

An energy maximum principle for vectorial higher order absolute minimisers in L^∞

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Abstract

We show that vectorial absolute minimisers of general k -th order L^∞ supremal functionals in $W^{k,\infty}(\Omega, \mathbb{R}^N)$ satisfy a maximum principle of the form

$$\max_{\bar{U}} \mathbb{H}(\cdot, u, Du, \dots, D^k u) = \max_{\partial U} \mathbb{H}(\cdot, u, Du, \dots, D^k u), \quad \forall U \subseteq \Omega \text{ open,}$$

suitably interpreted. This is only necessary for absolute minimisers, while it characterises a relevant weaker notion of absolute minimality involving compactly supported variations. Further, we obtain an existence result to the Dirichlet problem for such absolute minimisers. Finally, via different methods, we establish a gradient maximum principle for p -harmonic maps for $p < \infty$.

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1 Introduction

For $n, N, k \in \mathbb{N}$, let $\Omega \Subset \mathbb{R}^n$ be an open bounded set, and let

$$\mathbb{H} : \Omega \times \left(\mathbb{R}^N \times \mathbb{R}^{Nn} \times \dots \times \mathbb{R}^{Nn^k} \right) \longrightarrow \mathbb{R}$$

be a $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^\Lambda)$ -measurable function on the product, where for brevity we symbolise $\mathbb{R}^\Lambda \cong \mathbb{R}^N \times \mathbb{R}^{Nn} \times \dots \times \mathbb{R}^{Nn^k}$ (namely $\Lambda \equiv \Lambda(n, N, k) = N(n^{k+1} - 1)(n - 1)^{-1}$). Here $\mathcal{L}(\Omega)$ is the Lebesgue σ -algebra of Ω and $\mathcal{B}(\mathbb{R}^\Lambda)$ is the Borel σ -algebra of \mathbb{R}^Λ . Consider the supremal functional

$$E_\infty(u, U) := \operatorname{ess\,sup}_{x \in U} \mathbb{H}(x, D^{[k]}u(x)), \quad (1.1)$$

defined for $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$ and $U \subseteq \Omega$ an open subset. In the above, $D^{[k]}u := (u, Du, \dots, D^k u)$ denotes the *jet of order k* of u . One of the main aims of this paper is to establish a maximum principle property for the absolute minimisers of the functional E_∞ . We recall that $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$ is an *absolute minimiser* of E_∞ on Ω if

$$E_\infty(u, U) \leq E_\infty(u + \phi, U), \quad \forall \phi \in W_0^{k,\infty}(U, \mathbb{R}^N), \quad \forall U \subseteq \Omega \text{ open.} \quad (1.2)$$

We observe that this notion is slightly stronger than the standard absolute minimality commonly found in the literature, where minimality is required only on relatively compact open subsets $U \Subset \Omega$. Therefore, under relatively mild assumptions on the supremand \mathbb{H} , we prove that for an absolute minimiser the essential supremum in (1.1) is “attained on the boundary of U ”. In order to give a

rigorous meaning to this expression for general $W^{k,\infty}$ mappings, we will utilise the *essential limsup* of $H(\cdot, D^{[k]}u)$, in order to define the supremum of the supremand on ∂U . More generally, we define

$$H(x, D^{[k]}u(x))^* := \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x) \cap U} H(\cdot, D^{[k]}u), \quad \forall x \in \bar{U}. \quad (1.3)$$

The following is the notion of maximum principle that we will utilise herein for (absolute) minimisers of the supremal functional E_∞ .

Definition 1.1 (Energy maximum principle). *We say that $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$ satisfies the energy maximum principle for E_∞ in Ω if*

$$\operatorname{ess\,sup}_U H(\cdot, D^{[k]}u) = \sup_{\partial U} H(\cdot, D^{[k]}u)^*, \quad \forall U \subseteq \Omega \text{ open.}$$

Evidently, if $u \in C^k(\bar{\Omega}, \mathbb{R}^N)$ and $H \in C(\bar{\Omega} \times \mathbb{R}^\Lambda)$, the above reduces to the more classical pointwise statement

$$\max_{\bar{U}} H(\cdot, D^{[k]}u) = \max_{\partial U} H(\cdot, D^{[k]}u), \quad \forall U \subseteq \Omega \text{ open.}$$

Our first main result establishes that *this energy maximum principle property is satisfied by absolute minimisers of E_∞* . More precisely, this property holds on any set $U \subseteq \Omega$ for which u is a minimiser of $E_\infty(\cdot, U)$ with respect to its own boundary data. However, simple examples show that *it is not sufficient* for the minimality; for instance, the cone function $u(x) = |x|$ belongs to $W^{1,\infty}(\mathbb{R}^n)$ and clearly satisfies the energy (or, equivalently, gradient) maximum principle for $E_\infty(u, \Omega) := \operatorname{ess\,sup}_\Omega |Du|$, because $|Du| = 1$ a.e. on \mathbb{R}^n . But u is not a minimiser w.r.t. to its own data on any open set $U \subseteq \Omega$ which contains the origin (the vertex of the cone). In particular, u is not an absolute minimiser if Ω contains the origin. This raises the question whether we can characterise the class of maps that satisfy the energy maximum principle. We answer this affirmatively, by establishing that the energy maximum principle is a property that characterises a class of maps which is larger than the one of absolute minimisers, and are the ones that minimise $E_\infty(\cdot, U)$ with respect to *compactly supported variations* in U , for every $U \subseteq \Omega$.

Before stating our main results precisely, we underline that E_∞ in (1.1) is the general object of study in the L^∞ -Calculus of Variations, a field initiated by Aronsson in [4]. His pioneering work on the scalar first order case (namely when $N = k = 1$) has been well developed by now, and most challenges have been thoroughly analysed and understood (see [23] for a survey reference). More recently, the vectorial case ($N > 1$) and the higher order case ($k > 1$) have been approached respectively in [21] and [28, 33]. In these contexts, a complete theory is still far from reach. Without any pretension of being exhaustive, we refer also to [24, 25, 29] for a glimpse of the literature on vectorial first order problems and to [11, 16, 30–32] for higher order ones. Furthermore, the fractional order case ($k \notin \mathbb{N}$) has been recently explored in [12]. It is also appropriate pointing out that results concerning standard maximum and comparison principles for scalar first order absolute minimisers can be found for instance in [3, 20]. Lastly, further related results in the area appear in the following interesting papers: [2, 5, 7, 9, 10, 13, 34–36, 38–40].

It is worth mentioning that, given boundary data $u_0 \in W^{k,\infty}(\Omega, \mathbb{R}^N)$, the existence (and/or uniqueness) of absolute minimisers to the Dirichlet problem in $W_{u_0}^{k,\infty}(\Omega, \mathbb{R}^N) := u_0 + W^{k,\infty}(\Omega, \mathbb{R}^N)$ associated to E_∞ in (1.1) is an open problem already when $k = 1$ and $n, N \geq 2$ or when $N = 1$ and $n, k \geq 2$. There are some exceptions in the second order case, whenever H has special dependence in the second derivatives of u : relevant works are [11, 28, 31, 33]; surprisingly, similar results hold true also in the fractional case [12].

