# UNEXPECTED PHENOMENA FOR MEAN CURVATURE FUNCTIONALS IN THE HEISENBERG GROUP

MATTIA FOGAGNOLO, ANDREA PINAMONTI, AND SIMONE VERZELLESI

Abstract. The Euclidean paradigm that *spheres optimize mean curvature variational problems* breaks down in the sub-Riemannian Heisenberg group: neither the Pansu sphere nor the Korányi sphere is optimal for the variational problems associated with the Minkowski and Heintze-Karcher inequalities. Motivated by this phenomenon, we develop a variational theory for geometric problems driven by the horizontal mean curvature, focusing on the *total mean curvature* functional and the related *Minkowski inequality.* To investigate this phenomenon, we establish first and second variation formulas for general mean curvature functionals in arbitrary Riemannian manifolds, and then obtain corresponding formulas in Heisenberg groups through a Riemannian approximation scheme. We subsequently specialize to the optimization of total mean curvature under area constraint in the first Heisenberg group, introducing suitable notions of *non-characteristic* stationarity and stability. We identify a new one-parameter family of rotationally invariant critical surfaces, which we call *Pansu-Minkowski spheres.* Among them, we show that a distinguished member, the *optimal Pansu-Minkowski sphere*, emerges as the unique critical point of the Minkowski quotient, and uniquely minimizes it among Pansu-Minkowski spheres. We prove non-characteristic stability and local minimality of Pansu-Minkowski spheres under rotationally invariant perturbations, while showing their instability under unrestricted perturbations.

## 1. Introduction

In the Euclidean space, the most relevant geometric variational problems driven by mean curvature exhibit a remarkable rigidity in the shape of their optimal configurations. In their appropriate class of competitors, the *isoperimetric inequality* [27], the *Minkowski inequality* [14, 22, 29], and the *Heintze-Karcher inequality* [23, 32, 45, 47] all single out the Euclidean sphere as the unique optimal shape. As we shall see, this Euclidean paradigm breaks down in a rather unexpected way in sub-Riemannian geometry.

In the sub-Riemannian Heisenberg group $\mathbb{H}^1$, the prototypical model of sub-Riemannian [2] and pseudohermitian [8] geometry, the picture is far less understood. Considerable progress has been achieved for what concerns the isoperimetric problem [33, 30, 31, 25, 40, 44], although it is still unknown whether the closed, constant *horizontal mean curvature* surface known as *Pansu sphere* (Example 3.5) - the unique closed volume-preserving area-stationary surface - is indeed the isoperimetric set. On the other hand, essentially nothing is known for more general geometric variational problems driven by the *horizontal mean curvature.*

In this setting, a general mean curvature driven functional takes the form

$$\mathscr{H}_f^{\mathcal{H}}(S) = \int_S f\left(H^{\mathcal{H}}\right) d\sigma^{\mathcal{H}}. \tag{1.1}$$

Here $S \subseteq \mathbb{H}^1$ is a smooth, embedded, closed surface, $H^{\mathcal{H}}$ is its (horizontal) mean curvature, $d\sigma^{\mathcal{H}}$ is its (sub-Riemannian) area element and $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. We refer to Section 2 for the related definitions. Particularly relevant examples are the *(horizontal) area functional*, the *total (horizontal) mean curvature* and the *total inverse (horizontal) mean curvature*, defined respectively by

$$\sigma^{\mathcal{H}}(S) = \int_S d\sigma^{\mathcal{H}}, \qquad \mathscr{H}^{\mathcal{H}}(S) = \int_S H^{\mathcal{H}}\, d\sigma^{\mathcal{H}}, \qquad \left(\mathscr{H}^{\mathcal{H}}\right)^{-1}(S) = \int_S \frac{1}{H^{\mathcal{H}}}\, d\sigma^{\mathcal{H}}.$$

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As for the Euclidean case, the homogeneous structure of $\mathbb{H}^1$ induced by its *intrinsic dilations* relates the minimization of such functionals, under suitable geometric constraints, to the validity of the corresponding sharp geometric inequalities. Precisely, minimizing the total mean curvature under area constraints, i.e.

$$\inf\left\{\mathscr{H}^{\mathcal{H}}(S)\,:\,S \text{ is mean convex, } \sigma^{\mathcal{H}}(S)=k\right\}, \qquad k>0, \tag{1.2}$$

is equivalent to minimizing the (scaling invariant) *Minkowski quotient*

$$\mathcal{Q}^{\mathcal{H}}_{\text{mink}}(S)=\left(\sigma^{\mathcal{H}}(S)\right)^{-\frac{2}{3}}\mathscr{H}^{\mathcal{H}}(S). \tag{1.3}$$

Similarly, minimizing the total inverse mean curvature under volume constraints, i.e.

$$\inf\left\{\left(\mathscr{H}^{\mathcal{H}}\right)^{-1}(S)\,:\,S \text{ is strictly mean convex, } |\Omega(S)|=k\right\}, \qquad k>0. \tag{1.4}$$

corresponds to minimizing the (scaling invariant) *Heintze-Karcher quotient*

$$\mathcal{Q}^{\mathcal{H}}_{\text{hk}}(S)=|\Omega(S)|^{-1}\left(\mathscr{H}^{\mathcal{H}}\right)^{-1}(S), \tag{1.5}$$

where $|\Omega(S)|$ is the Lebesgue measure of the volume enclosed by $S$. In turn, sharp lower bounds for (1.3) and (1.5) would imply (sub-Riemannian analogs of) sharp Minkowski and Heintze-Karcher inequalities.

Beyond the analogy with the Euclidean framework, several reasons suggest that the Pansu sphere, henceforth denoted by $S_{\frac{1}{2}}$ for further convenience, should play a distinguished role in both optimization problems. Regarding the former problem, (1.3) is the correct first higher-order analogue of the *isoperimetric quotient*

$$\mathcal{Q}^{\mathcal{H}}_{\text{isop}}(S)=|\Omega(S)|^{-\frac{3}{4}}\sigma^{\mathcal{H}}(S). \tag{1.6}$$

The underlying conceptual picture is that area describes the first-order behavior of volume, while mean curvature governs the first-order behavior of area. Regarding (1.5), a structural connection with the isoperimetric problem is suggested by the behavior of (1.6) under evolution by *horizontal inverse mean curvature flow*. Loosely speaking, a family of closed embeddings $(S^t)_{t>0}$ evolves by horizontal inverse mean curvature flow if

$$\frac{\partial S^t}{\partial t}=\frac{1}{H^{\mathcal{H}}_t}\nu_t, \tag{1.7}$$

where $\nu_t$ denotes the *horizontal unit normal* to $S_t$. Recalling [24] that

$$\frac{d|\Omega(S^t)|}{dt}=\left(\mathscr{H}^{\mathcal{H}}\right)^{-1}(S^t), \qquad \frac{d\sigma^{\mathcal{H}}(S^t)}{dt}=\sigma^{\mathcal{H}}(S^t),$$

a formal computation combined with the forthcoming Theorem 3.7 yields

$$\frac{d}{dt}\left(\mathcal{Q}^{\mathcal{H}}_{\text{isop}}\right)(S^t)=\frac{3}{4}\mathcal{Q}^{\mathcal{H}}_{\text{isop}}\left(S^t\right)\left(\mathcal{Q}^{\mathcal{H}}_{\text{hk}}\left(S_{\frac{1}{2}}\right)-\mathcal{Q}^{\mathcal{H}}_{\text{hk}}\left(S^t\right)\right).$$

Were the Pansu sphere the optimal configuration for (1.5), the isoperimetric quotient would be non-decreasing along (1.7). This perspective is supported by another remarkable fact. The *Korányi sphere* (Example 3.6) - the boundary of the unit ball associated with the *Korányi gauge* - shares the same Heintze-Karcher quotient of the Pansu sphere (Theorem 3.8), and is in fact self-similar under inverse mean curvature flow. Finally, since many approaches to the Euclidean Heintze-Karcher inequality provide effective proofs of the Aleksandrov theorem [3] both in the smooth [32, 45] and in the measure-theoretic [16] framework, it would therefore appear natural to expect an analogous phenomenon in the Heisenberg setting.

The first surprising outcome of the present paper is that these intuitions are in fact incorrect.

**Theorem 1.1.** *Neither the Pansu sphere nor the Korányi sphere minimizes* (1.3) *or* (1.5).

This fact has several unexpected consequences. First, it exhibits a substantial lack of symmetry in the shape of optimal configurations for the main geometric variational problems in $\mathbb{H}^1$. Moreover, since the Pansu sphere is non-optimal for (1.5), Theorem 1.1 implies that a possible Heintze-Karcher inequality cannot be used as a direct tool toward the Aleksandrov theorem in the sub-Riemannian setting. The proof of Theorem 1.1 reveals an even more striking phenomenon. The Pansu and Korányi spheres are not merely non-minimizing: they are not even critical points, neither of the normalized functionals (1.3) and (1.5) (Theorem 3.7, Theorem 3.8), nor of the associated constrained problems (1.2) and (1.4) (Remark 3.9). The key observation is that (1.3) and (1.5), while being preserved by intrinsic dilations, are not invariant under Euclidean dilations. This fact allows one to produce suitable deformations along which both functionals strictly decrease.

It therefore becomes natural to investigate the structure of the actual equilibrium configurations of these problems. In the present work, we mainly focus on (1.2). Besides the intrinsic interest of (1.3), this choice is also motivated by analytical considerations. Indeed, as will become apparent from Theorem 4.2, the total mean curvature is, together with the area functional, the only mean curvature functional whose linearization is of zero-order in $H^{\mathcal{H}}$ and whose stability operator remains second-order. Nevertheless, the variational framework developed in this paper applies to a much broader class of mean curvature driven problems.

The study of geometric variations must account for the singular phenomena arising from the interaction between the differential and sub-Riemannian structures. Specialized to hypersurfaces, the latter gives rise to the appearance of the so-called *characteristic points*. Precisely, denoting by $\mathcal{H}$ the underlying *horizontal distribution* (Section 2), a point $p \in S$ is characteristic when $\mathcal{H}$ is tangent to $S$ at $p$. In the *characteristic set* $S_0$, the horizontal geometry collapses, and this typically creates substantial analytical difficulties.

Accordingly, to characterize critical points of (1.2), we introduce the notion of *non-characteristic variation* (Section 4.1), a smooth variation of the ambient space supported away from a neighborhood of the characteristic set. A fundamental ingredient in the whole variational analysis is the derivation of suitable first and second variation formulas along non-characteristic variations (Theorem 4.2). We shall return to these formulas in greater detail later in the introduction.

The natural first-order formulation of (1.2) consists in seeking critical points of the total mean curvature functionals under non-characteristic variations which preserve the area. In turn, exploiting the first variation formulas provided by Proposition 4.3 and Proposition 4.4, we show (Proposition 4.8) that this property is equivalent to the stationarity of the *penalized functional*

$$\mathcal{P}^{\mathcal{H}}_{\mathscr{L}}(S) = \mathscr{H}^{\mathcal{H}}(S) - \mathscr{L}\sigma^{\mathcal{H}}(S) \tag{1.8}$$

under arbitrary non-characteristic variations, as well as to the validity of the *Euler-Lagrange equation*

$$-4\left(J(\nu)\alpha + \alpha^2\right) - \mathscr{L}H^{\mathcal{H}} = 0. \tag{1.9}$$

In the above formulas, $J$ is the *complex structure* of $\mathbb{H}^1$, so that $J(\nu)$ generates the *horizontal tangent space* to $S$ (Section 2). Moreover, $\alpha$ is the *fundamental function* associated with $S$ (Section 2), and $\mathscr{L}$ is the appropriate Lagrange multiplier arising from the area constraint.

Motivated by the symmetries of the Pansu sphere, we look for solutions to (1.9) in the class of *rotationally invariant surfaces* (Section 3), which are symmetric with respect to rotations around a distinguished *vertical direction*. In this setting, the presence of such a symmetry is quite natural, as vertical rotations act as isometries on the underlying sub-Riemannian structure [13]. Combining topological arguments with a detailed analysis of (1.9), in Section 5 we completely characterize the rotationally invariant solutions to (1.9). Precisely, setting $4L = \mathscr{L}$, we show that every rotationally invariant solution to (1.9) is obtained by vertical rotation of the profile curve $\gamma_L = (x_L, t_L) : [-\arccos\sqrt{1-2L}, \arccos\sqrt{1-2L}] \to \mathbb{R}^2$, where $L \in \left(0, \frac{1}{2}\right]$ and

$$\begin{cases} x_L(s) = \dfrac{1}{2L}\left(\cos s - \dfrac{1-2L}{\cos s}\right), \\ t_L(s) = \dfrac{1}{4L^2}\left(\dfrac{s}{2} + \dfrac{\sin 2s}{4} - (1-2L)^2 \tan s\right), \end{cases} \tag{1.10}$$

thus obtaining a one-parameter family of rotationally invariant surfaces, say $S_L$ for $0 < L \leqslant \frac{1}{2}$, which are critical points of the total mean curvature along area-preserving non-characteristic variations.

This characterization carries with it a further series of consequences. First, unlike in the Euclidean setting, where the sphere is the unique equilibrium configuration of the total mean curvature under area-preserving variations [14, Theorem 5.3], in the Heisenberg group one finds infinitely many critical points. Most notably, one easily realizes (Example 3.5 and Remark 5.6) that the limiting configuration corresponding to $L = \frac{1}{2}$ is precisely the Pansu sphere $S_{\frac{1}{2}}$, whence the choice of notation. Surprisingly, although the Pansu sphere fails to be stationary for $\mathscr{H}^{\mathcal{H}}$ under *arbitrary* area-preserving variations (Remark 3.9), it is nevertheless stationary for $\mathscr{H}^{\mathcal{H}}$ along *non-characteristic* area-preserving variations. Thus, characteristic points are not merely a technical issue, but fundamentally affect the variational landscape. Besides confirming that the Pansu sphere still plays a role in this variational framework, this phenomenon also reveals a substantial difference with the sub-Riemannian isoperimetric problem: the Pansu sphere is indeed the unique closed, constant horizontal mean curvature surface, e.g. among rotationally invariant surfaces [43] or even among competitors with at least one isolated characteristic point [44]. Both because of their connection with the Minkowski problem and because the Pansu sphere itself arises as a limiting configuration within the family, we refer to the above surfaces as *Pansu-Minkowski spheres.*

Among Pansu-Minkowski spheres, one particular element nevertheless plays a privileged role. This phenomenon becomes apparent when comparing the *constrained* stationarity problem associated with (1.2) with the *unconstrained* stationarity problem associated with (1.3). Although being equivalent from the viewpoint of minimization, this equivalence breaks down at the *non-characteristic* first-order level. Indeed, while every critical point of $\mathcal{Q}^{\mathcal{H}}_{\mathrm{mink}}$ along non-characteristic variations is also stationary for $\mathscr{H}^{\mathcal{H}}$ along area-preserving non-characteristic variations (Proposition 4.9), the converse fails in general. More precisely, within the family of Pansu-Minkowski spheres, the only surface which is critical for (1.3) is the Pansu-Minkowski sphere corresponding to $L = \frac{1}{4}$. It is therefore not surprising that this configuration plays a distinguished role in the minimization of the Minkowski quotient: $S_{\frac{1}{4}}$ is the unique minimizer of (1.3) within the class of Pansu-Minkowski spheres. Accordingly, we refer to it as the *optimal Pansu-Minkowski sphere*, since the above facts naturally single it out as the primary candidate minimizer for the Minkowski quotient.

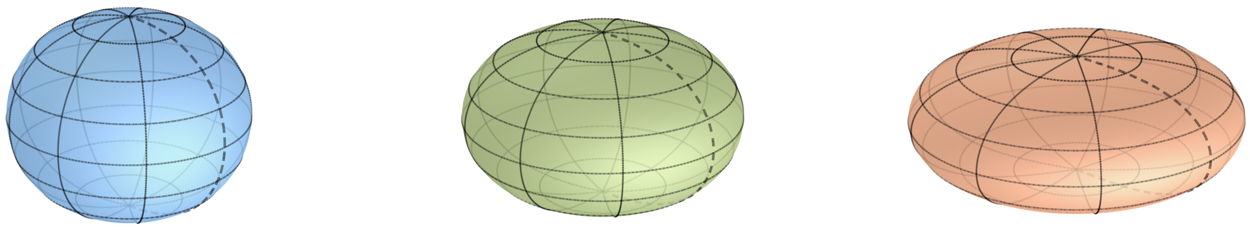

FIGURE 1. From left to right, the Pansu-Minkowski sphere for $L = \frac{1}{8}$, the optimal Pansu-Minkowski sphere $\left(L = \frac{1}{4}\right)$ and the Pansu sphere $\left(L = \frac{1}{2}\right)$. The three spheres are rescaled via intrinsic dilations to have unit sub-Riemannian area.

The next statement summarizes the above considerations.

**Theorem 1.2.** *For any $L \in \left(0, \frac{1}{2}\right]$, define $\gamma_L$ as in (1.10). Denote by $S_L$ its associated rotationally invariant surface. Let $S \subseteq \mathbb{H}^1$ be a rotationally invariant, closed, mean convex surface. The following are equivalent:*

- (i) *$S$ is a critical point for $\mathscr{H}^{\mathcal{H}}$ along area-preserving non-characteristic variations;*
- (ii) *$S$ is, up to dilations and vertical translations, a Pansu-Minkowski sphere $S_L$ for some $L \in \left(0, \frac{1}{2}\right]$.*

*Moreover, the following are equivalent:*

- (iii) *$S$ is a critical point for $\mathcal{Q}^{\mathcal{H}}_{\mathrm{mink}}$ along non-characteristic variations;*
- (iv) *$S$ is, up to dilations and vertical translations, the optimal Pansu-Minkowski sphere $S_{\frac{1}{4}}$.*

*Finally, the optimal Pansu-Minkowski sphere $S_{\frac{1}{4}}$ minimizes $\mathcal{Q}^{\mathcal{H}}_{\mathrm{mink}}$ within Pansu-Minkowski spheres:*

$$\mathcal{Q}^{\mathcal{H}}_{\mathrm{mink}}(S_L) \geqslant (18\pi)^{\frac{1}{3}}, \qquad \mathcal{Q}^{\mathcal{H}}_{\mathrm{mink}}(S_L) = (18\pi)^{\frac{1}{3}} \text{ if and only if } L = \frac{1}{4}.$$

A further natural question concerns the stability of Pansu-Minkowski spheres under non-characteristic perturbations. As for the isoperimetric problem [4, 5], the equivalence between the constrained problem (1.2) and the unconstrained problem associated with (1.8) breaks down at second-order. Exploiting the second variation formulas established in Theorem 4.2, we show that stability of critical points of $\mathscr{H}^{\mathcal{H}}$ under area-preserving non-characteristic variations is equivalent to stability of critical points of $\mathcal{P}^{\mathcal{H}}_L$ under non-characteristic variations which are area-preserving at first-order (Proposition 4.10). Consequently, our analysis is carried out at the level of the penalized functional $\mathcal{P}^{\mathcal{H}}_L$. Evaluated at the corresponding critical point $S_L$, its second variation along an *arbitrary* non-characteristic variation $\Phi$ is given (Section 6) by

$$\delta^2 \mathcal{P}^{\mathcal{H}}_L (S_L)[\Phi] = 4 \int_{S_L} \Big( \big( - \mathcal{S}\varphi\, J(\nu)\varphi - L\, (J(\nu)\varphi)^2 \big) + \big( 2\mathcal{S}\alpha - H^{\mathcal{H}} \left(\alpha^2 + 4L^2\right) \big) \varphi^2 \Big)\, d\sigma^{\mathcal{H}}. \tag{1.11}$$

In this formula, $\mathcal{S}$ is (up to normalization) the unique vector field which completes $J(\nu)$ to an orthogonal frame of $TS$ (cf. Section 2), while $\varphi$ is an appropriate projection of the *velocity* of $\Phi$ (Section 4).

Within the class of rotationally invariant non-characteristic perturbations, all Pansu-Minkowski spheres exhibit a much stronger property than mere stability. Indeed, we prove that the quadratic form associated with (1.11) is uniformly coercive with respect to *arbitrary* non-characteristic variations. More precisely,

$$\delta^2\mathcal{P}_L^{\mathcal{H}}(S_L)[\Phi] \geqslant 4(1-L)\int_{S_L}(J(\nu)\varphi)^2\, d\sigma^{\mathcal{H}} + \frac{8L(1-2L)}{1-L}\int_{S_L}\varphi^2\, d\sigma^{\mathcal{H}}. \tag{1.12}$$

We stress that the above lower bound does not require any first-order area constraint, and is therefore substantially stronger than standard stability. This is in sharp contrast with more familiar settings. For instance, Euclidean spheres do not satisfy such a strong stability property for the penalized functional associated with the Euclidean isoperimetric problem [4]. As a direct consequence of (1.12), all Pansu–Minkowski spheres locally minimize the total mean curvature under sufficiently small rotationally invariant non-characteristic perturbations. The above facts can be summarized as follows.

**Theorem 1.3.** *Let $L \in \left(0, \frac{1}{2}\right]$. Let $\Phi$ be a non-characteristic rotationally invariant variation (Section 6). Fix $\delta > 0$, and set $I(\delta) = \left(-\arccos\sqrt{1-2L}+\delta, \arccos\sqrt{1-2L}-\delta\right)$. Then:*

(i) *the lower bound (1.12) holds. In particular, $S_L$ is a stable critical point of $\mathscr{H}^{\mathcal{H}}$ along area-preserving, non-characteristic, rotationally invariant variations;*
(ii) *there exists $\varepsilon = \varepsilon(\delta, L) > 0$ such that, if $\varphi \in C_c^\infty(I(\delta))$ satisfies*

$$\sigma^{\mathcal{H}}\left(S_L^{\varphi}\right) = \sigma^{\mathcal{H}}\left(S_L\right), \qquad \|\varphi\|_{C^2(I(\delta))} \leqslant \varepsilon,$$

*where $S_L^{\varphi}$ is a (rotationally invariant) horizontally normal graph over $S_L$ (Section 7), then*

$$\mathscr{H}^{\mathcal{H}}\left(S_L\right) \leqslant \mathscr{H}^{\mathcal{H}}\left(S_L^{\varphi}\right).$$

The perhaps most surprising phenomenon emerges once rotational symmetry is removed. We show that Pansu-Minkowski spheres become unstable under general perturbations. Precisely, we construct highly oscillatory, first-order area-preserving angular variations along which the second variation of $\mathcal{P}_L^{\mathcal{H}}$ becomes negative. By Proposition 4.10, this yields instability of Pansu-Minkowski spheres for the constrained problem (1.2).

**Theorem 1.4.** *Let $L \in \left(0, \frac{1}{2}\right]$. There is a non-characteristic first-order area-preserving variation $\Phi$ such that*

$$\delta^2\mathcal{P}_L^{\mathcal{H}}(S_L)[\Phi] < 0.$$

*Pansu-Minkowski spheres are then unstable for $\mathscr{H}^{\mathcal{H}}$ along area-preserving non-characteristic variations.*

Theorem 1.4 implies that any global minimizer of (1.3), should it exist, cannot be rotationally invariant. Although sharing analogies with the sub-Riemannian *isodiametric problem* [26], this unexpected lack of symmetry casts serious doubts on the very existence - or at least on the uniqueness - of minimizers of (1.3) (cf. e.g. [9, 12] for similar issues in related settings).

Our analysis leaves several open questions. First, it is tempting to conjecture that the optimal Pansu-Minkowski sphere is the global minimizer of (1.3) within the class of rotationally invariant surfaces. On the other hand, although it is plausible that no global minimizer exists outside this symmetric class, these outcomes suggest that the search for sharp mean curvature driven inequalities requires abandoning symmetry as a guiding principle. We expect similarly exotic phenomena to arise in connection with the Heintze-Karcher problem (1.4). Indeed, although the Pansu sphere is not stationary for (1.4) under arbitrary volume-preserving variations, it nevertheless becomes stationary when one restricts to *non-characteristic* volume-preserving variations (Remark 5.7). It therefore remains somewhat mysterious that the Pansu sphere systematically emerges in these variational problems, despite not realizing the corresponding optimal configuration.

From the methodological viewpoint, a substantial part of the paper is devoted to the derivation of first and second variation formulas stated in Theorem 4.2, which we believe may be of independent interest. We establish them in full generality, considering an arbitrary mean curvature driven functional as in (1.1), arbitrary non-characteristic variations, and in Heisenberg groups of every dimension. Our approach relies on a well-established Riemannian approximation scheme [7, 11, 19, 34, 37], and is carried out in full detail in Appendix B. This approach, in turn, requires first and second variation formulas for the corresponding Riemannian analogue of (1.1). Since the available literature mostly addresses special cases - such as particular functionals [18], restricted classes of variations [21, 36], or additional geometric assumptions on the ambient manifold [21, 36, 39] - in Appendix A we derive these formulas in full generality, namely along arbitrary variations and in arbitrary Riemannian manifolds. We hope that this also provides a useful unified reference.

**Plan of the paper.** The paper is organized as follows. In Section 2 we collect some preliminaries on Heisenberg groups (Section 2.1) and on the geometry of hypersurfaces (Section 2.2). In Section 3 we introduce rotationally invariant surfaces, discuss their properties (Section 3.1), give some examples (Section 3.2), introduce the relevant functionals (Section 3.3) and prove Theorem 1.1 (Theorem 3.7, Theorem 3.8). In Section 4 we focus on the variation formulas for (1.1). After introducing variations (Section 4.1), we state Theorem 4.2 (Section 4.2) and we specialize it (Section 4.3) to (1.2) (Proposition 4.3, Proposition 4.4), discussing the relevant notions of stationarity (Proposition 4.8) and stability (Proposition 4.10). In Section 5 we characterize rotationally invariant critical points of (1.2), proving Theorem 1.2. In Section 6 and Section 7 we discuss their stability, instability and local minimality, proving Theorem 1.3 and Theorem 1.4. In Appendix A we establish arbitrary Riemannian variation formulas (Theorem A.8). In Appendix B, we prove Theorem 4.2.

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## 2. PRELIMINARIES

2.1. **Heisenberg groups.** Fix $n \geqslant 1$. The $n$-th Heisenberg group $(\mathbb{H}^n, \cdot)$ is $\mathbb{R}^{2n+1}$ with the group law

$$p \cdot p' = (x, y, t) \cdot (x', y', t') = \left( x + x', y + y', t + t' + \sum_{j=1}^{n} \left( x'_j y_j - x_j y'_j \right) \right), \tag{2.1}$$

where $p = (x, y, t) = (x_1, \dots, x_n, y_1, \dots, y_n, t)$. This group law realizes $\mathbb{H}^n$ as stratified Lie group. Its *horizontal distribution* $\mathcal{H}$ is generated by the left-invariant vector fields

$$Z_i = X_i = \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial t}, \qquad Z_{n+i} = Y_i = \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial t}, \qquad i = 1, \dots, n.$$

A vector field which is tangent to $\mathcal{H}$ at every point is called *horizontal.* Setting $Z_{2n+1} = T = \frac{\partial}{\partial t}$, then $Z_1, \dots, Z_{2n+1}$ is a global frame of left-invariant vector fields. We may identify a point $p \in \mathbb{H}^n$ with

$$\sum_{i=1}^{2n+1} p_j Z_j(p) \in T_p \mathbb{H}^n.$$

The only nontrivial commutation relations among $Z_1, \dots, Z_{2n+1}$ are

$$[X_i, Y_i] = -2T, \qquad i = 1, \dots, n.$$

The *complex structure* $J : \Gamma(T\mathbb{H}^n) \longrightarrow \Gamma(T\mathbb{H}^n)$ is the unique $C^\infty(\mathbb{H}^n)$-linear map which satisfies

$$J(X_i) = Y_i, \qquad J(Y_i) = -X_i \qquad \text{and} \qquad J(T) = 0, \qquad i = 1, \dots, n.$$

The triple $(\mathbb{H}^n, \mathcal{H}, J)$ is a prototypical *pseudohermitian manifold* (cf. [10, Appendix]). Moreover, $\mathbb{H}^n$ inherits a sub-Riemannian structure by fixing the unique Riemannian metric $\langle \cdot, \cdot \rangle$ making $X_1, \dots, X_n, Y_1, \dots, Y_n, T$ orthonormal. The *pseudohermitian connection* $\nabla$ (cf. [41]) is the unique metric connection with torsion

$$\nabla_{\mathbf{A}} \mathbf{B} - \nabla_{\mathbf{B}} \mathbf{A} - [\mathbf{A}, \mathbf{B}] = 2 \langle J(\mathbf{A}), \mathbf{B} \rangle T, \qquad \mathbf{A}, \mathbf{B} \in \Gamma(T\mathbb{H}^n). \tag{2.2}$$

We recall that $\nabla$ vanishes along left-invariant vector fields (cf. [40]), i.e.

$$\nabla Z_i = 0, \qquad i = 1, \dots, 2n+1. \tag{2.3}$$

In addition, $\mathbb{H}^n$ carries a *homogeneous structure* provided by *intrinsic dilations* (cf. [6]). Namely, we set

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t) \qquad \text{for every } \lambda \geqslant 0, \ (x, y, t) \in \mathbb{H}^n.$$

In this way, $\delta_\lambda$ is a Lie group isomorphism of $\mathbb{H}^n$ for any $\lambda > 0$. The Riemannian volume induced by $\langle \cdot, \cdot \rangle$ is the Haar measure of the group, i.e. the standard Lebesgue measure. It satisfies the homogeneity condition

$$|\delta_\lambda(E)| = \lambda^Q |E| \qquad \text{for every } E \subseteq \mathbb{H}^n \text{ measurable}, \ \lambda \geqslant 0,$$

where $Q := 2n + 2$ is known as *homogeneous dimension* of $(\mathbb{H}^n, \cdot)$ (cf. [46]). Therefore, the Riemannian divergence induced by $\langle \cdot, \cdot \rangle$ is the Euclidean divergence, and can be computed by

$$\operatorname{div} \mathbf{A} = \operatorname{div} \left( \sum_{i=1}^{2n+1} A^i Z_i \right) = \sum_{i=1}^{2n+1} Z_i A^i, \qquad \mathbf{A} \in \Gamma(T\mathbb{H}^n). \tag{2.4}$$

2.2. **Hypersurfaces.** Let $S \subseteq \mathbb{H}^n$ be a smooth, connected, embedded, two-sided hypersurface. If $S$ is closed, we denote by $\Omega(S)$ the boundary region it encloses. Recall that $p \in S$ is called a *characteristic point* if $\mathcal{H}_p = T_pS$. The set of characteristic points of $S$ is denoted by $S_0$. At non-characteristic points, the *horizontal tangent space* $\mathcal{H}TS$ is the smooth, $(2n-1)$-dimensional distribution defined by

$$\mathcal{H}T_pS = \mathcal{H}_p \cap T_pS, \qquad p \in S \setminus S_0.$$

Denote by $\mathbf{N}$ the Riemannian unit normal to $S$, and by $\mathbf{N}^{\mathcal{H}}$ its orthogonal projection onto $\mathcal{H}$. Then, the *horizontal unit normal*

$$\nu = \frac{\mathbf{N}^{\mathcal{H}}}{|\mathbf{N}^{\mathcal{H}}|}$$

is well-defined on $S \setminus S_0$, and is the unique, up to sign, horizontal unit vector field orthogonal to $\mathcal{H}TS$. Notice that $p$ is a characteristic point if and only if $|\mathbf{N}^{\mathcal{H}}| = 0$. Close to every non-characteristic point, it is always possible to extend $\nu$ to a full neighborhood in $\mathbb{H}^n$ by setting

$$\nu = \nabla^{\mathcal{H}} d, \tag{2.5}$$

where $d$ is the signed *Carnot-Carathéodory distance* from $S$ (cf. [20, 41]). Henceforth, $\nu$ is always extended as in (2.5). The *fundamental function* $\alpha$ is defined on $S \setminus S_0$ as the unique smooth function such that

$$\mathcal{S} := T - \alpha\nu \in \Gamma(TS).$$

It is known (cf. [20, 35]) that $\alpha = Td$, and moreover

$$\nabla_\nu \nu = -2\alpha J(\nu). \tag{2.6}$$

Denote by $\mathcal{H}'TS$ the distribution defined by

$$\mathcal{H}'T_pS = \mathcal{H}T_pS \cap J\left(\mathcal{H}T_pS\right), \qquad p \in S \setminus S_0.$$

Then, $\mathcal{H}'TS$ is a $(2n-2)$-dimensional sub-bundle of $\mathcal{H}TS$, and the latter can be orthogonally decomposed as $\mathcal{H}TS = \mathcal{H}'TS \oplus \operatorname{span} J(\nu)$. Notice that, in the first Heisenberg group, $\mathcal{H}'TS = \{0\}$. It is easy to check that

$$\mathbf{N} = \frac{1}{\sqrt{1+\alpha^2}}\nu + \frac{\alpha}{\sqrt{1+\alpha^2}}T, \qquad \alpha = \frac{\langle \mathbf{N}, T\rangle}{|\mathbf{N}^{\mathcal{H}}|}.$$

The *horizontal second fundamental form* $h^{\mathcal{H}}$ and the *symmetric horizontal second fundamental form* (cf. [20, 41]) $\tilde{h}^{\mathcal{H}}$ are defined on $S \setminus S_0$ respectively by

$$h^{\mathcal{H}}(\mathbf{A},\mathbf{B}) := \left\langle A^{\mathcal{H}}(\mathbf{A}), \mathbf{B}\right\rangle, \qquad \tilde{h}^{\mathcal{H}}(\mathbf{A},\mathbf{B}) := \frac{h^{\mathcal{H}}(\mathbf{A},\mathbf{B}) + h^{\mathcal{H}}(\mathbf{B},\mathbf{A})}{2}, \qquad \mathbf{A},\mathbf{B} \in \Gamma(\mathcal{H}TS),$$

where $A^{\mathcal{H}}(\mathbf{A}) := \nabla_{\mathbf{A}}\nu$ is the *horizontal shape operator*. The forms $h^{\mathcal{H}}$ and $\tilde{h}^{\mathcal{H}}$ are related (cf. [20]) by

$$\tilde{h}^{\mathcal{H}}(\mathbf{A},\mathbf{B}) = h^{\mathcal{H}}(\mathbf{A},\mathbf{B}) + \alpha\left\langle J(\mathbf{A}), \mathbf{B}\right\rangle, \qquad \mathbf{A},\mathbf{B} \in \Gamma(\mathcal{H}TS). \tag{2.7}$$

The *horizontal mean curvature* is then defined on $S \setminus S_0$ by

$$H^{\mathcal{H}} = \operatorname{trace} h^{\mathcal{H}} = \operatorname{trace} \tilde{h}^{\mathcal{H}}.$$

It is well-known that

$$H^{\mathcal{H}}_{\delta_\lambda(S)}(\delta_\lambda(p)) = \frac{1}{\lambda} H^{\mathcal{H}}_S(p), \qquad p \in S \setminus S_0, \lambda > 0.$$

We say that $S$ is *minimal* if $H^{\mathcal{H}} = 0$ on $S \setminus S_0$, *mean convex* if $H^{\mathcal{H}} \geqslant 0$ on $S \setminus S_0$, and *strictly mean convex* if $H^{\mathcal{H}} > 0$ on $S \setminus S_0$. Finally, the relevant sub-Riemannian surface measure $\sigma^{\mathcal{H}}$ is defined (cf. [15, 28]) by

$$\sigma^{\mathcal{H}} := \frac{1}{\sqrt{1+\alpha^2}}\sigma, \tag{2.8}$$

where $\sigma$ is the Riemannian surface measure induced by $\langle\cdot,\cdot\rangle$. Again, $\sigma^{\mathcal{H}}$ satisfies the homogeneity property

$$\sigma^{\mathcal{H}}(\delta_\lambda(S)) = \lambda^{Q-1}\sigma^{\mathcal{H}}(S).$$

The *tangent pseudohermitian connection* $\nabla^S$ is the affine connection defined on $S$ by

$$\nabla^S_{\mathbf{A}}\mathbf{B} = \nabla_{\mathbf{A}}\mathbf{B} - \langle \nabla_{\mathbf{A}}\mathbf{B}, \nu\rangle\nu, \qquad \mathbf{A},\mathbf{B} \in \Gamma(\mathcal{H}TS).$$

Let $\mathbf{E}_1, \dots, \mathbf{E}_{2n-1}$ be any local orthonormal frame of $\mathcal{H}TS$. If $\varphi \in C^\infty(S \setminus S_0)$ and $\mathbf{A} \in \Gamma(\mathcal{H}TS)$ is supported in $S \setminus S_0$, the *horizontal tangential gradient* of $\varphi$ and the *horizontal tangential divergence* of $\mathbf{A}$ are defined by

$$\nabla^{\mathcal{H},S}\varphi = \sum_{i=1}^{2n-1} (\mathbf{E}_i\varphi)\,\mathbf{E}_i, \qquad \operatorname{div}^{\mathcal{H},S}\mathbf{A} := \sum_{i=1}^{2n-1} \left\langle \nabla^S_{\mathbf{E}_i}\mathbf{A}, \mathbf{E}_i \right\rangle.$$

The *horizontal tangential Laplacian* and the *modified horizontal tangential Laplacian* are then defined by

$$\Delta^{\mathcal{H},S}\varphi = \operatorname{div}^{\mathcal{H},S}\nabla^{\mathcal{H},S}\varphi, \qquad \hat{\Delta}^{\mathcal{H},S}\varphi := \Delta^{\mathcal{H},S}\varphi + 2\alpha J(\nu)\varphi.$$

Finally, we denote by $\mathcal{J}^{\mathcal{H}}$ the *horizontal Jacobi operator*

$$\mathcal{J}^{\mathcal{H}}\varphi = -\hat{\Delta}^{\mathcal{H},S}\varphi - \varphi\left(|\tilde{h}^{\mathcal{H}}|^2 + 4J(\nu)\alpha + (2n+2)\alpha^2\right), \qquad \varphi \in C^\infty(S \setminus S_0). \tag{2.9}$$

Unlike $\Delta^{\mathcal{H},S}$ (cf. [15]), both $\hat{\Delta}^{\mathcal{H},S}$ and $\mathcal{J}^{\mathcal{H}}$ are self-adjoint on $C^\infty_c(S \setminus S_0)$ (cf. Proposition B.10).

