Energy maximum principle for vectorial higher order absolute minimisers

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Abstract

We show that vectorial absolute minimisers of higher order L^∞ variational problems satisfy an energy maximum principle. This property is only necessary for absolute minimisers, while it characterises a suitable weaker notion of absolute minimality involving compactly supported variations. Further, with different methods, we prove a gradient maximum principle for p-harmonic maps.

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1 Introduction

For $n, N, k \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $\mathrm{H} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \times \ldots \times \mathbb{R}^{Nn^k} \to \mathbb{R}$ be $\mathscr{L}(\Omega) \otimes \mathscr{B}(\mathbb{R}^\alpha)$-measurable, where we have identified $\mathbb{R}^\alpha \cong \mathbb{R}^N \times \mathbb{R}^{Nn} \times \ldots \times \mathbb{R}^{Nn^k}$ for $\alpha = \alpha(n, N, k) = N\frac{n^{k+1}-1}{n-1}$. here $\mathscr{L}(\Omega)$ is the Lebesgue σ-algebra of Ω and $\mathscr{B}(\mathbb{R}^\alpha)$ is the Borel σ-algebra of $\mathbb{R}^\alpha$. Let $U \subseteq \Omega$ be an open subset and consider the supremal functional

$$E_\infty(u, U) := \operatorname*{ess\,sup}_{x \in U} \mathrm{H}\big(x, \mathrm{D}^{[k]}u(x)\big), \tag{1.1}$$

where $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$ and $\mathrm{D}^{[k]}u := (u, \mathrm{D}u, \ldots, \mathrm{D}^k u)$ is the jet of order k of u.
The goal herein is to prove a maximum principle property for the absolute minimisers of the functional E_∞. We recall that $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$ is an absolute minimiser of E_∞ if

$$E_\infty(u, U) \leq E_\infty(u + \Phi, U) \qquad \forall \Phi \in W^{k,\infty}_0(U, \mathbb{R}^N) \quad \forall U \subseteq \Omega \text{ open.}$$

Under relatively mild assumptions on the supremand H, we prove that for an absolute minimiser the essential supremum in (1.1) is "attained at the boundary of U". In order to give a precise meaning to this expression, we refer to the supremum on ∂U of the *essential limsup* of $\mathrm{H}(\cdot, \mathrm{D}^{[k]}u)$, which is defined as

$$\mathrm{H}(x, \mathrm{D}^{[k]}u(x))^* := \lim_{\varepsilon \to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x) \cap U} \mathrm{H}(\cdot, \mathrm{D}^{[k]}u) \qquad \forall x \in \partial U. \tag{1.2}$$

Therefore, we give the following definition.

Definition 1.1 (Energy maximum principle)**.** *We say that $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$ satisfies the energy maximum principle for E_∞ in Ω if*

$$\operatorname*{ess\,sup}_{U} \mathrm{H}(\cdot, \mathrm{D}^{[k]}u) = \sup_{\partial U} \mathrm{H}(\cdot, \mathrm{D}^{[k]}u)^* \qquad \forall U \subseteq \Omega \text{ open.} \tag{1.3}$$

If $u \in C^k(\overline{\Omega}, \mathbb{R}^N)$ and $\mathrm{H} \in C(\Omega \times \mathbb{R}^\alpha)$, this can be rephrased as

$$\max_{\overline{U}} \mathrm{H}(\cdot, \mathrm{D}^{[k]}u) = \max_{\partial U} \mathrm{H}(\cdot, \mathrm{D}^{[k]}u) \qquad \forall U \subseteq \Omega \text{ open.}$$

We prove that this property is necessarily satisfied by absolute minimisers. Although it is not sufficient: the cone function $u(x) = |x|$ belongs to $W^{1,\infty}(\mathbb{R}^n)$ and clearly satisfies the energy (or, equivalently, gradient) maximum principle for $E_\infty(u, \Omega) := \operatorname{ess\,sup}_\Omega |\mathrm{D}u|$, but it is not an absolute minimiser if Ω contains the origin. However, we will see that the energy maximum principle is a property that characterises a class of maps which is larger than the one of absolute minimisers, and are the ones that minimise $E_\infty(\cdot, U)$ with respect to compactly supported variations in U, for every $U \subseteq \Omega$.

Before stating our main results, we underline that E_∞ in (1.1) is the general object of study in the L^∞-Calculus of Variations, a field initiated by Aronsson in [1]. his pioneering work on the scalar first order case (namely when $N = k = 1$) has been well developed by now, and most challenges have been thoroughly analysed and understood (see [13] for a survey reference). More recently, the vectorial case ($N > 1$) and the higher order case ($k > 1$) have been approached respectively in [11] and [16, 21]. In these contexts, a complete theory is still far from reach. Without any pretension of being exhaustive, we refer also to [14, 15, 17] for a glimpse of the literature on vectorial first order problems and to [5, 8, 18–20] for higher order ones. Furthermore, the fractional order case ($k \notin \mathbb{N}$) has been recently explored in [6].

It is worth mentioning that, given boundary data $u_0 \in W^{k,\infty}(\Omega, \mathbb{R}^N)$, the existence (and/or uniqueness) of absolute minimisers to the Dirichlet problem in $W^{k,\infty}_{u_0}(\Omega, \mathbb{R}^N)$ associated to E_∞ in (1.1) is an open problem already when $k = 1$ and $n, N > 1$ or when $N = 1$ and $n, k > 1$. There are some exceptions in the second order case, whenever H has special dependence in the second derivatives of u: relevant works are [5, 16, 19, 21]; surprisingly, similar results hold true also in the fractional case [6].

