right), \tag{3.56}$$

*where $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ is defined in* (3.5).

*Proof.* Recalling (3.5) and (3.7), identity (3.56) is immediately recognised as an obvious consequence of the definitions. □

In the second lemma, we take advantage of Lemma 3.16, Lemma 3.7, and Corollary 2.8 to show that, away from $\operatorname{spt} \mu_\star$, the energies $\mathcal{AC}_\varepsilon(\mathbf{u}_\varepsilon)$ and $E_\varepsilon(\mathbf{M}_\varepsilon)$ differ only by terms that vanish as $\varepsilon \to 0$. Then, we exploit this fact to obtain an upper bound for the length of $\mathscr{H}^1(\mathrm{S}_{\mathbf{M}_\star})$ (i.e., (3.59) below).

**Lemma 3.17.** *Let $\{(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\}$ be a sequence of critical points of $\mathscr{F}_\varepsilon$, subject to boundary conditions either as in* (6)–(7) *or as in* (8)–(9)*, and assume that* (10) *holds. Let $B := B(x_0, R) \subset\subset \Omega \setminus \operatorname{spt} \mu_\star$ be any ball. Then,*

$$\lim_{\varepsilon \to 0} \int_B \left\{ \frac{1}{\varepsilon} V(x, \mathbf{M}_\varepsilon(x)) - \frac{1}{\varepsilon} h(\mathbf{u}_\varepsilon) \right\} \mathrm{d}x = 0. \tag{3.57}$$

*Moreover,*

$$\mathcal{AC}_\varepsilon(\mathbf{u}_\varepsilon, B) = E_\varepsilon(\mathbf{M}_\varepsilon, B) + \mathrm{o}_{\varepsilon \to 0}(1), \tag{3.58}$$

*and, for any simply connected open set $G \subset\subset \Omega \setminus \operatorname{spt} \mu_\star$ with smooth boundary,*

$$c_\beta \mathscr{H}^1(\mathrm{S}_{\mathbf{M}_\star} \cap G) \leq \liminf_{\varepsilon \to 0} E_\varepsilon(\mathbf{M}_\varepsilon; G), \tag{3.59}$$

*where $c_\beta = \frac{2\sqrt{2}}{3}\left(\sqrt{2}\beta + 1\right)^{3/2}$ is the constant given by* (3.50).

*Proof.* Given the assumptions, for any $\varepsilon > 0$ small enough we can apply Lemma 3.16 with $(\mathbf{Q}, \mathbf{M}) = (\mathbf{Q}_\varepsilon(x), \mathbf{M}_\varepsilon(x))$ for any $x \in B$, so that (3.57) follows by integration of both sides of (3.56) over $B$ and taking the limit $\varepsilon \to 0$, and then using Lemma 3.7, Corollary 2.8, and (3.11) to show that the integral of the second term on the right-hand side of (3.56) tends to zero as $\varepsilon \to 0$.

Concerning (3.58), it follows from (3.57) and integrating both sides of (3.17) over $B$, using Hölder's inequality on the middle term in the right-hand side and the bound (2.62).

About (3.59), it follows from (3.58), a standard covering argument, Theorem 3.12 and the fact that, by (3.55), $\mathrm{S}_{\mathbf{u}_\star} = \mathrm{S}_{\mathbf{M}_\star}$. □

# 4 Proof of Theorem A

We are finally ready for the proof of Theorem A. In fact, all the claims have been already proved in the previous sections, but, for the reader's convenience, we collect the appropriate references below.

*Proof of Theorem A.* The first part (i.e., the part relative to the **Q**-component) of statement (i) follows from Proposition 3.5, while the second part comes from Theorem 3.14. About (ii), it follows from Lemma 3.4, Proposition 3.9, and (again) Theorem 3.14. From Lemma 3.3, we have the statement (iii). Concerning (iv), it follows from Lemma 3.3 Finally, statement (v) is part of Proposition 3.5. □

# A The pure Dirichlet case: $W^{1,2}_{\mathrm{loc}}$-convergence up to the boundary for $\mathbf{Q}_\varepsilon$

In this appendix, we assume that $\{(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\}$ is a sequence of critical points of $\mathscr{F}_\varepsilon$ subject to the pure Dirichlet boundary conditions (6), (7) and, arguing along the lines of [6, Theorem X.3], we prove that $\mathbf{Q}_\varepsilon \to \mathbf{Q}_\star$ strongly in $W^{1,2}_{\mathrm{loc}}\left(\overline{\Omega} \setminus \operatorname{spt} \mu_\star\right)$, i.e., locally up to the boundary but away from $\operatorname{spt} \mu_\star$. More precisely, we have the following proposition.

**Proposition A.1.** *Let $x_0 \in \partial\Omega$ and let $B(x_0,\, R)$ be any ball satisfying*

$$\overline{B(x_0,\, 2R)} \cap \operatorname{spt} \mu_\star = \emptyset. \tag{A.1}$$

*Then, as $\varepsilon \to 0$, we have $\mathbf{Q}_\varepsilon \to \mathbf{Q}_\star$ strongly in $W^{1,2}\,(B(x_0,\, R) \cap \Omega)$.*

As a first step towards the proof of Proposition A.1, we state a variant of the Pohozaev identity in Lemma 1.8.