Now we proceed to specify the assumptions on the function H . By denoting $X = (X_0, X_1, \dots, X_k) \in \mathbb{R}^N \times \mathbb{R}^{Nn} \times \dots \times \mathbb{R}^{Nn^k} = \mathbb{R}^\Lambda$, we assume that $H : \Omega \times \mathbb{R}^\Lambda \rightarrow \mathbb{R}$ is a Carathéodory function (namely

measurable with respect to the first variable and continuous with respect to the second), and also *strictly radially increasing* on \mathbb{R}^Λ , *essentially uniformly* with respect to a.e. $x \in \Omega$. In symbols, we assume that there exists a continuous function $c : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$, with $c(1, \cdot) = c(\cdot, 0) = 0$ and $c > 0$ in $(0, 1) \times (0, \infty)$, such that

$$\mathbf{H}(x, tX) \leq \mathbf{H}(x, X) - c(t, |X|), \quad \forall t \in [0, 1], \quad \forall X \in \mathbb{R}^\Lambda, \quad \text{a.e. } x \in \Omega. \quad (1.4)$$

In other words, $\mathbf{H}(x, \cdot)$ is a strictly radially increasing function with a modulus of monotonicity independent of x . Notice that the radial monotonicity implies that, for almost every $x \in \Omega$, $\mathbf{H}(x, \cdot)$ has a global minimum at $X = 0$. Without loss of generality we can assume that $\mathbf{H}(x, 0) = 0$ a.e. $x \in \Omega$, which implies that $\mathbf{H} \geq 0$. In this case, we can therefore write

$$\mathbf{E}_\infty(u, U) = \|\mathbf{H}(\cdot, \mathbf{D}^{[k]}u)\|_{L^\infty(U)}, \quad \text{for every } U \subseteq \Omega \text{ open.}$$

The strict radial monotonicity assumption may seem restrictive at first, however, the validity of our result extends to a wider class of supremands, even discontinuous ones: indeed, for any function $g : [0, \infty) \rightarrow [0, \infty)$ which is lower semicontinuous and non-decreasing, set $G = g \circ \mathbf{H}$ and $\tilde{\mathbf{E}}_\infty(u, U) := \text{ess sup}_U G(\cdot, \mathbf{D}^{[k]}u)$. It is not difficult to see that \mathbf{E}_∞ and $\tilde{\mathbf{E}}_\infty$ share the same minimisers. In addition, if $u \in W^{k, \infty}(\Omega, \mathbb{R}^N)$ satisfies the energy maximum principle for \mathbf{E}_∞ in Ω , then for every $U \subseteq \Omega$ open

$$\tilde{\mathbf{E}}_\infty(u, U) = g \left(\text{ess sup}_U \mathbf{H}(\cdot, \mathbf{D}^{[k]}u) \right) = g \left(\text{ess sup}_{\partial U} \mathbf{H}(\cdot, \mathbf{D}^{[k]}u)^* \right) = \left(\text{ess sup}_{\partial U} G(\cdot, \mathbf{D}^{[k]}u)^* \right),$$

i.e. u satisfies the energy maximum principle for $\tilde{\mathbf{E}}_\infty$ as well. Our second assumption on \mathbf{H} concerns the essential uniformity of its modulus of continuity, with respect to $x \in \Omega$. More precisely, we assume that for any $R > 0$, exists a non-decreasing $\omega_R \in C(0, \infty)$ such that $\omega_R(0^+) = 0$ and

$$|\mathbf{H}(x, X) - \mathbf{H}(x, X')| \leq \omega_R(|X - X'|), \quad \text{a.e. } x \in \Omega, \quad \forall X \in \overline{\mathbb{B}}_R(0). \quad (1.5)$$

Since $\mathbf{H}(x, \cdot)$ is continuous in \mathbb{R}^Λ , it is uniformly continuous on every closed ball $\overline{\mathbb{B}}_R(0) \subseteq \mathbb{R}^\Lambda$. Thus, we essentially require that the corresponding modulus of continuity on $\overline{\mathbb{B}}_R(0)$ is independent of $x \in \Omega$.

Now we may state our first main result.

Theorem 1.2 (Energy maximum principle for absolute minimisers). *Let $\Omega \Subset \mathbb{R}^n$ be a bounded open set and let $\mathbf{H} : \Omega \times \mathbb{R}^\Lambda \rightarrow [0, \infty)$ be a Carathéodory function satisfying (1.4) and (1.5), with $\mathbf{H}(x, 0) = 0$. Assume that $u \in W^{k, \infty}(\Omega, \mathbb{R}^N)$ is an absolute minimiser of the functional \mathbf{E}_∞ , defined in (1.1). Then u satisfies the energy maximum principle in Ω , namely*

$$\text{ess sup}_U \mathbf{H}(\cdot, \mathbf{D}^{[k]}u) = \sup_{\partial U} \mathbf{H}(\cdot, \mathbf{D}^{[k]}u)^*, \quad \forall U \subseteq \Omega \text{ open.} \quad (1.6)$$

Here $\mathbf{H}(\cdot, \mathbf{D}^{[k]}u)^*$ is the essential limsup of $\mathbf{H}(\cdot, \mathbf{D}^{[k]}u)$, defined in (1.3).

The reader should bear in mind that, more generally, the proof of Theorem 1.2 establishes that the property in (1.6) remains valid for every fixed open set $U \subseteq \Omega$ such that u is a minimiser for $\mathbf{E}_\infty(\cdot, U)$ among all competitors $v \in W_u^{k, \infty}(U, \mathbb{R}^N) := u + W_0^{k, \infty}(U, \mathbb{R}^N)$. Explicitly, we have the following more general result.

Proposition 1.3. *Let $U \Subset \mathbb{R}^n$ be a bounded open set and let \mathbf{E}_∞ be as in (1.1). Let also $\mathbf{H} : U \times \mathbb{R}^\Lambda \rightarrow [0, \infty)$ be a Carathéodory function satisfying (1.4) and (1.5), with $\mathbf{H}(x, 0) = 0$. Assume that $u \in W^{k, \infty}(U, \mathbb{R}^N)$ is such that*

$$\mathbf{E}_\infty(u, U) \leq \mathbf{E}_\infty(u + \phi, U), \quad \forall \phi \in W_0^{k, \infty}(U, \mathbb{R}^N).$$

Then

$$\operatorname{ess\,sup}_U \mathbf{H}(\cdot, \mathbf{D}^{[k]}u) = \sup_{\partial U} \mathbf{H}(\cdot, \mathbf{D}^{[k]}u)^*.$$

In view of the Proposition above, the energy maximum principle can be equivalently interpreted as stating that the *attainment set* $U(u)$ of a minimiser u must contain the boundary of U . The attainment set $U(u)$ is the collection of all points $x \in \bar{U}$ where $\mathbf{H}(x, \mathbf{D}^{[k]}u(x)) = \operatorname{ess\,sup}_U \mathbf{H}(\cdot, \mathbf{D}^{[k]}u)$, and it is well defined and non-empty in case $u \in C^k(\bar{U}, \mathbb{R}^N)$ and $\mathbf{H} \in C(\bar{U} \times \mathbb{R}^\Lambda)$. Since we will not exploit this point of view any further in this paper, we refrain from providing any further details. For a definition of $U(u)$ in the non-smooth setting we refer to [9], where the authors show also a minimality property of $U(u)$ for absolute minimisers in the scalar first order case.