Now we proceed to specify the assumptions on the function H. By denoting $X = (X_0, X_1, \dots, X_k) \in \mathbb{R}^N \times \mathbb{R}^{Nn} \times \dots \mathbb{R}^{Nn^k} = \mathbb{R}^\alpha$, we assume that $\mathrm{H} : \Omega \times \mathbb{R}^\alpha \to \mathbb{R}$ is Carathéodory and *strongly radially increasing* on $\mathbb{R}^\alpha$, uniformly w.r.t $x \in \Omega$. In symbols, we assume that there exists a continuous function $c : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$, with $c(1, \cdot) = c(\cdot, 0) = 0$ and $c > 0$ in $(0, 1) \times (0, \infty)$, such that

$$\mathrm{H}(x, tX) \leq \mathrm{H}(x, X) - c(t, |X|) \qquad \forall t \in [0, 1] \quad \forall X \in \mathbb{R}^\alpha \quad \text{a.e. } x \in \Omega. \tag{1.4}$$

In other words, $H(x, \cdot)$ is a strictly radially increasing function with a modulus of monotonicity independent of x. Notice that the radial monotonicity implies that, for almost every $x \in \Omega$, $\mathrm{H}(x, \cdot)$ has a global minimum at $X = 0$. Without loss of generality we can assume that $\mathrm{H}(x, 0) = 0$ a.e. $x \in \Omega$, so that H takes values in $\mathbb{R}^+$. In this case, $E_\infty(u, U) = \|\mathrm{H}(\cdot, \mathrm{D}^{[k]}u)\|_{L^\infty(U)}$ for every $U \subseteq \Omega$ open.

The strong radial monotonicity assumption may seem restrictive at first, however, the validity of our result extends to a wider class of supremands, even non-continuous ones: indeed, for a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ lower semicontinuous and non-decreasing, set $G = g \circ H$ and $E'_\infty(u, U) := \operatorname{ess\,sup}_U G(\cdot, \mathrm{D}^{[k]}u)$. It is not difficult to see that E_∞ and E'_∞ share the same minimisers. In addition, if $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$ satisfies the energy maximum principle for E_∞ in Ω, then for every $U \subseteq \Omega$ open

$$E'_\infty(u, U) = g\left(\operatorname*{ess\,sup}_U \mathrm{H}(\cdot, \mathrm{D}^{[k]}u)\right) = g\left(\operatorname*{ess\,sup}_{\partial U} \mathrm{H}(\cdot, \mathrm{D}^{[k]}u)^*\right) = \left(\operatorname*{ess\,sup}_{\partial U} G(\cdot, \mathrm{D}^{[k]}u)^*\right),$$

i.e. u satisfies the energy maximum principle for E'_∞ as well.
The second assumption on H concerns the uniformity w.r.t. x of its modulus of continuity: since

$\mathrm{H}(x,\cdot)$ is continuous in $\mathbb{R}^\alpha$, it is uniformly continuous on every closed ball $\overline{\mathbb{B}}_R(0) \subset \mathbb{R}^\alpha$. Thus, we require that the modulus of continuity on $\overline{\mathbb{B}}_R(0)$ is independent of x. More precisely, we assume

$$\begin{aligned} &\forall R > 0 \ \exists \omega_R \in C(\mathbb{R}^+) \text{ non-decreasing, with } \omega_R(0) = 0 : \\ &|\mathrm{H}(x,X) - \mathrm{H}(x,X')| \le \omega_R(|X - X'|), \qquad \text{a.e. } x \in \Omega, \quad \forall X \in \overline{\mathbb{B}}_R(0) \ . \end{aligned} \tag{1.5}$$

Now we can state our main result.

Theorem 1.2 (Energy maximum principle for absolute minimisers)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\mathrm{H} : \Omega \times \mathbb{R}^\alpha \to \mathbb{R}^+$ be a Carathéodory function satisfying* (1.4) *and* (1.5)*, with* $\mathrm{H}(x,0) = 0$*. Assume that $u \in W^{k,\infty}(\Omega,\mathbb{R}^N)$ is an absolute minimiser for the functional E_∞, defined in* (1.1)*. Then u satisfies the energy maximum principle for E_∞ in Ω, namely*

$$\operatorname*{ess\,sup}_U \mathrm{H}(\cdot,\mathrm{D}^{[k]}u) = \sup_{\partial U} \mathrm{H}(\cdot,\mathrm{D}^{[k]}u)^* \qquad \forall U \subseteq \Omega \text{ open.} \tag{1.6}$$

where $\mathrm{H}(\cdot,\mathrm{D}^{[k]}u)^*$ is the essential limsup of $\mathrm{H}(\cdot,\mathrm{D}^{[k]}u)$, defined in (1.2)*.*

Another way of interpreting the energy maximum principle property is by saying that, for every $U \subseteq \Omega$, the *attainment set* $U(u)$ must contain the boundary of U. The attainment set $U(u)$ is the collection of all points $x \in \overline{U}$ where $\mathrm{H}(x,\mathrm{D}^{[k]}u(x)) = \operatorname{ess\,sup}_U \mathrm{H}(\cdot,\mathrm{D}^{[k]}u)$, and it is well defined and non-empty in case $u \in C^k(\Omega,\mathbb{R}^N)$ and $\mathrm{H} \in C(\Omega \times \mathbb{R}^\alpha)$. For a definition of $U(u)$ in the non-smooth setting we refer to [4], where the authors show also a minimality property of $U(u)$ for absolute minimisers in the scalar first order case.

Before stating our second result, we give the definition of absolute minimiser w.r.t. compactly supported variations.

Definition 1.3 (The class $\mathrm{AM}_c(\Omega,\mathbb{R}^N)$)**.** *We say that $u \in W^{k,\infty}(\Omega,\mathbb{R}^N)$ is an absolute minimiser w.r.t. compactly supported variations, and we write $u \in \mathrm{AM}_c(\Omega,\mathbb{R}^N)$, if*

$$E_\infty(u,U) \le E_\infty(u+\Phi,U) \qquad \forall \Phi \in W^{k,\infty}_c(U,\mathbb{R}^N) \quad \forall U \subseteq \Omega \text{ open.}$$

As a consequence of Theorem 1.2, we obtain the following characterisation.

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\mathrm{H} : \Omega \times \mathbb{R}^\alpha \to \mathbb{R}^+$ be a Carathéodory function satisfying* (1.4) *and* (1.5)*, with* $\mathrm{H}(x,0) = 0$*. Consider the functional E_∞, defined in* (1.1)*. Then, the following are equivalent for a map $u \in W^{k,\infty}(\Omega,\mathbb{R}^N)$:*

i) u satisfies the energy maximum principle for E_∞ in Ω;

ii) $u \in \mathrm{AM}_c(\Omega,\mathbb{R}^N)$.

The proof of Theorem 1.2 and Corollary 1.4 are in Section 3, preceeded by a short preparatory section. Therein we discuss the different notions of radial monotonicity which are of interest for us and some properties of the essential limsup.