**Lemma A.2.** *Assume that $\Omega$ is star-shaped about a point $x_\star \in \Omega$. Then, there holds*

$$\begin{aligned} &\frac{2}{\varepsilon^2}\int_\Omega f_\varepsilon(\mathbf{Q}_\varepsilon, \mathbf{M}_\varepsilon)\,\mathrm{d}x + \frac{R}{2}\int_{\partial\Omega}\left(\left|\partial_{\boldsymbol{\nu}}\mathbf{Q}_\varepsilon\right|^2 + \varepsilon\left|\partial_{\boldsymbol{\nu}}\mathbf{M}_\varepsilon\right|^2\right)\mathrm{d}s \\ &\qquad \lesssim \frac{R}{2}\int_{\partial\Omega}\left(\left|\partial_{\boldsymbol{\tau}}\mathbf{Q}_\varepsilon\right|^2 + \varepsilon\left|\partial_{\boldsymbol{\tau}}\mathbf{M}_\varepsilon\right|^2 + \frac{2}{\varepsilon^2} f_\varepsilon(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)\right)\mathrm{d}s \end{aligned} \tag{A.2}$$

*where the implicit constant on the right-hand side depends only on $\Omega$ and the boundary data.*

*Proof.* Take $\mathbf{X} := x - x_\star$ in (1.48). Since $\Omega$ is star-shaped, there is a constant $\alpha_\Omega > 0$ such that

$$(x - x_\star)\cdot\boldsymbol{\nu} \geq \alpha_\Omega, \qquad \forall x \in \partial\Omega.$$

On the other hand, precisely as in [6, (12) of Chapter III]), we have

$$(\boldsymbol{\nu}\cdot\nabla\mathbf{Q}_\varepsilon)\cdot(\mathbf{X}\cdot\nabla\mathbf{Q}_\varepsilon) = ((x-x_\star)\cdot\boldsymbol{\nu})\left|\boldsymbol{\nu}\cdot\nabla\mathbf{Q}_\varepsilon\right|^2 + ((x-x_\star)\cdot\boldsymbol{\tau})\,(\partial_{\boldsymbol{\tau}}\mathbf{Q}_\varepsilon\cdot\partial_{\boldsymbol{\nu}}\mathbf{Q}_\varepsilon)$$

and the same holds for $\mathbf{M}_\varepsilon$. Combing these facts and applying suitably Young's inequality, we obtain (A.2), with a constant depending only on $\alpha_\Omega$ and the boundary data. ☐

*Remark* A.1. If $\{(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)\}$ satisfies the pure Dirichlet boundary conditions (6)–(7), then, in view of (6)–(7), and item (ii) of Lemma 1.18, the right-hand side of (A.2) is bounded by a constant depending only on $\alpha_\Omega$, $\beta$, and the boundary data.

*Proof of Proposition A.1.* As in the interior case, we argue in two steps, proving first that $\rho_\varepsilon \to 1$ and then that $\mathbf{j}_\varepsilon \to \mathbf{j}_\star$ strongly in $W^{1,2}\,(\Omega \cap B(x_0,\, R))$ as $\varepsilon \to 0$.

Let $\zeta \in C^\infty_c(B(x_0,\, 2R))$ be a smooth-cut off function such that

$$0 \leq \zeta \leq 1, \qquad \zeta \equiv 1 \quad \text{in } B(x_0,\, R).$$

*Step* 1 ($\rho_\varepsilon \to 1$ in $W^{1,2}\,(B(x_0,\, R)\cap\Omega)$). This step involves the Pohozaev-type inequality (A.2) and we will crucially rely on the Dirichlet boundary conditions (6)–(7) to control its right-hand side.

In view of (6)–(7), we may extend each pair $(\mathbf{Q}_\varepsilon,\, \mathbf{M}_\varepsilon)$ as explained in Remark 3.3 to $\Omega' \supset \Omega$, an open set containing a $(2R)$-neighbourhood of $\partial\Omega$. Since $\rho_\varepsilon \equiv 1$ on $\partial\Omega$, we then have $\rho_\varepsilon > 1/2$ eventually on $B(x_0,\, 3R/2)$. Therefore, arguing as in the proof of Lemma 3.7, we can take the equation (2.51) and multiply it by $\zeta(\rho_\varepsilon - 1 - \kappa_\star\varepsilon)$. Then, recalling that $\operatorname{spt}\zeta$ is far from $\operatorname{spt}\mu_\star$, we integrate over $\Omega$ to yield (we drop the subscripts $\varepsilon$ for notational simplicity)

$$\int_\Omega \left\{ \zeta\, |\nabla\rho|^2 + \frac{1}{2}\nabla\zeta\cdot\nabla\,(\rho - 1 - \kappa_\star\varepsilon)^2 + \zeta\rho(\rho - 1 - \kappa_\star\varepsilon)\,|\nabla\varphi|^2 \right.$$
$$\left. + \zeta\left(\frac{\rho-1}{\varepsilon} - \kappa_\star\right)\left(\frac{\rho-1}{\varepsilon}(\rho+1)\rho - \sigma\right)\right\}\,\mathrm{d}x = \int_{\partial\Omega} \zeta(\rho - 1 - \kappa_\star\varepsilon)\partial_{\boldsymbol{\nu}}\rho\,\mathrm{d}s.$$