## 3. ROTATIONALLY INVARIANT SURFACES

3.1. **Definition and properties.** We specialize Section 2.2 to rotationally invariant surfaces in $\mathbb{H}^1$. A rotationally invariant surface $S$ is, by definition (cf. [43]), a surface which is invariant under rotation around the $t$-axis. In particular, $S \setminus \{(0,0,t)\}$ can be smoothly parametrized, up to removing a meridian, by the map

$$P(s,\theta) = (x(s)\cos\theta, x(s)\sin\theta, t(s)), \qquad s \in I,\ \theta \in (0, 2\pi),$$

where $I \subseteq \mathbb{R}^n$ is an open, possibly unbounded, interval, and $\gamma(s) = (x(s), t(s))$ is a smooth, regular, embedded curve in $\mathbb{R}^2$, parametrized counterclockwise and such that $x > 0$. The latter is known as the *profile* of $S$. The parametrization $P$ induces local coordinates $(s,\theta)$ on $S$. The tangent space of $S$ at $P(s,\theta)$ is generated by

$$\begin{aligned} \frac{\partial}{\partial s} &= \left.\frac{\partial P}{\partial s}\right|_{(s,\theta)} = \dot{x}(s)\cos\theta\frac{\partial}{\partial x} + \dot{x}(s)\sin\theta\frac{\partial}{\partial y} + \dot{t}(s)\frac{\partial}{\partial t} = \dot{x}(s)\cos\theta X + \dot{x}(s)\sin\theta Y + \dot{t}(s)T,\\ \frac{\partial}{\partial \theta} &= \left.\frac{\partial P}{\partial \theta}\right|_{(s,\theta)} = -x(s)\sin\theta\frac{\partial}{\partial x} + x(s)\cos\theta\frac{\partial}{\partial y} = -x(s)\sin\theta X + x(s)\cos\theta Y + x(s)^2 T. \end{aligned} \tag{3.1}$$

In particular,

$$\mathbf{N} = \frac{1}{\sqrt{(1+x(s)^2)\dot{x}(s)^2 + \dot{t}(s)^2}}\left(\left(\dot{t}(s)\cos\theta - x(s)\dot{x}(s)\sin\theta\right)X + \left(\dot{t}(s)\sin\theta + x(s)\dot{x}(s)\cos\theta\right)Y - \dot{x}(s)T\right),$$

so that

$$|\mathbf{N}^{\mathcal{H}}| = \frac{\sqrt{\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2}}{\sqrt{(1+x(s)^2)\dot{x}(s)^2 + \dot{t}(s)^2}}. \tag{3.2}$$

By (3.2), we conclude that $S \setminus \{(0,0,t)\}$ is non-characteristic. On the other hand, every possible intersection between $S$ and the vertical axis is a characteristic point of $S$. In conclusion, $S_0 = S \cap \{(0,0,t)\}$. Therefore, by our previous computations, we deduce that, on $S \setminus S_0$,

$$\alpha = -\frac{\dot{x}(s)}{\sqrt{\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2}}, \tag{3.3}$$

$$\nu = \left(\frac{\dot{t}(s)\cos\theta - x(s)\dot{x}(s)\sin\theta}{\sqrt{\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2}}\right)X + \left(\frac{\dot{t}(s)\sin\theta + x(s)\dot{x}(s)\cos\theta}{\sqrt{\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2}}\right)Y, \tag{3.4}$$

$$J(\nu) = -\left(\frac{\dot{t}(s)\sin\theta + x(s)\dot{x}(s)\cos\theta}{\sqrt{\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2}}\right)X + \left(\frac{\dot{t}(s)\cos\theta - x(s)\dot{x}(s)\sin\theta}{\sqrt{\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2}}\right)Y, \tag{3.5}$$

$$\mathcal{S} = \left(\frac{\dot{x}(s)\dot{t}(s)\cos\theta - x(s)\dot{x}(s)^2\sin\theta}{\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2}\right)X + \left(\frac{\dot{x}(s)\dot{t}(s)\sin\theta + x(s)\dot{x}(s)^2\cos\theta}{\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2}\right)Y + T. \tag{3.6}$$

For further convenience, it is useful to express $J(\nu)$ and $\mathcal{S}$ in local coordinates.

**Lemma 3.1.** *It holds that, on $S \setminus S_0$,*

$$J(\nu) = -\left(\frac{x(s)}{\sqrt{\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2}}\right)\frac{\partial}{\partial s} + \left(\frac{\dot{t}(s)}{x(s)\sqrt{\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2}}\right)\frac{\partial}{\partial \theta}, \tag{3.7}$$

$$\mathcal{S} = \left(\frac{\dot{t}(s)}{\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2}\right)\frac{\partial}{\partial s} + \left(\frac{\dot{x}(s)^2}{\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2}\right)\frac{\partial}{\partial \theta}. \tag{3.8}$$

*Proof.* Let $a_1, a_2$ be such that $J(\nu) = a_1\frac{\partial}{\partial s} + a_2\frac{\partial}{\partial \theta}$. Then

$$J(\nu) = a_1\frac{\partial}{\partial s} + a_2\frac{\partial}{\partial \theta} \overset{(3.1)}{=} (a_1\dot{x}\cos\theta - a_2 x\sin\theta)X + (a_1\dot{x}\sin\theta + a_2 x\cos\theta)Y + (a_1\dot{t} + a_2x^2)T. \tag{3.9}$$

Therefore, comparing (3.5) with (3.9),

$$\begin{cases} a_1\dot{x}\cos\theta - a_2 x\sin\theta &= -\dfrac{\dot{t}\sin\theta + x\dot{x}\cos\theta}{\sqrt{\dot{t}^2 + x^2\dot{x}^2}}, \\ a_1\dot{x}\sin\theta + a_2 x\cos\theta &= \dfrac{\dot{t}\cos\theta - x\dot{x}\sin\theta}{\sqrt{\dot{t}^2 + x^2\dot{x}^2}}, \\ a_1\dot{t} + a_2x^2 &= 0. \end{cases} \tag{3.10}$$

By the third equation of (3.10), and since $x \neq 0$ on $S \setminus S_0$,

$$a_1\dot{x}\cos\theta - a_2 x\sin\theta = a_1\left(\frac{x\dot{x}\cos\theta + \dot{t}\sin\theta}{x}\right)$$

$$a_1\dot{x}\sin\theta + a_2 x\cos\theta = a_1\left(\frac{x\dot{x}\sin\theta - \dot{t}\cos\theta}{x}\right).$$

Therefore

$$\begin{cases} a_1\left(x\dot{x}\cos\theta + \dot{t}\sin\theta\right) &= -\dfrac{x}{\sqrt{\dot{t}^2 + x^2\dot{x}^2}}\left(x\dot{x}\cos\theta + \dot{t}\sin\theta\right), \\ a_1\left(x\dot{x}\sin\theta - \dot{t}\cos\theta\right) &= -\dfrac{x}{\sqrt{\dot{t}^2 + x^2\dot{x}^2}}\left(x\dot{x}\sin\theta - \dot{t}\cos\theta\right). \end{cases}$$

Since either $\langle J(\nu), X\rangle \neq 0$ or $\langle J(\nu), Y\rangle \neq 0$, (3.7) follows. Let $b_1, b_2$ be such that $\mathcal{S} = b_1\frac{\partial}{\partial s} + b_2\frac{\partial}{\partial \theta}$. Then

$$\mathcal{S} = b_1\frac{\partial}{\partial s} + b_2\frac{\partial}{\partial \theta} \overset{(3.1)}{=} (b_1\dot{x}\cos\theta - b_2 x\sin\theta)X + (b_1\dot{x}\sin\theta + b_2 x\cos\theta)Y + (b_1\dot{t} + b_2x^2)T. \tag{3.11}$$

Therefore, comparing (3.6) with (3.11),

$$\begin{cases} b_1\dot{x}\cos\theta - b_2 x\sin\theta &= \dfrac{\dot{x}\dot{t}\cos\theta - x\dot{x}^2\sin\theta}{\dot{t}^2 + x^2\dot{x}^2}, \\ b_1\dot{x}\sin\theta + b_2 x\cos\theta &= \dfrac{\dot{x}\dot{t}\sin\theta + x\dot{x}^2\cos\theta}{\dot{t}^2 + x^2\dot{x}^2}, \\ b_1\dot{t} + b_2x^2 &= 1. \end{cases} \tag{3.12}$$

By the third equation of (3.12),

$$b_1\dot{x}\cos\theta - b_2 x\sin\theta = b_1\left(\frac{x\dot{x}\cos\theta + \dot{t}\sin\theta}{x}\right) - \frac{\sin\theta}{x}$$

$$b_1\dot{x}\sin\theta + b_2 x\cos\theta = b_1\left(\frac{x\dot{x}\sin\theta - \dot{t}\cos\theta}{x}\right) + \frac{\cos\theta}{x}.$$

Therefore,

$$\begin{cases} b_1\left(x\dot{x}\cos\theta + \dot{t}\sin\theta\right) &= \dfrac{\dot{t}}{\dot{t}^2 + x^2\dot{x}^2}\left(x\dot{x}\cos\theta + \dot{t}\sin\theta\right), \\ b_1\left(x\dot{x}\sin\theta - \dot{t}\cos\theta\right) &= \dfrac{\dot{t}}{\dot{t}^2 + x^2\dot{x}^2}\left(x\dot{x}\sin\theta - \dot{t}\cos\theta\right). \end{cases}$$

Arguing as above, (3.8) follows. $\square$

Next, noticing that

$$\left\langle \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right\rangle \left\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right\rangle - \left\langle \frac{\partial}{\partial s}, \frac{\partial}{\partial \theta} \right\rangle^2 \overset{(3.1)}{=} \left(\dot{x}(s)^2 + \dot{t}(s)^2\right)\left(x(s)^2 + x(s)^4\right) - x(s)^4 \dot{t}(s)^2$$

and recalling that $x > 0$, we deduce

$$d\sigma = x(s)\sqrt{(1 + x(s)^2)\,\dot{x}(s)^2 + \dot{t}(s)^2}\, ds\, d\theta,$$

hence

$$d\sigma^{\mathcal{H}} \overset{(2.8),(3.3)}{=} x(s)\sqrt{\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2}\, ds\, d\theta. \tag{3.13}$$

If $S$ is closed, the divergence theorem yields

$$|\Omega(S)| \overset{(2.4)}{=} \frac{1}{2}\int_\Omega \operatorname{div}(xX + yY)\, d\mathcal{L}^3 = \left| \int_S \left(x\,\langle \mathbf{N}, X\rangle + y\,\langle \mathbf{N}, Y\rangle\right)\, d\sigma \right| = \pi \left| \int_I x(s)^2 \dot{t}(s)\, ds \right|.$$

In particular, when $\dot{t} \geqslant 0$ on $I$,

$$|\Omega(S)| = \pi \int_I x(s)^2 \dot{t}(s)\, ds. \tag{3.14}$$

Finally, the horizontal mean curvature of rotationally invariant surfaces can be expressed as follows (cf. [43]).

**Lemma 3.2.** *It holds that, on* $S \setminus S_0$,

$$H^{\mathcal{H}} = \frac{x(s)^3(\dot{x}(s)\ddot{t}(s) - \ddot{x}(s)\dot{t}(s)) + \dot{t}(s)^3}{x(s)\left(x(s)^2\dot{x}(s)^2 + \dot{t}(s)^2\right)^{3/2}}. \tag{3.15}$$

*Proof.* Notice that

$$H^{\mathcal{H}} = \left\langle \nabla_{J(\nu)}\nu, J(\nu)\right\rangle \overset{(2.3)}{=} \langle J(\nu), X\rangle\, J(\nu)\,\langle \nu, X\rangle + \langle J(\nu), Y\rangle\, J(\nu)\,\langle \nu, Y\rangle\,.$$

It suffices to show that

$$J(\nu)\,\langle \nu, X\rangle = \left(\frac{x(s)^3(\dot{x}(s)\ddot{t}(s) - \ddot{x}(s)\dot{t}(s)) + \dot{t}(s)^3}{x(s)\left(\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2\right)^{3/2}}\right)\langle J(\nu), X\rangle\,, \tag{3.16}$$

$$J(\nu)\,\langle \nu, Y\rangle = \left(\frac{x(s)^3(\dot{x}(s)\ddot{t}(s) - \ddot{x}(s)\dot{t}(s)) + \dot{t}(s)^3}{x(s)\left(\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2\right)^{3/2}}\right)\langle J(\nu), Y\rangle\,. \tag{3.17}$$

Indeed,

$$\begin{aligned}
J(\nu)\,\langle \nu, X\rangle &\overset{(3.4),(3.7)}{=} \left(\frac{1}{x\sqrt{\dot{t}^2 + x^2\dot{x}^2}}\right)\left(-x^2\frac{\partial}{\partial s}\left(\frac{\dot{t}\cos\theta - x\dot{x}\sin\theta}{\sqrt{\dot{t}^2 + x^2\dot{x}^2}}\right) + \dot{t}\frac{\partial}{\partial \theta}\left(\frac{\dot{t}\cos\theta - x\dot{x}\sin\theta}{\sqrt{\dot{t}^2 + x^2\dot{x}^2}}\right)\right)\\
&= \left(\frac{1}{x(\dot{t}^2 + x^2\dot{x}^2)^2}\right)\left(-x^2\left((\ddot{t}\cos\theta - \dot{x}^2\sin\theta - x\ddot{x}\sin\theta)(\dot{t}^2 + x^2\dot{x}^2) - (\dot{t}\cos\theta - x\dot{x}\sin\theta)(\dot{t}\ddot{t} + x\dot{x}^3 + x^2\dot{x}\ddot{x})\right)\right)\\
&\quad + \left(\frac{1}{x(\dot{t}^2 + x^2\dot{x}^2)^2}\right)\left(\dot{t}(-\dot{t}\sin\theta - x\dot{x}\cos\theta)(\dot{t}^2 + x^2\dot{x}^2)\right)\\
&= \left(\frac{1}{x(\dot{t}^2 + x^2\dot{x}^2)^2}\right)\left(x^3\ddot{x}\dot{t}^2\sin\theta - x^4\dot{x}^2\ddot{t}\cos\theta + x^4\dot{x}\ddot{x}\dot{t}\cos\theta - x^3\dot{x}\dot{t}\ddot{t}\sin\theta - \dot{t}^4\sin\theta - x\dot{x}\dot{t}^3\cos\theta\right)\\
&= \left(\frac{1}{x(\dot{t}^2 + x^2\dot{x}^2)^2}\right)\left(-\dot{t}\sin\theta - x\dot{x}\cos\theta\right)\left(x^3(\dot{x}\ddot{t} - \ddot{x}\dot{t}) + \dot{t}^3\right)\\
&\overset{(3.5)}{=} \langle J(\nu), X\rangle \left(\frac{x^3(\dot{x}\ddot{t} - \ddot{x}\dot{t}) + \dot{t}^3}{\left(\dot{t}^2 + x^2\dot{x}^2\right)^{\frac{3}{2}}}\right),
\end{aligned}$$

which is (3.16). Finally, (3.17) follows by a similar computation. □

3.2. **Examples.** We collect some relevant instances of rotationally invariant surfaces.

**Example 3.3.** (Horizontal plane) The horizontal plane is the rotationally invariant surface $S = \{(x, y, t) \in \mathbb{R}^3 \mid t = 0\}$. Its profile is the curve $\gamma : (0, \infty) \to \mathbb{R}^2$ given by

$$\gamma(s) = (s, 0).$$

The horizontal plane is a simple instance of minimal surface with characteristic points, because $S_0 = \{0\}$.

**Example 3.4** (Vertical cylinder)**.** The vertical cylinder $S = \{(x, y, t) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, t \in \mathbb{R}\}$ has profile $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by

$$\gamma(s) = (1, s).$$

It is a non-characteristic surface with constant horizontal mean curvature $H^{\mathcal{H}} \equiv 1$.

**Example 3.5** (Pansu sphere)**.** The Pansu sphere is the closed surface with profile $\gamma : (0, 2\pi) \to \mathbb{R}^2$ given by

$$\gamma(s) = \left(\sin\left(\frac{s}{2}\right), \frac{1}{4}\left(s - \sin(s) - \pi\right)\right).$$

The Pansu sphere has two characteristic points, $S_0 = \left\{\left(0, 0, -\frac{\pi}{4}\right), \left(0, 0, \frac{\pi}{4}\right)\right\}$, and it has constant horizontal mean curvature $H^{\mathcal{H}} \equiv 2$. Up to intrinsic dilations, it is the unique rotationally invariant closed surface of constant horizontal mean curvature of class $C^2$ in $\mathbb{H}^1$ (cf. [43]). For further convenience, we point out that the profile of the Pansu sphere can be equivalently parametrized by $\gamma : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}^2$, where

$$\gamma(s) = \left(\cos(s), \frac{s}{2} + \frac{\sin 2s}{4}\right).$$

**Example 3.6** (Korányi sphere)**.** The Korányi sphere is the 1-level set of the *Korányi norm*

$$\|(x, y, t)\| := \left(\left(x^2 + y^2\right)^2 + 4t^2\right)^{\frac{1}{4}}.$$

It is a closed, rotationally invariant surface such that $S_0 = \left\{\left(0, 0, -\frac{1}{2}\right), \left(0, 0, \frac{1}{2}\right)\right\}$, and its profile can be parametrized by $\gamma : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}^2$, where

$$\gamma(s) = \left(\sqrt{\cos s}, \frac{\sin s}{2}\right).$$

3.3. **Functionals driven by horizontal mean curvature.** By means of (3.13) and (3.15), functionals driven by the horizontal mean curvature of rotationally invariant surfaces are essentially one-dimensional. Precisely, the (sub-Riemannian) area functional reads as

$$\sigma^{\mathcal{H}}(S) = 2\pi \int_I x\sqrt{\dot{t}^2 + x^2\dot{x}^2}\, ds,$$

while the (horizontal) total mean curvature and total inverse mean curvature have the form

$$\mathscr{H}^{\mathcal{H}}(S) = 2\pi \int_I \frac{x^3(\dot{x}\ddot{t} - \ddot{x}\dot{t}) + \dot{t}^3}{\dot{t}^2 + x^2\dot{x}^2}\, \mathrm{d}s, \qquad \left(\mathscr{H}^{\mathcal{H}}\right)^{-1}(S) = 2\pi \int_I \frac{x^2\left(\dot{t}^2 + x^2\dot{x}^2\right)^2}{x^3(\dot{x}\ddot{t} - \ddot{x}\dot{t}) + \dot{t}^3}\, ds.$$

As already mentioned in the introduction, the above two quantities gain a geometric meaning provided that $S$ is, respectively, mean convex and strictly mean convex. Exploiting the particular shape of the functionals, we begin by showing that Pansu spheres are critical points neither for (1.3) nor for (1.5).

**Theorem 3.7.** *For any $R > 0$, let $S^R$ be the rotationally invariant surface whose profile $\gamma^R : (0, 2\pi) \to \mathbb{R}^2$ is*

$$\gamma^R(s) = \left(R\sin\left(\frac{s}{2}\right), \frac{1}{4}\left(s - \sin s - \pi\right)\right).$$

*Then $S^R$ is a closed, strictly mean convex surface. Moreover,*

$$\mathcal{Q}^{\mathcal{H}}_{\text{mink}}(S^1) = 2\pi^{\frac{2}{3}}, \qquad \left.\frac{d}{dR}\right|_{R=1} \left(\mathcal{Q}^{\mathcal{H}}_{\text{mink}}(S^R)\right) = \pi^{\frac{2}{3}} \tag{3.18}$$

*and*

$$\mathcal{Q}^{\mathcal{H}}_{\text{hk}}(S^1) = \frac{4}{3}, \qquad \left.\frac{d}{dR}\right|_{R=1} \left(\mathcal{Q}^{\mathcal{H}}_{\text{hk}}(S^R)\right) = -\frac{2}{3}. \tag{3.19}$$

*In particular, the Pansu sphere is not a minimum of* (1.3) *and* (1.5)*.*

*Proof.* By definition,

$$\dot{x} = \frac{R}{2}\cos\left(\frac{s}{2}\right), \qquad \dot{t} = \frac{1}{2}\sin^2\left(\frac{s}{2}\right), \qquad \ddot{x} = -\frac{R}{4}\sin\left(\frac{s}{2}\right), \qquad \ddot{t} = \frac{1}{2}\sin\left(\frac{s}{2}\right)\cos\left(\frac{s}{2}\right).$$

Notice that

$$\dot{x}\ddot{t} - \ddot{x}\dot{t} = \frac{R}{8}\sin\left(\frac{s}{2}\right)\left(1+\cos^2\left(\frac{s}{2}\right)\right), \qquad \dot{t}^2 + x^2\dot{x}^2 = \frac{R^4}{4}\sin^2\left(\frac{s}{2}\right)\cos^2\left(\frac{s}{2}\right) + \frac{1}{4}\sin^4\left(\frac{s}{2}\right).$$

Then

$$d\sigma^{\mathcal{H}} = \frac{R}{2}\sin^2\left(\frac{s}{2}\right)\sqrt{R^4\cos^2\left(\frac{s}{2}\right) + \sin^2\left(\frac{s}{2}\right)}, \qquad H^{\mathcal{H}} = \frac{R^4\left(1+\cos^2\left(\frac{s}{2}\right)\right) + \sin^2\left(\frac{s}{2}\right)}{R\left(R^4\cos^2\left(\frac{s}{2}\right) + \sin^2\left(\frac{s}{2}\right)\right)^{\frac{3}{2}}}.$$

By the above computation, $S^R$ is strictly mean convex, and moreover

$$\mathscr{H}^{\mathcal{H}}(S^R) = \pi \int_0^{2\pi} \frac{R^4\sin^2\left(\frac{s}{2}\right)\left(1+\cos^2\left(\frac{s}{2}\right)\right) + \sin^4\left(\frac{s}{2}\right)}{R^4\cos^2\left(\frac{s}{2}\right) + \sin^2\left(\frac{s}{2}\right)}\,\mathrm{d}s.$$

and

$$\sigma^{\mathcal{H}}(S^R) = \pi \int_0^{2\pi} R\sin^2\left(\frac{s}{2}\right)\sqrt{R^4\cos^2\left(\frac{s}{2}\right) + \sin^2\left(\frac{s}{2}\right)}\,\mathrm{d}s$$

Set

$$g(R,s) = \frac{R^4\sin^2\left(\frac{s}{2}\right)\left(1+\cos^2\left(\frac{s}{2}\right)\right) + \sin^4\left(\frac{s}{2}\right)}{R^4\cos^2\left(\frac{s}{2}\right) + \sin^2\left(\frac{s}{2}\right)}, \qquad h(R,s) = R\sin^2\left(\frac{s}{2}\right)\sqrt{R^4\cos^2\left(\frac{s}{2}\right) + \sin^2\left(\frac{s}{2}\right)}.$$

Since

$$R^4\cos^2\left(\frac{s}{2}\right) + \sin^2\left(\frac{s}{2}\right) \geqslant \frac{1}{2} \tag{3.20}$$

for any $R$ sufficiently close to 1, then

$$\frac{d}{dR}\sigma^{\mathcal{H}}(S^R)\bigg|_{R=1} = \pi\int_0^{2\pi} \frac{\partial h(R,s)}{\partial R}\bigg|_{R=1} ds = \pi\int_0^{2\pi}\left(\sin^2\left(\frac{s}{2}\right) + \frac{1}{2}\sin^2 s\right) ds = \frac{3}{2}\pi^2$$

and

$$\frac{d}{dR}\mathscr{H}^{\mathcal{H}}(S^R)\bigg|_{R=1} = \pi\int_0^{2\pi}\frac{\partial g(R,s)}{\partial R}\bigg|_{R=1} ds = 4\pi\int_0^{2\pi}\sin^4\left(\frac{s}{2}\right) ds = 3\pi^2.$$

Moreover,

$$\sigma^{\mathcal{H}}(S^1) = \pi\int_0^{2\pi}\sin^2\left(\frac{s}{2}\right) ds = \pi^2, \qquad \mathscr{H}^{\mathcal{H}}(S^1) = 2\pi\int_0^{2\pi}\sin^2\left(\frac{s}{2}\right) ds = 2\pi^2.$$

By the above computations, (3.18) follows. We prove (3.19). Indeed,

$$|\Omega(S^R)| \overset{(3.14)}{=} \frac{\pi R^2}{2}\int_0^{2\pi}\sin^4\left(\frac{s}{2}\right) ds = \frac{3\pi^2 R^2}{8}, \qquad |\Omega(S^1)| = \frac{3}{8}\pi^2 \qquad \frac{d}{dR}|\Omega(S^R)|\bigg|_{R=1} = \frac{3}{4}\pi^2$$

and

$$\left(\mathscr{H}^{\mathcal{H}}\right)^{-1}(S^R) = \pi\int_0^{2\pi}\frac{\sin^2\left(\frac{s}{2}\right)\left(R^5\cos^2\left(\frac{s}{2}\right) + R\sin^2\left(\frac{s}{2}\right)\right)^2}{R^4\left(1+\cos^2\left(\frac{s}{2}\right)\right) + \sin^2\left(\frac{s}{2}\right)}\,ds, \qquad \left(\mathscr{H}^{\mathcal{H}}\right)^{-1}(S^1) = \frac{1}{2}\pi^2.$$

Set

$$g(R,s) = \frac{\sin^2\left(\frac{s}{2}\right)\left(R^5\cos^2\left(\frac{s}{2}\right) + R\sin^2\left(\frac{s}{2}\right)\right)^2}{R^4\left(1+\cos^2\left(\frac{s}{2}\right)\right) + \sin^2\left(\frac{s}{2}\right)}.$$

By (3.20), we infer that

$$\frac{d}{dR}\left(\mathscr{H}^{\mathcal{H}}\right)^{-1}(S^R)\bigg|_{R=1} = \pi\int_0^{2\pi}\frac{\partial g(R,s)}{\partial R}\bigg|_{R=1} ds = \frac{3}{4}\pi\int_0^{2\pi}\sin^2 s\, ds = \frac{3}{4}\pi^2.$$

Combining the above computations, (3.19) follows. ☐

The same conclusions of Theorem 3.7 hold as well for the Korányi sphere.

**Theorem 3.8.** *Fix $R > 0$. Let $S^R$ be the rotationally invariant surface whose profile $\gamma_R : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}^2$ is*

$$\gamma^R(s) = \left(R\sqrt{\cos s}, \frac{1}{2}\sin s\right).$$

*Then $S^R$ is a closed, strictly mean convex surface. Moreover,*

$$\mathcal{Q}^{\mathcal{H}}_{\text{mink}}(S^1) = 6\left(\frac{4\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)^{-\frac{2}{3}}, \qquad \frac{d}{dR}\bigg|_{R=1} \mathcal{Q}^{\mathcal{H}}_{\text{mink}}(S^R) = \frac{52}{15}\left(\frac{4\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)^{-\frac{2}{3}}, \tag{3.21}$$

*where $\Gamma$ is the well-known* gamma function *(cf. [1]), and*

$$\mathcal{Q}^{\mathcal{H}}_{\text{hk}}(S^1) = \frac{4}{3}, \qquad \frac{d}{dR}\bigg|_{R=1} \mathcal{Q}^{\mathcal{H}}_{\text{hk}}(S^R) = \frac{8}{9}. \tag{3.22}$$

*In particular, the Korányi sphere is not a minimum of (1.3) and (1.5).*

*Proof.* By definition,

$$\dot{x} = -\frac{R^2 \sin s}{2x}, \qquad \dot{t} = \frac{1}{2}\cos s, \qquad \ddot{x} = \frac{-R^2 x\cos s + R^2 \dot{x}\sin s}{2x^2}, \qquad \ddot{t} = -\frac{1}{2}\sin s.$$

Then

$$\dot{x}\ddot{t} - \ddot{x}\dot{t} = \frac{R^2}{8x}\left(2 + \sin^2 s\right), \qquad \dot{t}^2 + x^2\dot{x}^2 = \frac{1}{4}\left(R^4\sin^2 s + \cos^2 s\right).$$

Therefore

$$d\sigma^{\mathcal{H}} = \frac{R}{2}\sqrt{\cos s}\sqrt{R^4\sin^2 s + \cos^2 s}, \qquad H^{\mathcal{H}} = \frac{\sqrt{\cos s}\left(R^4\left(2 + \sin^2 s\right) + \cos^2 s\right)}{R\left(R^4\sin^2 s + \cos^2 s\right)^{\frac{3}{2}}}.$$

In particular, $S^R$ is strictly mean convex. Moreover,

$$\mathscr{H}^{\mathcal{H}}(S^R) = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos s\left(R^4\left(2 + \sin^2 s\right) + \cos^2 s\right)}{R^4\sin^2 s + \cos^2 s}\,\mathrm{d}s$$

and

$$\sigma^{\mathcal{H}}(S^R) = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R\sqrt{\cos s}\sqrt{R^4\sin^2 s + \cos^2 s}\,\mathrm{d}s.$$

Set

$$g(R,s) = \frac{\cos s\left(R^4\left(2 + \sin^2 s\right) + \cos^2 s\right)}{R^4\sin^2 s + \cos^2 s}, \qquad h(R,s) = R\sqrt{\cos s}\sqrt{R^4\sin^2 s + \cos^2 s}.$$

Since

$$R^4\sin^2 s + \cos^2 s \geqslant \frac{1}{2} \tag{3.23}$$

for any $R$ sufficiently close to 1, then

$$\mathscr{H}^{\mathcal{H}}(S^1) = 3\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos s\,\mathrm{d}s = 6\pi$$

and

$$\frac{d}{dR}\mathscr{H}^{\mathcal{H}}(S^R)\bigg|_{R=1} = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial g(R,s)}{\partial R}\bigg|_{R=1} ds = 8\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 s\,ds = \frac{32}{3}\pi.$$

In order to deal with the area functional, we recall (cf. [1]) that the *beta function* is defined by

$$B(p,q) = 2\int_0^{\frac{\pi}{2}} (\sin s)^{2p-1}(\cos s)^{2q-1}\,ds, \qquad p, q \in \mathbb{R},\ p, q > 0,$$

and is related to the gamma function by the identity

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{3.24}$$

Therefore, recalling (cf. [1]) that

$$\Gamma(p+1)=p\Gamma(p), \qquad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}, \tag{3.25}$$

then

$$\sigma^{\mathcal{H}}(S^1)=\pi B\left(\frac{1}{2},\frac{3}{4}\right)\overset{(3.24)}{=}\pi\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}\right)\overset{(3.25)}{=}4\pi^{\frac{3}{2}}\left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right).$$

Moreover, since

$$\frac{d}{dR}\sigma^{\mathcal{H}}(S^R)\bigg|_{R=1}=\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\partial h(R,s)}{\partial R}\bigg|_{R=1}ds=\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos s}\left(1+2\sin^2 s\right)\,ds,$$

then

$$\frac{d}{dR}\sigma^{\mathcal{H}}(S^R)\bigg|_{R=1}=4\pi^{\frac{3}{2}}\left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)+2\pi B\left(\frac{3}{2},\frac{3}{4}\right)\overset{(3.24),(3.25)}{=}\frac{36}{5}\pi^{\frac{3}{2}}\left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right).$$

Combining the above computations, (3.21) follows. We prove (3.22). Indeed,

$$|\Omega(S^R)|=\frac{\pi R^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 s\,ds=\frac{\pi^2R^2}{4}, \qquad |\Omega(S^1)|=\frac{\pi^2}{4}, \qquad \frac{d}{dR}|\Omega(S^R)|\bigg|_{R=1}=\frac{\pi^2}{2}.$$

Moreover,

$$\left(\mathscr{H}^{\mathcal{H}}\right)^{-1}(S^R)=\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\left(R^5\sin^2 s+R\cos^2 s\right)^2}{\left(R^4\left(2+\sin^2 s\right)+\cos^2 s\right)}\,ds, \qquad \left(\mathscr{H}^{\mathcal{H}}\right)^{-1}(S^1)=\frac{1}{3}\pi^2.$$

Set

$$g(R,s)=\frac{\left(R^5\sin^2 s+R\cos^2 s\right)^2}{\left(R^4\left(2+\sin^2 s\right)+\cos^2 s\right)}.$$

By (3.23), we conclude that

$$\frac{d}{dR}\left(\mathscr{H}^{\mathcal{H}}\right)^{-1}(S^R)\bigg|_{R=1}=\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\partial g(R,s)}{\partial R}\bigg|_{R=1}ds=\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{20\sin^2 s-2}{9}\,ds=\frac{8}{9}\pi^2.$$

Combining the above computations, (3.22) follows. □

**Remark 3.9.** By Theorem 3.7 and Theorem 3.8, neither the Pansu sphere nor the Korányi sphere are critical points of $\mathcal{Q}^{\mathcal{H}}_{\mathrm{mink}}$ and $\mathcal{Q}^{\mathcal{H}}_{\mathrm{hk}}$. It is simple to exploit the previous constructions to show that, in addition, they are not critical points neither for (1.2) nor for (1.4). We prove this for the Pansu sphere, the Korányi case being analogous. To this end, we construct two families of competitors preserving respectively the area and the enclosed volume of the Pansu sphere. Let $\left(S^R\right)_{R>0}$ be as in the proof of Theorem 3.7. Set

$$\beta(R)=\left(\frac{\sigma^{\mathcal{H}}(S^1)}{\sigma^{\mathcal{H}}(S^R)}\right)^{\frac{1}{3}}, \qquad \gamma(R)=\left(\frac{|\Omega(S^1)|}{|\Omega(S^R)|}\right)^{\frac{1}{4}}, \qquad R>0.$$

Notice that $\beta(1)=\gamma(1)=1$. Moreover,

$$\sigma^{\mathcal{H}}\left(\delta_{\beta(R)}\left(S^R\right)\right)=\beta(R)^3\sigma^{\mathcal{H}}\left(S^R\right)=\sigma^{\mathcal{H}}\left(S^1\right)$$

and

$$\frac{\partial}{\partial R}\bigg|_{R=1}\mathscr{H}^{\mathcal{H}}\left(\delta_{\beta(R)}\left(S^R\right)\right)=\frac{\partial}{\partial R}\bigg|_{R=1}\left(\beta(R)^2\mathscr{H}^{\mathcal{H}}\left(S^R\right)\right)=\sigma^{\mathcal{H}}\left(S^1\right)^{\frac{2}{3}}\frac{\partial}{\partial R}\bigg|_{R=1}\mathcal{Q}^{\mathcal{H}}_{\mathrm{mink}}\left(S^R\right)\neq 0.$$

Similarly,

$$\left|\Omega\left(\delta_{\beta(R)}\left(S^R\right)\right)\right|=\left|\Omega\left(S^1\right)\right|, \qquad \frac{\partial}{\partial R}\bigg|_{R=1}\left(\mathscr{H}^{\mathcal{H}}\right)^{-1}\left(\delta_{\beta(R)}\left(S^R\right)\right)=\left|\left(S^1\right)\right|\frac{\partial}{\partial R}\bigg|_{R=1}\mathcal{Q}^{\mathcal{H}}_{\mathrm{hk}}\left(S^R\right)\neq 0.$$

## 4. VARIATION FORMULAS

Motivated by Theorem 3.7 and Theorem 3.8, we seek a general strategy to identify the correct critical configurations for problems such as (1.2) and (1.4). The key tool is provided by suitable sub-Riemannian first and second variation formulas. First, we state them in full generality, namely for arbitrary hypersurfaces and arbitrary functionals driven by the horizontal mean curvature. Subsequently, we specialize this framework to the total mean curvature functional in the first Heisenberg group.

4.1. **Variations.** A *variation* is a smooth map $\Phi : I \times \mathbb{H}^n \to \mathbb{H}^n$, where $I \subseteq \mathbb{R}$ is any open neighborhood of 0, such that:

- $p \mapsto \Phi(\tau, p)$ is a diffeomorphism for any $\tau \in I$;
- $\Phi(0,p) = p$ for any $p \in \mathbb{H}^n$.

We adopt the notation $\Phi_\tau(p) \coloneqq \Phi(\tau, p)$. A variation is *compactly supported* if $\Phi_\tau(p) = p$ outside a compact set $K(\Phi)$. In the following, we tacitly assume that a variation is compactly supported. This is clearly not restrictive when computing variations of a closed hypersurface. Define the time-dependent vector field $\mathcal{X}$ by

$$\mathcal{X}(\tau, q) = \frac{\partial}{\partial \sigma}\bigg|_{\sigma=0} \left(\Phi_{\sigma+\tau} \circ \Phi_\tau^{-1}\right)(q) \qquad \text{for any } \tau \in I,\ q \in \mathbb{H}^n.$$

Define

$$\mathbf{X}(q) = \mathcal{X}(0, q), \qquad \mathbf{Z}(q) = \frac{\partial}{\partial \tau}\bigg|_{\tau=0} \mathcal{X}(\tau, q) + \left(\nabla_{\mathbf{X}}\mathbf{X}\right)(q) \qquad \text{for any } q \in \mathbb{H}^n,$$

where in the above definition $\nabla$ is the pseudohermitian connection (2.2). We call the vector fields $\mathbf{X}$ and $\mathbf{Z}$ respectively *variational velocity field* and *variational acceleration field* of $\Phi$. Given any couple of vector fields $\mathbf{X}, \mathbf{Z}$ it is always possible to construct a variation having $\mathbf{X}$ and $\mathbf{Z}$ respectively as velocity and acceleration.