Finally, in Section 4, we give a proof of a gradient maximum principle for p-harmonic maps. The methods used therein are genuinely different from the ones in the previous sections, and are based on the theory of elliptic PDEs. For the scalar case, results of this kind are already present in the literature, e.g. in [9, Chapter 9]. We refer also to [22, Appendix B] for a proof of gradient maximum and minimum principles for the class of p-harmonic potentials on convex rings in the plane. However, to the best of our knowledge, the validity of a gradient maximum principle for vector-valued p-harmonic maps appears to be new.

2 Preparatory tools

2.1 On radial monotonicity

In this first preliminary section, we recall some notions of radial monotonicity for function on $\mathbb{R}^\alpha$, together with some relevant examples.

Definition 2.1. *A function $h : \mathbb{R}^\alpha \to \mathbb{R}$ is said to be*

i) radially non-decreasing if for every $X \in \mathbb{R}^\alpha$ the function $t \mapsto h(tX)$ is non-decreasing in $\mathbb{R}^+$;

ii) strictly radially increasing if for every $X \in \mathbb{R}^\alpha$ the function $t \mapsto h(tX)$ is strictly increasing in $\mathbb{R}^+$;

iii) strongly radially increasing if there exists a continuous function $c : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$, with $c(1,\cdot) = c(\cdot,0) = 0$ and $c > 0$ in $(0,1) \times (0,\infty)$, such that

$$h(tX) \leq h(X) - c(t,|X|) \qquad \forall t \in [0,1] \quad \forall X \in \mathbb{R}^\alpha.$$

Notice that *i)* $\Rightarrow$ *ii)* $\Rightarrow$ *iii)*, and implications do not invert in general. The above definition is in complete analogy with the notion of (level) convexity and its stronger versions (see e.g. [3]). In particular, any level convex (resp. strictly level convex) function with global minimum at the origin is radially non-decreasing (resp. strictly radially increasing). Of course, a radially non-decreasing function need not to be level convex, since its level subsets only need to be star shaped w.r.t. the origin.

Strongly radially increasing functions can be seen as strict radially increasing functions with a modulus of monotonicity.

Example 2.2. Relevant examples of strongly radially increasing functions are:

- Any strongly convex function with global minimum at the origin;
- Any convex C^1-function h with $h(0) = 0 < h(X)$ for all $X \neq 0$. Indeed, by [7, Thm 2.52], we have $\mathrm{D}h(X) \cdot X \geq h(X)$ for all $X \in \mathbb{R}^\alpha$, i.e. the radial speed of h has a modulus of positivity given by h itself. By integrating this inequality, one can see that h satisfies Definition 2.1 iii);
- The cone function associated to a (smooth) open set $S \subset \mathbb{R}^\alpha$ star shaped w.r.t. the origin with $\nu_S(X) \cdot X > 0$ for every $X \in \partial S$, where ν_S is the outward normal vector to ∂S.
- Functions of the form $h_{\lambda,\beta}(X) = h(X) + \lambda |X|^\beta$, with $\lambda, \beta > 0$ and h radially non-decreasing.

2.2 Properties of the essential limsup

In this second preliminary section, we give the definition of essential limsup, together with some properties.

Definition 2.3 (Essential limsup)**.** *Let $U \subseteq \mathbb{R}^n$ be a Borel set and $f \in L^\infty(U)$. We say that the function $f^* \in L^\infty(U)$, defined as*

$$f^*(x) := \lim_{\varepsilon \to 0} \operatorname*{ess\,sup}_{B_\varepsilon(x) \cap U} f \qquad \forall x \in U,$$

is the essential limsup of f.

Of course, if U is open and f is continuous in U, then $f^* = f$. However, in the more general setting of Definition 2.3, we may have $f \neq f^*$ on a set of positive measure: for instance, consider $f = \mathbb{1}_{\mathbb{R}^n \setminus K}$ for a nowhere dense compact set $K \subset \mathbb{R}^n$, with $\mathscr{L}^n(K) > 0$. In general, from [12, Proposition 9], we have that $f \leq f^*$ almost everywhere and f^* is upper semicontinuous on U. Moreover, again by [12, Proposition 9], the definition of f^* allows us to give a pointwise meaning to the essential supremum, indeed for any Borel set $U \subseteq \mathbb{R}^n$ we have

$$\sup_U f^* = \operatorname*{ess\,sup}_U f. \tag{2.1}$$

For a set $E \subset \mathbb{R}^n$ and $\rho > 0$, we denote by E^ρ the open ρ-neighbourhood of E, namely

$$E^\rho := \{x \in \mathbb{R}^n : \operatorname{dist}(x, E) < \rho\}.$$

The following lemma is in order.

Lemma 2.4. *Let $U \subset \mathbb{R}^n$ be on open bounded set. Then for any $K \subseteq \partial U$ compact subset, we have*

$$\sup_K f^* = \lim_{\rho \to 0} \operatorname*{ess\,sup}_{K^\rho \cap U} f.$$

Proof. We start with noticing that for every $x \in K$ there exists $\rho > 0$ such that $B_\rho(x) \cap U \subset K^\rho \cap U$, so that

$$\operatorname*{ess\,sup}_{B_\rho(x)\cap U} f \leq \operatorname*{ess\,sup}_{K^\rho \cap U} f.$$

By passing to the limit as $\rho \to 0$ in both sides and taking the supremum over $x \in K$ in the left hand side, we obtain

$$\sup_K f^* \leq \lim_{\rho \to 0} \operatorname*{ess\,sup}_{K^\rho \cap U} f.$$

It remains to prove the opposite inequality. By definition 2.3, for any fixed $x \in K$ and $\sigma > 0$, there exists $\rho_{x,\sigma} > 0$ such that

$$f^*(x) \geq \operatorname*{ess\,sup}_{B_{\rho_{x,\sigma}}(x)\cap U} f - \sigma. \tag{2.2}$$