Before stating our second result, we give the definition of absolute minimiser with respect to compactly supported variations.

Definition 1.4 (The class $\operatorname{AM}_c(\Omega, \mathbb{R}^N)$). *We say that $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$ is an absolute minimiser with respect to compactly supported variations, and we write $u \in \operatorname{AM}_c(\Omega, \mathbb{R}^N)$, if*

$$\mathbf{E}_\infty(u, U) \leq \mathbf{E}_\infty(u + \phi, U), \quad \forall \phi \in W_c^{k,\infty}(U, \mathbb{R}^N), \quad \forall U \subseteq \Omega \text{ open.} \quad (1.7)$$

As a consequence of Theorem 1.2, we obtain the following characterisation.

Theorem 1.5 (Characterisation of $\operatorname{AM}_c(\Omega, \mathbb{R}^N)$ via the energy maximum principle). *Let $\Omega \Subset \mathbb{R}^n$ be a bounded open set and let $\mathbf{H} : \Omega \times \mathbb{R}^\Lambda \rightarrow [0, \infty)$ be a Carathéodory function satisfying (1.4) and (1.5), with $\mathbf{H}(x, 0) = 0$. Consider the supremal functional \mathbf{E}_∞ , defined in (1.1). Then, for any map $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$, the following statements are equivalent:*

- (i) *u satisfies the energy maximum principle for \mathbf{E}_∞ in Ω , i.e. (1.6) holds true;*
- (ii) *u is an absolute minimiser w.r.t. compactly supported variations, i.e. (1.7) holds true.*

The proof of Theorems 1.2 and 1.5 are given in Section 3, preceded by a short preparatory section. In the latter, we recall the notions of strict radial monotonicity and essential limsup, along with some of their fundamental properties.

Finally, we consider the problem of proving a corresponding maximum principle result for the case of $p < \infty$. In the integral case the situation is quite different conceptually, and therefore for simplicity we restrict our attention to the case of $k = 1$ and \mathbf{H} being the Euclidean norm on the matrix space $\mathbb{R}^{N \times n}$. To this aim, we establish a gradient maximum principle for p -harmonic maps. To the best of our knowledge, a result of this kind is not known in the literature, at least in the vectorial case. In the scalar case, a gradient maximum principle for quasilinear elliptic equations can be found in [17, Chapter 15]. We refer also to [37, Appendix B] regarding gradient maximum and minimum principles for p -harmonic potentials on convex rings in the plane. In the case of general p -harmonic maps, however, one can only expect a gradient maximum principle to hold, since a minimum principle is not true in general, as existing examples show (see [6]).

Our third and final main result is therefore the following.

Theorem 1.6. *Let $\Omega \subseteq \mathbb{R}^n$ an open set and $p \in [2, \infty)$. Assume that $u : \Omega \rightarrow \mathbb{R}^N$ is a p -harmonic map, i.e. $u \in W_{\text{loc}}^{1,p}(\Omega, \mathbb{R}^N)$ satisfies*

$$\operatorname{div}(|Du|^{p-2}Du) = 0 \quad \text{in } \Omega, \quad (1.8)$$

in the sense of distributions. Then, for every open set $U \Subset \Omega$, we have

$$\max_{\bar{U}} |Du| = \max_{\partial U} |Du|. \quad (1.9)$$

We note that the above maxima are well defined since $u \in C_{\text{loc}}^{1,\alpha}(\Omega, \mathbb{R}^N)$, for some $\alpha > 0$, due to known regularity results for p -harmonic maps [41]. The methods used therein are genuinely different from the ones utilised to prove the previous results, and are based on the theory of elliptic PDEs.

Finally, we note that, even if we restrict Theorem 1.2 to the special case of $k = 1$ and \mathbb{H} being the Euclidean norm, which leads to $E_\infty(u, U) = \|Du\|_{L^\infty(U)}$, it still cannot be deduced from Theorem 1.6 by passing to the limit as $p \rightarrow \infty$. Indeed, it is not known if the gradient of vector valued absolute minimisers of E_∞ can be approximated locally uniformly by p -harmonic maps (and indeed this cannot be the case unless ∞ -harmonic maps are C^1 , which is known only in the scalar case and in two dimensions [18]). More generally, the lack of good approximation results for absolute minimisers via p -harmonic maps is one of the reasons why existence of absolute minimisers is still open in the vectorial case of $\min\{n, N\} \geq 2$. If $n = 1$ or $N = 1$, then special methods can be applied (see e.g. [1, 8, 26]).

Notwithstanding, unlike the case of general vectorial absolute minimisers, it is not difficult to establish existence results to the higher order Dirichlet problem for absolute minimisers with respect to compactly supported variations. The construction technique is based on the Dacorogna-Marcellini Baire Category method [15]. However, there is no hope for a uniqueness result in this class of variations, as this method (which can be seen as an analytic alternative to Gromov's convex integration), always yields infinitely-many solutions. Accordingly, as a consequence of Theorem 1.5, we have the following result.

Corollary 1.7 (Existence and non-uniqueness in $\text{AM}_c(\Omega, \mathbb{R}^N)$). *Let $\Omega \Subset \mathbb{R}^n$ be a bounded open set and let $\mathbb{H} : \Omega \times \mathbb{R}^\Lambda \rightarrow [0, \infty)$ be a continuous function satisfying (1.4) and (1.5), with $\mathbb{H}(x, 0) = 0$. Assume also that $\mathbb{H}(x, X_0, \dots, X_{k-1}, \cdot)$ is convex in \mathbb{R}^{Nn^k} . Consider the functional E_∞ , defined in (1.1) and let $u_0 \in W^{k,\infty}(\Omega, \mathbb{R}^N)$. Then, there exist infinitely many mappings $u \in W_{u_0}^{k,\infty}(\Omega, \mathbb{R}^N) \cap \text{AM}_c(\Omega, \mathbb{R}^N)$.*

The convexity assumption on \mathbb{H} in the last variable can be relaxed to Morrey quasiconvexity, provided that the boundary data u_0 is piecewise C^k (see [27]). However, we underline that the maps u constructed in Corollary 1.7 are not absolute minimisers in the sense of Definition 1.2. In fact, they are not global minimisers, since their energy is strictly greater than the minimal value of $E_\infty(\cdot, \Omega)$ in $W_{u_0}^{k,\infty}(\Omega, \mathbb{R}^N)$. On the other hand, one may wonder if a global minimiser which satisfies the energy maximum principle (or, equivalently, which is also an AM_c) is an absolute minimiser. The answer to this question is negative, as the following simple example shows.

Example 1.8. Let $n = N = 1$ and $\Omega = (-1, 0) \cup (0, 1)$. Consider the minimum problem for

$$E_\infty(u, U) := \|u'\|_{L^\infty(U)}, \quad U \subseteq \Omega,$$

among functions $u \in W^{1,\infty}(U)$ with Dirichlet boundary conditions

$$u(-1) = u(0) = 0, \quad u(1) = 1.$$

The McShane-Whitney extensions u^\pm are both global minimisers and they satisfy the energy maximum principle in Ω , since $|u^\pm| = 1$ a.e. in Ω . However, they differ from the absolute minimiser u , which clearly satisfies $u = 0$ in $(-1, 0)$.