**Proposition 4.1.** *Let $\mathbf{X}, \mathbf{Z} \in \Gamma(T\mathbb{H}^n)$. Assume that $\mathbf{X}$,$\mathbf{Z}$ are compactly supported in $\mathbb{H}^n$. Set*

$$\Phi_\tau(p) \coloneqq p \cdot \left(\tau \mathbf{X}(p) + \frac{\tau^2}{2}\mathbf{Z}(p)\right), \qquad \tau \in I,\ p \in \mathbb{H}^n. \tag{4.1}$$

*If $I$ is sufficiently small, then $\Phi$ is a smooth variation. Moreover, $\Phi$ has velocity $\mathbf{X}$ and acceleration $\mathbf{Z}$.*

*Proof.* First, $\Phi$ is a smooth map such that $\Phi_0(p) = p$. This and the fact that $\mathbf{X}, \mathbf{Z}$ are compactly supported imply that, for $I$ sufficiently small, $p \mapsto \Phi_\tau(p)$ is a diffeomorphism. Hence $\Phi$ is a smooth variation. Set $\Phi_\tau(q) = (\Phi_\tau(q)_1, \dots, \Phi_\tau(q)_{2n+1})$, $\Phi_\tau^{-1}(q) = \left(\Phi_\tau^{-1}(q)_1, \dots, \Phi_\tau^{-1}(q)_{2n+1}\right)$, and denote respectively by $(X^i)_{i=1}^{2n+1}$ and $(X^i)_{i=1}^{2n+1}$ the components of $\mathbf{X}$ and $\mathbf{Z}$ with respect to $Z_1, \dots, Z_{2n+1}$. Notice that

$$\begin{aligned}
\Phi_{\sigma+\tau}(q)_i &\overset{(2.1)}{=} q_i + (\sigma+\tau)X^i(q) + \frac{(\sigma+\tau)^2}{2}Z^i(q), \qquad i = 1, \dots, 2n,\\
\Phi_{\sigma+\tau}(q)_{2n+1} &\overset{(2.1)}{=} q_{2n+1} + (\sigma+\tau)X^{2n+1}(q) + \frac{(\sigma+\tau)^2}{2}Z^{2n+1}(q)\\
&\quad + \sum_{i=1}^n q_{n+i}\left((\sigma+\tau)X^i(q) + \frac{(\sigma+\tau)^2}{2}Z^i(q)\right) - \sum_{i=1}^n q_i\left((\sigma+\tau)X^{n+i}(q) + \frac{(\sigma+\tau)^2}{2}Z^{n+i}(q)\right),
\end{aligned}$$

whence

$$\begin{aligned}
\frac{\partial}{\partial\sigma}\bigg|_{\sigma=0}\Phi_{\sigma+\tau}(q) &= \sum_{i=1}^{2n}\left(X^i(q) + \tau Z^i(q)\right)\frac{\partial}{\partial z_i} + \left(X^{2n+1}(q) + \tau Z^{2n+1}(q)\right)\frac{\partial}{\partial t}\\
&\quad + \left(\sum_{i=1}^n q_{n+i}\left(X^i(q) + \tau Z^i(q)\right) - \sum_{i=1}^n q_i\left(X^{n+i}(q) + \tau Z^{n+i}(q)\right)\right)\frac{\partial}{\partial t}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{X}(\tau, p) &= \sum_{i=1}^{2n}\left(X^i\left(\Phi_\tau^{-1}(p)\right) + \tau Z^i\left(\Phi_\tau^{-1}(p)\right)\right)\frac{\partial}{\partial z_i} + \left(X^{2n+1}\left(\Phi_\tau^{-1}(p)\right) + \tau Z^{2n+1}\left(\Phi_\tau^{-1}(p)\right)\right)\frac{\partial}{\partial t}\\
&+ \left(\sum_{i=1}^n \Phi_\tau^{-1}(p)_{n+i}\left(X^i(\Phi_\tau^{-1}(p)) + \tau Z^i(\Phi_\tau^{-1}(p))\right) - \sum_{i=1}^n \Phi_\tau^{-1}(p)_i\left(X^{n+i}(\Phi_\tau^{-1}(p)) + \tau Z^{n+i}(\Phi_\tau^{-1}(p))\right)\right)\frac{\partial}{\partial t}.
\end{aligned}$$

In particular, recalling that $\Phi_0^{-1}(p)=p$, we deduce that $\mathscr{X}(0,p)=\mathbf{X}(p)$. Next, as $\Phi\left(\tau,\Phi_\tau^{-1}(p)\right)=p$,

$$0=\frac{\partial\Phi}{\partial\tau}(0,p)+\sum_{i,j=1}^{2n+1}\frac{\partial\Phi_i}{\partial z_j}(0,p)\left(\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\Phi_t^{-1}(p)_j\right)\frac{\partial}{\partial z_i}=\mathbf{X}(p)+\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\Phi_\tau^{-1}(p). \tag{4.2}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\mathscr{X}(\tau,p)=\mathbf{Z}(p)&+\sum_{i=1}^{2n}\left(\frac{\partial}{\partial\tau}\bigg|_{\tau=0}X^i\left(\Phi_\tau^{-1}(p)\right)\right)\frac{\partial}{\partial z_i}\\&+\left(\sum_{i=1}^{n}\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\left(\Phi_\tau^{-1}(p)_{n+i}\right)X^i(p)-\sum_{i=1}^{n}\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\left(\Phi_\tau^{-1}(p)_i\right)X^{n+i}(p)\right)\frac{\partial}{\partial t}\\&+\left(\sum_{i=1}^{n}p_{n+i}\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\left(X^i(\Phi_\tau^{-1}(p))\right)-\sum_{i=1}^{n}p_i\frac{\partial}{\partial\tau}\bigg|_{\tau=0}\left(X^{n+i}(\Phi_\tau^{-1}(p))\right)\right)\frac{\partial}{\partial t}\\&\overset{(4.2)}{=}\mathbf{Z}(p)-\sum_{i=1}^{2n+1}\mathbf{X}\left\langle\mathbf{X},\mathbf{Z}_i\right\rangle(p)Z_i(p)\\&\overset{(2.3)}{=}\mathbf{Z}(p)-\left(\nabla_{\mathbf{X}}\mathbf{X}\right)(p).\end{aligned}$$

□

If a hypersurface $S$ is fixed, we say that a smooth variation is *non-characteristic* whenever it is compactly supported outside $S_0$. Roughly speaking, non-characteristic variations do not move $S$ close to its characteristic points. Throughout this section we restrict ourselves to consider non-characteristic variations.

4.2. **First and second variation formulas.** The following crucial result establishes the variation formulas for (1.1). We postpone its proof to Appendix B. We state it for closed hypersurfaces. If instead one computes variations on non-compact hypersurfaces, it suffices to restrict the relevant functionals to $K(\Phi)$.

**Theorem 4.2.** *Let $S\subseteq\mathbb{H}^n$ be a smooth, embedded, closed hypersurface. Fix a function $f:\mathbb{R}\to\mathbb{R}$ which is smooth in a neighborhood of $\left\{H^{\mathcal{H}}(p)\,:\,p\in S\setminus S_0\right\}$. Let $\Phi$ be a smooth non-characteristic variation. Denote by $\mathbf{X}$ and $\mathbf{Z}$ its velocity and acceleration respectively. Define $\varphi,\psi\in C_c^\infty(S\setminus S_0)$ by*

$$\begin{aligned}&\varphi:=\left\langle\mathbf{X},\nu+\alpha T\right\rangle,\\&\psi:=\left\langle\mathbf{Z},\nu+\alpha T\right\rangle-2\left(\mathbf{X}-\varphi\nu\right)\varphi-\Big\langle\nabla_{\mathbf{X}-\varphi\nu}\left(\mathbf{X}-\varphi\nu\right),\nu+\alpha T\Big\rangle-2\alpha\varphi\left\langle\mathbf{X},J(\nu)\right\rangle.\end{aligned}\tag{4.3}$$

*Then*

$$\delta\mathscr{H}_f^{\mathcal{H}}(S)[\varphi]:=\delta\mathscr{H}_f^{\mathcal{H}}(S)[\Phi]=\int_S\varphi\Big(\mathcal{J}^{\mathcal{H}}f'(H^{\mathcal{H}})+f(H^{\mathcal{H}})H^{\mathcal{H}}\Big)\,d\sigma^{\mathcal{H}},\tag{4.4}$$

$$\delta^2\mathscr{H}_f^{\mathcal{H}}(S)[\Phi]=\delta\mathscr{H}_f^{\mathcal{H}}(S)\left[\psi\right]+\int_S\varphi\mathcal{L}^{\mathcal{H}}\varphi\,d\sigma^{\mathcal{H}},\tag{4.5}$$

*where $\mathcal{L}^{\mathcal{H}}$ is the self-adjoint operator defined on $S\setminus S_0$ by*

$$\begin{aligned}\mathcal{L}^{\mathcal{H}}\varphi=\,&\mathcal{J}^{\mathcal{H}}\left(f''(H^{\mathcal{H}})\mathcal{J}^{\mathcal{H}}\varphi\right)\\&+2\operatorname{div}^{\mathcal{H},S}\left\langle f'(H^{\mathcal{H}})A^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\varphi\right)\right)+4f'(H^{\mathcal{H}})\mathcal{S}J(\nu)\varphi-\left(f'(H^{\mathcal{H}})H^{\mathcal{H}}+f(H^{\mathcal{H}})\right)\hat{\Delta}^{\mathcal{H},S}\varphi\\&+4\alpha f'(H^{\mathcal{H}})\Big(\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\varphi,J(\nu)\right)-H^{\mathcal{H}}J(\nu)\varphi\Big)-4f''(H^{\mathcal{H}})\Big(\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}H^{\mathcal{H}},\nabla^{\mathcal{H},S}\varphi\right)+\mathcal{S}\varphi J(\nu)H^{\mathcal{H}}\Big)\\&+\left(f''(H^{\mathcal{H}})H^{\mathcal{H}}-2f'(H^{\mathcal{H}})\right)\left\langle\nabla^{\mathcal{H},S}H^{\mathcal{H}},\nabla^{\mathcal{H},S}\varphi\right\rangle\\&+\varphi f'(H^{\mathcal{H}})\Big(2\operatorname{trace}\left(\left(\tilde{h}^{\mathcal{H}}\right)^3\right)+12\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\alpha,J(\nu)\right)+8\mathcal{S}\alpha+6\tilde{h}^{\mathcal{H}}(J(\nu),J(\nu))\alpha^2+2H^{\mathcal{H}}\alpha^2\Big)\\&+\varphi\Big(f(H^{\mathcal{H}})\left(H^{\mathcal{H}}\right)^2-\left(2f'(H^{\mathcal{H}})H^{\mathcal{H}}+f(H^{\mathcal{H}})\right)\Big(|\tilde{h}^{\mathcal{H}}|^2+4J(\nu)\alpha+(2n+2)\alpha^2\Big)\Big).\end{aligned}$$

*Here $\mathcal{J}^{\mathcal{H}}$ and $A^{\mathcal{H}}$ are the horizontal Jacobi operator and the horizontal shape operator (cf. Section 2.2).*

4.3. **Total mean curvature: stationarity and stability.** In this section we focus on the total mean curvature functional

$$\mathscr{H}^{\mathcal{H}}(S) = \int_S H^{\mathcal{H}}\, d\sigma^{\mathcal{H}}$$

in the first Heisenberg group, describing the correct notions of stationarity and stability when area constraints are prescribed. First, we specialize Theorem 4.2 to area and total mean curvature. In both cases, (4.4) and (4.5) simplify drastically.

**Proposition 4.3** (Variation formulas - Area)**.** *Let $S \subseteq \mathbb{H}^1$ be a smooth, embedded, closed surface. Let $\Phi$ be a smooth non-characteristic variation. Then*

$$\delta\sigma^{\mathcal{H}}(S)[\Phi] = \int_S \varphi H^{\mathcal{H}}\, d\sigma^{\mathcal{H}}, \tag{4.6}$$

$$\delta^2\sigma^{\mathcal{H}}(S)[\Phi] = \delta\sigma^{\mathcal{H}}(S)\,[\psi] + \int_S \Big( (J(\nu)\varphi)^2 - 4\varphi^2 \left(J(\nu)\alpha + \alpha^2\right) \Big)\, d\sigma^{\mathcal{H}}. \tag{4.7}$$

Similar versions of Proposition 4.3 can be found e.g. in [15, 44].

**Proposition 4.4** (Variation formulas - Total mean curvature)**.** *Let $S \subseteq \mathbb{H}^1$ be a smooth, embedded, closed surface. Let $\Phi$ be a smooth non-characteristic variation. Then*

$$\delta\mathscr{H}^{\mathcal{H}}(S)[\Phi] = -4\int_S \varphi \left(J(\nu)\alpha + \alpha^2\right)\, d\sigma^{\mathcal{H}}, \tag{4.8}$$

$$\delta^2\mathscr{H}^{\mathcal{H}}(S)[\Phi] = \delta\mathscr{H}^{\mathcal{H}}(S)\,[\psi] + 4\int_S \Big( -\mathcal{S}\varphi\, J(\nu)\varphi + \varphi^2 \left(2\mathcal{S}\alpha - H^{\mathcal{H}}\alpha^2\right) \Big)\, d\sigma^{\mathcal{H}}. \tag{4.9}$$

*Proof.* Since $f(H^{\mathcal{H}}) \equiv H^{\mathcal{H}}$, $f'(H^{\mathcal{H}}) \equiv 1$ and $f''(H^{\mathcal{H}}) \equiv 0$. Moreover, as $n = 1$, $\mathcal{H}TS = \operatorname{span} J(\nu)$, whence

$$\nabla^{\mathcal{H},S} g = (J(\nu)g)\, J(\nu), \qquad \tilde{h}^{\mathcal{H}}(\mathbf{A},\mathbf{B}) = H^{\mathcal{H}} \langle J(\nu), \mathbf{A}\rangle \langle J(\nu), \mathbf{B}\rangle, \qquad |\tilde{h}^{\mathcal{H}}|^2 = \left(H^{\mathcal{H}}\right)^2$$

for $g \in C^\infty(S \setminus S_0)$ and $\mathbf{A}, \mathbf{B} \in \Gamma(\mathcal{H}TS)$. Then (4.8) directly follows. Finally, $\mathcal{L}^{\mathcal{H}}$ simplifies as

$$\begin{aligned}
\mathcal{L}^{\mathcal{H}}\varphi &= 2\operatorname{div}^{\mathcal{H},S}\left(H^{\mathcal{H}} J(\nu)\varphi J(\nu)\right) + 4\mathcal{S}J(\nu)\varphi - 2H^{\mathcal{H}}\hat{\Delta}^{\mathcal{H},S}\varphi - 2\left\langle \nabla^{\mathcal{H},S} H^{\mathcal{H}}, \nabla^{\mathcal{H},S}\varphi \right\rangle + 4\varphi\Big(2\mathcal{S}\alpha - H^{\mathcal{H}}\alpha^2\Big) \\
&= 2\operatorname{div}^{\mathcal{H},S}\left(H^{\mathcal{H}} J(\nu)\varphi J(\nu)\right) + 4\mathcal{S}J(\nu)\varphi - 2\operatorname{div}^{\mathcal{H},S}\left(H^{\mathcal{H}}\nabla^{\mathcal{H},S}\varphi\right) - 4\alpha H^{\mathcal{H}} J(\nu)\varphi + 4\varphi\Big(2\mathcal{S}\alpha - H^{\mathcal{H}}\alpha^2\Big) \\
&= 4\mathcal{S}J(\nu)\varphi - 4\alpha H^{\mathcal{H}} J(\nu)\varphi + 4\varphi\Big(2\mathcal{S}\alpha - H^{\mathcal{H}}\alpha^2\Big).
\end{aligned}$$

The thesis follows integrating by parts the first term of the above formula (cf. (B.23)). ☐

We are interested in variations which preserve the area. Accordingly, we say that a variation $\Phi$ is:

(i) *area-preserving* if $\sigma^{\mathcal{H}}(\Phi_\tau(S)) \equiv \sigma^{\mathcal{H}}(S)$ for every $\tau \in I$.
(ii) *first-order area-preserving* if $\delta\sigma^{\mathcal{H}}(S)[\Phi] = 0$.

Area-preserving variations are first-order area-preserving, while the latter condition is the first-order approximation of the area-preserving property. Nevertheless, owing to a classical argument (cf. [4, Lemma 2.1]), every first-order area-preserving variation can be modified to produce an area-preserving variation enjoying its same first-order behavior. Before doing this, we need to exclude the existence of closed minimal surfaces.

**Lemma 4.5.** *Let $S \subseteq \mathbb{H}^1$ be a smooth, embedded, closed surface. There exists $p \in S \setminus S_0$ such that $H^{\mathcal{H}}(p) \neq 0$.*

*Proof.* Assume not by contradiction. Let $\tilde{p} \in S \setminus S_0$. By [44, Theorem 4.8], and since $S$ is closed, there exists $s_1 \in (0, \infty)$ maximal with the property that the straight line segment $\{\tilde{p} \cdot \delta_s\left(J(\nu_{\tilde{p}})\right) : 0 \leqslant s \leqslant s_1\}$ is contained in $S$. Set $q = \tilde{p} \cdot \delta_{s_1}\left(J(\nu_{\tilde{p}})\right)$. Again by [44, Theorem 4.8] (cf. also [35]), $q \in S_0$. But in this case, [44, Theorem 4.17] (cf. also [10]) violates the maximality of $s_1$, a contradiction. ☐

**Proposition 4.6.** *Let $S \subseteq \mathbb{H}^1$ be a smooth, embedded, closed surface. Let $\Phi$ be a smooth, non-characteristic variation of $S$. Denote by $\mathbf{X}$ its velocity. If $\Phi$ is first-order area-preserving, then there exists a smooth, non-characteristic, area-preserving variation with velocity $\mathbf{X}$.*

*Proof.* Let $\mathbf{A}$ be a smooth vector field with compact support outside $S_0$. Define the smooth map $\tilde{\Phi}$ by

$$\tilde{\Phi}(\tau, \sigma, p) := p \cdot \left(\tau\mathbf{X}(p) + \sigma\mathbf{A}(p)\right), \qquad \tau, \sigma \in I,\, p \in \mathbb{H}^1.$$

Notice that $\sigma^{\mathcal{H}}\left(\tilde{\Phi}(0,0,S)\right) = \sigma^{\mathcal{H}}(S)$. Moreover, arguing as in the proof of Proposition 4.1 and by (4.6),

$$\frac{\partial}{\partial \tau}\bigg|_{\tau=0} \sigma^{\mathcal{H}}\left(\tilde{\Phi}(\tau,0,S)\right) = \delta\sigma^{\mathcal{H}}(S)[\mathbf{X}] = 0$$

and

$$\frac{\partial}{\partial \sigma}\bigg|_{\sigma=0} \sigma^{\mathcal{H}}\left(\tilde{\Phi}(0,\sigma,S)\right) = \delta\sigma^{\mathcal{H}}(S)[\mathbf{A}] = \int_S \left(\langle \mathbf{A},\nu\rangle + \alpha\langle \mathbf{A},T\rangle\right) H^{\mathcal{H}}\, d\sigma^{\mathcal{H}}.$$

By Lemma 4.5, $H^{\mathcal{H}}$ cannot vanish identically on $S \setminus S_0$. Therefore, we can choose $\mathbf{A}$ such that

$$\frac{\partial}{\partial \sigma}\bigg|_{\sigma=0} \sigma^{\mathcal{H}}\left(\tilde{\Phi}(0,\sigma,S)\right) \neq 0.$$

Therefore, the implicit function theorem yields the existence of a function $\sigma = \sigma(\tau)$, smooth in a neighborhood of 0, such that $\sigma(0) = 0$, $\sigma'(0) = 0$ and $\sigma^{\mathcal{H}}\left(\tilde{\Phi}(\tau,\sigma(\tau),S)\right) \equiv \sigma^{\mathcal{H}}(S)$. Then $\tilde{\Phi}(\tau,p) := \tilde{\Phi}(\tau,\sigma(\tau),p)$ satisfies the desired requirements. ☐

According to the above definitions, we say that $S$ is an *area-preserving critical point along non-characteristic variations* (for the total mean curvature) if $\delta\mathscr{H}^{\mathcal{H}}(S)[\Phi] = 0$ for any area-preserving, non-characteristic variation. As customary, stationarity is equivalent to the validity of an appropriate Euler-Lagrange equation, as well as to the stationarity of a suitable penalized functional. We just recall the following, elementary, linear algebra result.

**Lemma 4.7.** *Let $V$ be a vector space. Let $L_1, L_2 : V \to \mathbb{R}$ be linear functionals. Then, $\ker L_2 \subseteq \ker L_1$ if and only if there exists $\lambda \in \mathbb{R}$ such that $L_1 = \lambda L_2$.*

**Proposition 4.8** (Characterization of stationarity)**.** *Let $S \subseteq \mathbb{H}^1$ be a smooth, embedded, closed surface. The following are equivalent:*

(i) *$S$ is an area-preserving critical point along non-characteristic variations, i.e.*

$$\delta\mathscr{H}^{\mathcal{H}}(S)[\Phi] = 0$$

*for any area-preserving, non-characteristic variation $\Phi$;*

(ii) *there exist $\mathscr{L} \in \mathbb{R}$ such that*

$$\delta\mathscr{H}^{\mathcal{H}}(S)[\Phi] - \mathscr{L}\delta\sigma^{\mathcal{H}}(S)[\Phi] = 0 \tag{4.10}$$

*for any non-characteristic variation $\Phi$.*

*In these cases, $\mathscr{L}$ is unique, and*

$$-4\left(J(\nu)\alpha + \alpha^2\right) = \mathscr{L}H^{\mathcal{H}} \qquad \text{on } S \setminus S_0. \tag{4.11}$$

*Proof.* The proof of (ii) $\Longrightarrow$ (i) follows because area-preserving variations are first-order area-preserving. We prove (i) $\Longrightarrow$ (ii). Let $\Phi$ be any non-characteristic variation. Denote by $\mathbf{X}$ its velocity. Assume that $\delta\sigma^{\mathcal{H}}(S)[\Phi]$=0. By Proposition 4.6, there exists an area-preserving, non-characteristic variation $\tilde{\Phi}$ with velocity $\mathbf{X}$. By (i), $\delta\mathscr{H}^{\mathcal{H}}(S)[\Phi] = \delta\mathscr{H}^{\mathcal{H}}(S)[\tilde{\Phi}] = 0$. Therefore, (ii) follows by Lemma 4.7. In addition, (4.11) easily follows by (ii), (4.6) and (4.8). Finally, uniqueness of $\mathscr{L}$ follows by (4.11) and the already known fact that $H^{\mathcal{H}}$ cannot vanish identically on $S \setminus S_0$. ☐

One may wonder whether the above-mentioned notions of stationarity are equivalent to stationarity for the Minkowski quotient $\mathscr{Q}^{\mathcal{H}}_{\mathrm{mink}}$. One implication is trivial.

**Proposition 4.9.** *If $S$ is a critical point for $\mathscr{Q}^{\mathcal{H}}_{\mathrm{mink}}$ along non-characteristic variations, then $S$ is an area-preserving critical point for $\mathscr{H}^{\mathcal{H}}$ along non-characteristic variations.*

*Proof.* Let $\Phi$ be an area-preserving non-characteristic variation. Then

$$0 = \frac{d}{dt}\bigg|_{t=0} \mathscr{Q}^{\mathcal{H}}_{\mathrm{mink}}\left(\Phi_t(S)\right) = \sigma^{\mathcal{H}}(S)^{-\frac{2}{3}}\left(\delta^{\mathcal{H}}(S)[\Phi]\right),$$

whence $\delta^{\mathcal{H}}(S)[\Phi] = 0$, and the thesis follows. ☐

Nevertheless, as we will discuss in Section 5, the converse implication is surprisingly false. Finally, we discuss some equivalent notions of stability of critical points. As in many classical settings (cf. e.g. [4, 38]), while the first order behavior of $\mathscr{H}^{\mathcal{H}}$ agrees with that of $\mathscr{H}^{\mathcal{H}} - \mathscr{L}\sigma^{\mathcal{H}}$, these two formulations are no longer equivalent at second-order.

**Proposition 4.10** (Characterization of stability)**.** *Let $S \subseteq \mathbb{H}^1$ be a smooth, embedded, closed surface. Assume that $S$ is an area-preserving critical point along non-characteristic variations. The following are equivalent:*

(i) *$S$ is* area-preserving stable along non-characteristic variations, *i.e., by definition,*
$$\delta^2\mathscr{H}^{\mathcal{H}}(S)[\Phi] \geqslant 0$$
*for any area-preserving non-characteristic variation $\Phi$;*

(ii) *if $\mathscr{L}$ is as in (4.11), then*
$$\delta^2\mathscr{H}^{\mathcal{H}}(S)[\Phi] - \mathscr{L}\delta^2\sigma^{\mathcal{H}}(S)[\Phi] \geqslant 0$$
*for any first-order area-preserving non-characteristic variation $\Phi$.*

*Proof.* The implication (ii) $\implies$ (i) is trivial. We prove (i) $\implies$ (ii). Let $\Phi$ be a first-order area-preserving, non-characteristic variation. Denote by $\mathbf{X}$ and $\mathbf{Z}$ its velocity and acceleration respectively, and let $\varphi, \psi$ be as in (4.3). By Proposition 4.6, there exists an area-preserving non-characteristic variation $\tilde{\Phi}$ with velocity $\mathbf{X}$ and a suitable acceleration $\tilde{\mathbf{Z}}$. Accordingly, if $\tilde{\varphi}, \tilde{\psi}$ are as in (4.3), then $\tilde{\varphi} = \varphi$. Since $\tilde{\Phi}$ is area-preserving,

$$0 = \delta^2\sigma^{\mathcal{H}}(S)[\tilde{\Phi}] \overset{(4.7)}{=} \delta\sigma^{\mathcal{H}}(S)[\tilde{\psi}] + \int_S \Big( (J(\nu)\varphi)^2 - 4\varphi^2\left(J(\nu)\alpha + \alpha^2\right) \Big)\, d\sigma^{\mathcal{H}}. \tag{4.12}$$

Moreover, by Proposition 4.8,

$$\delta\mathscr{H}^{\mathcal{H}}(S)[\psi] - \mathscr{L}\delta\sigma^{\mathcal{H}}(S)[\psi] = 0 = \delta\mathscr{H}^{\mathcal{H}}(S)[\tilde{\psi}] - \mathscr{L}\delta\sigma^{\mathcal{H}}(S)[\tilde{\psi}]. \tag{4.13}$$

Therefore

$$\begin{aligned}
\delta^2\mathscr{H}^{\mathcal{H}}(S)[\Phi] - \mathscr{L}\delta^2\sigma^{\mathcal{H}}(S)[\Phi] &\overset{(4.7),(4.9)}{=} \delta\mathscr{H}^{\mathcal{H}}(S)\,[\psi] + 4\int_S \Big( -\mathcal{S}\varphi\, J(\nu)\varphi + \varphi^2\left(2\mathcal{S}\alpha - H^{\mathcal{H}}\alpha^2\right) \Big)\, d\sigma^{\mathcal{H}} \\
&\qquad - \mathscr{L}\delta\sigma^{\mathcal{H}}(S)\,[\psi] - \mathscr{L}\int_S \Big( (J(\nu)\varphi)^2 - 4\varphi^2\left(J(\nu)\alpha + \alpha^2\right) \Big)\, d\sigma^{\mathcal{H}} \\
&\overset{(4.13)}{=} \delta\mathscr{H}^{\mathcal{H}}(S)[\tilde{\psi}] + 4\int_S \Big( -\mathcal{S}\varphi\, J(\nu)\varphi + \varphi^2\left(2\mathcal{S}\alpha - H^{\mathcal{H}}\alpha^2\right) \Big)\, d\sigma^{\mathcal{H}} \\
&\qquad - \mathscr{L}\delta\sigma^{\mathcal{H}}(S)[\tilde{\psi}] - \mathscr{L}\int_S \Big( (J(\nu)\varphi)^2 - 4\varphi^2\left(J(\nu)\alpha + \alpha^2\right) \Big)\, d\sigma^{\mathcal{H}} \\
&\overset{(4.12)}{=} \delta\mathscr{H}^{\mathcal{H}}(S)[\tilde{\psi}] + 4\int_S \Big( -\mathcal{S}\varphi\, J(\nu)\varphi + \varphi^2\left(2\mathcal{S}\alpha - H^{\mathcal{H}}\alpha^2\right) \Big)\, d\sigma^{\mathcal{H}} \\
&\overset{(4.9)}{=} \delta^2\mathscr{H}^{\mathcal{H}}(S)[\tilde{\Phi}],
\end{aligned}$$

and the thesis follows by (i). □

## 5. TOTAL MEAN CURVATURE: ROTATIONALLY INVARIANT CRITICAL POINTS

In this section we characterize rotationally invariant surfaces in $\mathbb{H}^1$ which are mean convex area-preserving critical points of $\mathscr{H}^{\mathcal{H}}$ along non-characteristic variations. Precisely, we explicitly solve the Euler-Lagrange equation (4.11). To this aim, we specialize it to a rotationally invariant surface. Indeed, by (3.3) and (3.7),

$$J(\nu)\alpha + \alpha^2 = \dot{t}(s)\left( \frac{\dot{x}(s)^2\dot{t}(s) - x(s)\left(\dot{x}(s)\ddot{t}(s) - \ddot{x}(s)\dot{t}(s)\right)}{\left(\dot{t}(s)^2 + x(s)^2\dot{x}(s)^2\right)^2} \right). \tag{5.1}$$

Therefore, combining (3.15) with (5.1), then (4.11) reads as

$$-4\dot{t}\left( \frac{\dot{x}^2\dot{t} - x\left(\dot{x}\ddot{t} - \ddot{x}\dot{t}\right)}{\left(\dot{t}^2 + x^2\dot{x}^2\right)^2} \right) = \mathscr{L}\left( \frac{x^3(\dot{x}\ddot{t} - \ddot{x}\dot{t}) + \dot{t}^3}{x\left(x^2\dot{x}^2 + \dot{t}^2\right)^{3/2}} \right).$$

Equivalently, letting $L \in \mathbb{R}$ be such that $\mathscr{L} = 4L$, we need to solve

$$x\dot{t}\Big(\dot{x}^2\dot{t} - x\left(\dot{x}\ddot{t} - \ddot{x}\dot{t}\right)\Big) + L\sqrt{x^2\dot{x}^2 + \dot{t}^2}\left(x^3(\dot{x}\ddot{t} - \ddot{x}\dot{t}) + \dot{t}^3\right) = 0. \tag{5.2}$$

First, we show that there are no rotationally invariant closed surfaces satisfying (5.2) with $L = 0$.

**Proposition 5.1.** *Assume that $S$ satisfies (5.2) with $L = 0$. Then:*

(i) *either $S$ is, up to vertical translations, the horizontal plane of Example 3.3;*
(ii) *or $S$ is, up to dilations, the vertical cylinder of Example 3.4;*
(iii) *or the profile of $S$ is $\gamma(s) = (\sqrt{as+b}, \pm s)$ for some $a \neq 0$, $b \in \mathbb{R}$, and any $s \in \mathbb{R}$ with $as + b > 0$.*

*Proof.* Let $\gamma = (x,t) : I \to \mathbb{R}^2$ be the profile of $S$. Since $x > 0$ on $I$, then $\dot t\big(x^2 \dot t - x\left(\dot x \ddot t - \ddot x \dot t\right)\big) = 0$ on $I$. If $\dot t(\hat s) = 0$ for some $\hat s \in I$, then, by uniqueness, $\dot t \equiv 0$. In this case, $S$ is, up to vertical translation, the horizontal plane of Example 3.3. Assume instead $\dot t \neq 0$ on I. In this case we can choose either $\gamma(s) = (x(s), s)$ or $\gamma(s) = (x(s), -s)$. In both cases,

$$0 \overset{(5.2)}{=} \dot x^2 + x\ddot x = \frac{d}{ds}(x\dot x) = \frac{1}{2}\frac{d^2}{ds^2}\left(x^2\right).$$

Therefore, $x(s)^2 = as + b$, for some $a, b \in \mathbb{R}$. If $a = 0$, then $S$ is, up to dilations, the vertical cylinder of Example 3.4. If $a \neq 0$, then $x(s) = \sqrt{as+b}$, where $I$ is defined by $as + b > 0$. The thesis follows. ☐

Next, we provide an *a priori* bound for $L$ in order for (5.2) to be satisfied by a closed surface. Moreover, we show that closed solutions to (5.2) are the union of two vertical graphs.

**Proposition 5.2.** *Let $S$ be a rotationally invariant closed surface. Since $S$ is closed, there exists $\hat s$ such that*

$$x(\hat s) = \max\{x(s), s \in \overline{I}\}.$$

*Up to dilations, we assume that $x(\hat s) = 1$. If $S$ solves (5.2), then $L \in (0,1)$. If in addition $S$ is mean convex, $S$ is the union of two vertical graphs.*

*Proof.* Let $\hat s$ be as in the statement. In particular, $\dot x(\hat s) = 0$ and $\ddot x(\hat s) \leqslant 0$. Since the parametrization is counterclockwise, then $\dot t(\hat s) > 0$. By evaluating (5.2) at $\hat s$, we infer that

$$\ddot x(\hat s)\dot t(\hat s)^2 + L\dot t(\hat s)\left(-\ddot x(\hat s)\dot t(\hat s) + \dot t(\hat s)^3\right) = 0.$$

Since $\dot t(\hat s) \neq 0$, then $\ddot x(\hat s) + L\left(-\ddot x(\hat s) + \dot t(\hat s)^2\right) = 0$, that is

$$(1 - L)\ddot x(\hat s) = -L\dot t(\hat s)^2. \tag{5.3}$$

By Proposition 5.1, the right hand side of (5.3) is not zero. Therefore $L \neq 1$, and moreover

$$\ddot x(\hat s) = -\left(\frac{L}{1-L}\right)\dot t(\hat s)^2.$$

Since $\ddot x(\hat s) \leqslant 0$, then $\frac{L}{1-L} \geqslant 0$. Recalling that $L \neq 0$ and $L \neq 1$, we conclude that $L \in (0,1)$. Assume that $S$ is mean convex. We prove that it is the union of two vertical graphs. Without loss of generality, assume that $\gamma$ is parametrized by arc-length, so that $\kappa = \dot x\ddot t - \ddot x\dot t$ is the curvature of $\gamma$. Notice that a rotationally invariant, closed surface is homeomorphic either to a sphere or to a torus. Assume first that $S$ is homeomorphic to a sphere. We claim that $\hat s$ is the unique point in $I$ such that $\dot x(\hat s) = 0$. Assume not by contradiction. Let $s_1 \neq \hat s$ be such that $\dot x(s_1) = 0$. Then there exists $\tilde s \in I$ such that $\dot x(\tilde s) = 0$ and $\kappa(\tilde s) \leqslant 0$. Indeed, if $\kappa(s_1) \leqslant 0$, just choose $\tilde s = s_1$. Otherwise, there exists $s_2 \in I$ satisfying the desired properties, and in this case we set $\tilde s = s_2$ Since $\dot x(\tilde s) = 0$, then $\dot t(\tilde s) \neq 0$. We claim that $\dot t(\tilde s) > 0$. If not, then

$$H^{\mathcal{H}}(\tilde s) = \frac{x^3(\tilde s)\kappa(\tilde s) + \dot t^3(\tilde s)}{x(\tilde s)|\dot t(\tilde s)|^3} < 0,$$

a contradiction with the fact that $S$ is mean convex. In particular, $0 \geqslant \kappa(\tilde s) = -\ddot x(\tilde s)\dot t(\tilde s)$, whence $\ddot x(\tilde s) \geqslant 0$. Evaluating (5.2) at $\tilde s$, and since $\dot x(\tilde s) = 0$ and $\dot t(\tilde s) > 0$, we get

$$x(\tilde s)^2\ddot x(\tilde s)\dot t(\tilde s)^2 + L\dot t(\tilde s)\left(-x(\tilde s)^3\ddot x(\tilde s)\dot t(\tilde s) + \dot t(\tilde s)^3\right) = 0,$$

whence

$$x(\tilde s)^2\ddot x(\tilde s)\left(1 - Lx(\tilde s)\right) = -L\dot t(\tilde s)^2. \tag{5.4}$$

By definition of $\hat s$, then $x(\tilde s) \leqslant 1$. Since $L \in (0,1)$, then $1 - Lx(\tilde s) > 0$. Therefore, (5.4) implies that $\ddot x(\tilde s) < 0$, a contradiction. Therefore, the profile of $S$ is the union of two vertical graphs over the interval $(0, \hat s)$. Finally, assume that $S$ is homeomorphic to a torus. Let $\check s \in I$ be such that $x(\check s) = \min\{x(s) : s \in \overline{I}\}$. We claim that $\check s$ and $\hat s$ are the unique points where $\dot x = 0$. If not, one can argue as above to infer the existence of $\tilde s \in I$ such that $\dot x(\tilde s) = 0$ and $\kappa(\tilde s) \leqslant 0$. Then, since $S$ is mean convex, we deduce as above that $\dot t(\tilde s) > 0$. In particular, $\ddot x(\tilde s) \geqslant 0$. Arguing *verbatim* as above, we conclude that $\ddot x(\tilde s) < 0$, reaching a contradiction. Again, the profile of $S$ is the union of two vertical graphs over the interval $(\check s, \hat s)$. The thesis follows. $\square$

By Proposition 5.2, mean convex solutions to (5.2) are the union of an upper vertical graph and a lower vertical graph. Denote respectively by $\gamma^+ = (s, t^+(s))$ and $\gamma^- = (s, t^-(s))$ their profile. Notice that $\gamma^+$ is parametrized clockwise, while $\gamma^-$ is parametrized counterclockwise. When $S$ is homeomorphic to a sphere, $\gamma^+, \gamma^- : (0,1) \to \mathbb{R}^2$. Moreover, since $S$ is at least of class $C^1$, then

$$\lim_{s\to 1^-} \dot t^-(s) = +\infty, \qquad \lim_{s\to 1^-} \dot t^+(s) = -\infty.$$

Instead, when $S$ is homeomorphic to a torus, then $\gamma^+, \gamma^- : (\check s, 1) \to \mathbb{R}^2$ for some $\check s \in (0,1)$, and moreover

$$\lim_{s\to 1^-} \dot t^-(s) = +\infty, \qquad \lim_{s\to \check s^+} \dot t^-(s) = -\infty, \qquad \lim_{s\to 1^-} \dot t^+(s) = -\infty, \qquad \lim_{s\to \check s^+} \dot t^+(s) = +\infty.$$

Notice that, when a profile $\gamma$ admits a graphical parametrization $\gamma(s) = (s, t(s))$, (5.2) simplifies as

$$s\dot t(s)^2 - s^2 \dot t(s)\ddot t(s) + L\sqrt{\dot t(s)^2 + s^2}\,\left(s^3 \ddot t(s) + \dot t(s)^3\right) = 0. \tag{5.5}$$

In particular, setting $w(s) = \dot t(s)$, (5.5) reads as

$$s w(s)^2 - s^2 w(s)\dot w(s) + L\sqrt{w(s)^2 + s^2}\,\left(s^3 \dot w(s) + w(s)^3\right) = 0. \tag{5.6}$$

Our final step consists in characterizing all possible solutions to (5.6) under the above-mentioned boundary verticality conditions. We focus on $\gamma_-$, being the characterization of $\gamma_+$ completely analogous. First, we consider the case in which $S$ is homeomorphic to a sphere.

**Lemma 5.3.** *Let $L \in (0,1)$. Then the system*

$$\begin{cases} s\, w(s)^2 - s^2 w(s)\, \dot w(s) + L\sqrt{w(s)^2 + s^2}\, \left(s^3 \dot w(s) + w(s)^3\right) = 0, \\ \lim\limits_{s\to 1^-} w(s) = +\infty \end{cases} \tag{5.7}$$

*is solvable in $C^1(0,1)$ if and only if $L \in \left(0, \frac{1}{2}\right]$. The solution is unique, and it is given explicitly by*

$$w_L^-(s) = \frac{s}{\sqrt{2L}}\sqrt{\frac{Ls^2 - 2L + 1 + s\sqrt{L^2 s^2 - 2L + 1}}{1 - s^2}}.$$

*Proof.* When $L \in \left(0, \frac{1}{2}\right]$, simple computation shows that $w_L^-$ is well-defined and solves (5.7). Conversely, fix $L \in (0,1)$ and let $w$ be a solution to (5.7). Set

$$z(s) = \frac{w(s)}{\sqrt{w(s)^2 + s^2}}, \qquad s \in (0,1).$$

Then $z \in C^1(0,1)$, and

$$\dot z(s) = \frac{1}{w(s)^2 + s^2}\left(\dot w(s)\sqrt{w(s)^2 + s^2} - w(s)\left(\frac{w(s)\dot w(s) + s}{\sqrt{w(s)^2 + s^2}}\right)\right) = s\left(\frac{\dot w(s) s - w(s)}{\left(w(s)^2 + s^2\right)^{\frac{3}{2}}}\right).$$

In particular,

$$w(s) = z(s)\sqrt{w(s)^2 + s^2}, \qquad \dot w(s) s - w(s) = \frac{\dot z(s)\left(w(s)^2 + s^2\right)^{\frac{3}{2}}}{s}. \tag{5.8}$$

Therefore,

$$\begin{aligned} 0 &\overset{(5.7)}{=} s w(s)^2 - s^2 w(s)\dot w(s) + L\sqrt{w(s)^2 + s^2}\, \left(s^3 \dot w(s) + w(s)^3\right) \\ &= s w(s)\left(w(s) - s\dot w(s)\right) + L\sqrt{w(s)^2 + s^2}\, \left(s^2\left(s\dot w(s) - w(s)\right) + w(s)\left(w(s)^2 + s^2\right)\right) \\ &\overset{(5.8)}{=} -z(s)\dot z(s)\left(w(s)^2 + s^2\right)^2 + L\sqrt{w(s)^2 + s^2}\left(s\dot z(s)\left(w(s)^2 + s^2\right)^{\frac{3}{2}} + z(s)\left(w(s)^2 + s^2\right)^{\frac{3}{2}}\right) \\ &= \left(w(s)^2 + s^2\right)^2\Big(-z(s)\dot z(s) + L\left(s\dot z(s) + z(s)\right)\Big) \\ &= -\frac{1}{2}\left(w(s)^2 + s^2\right)^2 \frac{d}{ds}\Big(z(s)^2 - 2Lsz(s)\Big). \end{aligned}$$

Since $\big(w(s)^2+s^2\big)^2 \neq 0$ on $(0,1)$, then there exists $a\in\mathbb{R}$ such that

$$z(s)^2-2Lsz(s)+a=0, \qquad s\in(0,1). \tag{5.9}$$

By the verticality condition of (5.7), we deduce that $\lim_{s\to1^-} z(s)=1$. Combining this information with (5.9), we deduce that $a=2L-1$, whence

$$z(s)^2-2Lsz(s)+2L-1=0, \qquad s\in(0,1). \tag{5.10}$$

Notice that (5.10) is a quadratic equation in $z(s)$, whose discriminant is given by $4L^2s^2-4(2L-1)$. By (5.10), then, $4L^2s^2-4(2L-1)\geqslant 0$ for any $s\in(0,1)$, whence we deduce that $L\leqslant\frac{1}{2}$. We conclude that in this case $w=w_L$. Indeed, since $L\leqslant\frac12$, the discriminant $4L^2s^2-4(2L-1)$ is positive for any $s\in(0,1)$, whence

$$z(s)\equiv Ls+\sqrt{L^2s^2-2L+1} \qquad \text{or} \qquad z(s)\equiv Ls-\sqrt{L^2s^2-2L+1}, \qquad s\in(0,1).$$

The second possibility can be discarded, because in that case $\lim_{s\to1^-} z(s)=2L-1\neq1$. We conclude that

$$z(s)=Ls+\sqrt{L^2s^2-2L+1}, \qquad s\in(0,1).$$

Since $0<z(s)<1$ for any $s\in(0,1)$, then $w(s)>0$ for any $s\in(0,1)$, and, by simple computations,

$$w(s)=\frac{sz(s)}{\sqrt{1-z(s)^2}}=w_L^-(s).$$

□

Finally, we show that $S$ cannot be homeomorphic to a torus.