The family of balls $\{B_{\rho_{x,\sigma}}(x);\, x \in K\}$ is an open covering of K. Thus, we can extract a finite subcovering $\{B_{\rho_{x_i,\sigma}}(x_i);\, x_i \in K\}_{i=1,\dots,m}$. For seek of brevity, let us denote $B_{i,\sigma} = B_{\rho_{x_i,\sigma}}(x_i)$.
We claim that there exists $\rho_\sigma > 0$ such that $K^{\rho_\sigma} \subset \bigcup_{i=1}^m B_{i,\sigma}$.
By contradiction, suppose that for every $j \in \mathbb{N}$ there exists $y_j \in K^{1/j} \setminus \bigcup_{i=1}^m B_{i,\sigma}$. This is equivalent to say that

$$\forall j \in \mathbb{N} \quad \exists y_j \in \mathbb{R}^n : \begin{cases} y_j \notin B_{i,\sigma} \quad \forall i = 1, \dots, m, \\ \operatorname{dist}(y_j, K) \leq 1/j. \end{cases} \tag{2.3}$$

Let $\bar{y}_j \in K$ be a point realising $d(y_j, K)$. Since K is compact, there exists a subsequence $(\bar{y}_{j_k}) \subset (\bar{y}_j)$ such that $\bar{y}_{j_k} \to \bar{x}$ as $k \to \infty$, for some $\bar{x} \in K$. Moreover, since $\operatorname{dist}(y_{j_k}, \bar{y}_{j_k}) \leq 1/j_k$, we have also that $y_{j_k} \to \bar{x}$ as $k \to \infty$. But being $\{B_{i,\sigma}\}_{i=1,\dots,m}$ a covering of K, we must have that eventually $y_{j_k} \in B_{i,\sigma}$ for some $i \in \{1, \dots, m\}$, which contradicts (2.3).

Now, define $U_\sigma = \bigcup_{i=1}^m (B_{i,\sigma} \cap U)$ and notice that $K^{\rho_\sigma} \cap U \subset U_\sigma$. Thus, using also (2.2), we have

$$\begin{aligned}\lim_{\rho\to 0} \operatorname*{ess\,sup}_{K^\rho \cap U} f &\leq \operatorname*{ess\,sup}_{K^{\rho_\sigma}\cap U} f \\ &\leq \operatorname*{ess\,sup}_{U_\sigma} f \\ &\leq \max_{i=1,\dots,m} \operatorname*{ess\,sup}_{B_{i,\sigma}\cap U} f \\ &\leq \max_{i=1,\dots,m} f^*(x_i) + \sigma \\ &\leq \sup_K f^* + \sigma.\end{aligned}$$

By letting $\sigma \to 0$, we conclude the proof. □

3 Proof of the main results

This section is devoted to the proof of Theorems 1.2 and Corollary 1.4. The proof of the latter is basically a consequence of the argument in the proof of the former. Indeed, the key idea in the proof of Theorem 1.2 is to find, for every $U \subseteq \Omega$, a suitable competitor in $u_\lambda \in W_u^{k,\infty}(U, \mathbb{R}^N)$ to compare with the absolute minimiser u. Actually, it turns out that $u_\lambda \in u + W_c^{k,\infty}(U, \mathbb{R}^N)$, and this will be the crucial observation in the proof of Corollary 1.4.
We start with proving Theorem 1.2: the key argument is based on the case $\mathrm{H}(x, \cdot)$ strongly radially increasing on $\mathbb{R}^\alpha$, uniformly w.r.t $x \in \Omega$. The result in the general case, i.e. when $\mathrm{H}(x, \cdot)$ is radially non-decreasing, is recovered by an approximation argument.

Proof of Theorem 1.2:
Fix $U \subseteq \Omega$ an open subset. First, notice that for every $x \in \partial U$ and every $\rho > 0$, we clearly have

$$\operatorname*{ess\,sup}_{B_\rho(x)\cap U} \mathrm{H}(\cdot, \mathrm{D}^{[k]}u) \leq \operatorname*{ess\,sup}_{U} \mathrm{H}(\cdot, \mathrm{D}^{[k]}u).$$

From definition 2.3, by passing to the limit as $\rho \to 0$ and then to the supremum over all $x \in \partial U$ in the left hand side, we obtain

$$\sup_{\partial U} \mathrm{H}(\cdot, \mathrm{D}^{[k]}u)^* \leq \operatorname*{ess\,sup}_{U} \mathrm{H}(\cdot, \mathrm{D}^{[k]}u). \tag{3.1}$$

Let us prove the opposite inequality. To this purpose we need to use assumptions (1.4) and (1.5). We start by setting

$$M := \operatorname*{ess\,sup}_{U} \mathrm{H}(\cdot, \mathrm{D}^{[k]}u). \tag{3.2}$$

If $M = 0$, there is nothing to prove, so we can assume that $M > 0$. For seek of contradiction, assume that (3.1) holds strict. Then, by applying Lemma 2.4 to $K = \partial U$, one can find $\delta > 0$ such that

$$\lim_{\rho\to 0} \operatorname*{ess\,sup}_{(\partial U)^\rho \cap U} \mathrm{H}(\cdot, \mathrm{D}^{[k]}u) + 2\delta \leq M.$$

In particular, there exists $\varepsilon > 0$ such that

$$\operatorname*{ess\,sup}_{(\partial U)^\varepsilon \cap U} \mathrm{H}(\cdot, \mathrm{D}^{[k]}u) + \delta \leq M. \tag{3.3}$$

By possibly reducing ε in size, we can choose a function $\varphi_\varepsilon \in C^\infty(\mathbb{R}^n)$, with $0 \le \varphi_\varepsilon \le 1$, such that $\varphi_\varepsilon = 0$ in $U \setminus (\partial U)^\varepsilon$ and $\varphi_\varepsilon = 1$ in $(\partial U)^{\varepsilon/2}$. Now, for $\lambda \in (0,1)$, set

$$u_\lambda := \lambda u + (1-\lambda)\varphi_\varepsilon u.$$

Notice that $u_\lambda \in W^{k,\infty}(U,\mathbb{R}^N)$. Moreover, $u_\lambda = u$ on $(\partial U)^{\varepsilon/2} \cap U$, so that

$$u_\lambda \in u + W^{k,\infty}_c(U,\mathbb{R}^N) \subset W^{k,\infty}_u(U,\mathbb{R}^N). \tag{3.4}$$

Let us compute $\mathrm{D}^{[k]}u_\lambda$. For every $h = 1,\dots,k$, we have

$$\mathrm{D}^h u_\lambda = \lambda \mathrm{D}^h u + (1-\lambda)\mathrm{D}^h(\varphi_\varepsilon u). \tag{3.5}$$