2 Preparatory tools

2.1 On radial monotonicity

In this first preliminary section, we recall the notions of weak and strict radial monotonicity for a continuous function on \mathbb{R}^Λ , together with some properties and relevant examples.

Definition 2.1. A continuous function $h : \mathbb{R}^\Lambda \rightarrow \mathbb{R}$ is said to be weakly (resp. strictly) radially increasing if for every $X \in \mathbb{R}^\Lambda \setminus \{0\}$ the function $t \mapsto h(tX)$ is weakly (resp. strictly) increasing in $[0, \infty)$.

Similarly to Example 2.3 below, it can be shown that any level convex function with global minimum at the origin is weakly radially increasing. Of course, a weakly radially increasing function need not be level convex, since its level subsets only need to be star-shaped with respect to the origin.

A continuous function which is strictly radially increasing possesses a modulus of monotonicity, as we can show in the following.

Proposition 2.2. Let $h : \mathbb{R}^\Lambda \rightarrow \mathbb{R}$ be continuous. Then h is strictly radially increasing if and only if there exists a continuous function $c : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$, with $c(1, \cdot) = c(\cdot, 0) = 0$ and $c > 0$ in $(0, 1) \times (0, \infty)$, such that

$$h(tX) \leq h(X) - c(t, |X|) \quad \forall t \in [0, 1] \quad \forall X \in \mathbb{R}^\Lambda.$$

Proof. Since the strict radial monotonicity for h reads as

$$t \in (0, 1), X \in \mathbb{R}^\Lambda \setminus \{0\} \implies h(tX) < h(X), \quad (2.1)$$

it is evident that the existence of a function c as above implies that h is strictly radially increasing. On the other hand, if h is strictly radially increasing, let us define $c : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ by

$$c(t, s) := \inf\{h(X) - h(tX) : |X| = s, X \in \mathbb{R}^\Lambda\}.$$

Then, c is continuous, $c(1, s) = 0$ for all $s \in [0, \infty)$, and $c(t, 0) = 0$ for all $t \in [0, 1]$. Moreover, (2.1) and the compactness of the s -spheres in \mathbb{R}^Λ imply $c(t, s) > 0$ for all $t \in (0, 1)$ and $s > 0$. \square

We note that the existence of a modulus of monotonicity naturally leads to, and justifies, Definition 1.4.

Example 2.3. Relevant examples of strictly radially increasing functions are:

- Any strictly level convex function with global minimum at the origin. Indeed, a function $h : \mathbb{R}^\Lambda \rightarrow \mathbb{R}$ is said to be strictly level convex¹ if

$$h(tX + (1-t)Y) < \max\{h(X), h(Y)\} \quad \forall X, Y \in \mathbb{R}^\Lambda : X \neq Y, \quad \forall t \in (0, 1).$$

By choosing $Y = 0$, we obtain

$$h(tX) < h(X) \quad \forall X \in \mathbb{R}^\Lambda \setminus \{0\} \quad \forall t \in (0, 1),$$

which is equivalent to strict radial monotonicity (see (2.1)).

- Any convex C^1 -function h with $h(0) = 0 < h(X)$ for all $X \neq 0$. Indeed, by [14, Thm 2.52], we have $Dh(X) \cdot X \geq h(X)$ for all $X \in \mathbb{R}^\Lambda$, i.e. the radial speed of h has a modulus of positivity given by h itself. In particular, h is weakly radially increasing. By integration, for $t \in [0, 1]$ and $X \in \mathbb{R}^\Lambda$, we obtain

$$\begin{aligned} h(X) - h(tX) &= \int_t^1 Dh(sX) \cdot X ds \\ &\geq \int_t^1 \frac{1}{s} h(sX) ds \\ &\geq (1-t)h(tX) \\ &\geq (1-t) \min_{|Y|=t|X|} h(Y) =: c(t, |X|). \end{aligned}$$

¹Note that the notion of (strict) level convexity is called (strict) quasi-convexity in [8].

The function c is a modulus of monotonicity in the sense of Proposition 2.2.

- The cone function h associated to a bounded open set $S \subseteq \mathbb{R}^\Lambda$ star shaped with respect to the origin. This can be seen in a similar way to the previous example, by simply observing that since h is 1-homogeneous, one has $\frac{d}{dt}h(tX) = h(X)$ for all $X \in \mathbb{R}^\Lambda$.

2.2 Properties of the essential limsup

In this second preliminary section, we give the definition of essential limsup, together with some properties.

Definition 2.4 (Essential limsup). *Let $U \subseteq \mathbb{R}^n$ be a Borel set and $f \in L^\infty(U)$. We say that the function $f^* \in L^\infty(U)$, defined as*

$$f^*(x) := \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x) \cap U} f \quad \forall x \in U,$$

is the essential limsup of f .

Of course, if U is open and f is continuous in U , then $f^* = f$. However, in the more general setting of Definition 2.4, we may have $f \neq f^*$ on a set of positive measure: for instance, consider $f = \mathbb{1}_{\mathbb{R}^n \setminus K}$ for a nowhere dense compact set $K \subseteq \mathbb{R}^n$, with $\mathcal{L}^n(K) > 0$. In general, from [22, Proposition 9], we have that $f \leq f^*$ almost everywhere and f^* is upper semicontinuous on U . Moreover, again by [22, Proposition 9], the definition of f^* allows us to give a pointwise meaning to the essential supremum, indeed for any Borel set $U \subseteq \mathbb{R}^n$ we have

$$\sup_U f^* = \operatorname{ess\,sup}_U f. \quad (2.2)$$

For a set $E \subseteq \mathbb{R}^n$ and $\rho > 0$, we denote by E^ρ the open ρ -neighbourhood of E , namely

$$E^\rho := \{x \in \mathbb{R}^n : \operatorname{dist}(x, E) < \rho\}.$$

The following lemma is in order.

Lemma 2.5. *Let $U \subseteq \mathbb{R}^n$ be a bounded open set. Then, for any $K \subseteq \partial U$ compact subset, we have*

$$\sup_K f^* = \lim_{\rho \rightarrow 0} \operatorname{ess\,sup}_{K^\rho \cap U} f.$$

Proof. We start with noticing that for every $x \in K$ there exists $\rho > 0$ such that $B_\rho(x) \cap U \subseteq K^\rho \cap U$, so that

$$\operatorname{ess\,sup}_{B_\rho(x) \cap U} f \leq \operatorname{ess\,sup}_{K^\rho \cap U} f.$$

By passing to the limit as $\rho \rightarrow 0$ in both sides and taking the supremum over $x \in K$ in the left hand side, we obtain

$$\sup_K f^* \leq \lim_{\rho \rightarrow 0} \operatorname{ess\,sup}_{K^\rho \cap U} f.$$

It remains to prove the opposite inequality. By definition 2.4, for any fixed $x \in K$ and $\sigma > 0$, there exists $\rho_{x,\sigma} > 0$ such that

$$f^*(x) \geq \operatorname{ess\,sup}_{B_{\rho_{x,\sigma}}(x) \cap U} f - \sigma. \quad (2.3)$$

The family of balls $\{B_{\rho_{x_i}, \sigma}(x_i); x_i \in K\}$ is an open covering of K . Thus, we can extract a finite subcovering $\{B_{\rho_{x_i}, \sigma}(x_i); x_i \in K\}_{i=1, \dots, m}$. For brevity, let us denote $B_{i, \sigma} = B_{\rho_{x_i}, \sigma}(x_i)$.