**Lemma 5.4.** *Let $L\in(0,1)$ and $\check{s}\in(0,1)$. The system*

$$\begin{cases} s\,w(s)^2-s^2w(s)\,\dot w(s)+L\sqrt{w(s)^2+s^2}\,\left(s^3\dot w(s)+w(s)^3\right)=0,\\ \lim\limits_{s\to\check{s}^+} w(s)=-\infty\\ \lim\limits_{s\to1^-} w(s)=+\infty \end{cases} \tag{5.11}$$

*is not solvable in $C^1(\check{s},1)$.*

*Proof.* Assume by contradiction that $w$ solves (5.11) for some $L,\check{s}\in(0,1)$. Let $z$ be as in the proof of Lemma 5.3. Arguing *verbatim* as above, the equation and the verticality condition at 1 imply

$$z(s)^2-2Lsz(s)+2L-1=0, \qquad s\in(\check{s},1). \tag{5.12}$$

Moreover, the verticality condition at $\check{s}$ and the definition of $z$ imply

$$\lim_{s\to\check{s}^+} z(s)=-1. \tag{5.13}$$

By (5.12) and (5.13) it follows that $2L(\check{s}+1)=0$, whence either $L=0$ or $\check{s}=-1$, a contradiction. □

In the same way, the unique possible profiles of vertical upper graphs are of the form $\gamma_L^+(s)=(s,-t_L^-(s))$ for $L\in\left(0,\frac12\right]$. Therefore, $S$ is the union of two vertical graphs with profiles $\gamma_L^+$ and $\gamma_L^-$ for some $L\in\left(0,\frac12\right]$. We then proved the following rigidity statement.

**Proposition 5.5.** *Let $S$ be rotationally invariant, closed, mean convex. The following are equivalent:*

(i) *$S$ is an area-preserving critical point for $\mathscr{H}^{\mathcal{H}}$ along non-characteristic variations;*
(ii) *$S$ is the union of two vertical graphs with profiles $\gamma_L^+$ and $\gamma_L^-$ for some $L\in\left(0,\frac12\right]$.*

For $L\in\left(0,\frac12\right]$, denote the corresponding critical point by $S_L$. We first show that it is possible to describe them by means of a global parameterization. Let $\gamma_L=(x_L,t_L):[-\arccos\sqrt{1-2L},\arccos\sqrt{1-2L}]\to\mathbb{R}^2$ be the regular parametrization given by

$$\begin{cases} x_L(s)=\dfrac{1}{2L}\left(\cos s-\dfrac{1-2L}{\cos s}\right),\\[2ex] t_L(s)=\dfrac{1}{4L^2}\left(\dfrac{s}{2}+\dfrac{\sin 2s}{4}-(1-2L)^2\tan s\right). \end{cases}$$

We show that $\gamma_L$ is indeed a parametrization of $S_L$. We write $x_L = x$ and $t_L = t$, and we set $k = 1 - 2L$. Set

$$h(s) = \frac{1}{2L}\left(\cos s + \frac{k}{\cos s}\right).$$

Notice that both $x$ and $h$ are positive on $(-\arccos\sqrt{k}, \arccos\sqrt{k})$. Moreover,

$$\dot{h}(s) = \frac{1}{2L}\left(-\sin s + \frac{k\sin s}{\cos^2 s}\right) = -\tan s\, x(s).$$

Therefore

$$\begin{cases} \dot{x}(s) = \dfrac{1}{2L}\left(-\sin s - \dfrac{k\sin s}{\cos^2 s}\right) = -\tan s\, h(s), \\ \dot{t}(s) = \dfrac{1}{4L^2}\left(\cos^2 s - \dfrac{k^2}{\cos^2 s}\right) = x(s)\, h(s), \end{cases} \qquad \begin{cases} \ddot{x}(s) = -\dfrac{h(s)}{\cos^2 s} + \tan^2 s\, x(s), \\ \ddot{t}(s) = -\tan s\, h(s)^2 - \tan s\, x(s)^2. \end{cases}$$

In particular

$$x(s)^2\dot{x}(s)^2 + \dot{t}(s)^2 = \tan^2 s\, x(s)^2\, h(s)^2 + x(s)^2\, h(s)^2 = \frac{x(s)^2\, h(s)^2}{\cos^2 s} \tag{5.14}$$

and

$$\dot{x}(s)\ddot{t}(s) - \ddot{x}(s)\dot{t}(s) = \tan(s)^2 h(s)^3 + \frac{x(s)h(s)^2}{\cos^2 s}. \tag{5.15}$$

A simple check reveals that $\gamma_L$ solves (5.2). By Proposition 5.5, we conclude that $\gamma_L$ is the profile of $S_L$.

**Remark 5.6.** Notice that

$$\gamma_{\frac{1}{2}}(s) = \left(\cos s, \frac{s}{2} + \frac{\sin 2s}{4}\right), \qquad s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

is precisely the Pansu sphere as in Example 3.5. In particular, Proposition 5.5 shows that the Pansu sphere is an area-preserving critical point for $\mathscr{H}^{\mathcal{H}}$ under non-characteristic variations. However, Theorem 3.7 demonstrates that the non-characteristic condition is essential: it constructs a variation of the Pansu sphere that fixes the characteristic points but is not non-characteristic, and for which the first variation does not vanish.

**Remark 5.7.** It is interesting to observe that the Pansu sphere is also a critical point for the volume-constrained Heintze-Karcher problem (1.4). Indeed, by Theorem 4.2 and [42, Proposition 1.19], the Euler-Lagrange equation associated with the minimization of $\left(\mathscr{H}^{\mathcal{H}}\right)^{-1}$ under volume constraint reads as

$$\hat{\Delta}^{\mathcal{H},S}\left(\frac{1}{(H^{\mathcal{H}})^2}\right) + \frac{4\left(J(\nu)\alpha + \alpha^2\right)}{(H^{\mathcal{H}})^2} + 2 = \mathscr{L}, \tag{5.16}$$

where $\mathscr{L} \in \mathbb{R}$ is the Lagrange multiplier arising from the volume constraint. Since the Pansu sphere has constant mean curvature, (4.11) implies that $J(\nu)\alpha + \alpha^2$ is constant. Hence, the Pansu sphere satisfies (5.16).

Motivated by Remark 5.6, we refer to our critical points as *Pansu-Minkowski spheres*. It is then natural to understand, among all Pansu-Minkowski spheres, which is a critical configuration for (1.3), as well as which is the optimal shape for the same problem. To this end, notice that

$$d\sigma^{\mathcal{H}} \overset{(5.14)}{=} \frac{x(s)^2 h(s)}{\cos s}\, ds\, d\theta = \frac{1}{8L^3}\left(\cos^2 s - k - \frac{k^2}{\cos^2 s} + \frac{k^3}{\cos^4 s}\right) ds\, d\theta, \tag{5.17}$$

and moreover

$$H^{\mathcal{H}} \overset{(5.15)}{=} \frac{\cos(s)^3}{x(s)h(s)^3}\left(\tan^2 s\, h(s)^3 + \frac{x(s)h(s)^2}{\cos(s)^2} + h(s)^3\right) = \frac{x(s) + h(s)}{x(s)h(s)}\cos(s) = \frac{4L\cos^4 s}{\cos^4 s - k^2} \tag{5.18}$$

and

$$H^{\mathcal{H}}\, d\sigma^{\mathcal{H}} = x(s)\left(x(s) + h(s)\right) = \frac{1}{2L^2}\left(cos^2 s - k\right)\, ds\, d\theta. \tag{5.19}$$

Therefore

$$
\begin{aligned}
\sigma^{\mathcal{H}}(S_L) &= \frac{\pi}{4L^3}\int_{-\arccos\sqrt{k}}^{\arccos\sqrt{k}}\left(\cos^2 s - k - \frac{k^2}{\cos^2 s} + \frac{k^3}{\cos^4 s}\right) ds\\
&= \frac{\pi}{4L^3}\int_{-\arccos\sqrt{k}}^{\arccos\sqrt{k}}\left(\frac{1+\cos 2s}{2} - k - k^2\frac{d}{ds}(\tan s) + k^3\frac{d}{ds}(\tan s)\left(1+\tan^2 s\right)\right) ds\\
&= \frac{\pi}{2L^3}\left[\frac{s}{2} + \frac{\sin s\cos s}{2} - ks - k^2\tan s + k^3\tan s + \frac{k^3}{3}\tan^3 s\right]_0^{\arccos\sqrt{k}} \qquad (5.20)\\
&= \frac{\pi}{2L^3}\left(\frac{(1-2k)\arccos\sqrt{k}}{2} + \frac{\sqrt{k(1-k)}}{2} - k^2(1-k)\sqrt{\frac{1-k}{k}} + \frac{k^3}{3}\left(\frac{1-k}{k}\right)^{\frac{3}{2}}\right)\\
&= \frac{\pi}{4L^3}\left((4L-1)\arccos\sqrt{1-2L} + \left(1 - \frac{8L(1-2L)}{3}\right)\sqrt{2L(1-2L)}\right).
\end{aligned}
$$

Moreover,

$$
\mathscr{H}^{\mathcal{H}}(S_L) = \frac{\pi}{L^2}\int_{-\arccos\sqrt{k}}^{\arccos\sqrt{k}}\left(\cos^2 s - k\right) ds = \frac{\pi}{L^2}\left((4L-1)\arccos\sqrt{1-2L} + \sqrt{2L(1-2L)}\right) \tag{5.21}
$$

Therefore, recalling that $\mathscr{Q}^{\mathcal{H}}_{\mathrm{mink}} := \left(\sigma^{\mathcal{H}}\right)^{-2/3}\mathscr{H}^{\mathcal{H}}$, we conclude that

$$
\mathscr{Q}^{\mathcal{H}}(S_L) = 2(2\pi)^{\frac{1}{3}}\frac{(4L-1)\arccos\sqrt{1-2L} + \sqrt{2L(1-2L)}}{\left((4L-1)\arccos\sqrt{1-2L} + \frac{1}{3}(3-8L+16L^2)\sqrt{2L(1-2L)}\right)^{\frac{2}{3}}}.
$$

The next result highlights the distinguished role played by the *optimal Pansu-Minkowski sphere* $S_{\frac{1}{4}}$.

**Proposition 5.8.** *The following holds.*

$$
\mathscr{Q}^{\mathcal{H}}_{\mathrm{mink}}(S_L) \geqslant (18\pi)^{\frac{1}{3}}, \qquad \mathscr{Q}^{\mathcal{H}}_{\mathrm{mink}}(S_L) = (18\pi)^{\frac{1}{3}} \text{ if and only if } L = \frac{1}{4}.
$$

*Moreover,*

$$
\mathscr{L} = \frac{2\mathscr{H}^{\mathcal{H}}(S_L)}{3\sigma^{\mathcal{H}}(S_L)} \text{ if and only if } L = \frac{1}{4}, \tag{5.22}
$$

*and* $S_{\frac{1}{4}}$ *is the unique rotationally invariant critical point of* $\mathscr{Q}^{\mathcal{H}}_{\mathrm{mink}}$ *under non-characteristic variations.*

*Proof.* We show that $S_{\frac{1}{4}}$ is the unique minimum point of $\mathscr{Q}^{\mathcal{H}}_{\mathrm{mink}}$ within Pansu-Minkowski spheres. Consider the re-parametrization

$$
\ell = 2\arccos\sqrt{1-2L} \in (0,\pi].
$$

A simple computation shows that

$$
(4L-1)\arccos\sqrt{1-2L} = -\frac{1}{2}\ell\cos\ell, \qquad \sqrt{2L(1-2L)} = \frac{1}{2}\sin\ell, \qquad \frac{1}{3}(3-8L+16L^2) = 1 - \frac{1}{3}\sin^2\ell.
$$

Therefore

$$
\mathscr{Q}^{\mathcal{H}}_{\mathrm{mink}}\left(S_{L(\ell)}\right) = (2\pi)^{\frac{1}{3}}\frac{\sin\ell - \ell\cos\ell}{\left(\frac{1}{2}\left(1-\frac{1}{3}\sin^2\ell\right)\sin\ell - \frac{1}{2}\ell\cos\ell\right)^{\frac{2}{3}}} = 4(2\pi)^{\frac{1}{3}}\frac{\sin\ell - \ell\cos\ell}{\left(3\sin\ell + \frac{1}{3}\sin 3\ell - 4\ell\cos\ell\right)^{\frac{2}{3}}},
$$

where in the last equality we exploited the identity $-4\sin^3\ell = \sin 3\ell - 3\sin\ell$. Notice that $L\left(\frac{\pi}{2}\right) = \frac{1}{4}$. To conclude, it suffices to check that

$$
\frac{d\mathscr{Q}^{\mathcal{H}}_{\mathrm{mink}}\left(S_{L(\ell)}\right)}{d\ell} < 0 \text{ on } \left(0,\frac{\pi}{2}\right), \qquad \left.\frac{d\mathscr{Q}^{\mathcal{H}}_{\mathrm{mink}}\left(S_{L(\ell)}\right)}{d\ell}\right|_{\ell=\frac{\pi}{2}} = 0, \qquad \frac{d\mathscr{Q}^{\mathcal{H}}_{\mathrm{mink}}\left(S_{L(\ell)}\right)}{d\ell} > 0 \text{ on } \left(\frac{\pi}{2},\pi\right). \tag{5.23}
$$

To this aim,

$$
\begin{aligned}
\frac{d\mathcal{Q}^{\mathcal{H}}_{\text{mink}}\left(S_{L(\ell)}\right)}{d\ell} &= 4(2\pi)^{\frac{1}{3}}\frac{\ell\sin\ell\left(3\sin\ell+\frac{1}{3}\sin 3\ell-4\ell\cos\ell\right)-\frac{2}{3}\left(\sin\ell-\ell\cos\ell\right)\left(\cos 3\ell-\cos\ell+4\ell\sin\ell\right)}{\left(3\sin\ell+\frac{1}{3}\sin 3\ell-4\ell\cos\ell\right)^{\frac{5}{3}}}\\
&=\frac{4(2\pi)^{\frac{1}{3}}}{3}\frac{\sin 2\ell\left(-\frac{1}{2}\ell\sin 2\ell-2\ell^2-\cos 2\ell+2\sin^2\ell+1\right)+\ell\left(2\sin^2\ell\left(\frac{1+\cos 2\ell}{2}\right)-4\cos^2\ell\left(\frac{1-\cos 2\ell}{2}\right)\right)}{\left(3\sin\ell+\frac{1}{3}\sin 3\ell-4\ell\cos\ell\right)^{\frac{5}{3}}}\\
&=-\frac{4(2\pi)^{\frac{1}{3}}}{3}\sin 2\ell\left(\frac{\ell\sin 2\ell+2\ell^2+\cos 2\ell-2\sin^2\ell-1}{\left(3\sin\ell+\frac{1}{3}\sin 3\ell-4\ell\cos\ell\right)^{\frac{5}{3}}}\right).
\end{aligned}
$$

Then, (5.23) follows if $f(\ell)=\ell\sin 2\ell+2\ell^2+\cos 2\ell-2\sin^2\ell-1$ is positive on $(0,\pi)$. Observe that

$$\dot f(\ell)=2\ell\cos 2\ell+4\ell-3\sin 2\ell.$$

In particular, $f(0)=\dot f(0)=0$. Then $f>0$ on $(0,\pi)$ provided that $\ddot f>0$ on $(0,\pi)$. To this aim,

$$\ddot f(\ell)=4\left(1-\cos 2\ell-\ell\sin 2\ell\right)=8\sin\ell\left(\sin\ell-\ell\cos\ell\right).$$

Therefore $\ddot f$ is positive. Then (5.23) follows. We prove (5.22). Combining (5.20) and (5.21),

$$\mathscr{L}-\frac{2\mathscr{H}^{\mathcal{H}}(S_L)}{3\sigma^{\mathcal{H}}(S_L)}=4L(4L-1)\left(\frac{\arccos\sqrt{1-2L}+(4L-1)\sqrt{2L(1-2L)}}{3\left(4L-1\right)\arccos\sqrt{1-2L}+(3-8L+16L^2)\sqrt{2L(1-2L)}}\right).$$

Then (5.22) follows provided that $\arccos\sqrt{1-2L}+(4L-1)\sqrt{2L(1-2L)}\neq 0$ for any $L\neq\frac{1}{4}$. Changing variables as above,

$$\arccos\sqrt{1-2L}+(4L-1)\sqrt{2L(1-2L)}=\frac{1}{4}\left(2\ell-\sin 2\ell\right),\qquad \ell\in(0,\pi],$$

whence (5.22) easily follows. Finally, fix an arbitrary non-characteristic variation. Then, recalling Proposition 4.9,

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0}\mathcal{Q}^{\mathcal{H}}_{\text{mink}}\left(\Phi_t(S_L)\right) &= \sigma^{\mathcal{H}}(S_L)^{-\frac{2}{3}}\left(-\frac{2\mathscr{H}^{\mathcal{H}}(S_L)}{3\sigma^{\mathcal{H}}(S_L)}\left(\delta\sigma^{\mathcal{H}}(S_L)[\Phi]\right)+\left(\delta\mathscr{H}^{\mathcal{H}}(S_L)[\Phi]\right)\right)\\
&\overset{(4.10)}{=}\sigma^{\mathcal{H}}(S_L)^{-\frac{2}{3}}\left(\delta\sigma^{\mathcal{H}}(S_L)[\Phi]\right)\left(-\frac{2\mathscr{H}^{\mathcal{H}}(S_L)}{3\sigma^{\mathcal{H}}(S_L)}+\mathscr{L}\right).
\end{aligned}
$$

By (5.22), $S_{\frac{1}{4}}$ is a critical point. Moreover, again by (5.22) and choosing a non-characteristic variation for which $\delta\sigma^{\mathcal{H}}(S_L)[\Phi]\neq 0$, $S_{\frac{1}{4}}$ is the unique critical point. The thesis follows. ☐

*Proof of Theorem 1.2.* It follows by Proposition 5.5 (and subsequent remarks) and Proposition 5.8. ☐

## 6. Total mean curvature: stability of Pansu-Minkowski spheres

Next, we discuss the stability of Pansu-Minkowski spheres. By Proposition 4.10, we focus on the penalized functional $\mathcal{P}^{\mathcal{H}}_L$ as introduced in (1.8). Fix a smooth non-characteristic variation $\Phi$. Let $\varphi$ and $\psi$ be as in (4.3). First, we evaluate the second variation of $\mathcal{P}^{\mathcal{H}}_L$ at the Pansu-Minkowski sphere $S_L$. By (4.7) and (4.9),

$$
\begin{aligned}
\delta^2\mathcal{P}^{\mathcal{H}}_L(S_L)[\Phi] &= \delta\mathscr{H}^{\mathcal{H}}(S_L)\left[\psi\right]-4L\delta\sigma^{\mathcal{H}}(S_L)\left[\psi\right]\\
&\quad+4\int_{S_L}\Big(-\mathcal{S}\varphi\,J(\nu)\varphi+\varphi^2\left(2\mathcal{S}\alpha-H^{\mathcal{H}}\alpha^2\right)\Big)\,d\sigma^{\mathcal{H}}-4L\int_{S_L}\Big(\left(J(\nu)\varphi\right)^2-4\varphi^2\left(J(\nu)\alpha+\alpha^2\right)\Big)\,d\sigma^{\mathcal{H}}\\
&\overset{(4.10),(4.11)}{=}4\int_{S_L}\Big(\left(-\mathcal{S}\varphi\,J(\nu)\varphi-L\left(J(\nu)\varphi\right)^2\right)+\left(2\mathcal{S}\alpha-H^{\mathcal{H}}\left(\alpha^2+4L^2\right)\right)\varphi^2\Big)\,d\sigma^{\mathcal{H}}.
\end{aligned}
$$

We make the zero-order term explicit, and we provide a sharp lower bound.

**Lemma 6.1.** *It holds that*

$$2\mathcal{S}\alpha-H^{\mathcal{H}}\left(\alpha^2+4L^2\right)=\frac{2(1-2L)}{x(s)^3h(s)}.$$

*In particular,*

$$2\mathcal{S}\alpha-H^{\mathcal{H}}\left(\alpha^2+4L^2\right)\geqslant\left.\left[2\mathcal{S}\alpha-H^{\mathcal{H}}\left(\alpha^2+4L^2\right)\right]\right|_{s=0}=\frac{2L(1-2L)}{1-L}.$$

*Proof.* Notice that

$$\alpha \overset{(3.3)}{=} -\frac{\dot{x}}{\sqrt{\dot{t}^2 + x^2\dot{x}^2}} = \frac{\sin s}{x}, \qquad \mathcal{S}\alpha \overset{(3.8)}{=} \frac{\dot{t}\dot{\alpha}}{\dot{t}^2 + x^2\dot{x}^2} = \frac{\cos s}{x^2 h}\left(\cos^2 s + \frac{h}{x}\sin^2 s\right).$$

Therefore, recalling (5.18),

$$\begin{aligned} 2\mathcal{S}\alpha - H^{\mathcal{H}}\left(\alpha^2 + 4L^2\right) &= \frac{2\cos s}{x^2 h}\left(\cos^2 s + \frac{h}{x}\sin^2 s\right) - \frac{2\cos^2 s}{2Lxh}\left(\frac{\sin^2 s}{x^2} + 4L^2\right) \\ &= \frac{2\cos s}{x^3 h}\left(x\cos^2 s + h\sin^2 s - \frac{\sin^2 s\cos s}{2L} - 2Lx^2\cos s\right) \\ &= \frac{2\cos s}{x^3 h}\left((x-h)\cos^2 s + h - \frac{\sin^2 s\cos s}{2L} - \frac{\cos^3}{2L} + \frac{2(1-2L)\cos s}{2L} - \frac{(1-2L)^2}{2L\cos s}\right) \\ &= \frac{\cos s}{Lx^3 h}\left(\cos s + \frac{1-2L}{\cos s} - \sin^2 s\cos s - \cos^3 - \frac{(1-2L)^2}{\cos s}\right) \\ &= \frac{2(1-2L)}{x^3 h}. \end{aligned}$$

To conclude, recall that $x$ and $h$ achieve their maximum at $s = 0$, and moreover $x(0) = 1$ and $h(0) = \frac{1-L}{L}$. ☐

Next, we deal with the first-order term.

**Lemma 6.2.** *It holds that*

$$\begin{aligned} -\mathcal{S}\varphi\, J(\nu)\varphi - L\left(J(\nu)\varphi\right)^2 &= \left(\frac{\cos^3 s}{xh^2} - \frac{L\cos^2 s}{h^2}\right)\left(\frac{\partial\varphi}{\partial s}\right)^2 \\ &\quad + \left(\frac{2L\cos^2 s}{xh} - \frac{\cos s\cos 2s}{x^2 h}\right)\frac{\partial\varphi}{\partial s}\frac{\partial\varphi}{\partial\theta} - \left(\frac{\sin^2 s\cos s}{x^3} + \frac{L\cos^2 s}{x^2}\right)\left(\frac{\partial\varphi}{\partial\theta}\right)^2. \end{aligned} \tag{6.1}$$

*If in addition $\varphi$ is independent of $\theta$, then the following sharp lower bound holds:*

$$-\mathcal{S}\varphi\, J(\nu)\varphi - L\left(J(\nu)\varphi\right)^2 \geqslant (1-L)\left(J(\nu)\varphi\right)^2.$$

*Proof.* Notice that

$$\begin{aligned} J(\nu)\varphi &\overset{(3.7)}{=} -\left(\frac{x}{\sqrt{\dot{t}^2 + x^2\dot{x}^2}}\right)\frac{\partial\varphi}{\partial s} + \left(\frac{\dot{t}}{x\sqrt{\dot{t}^2 + x^2\dot{x}^2}}\right)\frac{\partial\varphi}{\partial\theta} = -\left(\frac{\cos s}{h}\right)\frac{\partial\varphi}{\partial s} + \frac{\cos s}{x}\frac{\partial\varphi}{\partial\theta}, \\ \mathcal{S}\varphi &\overset{(3.8)}{=} \left(\frac{\dot{t}}{\dot{t}^2 + x^2\dot{x}^2}\right)\frac{\partial\varphi}{\partial s} + \left(\frac{\dot{x}^2}{\dot{t}^2 + x^2\dot{x}^2}\right)\frac{\partial\varphi}{\partial\theta} = \left(\frac{\cos^2 s}{xh}\right)\frac{\partial\varphi}{\partial s} + \left(\frac{\sin^2 s}{x^2}\right)\frac{\partial\varphi}{\partial\theta}. \end{aligned}$$

Therefore

$$\left(J(\nu)\varphi\right)^2 = \left(\frac{\cos^2 s}{h^2}\right)\left(\frac{\partial\varphi}{\partial s}\right)^2 - \left(\frac{2\cos^2 s}{xh}\right)\frac{\partial\varphi}{\partial s}\frac{\partial\varphi}{\partial\theta} + \left(\frac{\cos^2 s}{x^2}\right)\left(\frac{\partial\varphi}{\partial\theta}\right)^2$$

and

$$-\mathcal{S}\varphi J(\nu)\varphi = \left(\frac{\cos^3 s}{xh^2}\right)\left(\frac{\partial\varphi}{\partial s}\right)^2 - \left(\frac{\cos s\cos 2s}{x^2 h}\right)\frac{\partial\varphi}{\partial s}\frac{\partial\varphi}{\partial\theta} - \left(\frac{\sin^2 s\cos s}{x^3}\right)\left(\frac{\partial\varphi}{\partial\theta}\right)^2,$$

whence (6.1) follows. Next, assume that $\varphi$ is independent of $\theta$. Let $\mu \geqslant 0$. Then

$$\begin{aligned} -\mathcal{S}\varphi\, J(\nu)\varphi - (L+\mu)\left(J(\nu)\varphi\right)^2 &\overset{(6.1)}{=} \left(\frac{\cos^3 s}{xh^2} - \frac{(L+\mu)\cos^2 s}{h^2}\right)\left(\frac{\partial\varphi}{\partial s}\right)^2 \\ &= \left(\frac{\cos s}{2Lxh^2}\right)\left((L-\mu)\cos^2 s + (L+\mu)(1-2L)\right). \end{aligned}$$

Assume first that $L - \mu \geqslant 0$. Since $\cos^2 s > (1-2L)$ on $(-\arccos\sqrt{1-2L}, \arccos\sqrt{1-2L})$, then

$$(L-\mu)\cos^2 s + (L+\mu)(1-2L) \geqslant 2L(1-2L) \geqslant 0.$$

Assume instead $L - \mu < 0$. Since $\cos^2 s \leqslant \cos^2(0) = 1$ on $(-\arccos\sqrt{1-2L}, \arccos\sqrt{1-2L})$, then

$$(L-\mu)\cos^2 s + (L+\mu)(1-2L) \geqslant (L-\mu) + (L+\mu)(1-2L) = 2L(1-L) - 2L\mu,$$

whence $-\mathcal{S}\varphi\, J(\nu)\varphi - (L+\mu)\left(J(\nu)\varphi\right)^2 \geqslant 0$ provided that $\mu \leqslant 1 - L$. ☐

Motivated by Lemma 6.2, we introduce a relevant class of variations. Precisely, we say that a smooth non-characteristic variation of $S_L$ is *rotationally invariant* if $\varphi$ is independent of $\theta$.

**Remark 6.3.** We stress that requiring a variation to be rotationally invariant is much weaker than restricting to the class of rotationally invariant competitors, as no restriction on $\psi$ is imposed.

Combining Lemma 6.1, Lemma 6.2 and Proposition 4.10, we deduce that Pansu-Minkowski spheres are (much more than) stable for $\mathscr{H}^{\mathcal{H}}$ along area-preserving, rotationally invariant, non-characteristic variations.

**Proposition 6.4.** *Let $L \in \left(0, \frac{1}{2}\right]$. Let $\Phi$ be a non-characteristic rotationally invariant variation. Then*

$$\delta^2 \mathcal{P}_L^{\mathcal{H}}(S_L)[\Phi] \geqslant 4(1-L) \int_{S_L} (J(\nu)\varphi)^2 \, d\sigma^{\mathcal{H}} + \frac{8L(1-2L)}{1-L} \int_{S_L} \varphi^2 \, d\sigma^{\mathcal{H}}. \tag{6.2}$$

*In particular, $S_L$ is stable for $\mathscr{H}^{\mathcal{H}}$ along area-preserving non-characteristic rotationally invariant variations.*

**Remark 6.5.** When $L = \frac{1}{2}$, i.e. when $S_L$ is the standard Pansu sphere, the coefficient multiplying the zero-order term in the right hand side of (6.2) vanishes. Nevertheless, by the Poincaré inequality, it is still possible to provide a lower bound of the form

$$\delta^2 \mathcal{P}_L^{\mathcal{H}}(S_L)[\Phi] \geqslant c_{\mathrm{I}}(\delta, L) \left( \|\varphi\|^2_{L^2(I(\delta))} + \|\dot{\varphi}\|^2_{L^2(I(\delta))} \right), \tag{6.3}$$

where $\operatorname{supp} \varphi \subseteq I(\delta) := \left(-\arccos\sqrt{1-2L} + \delta, \arccos\sqrt{1-2L} - \delta\right)$ and $c_I(\delta, L) > 0$, and $c_I(\delta, L)$ tends to 0 as $\delta \to 0^+$.

We conclude this section with the proof of Theorem 1.4: the rotational invariance constraint cannot be removed, as Pansu-Minkowski spheres are unstable under more general variations.

*Proof of Theorem 1.4.* Fix $L \in \left(0, \frac{1}{2}\right]$. Let $0 < \delta < \arccos\sqrt{1-2L}$. Let $M \in \mathbb{N}_+$. Let $\psi \in C_c^\infty(-\delta, \delta)$ be not identically vanishing. Define

$$\varphi(s, \theta) = \psi(s) \sin M\theta, \qquad s \in \left(-\arccos\sqrt{1-2L}, \arccos\sqrt{1-2L}\right), \theta \in (0, 2\pi).$$

Let $\mathbf{X}$ be such that $\mathbf{X}|_S = \varphi\nu$. Set $\mathbf{Z} = 0$. Let $\Phi$ be a variation as in (4.1). Then $\Phi$ is a smooth, non-characteristic, horizontally normal variation with velocity $\mathbf{X}$ and with no acceleration. Moreover,

$$\delta\sigma^{\mathcal{H}}(S_L)[\Phi] \overset{(4.6)}{=} \int_{S_L} \varphi H^{\mathcal{H}} \, d\sigma^{\mathcal{H}} \overset{(5.19)}{=} \left( \int_0^{2\pi} \sin M\theta \, d\theta \right) \left( \int_{-\delta}^{\delta} \psi x(x+h) \, ds \right) = 0,$$

whence $\Phi$ is first-order area-preserving. Notice that

$$\frac{\partial \varphi}{\partial s} = \dot{\psi} \sin M\theta, \qquad \frac{\partial \varphi}{\partial \theta} = M\psi \cos M\theta.$$

Since

$$\int_0^{2\pi} \sin M\theta \cos M\theta \, d\theta = 0, \qquad \int_0^{2\pi} \sin^2\theta \, d\theta = \int_0^{2\pi} \cos^2\theta \, d\theta = \pi,$$

then (5.17) , Lemma 6.1 and Lemma 6.2 imply that $\delta^2 \mathcal{P}_L^{\mathcal{H}}(S_L)[\Phi]$ equals

$$4\pi \int_{-\delta}^{\delta} \left[ \left( \frac{x\cos^2 s}{h} - \frac{Lx^2 \cos s}{h} \right) \dot{\psi}^2 + \left( \frac{2(1-2L)}{x \cos s} \right) \psi^2 \right] ds - 4\pi M^2 \int_{-\delta}^{\delta} \left( \frac{h \sin^2 s}{x} + Lh \cos s \right) \psi^2 \, ds.$$

Since

$$\int_{-\delta}^{\delta} \left( \frac{h \sin^2 s}{x} + Lh \cos s \right) \psi^2 \, ds > 0,$$

then $\delta^2 \mathcal{P}_L^{\mathcal{H}}(S_L)[\Phi] < 0$ provided that $M$ is sufficiently large. The thesis follows by Proposition 4.10. □

## 7. Total mean curvature: local minimality of Pansu-Minkowski spheres

In this section we prove that Pansu-Minkowski spheres are local minimizers, in the sense of (1.2), in the class of rotationally invariant surfaces. To this aim, fix $L \in \left(0, \frac{1}{2}\right]$, and denote the profile of $S_L$ simply by $(x, t)$. Fix $\delta > 0$, recall that $I(\delta) = (-\arccos\sqrt{1-2L} + \delta, \arccos\sqrt{1-2L} - \delta)$. Fix $\varphi \in C^\infty_c(I(\delta))$, and set $S_L^\varphi := S \cdot \varphi\nu$. By (2.1) and (3.4), $S_L^\varphi$ can be parametrized by $(\xi^\varphi(s,\theta), \eta^\varphi(s,\theta), t^\varphi(s,\theta))$, where

$$\begin{aligned}
\xi^\varphi &:= \left(x + \frac{\varphi\dot t}{\sqrt{\dot t^2 + x^2\dot x^2}}\right)\cos\theta - \left(\frac{\varphi x\dot x}{\sqrt{\dot t^2 + x^2\dot x^2}}\right)\sin\theta,\\
\eta^\varphi &:= \left(x + \frac{\varphi\dot t}{\sqrt{\dot t^2 + x^2\dot x^2}}\right)\sin\theta + \left(\frac{\varphi x\dot x}{\sqrt{\dot t^2 + x^2\dot x^2}}\right)\cos\theta,\\
t^\varphi &:= t - \frac{\varphi x^2\dot x}{\sqrt{\dot t^2 + x^2\dot x^2}}.
\end{aligned}$$

Since

$$\sqrt{\left(\xi^\varphi(s,\theta)\right)^2 + \left(\eta^\varphi(s,\theta)\right)^2} = \sqrt{\left(x + \frac{\varphi\dot t}{\sqrt{\dot t^2 + x^2\dot x^2}}\right)^2 + \left(\frac{\varphi x\dot x}{\sqrt{\dot t^2 + x^2\dot x^2}}\right)^2} =: x^\varphi(s),$$

then $S_L^\varphi$ is a rotationally invariant surface, with profile $(x^\varphi, t^\varphi)$.