By the Leibniz formula,

$$\partial^h_{i_1,\dots,i_h}(\varphi_\varepsilon u) = \sum_{j=0}^{h} \binom{h}{j} (\partial^{h-j}_{i_{j+1},\dots,i_h}\varphi_\varepsilon)(\partial^j_{i_1,\dots,i_j} u).$$

In particular, for every $h = 0,\dots,k$, we can estimate

$$|\mathrm{D}^h(\varphi_\varepsilon u)| \le 2^h \|\varphi_\varepsilon\|_{W^{h,\infty}(U)} \|u\|_{W^{h,\infty}(U,\mathbb{R}^N)} \qquad \text{in } U,$$

which gives

$$|\mathrm{D}^{[k]}(\varphi_\varepsilon u)| \le 2^{k+1} \|\varphi_\varepsilon\|_{W^{k,\infty}(U)} \|u\|_{W^{k,\infty}(U,\mathbb{R}^N)} \qquad \text{in } U. \tag{3.6}$$

Putting together (3.5) and (3.6), we obtain

$$|\mathrm{D}^{[k]}u_\lambda - \lambda \mathrm{D}^{[k]}u| \le (1-\lambda)2^{k+1} \|\varphi_\varepsilon\|_{W^{k,\infty}(U)} \|u\|_{W^{k,\infty}(U,\mathbb{R}^N)} \qquad \text{in } U. \tag{3.7}$$

In particular, if we set

$$C_1 = C_1(\varepsilon,k,\lambda) = (1-\lambda)2^{k+1}\|\varphi_\varepsilon\|_{W^{k,\infty}(U)} + \lambda,$$

then

$$|\mathrm{D}^{[k]}u_\lambda| \le C_1 \|u\|_{W^{k,\infty}(U,\mathbb{R}^N)} \qquad \text{in } U.$$

Now, we recall assumption (1.5) and apply it to

$$R := C_1 \|u\|_{W^{k,\infty}(U,\mathbb{R}^N)}, \qquad \mathrm{D}^{[k]}u_\lambda(x),\ \lambda \mathrm{D}^{[k]}u(x) \in \overline{\mathbb{B}}_R(0) \quad \text{for a.e. } x \in U,$$

obtaining the existence of a continuous function $\omega = \omega_R : \mathbb{R}^+ \to \mathbb{R}^+$, non-decreasing, with $\omega(0) = 0$, such that

$$|\mathrm{H}(x, \mathrm{D}^{[k]}u_\lambda(x)) - \mathrm{H}(x, \lambda \mathrm{D}^{[k]}u(x))| \le \omega(|\mathrm{D}^{[k]}u_\lambda(x) - \lambda \mathrm{D}^{[k]}u(x)|) \qquad \text{a.e. } x \in U.$$

From (3.7) and the monotonicity of ω, setting

$$C_2 = C_2(\varepsilon,k) = 2^{k+1}\|\varphi_\varepsilon\|_{W^{k,\infty}(U)} \|u\|_{W^{k,\infty}(U,\mathbb{R}^N)},$$

we get

$$|\mathrm{H}(\cdot, \mathrm{D}^{[k]}u_\lambda) - \mathrm{H}(\cdot, \lambda \mathrm{D}^{[k]}u)| \le \omega\left((1-\lambda)C_2\right) \qquad \text{a.e. in } U.$$

By restricting the previous estimate to $(\partial U)^\varepsilon \cap U$ and passing to the essential supremum on this set, we have

$$\begin{aligned}\operatorname*{ess\,sup}_{(\partial U)^\varepsilon \cap U} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u_\lambda) &\leq \operatorname*{ess\,sup}_{(\partial U)^\varepsilon \cap U} \mathrm{H}(\cdot, \lambda \mathrm{D}^{[k]} u) + \omega\left((1-\lambda) C_2\right) \\ &\leq M - \delta + \omega\left((1-\lambda)C_2\right),\end{aligned}$$

where in the last inequality we have applied (3.3). Now, for λ sufficiently close to 1, by continuity of ω, we have $\omega\left((1-\lambda)C_2\right) \leq \delta/2$, giving

$$\operatorname*{ess\,sup}_{(\partial U)^\varepsilon \cap U} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u_\lambda) \leq M - \frac{\delta}{2} < M. \tag{3.8}$$

On the other hand, let us prove that

$$\operatorname*{ess\,sup}_{U \setminus (\partial U)^\varepsilon} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u_\lambda) < M. \tag{3.9}$$

To this purpose, notice that, since $\varphi_\varepsilon = 0$ in $U \setminus (\partial U)^\varepsilon$, we have $u_\lambda = \lambda u$ in $U \setminus (\partial U)^\varepsilon$. Thus, set

$$\overline{M} := \operatorname*{ess\,sup}_{U} \mathrm{H}(\cdot, \lambda \mathrm{D}^{[k]} u)$$

and aim at proving $\overline{M} < M$. Without loss of generality, we can assume $\overline{M} > 0$. So, by definition of essential supremum, for every $\delta \in (0, \overline{M}/2)$, there exists $x_\delta \in U$ such that

$$\mathrm{H}(x_\delta, \lambda \mathrm{D}^{[k]} u(x_\delta)) \geq \overline{M} - \delta. \tag{3.10}$$

On the other hand, by recalling assumption (1.4) and (3.2), we have

$$\begin{aligned}\mathrm{H}(x_\delta, \lambda \mathrm{D}^{[k]} u(x_\delta)) &\leq \mathrm{H}(x_\delta, \mathrm{D}^{[k]} u(x_\delta)) - c(\lambda, |\mathrm{D}^{[k]} u(x_\delta)|) \\ &\leq M - c(\lambda, |\mathrm{D}^{[k]} u(x_\delta)|).\end{aligned} \tag{3.11}$$

Now we claim that there exists $\sigma > 0$ independent of δ such that

$$c(\lambda, |\mathrm{D}^{[k]} u(x_\delta)|) \geq \sigma. \tag{3.12}$$

Indeed, from (3.10) and the continuity of H at $X = 0$, we infer that $|\mathrm{D}^{[k]} u(x_\delta)|$ is bounded away from 0. Moreover, since $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$, we have that $|\mathrm{D}^{[k]} u(x_\delta)|$ is also bounded from above. In other words, there exist two constants $R \geq r > 0$, independent of δ, such that

$$r \leq |\mathrm{D}^{[k]} u(x_\delta)| \leq R.$$

Hence, since c is continuous and strictly positive in $(0,1) \times (0, \infty)$, we have

$$c(\lambda, |\mathrm{D}^{[k]} u(x_\delta)|) \geq \min_{\rho \in [r,R]} c(\lambda, \rho) =: \sigma > 0.$$