We claim that there exists $\rho_\sigma > 0$ such that $K^{\rho_\sigma} \subseteq \bigcup_{i=1}^m B_{i, \sigma}$.

By contradiction, suppose that for every $j \in \mathbb{N}$ there exists $y_j \in K^{1/j} \setminus \bigcup_{i=1}^m B_{i, \sigma}$. This is equivalent to say that

$$\forall j \in \mathbb{N} \quad \exists y_j \in \mathbb{R}^n : \begin{cases} y_j \notin B_{i, \sigma} & \forall i = 1, \dots, m, \\ \text{dist}(y_j, K) \leq 1/j. \end{cases} \quad (2.4)$$

Let $\bar{y}_j \in K$ be a point realising $d(y_j, K)$. Since K is compact, there exists a subsequence $(\bar{y}_{j_k}) \subseteq (\bar{y}_j)$ such that $\bar{y}_{j_k} \rightarrow \bar{x}$ as $k \rightarrow \infty$, for some $\bar{x} \in K$. Moreover, since $\text{dist}(y_{j_k}, \bar{y}_{j_k}) \leq 1/j_k$, we have also that $y_{j_k} \rightarrow \bar{x}$ as $k \rightarrow \infty$. But being $\{B_{i, \sigma}\}_{i=1, \dots, m}$ a covering of K , we must have that eventually $y_{j_k} \in B_{i, \sigma}$ for some $i \in \{1, \dots, m\}$, which contradicts (2.4).

Now, define $U_\sigma = \bigcup_{i=1}^m (B_{i, \sigma} \cap U)$ and notice that $K^{\rho_\sigma} \cap U \subseteq U_\sigma$. Thus, using also (2.3), we have

$$\begin{aligned} \lim_{\rho \rightarrow 0} \text{ess sup}_{K^\rho \cap U} f &\leq \text{ess sup}_{K^{\rho_\sigma} \cap U} f \\ &\leq \text{ess sup}_{U_\sigma} f \\ &\leq \max_{i=1, \dots, m} \text{ess sup}_{B_{i, \sigma} \cap U} f \\ &\leq \max_{i=1, \dots, m} f^*(x_i) + \sigma \\ &\leq \sup_K f^* + \sigma. \end{aligned}$$

By letting $\sigma \rightarrow 0$, we conclude the proof. \square

3 Proofs of the main results

This section is devoted to the proof of Theorems 1.2 and 1.5. The proof of the latter is basically a consequence of the argument in the proof of the former. Indeed, the key idea in the proof of Theorem 1.2 is to find, for every $U \subseteq \Omega$, a suitable competitor in $u_\lambda \in W_u^{k, \infty}(U, \mathbb{R}^N)$ to compare with the absolute minimiser u . Actually, it turns out that $u_\lambda \in u + W_c^{k, \infty}(U, \mathbb{R}^N)$, and this will be the crucial observation in the proof of Theorem 1.5.

We start with proving Theorem 1.2.

Proof of Theorem 1.2: Fix $U \subseteq \Omega$ an open subset. First, notice that for every $x \in \partial U$ and every $\rho > 0$, we clearly have

$$\text{ess sup}_{B_\rho(x) \cap U} \mathbf{H}(\cdot, D^{[k]}u) \leq \text{ess sup}_U \mathbf{H}(\cdot, D^{[k]}u).$$

From Definition 2.4, by passing to the limit as $\rho \rightarrow 0$ and then to the supremum over all $x \in \partial U$ in the left hand side, we obtain

$$\sup_{\partial U} \mathbf{H}(\cdot, D^{[k]}u)^* \leq \text{ess sup}_U \mathbf{H}(\cdot, D^{[k]}u). \quad (3.1)$$

Let us prove the opposite inequality. To this purpose we need to use assumptions (1.4) and (1.5). We start by setting

$$M := \text{ess sup}_U \mathbf{H}(\cdot, D^{[k]}u). \quad (3.2)$$

If $M = 0$, there is nothing to prove, so we can assume that $M > 0$. For the sake of contradiction, assume that (3.1) holds strictly. Then, by applying Lemma 2.5 to $K = \partial U$, one can find $\delta > 0$ such that

$$\lim_{\rho \rightarrow 0} \operatorname{ess\,sup}_{(\partial U)^\rho \cap U} \mathsf{H}(\cdot, \mathsf{D}^{[k]}u) + 2\delta \leq M.$$

In particular, there exists $\varepsilon > 0$ such that

$$\operatorname{ess\,sup}_{(\partial U)^\varepsilon \cap U} \mathsf{H}(\cdot, \mathsf{D}^{[k]}u) + \delta \leq M. \quad (3.3)$$

By possibly reducing ε in size, we can choose a function $\varphi_\varepsilon \in C^\infty(\mathbb{R}^n)$, with $0 \leq \varphi_\varepsilon \leq 1$, such that $\varphi_\varepsilon = 0$ in $U \setminus (\partial U)^\varepsilon$ and $\varphi_\varepsilon = 1$ in $(\partial U)^{\varepsilon/2}$. Now, for $\lambda \in (0, 1)$, set

$$u_\lambda := \lambda u + (1 - \lambda)\varphi_\varepsilon u.$$

Notice that $u_\lambda \in W^{k,\infty}(U, \mathbb{R}^N)$. Moreover, $u_\lambda = u$ on $(\partial U)^{\varepsilon/2} \cap U$, so that

$$u_\lambda \in u + W_c^{k,\infty}(U, \mathbb{R}^N) \subseteq W_u^{k,\infty}(U, \mathbb{R}^N). \quad (3.4)$$

Let us compute $\mathsf{D}^{[k]}u_\lambda$. For every $h = 1, \dots, k$, we have

$$\mathsf{D}^h u_\lambda = \lambda \mathsf{D}^h u + (1 - \lambda)\mathsf{D}^h(\varphi_\varepsilon u). \quad (3.5)$$

By the Leibniz formula,

$$\partial_{i_1, \dots, i_h}^h(\varphi_\varepsilon u) = \sum_{j=0}^h \binom{h}{j} (\partial_{i_{j+1}, \dots, i_h}^{h-j} \varphi_\varepsilon) (\partial_{i_1, \dots, i_j}^j u).$$

In particular, for every $h = 0, \dots, k$, we can estimate

$$|\mathsf{D}^h(\varphi_\varepsilon u)| \leq 2^h \|\varphi_\varepsilon\|_{W^{h,\infty}(U)} \|u\|_{W^{h,\infty}(U, \mathbb{R}^N)} \quad \text{in } U,$$

which gives

$$|\mathsf{D}^{[k]}(\varphi_\varepsilon u)| \leq 2^{k+1} \|\varphi_\varepsilon\|_{W^{k,\infty}(U)} \|u\|_{W^{k,\infty}(U, \mathbb{R}^N)} \quad \text{in } U. \quad (3.6)$$

Putting together (3.5) and (3.6), we obtain

$$|\mathsf{D}^{[k]}u_\lambda - \lambda \mathsf{D}^{[k]}u| \leq (1 - \lambda) 2^{k+1} \|\varphi_\varepsilon\|_{W^{k,\infty}(U)} \|u\|_{W^{k,\infty}(U, \mathbb{R}^N)} \quad \text{in } U. \quad (3.7)$$