**Proposition 7.1.** *Fix $L \in \left(0, \frac{1}{2}\right]$ and $\delta > 0$. There exists $\varepsilon = \varepsilon(\delta, L) > 0$ such that, if $\varphi \in C^\infty_c(I(\delta))$ satisfies*

$$\sigma^{\mathcal{H}}\left(S_L^\varphi\right) = \sigma^{\mathcal{H}}\left(S_L\right), \qquad \|\varphi\|_{C^2(I(\delta))} \leqslant \varepsilon,$$

*then*

$$\mathscr{H}^{\mathcal{H}}\left(S_L\right) \leqslant \mathscr{H}^{\mathcal{H}}\left(S_L^\varphi\right).$$

*Proof.* For $p \in S_L$ and $\tau \in [-1, 1]$, set

$$\Phi(\tau, p(s,\theta)) = \left(\xi^{\tau\varphi}(s,\theta), \eta^{\tau\varphi}(s,\theta), t^{\tau\varphi}(s,\theta)\right),$$

and extend it smoothly into a smooth variation of $\mathbb{H}^1$. For any $\tau \in [0, 1]$, $\Phi(\tau, S)$ is the rotationally invariant surface with profile $(x^{\tau\varphi}, t^{\tau\varphi})$. Moreover, denoting by $\mathbf{X}$ the normal velocity of $\Phi$, then $\mathbf{X}|_S = \varphi\nu$. Define $f : \mathbb{R}^5 \to \mathbb{R}$ by

$$f(p_1, q_1, q_2, r_1, r_2) := \frac{p_1^3\left(q_1 r_2 - q_2 r_1\right) + q_2^3}{\left(q_2^2 + p_1^2 q_1^2\right)^2} - 4Lp_1\sqrt{q_2^2 + p_1^2 q_1^2}.$$

Moreover, define $Q : [0, 1] \to \mathbb{R}^5$ by

$$Q(\tau) := \left(x^{\tau\varphi}, \dot x^{\tau\varphi}, \dot t^{\tau\varphi}, \ddot x^{\tau\varphi}, \ddot t^{\tau\varphi}\right).$$

With these definitions,

$$\mathcal{P}_L^{\mathcal{H}}\left(\Phi(\tau, S)\right) = 2\pi\int_{I(\delta)} f\left(Q(\tau)\right)\, ds.$$

By Taylor's formula with Lagrange remainder, and since $S_L$ is an area-preserving critical point along non-characteristic variations, there exists $\tilde\tau \in (0, 1)$ such that

$$\begin{aligned}
\mathcal{P}_L^{\mathcal{H}}\left(S_L^\varphi\right) &= \mathcal{P}_L^{\mathcal{H}}\left(S_L\right) + \left.\frac{d}{d\tau}\right|_{\tau=0}\mathcal{P}_L^{\mathcal{H}}\left(\Phi(\tau, S)\right) + \frac{1}{2}\left.\frac{d^2}{d\tau}\right|_{\tau=\tilde\tau}\mathcal{P}_L^{\mathcal{H}}\left(\Phi(\tau, S)\right)\\
&= \mathcal{P}_L^{\mathcal{H}}\left(S_L\right) + \delta\mathcal{P}_L^{\mathcal{H}}(S)[\Phi] + \frac{1}{2}\delta^2\mathcal{P}_L^{\mathcal{H}}(S)[\Phi] + \pi\int_{I(\delta)}\left(\left.\frac{\partial^2}{\partial\tau}\right|_{\tau=\tilde\tau} f(Q(\tau)) - \left.\frac{d^2}{d\tau}\right|_{\tau=0} f(Q(\tau))\right) ds\\
&\overset{(4.10)}{=} \mathcal{P}_L^{\mathcal{H}}\left(S_L\right) + \underbrace{\frac{1}{2}\delta^2\mathcal{P}_L^{\mathcal{H}}(S)[\Phi]}_{\mathrm{I}} + \underbrace{\pi\int_{I(\delta)}\left(\left.\frac{\partial^2}{\partial\tau}\right|_{\tau=\tilde\tau} f(Q(\tau)) - \left.\frac{d^2}{d\tau}\right|_{\tau=0} f(Q(\tau))\right) ds}_{\mathrm{II}}.
\end{aligned}$$

Since $f$ is affine in the variables $r_1$ and $r_2$, a simple computation yields that

$$\frac{\partial^2}{\partial\tau^2} f(Q(\tau)) = A^{\tau\varphi}\varphi^2 + B^{\tau\varphi}\varphi\dot\varphi + C^{\tau\varphi}\varphi\ddot\varphi + D^{\tau\varphi}\dot\varphi^2 + E^{\tau\varphi}\dot\varphi\ddot\varphi,$$

where $A^{\tau\varphi}, B^{\tau\varphi}, D^{\tau\varphi}$ depend smoothly on $\tau\varphi, \tau\dot{\varphi}, \tau\ddot{\varphi}$ and $C^{\tau\varphi}, E^{\tau\varphi}$ depend smoothly on $\tau\varphi, \tau\dot{\varphi}$. Therefore, integrating by parts,

$$\int_{I(\delta)} \frac{\partial^2}{\partial\tau^2} f(Q(\tau))\, ds = \int_{I(\delta)} \left( A^{\tau\varphi}\varphi^2 + \tilde{B}^{\tau\varphi}\varphi\dot{\varphi} + \tilde{D}^{\tau\varphi}\dot{\varphi}^2 \right) ds,$$

where

$$\tilde{B}^{\tau\varphi} = B^{\tau\varphi} - \frac{\partial}{\partial s} C^{\tau\varphi}, \qquad \tilde{D}^{\tau\varphi} = D^{\tau\varphi} - C^{\tau\varphi} - \frac{1}{2}\frac{\partial}{\partial s} E^{\tau\varphi}.$$

In particular, $A^{\tau\varphi}, \tilde{B}^{\tau\varphi}, \tilde{D}^{\tau\varphi}$ depend smoothly on $\tau\varphi, \tau\dot{\varphi}, \tau\ddot{\varphi}$. Therefore, since $\operatorname{supp}\varphi \subseteq I(\delta)$, there exists $c_{\mathrm{II}}(\delta, L)$ such that

$$|\mathrm{II}| \leqslant c_{\mathrm{II}}(\delta, L) \|\varphi\|_{C^2(I(\delta))} \left( \|\varphi\|^2_{L^2(I(\delta))} + \|\dot{\varphi}\|^2_{L^2(I(\delta))} \right). \tag{7.1}$$

Combining (7.1) and (6.3), we conclude that

$$\mathcal{P}^{\mathcal{H}}_L \left(S^{\varphi}_L\right) \geqslant \mathcal{P}^{\mathcal{H}}_L \left(S_L\right) + \left( \frac{1}{2} c_{\mathrm{I}}(\delta, L) - c_{\mathrm{II}}(\delta, L) \|\varphi\|_{C^2(I(\delta))} \right) \left( \|\varphi\|^2_{L^2(I(\delta))} + \|\dot{\varphi}\|^2_{L^2(I(\delta))} \right).$$

Since $\sigma^{\mathcal{H}}(S^{\varphi}_L) = \sigma^{\mathcal{H}}(S_L)$, then $\mathcal{P}^{\mathcal{H}}_L \left(S^{\varphi}_L\right) - \mathcal{P}^{\mathcal{H}}_L \left(S_L\right) = \mathscr{H}^{\mathcal{H}}\left(S^{\varphi}_L\right) - \mathscr{H}^{\mathcal{H}}\left(S_L\right)$, whence the thesis follows. ☐

*Proof of Theorem 1.3.* It follows combining Proposition 6.4 and Proposition 7.1. ☐

## Appendix A. Variation formulas for Riemannian mean curvature functionals

In this first appendix, we establish variation formulas for functionals driven by mean curvature in arbitrary Riemannian manifolds; see Theorem A.8. For ease of reference, the presentation is entirely self-contained. Much of the notation and several results are borrowed from [42]. We refer to [17] for a general account of Riemannian geometry.

A.1. **Preliminaries.** Here and hereafter, $M$ is a fixed $(n+1)$-dimensional Riemannian manifold for some $n \geqslant 1$, $\langle\cdot,\cdot\rangle$ is its Riemannian metric and $\nabla$ is its Levi-Civita connection. In the following, Einstein's summation convention is assumed. Denote by $R$ both the $(3,1)$ and the $(4,0)$ Riemann tensor, namely

$$\mathrm{R}(\mathbf{A},\mathbf{B})\mathbf{C} = \nabla_{\mathbf{A}}\nabla_{\mathbf{B}}\mathbf{C} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}}\mathbf{C} - \nabla_{[\mathbf{A},\mathbf{B}]}\mathbf{C}, \quad \mathrm{R}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = \langle \mathrm{R}(\mathbf{A},\mathbf{B})\mathbf{C}, \mathbf{D}\rangle, \quad \mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \Gamma(TM).$$

If $\mathbf{A}, \mathbf{C} \in \Gamma(TM)$ are fixed, denote by $\mathcal{C}$ the $(2,0)$-tensor field defined by

$$\mathcal{C}(\mathbf{A},\mathbf{C})(\mathbf{B},\mathbf{D}) \coloneqq \mathrm{R}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}), \qquad \mathbf{B},\mathbf{D} \in \Gamma(TM).$$

In this way,

$$\mathrm{Ric}(\mathbf{A},\mathbf{C}) = -\operatorname{trace}\mathcal{C}(\mathbf{A},\mathbf{C}), \qquad \mathbf{A},\mathbf{C} \in \Gamma(TM).$$

We recall that, if $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \Gamma(TM)$, then

$$\left(\nabla_{\mathbf{A}} \mathrm{Ric}\right)(\mathbf{B},\mathbf{C}) = -\operatorname{trace}\left(\nabla_{\mathbf{A}} \mathrm{R}\right)(\mathbf{B},\cdot,\mathbf{C},\cdot), \qquad \left(\nabla_{\mathbf{A}} \mathrm{R}\right)(\mathbf{B},\mathbf{C},\mathbf{D},\mathbf{D}) = 0.$$

Let $p \in M$. Let $x_1, \dots, x_{n+1}$ be local coordinates in $M$ near $p$. Write

$$\mathrm{R}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \mathrm{R}^l_{ijk}\frac{\partial}{\partial x_l}, \qquad \left(\nabla_{\frac{\partial}{\partial x_\alpha}} \mathrm{R}\right)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = (\nabla \mathrm{R})^l_{\alpha ijk}\frac{\partial}{\partial x_l}, \qquad \alpha, i, j, k = 1, \dots, n+1.$$

Notice that

$$\mathrm{R}^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x_i} - \frac{\partial \Gamma^l_{ik}}{\partial x_j} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}, \qquad i, j, k, l = 1, \dots, n+1, \tag{A.1}$$

where $\Gamma^k_{ij}$ are the Christoffel symbols with respect to $x_1, \dots, x_{n+1}$. In particular, when $x_1, \dots, x_{n+1}$ are normal coordinates centered at $p$,

$$\Gamma^k_{ij}(p) = 0 \qquad \text{for any } i, j, k = 1, \dots, n+1, \tag{A.2}$$

so that

$$\mathrm{R}^l_{ijk}(p) = \frac{\partial \Gamma^l_{jk}}{\partial x_i}(p) - \frac{\partial \Gamma^l_{ik}}{\partial x_j}(p) \qquad \text{for any } i, j, k, l = 1, \dots, n+1 \tag{A.3}$$

and

$$(\nabla\,\mathrm{R})^l_{\alpha ijk}\,(p) = \frac{\partial^2\Gamma^l_{jk}}{\partial x_\alpha \partial x_i}(p) - \frac{\partial^2\Gamma^l_{ik}}{\partial x_\alpha \partial x_j}(p) \qquad \text{for any } \alpha, i, j, k, l = 1, \dots, n+1. \tag{A.4}$$

The following technical lemma is the main computational tool used below.

**Lemma A.1.** *Let $p \in M$. Let $x_1, \dots, x_{n+1}$ be local coordinates in $M$ near $p$. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \Gamma(TM)$. Then, near $p$,*

$$\begin{aligned}\nabla_{\mathbf{A}}\nabla_{\mathbf{B}}\mathbf{C} = A^\alpha B^\beta \left(\frac{\partial^2 C^\gamma}{\partial x_\alpha \partial x_\beta}\frac{\partial}{\partial x_\gamma} + \frac{\partial C^\gamma}{\partial x_\alpha}\Gamma^\delta_{\beta\gamma}\frac{\partial}{\partial x_\delta} + C^\gamma \frac{\partial \Gamma^\delta_{\alpha\gamma}}{\partial x_\beta}\frac{\partial}{\partial x_\delta} + \frac{\partial C^\gamma}{\partial x_\beta}\Gamma^\delta_{\alpha\gamma}\frac{\partial}{\partial x_\delta} + C^\gamma\Gamma^\delta_{\alpha\gamma}\Gamma^\eta_{\beta\delta}\frac{\partial}{\partial x_\eta}\right) \\ + A^\alpha \frac{\partial B^\beta}{\partial x_\alpha}\left(\frac{\partial C^\gamma}{\partial x_\beta}\frac{\partial}{\partial x_\gamma} + C^\gamma\Gamma^\delta_{\beta\gamma}\frac{\partial}{\partial x_\delta}\right) + \mathrm{R}(\mathbf{A},\mathbf{B})\mathbf{C}.\end{aligned} \tag{A.5}$$

*Assume in addition that $x_1, \dots, x_{n+1}$ are normal coordinates centered at $p$. Then, at $p$,*

$$\nabla_{\mathbf{A}}\nabla_{\mathbf{B}}\mathbf{C} = \mathrm{R}(\mathbf{A},\mathbf{B})\mathbf{C} + A^\alpha B^\beta C^\gamma \frac{\partial\Gamma^\delta_{\alpha\gamma}}{\partial x_\beta}\frac{\partial}{\partial x_\delta} + A^\alpha B^\beta \frac{\partial^2 C^\gamma}{\partial x_\alpha \partial x_\beta}\frac{\partial}{\partial x_\gamma} + A^\alpha \frac{\partial B^\beta}{\partial x_\alpha}\frac{\partial C^\gamma}{\partial x_\beta}\frac{\partial}{\partial x_\gamma} \tag{A.6}$$

*and*

(A.7)

$$\begin{aligned}\nabla_{\mathbf{A}}\nabla_{\mathbf{B}}\nabla_{\mathbf{C}}\mathbf{D} = (\nabla_{\mathbf{B}}\,\mathrm{R})\,(\mathbf{A},\mathbf{C})\mathbf{D} + \mathrm{R}(\mathbf{A},\mathbf{B})\nabla_{\mathbf{C}}\mathbf{D} + \mathrm{R}\,(\mathbf{A},\mathbf{C})\,\nabla_{\mathbf{B}}\mathbf{D} + \mathrm{R}\,(\mathbf{A},\nabla_{\mathbf{B}}\mathbf{C})\,\mathbf{D} + \mathrm{R}\,(\nabla_{\mathbf{A}}\mathbf{B},\mathbf{C})\,\mathbf{D} \\ + A^\alpha B^\beta\left(\frac{\partial^2 C^\delta}{\partial x_\alpha\partial x_\beta}\frac{\partial D^\gamma}{\partial x_\delta} + \frac{\partial C^\delta}{\partial x_\beta}\frac{\partial^2 D^\gamma}{\partial x_\alpha \partial x_\delta} + \frac{\partial C^\delta}{\partial x_\alpha}\frac{\partial^2 D^\gamma}{\partial x_\beta\partial x_\delta} + C^\delta\frac{\partial^3 D^\gamma}{\partial x_\alpha\partial x_\beta\partial x_\delta}\right)\frac{\partial}{\partial x_\gamma} \\ + A^\alpha B^\beta\left(C^\eta\frac{\partial D^\delta}{\partial x_\eta}\frac{\partial\Gamma^\gamma_{\alpha\delta}}{\partial x_\beta} + \frac{\partial C^\delta}{\partial x_\beta}D^\eta\frac{\partial\Gamma^\gamma_{\alpha\eta}}{\partial x_\delta} + C^\delta\frac{\partial D^\eta}{\partial x_\beta}\frac{\partial\Gamma^\gamma_{\alpha\eta}}{\partial x_\delta} + \frac{\partial C^\delta}{\partial x_\alpha}D^\eta\frac{\partial\Gamma^\gamma_{\delta\eta}}{\partial x_\beta} + C^\delta\frac{\partial D^\eta}{\partial x_\alpha}\frac{\partial\Gamma^\gamma_{\delta\eta}}{\partial x_\beta}\right)\frac{\partial}{\partial x_\gamma} \\ + A^\alpha B^\beta C^\delta D^\eta\frac{\partial^2\Gamma^\gamma_{\alpha\eta}}{\partial x_\beta\partial x_\delta}\frac{\partial}{\partial x_\gamma} + A^\alpha\frac{\partial B^\beta}{\partial x_\alpha}\left(\frac{\partial C^\delta}{\partial x_\beta}\frac{\partial D^\gamma}{\partial x_\delta} + C^\delta\frac{\partial^2 D^\gamma}{\partial x_\beta\partial x_\delta} + C^\delta D^\eta\frac{\partial\Gamma^\gamma_{\beta\eta}}{\partial x_\delta}\right)\frac{\partial}{\partial x_\gamma}.\end{aligned}$$

*Proof.* By a direct computation, near $p$,

$$\begin{aligned}\nabla_{\mathbf{A}}\nabla_{\mathbf{B}}\mathbf{C} &= A^\alpha\nabla_{\frac{\partial}{\partial x_\alpha}}\left(B^\beta\nabla_{\frac{\partial}{\partial x_\beta}}\left(C^\gamma\frac{\partial}{\partial x_\gamma}\right)\right) \\ &= A^\alpha B^\beta\nabla_{\frac{\partial}{\partial x_\alpha}}\left(\nabla_{\frac{\partial}{\partial x_\beta}}\left(C^\gamma\frac{\partial}{\partial x_\gamma}\right)\right) + A^\alpha\frac{\partial B^\beta}{\partial x_\alpha}\nabla_{\frac{\partial}{\partial x_\beta}}\left(C^\gamma\frac{\partial}{\partial x_\gamma}\right) \\ &= A^\alpha B^\beta\nabla_{\frac{\partial}{\partial x_\alpha}}\left(\frac{\partial C^\gamma}{\partial x_\beta}\frac{\partial}{\partial x_\gamma} + C^\gamma\Gamma^\delta_{\beta\gamma}\frac{\partial}{\partial x_\delta}\right) + A^\alpha\frac{\partial B^\beta}{\partial x_\alpha}\left(\frac{\partial C^\gamma}{\partial x_\beta}\frac{\partial}{\partial x_\gamma} + C^\gamma\Gamma^\delta_{\beta\gamma}\frac{\partial}{\partial x_\delta}\right) \\ &= A^\alpha B^\beta\left(\frac{\partial^2 C^\gamma}{\partial x_\alpha\partial x_\beta}\frac{\partial}{\partial x_\gamma} + \frac{\partial C^\gamma}{\partial x_\alpha}\Gamma^\delta_{\beta\gamma}\frac{\partial}{\partial x_\delta} + C^\gamma\frac{\partial\Gamma^\delta_{\beta\gamma}}{\partial x_\alpha}\frac{\partial}{\partial x_\delta} + \frac{\partial C^\gamma}{\partial x_\beta}\Gamma^\delta_{\alpha\gamma}\frac{\partial}{\partial x_\delta} + C^\gamma\Gamma^\delta_{\beta\gamma}\Gamma^\eta_{\alpha\delta}\frac{\partial}{\partial x_\eta}\right) \\ &\quad + A^\alpha\frac{\partial B^\beta}{\partial x_\alpha}\left(\frac{\partial C^\gamma}{\partial x_\beta}\frac{\partial}{\partial x_\gamma} + C^\gamma\Gamma^\delta_{\beta\gamma}\frac{\partial}{\partial x_\delta}\right).\end{aligned}$$

Since

$$\begin{aligned}A^\alpha B^\beta C^\gamma\left(\frac{\partial\Gamma^\delta_{\beta\gamma}}{\partial x_\alpha}\frac{\partial}{\partial x_\delta} + \Gamma^\delta_{\beta\gamma}\Gamma^\eta_{\alpha\delta}\frac{\partial}{\partial x_\eta}\right) &= A^\alpha B^\beta C^\gamma\left(\frac{\partial\Gamma^\delta_{\beta\gamma}}{\partial x_\alpha} + \Gamma^\eta_{\beta\gamma}\Gamma^\delta_{\alpha\eta}\right)\frac{\partial}{\partial x_\delta} \\ &\overset{(A.1)}{=} \mathrm{R}(\mathbf{A},\mathbf{B})\mathbf{C} + A^\alpha B^\beta C^\gamma\left(\frac{\partial\Gamma^\delta_{\alpha\gamma}}{\partial x_\beta} + \Gamma^\eta_{\alpha\gamma}\Gamma^\delta_{\beta\eta}\right)\frac{\partial}{\partial x_\delta},\end{aligned}$$

then (A.5) follows. Moreover, (A.6) follows by (A.5) and (A.2). Finally, at $p$,

$$\begin{aligned}\nabla_{\mathbf{A}}\nabla_{\mathbf{B}}\nabla_{\mathbf{C}}\mathbf{D} &= \nabla_{\mathbf{A}}\nabla_{\mathbf{B}}\left(\nabla_{\mathbf{C}}\mathbf{D}\right) \\ &\overset{(A.6)}{=} \mathrm{R}(\mathbf{A},\mathbf{B})\nabla_{\mathbf{C}}\mathbf{D} + A^\alpha B^\beta\left(\nabla_{\mathbf{C}}\mathbf{D}\right)^\gamma\frac{\partial\Gamma^\delta_{\alpha\gamma}}{\partial x_\beta}\frac{\partial}{\partial x_\delta} + A^\alpha B^\beta\frac{\partial^2\left(\nabla_{\mathbf{C}}\mathbf{D}\right)^\gamma}{\partial x_\alpha\partial x_\beta}\frac{\partial}{\partial x_\gamma} + A^\alpha\frac{\partial B^\beta}{\partial x_\alpha}\frac{\partial\left(\nabla_{\mathbf{C}}\mathbf{D}\right)^\gamma}{\partial x_\beta}\frac{\partial}{\partial x_\gamma}.\end{aligned}$$

Fix $\alpha, \beta, \gamma = 1, \dots, n+1$. Recall that

$$(\nabla_{\mathbf{C}} \mathbf{D})^\gamma = C^\delta \frac{\partial D^\gamma}{\partial x_\delta} + C^\delta D^\eta \Gamma^\gamma_{\delta\eta}.$$

Then, at $p$,

$$\begin{aligned}\frac{\partial\left(\nabla_{\mathbf{C}} \mathbf{D}\right)^\gamma}{\partial x_\beta} &= \frac{\partial C^\delta}{\partial x_\beta}\frac{\partial D^\gamma}{\partial x_\delta} + C^\delta \frac{\partial^2 D^\gamma}{\partial x_\beta \partial x_\delta} + \frac{\partial C^\delta}{\partial x_\beta} D^\eta \Gamma^\gamma_{\delta\eta} + C^\delta \frac{\partial D^\eta}{\partial x_\beta}\Gamma^\gamma_{\delta\eta} + C^\delta D^\eta \frac{\partial \Gamma^\gamma_{\delta\eta}}{\partial x_\beta} \\ &\overset{(A.2)}{=} \frac{\partial C^\delta}{\partial x_\beta}\frac{\partial D^\gamma}{\partial x_\delta} + C^\delta \frac{\partial^2 D^\gamma}{\partial x_\beta \partial x_\delta} + C^\delta D^\eta \frac{\partial \Gamma^\gamma_{\delta\eta}}{\partial x_\beta},\end{aligned}$$

and moreover

$$\begin{aligned}\frac{\partial^2\left(\nabla_{\mathbf{C}} \mathbf{D}\right)^\gamma}{\partial x_\alpha \partial x_\beta} &\overset{(A.2)}{=} \frac{\partial^2 C^\delta}{\partial x_\alpha \partial x_\beta}\frac{\partial D^\gamma}{\partial x_\delta} + \frac{\partial C^\delta}{\partial x_\beta}\frac{\partial^2 D^\gamma}{\partial x_\alpha \partial x_\delta} + \frac{\partial C^\delta}{\partial x_\alpha}\frac{\partial^2 D^\gamma}{\partial x_\beta \partial x_\delta} + C^\delta \frac{\partial^3 D^\gamma}{\partial x_\alpha \partial x_\beta \partial x_\delta} \\ &\quad + \frac{\partial C^\delta}{\partial x_\beta} D^\eta \frac{\partial \Gamma^\gamma_{\delta\eta}}{\partial x_\alpha} + C^\delta \frac{\partial D^\eta}{\partial x_\beta}\frac{\partial \Gamma^\gamma_{\delta\eta}}{\partial x_\alpha} + \frac{\partial C^\delta}{\partial x_\alpha} D^\eta \frac{\partial \Gamma^\gamma_{\delta\eta}}{\partial x_\beta} + C^\delta \frac{\partial D^\eta}{\partial x_\alpha}\frac{\partial \Gamma^\gamma_{\delta\eta}}{\partial x_\beta} + C^\delta D^\eta \frac{\partial^2 \Gamma^\gamma_{\delta\eta}}{\partial x_\alpha \partial x_\beta}.\end{aligned}$$

Therefore

$$\begin{aligned}\nabla_{\mathbf{A}}\nabla_{\mathbf{B}}\nabla_{\mathbf{C}}\mathbf{D} &= \mathrm{R}(\mathbf{A},\mathbf{B})\nabla_{\mathbf{C}}\mathbf{D} + A^\alpha B^\beta C^\eta \frac{\partial D^\gamma}{\partial x_\eta}\frac{\partial \Gamma^\delta_{\alpha\gamma}}{\partial x_\beta}\frac{\partial}{\partial x_\delta} \\ &+ A^\alpha B^\beta \left(\frac{\partial^2 C^\delta}{\partial x_\alpha \partial x_\beta}\frac{\partial D^\gamma}{\partial x_\delta} + \frac{\partial C^\delta}{\partial x_\beta}\frac{\partial^2 D^\gamma}{\partial x_\alpha \partial x_\delta} + \frac{\partial C^\delta}{\partial x_\alpha}\frac{\partial^2 D^\gamma}{\partial x_\beta \partial x_\delta} + C^\delta \frac{\partial^3 D^\gamma}{\partial x_\alpha \partial x_\beta \partial x_\delta}\right)\frac{\partial}{\partial x_\gamma} \\ &+ \underbrace{A^\alpha B^\beta \left(\frac{\partial C^\delta}{\partial x_\beta} D^\eta \frac{\partial \Gamma^\gamma_{\delta\eta}}{\partial x_\alpha} + C^\delta \frac{\partial D^\eta}{\partial x_\beta}\frac{\partial \Gamma^\gamma_{\delta\eta}}{\partial x_\alpha}\right)\frac{\partial}{\partial x_\gamma}}_{\mathrm{I}} + A^\alpha B^\beta \left(\frac{\partial C^\delta}{\partial x_\alpha} D^\eta \frac{\partial \Gamma^\gamma_{\delta\eta}}{\partial x_\beta} + C^\delta \frac{\partial D^\eta}{\partial x_\alpha}\frac{\partial \Gamma^\gamma_{\delta\eta}}{\partial x_\beta}\right)\frac{\partial}{\partial x_\gamma} \\ &+ \underbrace{A^\alpha B^\beta C^\delta D^\eta \frac{\partial^2 \Gamma^\gamma_{\delta\eta}}{\partial x_\alpha \partial x_\beta}\frac{\partial}{\partial x_\gamma}}_{\mathrm{II}} + A^\alpha \frac{\partial B^\beta}{\partial x_\alpha}\left(\frac{\partial C^\delta}{\partial x_\beta}\frac{\partial D^\gamma}{\partial x_\delta} + C^\delta \frac{\partial^2 D^\gamma}{\partial x_\beta \partial x_\delta}\right)\frac{\partial}{\partial x_\gamma} + \underbrace{A^\alpha \frac{\partial B^\beta}{\partial x_\alpha} C^\delta D^\eta \frac{\partial \Gamma^\gamma_{\delta\eta}}{\partial x_\beta}\frac{\partial}{\partial x_\gamma}}_{\mathrm{III}}.\end{aligned}$$

Since

$$\begin{aligned}\mathrm{I} &\overset{(A.3)}{=} \mathrm{R}\left(\mathbf{A},\nabla_{\mathbf{B}}\mathbf{C}\right)\mathbf{D} + A^\alpha B^\beta \frac{\partial C^\delta}{\partial x_\beta} D^\eta \frac{\partial \Gamma^\gamma_{\alpha\eta}}{\partial x_\delta}\frac{\partial}{\partial x_\gamma} + \mathrm{R}\left(\mathbf{A},\mathbf{C}\right)\nabla_{\mathbf{B}}\mathbf{D} + A^\alpha B^\beta C^\delta \frac{\partial D^\eta}{\partial x_\beta}\frac{\partial \Gamma^\gamma_{\alpha\eta}}{\partial x_\delta}\frac{\partial}{\partial x_\gamma}, \\ \mathrm{II} &= A^\alpha B^\beta C^\delta D^\eta \frac{\partial^2 \Gamma^\gamma_{\delta\eta}}{\partial x_\beta \partial x_\alpha}\frac{\partial}{\partial x_\gamma} \overset{(A.4)}{=} \left(\nabla_{\mathbf{B}}\,\mathrm{R}\right)\left(\mathbf{A},\mathbf{C}\right)\mathbf{D} + A^\alpha B^\beta C^\delta D^\eta \frac{\partial^2 \Gamma^\gamma_{\alpha\eta}}{\partial x_\beta \partial x_\delta}\frac{\partial}{\partial x_\gamma}, \\ \mathrm{III} &\overset{(A.3)}{=} \mathrm{R}\left(\nabla_{\mathbf{A}}\mathbf{B},\mathbf{C}\right)\mathbf{D} + A^\alpha \frac{\partial B^\beta}{\partial x_\alpha} C^\delta D^\eta \frac{\partial \Gamma^\gamma_{\beta\eta}}{\partial x_\delta}\frac{\partial}{\partial x_\gamma}\end{aligned}$$

then (A.7) follows. □

A.2. **Hypersurfaces.** Let $S \subseteq M$ be a smooth, closed, embedded hypersurface. Denote by $\mathbf{N}$ a globally defined unit normal to $S$. If $\mathbf{A} \in \Gamma(TM)$, denote by $\mathbf{A}^T$ its orthogonal projection onto $TS$. Denote by $A$ and $h$ respectively the shape operator and the second fundamental form computed with respect to $\mathbf{N}$, i.e.

$$A\left(\mathbf{A}\right) = \nabla_{\mathbf{A}}\mathbf{N}, \qquad h(\mathbf{A},\mathbf{B}) = \left\langle A(\mathbf{A}),\mathbf{B}\right\rangle, \qquad \mathbf{A},\mathbf{B} \in \Gamma(TS).$$

Denote by $H$ the (non-normalized) mean curvature of $S$, and by $\nabla^S$ the Levi-Civita connection of $S$. If $B$ is a $(2,0)$-tensor field on $S$, set

$$B^t(\mathbf{A},\mathbf{B}) := B(\mathbf{B},\mathbf{A}), \qquad \langle L_B(\mathbf{A}),\mathbf{B}\rangle = B(\mathbf{A},\mathbf{B}), \qquad \mathbf{A},\mathbf{B} \in \Gamma(TS).$$

For any $k \in \mathbb{N}_+$, define

$$B^k(\mathbf{A},\mathbf{B}) = \left\langle L_B^k(\mathbf{A}),\mathbf{B}\right\rangle, \qquad \mathbf{A},\mathbf{B} \in \Gamma(TS).$$

If $B$ is symmetric, then $B^k$ is symmetric. If $p \in S$ and $e_1, \dots, e_n$ is an orthonormal basis of $T_pS$, we write

$$\left(B^k\right)_{ij} \coloneqq B^k(e_i, e_j) = \sum_{l_1, \dots, l_{k-1}=1}^{n} B_{il_1} B_{l_! l_2} \cdots B_{l_{k-1} l_{k-1}} B_{l_{k-1} j}, \qquad i, j = 1, \dots, n.$$

If $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \Gamma(TS)$, the *Codazzi equation* and the *traced Codazzi equation* read as

$$\begin{aligned} \left(\nabla^S_{\mathbf{B}} h\right)(\mathbf{A}, \mathbf{C}) &= \left(\nabla^S_{\mathbf{A}} h\right)(\mathbf{B}, \mathbf{C}) + \mathrm{R}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{N}), \\ \left(\operatorname{div}^S h\right)(\mathbf{A}) &= \mathbf{A} H + \operatorname{Ric}(\mathbf{A}, \mathbf{N}) \end{aligned} \tag{A.8}$$

We will make use of the following consequence of the traced Codazzi equation (A.8).

**Lemma A.2.** *Let* $\mathbf{A} \in \Gamma(TM)$. *Then*

$$\left\langle h, \nabla^S \mathbf{A} \right\rangle + \operatorname{Ric}(\mathbf{A}, \mathbf{N}) = \operatorname{div}^S A \left(\mathbf{A}^T\right) - \mathbf{A}^T H + \langle \mathbf{A}, \mathbf{N} \rangle \left(|h|^2 + \operatorname{Ric}(\mathbf{N}, \mathbf{N})\right). \tag{A.9}$$

*Proof.* Let $p \in S$. Let $\mathbf{E}_1, \dots, \mathbf{E}_n$ be a geodesic frame of $S$ at $p$. Then

$$\begin{aligned} \left\langle h, \nabla^S \mathbf{A} \right\rangle &= \sum_{i,j=1}^{n} h(\mathbf{E}_i, \mathbf{E}_j) \left\langle \nabla_{\mathbf{E}_i} \mathbf{A}, \mathbf{E}_j \right\rangle \\ &= \sum_{i,j=1}^{n} h(\mathbf{E}_i, \mathbf{E}_j) \mathbf{E}_i \left\langle \mathbf{A}, \mathbf{E}_j \right\rangle - \sum_{i,j=1}^{n} h(\mathbf{E}_i, \mathbf{E}_j) \left\langle \mathbf{A}, \nabla_{\mathbf{E}_i} \mathbf{E}_j \right\rangle \\ &= \sum_{i,j=1}^{n} \mathbf{E}_i \left(h(\mathbf{E}_i, \mathbf{E}_j) \left\langle \mathbf{A}, \mathbf{E}_j \right\rangle\right) - \sum_{i,j=1}^{n} \mathbf{E}_i \left(h(\mathbf{E}_i, \mathbf{E}_j)\right) \left\langle \mathbf{A}, \mathbf{E}_j \right\rangle - \langle \mathbf{A}, \mathbf{N} \rangle \sum_{i,j=1}^{n} h(\mathbf{E}_i, \mathbf{E}_j) \left\langle \mathbf{N}, \nabla_{\mathbf{E}_i} \mathbf{E}_j \right\rangle \\ &= \sum_{i=1}^{n} \mathbf{E}_i \left(h\left(\mathbf{E}_i, \mathbf{A}^T\right)\right) - \sum_{i,j=1}^{n} \left(\nabla^S_{\mathbf{E}_i} h\right)(\mathbf{E}_i, \mathbf{E}_j) \left\langle \mathbf{A}, \mathbf{E}_j \right\rangle + \langle \mathbf{A}, \mathbf{N} \rangle |h|^2 \\ &= \operatorname{div}^S A \left(\mathbf{A}^T\right) - \left(\operatorname{div}^S h\right)\left(\mathbf{A}^T\right) + \langle \mathbf{A}, \mathbf{N} \rangle |h|^2. \end{aligned}$$

Therefore

$$\begin{aligned} \left\langle h, \nabla^S \mathbf{A} \right\rangle + \operatorname{Ric}(\mathbf{A}, \mathbf{N}) &= \operatorname{div}^S A \left(\mathbf{A}^T\right) - \left(\operatorname{div}^S h\right)\left(\mathbf{A}^T\right) + \operatorname{Ric}\left(\mathbf{A}^T, \mathbf{N}\right) + \langle \mathbf{A}, \mathbf{N} \rangle \left(|h|^2 + \operatorname{Ric}(\mathbf{N}, \mathbf{N})\right) \\ &\overset{(A.8)}{=} \operatorname{div}^S A \left(\mathbf{A}^T\right) - \mathbf{A}^T H + \langle \mathbf{A}, \mathbf{N} \rangle \left(|h|^2 + \operatorname{Ric}(\mathbf{N}, \mathbf{N})\right). \end{aligned}$$

□

A.3. **Variations.** A smooth variation is a smooth map $\Phi : I \times M \to M$, where $I \subseteq \mathbb{R}$ is any open neighborhood of 0, such that:

- $p \mapsto \Phi(t, p)$ is a diffeomorphism for any $t \in I$;
- $\Phi(0, p) = p$ for any $p \in M$.

We may adopt the notation $\Phi_t(p) \coloneqq \Phi(t, p)$. Define the time-dependent vector field $\mathscr{X}$ by

$$\mathscr{X}(t, q) = \left.\frac{\partial}{\partial s}\right|_{s=0} \left(\Phi_{s+t} \circ \Phi_t^{-1}\right)(q) \qquad \text{for any } t \in I \text{ and } q \in M.$$

Define

$$\mathbf{X}(q) = \mathscr{X}(0, q), \qquad \mathbf{X}'(q) = \left.\frac{\partial}{\partial t}\right|_{t=0} \mathscr{X}(t, q), \qquad \mathbf{Z}(q) = \left(\mathbf{X}' + \nabla_{\mathbf{X}} \mathbf{X}\right)(q) \qquad \text{for any } q \in M.$$

The vector fields $\mathbf{X}$ and $\mathbf{Z}$ are known as *variational velocity field* and *variational acceleration field* of $\Phi$. We point out that, while the velocity depends only on the underlying differential structure, the acceleration does depend on the metric via the connection term $\nabla_{\mathbf{X}} \mathbf{X}$. Fix $\varphi$ and $\mathbf{X}^T \in \Gamma(TS)$ such that

$$\mathbf{X}|_S = \mathbf{X}^T + \varphi \mathbf{N}.$$

When $X^T \equiv 0$, $\Phi$ is called *normal variation.* Set $S_t \coloneqq \Phi_t(S)$ for any $t \in \mathbb{R}$. If $p \in S$ is fixed, set $\beta_p(t) \coloneqq \Phi_t(p)$ for every $t \in I$. If there is no ambiguity, we write $\beta = \beta_p$. By definition, $\beta(t) \in S_t$ for every $t \in I$. Moreover,

$$\mathscr{X}(t, \beta(t)) = \mathscr{X}(t, \Phi_t(p)) = \left.\frac{\partial}{\partial s}\right|_{s=0} \Phi_{s+t}(p) = \left.\frac{\partial}{\partial s}\right|_{s=t} \Phi_s(p) = \dot{\beta}(t) \qquad \text{for any } t \in \mathbb{R}.$$

Therefore, $\Phi$ is the (time-dependent) flow of $\mathcal{X}$. For any $t \in \mathbb{R}$, denote by $\mathcal{N}(t, \cdot)$ a smooth choice of arbitrary local extensions, around $S_t$, of unit normals to $S_t$, in such a way that $\mathcal{N}(0, q) = \mathbf{N}(q)$ locally around $S$. Moreover, if $p \in S$, denote by $H_t\left(\Phi_t(p)\right)$ the mean curvature of $S_t$ at $\Phi_t(p)$. When $p \in S$ is fixed and local coordinates $x_1, \ldots, x_{n+1}$ around $p$ are given, we will write

$$\mathcal{X}(t, q) = a^i(t, q) \left.\frac{\partial}{\partial x_i}\right|_q, \qquad \mathcal{N}(t, q) = c^k(t, q) \left.\frac{\partial}{\partial x_k}\right|_q$$

and

$$\mathbf{X}(q) = X^i(q) \left.\frac{\partial}{\partial x_i}\right|_q, \qquad \mathbf{X}'(q) = \left(X'\right)^i(q) \left.\frac{\partial}{\partial x_i}\right|_q, \qquad \mathbf{N}(q) = N^i(q) \left.\frac{\partial}{\partial x_i}\right|_q.$$

Notice that, by definition,

$$X^i(q) = a^i(0, q), \qquad (X')^i(q) = \left.\frac{\partial}{\partial t}\right|_{t=0} a^i(t, q), \qquad N^i(q) = c^i(0, q).$$

We denote by $\frac{D}{dt}$ the covariant derivative operator, along $\beta$, induced by $\nabla$. We recall that

$$\frac{D}{dt}\left(f^j(t) \left.\frac{\partial}{\partial x_j}\right|_{\beta(t)}\right) = \dot{f}^i(t) \left.\frac{\partial}{\partial x_j}\right|_{\beta(t)} + a^i(t, \beta(t)) f^j(t) \Gamma_{ij}^k(\beta(t)) \left.\frac{\partial}{\partial x_k}\right|_{\beta(t)}. \tag{A.10}$$

Finally, when $p \in S$ and $e \in T_p S$ are fixed, we set

$$E(t) := \left(d \Phi_t\right)|_p(e) \qquad \text{for every } t \in I. \tag{A.11}$$

$E$ is a vector field along $\beta$, and $E(t) \in T_{\beta(t)} S_t$ for every $t \in I$. In local coordinates around $p$, we write

$$E(t) = b^j(t) \left.\frac{\partial}{\partial x_j}\right|_{\beta(t)}$$

Fix $p \in S$. Let $e_1, \ldots, e_n$ be an orthonormal basis of $T_p S$. The *Jacobian* of $\Phi_t|_S$ at $p$ is defined by

$$\operatorname{Jac}\left(\Phi_t|_S\right)(p) := \sqrt{\det \mathcal{G}(t)}$$

where the symmetric matrix $\mathcal{G}$ is defined by

$$\mathcal{G}(t)_{ij} := \langle E_i(t), E_j(t)\rangle, \qquad i, j = 1, \ldots, n. \tag{A.12}$$

If $p \in S$ and we fix local coordinates $x_1, \ldots, x_{n+1}$ in a neighborhood of $p$, say $U$, the continuity of $\Phi$ ensures that $\Phi_t(q) \in U$ for any $t$ small and any $q$ sufficiently close to $p$. We will tacitly assume this.