Putting together (3.10), (3.11), and (3.12), we obtain

$$\overline{M} - \delta \leq M - c(\lambda, |\mathrm{D}^{[k]} u(x_\delta)|) \leq M - \sigma,$$

for $\sigma > 0$ independent of δ. We conclude that $\overline{M} \leq M - \sigma < M$, by letting $\delta \to 0$.
As a consequence, we deduce (3.9), indeed

$$\begin{aligned} \operatorname*{ess\,sup}_{U \setminus (\partial U)^\varepsilon} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u_\lambda) &= \operatorname*{ess\,sup}_{U \setminus (\partial U)^\varepsilon} \mathrm{H}(\cdot, \lambda \mathrm{D}^{[k]} u) \\ &\leq \operatorname*{ess\,sup}_{U} \mathrm{H}(\cdot, \lambda \mathrm{D}^{[k]} u) \\ &= \overline{M} < M \end{aligned}$$

Finally, by recalling (3.2), and the inequalities (3.8) and (3.9), we obtain

$$\operatorname*{ess\,sup}_{U} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u_\lambda) \leq \max \left\{ \operatorname*{ess\,sup}_{(\partial U)^\varepsilon \cap U} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u_\lambda), \operatorname*{ess\,sup}_{U \setminus (\partial U)^\varepsilon} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u_\lambda) \right\} < M.$$

This inequality and (3.4) contradict the absolute minimality of u.
The proof is complete. □

A straightforward consequence is Corollary 1.4, whose proof is detailed below.

Proof of Corollary 1.4:
Let us show that satisfying the energy maximum principle implies the membership to the class $\mathrm{AM}_c(\Omega, \mathbb{R}^N)$. Fix $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$, an open set $U \subseteq \Omega$ and a map $\Phi \in W^{k,\infty}_c(U, \mathbb{R}^N)$. Let $U' \Subset U$ be an open subset with $\mathrm{supp}(\Phi) \Subset U'$. Then we have

$$\begin{aligned} E_\infty(u + \Phi, U) &= \operatorname*{ess\,sup}_{U} \mathrm{H}(\cdot, \mathrm{D}^{[k]}(u + \Phi)) \\ &\geq \operatorname*{ess\,sup}_{U \setminus U'} \mathrm{H}(\cdot, \mathrm{D}^{[k]}(u + \Phi)) \\ &= \operatorname*{ess\,sup}_{U \setminus U'} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u) \end{aligned} \tag{3.13}$$

Notice that $(\partial U)^\rho \cap U \subset U \setminus U'$ for a sufficiently small $\rho > 0$. Hence

$$\operatorname*{ess\,sup}_{U \setminus U'} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u) \geq \operatorname*{ess\,sup}_{(\partial U)^\rho \cap U} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u).$$

By passing to the limit as $\rho \to 0$ in the right hand side and invoking Lemma 2.4, we get

$$\operatorname*{ess\,sup}_{U \setminus U'} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u) \geq \lim_{\rho \to 0} \operatorname*{ess\,sup}_{(\partial U)^\rho \cap U} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u) = \sup_{\partial U} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u)^*. \tag{3.14}$$

Thus, if u satisfies the energy maximum principle in (1.6), by putting together (3.13) and (3.14), we get

$$\begin{aligned} E_\infty(u + \Phi, U) &\geq \sup_{\partial U} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u)^* \\ &= \operatorname*{ess\,sup}_{U} \mathrm{H}(\cdot, \mathrm{D}^{[k]} u) \\ &= E_\infty(u, U), \end{aligned}$$

so that $u \in \mathrm{AM}_c(\Omega, \mathbb{R}^N)$.
For the reverse implication, one can just replicate the proof of Theorem 1.2. Indeed, inequality (3.1) holds for every $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$. The opposite inequality is obtained exactly with the same reasoning, since the competitor u_λ belongs to the class $u + W^{k,\infty}_c(U, \mathbb{R}^N)$ (recall (3.4)). □

4 On the gradient maximum principle for p-harmonic maps

In this section, we prove a gradient maximum principle for p-harmonic maps. To our best knowledge, a result of this kind is not present in the literature, at least in the vectorial case. For the scalar one, a gradient maximum principle for quasilinear elliptic equations can be found in [9, Chapter 15]. We refer also to [22, Appendix B] regarding gradient maximum and minimum principles for p-harmonic potentials on convex rings in the plane. In the case of general p-harmonic maps, however, one can only expect a gradient maximum principle to hold, since a minimum principle is not true in general, as existing examples show (see [2]).
We show the following result.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ an open set and $p \in [2, \infty)$. Assume that $u : \Omega \to \mathbb{R}^N$ is a p-harmonic map, i.e. $u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N)$ satisfying*

$$\operatorname{div}(|\mathrm{D}u|^{p-2}\mathrm{D}u) = 0 \qquad \text{in } \Omega, \tag{4.1}$$

in the sense of distributions. Then, for every open set $U \Subset \Omega$, we have

$$\max_{\overline{U}} |\mathrm{D}u| = \max_{\partial U} |\mathrm{D}u|. \tag{4.2}$$

Remark 4.2. We observe that the above maxima are well defined since $u \in C^{1,\alpha}_{\text{loc}}(\Omega, \mathbb{R}^N)$, from regularity theory for p-harmonic maps [23].
Moreover, we underline that Theorem 1.2, in the particular case when $E_\infty(u, U) = \operatorname{ess\,sup}_U |\mathrm{D}u|$, cannot be deduced from Theorem 4.1 by passing to the limit as $p \to \infty$. Indeed, it is not known if vector valued absolute minimisers of E_∞ can be approximated locally uniformly by p-harmonic maps. More in general, the lack of good approximation results for absolute minimisers via p-harmonic maps is one of the reasons why existence of absolute minimisers is still open in the vectorial case.