In particular, if we set

$$C_1 = C_1(\varepsilon, k, \lambda) = (1 - \lambda) 2^{k+1} \|\varphi_\varepsilon\|_{W^{k,\infty}(U)} + \lambda,$$

then

$$|\mathsf{D}^{[k]}u_\lambda| \leq C_1 \|u\|_{W^{k,\infty}(U, \mathbb{R}^N)} \quad \text{in } U.$$

Now, we recall assumption (1.5) and apply it to

$$R := C_1 \|u\|_{W^{k,\infty}(U, \mathbb{R}^N)}, \quad \mathsf{D}^{[k]}u_\lambda(x), \lambda \mathsf{D}^{[k]}u(x) \in \overline{\mathbb{B}}_R(0) \quad \text{for a.e. } x \in U,$$

obtaining the existence of a continuous function $\omega = \omega_R : (0, \infty) \rightarrow (0, \infty)$, non-decreasing, with $\omega(0^+) = 0$, such that

$$|\mathsf{H}(x, \mathsf{D}^{[k]}u_\lambda(x)) - \mathsf{H}(x, \lambda \mathsf{D}^{[k]}u(x))| \leq \omega(|\mathsf{D}^{[k]}u_\lambda(x) - \lambda \mathsf{D}^{[k]}u(x)|) \quad \text{a.e. } x \in U.$$

From (3.7) and the monotonicity of ω , setting

$$C_2 = C_2(\varepsilon, k) = 2^{k+1} \|\varphi_\varepsilon\|_{W^{k,\infty}(U)} \|u\|_{W^{k,\infty}(U, \mathbb{R}^N)},$$

we get

$$|\mathbf{H}(\cdot, \mathbf{D}^{[k]}u_\lambda) - \mathbf{H}(\cdot, \lambda \mathbf{D}^{[k]}u)| \leq \omega((1-\lambda)C_2) \quad \text{a.e. in } U.$$

By restricting this estimate to $(\partial U)^\varepsilon \cap U$ and passing to the essential supremum on this set, we have

$$\begin{aligned} \operatorname{ess\,sup}_{(\partial U)^\varepsilon \cap U} \mathbf{H}(\cdot, \mathbf{D}^{[k]}u_\lambda) &\leq \operatorname{ess\,sup}_{(\partial U)^\varepsilon \cap U} \mathbf{H}(\cdot, \lambda \mathbf{D}^{[k]}u) + \omega((1-\lambda)C_2) \\ &\leq M - \delta + \omega((1-\lambda)C_2), \end{aligned}$$

where in the last inequality we have applied (3.3). Now, for λ sufficiently close to 1, by continuity of ω , we have $\omega((1-\lambda)C_2) \leq \delta/2$, giving

$$\operatorname{ess\,sup}_{(\partial U)^\varepsilon \cap U} \mathbf{H}(\cdot, \mathbf{D}^{[k]}u_\lambda) \leq M - \frac{\delta}{2} < M. \quad (3.8)$$

On the other hand, let us prove that

$$\operatorname{ess\,sup}_{U \setminus (\partial U)^\varepsilon} \mathbf{H}(\cdot, \mathbf{D}^{[k]}u_\lambda) < M. \quad (3.9)$$

To this purpose, notice that, since $\varphi_\varepsilon = 0$ in $U \setminus (\partial U)^\varepsilon$, we have $u_\lambda = \lambda u$ in $U \setminus (\partial U)^\varepsilon$. Thus, set

$$M_\lambda := \operatorname{ess\,sup}_U \mathbf{H}(\cdot, \lambda \mathbf{D}^{[k]}u)$$

and aim at proving $M_\lambda < M$. Without loss of generality, we can assume $M_\lambda > 0$. So, by definition of essential supremum, for every $\delta \in (0, M_\lambda/2)$, there exists $x_\delta \in U$ such that

$$\mathbf{H}(x_\delta, \lambda \mathbf{D}^{[k]}u(x_\delta)) \geq M_\lambda - \delta. \quad (3.10)$$

On the other hand, by recalling assumption (1.4) and (3.2), we have

$$\begin{aligned} \mathbf{H}(x_\delta, \lambda \mathbf{D}^{[k]}u(x_\delta)) &\leq \mathbf{H}(x_\delta, \mathbf{D}^{[k]}u(x_\delta)) - c(\lambda, |\mathbf{D}^{[k]}u(x_\delta)|) \\ &\leq M - c(\lambda, |\mathbf{D}^{[k]}u(x_\delta)|). \end{aligned} \quad (3.11)$$

Now we claim that there exists $\sigma > 0$ independent of δ such that

$$c(\lambda, |\mathbf{D}^{[k]}u(x_\delta)|) \geq \sigma. \quad (3.12)$$

Indeed, from (3.10) and the continuity of \mathbf{H} at $X = 0$, we infer that $|\mathbf{D}^{[k]}u(x_\delta)|$ is bounded away from 0. Moreover, since $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$, we have that $|\mathbf{D}^{[k]}u(x_\delta)|$ is also bounded from above. In other words, there exist two constants $R \geq r > 0$, independent of δ , such that

$$r \leq |\mathbf{D}^{[k]}u(x_\delta)| \leq R.$$

Hence, since c is continuous and strictly positive in $(0, 1) \times (0, \infty)$, we have

$$c(\lambda, |\mathbf{D}^{[k]}u(x_\delta)|) \geq \min_{\rho \in [r, R]} c(\lambda, \rho) =: \sigma > 0.$$

Putting together (3.10), (3.11), and (3.12), we obtain

$$M_\lambda - \delta \leq M - c(\lambda, |D^{[k]}u(x_\delta)|) \leq M - \sigma,$$

for $\sigma > 0$ independent of δ . We conclude that $M_\lambda \leq M - \sigma < M$, by letting $\delta \rightarrow 0$. As a consequence, we deduce (3.9), indeed

$$\begin{aligned} \operatorname{ess\,sup}_{U \setminus (\partial U)^\varepsilon} H(\cdot, D^{[k]}u_\lambda) &= \operatorname{ess\,sup}_{U \setminus (\partial U)^\varepsilon} H(\cdot, \lambda D^{[k]}u) \\ &\leq \operatorname{ess\,sup}_U H(\cdot, \lambda D^{[k]}u) \\ &= M_\lambda < M \end{aligned}$$

Finally, by recalling (3.2), and the inequalities (3.8) and (3.9), we obtain

$$\operatorname{ess\,sup}_U H(\cdot, D^{[k]}u_\lambda) \leq \max \left\{ \operatorname{ess\,sup}_{(\partial U)^\varepsilon \cap U} H(\cdot, D^{[k]}u_\lambda), \operatorname{ess\,sup}_{U \setminus (\partial U)^\varepsilon} H(\cdot, D^{[k]}u_\lambda) \right\} < M.$$

This inequality and (3.4) contradict the absolute minimality of u .

The proof is complete. \square

A straightforward consequence is Theorem 1.5, whose proof is detailed below.