A.4. **Pointwise variations.** In this section we deduce the pointwise evolution of the relevant geometric quantities. The first lemma (cf. [42, Lemma 1.13] and [42, Lemma 1.22]) describes the behavior of $E$.

**Lemma A.3.** *Let $p \in S$. Let $e \in T_p S$. Then*

$$\left.\frac{D}{dt}\right|_{t=0} E(t) = \nabla_e \mathbf{X}; \tag{A.13}$$

$$\left.\frac{D^2}{dt^2}\right|_{t=0} E(t) = \nabla_e \mathbf{Z} + \mathrm{R}(\mathbf{X}, e)\mathbf{X}. \tag{A.14}$$

By Lemma A.3, we can compute the evolution of the Jacobian of $\Phi$. Lemma A.4 is surely well-known (cf. [42, Theorem 1.11] and [42, Theorem 1.21]). We include its proof for the sake of completeness.

**Lemma A.4.** *Let $p \in S$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Jac}\left(\Phi_t|_S\right)(p) = \varphi H + \operatorname{div}^S \mathbf{X}^T, \tag{A.15}$$

$$\begin{aligned} \left.\frac{d^2}{dt^2}\right|_{t=0} \operatorname{Jac}\left(\Phi_t|_S\right)(p) = {} & \left(\operatorname{div}^S \mathbf{X}\right)^2 - \left\langle \nabla^S \mathbf{X}, \left(\nabla^S \mathbf{X}\right)^t \right\rangle - \operatorname{Ric}(\mathbf{X}, \mathbf{X}) - \mathrm{R}\left(\mathbf{X}^T, \mathbf{N}, \mathbf{X}^T, \mathbf{N}\right) \\ & + \left|\nabla^S \varphi\right|^2 - 2h\left(\nabla^S \varphi, \mathbf{X}^T\right) + h^2\left(\mathbf{X}^T, \mathbf{X}^T\right) + \operatorname{div}^S \mathbf{Z}. \end{aligned} \tag{A.16}$$

*In particular, if $\Phi$ is a normal variation,*

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \operatorname{Jac}\left(\Phi_t|_S\right)(p) = \varphi^2 H^2 - \varphi^2 |h|^2 - \varphi^2 \operatorname{Ric}(\mathbf{N}, \mathbf{N}) + \left|\nabla^S \varphi\right|^2 + \operatorname{div}^S \mathbf{Z}. \tag{A.17}$$

*Proof.* Since $\mathcal{G}(t)$ has full rank for every $t$ sufficiently small, Jacobi's formula grants that, for any such $t$,

$$\frac{d \det \mathcal{G}(t)}{dt} = \det \mathcal{G}(t) \operatorname{trace}\left(\mathcal{G}(t)^{-1} \frac{d\mathcal{G}(t)}{dt}\right),$$

whence

$$\frac{d}{dt} \operatorname{Jac}\left(\Phi_t|_S\right)(p) = \frac{1}{2} \operatorname{Jac}\left(\Phi_t|_S\right)(p) \operatorname{trace}\left(\mathcal{G}(t)^{-1} \frac{d\mathcal{G}(t)}{dt}\right).$$

Recalling that $\mathcal{G}(0)_{ij} = \delta_{ij}$ for $i, j = 1, \dots, n$ by construction,

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Jac}\left(\Phi_t|_S\right)(p) = \frac{1}{2} \operatorname{trace}\left(\left.\frac{d}{dt}\right|_{t=0} \mathcal{G}(t)\right) \overset{(\text{A.13})}{=} \sum_{i=1}^{n} \left\langle \nabla_{e_i} \mathbf{X}, e_i \right\rangle = \operatorname{div}^S \mathbf{X}.$$

Since

$$\operatorname{div}^S \mathbf{X} = \operatorname{div}^S(\varphi \mathbf{N}) + \operatorname{div}^S \mathbf{X}^T = \varphi H + \operatorname{div}^S \mathbf{X}^T,$$

(A.15) follows. Moreover, since

$$\frac{d\mathcal{G}(t)^{-1}}{dt} = -\mathcal{G}(t)^{-1} \frac{d\mathcal{G}(t)}{dt} \mathcal{G}(t)^{-1}, \tag{A.18}$$

then

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \operatorname{Jac}\left(\Phi_t|_S\right)(p) \overset{(\text{A.15})}{=} \left(\operatorname{div}^S \mathbf{X}\right)^2 + \underbrace{\frac{1}{2} \operatorname{trace}\left(-\left(\left.\frac{d}{dt}\right|_{t=0} \mathcal{G}(t)\right)^2\right)}_{\text{I}} + \underbrace{\frac{1}{2} \operatorname{trace}\left(\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{G}(t)\right)}_{\text{II}}.$$

First,

$$\text{I} \overset{(\text{A.13})}{=} -\frac{1}{2} \sum_{i,j=1}^{n} \left(\left\langle \nabla_{e_i} \mathbf{X}, e_j \right\rangle + \left\langle \nabla_{e_j} \mathbf{X}, e_i \right\rangle\right)^2 = -\sum_{i,j=1}^{n} \left\langle \nabla_{e_i} \mathbf{X}, e_j \right\rangle^2 - \sum_{i,j=1}^{n} \left\langle \nabla_{e_i} \mathbf{X}, e_j \right\rangle \left\langle \nabla_{e_j} \mathbf{X}, e_i \right\rangle.$$

Moreover,

$$\begin{aligned}
\text{II} &\overset{(\text{A.13}),(\text{A.14})}{=} \sum_{i=1}^{n} \left\langle \nabla_{e_i} \mathbf{X}, \nabla_{e_i} \mathbf{X} \right\rangle + \sum_{i=1}^{n} \left\langle \nabla_{e_i} \mathbf{Z} + \mathrm{R}(\mathbf{X}, e_i)\mathbf{X}, e_i \right\rangle \\
&= \sum_{i,j=1}^{n} \left\langle \nabla_{e_i} \mathbf{X}, e_j \right\rangle^2 + \sum_{i=1}^{n} \left\langle \nabla_{e_i} \mathbf{X}, \mathbf{N} \right\rangle^2 + \operatorname{div}^S \mathbf{Z} - \operatorname{Ric}(\mathbf{X}, \mathbf{X}) - \mathrm{R}(\mathbf{X}, \mathbf{N}, \mathbf{X}, \mathbf{N}) \\
&= \sum_{i,j=1}^{n} \left\langle \nabla_{e_i} \mathbf{X}, e_j \right\rangle^2 + \sum_{i=1}^{n} \left(e_i(\varphi) - \left\langle A\left(\mathbf{X}^T\right), e_i \right\rangle\right)^2 + \operatorname{div}^S \mathbf{Z} - \operatorname{Ric}(\mathbf{X}, \mathbf{X}) - \mathrm{R}(\mathbf{X}^T, \mathbf{N}, \mathbf{X}^T, \mathbf{N}) \\
&= \sum_{i,j=1}^{n} \left\langle \nabla_{e_i} \mathbf{X}, e_j \right\rangle^2 + \left|\nabla^S \varphi\right|^2 - 2h\left(\nabla^S \varphi, \mathbf{X}^T\right) + h^2\left(\mathbf{X}^T, \mathbf{X}^T\right) + \operatorname{div}^S \mathbf{Z} - \operatorname{Ric}(\mathbf{X}, \mathbf{X}) - \mathrm{R}(\mathbf{X}^T, \mathbf{N}, \mathbf{X}^T, \mathbf{N}).
\end{aligned}$$

In this way, (A.16) follows. Finally, if $\Phi$ is a normal variation, then $\mathbf{X} = \varphi \mathbf{N}$ and $\mathbf{X}^T = 0$. In particular, $\nabla^S \mathbf{X} = \left(\nabla^S \mathbf{X}\right)^t = \varphi h$, and (A.17) by (A.16). □

Define $V(t) \coloneqq \mathcal{N}(t, \beta(t))$. By definition, $V$ is a vector field along $\beta$, and in particular $V(t)$ is normal to $S_t$ at $\beta(t)$ for every $t \in I$. $V$ evolves as follows.

**Lemma A.5.** *Let $p \in S$. Let $e_1, \dots, e_n$ be an orthonormal basis of $T_pS$. Then*

$$V'(p) \coloneqq \left.\frac{D}{dt}\right|_{t=0} V(t) = -\nabla^S \varphi + A\left(\mathbf{X}^T\right); \tag{A.19}$$

$$\begin{aligned}
V''(p) \coloneqq \left.\frac{D^2}{dt^2}\right|_{t=0} V(t) &= \left(-\left|\nabla^S \varphi\right|^2 - \left|A\left(\mathbf{X}^T\right)\right|^2 + 2h\left(\mathbf{X}^T, \nabla^S \varphi\right)\right) \mathbf{N} \\
&\quad + \sum_{j=1}^{n} \left(-\left\langle \mathbf{N}, \nabla_{e_j} \mathbf{Z} + \mathrm{R}(\mathbf{X}, e_j)\, \mathbf{X}^T \right\rangle + 2\left\langle \nabla^S \varphi - A\left(\mathbf{X}^T\right), \varphi A(e_j) + \nabla_{e_j}\, \mathbf{X}^T \right\rangle\right) e_j.
\end{aligned} \tag{A.20}$$

*Moreover, fix local coordinates* $x_1, \dots, x_{n+1}$ *in* $M$ *near* $p$*, and set*

$$\mathbf{N}'(q) = \frac{\partial c^k}{\partial t}(0,q)\, \frac{\partial}{\partial x_k}\Big|_q, \qquad \mathbf{N}''(q) = \frac{\partial^2 c^k}{\partial t^2}(0,q)\, \frac{\partial}{\partial x_k}\Big|_q \qquad \textit{locally around } p.$$

*Then, when* $q \in S$ *is close to* $p$*,*

$$\mathbf{N}'(q) = V'(q) - \nabla_{\mathbf{X}} \mathbf{N}, \tag{A.21}$$

$$\mathbf{N}''(q) = V''(q) - 2\nabla_{\mathbf{X}} \mathbf{N}' - \nabla_{\mathbf{X}'} \mathbf{N} - \nabla_{\mathbf{X}} \nabla_{\mathbf{X}} \mathbf{N} \tag{A.22}$$

*Proof.* First, (A.19) follows by [42, Lemma 1.25]. We prove (A.20). As $|V(t)| \equiv 1$, then

$$\left\langle \frac{D}{dt} V(t), V(t) \right\rangle \equiv 0. \tag{A.23}$$

In particular, by (A.19) and (A.23),

$$\left\langle \frac{D^2}{dt^2}\Big|_{t=0} V(t), \mathbf{N}|_p \right\rangle = \frac{d}{dt}\Big|_{t=0} \left\langle \frac{D}{dt} V(t), V(t) \right\rangle - \left|V'(p)\right|^2 = -\left|\nabla^S \varphi\right|^2 - \left|A\left(\mathbf{X}^T\right)\right|^2 + 2h\left(\mathbf{X}^T, \nabla^S \varphi\right).$$

Let $f \in T_p S$. Set $F(t) \coloneqq (d\Phi_t)\,|_p(f)$. Recall that $F(t)$ is a vector field along $\beta$, and $F(t) \in T_{\beta(t)} S_t$. Therefore

$$\begin{aligned}
\left\langle \frac{D^2}{dt^2}\Big|_{t=0} V(t), f \right\rangle &= \frac{d}{dt}\Big|_{t=0} \left\langle \frac{D}{dt} V(t), F(t) \right\rangle - \left\langle \frac{D}{dt}\Big|_{t=0} V(t), \frac{D}{dt}\Big|_{t=0} F(t) \right\rangle \\
&= -\frac{d}{dt}\Big|_{t=0} \left\langle V(t), \frac{D}{dt} F(t) \right\rangle - \left\langle \frac{D}{dt}\Big|_{t=0} V(t), \frac{D}{dt}\Big|_{t=0} F(t) \right\rangle \\
&= -\left\langle \mathbf{N}, \frac{D^2}{dt^2}\Big|_{t=0} F(t) \right\rangle - 2\left\langle \frac{D}{dt}\Big|_{t=0} V(t), \frac{D}{dt}\Big|_{t=0} F(t) \right\rangle \\
&\overset{(A.13),(A.14),(A.19)}{=} -\left\langle \mathbf{N}, \nabla_f\left(\nabla_{\mathbf{X}} \mathbf{X} + \mathbf{X}'\right) + \mathrm{R}(\mathbf{X}, f)\mathbf{X} \right\rangle + 2\left\langle \nabla^S \varphi - A\left(\mathbf{X}^T\right), \nabla_f \mathbf{X} \right\rangle \\
&= -\left\langle \mathbf{N}, \nabla_f \mathbf{Z} + \mathrm{R}(\mathbf{X}, f)\ \mathbf{X}^T \right\rangle + 2\left\langle \nabla^S \varphi - A\left(\ \mathbf{X}^T\right), \varphi A(f) + \nabla_f\ \mathbf{X}^T \right\rangle.
\end{aligned}$$

In this way, (A.20) follows. Next we prove (A.21). Indeed, if $q \in S$ is close to $p$,

$$\frac{D}{dt}\Big|_{t=0} \mathcal{N}(t, \beta_q(t)) \overset{(A.10)}{=} \frac{\partial c^k}{\partial t}(0,q)\, \frac{\partial}{\partial x_k}\Big|_q + X^i(q) \frac{\partial N^k}{\partial x_i}(q)\, \frac{\partial}{\partial x_k}\Big|_q + X^i(q) N^k(q) \Gamma^l_{ik}(q)\, \frac{\partial}{\partial x_l}\Big|_q = \mathbf{N}' + \nabla_{\mathbf{X}} \mathbf{N}.$$

Finally, for $q$ close to $p$,

$$\begin{aligned}
&\frac{D^2}{dt^2}\Big|_{t=0} \mathcal{N}(t, \beta_q(t)) = \frac{D^2}{dt^2}\Big|_{t=0} \left( c^k(t, \beta_q(t))\, \frac{\partial}{\partial x_k}\Big|_{\beta_q(t)} \right) \\
&= \frac{D}{dt}\Big|_{t=0} \left( \frac{\partial c^k}{\partial t}(t, \beta_q(t))\, \frac{\partial}{\partial x_k}\Big|_{\beta_q(t)} + a^i(t, \beta_q(t)) \frac{\partial c^k}{\partial x_i}(t, \beta_q(t))\, \frac{\partial}{\partial x_k}\Big|_{\beta_q(t)} \right) \\
&\quad + \frac{D}{dt}\Big|_{t=0} \left( a^i(t, \beta_q(t)) c^k(t, \beta_q(t)) \Gamma^l_{ik}(\beta_q(t))\, \frac{\partial}{\partial x_l}\Big|_{\beta_q(t)} \right) \\
&= \mathbf{N}''(q) + 2X^i(q) \frac{\partial^2 c^k}{\partial x_i \partial t}(0,q)\, \frac{\partial}{\partial x_k}\Big|_q + 2X^i(q) \frac{\partial c^k}{\partial t}(0,q) \Gamma^l_{ik}(q)\, \frac{\partial}{\partial x_l}\Big|_q + (X')^i(q) \frac{\partial N^k}{\partial x_i}(q)\, \frac{\partial}{\partial x_k}\Big|_q \\
&\quad + X^l(q) \frac{\partial X^i}{\partial x_l}(q) \frac{\partial N^k}{\partial x_i}(q)\, \frac{\partial}{\partial x_k}\Big|_q + X^i(q) X^l(q) \frac{\partial^2 N^k}{\partial x_l \partial x_i}(q)\, \frac{\partial}{\partial x_k}\Big|_q + 2X^i(q) X^l(q) \frac{\partial N^k}{\partial x_i}(q) \Gamma^m_{lk}(q)\, \frac{\partial}{\partial x_m}\Big|_q \\
&\quad + \left(X'\right)^i(q) N^k(q) \Gamma^l_{ik}(q)\, \frac{\partial}{\partial x_l}\Big|_q + X^m(q) \frac{\partial X^i}{\partial x_m}(q) N^k(q) \Gamma^l_{ik}(q)\, \frac{\partial}{\partial x_l}\Big|_q \\
&\quad + X^i(q) X^m(q) N^k(q) \frac{\partial \Gamma^l_{ik}}{\partial x_m}(q)\, \frac{\partial}{\partial x_l}\Big|_q + X^i(q) X^m(q) N^k(q) \Gamma^l_{ik}(q) \Gamma^s_{ml}\, \frac{\partial}{\partial x_s}\Big|_q,
\end{aligned}$$

whence (A.22) follows by (A.5). □

Next, we describe the evolution of the shape operator. The following proof relies crucially on Lemma A.1.

**Lemma A.6.** *Let $p \in S$. Let $e \in T_pS$. Then*

$$\frac{D}{dt}\bigg|_{t=0}\left(\nabla_{E(t)}\mathcal{N}(t,\cdot)\big|_{\beta(t)}\right) = \nabla_e V' + \mathrm{R}(\mathbf{X}, e)\mathbf{N}, \tag{A.24}$$

$$\frac{D^2}{dt^2}\bigg|_{t=0}\left(\nabla_{E(t)}\mathcal{N}(t,\cdot)\big|_{\beta(t)}\right) = \nabla_e V'' + \mathrm{R}\left(\mathbf{X}, \nabla_e\mathbf{X}\right)\mathbf{N} + 2\,\mathrm{R}(\mathbf{X}, e)V' + \mathrm{R}\left(\mathbf{Z}, e\right)\mathbf{N} + \left(\nabla_{\mathbf{X}}\mathrm{R}\right)(\mathbf{X}, e)\mathbf{N}, \tag{A.25}$$

*where $V'$ and $V''$ are given respectively by* (A.19) *and* (A.20)*.*

*Proof.* Consider local normal coordinates $x_1, \dots, x_{n+1}$ centered at $p$. Since $E(t) \in T_{\beta(t)}S_t$, then

$$\nabla_{E(t)}\mathcal{N}(t,\cdot)\big|_{\beta(t)} = b^j(t)\frac{\partial c^k}{\partial x_j}(t,\beta(t))\,\frac{\partial}{\partial x_k}\bigg|_{\beta(t)} + b^j(t)c^k(t,\beta(t))\Gamma^l_{jk}(\beta(t))\,\frac{\partial}{\partial x_l}\bigg|_{\beta(t)}.$$

Therefore

$$\begin{aligned}
\frac{D}{dt}\Big(\nabla_{E(t)}\mathcal{N}(t,\cdot)\big|_{\beta(t)}\Big) &= \dot{b}^j(t)\frac{\partial c^k}{\partial x_j}(t,\beta(t))\,\frac{\partial}{\partial x_k}\bigg|_{\beta(t)} + b^j(t)\frac{\partial^2 c^k}{\partial t\partial x_j}(t,\beta(t))\,\frac{\partial}{\partial x_k}\bigg|_{\beta(t)}\\
&+ a^i(t,\beta(t))b^j(t)\frac{\partial^2 c^k}{\partial x_i\partial x_j}(t,\beta(t))\,\frac{\partial}{\partial x_k}\bigg|_{\beta(t)} + a^i(t,\beta(t))b^j(t)\frac{\partial c^k}{\partial x_j}(t,\beta(t))\Gamma^l_{ik}(\beta(t))\,\frac{\partial}{\partial x_l}\bigg|_{\beta(t)}\\
&+ \dot{b}^j(t)c^k(t,\beta(t))\Gamma^l_{jk}(\beta(t))\,\frac{\partial}{\partial x_l}\bigg|_{\beta(t)} + b^j(t)\frac{\partial c^k}{\partial t}(t,\beta(t))\Gamma^l_{jk}(\beta(t))\,\frac{\partial}{\partial x_l}\bigg|_{\beta(t)}\\
&+ a^i(t,\beta(t))b^j(t)\frac{\partial c^k}{\partial x_i}(t,\beta(t))\Gamma^l_{jk}(\beta(t))\,\frac{\partial}{\partial x_l}\bigg|_{\beta(t)} + a^i(t,\beta(t))b^j(t)c^k(t,\beta(t))\frac{\partial \Gamma^l_{jk}}{\partial x_i}(\beta(t))\,\frac{\partial}{\partial x_l}\bigg|_{\beta(t)}\\
&+ a^i(t,\beta(t))b^j(t)c^k(t,\beta(t))\Gamma^l_{jk}(\beta(t))\Gamma^m_{il}(\beta(t))\,\frac{\partial}{\partial x_m}\bigg|_{\beta(t)}.
\end{aligned} \tag{A.26}$$

In particular, by (A.2),

$$\begin{aligned}
\frac{D}{dt}\bigg|_{t=0}\Big(\nabla_{E(t)}\mathcal{N}(t,\cdot)\big|_{\beta(t)}\Big) = \dot{b}^j(0)\frac{\partial N^k}{\partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p &+ e^j\frac{\partial^2 c^k}{\partial t\partial x_j}(0,p)\,\frac{\partial}{\partial x_k}\bigg|_p\\
&+ X^i(p)e^j\frac{\partial^2 N^k}{\partial x_i\partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p + X^i(p)e^jN^k(p)\frac{\partial \Gamma^l_{jk}}{\partial x_i}(p)\,\frac{\partial}{\partial x_l}\bigg|_p.
\end{aligned}$$

Observe that

$$\dot{b}^i(0)\,\frac{\partial}{\partial x_i}\bigg|_p \overset{(A.2)}{=} \frac{D}{dt}\bigg|_{t=0} E(t) \overset{(A.13)}{=} \nabla_e\mathbf{X} \overset{(A.2)}{=} \frac{\partial X^i}{\partial x_j}(p)e^j\,\frac{\partial}{\partial x_i}\bigg|_p, \tag{A.27}$$

whence (A.24) follows by

$$\begin{aligned}
\frac{D}{dt}\bigg|_{t=0}\Big(\nabla_{E(t)}\mathcal{N}(t,\cdot)\big|_{\beta(t)}\Big) = \frac{\partial X^i}{\partial x_j}(p)e^j\frac{\partial N^k}{\partial x_i}(p)\,\frac{\partial}{\partial x_k}\bigg|_p &+ e^j\frac{\partial^2 c^k}{\partial x_j\partial t}(0,p)\,\frac{\partial}{\partial x_k}\bigg|_p\\
&+ X^i(p)e^j\frac{\partial^2 N^k}{\partial x_i\partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p + X^i(p)e^jN^k(p)\frac{\partial \Gamma^l_{jk}}{\partial x_i}(p)\,\frac{\partial}{\partial x_l}\bigg|_p\\
&\overset{(A.6)}{=} e^j\frac{\partial^2 c^k}{\partial x_j\partial t}(0,p)\,\frac{\partial}{\partial x_k}\bigg|_p + \nabla_e\nabla_{\mathbf{X}}\mathbf{N} - \mathrm{R}(e,\mathbf{X})\mathbf{N}\\
&\overset{(A.2)}{=} \nabla_e\mathbf{N}' + \nabla_e\nabla_{\mathbf{X}}\mathbf{N} - \mathrm{R}(e,\mathbf{X})\mathbf{N}\\
&\overset{(A.21)}{=} \nabla_e V' - \mathrm{R}(e,\mathbf{X})\mathbf{N}.
\end{aligned}$$

We prove (A.25). By (A.26) and (A.2),

$$\begin{aligned}
\frac{D^2}{dt^2}\bigg|_{t=0}\Big(\nabla_{E(t)}\mathcal{N}(t,\cdot)\big|_{\beta(t)}\Big) &= \ddot{b}^j(0)\frac{\partial N^k}{\partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p + 2\dot{b}^j(0)\frac{\partial^2 c^k}{\partial x_j\partial t}(0,p)\,\frac{\partial}{\partial x_k}\bigg|_p \\
&+ 2X^i(p)\dot{b}^j(0)\frac{\partial^2 N^k}{\partial x_i\partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p + e^j\frac{\partial^3 c^k}{\partial x_j\partial t^2}(0,p)\,\frac{\partial}{\partial x_k}\bigg|_p + 2X^i(p)e^j\frac{\partial^3 c^k}{\partial x_i\partial x_j\partial t}(0,p)\,\frac{\partial}{\partial x_k}\bigg|_p \\
&+ (X')^i(p)e^j\frac{\partial^2 N^k}{\partial x_i\partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p + X^l(p)\frac{\partial X^i}{\partial x_l}(p)e^j\frac{\partial^2 N^k}{\partial x_i\partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p + X^i(p)X^l(p)e^j\frac{\partial^3 N^k}{\partial x_l\partial x_i\partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p \\
&+ X^i(p)X^m(p)e^j\frac{\partial N^k}{\partial x_j}(p)\frac{\partial \Gamma^l_{ik}}{\partial x_m}(p)\,\frac{\partial}{\partial x_l}\bigg|_p + 2X^i(p)\dot{b}^j(0)N^k(p)\frac{\partial \Gamma^l_{jk}}{\partial x_i}(p)\,\frac{\partial}{\partial x_l}\bigg|_p \\
&+ 2X^i(p)e^j\frac{\partial c^k}{\partial t}(0,p)\frac{\partial \Gamma^l_{jk}}{\partial x_i}(p)\,\frac{\partial}{\partial x_l}\bigg|_p + 2X^i(p)X^m(p)e^j\frac{\partial N^k}{\partial x_i}(p)\frac{\partial \Gamma^l_{jk}}{\partial x_m}(p)\,\frac{\partial}{\partial x_l}\bigg|_p \\
&+ (X')^i(p)e^jN^k(p)\frac{\partial \Gamma^l_{jk}}{\partial x_i}(p)\,\frac{\partial}{\partial x_l}\bigg|_p + X^m(p)\frac{\partial X^i}{\partial x_m}(p)e^jN^k(p)\frac{\partial \Gamma^l_{jk}}{\partial x_i}(p)\,\frac{\partial}{\partial x_l}\bigg|_p \\
&+ X^i(p)X^m(p)e^jN^k(p)\frac{\partial^2 \Gamma^l_{jk}}{\partial x_m\partial x_i}(p)\,\frac{\partial}{\partial x_l}\bigg|_p .
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{D^2}{dt^2}\bigg|_{t=0}E(t) &\overset{(A.10)}{=} \frac{D}{dt}\bigg|_{t=0}\left(\dot{b}^j(t)\,\frac{\partial}{\partial x_j}\bigg|_{\beta(t)} + a^i(t,\beta(t))b^j(t)\Gamma^l_{ij}(\beta(t))\,\frac{\partial}{\partial x_l}\bigg|_{\beta(t)}\right) \\
&\overset{(A.2)}{=} \ddot{b}^j(0)\,\frac{\partial}{\partial x_j}\bigg|_p + X^i(p)X^m(p)e^j\frac{\partial \Gamma^l_{ij}}{\partial x_m}(p)\,\frac{\partial}{\partial x_l}\bigg|_p \\
&= \left(\ddot{b}^j(0) + X^i(p)X^m(p)e^l\frac{\partial \Gamma^j_{il}}{\partial x_m}(p)\right)\frac{\partial}{\partial x_j}\bigg|_p .
\end{aligned}$$

Therefore, by (A.2), (A.6), (A.14) and for any $j = 1, \dots, n+1$,

(A.28)
$$\begin{aligned}
\ddot{b}^j(0) &= (\nabla_e\nabla_{\mathbf{X}}\mathbf{X})^j(p) + \big(\nabla_e\mathbf{X}'\big)^j(p) + (\mathrm{R}(\mathbf{X},e)\mathbf{X})^j(p) - X^i(p)X^m(p)e^l\frac{\partial \Gamma^j_{il}}{\partial x_m}(p) \\
&= e^lX^i(p)X^m(p)\frac{\partial \Gamma^j_{lm}}{\partial x_i} + e^lX^i(p)\frac{\partial^2 X^j}{\partial x_l\partial x_i}(p) + e^l\frac{\partial X^i}{\partial x_l}(p)\frac{\partial X^j}{\partial x_i}(p) + e^l\frac{\partial (X')^j}{\partial x_l}(p) - X^i(p)X^m(p)e^l\frac{\partial \Gamma^j_{il}}{\partial x_m}(p) \\
&= e^lX^i(p)\frac{\partial^2 X^j}{\partial x_l\partial x_i}(p) + e^l\frac{\partial X^i}{\partial x_l}(p)\frac{\partial X^j}{\partial x_i}(p) + e^l\frac{\partial (X')^j}{\partial x_l}(p),
\end{aligned}$$

where the last equality follows by the symmetry of $\nabla$. Then, by (A.27) and (A.28),

$$\begin{aligned}
\frac{D^2}{dt^2}\bigg|_{t=0}\Big(\nabla_{E(t)}\mathcal{N}(t,\cdot)\big|_{\beta(t)}\Big) &= e^l X^i(p)\frac{\partial^2 X^j}{\partial x_l \partial x_i}(p)\frac{\partial N^k}{\partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p + e^l\frac{\partial X^i}{\partial x_l}(p)\frac{\partial X^j}{\partial x_i}(p)\frac{\partial N^k}{\partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p \\
&+ e^l\frac{\partial (X')^j}{\partial x_l}(p)\frac{\partial N^k}{\partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p + 2\frac{\partial X^i}{\partial x_j}(p)e^j\frac{\partial^2 c^k}{\partial x_i \partial t}(0,p)\,\frac{\partial}{\partial x_k}\bigg|_p \\
&+ 2X^i(p)\frac{\partial X^m}{\partial x_j}(p)e^j\frac{\partial^2 N^k}{\partial x_i \partial x_m}(p)\,\frac{\partial}{\partial x_k}\bigg|_p + e^j\frac{\partial^3 c^k}{\partial x_j \partial t^2}(0,p)\,\frac{\partial}{\partial x_k}\bigg|_p + 2X^i(p)e^j\frac{\partial^3 c^k}{\partial x_i \partial x_j \partial t}(0,p)\,\frac{\partial}{\partial x_k}\bigg|_p \\
&+ (X')^i(p)e^j\frac{\partial^2 N^k}{\partial x_i \partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p + X^l(p)\frac{\partial X^i}{\partial x_l}(p)e^j\frac{\partial^2 N^k}{\partial x_i \partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p + X^i(p)X^l(p)e^j\frac{\partial^3 N^k}{\partial x_l \partial x_i \partial x_j}(p)\,\frac{\partial}{\partial x_k}\bigg|_p \\
&+ X^i(p)X^m(p)e^j\frac{\partial N^k}{\partial x_j}(p)\frac{\partial \Gamma^l_{ik}}{\partial x_m}(p)\,\frac{\partial}{\partial x_l}\bigg|_p + 2X^i(p)\frac{\partial X^m}{\partial x_j}(p)e^j N^k(p)\frac{\partial \Gamma^l_{mk}}{\partial x_i}(p)\,\frac{\partial}{\partial x_l}\bigg|_p \\
&+ 2X^i(p)e^j\frac{\partial c^k}{\partial t}(0,p)\frac{\partial \Gamma^l_{jk}}{\partial x_i}(p)\,\frac{\partial}{\partial x_l}\bigg|_p + 2X^i(p)X^m(p)e^j\frac{\partial N^k}{\partial x_i}(p)\frac{\partial \Gamma^l_{jk}}{\partial x_m}(p)\,\frac{\partial}{\partial x_l}\bigg|_p \\
&+ (X')^i(p)e^j N^k(p)\frac{\partial \Gamma^l_{jk}}{\partial x_i}(p)\,\frac{\partial}{\partial x_l}\bigg|_p + X^m(p)\frac{\partial X^i}{\partial x_m}(p)e^j N^k(p)\frac{\partial \Gamma^l_{jk}}{\partial x_i}(p)\,\frac{\partial}{\partial x_l}\bigg|_p \\
&+ X^i(p)X^m(p)e^j N^k(p)\frac{\partial^2 \Gamma^l_{jk}}{\partial x_m \partial x_i}(p)\,\frac{\partial}{\partial x_l}\bigg|_p .
\end{aligned}$$

A careful comparison between the above expression and (A.6) and (A.7) grants that

$$\begin{aligned}
\frac{D^2}{dt^2}\bigg|_{t=0}\Big(\nabla_{E(t)}\mathcal{N}(t,\cdot)\big|_{\beta(t)}\Big) &= \nabla_e\nabla_{\mathbf{X}}\nabla_{\mathbf{X}}\mathbf{N} - \left(\nabla_{\mathbf{X}}\mathrm{R}\right)(e,\mathbf{X})\mathbf{N} - 2\,\mathrm{R}(e,\mathbf{X})\nabla_{\mathbf{X}}\mathbf{N} - \mathrm{R}\left(e,\nabla_{\mathbf{X}}\mathbf{X}\right)\mathbf{N} \\
&- \mathrm{R}\left(\nabla_e\mathbf{X},\mathbf{X}\right)\mathbf{N} + 2\nabla_e\nabla_{\mathbf{X}}\mathbf{N}' - 2\,\mathrm{R}(e,\mathbf{X})\mathbf{N}' + \nabla_e\mathbf{N}'' + \nabla_e\nabla_{\mathbf{X}'}\mathbf{N} - \mathrm{R}(e,\mathbf{X}')\mathbf{N}.
\end{aligned}$$

In addition,

$$-2\,\mathrm{R}(e,\mathbf{X})\mathbf{N}' - 2\,\mathrm{R}(e,\mathbf{X})\nabla_{\mathbf{X}}\mathbf{N} \overset{(A.21)}{=} -2\,\mathrm{R}(e,\mathbf{X})V'$$

and

$$\nabla_e\mathbf{N}'' + 2\nabla_e\nabla_{\mathbf{X}}\mathbf{N}' + \nabla_e\nabla_{\mathbf{X}'}\mathbf{N} + \nabla_e\nabla_{\mathbf{X}}\nabla_{\mathbf{X}}\mathbf{N} \overset{(A.22)}{=} \nabla_e V'',$$

whence

$$\frac{D^2}{dt^2}\bigg|_{t=0}\Big(\nabla_{E(t)}\mathcal{N}(t,\cdot)\big|_{\beta(t)}\Big) = -\left(\nabla_{\mathbf{X}}\mathrm{R}\right)(e,\mathbf{X})\mathbf{N} - \mathrm{R}\left(e,\mathbf{Z}\right)\mathbf{N} + \nabla_e V'' - 2\,\mathrm{R}(e,\mathbf{X})V' - \mathrm{R}\left(\nabla_e\mathbf{X},\mathbf{X}\right)\mathbf{N},$$

and (A.25) follows. ☐

Lemma A.6 allows to compute the pointwise evolution of the mean curvature at first and second-order.