Since $\rho \equiv 1$ on $\partial\Omega$, from the Pohozaev identity (A.2), the imposed boundary conditions (6)–(7), and Remark A.1 it follows that

$$\int_{\partial\Omega} \zeta(\rho - 1 - \kappa_\star\varepsilon)\partial_{\boldsymbol{\nu}}\rho\,\mathrm{d}s = -\varepsilon\kappa_\star\int_{\partial\Omega}\zeta\partial_{\boldsymbol{\nu}}\rho = \mathrm{o}_{\varepsilon\to 0}(1).$$

By Remark 3.3, Theorem 2.11, and an argument via Gagliardo-Nirenberg interpolation completely analogous to that in Corollary 3.8, we have

$$\rho_\varepsilon \to 1 \qquad \text{uniformly on } \overline{B(x_0,\, 2R)}, \text{ as } \varepsilon \to 0. \tag{A.3}$$

Using these facts, and recalling that $\operatorname{spt}\zeta \subset B(x_0,\, 2R)$, we can argue, from now on, as in Lemma 3.7, with no changes.

*Step* 2 ($\mathbf{j}_\varepsilon \to \mathbf{j}_\star$ in $W^{1,2}\,(B(x_0,\, R)\cap\Omega)$). As in the interior case dealt with in Lemma 3.6, we consider a suitable, 'Hodge-type' decomposition of $\mathbf{j}_\varepsilon := j(\mathbf{Q}_\varepsilon)$, i.e., we write

$$\mathbf{j}_\varepsilon = \nabla H_\varepsilon + \nabla^\perp\phi_\varepsilon + \nabla^\perp\xi_\varepsilon \qquad \text{in } \Omega,$$

where $H_\varepsilon$, $\phi_\varepsilon$, and $\xi_\varepsilon$ solve, respectively, (2.13)-(2.14), (2.9), and (2.10), and we study the convergence of $H_\varepsilon$, $\phi_\varepsilon$, and $\xi_\varepsilon$ separately.

Since $H_\varepsilon = 0$ and $\partial_{\boldsymbol{\nu}}\xi_\varepsilon = 0$ on $\partial\Omega$ for any $\varepsilon > 0$, in these cases there are no boundary terms and we can argue exactly as in the interior case, using the cut-off function $\zeta$ above and (A.3), without other changes.

Concerning $\phi_\varepsilon$, by (A.1), (A.3), and (1.12), we may assume that $\operatorname{curl}(\alpha_\varepsilon^2\mathbf{j}_\varepsilon) \equiv 0$ on $\operatorname{spt}\zeta$ for any $\varepsilon > 0$ small enough. Thus, multiplying (2.9) by $\zeta(\phi_\varepsilon - \phi_\star)$, and integrating over $\Omega$, we have

$$\int_\Omega \zeta\nabla(\phi_\varepsilon - \phi_\star)\cdot\nabla\phi_\varepsilon\,\mathrm{d}x = \frac{1}{2}\int_{\partial\Omega}\zeta(\phi_\varepsilon - \phi_\star)\,(\mathbf{Q}_{\mathrm{bd}}\times\partial_{\boldsymbol{\tau}}\mathbf{Q}_{\mathrm{bd}})\,\mathrm{d}s + \mathrm{o}_{\varepsilon\to 0}(1). \tag{A.4}$$

Next, we observe that from (2.15) and Sobolev embedding (since $\partial\Omega$ is Lipschitz and compact), we have

$$\phi_\varepsilon \rightharpoonup \phi_\star \qquad \text{weakly in } W^{1,p}(\Omega),$$

for any $1 \le p < 2$ as $\varepsilon \to 0$, so that, again by Sobolev embedding and the Lipschitz regularity of $\partial\Omega$, we obtain

$$\phi_\varepsilon \to \phi_\star \qquad \text{strongly in } L^p(\partial\Omega). \tag{A.5}$$

for any $1 \le p < 2$ as $\varepsilon \to 0$. From here, it follows that

$$\|\nabla\phi_\varepsilon\|_{L^2(\Omega\cap B(x_0,\,R))} \le C,$$

where $C$ does not depend on $\varepsilon$, and thus

$$\phi_\varepsilon \rightharpoonup \phi_\star \qquad \text{weakly in } W^{1,2}(\Omega \cap B(x_0, R))$$

as $\varepsilon \to 0$. On the other hand, using (A.5) together with Hölder's inequality, we see that the right-hand side of (A.4) vanishes as $\varepsilon \to 0$, so that

$$\phi_\varepsilon \to \phi_\star \qquad \text{strongly in } W^{1,2}(\Omega \cap B(x_0, R))$$

as $\varepsilon \to 0$.

*Step* 3 (Conclusion). Putting together Step 1 and Step 2 leads directly to the conclusion of the proof. □

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