Proof of Theorem 1.5: Let us show that satisfying the energy maximum principle implies the membership to the class $\operatorname{AM}_c(\Omega, \mathbb{R}^N)$. Fix $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$, an open set $U \subseteq \Omega$ and a map $\phi \in W_c^{k,\infty}(U, \mathbb{R}^N)$. Let $U' \subseteq U$ be an open subset with $\operatorname{supp}(\phi) \subseteq U'$. Then we have

$$\begin{aligned} E_\infty(u + \phi, U) &= \operatorname{ess\,sup}_U H(\cdot, D^{[k]}(u + \phi)) \\ &\geq \operatorname{ess\,sup}_{U \setminus U'} H(\cdot, D^{[k]}(u + \phi)) \\ &= \operatorname{ess\,sup}_{U \setminus U'} H(\cdot, D^{[k]}u) \end{aligned} \tag{3.13}$$

Notice that $(\partial U)^\rho \cap U \subseteq U \setminus U'$ for a sufficiently small $\rho > 0$. Hence

$$\operatorname{ess\,sup}_{U \setminus U'} H(\cdot, D^{[k]}u) \geq \operatorname{ess\,sup}_{(\partial U)^\rho \cap U} H(\cdot, D^{[k]}u).$$

By passing to the limit as $\rho \rightarrow 0$ in the right hand side and invoking Lemma 2.5, we get

$$\operatorname{ess\,sup}_{U \setminus U'} H(\cdot, D^{[k]}u) \geq \lim_{\rho \rightarrow 0} \operatorname{ess\,sup}_{(\partial U)^\rho \cap U} H(\cdot, D^{[k]}u) = \sup_{\partial U} H(\cdot, D^{[k]}u)^*. \tag{3.14}$$

Thus, if u satisfies the energy maximum principle (1.6), by putting together (3.13) and (3.14), we get

$$\begin{aligned} E_\infty(u + \phi, U) &\geq \sup_{\partial U} H(\cdot, D^{[k]}u)^* \\ &= \operatorname{ess\,sup}_U H(\cdot, D^{[k]}u) \\ &= E_\infty(u, U), \end{aligned}$$

so that $u \in \operatorname{AM}_c(\Omega, \mathbb{R}^N)$.

For the reverse implication, one can just replicate the proof of Theorem 1.2. Indeed, inequality

(3.1) holds for every $u \in W^{k,\infty}(\Omega, \mathbb{R}^N)$. The opposite inequality is obtained exactly with the same reasoning, since the competitor u_λ belongs to the class $u + W_c^{k,\infty}(U, \mathbb{R}^N)$ (recall (3.4)). \square

As a consequence of Theorem 1.5, we obtain Corollary 1.7.

Proof of Corollary 1.7: For $\Lambda > E_\infty(u_0, \Omega)$, consider the problem

$$\begin{cases} H(\cdot, D^{[k]}u) = \Lambda, & \text{a.e. in } \Omega, \\ u \in W_{u_0}^{k,\infty}(\Omega, \mathbb{R}^N). \end{cases}$$

By [27, Proposition 1], it admits infinitely many solutions $u \in W_{u_0}^{k,\infty}(\Omega, \mathbb{R}^N)$. Each one of them trivially satisfies the energy maximum principle for E_∞ in Ω :

$$\operatorname{ess\,sup}_U H(\cdot, D^{[k]}u) = \sup_{\partial U} H(\cdot, D^{[k]}u)^* = \Lambda \quad \forall U \subseteq \Omega \text{ open.}$$

By Corollary 1.5, we have also that $u \in \operatorname{AM}_c(\Omega, \mathbb{R}^N)$. \square

Now we turn our attention to Theorem 1.6 in the case of $p < \infty$. The content of the following lemma is probably well known, but we present its proof for sake of completeness.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $u : \Omega \rightarrow \mathbb{R}^N$ be a p -harmonic map, $p \in [2, \infty)$. Set $\mathcal{C} := \{x \in \Omega : Du(x) = 0\}$. Then $u \in C^\infty(\Omega \setminus \mathcal{C}, \mathbb{R}^N)$.*

Proof. Define $F : \mathbb{R}^{N \times n} \rightarrow \mathbb{R}^{N \times n}$ as $F(X) := |X|^{p-2}X$. Then, we have

$$DF(X) = (p-2)|X|^{p-4}X \otimes X + |X|^{p-2}I,$$

where I is the identity map in $(\mathbb{R}^{N \times n})^{\otimes 2}$. Using the following identity

$$(X \otimes X)Y : Z = (X : Y)(X : Z) \quad \forall X, Y, Z \in \mathbb{R}^{N \times n},$$

we have that

$$DF(X)Y : Z = Y : DF(X)Z \quad \forall X, Y, Z \in \mathbb{R}^{N \times n},$$

i.e. $DF(X)$ is a symmetric 4-tensor. Moreover, for every $X, Y \in \mathbb{R}^{N \times n}$, we have

$$|X|^{p-2}|Y|^2 \leq DF(X)Y : Y = (p-2)|X|^{p-4}(X : Y)^2 + |X|^{p-2}|Y|^2 \leq (p-1)|X|^{p-2}|Y|^2.$$

In particular, if we assume $0 < \lambda \leq |X| \leq \Lambda$, then $DF(X)$ is elliptic in the Legendre sense (see [19]), since

$$\lambda^{p-2}|Y|^2 \leq DF(X)Y : Y \leq (p-1)\Lambda^{p-2}|Y|^2. \quad (3.15)$$

Now observe that u solves (distributionally)

$$\operatorname{div}(F(Du)) = 0 \quad \text{in } \Omega. \quad (3.16)$$

For every fixed $U' \Subset U \Subset \Omega \setminus \mathcal{C}$ and $i \in \mathbb{N}$, the difference quotient of a function $f : U \rightarrow \mathbb{R}$

$$D_h^i f(x) := \frac{f(x + he_i) - f(x)}{h}$$

is well defined on U' , for every $h \in \mathbb{R}$ with $0 < |h| < \operatorname{dist}(U', \partial U)$.

By restricting (3.16) to U , we can apply the difference quotient and obtain

$$\operatorname{div}(D_h^i(F(Du))) = 0 \quad \text{in } U'.$$

For $x \in U'$, we write

$$\begin{aligned} D_h^i(F(Du)) &= \frac{1}{h} \int_0^1 \frac{d}{dt} [F(Du(x) + thD_h^i Du(x))] dt \\ &= \left(\int_0^1 DF(Du(x) + thD_h^i Du(x)) dt \right) D_h^i Du(x). \end{aligned}$$

Observe that

$$\mathbb{A}(x) := \int_0^1 DF(Du(x) + thD_h^i Du(x)) dt$$

is a symmetric 4-tensor field on U' . Let us prove that it is Legendre elliptic: first notice that there exists $\lambda > 0$ such that $|Du| \geq \lambda$ in U' . In addition, since $u \in C^{1,\alpha}(U, \mathbb{R}^N)$ (see [41]), for $|h|$ sufficiently small we have

$$|Du(x + he_i) - Du(x)| \leq \frac{\lambda}{2} \quad \forall x \in U'.$$

Thus, for $t \in [0, 1]$, we estimate

$$|Du(x) + thD_h^i Du(x)| = |Du(x) + t(Du(x + he_i) - Du(x))| \geq \lambda - t\frac{\lambda}{2} \geq \frac{\lambda}{2} \quad \forall x \in U'.$$

On the other hand,

$$|Du(x) + thD_h^i Du(x)| \leq 3\|Du\|_{L^\infty(U)} \quad \forall x \in U'.$$

From (3.15), we deduce the Legendre ellipticity of \mathbb{A} in U' (with ellipticity constants independent of h).