**Lemma A.7.** *Let $p \in S$. Then*

$$\frac{d}{dt}\bigg|_{t=0} H_t\left(\Phi_t(p)\right) = -\Delta^S\varphi - \varphi\left(|h|^2 + \mathrm{Ric}(\mathbf{N},\mathbf{N})\right) + \mathbf{X}^T H \tag{A.29}$$

$$\begin{aligned}
\frac{d^2}{dt^2}\bigg|_{t=0} H_t\left(\Phi_t(p)\right) &= 2\left\langle h,\left(\nabla^S X\right)^2\right\rangle - 2\left\langle \nabla^S X,\left(\nabla^S V'\right)^t\right\rangle - \langle h,\mathcal{C}(\mathbf{X},\mathbf{X})\rangle - \langle \nabla^S\mathbf{X},\mathcal{C}(\mathbf{N},\mathbf{X})\rangle \\
&+ \left(-\left|\nabla^S\varphi\right|^2 + 2h\left(\nabla^S\varphi,\mathbf{X}^T\right) - h^2\left(\mathbf{X}^T,\mathbf{X}^T\right)\right)H + \mathrm{div}^S\left(\tilde{V}''\right)^T \\
&- 2\,\mathrm{Ric}(\mathbf{X},V') - \mathrm{R}\left(\mathbf{X}^T,\mathbf{N},V',\mathbf{N}\right) - \left(\nabla_{\mathbf{X}}\mathrm{Ric}\right)(\mathbf{X},\mathbf{N}) \\
&- \Delta^S\langle\mathbf{Z},\mathbf{N}\rangle - \langle\mathbf{Z},\mathbf{N}\rangle\left(|h|^2 + \mathrm{Ric}(\mathbf{N},\mathbf{N})\right) + \mathbf{Z}^T H,
\end{aligned} \tag{A.30}$$

*where*

$$\left(\tilde{V}''\right)^T \coloneqq \sum_{j=1}^{n}\left(\mathrm{R}(\mathbf{X},e_j,\mathbf{N},\mathbf{X}^T) + 2\left\langle\nabla^S\varphi - A\left(\mathbf{X}^T\right),\varphi A(e_j) + \nabla_{e_j}\,\mathbf{X}^T\right\rangle\right)e_j$$

*In particular, if* $\Phi$ *is a normal variation,*

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} H_t\left(\Phi_t(p)\right) = {} & \operatorname{div}^S\left(A\left(\nabla^S\varphi^2\right)\right) + 2\varphi \operatorname{div}^S A\left(\nabla^S\varphi\right) - 2\varphi\left\langle\nabla^S H, \nabla^S\varphi\right\rangle - \left|\nabla^S\varphi\right|^2 H \\ & + 2\varphi^2 \operatorname{trace}\left(h^3\right) - 2\varphi^2\langle h, \mathcal{C}(\mathbf{N},\mathbf{N})\rangle - \varphi^2\left(\nabla_{\mathbf{N}}\operatorname{Ric}\right)(\mathbf{N},\mathbf{N}) \\ & - \Delta^S\langle\mathbf{Z},\mathbf{N}\rangle - \langle\mathbf{Z},\mathbf{N}\rangle\left(|h|^2 + \operatorname{Ric}(\mathbf{N},\mathbf{N})\right) + \mathbf{Z}^T H.\end{aligned} \tag{A.31}$$

*Proof.* Formula (A.29) is well-known (cf. [42, Lemma 1.26]). We prove it for the sake of completeness. Let $e_1, \dots, e_n$ be any orthonormal basis of $T_pS$. For $j = 1, \dots, n$, let $E_j(t)$ be as in (A.11). Define $\S(t)$ by

$$\S(t)_{ij} = \left\langle \nabla_{E_i(t)}\mathcal{N}(t,\cdot)\big|_{\beta(t)}, E_j(t)\right\rangle, \qquad i,j = 1,\dots,n.$$

Then $\S(t)$ is symmetric, and $H(\Phi_t(p)) = \operatorname{trace}\left(\mathcal{G}(t)^{-1}\S(t)\right)$, where $\mathcal{G}$ is defined in (A.12). Recalling (A.18),

$$\frac{d}{dt}H_t(\Phi_t(p)) = \operatorname{trace}\left(-\mathcal{G}(t)^{-1}\frac{d\mathcal{G}(t)}{dt}\mathcal{G}(t)^{-1}\S(t) + \mathcal{G}(t)^{-1}\frac{d\S(t)}{dt}\right).$$

Since $e_1, \dots, e_n$ is orthonormal, then $\mathcal{G}(0)_{ij} = \delta_{ij}$ for $i, j = 1, \dots, n$, whence

$$\left.\frac{d}{dt}\right|_{t=0} H_t(\Phi_t(p)) = \operatorname{trace}\left(-\left(\left.\frac{d}{dt}\right|_{t=0}\mathcal{G}(t)\right)\S(0) + \left.\frac{d}{dt}\right|_{t=0}\S(t)\right), \tag{A.32}$$

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} H_t(\Phi_t(p)) = {} & \underbrace{\operatorname{trace}\left(2\left(\left.\frac{d}{dt}\right|_{t=0}\mathcal{G}(t)\right)^2\S(0)\right)}_{\mathrm{I}} + \underbrace{\operatorname{trace}\left(-\left(\left.\frac{d^2}{dt^2}\right|_{t=0}\mathcal{G}(t)\right)\S(0)\right)}_{\mathrm{II}} \\ & + \underbrace{\operatorname{trace}\left(-2\left(\left.\frac{d}{dt}\right|_{t=0}\mathcal{G}(t)\right)\left(\left.\frac{d}{dt}\right|_{t=0}\S(t)\right)\right)}_{\mathrm{III}} + \underbrace{\operatorname{trace}\left(\left.\frac{d^2}{dt^2}\right|_{t=0}\S(t)\right)}_{\mathrm{IV}}.\end{aligned} \tag{A.33}$$

First, by (A.32),

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} H_t(\Phi_t(p)) & = -\sum_{i,j=1}^n \left(\left.\frac{d}{dt}\right|_{t=0}\mathcal{G}(t)_{ij}\right)\S(0)_{ij} + \sum_{i=1}^n \left.\frac{d}{dt}\right|_{t=0}\S(t)_{ii} \\ & \overset{(A.13),(A.24)}{=} -\sum_{i,j=1}^n h_{ij}\left(\langle\nabla_{e_i}\mathbf{X}, e_j\rangle + \langle\nabla_{e_j}\mathbf{X}, e_i\rangle\right) + \sum_{i=1}^n\langle\nabla_{e_i}V', e_i\rangle - \operatorname{Ric}(\mathbf{X},\mathbf{N}) + \sum_{i=1}^n\langle\nabla_{e_i}\mathbf{N}, \nabla_{e_i}\mathbf{X}\rangle.\end{aligned}$$

Notice that, for $i, j = 1, \dots, n$,

$$\langle\nabla_{e_i}\mathbf{X}, e_j\rangle + \langle\nabla_{e_j}\mathbf{X}, e_i\rangle = 2\varphi h_{ij} + \left\langle\nabla_{e_i}\mathbf{X}^T, e_j\right\rangle + \left\langle\nabla_{e_j}\mathbf{X}^T, e_i\right\rangle = 2\varphi h_{ij} + 2\operatorname{sym}\left(\nabla^S\mathbf{X}^T\right)_{ij} \tag{A.34}$$

and

$$\left\langle\nabla_{e_i}V', e_j\right\rangle \overset{(A.19)}{=} -\left(\operatorname{Hess}^S\varphi\right)_{ij} + \left\langle\nabla_{e_i}A\left(\mathbf{X}^T\right), e_j\right\rangle. \tag{A.35}$$

Moreover,

$$\sum_{i=1}^n\langle\nabla_{e_i}\mathbf{N}, \nabla_{e_i}\mathbf{X}\rangle = \varphi\sum_{i=1}^n\langle\nabla_{e_i}\mathbf{N}, \nabla_{e_i}\mathbf{N}\rangle + \sum_{i=1}^n\left\langle\nabla_{e_i}\mathbf{N}, \nabla_{e_i}\mathbf{X}^T\right\rangle = \varphi|h|^2 + \left\langle h, \operatorname{sym}\left(\nabla^S\mathbf{X}^T\right)\right\rangle \tag{A.36}$$

Therefore, by (A.34), (A.35) and (A.36),

$$\left.\frac{d}{dt}\right|_{t=0} H_t(\Phi_t(p)) = -\Delta^S\varphi - \varphi\left(|h|^2 + \operatorname{Ric}(\mathbf{N},\mathbf{N})\right) - \left\langle h, \operatorname{sym}\left(\nabla^S\mathbf{X}^T\right)\right\rangle + \operatorname{div}^S A\left(\mathbf{X}^T\right) - \operatorname{Ric}(\mathbf{X}^T, N).$$

Next, extend $\mathbf{X}^T$ to a smooth vector field with compact support in $M$. Denote by $\Psi_t : M \to M$ its flow. Then $\Psi$ is a variation of $S$ with velocity $\mathbf{X}^T$. By the above formula,

$$\mathbf{X}^T H(p) = \left.\frac{d}{dt}\right|_{t=0} H_t(\Psi_t(p)) = -\left\langle h, \operatorname{sym}\left(\nabla^S\mathbf{X}^T\right)\right\rangle + \operatorname{div}^S A\left(\mathbf{X}^T\right) - \operatorname{Ric}(\mathbf{X}^T, \mathbf{N}).$$

Therefore, (A.29) follows. Next, we prove (A.30). First we compute I. Indeed,

$$\begin{aligned}\mathrm{I} &= 2\sum_{i,j,k=1}^{n}\left(\frac{d}{dt}\bigg|_{t=0}\mathcal{G}(t)_{ik}\right)\left(\frac{d}{dt}\bigg|_{t=0}\mathcal{G}(t)_{jk}\right)\S(0)_{ij}\\ &\overset{\text{(A.13)}}{=} 2\sum_{i,j,k=1}^{n} h_{ij}\left(\langle\nabla_{e_i}\mathbf{X},e_k\rangle+\langle\nabla_{e_k}\mathbf{X},e_i\rangle\right)\left(\langle\nabla_{e_j}\mathbf{X},e_k\rangle+\langle\nabla_{e_k}\mathbf{X},e_j\rangle\right).\end{aligned}$$

Next we compute II. Noticing that

$$\langle\nabla_{e_i}\mathbf{X},\mathbf{N}\rangle = e_i(\varphi)+\left\langle\nabla_{e_i}\mathbf{X}^T,\mathbf{N}\right\rangle = e_i(\varphi)-\left\langle\nabla_{e_i}\mathbf{N},\mathbf{X}^T\right\rangle = -\left\langle V',e_i\right\rangle \qquad i=1,\dots,n, \tag{A.37}$$

we deduce that

$$\begin{aligned}\mathrm{II} &= -\sum_{i,j=1}^{n}\left(\frac{d^2}{dt^2}\bigg|_{t=0}\mathcal{G}(t)_{ij}\right)\S(0)_{ij}\\ &= -\sum_{i,j=1}^{n} h_{ij}\,\frac{d^2}{dt^2}\bigg|_{t=0}\langle E_i(t),E_j(t)\rangle\\ &= -2\sum_{i,j=1}^{n} h_{ij}\,\frac{d}{dt}\bigg|_{t=0}\left\langle\frac{D}{dt}E_i(t),E_j(t)\right\rangle\\ &= -2\sum_{i,j=1}^{n} h_{ij}\left\langle\frac{D^2}{dt^2}\bigg|_{t=0}E_i(t),E_j(t)\right\rangle-2\sum_{i,j=1}^{n} h_{ij}\left\langle\frac{D}{dt}\bigg|_{t=0}E_i(t),\frac{D}{dt}\bigg|_{t=0}E_j(t)\right\rangle\\ &\overset{\text{(A.13),(A.14)}}{=} -2\sum_{i,j=1}^{n} h_{ij}\,\langle\nabla_{e_i}\mathbf{Z},e_j\rangle-2\sum_{i,j=1}^{n} h_{ij}\,\mathrm{R}(\mathbf{X},e_i,\mathbf{X},e_j)-2\sum_{i,j=1}^{n} h_{ij}\,\langle\nabla_{e_i}\mathbf{X},\nabla_{e_j}\mathbf{X}\rangle\\ &= -2\left\langle h,\nabla^S\mathbf{Z}\right\rangle-2\,\langle h,\mathcal{C}(\mathbf{X},\mathbf{X})\rangle-2\sum_{i,j=1}^{n} h_{ij}\,\langle\nabla_{e_i}\mathbf{X},e_k\rangle\langle\nabla_{e_j}\mathbf{X},e_k\rangle-2\sum_{i,j=1}^{n} h_{ij}\,\langle\nabla_{e_i}\mathbf{X},\mathbf{N}\rangle\langle\nabla_{e_j}\mathbf{X},\mathbf{N}\rangle\\ &\overset{\text{(A.37)}}{=} -2\left\langle h,\nabla^S\mathbf{Z}\right\rangle-2\,\langle h,\mathcal{C}(\mathbf{X},\mathbf{X})\rangle-2\sum_{i,j=1}^{n} h_{ij}\,\langle\nabla_{e_i}\mathbf{X},e_k\rangle\langle\nabla_{e_j}\mathbf{X},e_k\rangle-2h\left(V',V'\right).\end{aligned}$$

We compute III. To this aim,

$$\begin{aligned}\mathrm{III} &= -2\sum_{i,j=1}^{n}\left(\frac{d}{dt}\bigg|_{t=0}\mathcal{G}(t)_{ij}\right)\left(\frac{d}{dt}\bigg|_{t=0}\S(t)_{ij}\right)\\ &\overset{\text{(A.13),(A.24)}}{=} -2\sum_{i,j=1}^{n}\left(\langle\nabla_{e_i}\mathbf{X},e_j\rangle+\langle\nabla_{e_j}\mathbf{X},e_i\rangle\right)\left(\langle\nabla_{e_i}V'+\mathrm{R}(\mathbf{X},e_i)\mathbf{N},e_j\rangle+\langle\nabla_{e_i}\mathbf{N},\nabla_{e_j}\mathbf{X}\rangle\right)\\ &= -2\sum_{i,j=1}^{n}\left(\langle\nabla_{e_i}\mathbf{X},e_j\rangle+\langle\nabla_{e_j}\mathbf{X},e_i\rangle\right)\langle\nabla_{e_i}V',e_j\rangle-2\sum_{i,j,k=1}^{n} h_{ij}\left(\langle\nabla_{e_i}\mathbf{X},e_k\rangle+\langle\nabla_{e_k}\mathbf{X},e_i\rangle\right)\langle\nabla_{e_k}\mathbf{X},e_j\rangle\\ &\quad-2\left\langle\nabla^S\mathbf{X},\mathcal{C}(\mathbf{X},\mathbf{N})\right\rangle-2\left\langle\nabla^S\mathbf{X},\mathcal{C}(\mathbf{N},\mathbf{X})\right\rangle.\end{aligned}$$

Finally, we compute IV. Indeed, by (A.14), (A.24) and (A.25),

$$\begin{aligned}\mathrm{IV} &= \sum_{i=1}^{n}\frac{d^2}{dt^2}\bigg|_{t=0}\S(t)_{ii}\\ &= \underbrace{\mathrm{div}^S V''}_{\mathrm{IV.1}}+\underbrace{\sum_{i=1}^{n}\left\langle\mathrm{R}\left(\mathbf{X},\nabla_{e_i}\mathbf{X}\right)\mathbf{N}+2\,\mathrm{R}(\mathbf{X},e_i)V'+\mathrm{R}\left(\mathbf{Z},e_i\right)\mathbf{N}+\left(\nabla_{\mathbf{X}}\mathrm{R}\right)(\mathbf{X},e_i)\mathbf{N},e_i\right\rangle}_{\mathrm{IV.2}}\\ &\quad+\underbrace{2\sum_{i=1}^{n}\left\langle\nabla_{e_i}V'+\mathrm{R}(\mathbf{X},e_i)\mathbf{N},\nabla_{e_i}\mathbf{X}\right\rangle}_{\mathrm{IV.3}}+\underbrace{\sum_{i=1}^{n}\langle\nabla_{e_i}\mathbf{N},\nabla_{e_i}\mathbf{Z}+\mathrm{R}(\mathbf{X},e_i)\mathbf{X}\rangle}_{\mathrm{IV.4}}.\end{aligned}$$

First,

$$\mathrm{IV}.1 \overset{(A.20)}{=} \left(-\left|\nabla^S\varphi\right|^2 + 2h\left(\nabla^S\varphi, \mathbf{X}^T\right) - h^2\left(\mathbf{X}^T, \mathbf{X}^T\right)\right) H + \operatorname{div}^S\left(V''\right)^T.$$

Moreover,

$$\begin{aligned}\mathrm{IV}.2 = & \sum_{i,j=1}^n \langle \nabla_{e_i}\mathbf{X}, e_j\rangle \,\mathrm{R}(\mathbf{X}, e_j, \mathbf{N}, e_i) + \sum_{i=1}^n \langle \nabla_{e_i}\mathbf{X}, \mathbf{N}\rangle \,\mathrm{R}(\mathbf{X}, \mathbf{N}, \mathbf{N}, e_i) \\ & - 2\operatorname{Ric}(\mathbf{X}, V') - 2\,\mathrm{R}(\mathbf{X}, \mathbf{N}, V', \mathbf{N}) - \operatorname{Ric}(\mathbf{Z}, \mathbf{N}) - \left(\nabla_{\mathbf{X}}\operatorname{Ric}\right)(\mathbf{X}, \mathbf{N}) \\ \overset{(A.37)}{=} & \left\langle \nabla^S\mathbf{X}, \mathcal{C}(\mathbf{N}, \mathbf{X})\right\rangle - 2\operatorname{Ric}(\mathbf{X}, V') - \mathrm{R}(\mathbf{X}^T, \mathbf{N}, V', \mathbf{N}) - \operatorname{Ric}(\mathbf{Z}, \mathbf{N}) - \left(\nabla_{\mathbf{X}}\operatorname{Ric}\right)(\mathbf{X}, \mathbf{N}).\end{aligned}$$

In addition,

$$\begin{aligned}\mathrm{IV}.3 = & \, 2\sum_{i,j=1}^n \left\langle \nabla_{e_i}V', e_j\right\rangle \langle \nabla_{e_i}\mathbf{X}, e_j\rangle + 2\sum_{i=1}^n \left\langle \nabla_{e_i}V', \mathbf{N}\right\rangle \langle \nabla_{e_i}\mathbf{X}, \mathbf{N}\rangle + 2\sum_{i,j=1}^n \mathrm{R}(\mathbf{X}, e_i, \mathbf{N}, e_j) \langle \nabla_{e_i}\mathbf{X}, e_j\rangle \\ \overset{(A.37)}{=} & \, 2\sum_{i,j=1}^n \left\langle \nabla_{e_i}V', e_j\right\rangle \langle \nabla_{e_i}\mathbf{X}, e_j\rangle + 2h\left(V', V'\right) + 2\left\langle \nabla^S X, \mathcal{C}(\mathbf{X}, \mathbf{N})\right\rangle\end{aligned}$$

Finally,

$$\mathrm{IV}.4 = \sum_{i,j=1}^n h_{ij}\langle \nabla_{e_i}\mathbf{Z}, e_j\rangle + \sum_{i,j=1}^n h_{ij}\,\mathrm{R}(\mathbf{X}, e_i, \mathbf{X}, e_j) = \left\langle h, \nabla^S\mathbf{Z}\right\rangle + \langle h, \mathcal{C}(\mathbf{X}, \mathbf{X})\rangle.$$

Recalling (A.33) and combining the above computations,

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} H_t\left(\Phi_t(p)\right) = & \, 2\left\langle h, \left(\nabla^S X\right)^2\right\rangle - 2\left\langle \nabla^S X, \left(\nabla^S V'\right)^t\right\rangle - \langle h, \mathcal{C}(\mathbf{X}, \mathbf{X})\rangle - \left\langle \nabla^S\mathbf{X}, \mathcal{C}(\mathbf{N}, \mathbf{X})\right\rangle \\ & + \left(-\left|\nabla^S\varphi\right|^2 + 2h\left(\nabla^S\varphi, \mathbf{X}^T\right) - h^2\left(\mathbf{X}^T, \mathbf{X}^T\right)\right) H + \operatorname{div}^S\left(V''\right)^T \\ & - 2\operatorname{Ric}(\mathbf{X}, V') - \mathrm{R}\left(\mathbf{X}^T, \mathbf{N}, V', \mathbf{N}\right) - \left(\nabla_{\mathbf{X}}\operatorname{Ric}\right)(\mathbf{X}, \mathbf{N}) - \left\langle h, \nabla^S\mathbf{Z}\right\rangle - \operatorname{Ric}(\mathbf{Z}, \mathbf{N}).\end{aligned}$$

Next, fix $p \in S$ and a geodesic frame $\mathbf{E}_1, \ldots, \mathbf{E}_n$ at $p$. Recall that

$$\left(V''\right)^T = -\sum_{i=1}^n \langle \mathbf{N}, \nabla_{\mathbf{E}_i}\mathbf{Z}\rangle \,\mathbf{E}_i + \left(\tilde{V}''\right)^T.$$

Therefore

$$\begin{aligned}\operatorname{div}^S\left(V''\right)^T & = -\sum_{i=1}^n \mathbf{E}_i \langle \mathbf{N}, \nabla_{\mathbf{E}_i}\mathbf{Z}\rangle + \operatorname{div}^S\left(\tilde{V}''\right)^T \\ & = -\sum_{i=1}^n \mathbf{E}_i \langle \mathbf{N}, \nabla_{\mathbf{E}_i}\left(\langle \mathbf{Z}, \mathbf{N}\rangle \mathbf{N}\right)\rangle - \sum_{i=1}^n \mathbf{E}_i \left\langle \mathbf{N}, \nabla_{\mathbf{E}_i}\mathbf{Z}^T\right\rangle + \operatorname{div}^S\left(\tilde{V}''\right)^T \\ & = -\sum_{i=1}^n \mathbf{E}_i\left(\mathbf{E}_i\langle \mathbf{Z}, \mathbf{N}\rangle\right) + \sum_{i=1}^n \mathbf{E}_i\left\langle A\left(\mathbf{Z}^T\right), \mathbf{E}_i\right\rangle + \operatorname{div}^S\left(\tilde{V}''\right)^T \\ & = -\Delta^S\langle \mathbf{Z}, \mathbf{N}\rangle + \operatorname{div}^S A\left(\mathbf{Z}^T\right) + \operatorname{div}^S\left(\tilde{V}''\right)^T.\end{aligned}$$

Moreover,

$$-\left\langle h, \nabla^S\mathbf{Z}\right\rangle - \operatorname{Ric}(\mathbf{Z}, \mathbf{N}) \overset{(A.9)}{=} -\operatorname{div}^S A\left(\mathbf{Z}^T\right) + \mathbf{Z}^T H - \langle \mathbf{Z}, \mathbf{N}\rangle\left(|h|^2 + \operatorname{Ric}(\mathbf{N}, \mathbf{N})\right).$$

Therefore (A.30) follows. Finally, assume that $\Phi$ is normal. Then $\mathbf{X} = \varphi\mathbf{N}$, $\mathbf{X}^T = 0$ and $V' = -\nabla^S\varphi$, so that

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} H_t\left(\Phi_t(p)\right) = & \, 2\left\langle h, \left(\nabla^S(\varphi\mathbf{N})\right)^2\right\rangle + 2\left\langle \nabla^S(\varphi\mathbf{N}), \left(\nabla^S\left(\nabla^S\varphi\right)\right)^t\right\rangle - \varphi^2\langle h, \mathcal{C}(\mathbf{N}, \mathbf{N})\rangle \\ & - \varphi\left\langle \nabla^S(\varphi\mathbf{N}), \mathcal{C}(\mathbf{N}, \mathbf{N})\right\rangle - \left|\nabla^S\varphi\right|^2 H + \operatorname{div}^S\left(\tilde{V}''\right)^T + 2\varphi\operatorname{Ric}(\mathbf{N}, \nabla^S\varphi) - \varphi^2\left(\nabla_{\mathbf{N}}\operatorname{Ric}\right)(\mathbf{N}, \mathbf{N}) \\ & - \Delta^S\langle \mathbf{Z}, \mathbf{N}\rangle - \langle \mathbf{Z}, \mathbf{N}\rangle\left(|h|^2 + \operatorname{Ric}(\mathbf{N}, \mathbf{N})\right) + \mathbf{Z}^T H.\end{aligned}$$

Noticing that $\nabla^S(\varphi \mathbf{N}) = \varphi h$ and $\left(\nabla^S\left(\nabla^S\varphi\right)\right)^t = \operatorname{Hess}^S\varphi$, we deduce that

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0} H_t\left(\Phi_t(p)\right) &= 2\varphi^2 \operatorname{trace}\left(h^3\right) + 2\varphi\left\langle h, \operatorname{Hess}^S\varphi\right\rangle - 2\varphi^2\left\langle h, \mathcal{C}(\mathbf{N},\mathbf{N})\right\rangle - \left|\nabla^S\varphi\right|^2 H + \operatorname{div}^S\left(\tilde{V}''\right)^T \\ &\quad + 2\varphi \operatorname{Ric}\left(\mathbf{N}, \nabla^S\varphi\right) - \varphi^2\left(\nabla_{\mathbf{N}}\operatorname{Ric}\right)(\mathbf{N},\mathbf{N}) - \Delta^S\langle \mathbf{Z},\mathbf{N}\rangle - \langle \mathbf{Z},\mathbf{N}\rangle\left(|h|^2 + \operatorname{Ric}(\mathbf{N},\mathbf{N})\right) + \mathbf{Z}^T H.\end{aligned}$$

Moreover,

$$2\varphi\left\langle h, \operatorname{Hess}^S\varphi\right\rangle + 2\varphi\operatorname{Ric}\left(\mathbf{N},\nabla^S\varphi\right) \overset{(A.9)}{=} 2\varphi\operatorname{div}^S A\left(\nabla^S\varphi\right) - 2\varphi\left\langle\nabla^S H, \nabla^S\varphi\right\rangle.$$

Finally, since

$$\left(\tilde{V}''\right)^T = 2\sum_{i=1}^n\left\langle\nabla^S\varphi, \varphi A(e_i)\right\rangle e_i = \sum_{i=1}^n\left\langle\nabla^S\left(\varphi^2\right), A(e_i)\right\rangle e_i = \sum_{i=1}^n\left\langle A\left(\nabla^S\left(\varphi^2\right)\right), e_i\right\rangle e_i = A\left(\nabla^S\left(\varphi^2\right)\right),$$

(A.31) follows. ☐

A.5. **Riemannian variation formulas.** Let $S$ be a smooth, embedded, closed hypersurface. If $f:\mathbb{R}\to\mathbb{R}$ is smooth in a neighborhood of $\{H(p)\,:\,p\in S\}$, set

$$\mathscr{H}_f(S) = \int_S f(H)\,dS. \tag{A.38}$$

If $\Phi$ is a smooth variation, the area formula implies that

$$\mathscr{H}_f\left(\Phi_t(S)\right) = \int_S f\left(H_t\left(\Phi_t(p)\right)\right)\operatorname{Jac}\left(\Phi_t|_S\right)(p)\ dS. \tag{A.39}$$

Denote by $\mathscr{J}$ the *Jacobi operator* associated to $S$,

$$\mathscr{J}\varphi = -\Delta^S\varphi - \varphi\left(|h|^2 + \operatorname{Ric}(\mathbf{N},\mathbf{N})\right), \qquad \varphi\in C^\infty(S). \tag{A.40}$$

Recall that $\mathscr{J}$ is self-adjoint, namely

$$\int_S \varphi\mathscr{J}\psi\,dS = \int_S \psi\mathscr{J}\varphi\,dS, \qquad \varphi,\psi\in C^\infty(S). \tag{A.41}$$

The first and second variation formulas for (A.38) read as follows.

**Theorem A.8.** *Let $S\subseteq M$ be a smooth, closed, embedded hypersurface. Let $\Phi$ be a smooth variation. Denote by $\mathbf{X}$ and $\mathbf{Z}$ its velocity and acceleration respectively. On $S$, decompose $\mathbf{X}$ as $\mathbf{X}|_S = \varphi\mathbf{N} + \mathbf{X}^T$. Then*

$$\delta\mathscr{H}_f(S)[\varphi] := \delta\mathscr{H}_f(S)[\Phi] = \int_S \varphi\Big(-\Delta^S f'(H) - f'(H)\left(|h|^2 + \operatorname{Ric}(\mathbf{N},\mathbf{N})\right) + f(H)H\Big)\,dS, \tag{A.42}$$

$$\delta^2\mathscr{H}_f(S)[\Phi] = \delta\mathscr{H}_f(S)\left[\langle\mathbf{Z}|_S,\mathbf{N}\rangle - 2\mathbf{X}^T\varphi + h\left(\mathbf{X}^T,\mathbf{X}^T\right)\right] + \int_S \varphi\,\mathscr{L}\varphi\,dS, \tag{A.43}$$

*where*

$$\begin{aligned}\mathscr{L}\varphi = \mathscr{J}&\left(f''(H)\mathscr{J}\varphi\right) + 2f'(H)\operatorname{div}^S A\left(\nabla^S\varphi\right) - \left(f'(H)H + f(H)\right)\Delta^S\varphi \\ &- 2f''(H)\left\langle\nabla^S H, A\left(\nabla^S\varphi\right)\right\rangle + \left(f''(H)H - 2f'(H)\right)\left\langle\nabla^S H,\nabla^S\varphi\right\rangle \\ &+ \varphi f'(H)\left(2\operatorname{trace}\left(h^3\right) - 2\left\langle h,\mathcal{C}(\mathbf{N},\mathbf{N})\right\rangle - \left(\nabla_{\mathbf{N}}\operatorname{Ric}\right)(\mathbf{N},\mathbf{N})\right) \\ &+ \varphi\left(f(H)H^2 - \left(2f'(H)H + f(H)\right)\left(|h|^2 + \operatorname{Ric}(\mathbf{N},\mathbf{N})\right)\right).\end{aligned}$$

*Proof.* First,

$$\begin{aligned}\frac{d\mathscr{H}_f\left(\Phi_t(S)\right)}{dt} &\overset{(A.39)}{=} \int_S \frac{d}{dt}\Big(f\left(H_t\left(\Phi_t(p)\right)\right)\operatorname{Jac}\left(\Phi_t|_S\right)(p)\Big)\,dS \\ &= \int_S\left(f'\left(H_t\left(\Phi_t(p)\right)\right)\frac{dH_t\left(\Phi_t(p)\right)}{dt}\operatorname{Jac}\left(\Phi_t|_S\right)(p) + f\left(H_t\left(\Phi_t(p)\right)\right)\frac{d\operatorname{Jac}\left(\Phi_t|_S\right)(p)}{dt}\right)\,dS.\end{aligned}$$

In particular, by (A.15) and (A.29),

$$\delta\mathscr{H}_f[\mathbf{X}] = \int_S\Big(f'(H)\left(-\Delta^S\varphi - \varphi\left(|h|^2 + \operatorname{Ric}(\mathbf{N},\mathbf{N})\right) + \mathbf{X}^T H\right) + f(H)\left(\varphi H + \operatorname{div}^S\mathbf{X}^T\right)\Big)\,dS.$$

Notice that, being $S$ closed,

$$\int_S \left(f'(H)\mathbf{X}^T H + f(H)\operatorname{div}^S \mathbf{X}^T\right) dS = \int_S \operatorname{div}^S \left(f(H)\mathbf{X}^T\right)\, dS = 0. \tag{A.44}$$

The divergence theorem and (A.44) imply (A.42). We prove (A.43). We reduce to deal with the case of normal variations. Let $\mathbf{Y} \in \Gamma(TM)$ be such that $\mathbf{Y}|_S = \mathbf{X}^T$. Denote by $\Psi$ the flow of $-\mathbf{Y}$. Since $\mathbf{Y}|_S \in \Gamma(TS)$, then $\Psi(t,S) = S$ for any small $t$. Set $\tilde\Phi(t,p) := \Phi(t,\Psi(t,p))$. Then $\tilde\Phi$ is a smooth variation, and moreover

$$\tilde S_t := \tilde\Phi(t,S) = \Phi(t,\Psi(t,S)) = \Phi(t,S) = S_t \tag{A.45}$$

for any $t$ sufficiently small. The area formula and (A.45) imply that

$$\delta^2\mathscr{H}_f(S)[\Phi] = \delta^2\mathscr{H}_f(S)[\tilde\Phi]. \tag{A.46}$$

We compute the velocity $\tilde X$ and the acceleration $\tilde Z$ of $\tilde\Phi$. Fix $p \in M$. Fix local normal coordinates $x_1,\dots,x_{n+1}$ centered at $p$. Set $q(t) = \Phi_t^{-1}(p)$ and $r(t) = \Psi_t^{-1}(q(t))$. Then, since $\Psi$ is the flow of $-\mathbf{Y}$,

$$\begin{aligned}\tilde{\mathscr{X}}(t,p) &= \frac{\partial}{\partial s}\bigg|_{s=0} \Phi(t+s,\Psi(t+s,r(t)))\\ &= \sum_{i=1}^{n+1} \frac{\partial\Phi^i}{\partial t}(t,q(t))\frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n+1} \frac{\partial\Phi^i}{\partial x_j}(t,q(t))\frac{\partial\Psi^j}{\partial t}(t,r(t))\frac{\partial}{\partial x_i}\\ &= \mathscr{X}(t,p) - \sum_{i,j=1}^{n+1} \frac{\partial\Phi^i}{\partial x_j}(t,q(t))\mathbf{Y}(q(t))\frac{\partial}{\partial x_i}.\end{aligned} \tag{A.47}$$

In particular, $\tilde X = \mathbf{X} - \mathbf{Y}$, and $\tilde{\mathbf{X}}|_S = \varphi\mathbf{N}$, whence $\tilde\Phi$ is a normal variation. Moreover, as $\Phi\left(t,q(t)\right) = p$,

$$0 = \frac{\partial\Phi}{\partial t}(0,p) + \sum_{i,j=1}^{n+1} \frac{\partial\Phi_i}{\partial z_j}(0,p)\dot q^j(0)\frac{\partial}{\partial x_i} = \frac{\partial\Phi}{\partial t}(0,p) + \dot q(0),$$

whence $\dot q(0) = -\mathbf{X}(p)$. Therefore

$$\begin{aligned}\tilde{\mathbf{X}}'(p) &\overset{(A.47)}{=} \mathbf{X}'(p) - \sum_{i,j,k=1}^{n+1}\left(\delta_{ik}\frac{\partial^2\Phi^i}{\partial t\partial x_j}(0,p)\mathbf{Y}^j(p) + \frac{\partial^2\Phi^i}{\partial x_k\partial x_j}(0,p)\dot q^k(0)\mathbf{Y}^j(p) + \frac{\partial\Phi^i}{\partial x_j}(0,p)\frac{\partial\mathbf{Y}^j}{\partial x_k}(p)\dot q^k(0)\right)\frac{\partial}{\partial x_i}\\ &= \mathbf{X}'(p) - \sum_{i,j=1}^{n+1}\frac{\partial\mathbf{X}^i}{\partial x_j}(p)\mathbf{Y}^j(p)\frac{\partial}{\partial x_i} + \sum_{i,k=1}^{n+1}\frac{\partial\mathbf{Y}^i}{\partial x_k}(p)\mathbf{X}^k(p)\frac{\partial}{\partial x_i}\\ &= \mathbf{X}'(p) - \left(\nabla_{\mathbf{Y}}\mathbf{X}\right)(p) + \left(\nabla_{\mathbf{X}}\mathbf{Y}\right)(p).\end{aligned}$$

Finally,

$$\tilde{\mathbf{Z}} = \tilde{\mathbf{X}}' + \nabla_{\tilde{\mathbf{X}}}\tilde{\mathbf{X}} = \mathbf{X}' - \nabla_{\mathbf{Y}}\mathbf{X} + \nabla_{\mathbf{X}}\mathbf{Y} + \nabla_{\mathbf{X}-\mathbf{Y}}\left(\mathbf{X}-\mathbf{Y}\right) = \mathbf{Z} - 2\nabla_{\mathbf{Y}}\mathbf{X} + \nabla_{\mathbf{Y}}\mathbf{Y},$$

so that, recalling that $\mathbf{Y}|_S = \mathbf{X}^T$ ,

$$\left\langle \tilde{\mathbf{Z}}|_S, \mathbf{N}\right\rangle = \langle \mathbf{Z}|_S, \mathbf{N}\rangle - 2\mathbf{X}^T\varphi + h\left(\mathbf{X}^T,\mathbf{X}^T\right). \tag{A.48}$$

Then, by (A.15), (A.17), (A.29), (A.31) and (A.46),

$$\begin{aligned}\delta^2\mathscr{H}_f(S)[\Phi] = &\int_S f''(H)\left(\Delta^S\varphi+\varphi\left(|h|^2+\operatorname{Ric}(\mathbf{N},\mathbf{N})\right)\right)^2\,dS\\&+\int_S f'(H)\Big(\operatorname{div}^S\left(A\left(\nabla^S\varphi^2\right)\right)+2\varphi\operatorname{div}^S A\left(\nabla^S\varphi\right)-2\varphi\left\langle\nabla^S H,\nabla^S\varphi\right\rangle-\left|\nabla^S\varphi\right|^2 H\Big)\,dS\\&+\int_S f'(H)\Big(2\varphi^2\operatorname{trace}\left(h^3\right)-2\varphi^2\left\langle h,\mathcal{C}(\mathbf{N},\mathbf{N})\right\rangle-\varphi^2\left(\nabla_{\mathbf{N}}\operatorname{Ric}\right)(\mathbf{N},\mathbf{N})\Big)\,dS\\&+\int_S f'(H)\Big(-\Delta^S\left\langle\tilde{\mathbf{Z}},\mathbf{N}\right\rangle-\left\langle\tilde{\mathbf{Z}},\mathbf{N}\right\rangle\left(|h|^2+\operatorname{Ric}(\mathbf{N},\mathbf{N})\right)+\tilde{\mathbf{Z}}^T H\Big)\,dS\\&+\int_S 2\varphi H f'(H)\Big(-\Delta^S\varphi-\varphi\left(|h|^2+\operatorname{Ric}(\mathbf{N},\mathbf{N})\right)\Big)\\&+\int_S f(H)\Big(\varphi^2H^2-\varphi^2|h|^2-\varphi^2\operatorname{Ric}(\mathbf{N},\mathbf{N})+\left|\nabla^S\varphi\right|^2+\operatorname{div}^S\tilde{\mathbf{Z}}\Big)\,dS.\end{aligned}$$

First, combining (A.42) and (A.44), and recalling (A.40),

$$\begin{aligned}\delta^2\mathscr{H}_f(S)[\Phi] = &\underbrace{\int_S \left(f''(H)\mathcal{J}\varphi\right)\mathcal{J}\varphi\,dS}_{\mathrm{I}}\\&+\underbrace{\int_S f'(H)\Big(\operatorname{div}^S\left(A\left(\nabla^S\varphi^2\right)\right)-\left|\nabla^S\varphi\right|^2 H\Big)\,dS}_{\mathrm{II}}+\int_S f'(H)\Big(2\varphi\operatorname{div}^S A\left(\nabla^S\varphi\right)-2\varphi\left\langle\nabla^S H,\nabla^S\varphi\right\rangle\Big)\,dS\\&+\int_S f'(H)\Big(2\varphi^2\operatorname{trace}\left(h^3\right)-2\varphi^2\left\langle h,\mathcal{C}(\mathbf{N},\mathbf{N})\right\rangle-\varphi^2\left(\nabla_{\mathbf{N}}\operatorname{Ric}\right)(\mathbf{N},\mathbf{N})\Big)\,dS\\&+\int_S 2\varphi H f'(H)\Big(-\Delta^S\varphi-\varphi\left(|h|^2+\operatorname{Ric}(\mathbf{N},\mathbf{N})\right)\Big)\\&+\underbrace{\int_S f(H)\left|\nabla^S\varphi\right|^2\,dS}_{\mathrm{III}}+\int_S f(H)\Big(\varphi^2H^2-\varphi^2|h|^2-\varphi^2\operatorname{Ric}(\mathbf{N},\mathbf{N})\Big)\,dS+\delta\mathscr{H}_f(S)\left[\left\langle\tilde{\mathbf{Z}}|_S,\mathbf{N}\right\rangle\right].\end{aligned}$$

First,

$$\mathrm{I}\overset{(A.41)}{=}\int_S\varphi\,\mathcal{J}\left(f''(H)\mathcal{J}\varphi\right)\,dS.$$

Moreover,

$$\begin{aligned}\mathrm{II}&=\int_S\Big(-\left\langle\nabla^S f'(H),A\left(\nabla^S\varphi^2\right)\right\rangle-\left\langle f'(H)H\nabla^S\varphi,\nabla^S\varphi\right\rangle\Big)\,dS\\&=\int_S\varphi\Big(-2f''(H)\left\langle\nabla^S H,A\left(\nabla^S\varphi\right)\right\rangle+\operatorname{div}^S\left(f'(H)H\nabla^S\varphi\right)\Big)\,dS\\&=\int_S\varphi\Big(-2f''(H)\left\langle\nabla^S H,A\left(\nabla^S\varphi\right)\right\rangle+\left(f'(H)+f''(H)H\right)\left\langle\nabla^S H,\nabla^S\varphi\right\rangle+f'(H)H\Delta^S\varphi\Big)\,dS.\end{aligned}$$

Finally,

$$\mathrm{III}=-\int_S\varphi\operatorname{div}^S\left(f(H)\nabla^S\varphi\right)\,dS=-\int_S\varphi\Big(f'(H)\left\langle\nabla^S H,\nabla^S\varphi\right\rangle+f(H)\Delta^S\varphi\Big)\,dS.$$

In particular,

$$\mathrm{II}+\mathrm{III}=\int_S\varphi\Big(-2f''(H)\left\langle\nabla^S H,A\left(\nabla^S\varphi\right)\right\rangle+f''(H)H\left\langle\nabla^S H,\nabla^S\varphi\right\rangle+\left(f'(H)H-f(H)\right)\Delta^S\varphi\Big)\,dS.$$

The thesis follows combining the above computations with (A.48). □

## APPENDIX B. VARIATION FORMULAS FOR SUB-RIEMANNIAN MEAN CURVATURE FUNCTIONALS

In this second appendix we prove Theorem 4.2 by means of a Riemannian approximation scheme. Namely, we approximate the sub-Riemannian structure of $\mathbb{H}^n$ with a sequence of Riemannian manifolds $(\mathbb{H}^n,\langle\cdot,\cdot\rangle_\varepsilon)_{\varepsilon>0}$, and we exploit Theorem A.8 to deduce the analogous sub-Riemannian variation formulas.

B.1. **Preliminaries.** For any $\varepsilon > 0$, denote by $\langle\cdot,\cdot\rangle_\varepsilon$ the unique Riemannian metric on $\mathbb{H}^n$ such that $(Z_1^\varepsilon,\dots,Z_{2n+1}^\varepsilon) = (X_1,\dots,X_n,Y_1,\dots,Y_n,\varepsilon T)$ is a global orthonormal frame. If $\mathbf{A} \in \Gamma(T\mathbb{H}^n)$, denote by $A^{1,\varepsilon},\dots,A^{2n,\varepsilon},A^{2n+1,\varepsilon}$ its components with respect to the above frame. Clearly $A^j := A^{j,\varepsilon}$ is independent of $\varepsilon$ for $j = 1,\dots,2n$. We may write $A^{2n+1} := A^{2n+1,\varepsilon}$ when there is no ambiguity. Since $\langle\mathbf{A},\mathbf{B}\rangle_\varepsilon$ does not depend on $\varepsilon$ when $\mathbf{A},\mathbf{B} \in \Gamma(\mathcal{H})$, we set $\langle\mathbf{A},\mathbf{B}\rangle := \langle\mathbf{A},\mathbf{B}\rangle_\varepsilon$. Denote by $\nabla^\varepsilon$ and $\mathrm{R}^\varepsilon$ the induced Levi-Civita connection and curvature tensor respectively. We recall the following relations (cf. [37]):

$$\begin{aligned}
&\nabla^\varepsilon_{X_i}X_j = 0, && \nabla^\varepsilon_{Y_i}Y_j = 0, && \nabla^\varepsilon_T T = 0,\\
&\nabla^\varepsilon_{X_i}Y_j = -\delta_{i,j}T, && \nabla^\varepsilon_{X_i}\varepsilon T = \frac{Y_i}{\varepsilon}, && \nabla^\varepsilon_{Y_i}\varepsilon T = -\frac{X_i}{\varepsilon},\\
&\nabla^\varepsilon_{Y_i}X_j = \delta_{i,j}T, && \nabla^\varepsilon_{\varepsilon T}X_i = \frac{Y_i}{\varepsilon}, && \nabla^\varepsilon_{\varepsilon T}Y_i = -\frac{X_i}{\varepsilon}.
\end{aligned} \tag{B.1}$$

The Levi-Civita connection $\nabla^\varepsilon$ relates to the pseudohermitian connection $\nabla$ (cf. (2.2)) as follows.