The content of the following lemma is probably well known, but we present its proof for sake of completeness.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}^N$ be a p-harmonic map, $p \in [2, \infty)$. Set $\mathscr{C} := \{x \in \Omega : \mathrm{D}u(x) = 0\}$. Then $u \in C^\infty(\Omega \setminus \mathscr{C})$.*

Proof. Define $F : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ as $F(X) := |X|^{p-2}X$. Then, we have

$$\mathrm{D}F(X) = (p-2)|X|^{p-4}X \otimes X + |X|^{p-2}I,$$

where I is the identity map in $(\mathbb{R}^{N\times n})^{\otimes 2}$. Using the following identity

$$(X \otimes X)Y : Z = (X : Y)(X : Z) \qquad \forall X, Y, Z \in \mathbb{R}^{N\times n},$$

we have that

$$\mathrm{D}F(X)Y : Z = Y : \mathrm{D}F(X)Z \qquad \forall X, Y, Z \in \mathbb{R}^{N\times n},$$

i.e. $\mathrm{D}F(X)$ is a symmetric 4-tensor. Moreover, for every $X, Y \in \mathbb{R}^{N\times n}$, we have

$$|X|^{p-2}|Y|^2 \le \mathrm{D}F(X)Y : Y = (p-2)|X|^{p-4}(X : Y)^2 + |X|^{p-2}|Y|^2 \le (p-1)|X|^{p-2}|Y|^2.$$

In particular, if we assume $0 < \lambda \le |X| \le \Lambda$, then

$$\lambda^{p-2}|Y|^2 \le \mathrm{D}F(X)Y : Y \le (p-1)\Lambda^{p-2}|Y|^2, \tag{4.3}$$

so that, $DF(X)$ is elliptic in the Legendre sense (see [10]).
Now, fix $U \Subset \Omega \setminus \mathscr{C}$ and observe that u solves

$$\operatorname{div}(F(\mathrm{D}u)) = 0 \qquad \text{in } U.$$

For every fixed $U' \Subset U$ and $i \in \mathbb{N}$, the difference quotient of a function $f : U \to \mathbb{R}$

$$\mathrm{D}^i_h f(x) := \frac{f(x + he_i) - f(x)}{h}$$

is well defined on U', for every $h \in \mathbb{R}$ with $0 < |h| < \operatorname{dist}(U', \partial U)$.
By applying the difference quotient to the differential system solved by u, we have

$$\operatorname{div}(\mathrm{D}^i_h(F(\mathrm{D}u))) = 0 \qquad \text{in } U'.$$

For $x \in U'$, we write

$$\begin{aligned}\mathrm{D}^i_h(F(\mathrm{D}u)) &= \frac{1}{h}\int_0^1 \frac{d}{dt}\left[F\left(\mathrm{D}u(x) + th\mathrm{D}^i_h\mathrm{D}u(x)\right)\right] dt \\ &= \left(\int_0^1 \mathrm{D}F\left(\mathrm{D}u(x) + th\mathrm{D}^i_h\mathrm{D}u(x)\right) dt\right)\mathrm{D}^i_h\mathrm{D}u(x).\end{aligned}$$

Observe that

$$\mathbb{A}(x) := \int_0^1 \mathrm{D}F\left(\mathrm{D}u(x) + th\mathrm{D}^i_h\mathrm{D}u(x)\right) dt$$

is a symmetric 4-tensor field on U'. Let us prove that it is Legendre elliptic: first notice that there exists $\lambda > 0$ such that $|\mathrm{D}u| \geq \lambda$ in U'. In addition, since $u \in C^{1,\alpha}(U)$, for $|h|$ sufficiently small we have

$$|\mathrm{D}u(x + he_i) - \mathrm{D}u(x)| \leq \frac{\lambda}{2} \qquad \forall x \in U'.$$

Thus, for $t \in [0, 1]$, we estimate

$$|\mathrm{D}u(x) + th\mathrm{D}^i_h\mathrm{D}u(x)| = |\mathrm{D}u(x) + t(\mathrm{D}u(x + he_i) - \mathrm{D}u(x))| \geq \lambda - t\frac{\lambda}{2} \geq \frac{\lambda}{2} \qquad \forall x \in U'.$$

On the other hand,

$$|\mathrm{D}u(x) + th\mathrm{D}^i_h\mathrm{D}u(x)| \leq 3\|\mathrm{D}u\|_{L^\infty(U)} \qquad \forall x \in U'.$$

From (4.3), we deduce the Legendre ellipticity of $\mathbb{A}$ in U' (with ellipticity constants independent of h).
Therefore, we have proved that $w := \mathrm{D}^i_h\mathrm{D}u$ solves the linear elliptic system

$$\operatorname{div}(\mathbb{A}w) = 0 \qquad \text{in } U'.$$

By Schauder estimates [10, Thm. 5.19], since $\mathbb{A}$ has α-Hölder coefficients, we have $w \in C^{1,\alpha}(U')$. From the properties of the difference quotient, we infer $\mathrm{D}u \in C^{2,\alpha}(U')$. By a bootstrap argument and arbitrariness of U', we conclude $u \in C^\infty(\Omega \setminus \mathscr{C})$. □

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1:
First, we observe that if $p = 2$, the result is straightforward: every harmonic map $u : \Omega \to \mathbb{R}^N$ is smooth and satisfies $\Delta|\mathrm{D}u|^2 \geq 0$ on Ω. So, we can assume $p > 2$ in the following.

We divide the proof in two steps: firstly, we prove the gradient maximum principle outside the closed set $\mathscr{C} = \{x \in \Omega : \mathrm{D}u(x) = 0\}$; second, we show the argument for a generic open set $U \Subset \Omega$. Of course, we can assume that $\Omega \setminus \mathscr{C} \neq \varnothing$, otherwise u is constant on Ω.