Therefore, we have proved that $w := D_h^i Du$ solves the linear elliptic system

$$\operatorname{div}(\mathbb{A}w) = 0 \quad \text{in } U'.$$

By Schauder estimates [19, Thm. 5.19], since \mathbb{A} has α -Hölder coefficients (recall $u \in C^{1,\alpha}(U, \mathbb{R}^N)$), we have $w \in C^{1,\alpha}(U', \mathbb{R}^{N \times n})$. The properties of the difference quotient ensure $Du \in C^{2,\alpha}(U', \mathbb{R}^{N \times n})$. By a bootstrap argument and arbitrariness of U' , we can conclude that $u \in C^\infty(\Omega \setminus \mathcal{C}, \mathbb{R}^N)$. \square

Now, we are ready to prove Theorem 1.6.

Proof of Theorem 1.6: First, we observe that if $p = 2$, the result is straightforward: every harmonic map $u : \Omega \rightarrow \mathbb{R}^N$ is smooth and satisfies $\Delta|Du|^2 \geq 0$ on Ω . So, we may assume $p > 2$ in the following.

We divide the proof in two steps: firstly, we prove the gradient maximum principle outside the closed set $\mathcal{C} = \{x \in \Omega : Du(x) = 0\}$; second, we show the argument for a generic open set $U \Subset \Omega$. Of course, we can assume that $\Omega \setminus \mathcal{C} \neq \emptyset$, otherwise u is constant on Ω .

Step 1: assume $U \Subset \Omega \setminus \mathcal{C}$.

In the following, the symbols \cdot and $\langle \cdot, \cdot \rangle$ stand for the inner products in \mathbb{R}^n and \mathbb{R}^N respectively, while $:$ stands for the Hilbert-Schmidt inner product in $\mathbb{R}^{N \times n}$.

By Lemma 3.1, we have $u \in C^\infty(\bar{U}, \mathbb{R}^N)^2$. So, we can differentiate (1.8) with respect to x_i , obtaining

$$0 = \partial_i [\operatorname{div}(|Du|^{p-2} Du)] = \operatorname{div}((p-2)|Du|^{p-4} (Du : D\partial_i u) Du) + |Du|^{p-2} D\partial_i u \quad \text{in } U.$$

²By this notation, we mean that $u \in C^\infty(U', \mathbb{R}^N)$ for some $U \Subset U' \subseteq \Omega \setminus \mathcal{C}$.

By taking the inner product with $\partial_i u$, we have

$$\begin{aligned} 0 &= \langle \operatorname{div}((p-2)|Du|^{p-4}(Du : D\partial_i u)Du + |Du|^{p-2}D\partial_i u), \partial_i u \rangle \\ &= \operatorname{div}((p-2)|Du|^{p-4}(Du : D\partial_i u)\langle Du, \partial_i u \rangle + |Du|^{p-2}\langle D\partial_i u, \partial_i u \rangle) - \\ &\quad - (p-2)|Du|^{p-4}(Du : D\partial_i u)^2 - |Du|^{p-2}|D\partial_i u|^2, \quad \text{in } U. \end{aligned} \quad (3.17)$$

From (3.17), we deduce

$$\operatorname{div}((p-2)|Du|^{p-4}(Du : D\partial_i u)\langle Du, \partial_i u \rangle + |Du|^{p-2}\langle D\partial_i u, \partial_i u \rangle) \geq 0, \quad \text{in } U. \quad (3.18)$$

By taking the sum over $i = 1, \dots, n$ in (3.18) and observing that

$$\sum_{i=1}^n (Du : D\partial_i u)\langle Du, \partial_i u \rangle = (Du^T Du) \cdot \langle D^2 u, Du \rangle,$$

we obtain

$$\operatorname{div}((p-2)|Du|^{p-4}(Du^T Du) \cdot \langle D^2 u, Du \rangle + |Du|^{p-2}\langle D^2 u, Du \rangle) \geq 0, \quad \text{in } U. \quad (3.19)$$

Now, we define

$$f := \frac{1}{p}|Du|^p, \quad A := I + (p-2)\frac{Du^T Du}{|Du|^2},$$

where I is the identity matrix of $\mathbb{R}^{n \times n}$. Thus,

$$Df = |Du|^{p-2}\langle D^2 u, Du \rangle,$$

so that we can rewrite (3.19) as

$$\operatorname{div}(ADf) \geq 0, \quad \text{in } U. \quad (3.20)$$

Since $u \in C^\infty(\bar{U}, \mathbb{R}^N)$, the function f is a classical subsolution to a linear elliptic partial differential inequality in divergence form, with smooth and bounded coefficients. By the maximum principle for elliptic PDEs, we have

$$\max_{\bar{U}} f = \max_{\partial U} f,$$

yielding (1.9).

Step 2: general case.

Let $U \subseteq \Omega$ be an open subset. Then, without loss of generality, we can assume that $U \not\subseteq \mathcal{C}$. Indeed, if $U \subseteq \mathcal{C}$, then also $\bar{U} \subseteq \mathcal{C}$, so that (1.9) trivially holds. Therefore, since $U \setminus \mathcal{C}$ is a non-empty open subset of Ω , there exists $\varepsilon > 0$ such that $U_\varepsilon := U \setminus \bar{\mathcal{C}}^\varepsilon$ is a non-empty open subset of Ω , where \mathcal{C}^ε is the open ε -neighbourhood of \mathcal{C} . Thanks to *Step 1*, since $U_\varepsilon \Subset \Omega \setminus \mathcal{C}$, we have

$$\max_{\bar{U}_\varepsilon} |Du| = \max_{\partial U_\varepsilon} |Du|. \quad (3.21)$$

Now, set $M := \max_{\bar{U}} |Du| > 0$. By possibly reducing ε in size, the continuity of Du ensures

$$\max_{\bar{\mathcal{C}}^\varepsilon} |Du| < M. \quad (3.22)$$

Hence

$$M = \sup_U |Du| = \max \left\{ \sup_{U_\varepsilon} |Du|, \max_{\bar{\mathcal{C}}^\varepsilon} |Du| \right\} = \sup_{U_\varepsilon} |Du| = \max_{\bar{U}_\varepsilon} |Du|.$$

In particular, from (3.21) we have

$$\max_{\partial U_\varepsilon} |Du| = M.$$

On the other hand, we can write

$$\partial U_\varepsilon = (\partial U \setminus \overline{\mathcal{C}^\varepsilon}) \cup (U \cap \partial \mathcal{C}^\varepsilon),$$

so that, using (3.22),

$$M = \sup_{\partial U_\varepsilon} |Du| = \max \left\{ \sup_{\partial U \setminus \overline{\mathcal{C}^\varepsilon}} |Du|, \sup_{U \cap \partial \mathcal{C}^\varepsilon} |Du| \right\} \leq \max \left\{ \max_{\partial U} |Du|, \max_{\overline{\mathcal{C}^\varepsilon}} |Du| \right\} = \max_{\partial U} |Du|,$$

which ensures the validity of (1.9) . □

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