Step 1: assume $U \Subset \Omega \setminus \mathscr{C}$.
In the following, the symbols $\cdot$ and $\langle \cdot, \cdot \rangle$ stand for the inner products in $\mathbb{R}^n$ and $\mathbb{R}^N$ respectively, while : stands for the Hilbert-Schmidt inner product in $\mathbb{R}^{N \times n}$.
By Lemma 4.3, we have $u \in C^\infty(\overline{U}, \mathbb{R}^N)$[1]. So, we can differentiate (4.1) with respect to x_i, obtaining

$$0 = \partial_i \left[\mathrm{div}\big(|\mathrm{D}u|^{p-2}\mathrm{D}u\big) \right] = \mathrm{div}\left((p-2)|\mathrm{D}u|^{p-4}(\mathrm{D}u : \mathrm{D}\partial_i u)\mathrm{D}u + |\mathrm{D}u|^{p-2}\mathrm{D}\partial_i u \right) \qquad \text{in } U.$$

By taking the inner product with $\partial_i u$, we have

$$\begin{aligned} 0 &= \left\langle \mathrm{div}\left((p-2)|\mathrm{D}u|^{p-4}(\mathrm{D}u : \mathrm{D}\partial_i u)\mathrm{D}u + |\mathrm{D}u|^{p-2}\mathrm{D}\partial_i u \right), \partial_i u \right\rangle \\ &= \mathrm{div}\left((p-2)|\mathrm{D}u|^{p-4}(\mathrm{D}u : \mathrm{D}\partial_i u)\langle \mathrm{D}u, \partial_i u\rangle + |\mathrm{D}u|^{p-2}\langle \mathrm{D}\partial_i u, \partial_i u\rangle \right) - \\ &\quad - (p-2)|\mathrm{D}u|^{p-4}(\mathrm{D}u : \mathrm{D}\partial_i u)^2 - |\mathrm{D}u|^{p-2}|\mathrm{D}\partial_i u|^2, \qquad \text{in } U. \end{aligned} \tag{4.4}$$

From (4.4), we deduce

$$\mathrm{div}\left((p-2)|\mathrm{D}u|^{p-4}(\mathrm{D}u : \mathrm{D}\partial_i u)\langle \mathrm{D}u, \partial_i u\rangle + |\mathrm{D}u|^{p-2}\langle \mathrm{D}\partial_i u, \partial_i u\rangle \right) \geq 0, \qquad \text{in } U. \tag{4.5}$$

By taking the sum over $i = 1, \dots, n$ in (4.5) and observing that

$$\sum_{i=1}^n (\mathrm{D}u : \mathrm{D}\partial_i u)\langle \mathrm{D}u, \partial_i u\rangle = \left(\mathrm{D}u^{\mathrm{T}}\mathrm{D}u\right) \cdot \langle \mathrm{D}^2 u, \mathrm{D}u\rangle,$$

we obtain

$$\mathrm{div}\left((p-2)|\mathrm{D}u|^{p-4}\left(\mathrm{D}u^{\mathrm{T}}\mathrm{D}u\right) \cdot \langle \mathrm{D}^2 u, \mathrm{D}u\rangle + |\mathrm{D}u|^{p-2}\langle \mathrm{D}^2 u, \mathrm{D}u\rangle \right) \geq 0, \qquad \text{in } U. \tag{4.6}$$

Now, we define

$$f := \frac{1}{p}|\mathrm{D}u|^p, \qquad \mathrm{A} := I + (p-2)\frac{\mathrm{D}u^{\mathrm{T}}\mathrm{D}u}{|\mathrm{D}u|^2},$$

where I is the identity matrix of $\mathbb{R}^{n \times n}$. Thus,

$$\mathrm{D}f = |\mathrm{D}u|^{p-2}\langle \mathrm{D}^2 u, \mathrm{D}u\rangle,$$

so that we can rewrite (4.6) as

$$\mathrm{div}(\mathrm{A}\mathrm{D}f) \geq 0, \qquad \text{in } U. \tag{4.7}$$

Since $u \in C^\infty(\overline{U}, \mathbb{R}^N)$, the function f is a classical subsolution to a linear elliptic partial differential inequality in divergence form, with smooth and bounded coefficients. By the maximum principle for elliptic PDEs, we have

$$\max_{\overline{U}} f = \max_{\partial U} f,$$

yielding (4.2).

Step 2: general case.
Let $U \Subset \Omega$ be an open subset. Then, without loss of generality, we can assume that $U \not\subset \mathscr{C}$. Indeed,

[1]By this notation, we mean that $u \in C^\infty(U', \mathbb{R}^N)$ for some $U \Subset U' \Subset \Omega \setminus \mathscr{C}$.

if $U \subset \mathscr{C}$, then also $\overline{U} \subset \mathscr{C}$, so that (4.2) trivially holds. Therefore, since $U \setminus \mathscr{C}$ is a non-empty open subset of Ω, there exists $\varepsilon > 0$ such that $U_\varepsilon := U \setminus \overline{\mathscr{C}^\varepsilon}$ is a non-empty open subset of Ω, where $\mathscr{C}^\varepsilon$ is the open ε-neighbourhood of $\mathscr{C}$. Thanks to *Step 1*, since $U_\varepsilon \Subset \mathbb{R}^n \setminus \mathscr{C}$, we have

$$\max_{\overline{U}_\varepsilon} |Du| = \max_{\partial U_\varepsilon} |Du|. \tag{4.8}$$

Now, set $M := \max_{\overline{U}} |\mathrm{D}u| > 0$. By possibly reducing ε in size, the continuity of $\mathrm{D}u$ ensures

$$\max_{\overline{\mathscr{C}^\varepsilon}} |\mathrm{D}u| < M. \tag{4.9}$$

Hence

$$M = \sup_{U} |\mathrm{D}u| = \max\left\{ \sup_{U_\varepsilon} |\mathrm{D}u|, \max_{\overline{\mathscr{C}^\varepsilon}} |\mathrm{D}u| \right\} = \sup_{U_\varepsilon} |\mathrm{D}u| = \max_{\overline{U}_\varepsilon} |\mathrm{D}u|.$$

In particular, from (4.8) we have

$$\max_{\partial U_\varepsilon} |Du| = M.$$

On the other hand, we can write

$$\partial U_\varepsilon = (\partial U \setminus \overline{\mathscr{C}^\varepsilon}) \cup (U \cap \partial \mathscr{C}^\varepsilon),$$

so that, using (4.9),

$$M = \sup_{\partial U_\varepsilon} |Du| = \max\left\{ \sup_{\partial U \setminus \overline{\mathscr{C}^\varepsilon}} |\mathrm{D}u|, \sup_{U \cap \partial \mathscr{C}^\varepsilon} |\mathrm{D}u| \right\} \leq \max\left\{ \max_{\partial U} |\mathrm{D}u|, \max_{\overline{\mathscr{C}^\varepsilon}} |\mathrm{D}u| \right\} = \max_{\partial U} |\mathrm{D}u|,$$

which ensures the validity of (4.2) . □

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