**Lemma B.1.** *Let $\mathbf{A},\mathbf{B} \in \Gamma(T\mathbb{H}^n)$. Then*

$$\nabla^\varepsilon_{\mathbf{A}}\mathbf{B} = \nabla_{\mathbf{A}}\mathbf{B} + \frac{\langle\mathbf{A},\varepsilon T\rangle_\varepsilon}{\varepsilon}J(\mathbf{B}) + \frac{\langle\mathbf{B},\varepsilon T\rangle_\varepsilon}{\varepsilon}J(\mathbf{A}) + \langle\mathbf{A},J(\mathbf{B})\rangle T. \tag{B.2}$$

*In particular,*

$$\nabla^\varepsilon_{\mathbf{A}}\varepsilon T = \frac{1}{\varepsilon}J(\mathbf{A}). \tag{B.3}$$

*Moreover, if $\mathbf{A},\mathbf{B},\mathbf{C} \in \Gamma(\mathcal{H})$, then*

$$\langle\nabla^\varepsilon_{\mathbf{A}}\mathbf{B},\mathbf{C}\rangle_\varepsilon = \langle\nabla_{\mathbf{A}}\mathbf{B},\mathbf{C}\rangle. \tag{B.4}$$

*Proof.* By a direct computation,

$$\begin{aligned}
\nabla^\varepsilon_{\mathbf{A}}\mathbf{B} &= \sum_{j,k=1}^{2n+1} A^{j,\varepsilon}Z_j^\varepsilon B^{k,\varepsilon}Z_k^\varepsilon + \sum_{j,k=1}^{2n+1} A^{j,\varepsilon}B^{k,\varepsilon}\nabla^\varepsilon_{Z_j^\varepsilon}Z_k^\varepsilon\\
&\overset{(B.1)}{=} \nabla_{\mathbf{A}}\mathbf{B} + \sum_{j=1}^n A^jB^{n+j}\nabla^\varepsilon_{X_j}Y_j + \langle\mathbf{B},\varepsilon T\rangle_\varepsilon\sum_{j=1}^n A^j\nabla^\varepsilon_{X_j}\varepsilon T + \sum_{j=1}^n A^{n+j}B^j\nabla^\varepsilon_{Y_j}X_j\\
&\quad + \langle\mathbf{B},\varepsilon T\rangle_\varepsilon\sum_{j=1}^n A^{n+j}\nabla^\varepsilon_{Y_j}\varepsilon T + \langle\mathbf{A},\varepsilon T\rangle_\varepsilon\sum_{j=1}^n B^j\nabla^\varepsilon_{\varepsilon T}X_j + \langle\mathbf{A},\varepsilon T\rangle_\varepsilon\sum_{j=1}^n B^{n+j}\nabla^\varepsilon_{\varepsilon T}Y_j\\
&\overset{(B.1)}{=} \nabla_{\mathbf{A}}\mathbf{B} + \frac{\langle\mathbf{A},\varepsilon T\rangle_\varepsilon}{\varepsilon}J(\mathbf{B}) + \frac{\langle\mathbf{B},\varepsilon T\rangle_\varepsilon}{\varepsilon}J(\mathbf{A}) + \langle\mathbf{A},J(\mathbf{B})\rangle T.
\end{aligned}$$

□

B.2. **Ambient curvatures.** First, we compute the ambient curvatures of $(\mathbb{H}^n,\langle\cdot,\cdot\rangle_\varepsilon)$, starting from $\mathrm{R}^\varepsilon$.

**Proposition B.2.** *Let $\varepsilon > 0$. Let $\mathbf{A},\mathbf{B},\mathbf{C} \in \Gamma(T\mathbb{H}^n)$. Then*

$$\begin{aligned}
\mathrm{R}^\varepsilon(\mathbf{A},\mathbf{B})\mathbf{C} &= \frac{2}{\varepsilon^2}\langle J(\mathbf{A}),\mathbf{B}\rangle_\varepsilon J(\mathbf{C}) + \frac{1}{\varepsilon^2}\langle J(\mathbf{A}),\mathbf{C}\rangle_\varepsilon J(\mathbf{B}) - \frac{1}{\varepsilon^2}\langle J(\mathbf{B}),\mathbf{C}\rangle_\varepsilon J(\mathbf{A})\\
&\quad + \frac{A^{2n+1,\varepsilon}}{\varepsilon^2}\langle\mathbf{B},\mathbf{C}\rangle_\varepsilon\varepsilon T - \frac{B^{2n+1,\varepsilon}}{\varepsilon^2}\langle\mathbf{A},\mathbf{C}\rangle_\varepsilon\varepsilon T + \frac{B^{2n+1,\varepsilon}C^{2n+1,\varepsilon}}{\varepsilon^2}\mathbf{A} - \frac{A^{2n+1,\varepsilon}C^{2n+1,\varepsilon}}{\varepsilon^2}\mathbf{B}.
\end{aligned}$$

*In particular, if $\mathbf{D} \in \Gamma(T\mathbb{H}^n)$,*

$$\begin{aligned}
\mathrm{R}^\varepsilon(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) &= \frac{2}{\varepsilon^2}\langle J(\mathbf{A}),\mathbf{B}\rangle_\varepsilon\langle J(\mathbf{C}),\mathbf{D}\rangle_\varepsilon + \frac{1}{\varepsilon^2}\langle J(\mathbf{A}),\mathbf{C}\rangle_\varepsilon\langle J(\mathbf{B}),\mathbf{D}\rangle_\varepsilon - \frac{1}{\varepsilon^2}\langle J(\mathbf{B}),\mathbf{C}\rangle_\varepsilon\langle J(\mathbf{A}),\mathbf{D}\rangle_\varepsilon\\
&\quad + \frac{A^{2n+1,\varepsilon}D^{2n+1,\varepsilon}}{\varepsilon^2}\langle\mathbf{B},\mathbf{C}\rangle_\varepsilon - \frac{B^{2n+1,\varepsilon}D^{2n+1,\varepsilon}}{\varepsilon^2}\langle\mathbf{A},\mathbf{C}\rangle_\varepsilon + \frac{B^{2n+1,\varepsilon}C^{2n+1,\varepsilon}}{\varepsilon^2}\langle\mathbf{A},\mathbf{D}\rangle_\varepsilon - \frac{A^{2n+1,\varepsilon}C^{2n+1,\varepsilon}}{\varepsilon^2}\langle\mathbf{B},\mathbf{D}\rangle_\varepsilon.
\end{aligned} \tag{B.5}$$

*Proof.* Fix $i, j, k = 1, \dots, n$. Then, by (B.1),

$$\begin{aligned}
&\mathrm{R}^{\varepsilon}(X_i, X_j)X_k = 0,\\
&\mathrm{R}^{\varepsilon}(X_i, X_j)Y_k = \nabla^{\varepsilon}_{X_i}\nabla^{\varepsilon}_{X_j}Y_k - \nabla^{\varepsilon}_{X_j}\nabla^{\varepsilon}_{X_i}Y_k - \nabla^{\varepsilon}_{[X_i,X_j]}Y_k = -\frac{\delta_{jk}}{\varepsilon}\nabla^{\varepsilon}_{X_i}\varepsilon T + \frac{\delta_{ik}}{\varepsilon}\nabla^{\varepsilon}_{X_j}\varepsilon T = \frac{1}{\varepsilon^2}\left(\delta_{ik}Y_j - \delta_{jk}Y_i\right),\\
&\mathrm{R}^{\varepsilon}(X_i, X_j)\varepsilon T = \nabla^{\varepsilon}_{X_i}\nabla^{\varepsilon}_{X_j}\varepsilon T - \nabla^{\varepsilon}_{X_j}\nabla^{\varepsilon}_{X_i}\varepsilon T - \nabla^{\varepsilon}_{[X_i,X_j]}\varepsilon T = \frac{1}{\varepsilon}\nabla^{\varepsilon}_{X_i}Y_j - \frac{1}{\varepsilon}\nabla^{\varepsilon}_{X_j}Y_i = 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathrm{R}^{\varepsilon}(X_i, Y_j)X_k = \nabla^{\varepsilon}_{X_i}\nabla^{\varepsilon}_{Y_j}X_k - \nabla^{\varepsilon}_{Y_j}\nabla^{\varepsilon}_{X_i}X_k - \nabla^{\varepsilon}_{[X_i,Y_j]}X_k = \frac{\delta_{jk}}{\varepsilon}\nabla^{\varepsilon}_{X_i}\varepsilon T + \frac{2\delta_{ij}}{\varepsilon}\nabla^{\varepsilon}_{\varepsilon T}X_k = \frac{1}{\varepsilon^2}\left(\delta_{jk}Y_i + 2\delta_{ij}Y_k\right),\\
&\mathrm{R}^{\varepsilon}(X_i, Y_j)Y_k = \nabla^{\varepsilon}_{X_i}\nabla^{\varepsilon}_{Y_j}Y_k - \nabla^{\varepsilon}_{Y_j}\nabla^{\varepsilon}_{X_i}Y_k - \nabla^{\varepsilon}_{[X_i,Y_j]}Y_k = \frac{\delta_{ik}}{\varepsilon}\nabla^{\varepsilon}_{Y_j}\varepsilon T + \frac{2\delta_{ij}}{\varepsilon}\nabla^{\varepsilon}_{\varepsilon T}Y_k = -\frac{1}{\varepsilon^2}\left(\delta_{ik}X_j + 2\delta_{ij}X_k\right),\\
&\mathrm{R}^{\varepsilon}(X_i, Y_j)\varepsilon T = \nabla^{\varepsilon}_{X_i}\nabla^{\varepsilon}_{Y_j}\varepsilon T - \nabla^{\varepsilon}_{Y_j}\nabla^{\varepsilon}_{X_i}\varepsilon T - \nabla^{\varepsilon}_{[X_i,Y_j]}\varepsilon T = 0.
\end{aligned}$$

In addition,

$$\begin{aligned}
\mathrm{R}^{\varepsilon}(X_i, \varepsilon T)X_k &= \nabla^{\varepsilon}_{X_i}\nabla^{\varepsilon}_{\varepsilon T}X_k - \nabla^{\varepsilon}_{\varepsilon T}\nabla^{\varepsilon}_{X_i}X_k - \nabla^{\varepsilon}_{[X_i,\varepsilon T]}X_k = \frac{1}{\varepsilon}\nabla^{\varepsilon}_{X_i}Y_k = -\frac{\delta_{ik}}{\varepsilon^2}\varepsilon T,\\
\mathrm{R}^{\varepsilon}(X_i, \varepsilon T)Y_k &= \nabla^{\varepsilon}_{X_i}\nabla^{\varepsilon}_{\varepsilon T}Y_k - \nabla^{\varepsilon}_{\varepsilon T}\nabla^{\varepsilon}_{X_i}Y_k - \nabla^{\varepsilon}_{[X_i,\varepsilon T]}Y_k = 0,\\
\mathrm{R}^{\varepsilon}(X_i, \varepsilon T)\varepsilon T &= \nabla^{\varepsilon}_{X_i}\nabla^{\varepsilon}_{\varepsilon T}\varepsilon T - \nabla^{\varepsilon}_{\varepsilon T}\nabla^{\varepsilon}_{X_i}\varepsilon T - \nabla^{\varepsilon}_{[X_i,\varepsilon T]}\varepsilon T = -\frac{1}{\varepsilon}\nabla^{\varepsilon}_{\varepsilon T}Y_i = \frac{1}{\varepsilon^2}X_i.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\mathrm{R}^{\varepsilon}(Y_i, Y_j)X_k = \nabla^{\varepsilon}_{Y_i}\nabla^{\varepsilon}_{Y_j}X_k - \nabla^{\varepsilon}_{Y_j}\nabla^{\varepsilon}_{Y_i}X_k - \nabla^{\varepsilon}_{[Y_i,Y_j]}X_k = \frac{\delta_{jk}}{\varepsilon}\nabla^{\varepsilon}_{Y_i}\varepsilon T - \frac{\delta_{ik}}{\varepsilon}\nabla^{\varepsilon}_{Y_j}\varepsilon T = \frac{1}{\varepsilon^2}\left(\delta_{ik}X_j - \delta_{jk}X_i\right),\\
&\mathrm{R}^{\varepsilon}(Y_i, Y_j)Y_k = 0,\\
&\mathrm{R}^{\varepsilon}(Y_i, Y_j)\varepsilon T = \nabla^{\varepsilon}_{Y_i}\nabla^{\varepsilon}_{Y_j}\varepsilon T - \nabla^{\varepsilon}_{Y_j}\nabla^{\varepsilon}_{Y_i}\varepsilon T - \nabla^{\varepsilon}_{[Y_i,Y_j]}\varepsilon T = -\frac{1}{\varepsilon}\nabla^{\varepsilon}_{Y_i}X_j + \frac{1}{\varepsilon}\nabla^{\varepsilon}_{Y_j}X_i = 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{R}^{\varepsilon}(Y_i, \varepsilon T)X_k &= \nabla^{\varepsilon}_{Y_i}\nabla^{\varepsilon}_{\varepsilon T}X_k - \nabla^{\varepsilon}_{\varepsilon T}\nabla^{\varepsilon}_{Y_i}X_k - \nabla^{\varepsilon}_{[Y_i,\varepsilon T]}X_k = 0,\\
\mathrm{R}^{\varepsilon}(Y_i, \varepsilon T)Y_k &= \nabla^{\varepsilon}_{Y_i}\nabla^{\varepsilon}_{\varepsilon T}Y_k - \nabla^{\varepsilon}_{\varepsilon T}\nabla^{\varepsilon}_{Y_i}Y_k - \nabla^{\varepsilon}_{[Y_i,\varepsilon T]}Y_k = -\frac{1}{\varepsilon}\nabla^{\varepsilon}_{Y_i}X_k = -\frac{\delta_{ik}}{\varepsilon^2}\varepsilon T,\\
\mathrm{R}^{\varepsilon}(Y_i, \varepsilon T)\varepsilon T &= \nabla^{\varepsilon}_{Y_i}\nabla^{\varepsilon}_{\varepsilon T}\varepsilon T - \nabla^{\varepsilon}_{\varepsilon T}\nabla^{\varepsilon}_{Y_i}\varepsilon T - \nabla^{\varepsilon}_{[Y_i,\varepsilon T]}\varepsilon T = \frac{1}{\varepsilon}\nabla^{\varepsilon}_{\varepsilon T}X_i = \frac{1}{\varepsilon^2}Y_i.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{R}^{\varepsilon}(\mathbf{A}, \mathbf{B})\mathbf{C} &= \sum_{i,j,k=1}^{2n+1} A^i B^j C^k\, \mathrm{R}^{\varepsilon}(Z_i, Z_j)Z_k\\
&= \sum_{i=1}^{n}\sum_{j,k=1}^{2n+1} A^i B^j C^k\, \mathrm{R}^{\varepsilon}(X_i, Z_j)Z_k + \sum_{i=1}^{n}\sum_{j,k=1}^{2n+1} A^{n+i} B^j C^k\, \mathrm{R}^{\varepsilon}(Y_i, Z_j)Z_k + A^{2n+1}\sum_{j,k=1}^{2n+1} B^j C^k\, \mathrm{R}^{\varepsilon}(\varepsilon T, Z_j)Z_k\\
&= \sum_{i,j=1}^{n}\sum_{k=1}^{2n+1} A^i B^j C^k\, \mathrm{R}^{\varepsilon}(X_i, X_j)Z_k + \sum_{i,j=1}^{n}\sum_{k=1}^{2n+1} A^i B^{n+j} C^k\, \mathrm{R}^{\varepsilon}(X_i, Y_j)Z_k + B^{2n+1}\sum_{i=1}^{n}\sum_{k=1}^{2n+1} A^i C^k\, \mathrm{R}^{\varepsilon}(X_i, \varepsilon T)Z_k\\
&\quad + \sum_{i,j=1}^{n}\sum_{k=1}^{2n+1} A^{n+i} B^j C^k\, \mathrm{R}^{\varepsilon}(Y_i, X_j)Z_k + \sum_{i,j=1}^{n}\sum_{k=1}^{2n+1} A^{n+i} B^{n+j} C^k\, \mathrm{R}^{\varepsilon}(Y_i, Y_j)Z_k\\
&\quad + B^{2n+1}\sum_{i=1}^{n}\sum_{k=1}^{2n+1} A^{n+i} C^k\, \mathrm{R}^{\varepsilon}(Y_i, \varepsilon T)Z_k + A^{2n+1}\sum_{j=1}^{n}\sum_{k=1}^{2n+1} B^j C^k\, \mathrm{R}^{\varepsilon}(\varepsilon T, X_j)Z_k\\
&\quad + A^{2n+1}\sum_{j=1}^{n}\sum_{k=1}^{2n+1} B^{n+j} C^k\, \mathrm{R}^{\varepsilon}(\varepsilon T, Y_j)Z_k.
\end{aligned}$$

By the symmetry of $\mathrm{R}^\varepsilon$,

$$\begin{aligned}\mathrm{R}^\varepsilon(\mathbf{A},\mathbf{B})\mathbf{C} = {} & \underbrace{\sum_{i,j=1}^{n}\sum_{k=1}^{2n+1} A^iB^jC^k\,\mathrm{R}^\varepsilon(X_i,X_j)Z_k}_{\mathrm{I}} + \underbrace{\sum_{i,j=1}^{n}\sum_{k=1}^{2n+1}\left(A^iB^{n+j}-A^{n+j}B^i\right)C^k\,\mathrm{R}^\varepsilon(X_i,Y_j)Z_k}_{\mathrm{II}} \\ & + \underbrace{\sum_{i=1}^{n}\sum_{k=1}^{2n+1}\left(A^iB^{2n+1}-A^{2n+1}B^i\right)C^k\,\mathrm{R}^\varepsilon(X_i,\varepsilon T)Z_k}_{\mathrm{III}} + \underbrace{\sum_{i,j=1}^{n}\sum_{k=1}^{2n+1} A^{n+i}B^{n+j}C^k\,\mathrm{R}^\varepsilon(Y_i,Y_j)Z_k}_{\mathrm{IV}} \\ & + \underbrace{\sum_{i=1}^{n}\sum_{k=1}^{2n+1}\left(A^{n+i}B^{2n+1}-A^{2n+1}B^{n+i}\right)C^k\,\mathrm{R}^\varepsilon(Y_i,\varepsilon T)Z_k}_{\mathrm{V}}\,.\end{aligned}$$

First,

$$\mathrm{I} = \frac{1}{\varepsilon^2}\sum_{i,j=1}^{n} A^iB^jC^{n+i}Y_j - \frac{1}{\varepsilon^2}\sum_{i,j=1}^{n} A^iB^jC^{n+j}Y_i = \frac{1}{\varepsilon^2}\sum_{i,j=1}^{n} A^jB^iC^{n+j}Y_i - \frac{1}{\varepsilon^2}\sum_{i,j=1}^{n} A^iB^jC^{n+j}Y_i.$$

Moreover,

$$\begin{aligned}\mathrm{II} &= \sum_{i,j,k=1}^{n}\left(A^iB^{n+j}-A^{n+j}B^i\right)C^k\,\mathrm{R}^\varepsilon(X_i,Y_j)X_k + \sum_{i,j,k=1}^{n}\left(A^iB^{n+j}-A^{n+j}B^i\right)C^{n+k}\,\mathrm{R}^\varepsilon(X_i,Y_j)Y_k \\ &= \frac{1}{\varepsilon^2}\sum_{i,j=1}^{n}\left(A^iB^{n+j}-A^{n+j}B^i\right)C^jY_i + \frac{2}{\varepsilon^2}\sum_{i,k=1}^{n}\left(A^iB^{n+i}-A^{n+i}B^i\right)C^kY_k \\ &\quad - \frac{1}{\varepsilon^2}\sum_{i,j=1}^{n}\left(A^iB^{n+j}-A^{n+j}B^i\right)C^{n+i}X_j - \frac{2}{\varepsilon^2}\sum_{i,k=1}^{n}\left(A^iB^{n+i}-A^{n+i}B^i\right)C^{n+k}X_i \\ &= \frac{1}{\varepsilon^2}\sum_{i,j=1}^{n}\left(A^iB^{n+j}-A^{n+j}B^i\right)C^jY_i - \frac{1}{\varepsilon^2}\sum_{i,j=1}^{n}\left(A^jB^{n+i}-A^{n+i}B^j\right)C^{n+j}X_i + \frac{2}{\varepsilon^2}\left\langle J(\mathbf{A}),\mathbf{B}\right\rangle J(\mathbf{C}).\end{aligned}$$

In addition,

$$\begin{aligned}\mathrm{III} &= \sum_{i,k=1}^{n}\left(A^iB^{2n+1}-A^{2n+1}B^i\right)C^k\,\mathrm{R}^\varepsilon(X_i,\varepsilon T)X_k + C^{2n+1}\sum_{i=1}^{n}\left(A^iB^{2n+1}-A^{2n+1}B^i\right)\mathrm{R}^\varepsilon(X_i,\varepsilon T)\varepsilon T \\ &= -\frac{1}{\varepsilon^2}\sum_{i=1}^{n}\left(A^iB^{2n+1}-A^{2n+1}B^i\right)C^i\varepsilon T + \frac{C^{2n+1}}{\varepsilon^2}\sum_{i=1}^{n}\left(A^iB^{2n+1}-A^{2n+1}B^i\right)X_i \\ &= -\frac{B^{2n+1}}{\varepsilon^2}\sum_{j=1}^{n}A^jC^j\varepsilon T + \frac{A^{2n+1}}{\varepsilon^2}\sum_{j=1}^{n}B^jC^j\varepsilon T + \frac{B^{2n+1}C^{2n+1}}{\varepsilon^2}\sum_{i=1}^{n}A^iX_i - \frac{A^{2n+1}C^{2n+1}}{\varepsilon^2}\sum_{i=1}^{n}B^iX_i.\end{aligned}$$

Furthermore,

$$\mathrm{IV} = \frac{1}{\varepsilon^2}\sum_{i,j=1}^{n} A^{n+i}B^{n+j}C^iX_j - \frac{1}{\varepsilon^2}\sum_{i,j=1}^{n} A^{n+i}B^{n+j}C^jX_i = \frac{1}{\varepsilon^2}\sum_{i,j=1}^{n} A^{n+j}B^{n+i}C^jX_i - \frac{1}{\varepsilon^2}\sum_{i,j=1}^{n} A^{n+i}B^{n+j}C^jX_i,$$

Finally,

$$\begin{aligned}
\mathrm{V} &= \sum_{i,k=1}^{n} \left(A^{n+i}B^{2n+1} - A^{2n+1}B^{n+i}\right) C^{n+k}\, \mathrm{R}^{\varepsilon}(Y_i, \varepsilon T)Y_k \\
&\quad + C^{2n+1}\sum_{i=1}^{n} \left(A^{n+i}B^{2n+1} - A^{2n+1}B^{n+i}\right) \mathrm{R}^{\varepsilon}(Y_i, \varepsilon T)\varepsilon T \\
&= -\frac{1}{\varepsilon^2}\sum_{i=1}^{n} \left(A^{n+i}B^{2n+1} - A^{2n+1}B^{n+i}\right) C^{n+i}\varepsilon T + \frac{C^{2n+1}}{\varepsilon^2}\sum_{i=1}^{n} \left(A^{n+i}B^{2n+1} - A^{2n+1}B^{n+i}\right) Y_i \\
&= -\frac{B^{2n+1}}{\varepsilon^2}\sum_{i=1}^{n} A^{n+i}C^{n+i}\varepsilon T + \frac{A^{2n+1}}{\varepsilon^2}\sum_{i=1}^{n} B^{n+i}C^{n+i}\varepsilon T \\
&\quad + \frac{B^{2n+1}C^{2n+1}}{\varepsilon^2}\sum_{i=1}^{n} A^{n+i}Y_i - \frac{A^{2n+1}C^{2n+1}}{\varepsilon^2}\sum_{i=1}^{n} B^{n+i}Y_i.
\end{aligned}$$

By the above computations,

$$\begin{aligned}
\mathrm{I} + \mathrm{II} + \mathrm{IV} &= \frac{2}{\varepsilon^2}\left\langle J(\mathbf{A}), \mathbf{B}\right\rangle J(\mathbf{C}) + \frac{1}{\varepsilon^2}\left\langle J(\mathbf{A}), \mathbf{C}\right\rangle \sum_{i=1}^{n} B^i Y_i - \frac{1}{\varepsilon^2}\left\langle J(\mathbf{A}), \mathbf{C}\right\rangle \sum_{i=1}^{n} B^{n+i} X_i \\
&\quad + \frac{1}{\varepsilon^2}\left\langle J(\mathbf{B}), \mathbf{C}\right\rangle \sum_{i=1}^{n} A^{n+i} X_i - \frac{1}{\varepsilon^2}\left\langle J(\mathbf{B}), \mathbf{C}\right\rangle \sum_{i=1}^{n} A^i Y_i \\
&= \frac{2}{\varepsilon^2}\left\langle J(\mathbf{A}), \mathbf{B}\right\rangle J(\mathbf{C}) + \frac{1}{\varepsilon^2}\left\langle J(\mathbf{A}), \mathbf{C}\right\rangle J(\mathbf{B}) - \frac{1}{\varepsilon^2}\left\langle J(\mathbf{B}), \mathbf{C}\right\rangle J(\mathbf{A})
\end{aligned}$$

and

$$\mathrm{III} + \mathrm{V} = \frac{A^{2n+1}}{\varepsilon^2}\langle \mathbf{B}, \mathbf{C}\rangle \varepsilon T - \frac{B^{2n+1}}{\varepsilon^2}\langle \mathbf{A}, \mathbf{C}\rangle \varepsilon T + \frac{B^{2n+1}C^{2n+1}}{\varepsilon^2}\mathbf{A} - \frac{A^{2n+1}C^{2n+1}}{\varepsilon^2}\mathbf{B}.$$

The thesis follows combining the above computations. ☐

The explicit expression of the Ricci curvature $\mathrm{Ric}^{\varepsilon}$ is a simple consequence of Proposition B.2.

**Proposition B.3.** *Let $\varepsilon > 0$. Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \Gamma(T\mathbb{H}^n)$. Then*

$$\mathrm{Ric}^{\varepsilon}(\mathbf{B}, \mathbf{C}) = -\frac{2}{\varepsilon^2}\left\langle \mathbf{B}, \mathbf{C}\right\rangle_{\varepsilon} + \frac{(2n+2)}{\varepsilon^2}\left\langle \mathbf{B}, \varepsilon T\right\rangle_{\varepsilon}\left\langle \mathbf{C}, \varepsilon T\right\rangle_{\varepsilon} \tag{B.6}$$

*and*

$$\left(\nabla^{\varepsilon}_{\mathbf{A}} \mathrm{Ric}^{\varepsilon}\right)(\mathbf{B}, \mathbf{C}) = \frac{2n+2}{\varepsilon^3}\left\langle J(\mathbf{A}), \mathbf{B}\right\rangle_{\varepsilon}\left\langle \mathbf{C}, \varepsilon T\right\rangle_{\varepsilon} + \frac{2n+2}{\varepsilon^3}\left\langle J(\mathbf{A}), \mathbf{C}\right\rangle_{\varepsilon}\left\langle \mathbf{B}, \varepsilon T\right\rangle_{\varepsilon}$$

*In particular,*

$$\left(\nabla^{\varepsilon}_{\mathbf{A}} \mathrm{Ric}^{\varepsilon}\right)(\mathbf{A}, \mathbf{A}) = 0. \tag{B.7}$$

*Proof.* By definition,

$$\begin{aligned}
\varepsilon^2\,\mathrm{Ric}(\mathbf{B}, \mathbf{C}) &= \sum_{i=1}^{2n}\Big(2\left\langle J(Z_i), \mathbf{B}\right\rangle_{\varepsilon}\left\langle J(\mathbf{C}), Z_i\right\rangle_{\varepsilon} + \left\langle J(Z_i), \mathbf{C}\right\rangle_{\varepsilon}\left\langle J(\mathbf{B}), Z_i\right\rangle_{\varepsilon} - \left\langle J(\mathbf{B}), \mathbf{C}\right\rangle_{\varepsilon}\left\langle J(Z_i), Z_i\right\rangle_{\varepsilon}\Big) \\
&\quad + \left\langle \mathbf{B}, \mathbf{C}\right\rangle_{\varepsilon} - B^{2n+1,\varepsilon}C^{2n+1,\varepsilon} + (2n+1)B^{2n+1,\varepsilon}C^{2n+1,\varepsilon} - B^{2n+1,\varepsilon}C^{2n+1,\varepsilon}. \\
&= -3\left\langle J(\mathbf{B}), J(\mathbf{C})\right\rangle + \left\langle \mathbf{B}, \mathbf{C}\right\rangle_{\varepsilon} + (2n-1)B^{2n+1,\varepsilon}C^{2n+1,\varepsilon} \\
&= -2\left\langle \mathbf{B}, \mathbf{C}\right\rangle + (2n+2)B^{2n+1,\varepsilon}C^{2n+1,\varepsilon}.
\end{aligned}$$

Therefore, since $\nabla^{\varepsilon}\left\langle \cdot, \cdot\right\rangle_{\varepsilon} \equiv 0$,

$$\begin{aligned}
\varepsilon^2\left(\nabla_{\mathbf{A}} \mathrm{Ric}^{\varepsilon}\right)(\mathbf{B}, \mathbf{C}) &= (2n+2)\big(\mathbf{A}\left(\left\langle \mathbf{B}, \varepsilon T\right\rangle_{\varepsilon}\left\langle \mathbf{C}, \varepsilon T\right\rangle_{\varepsilon}\right) - \left\langle \nabla^{\varepsilon}_{\mathbf{A}}\mathbf{B}, \varepsilon T\right\rangle_{\varepsilon}\left\langle \mathbf{C}, \varepsilon T\right\rangle_{\varepsilon} - \left\langle \mathbf{B}, \varepsilon T\right\rangle_{\varepsilon}\left\langle \nabla^{\varepsilon}_{\mathbf{A}}\mathbf{C}, \varepsilon T\right\rangle_{\varepsilon}\big) \\
&= (2n+2)\big(\left\langle \mathbf{B}, \nabla^{\varepsilon}_{\mathbf{A}}\varepsilon T\right\rangle_{\varepsilon}\left\langle \mathbf{C}, \varepsilon T\right\rangle_{\varepsilon} + \left\langle \mathbf{B}, \varepsilon T\right\rangle_{\varepsilon}\left\langle \mathbf{C}, \nabla^{\varepsilon}_{\mathbf{A}}\varepsilon T\right\rangle_{\varepsilon}\big) \\
&\overset{(B.3)}{=} \frac{2n+2}{\varepsilon}\left\langle J(\mathbf{A}), \mathbf{B}\right\rangle_{\varepsilon}\left\langle \mathbf{C}, \varepsilon T\right\rangle_{\varepsilon} + \frac{2n+2}{\varepsilon}\left\langle J(\mathbf{A}), \mathbf{C}\right\rangle_{\varepsilon}\left\langle \mathbf{B}, \varepsilon T\right\rangle_{\varepsilon}.
\end{aligned}$$

Finally, (B.7) is straightforward. ☐

B.3. **Extrinsic curvatures.** Fix $\varepsilon > 0$. We adapt some aspects of Section 2.2 to the Riemannian structure $(\mathbb{H}^n, \langle\cdot,\cdot\rangle_\varepsilon)$. We fix a smooth, closed, embedded hypersurface $S \subseteq \mathbb{H}^n$, and we assign an upper index $\varepsilon$ to the geometric quantities associated with $S$ and induced by $\langle\cdot,\cdot\rangle_\varepsilon$. Accordingly,

$$\mathbf{N}^\varepsilon = \frac{1}{\sqrt{1+\varepsilon^2\alpha^2}}\nu + \frac{\varepsilon\alpha}{\sqrt{1+\varepsilon^2\alpha^2}}\varepsilon T. \tag{B.8}$$

Therefore, $TS$ admits the $\langle\cdot,\cdot\rangle_\varepsilon$-orthonormal decomposition $TS = \mathcal{H}'TS \oplus \operatorname{span} J(\nu) \oplus \operatorname{span} \mathcal{S}^\varepsilon$, where

$$\mathcal{S}^\varepsilon = -\frac{\varepsilon\alpha}{\sqrt{1+\varepsilon^2\alpha^2}}\nu + \frac{1}{\sqrt{1+\varepsilon^2\alpha^2}}\varepsilon T. \tag{B.9}$$

If $\sigma^\varepsilon$ is the Riemannian surface measure induced by $\langle\cdot,\cdot\rangle_\varepsilon$, it is easy to check that

$$\sigma^\varepsilon = \frac{\sqrt{1+\varepsilon^2\alpha^2}}{\varepsilon\sqrt{1+\alpha^2}}\sigma^1. \tag{B.10}$$

In the rest of this section, we express some Riemannian extrinsic quantities with respect to the relevant sub-Riemannian geometric objects. We begin by the second fundamental form.

**Lemma B.4.** *Let* $\varepsilon > 0$. *Let* $\mathbf{A}, \mathbf{B} \in \Gamma(\mathcal{H}TS)$. *Let* $\mathbf{C} \in \Gamma(\mathcal{H}'TS)$. *Then*

$$\begin{aligned} h^\varepsilon(\mathbf{A},\mathbf{B}) &= \frac{1}{\sqrt{1+\varepsilon^2\alpha^2}}\tilde{h}^{\mathcal{H}}(\mathbf{A},\mathbf{B}), & h^\varepsilon(\mathbf{C},\mathcal{S}^\varepsilon) &= \frac{\varepsilon\mathbf{C}\alpha}{1+\varepsilon^2\alpha^2},\\ h^\varepsilon(J(\nu),\mathcal{S}^\varepsilon) &= \frac{1}{\varepsilon} + \frac{\varepsilon J(\nu)\alpha}{1+\varepsilon^2\alpha^2}, & h^\varepsilon(\mathcal{S}^\varepsilon,\mathcal{S}^\varepsilon) &= \frac{\varepsilon^2\mathcal{S}\alpha}{(1+\varepsilon^2\alpha^2)^{\frac{3}{2}}}. \end{aligned} \tag{B.11}$$

*Proof.* First, since

$$\langle \nu, \mathbf{B}\rangle = \langle \varepsilon T, \mathbf{B}\rangle_\varepsilon = 0, \tag{B.12}$$

then

$$\begin{aligned} h^\varepsilon(\mathbf{A},\mathbf{B}) &\overset{(B.8)}{=} \left\langle \nabla^\varepsilon_{\mathbf{A}}\left(\frac{1}{\sqrt{1+\varepsilon^2\alpha^2}}\nu\right), \mathbf{B}\right\rangle_\varepsilon + \left\langle \nabla^\varepsilon_{\mathbf{A}}\left(\frac{\varepsilon\alpha}{\sqrt{1+\varepsilon^2\alpha^2}}\varepsilon T\right), \mathbf{B}\right\rangle_\varepsilon\\ &\overset{(B.12)}{=} \frac{1}{\sqrt{1+\varepsilon^2\alpha^2}}\Big( \langle \nabla^\varepsilon_{\mathbf{A}}\nu, \mathbf{B}\rangle_\varepsilon + \varepsilon\alpha \langle \nabla^\varepsilon_{\mathbf{A}}\varepsilon T, \mathbf{B}\rangle_\varepsilon \Big)\\ &\overset{(B.3),(B.4)}{=} \frac{1}{\sqrt{1+\varepsilon^2\alpha^2}}\Big( h^{\mathcal{H}}(\mathbf{A},\mathbf{B}) + \alpha\langle J(\mathbf{A}),\mathbf{B}\rangle \Big)\\ &\overset{(2.7)}{=} \frac{1}{\sqrt{1+\varepsilon^2\alpha^2}}\tilde{h}^{\mathcal{H}}(\mathbf{A},\mathbf{B}). \end{aligned}$$

Let $\mathbf{D} \in \Gamma(TS)$. Then

$$\begin{aligned} &h^\varepsilon(\mathbf{D},\mathcal{S}^\varepsilon) = \langle \nabla^\varepsilon_{\mathbf{D}}\mathbf{N}^\varepsilon, \mathcal{S}^\varepsilon\rangle_\varepsilon\\ &\overset{(B.8)}{=} \frac{1}{\sqrt{1+\varepsilon^2\alpha^2}}\left\langle \nabla^\varepsilon_{\mathbf{D}}\left(\frac{1}{\sqrt{1+\varepsilon^2\alpha^2}}\nu\right), -\varepsilon\alpha\nu + \varepsilon T\right\rangle_\varepsilon + \frac{1}{\sqrt{1+\varepsilon^2\alpha^2}}\left\langle \nabla^\varepsilon_{\mathbf{D}}\left(\frac{\varepsilon\alpha}{\sqrt{1+\varepsilon^2\alpha^2}}\varepsilon T\right), -\varepsilon\alpha\nu + \varepsilon T\right\rangle_\varepsilon\\ &= -\frac{\varepsilon\alpha}{\sqrt{1+\varepsilon^2\alpha^2}}\mathbf{D}\left(\frac{1}{\sqrt{1+\varepsilon^2\alpha^2}}\right) + \frac{1}{1+\varepsilon^2\alpha^2}\left\langle \nabla^\varepsilon_{\mathbf{D}}\nu, -\varepsilon\alpha\nu + \varepsilon T\right\rangle_\varepsilon\\ &+ \frac{\varepsilon\alpha}{\sqrt{1+\varepsilon^2\alpha^2}}\mathbf{D}\left(\frac{1}{\sqrt{1+\varepsilon^2\alpha^2}}\right) + \frac{\varepsilon\mathbf{D}\alpha}{1+\varepsilon^2\alpha^2} + \frac{\varepsilon\alpha}{1+\varepsilon^2\alpha^2}\left\langle \nabla^\varepsilon_{\mathbf{D}}\varepsilon T, -\varepsilon\alpha\nu + \varepsilon T\right\rangle_\varepsilon\\ &= \frac{1}{1+\varepsilon^2\alpha^2}\left\langle \nabla^\varepsilon_{\mathbf{D}}\nu, \varepsilon T\right\rangle_\varepsilon - \frac{\varepsilon^2\alpha^2}{1+\varepsilon^2\alpha^2}\left\langle \nabla^\varepsilon_{\mathbf{D}}\varepsilon T, \nu\right\rangle_\varepsilon + \frac{\varepsilon\mathbf{D}\alpha}{1+\varepsilon^2\alpha^2}\\ &\overset{(B.3)}{=} -\frac{1}{\varepsilon(1+\varepsilon^2\alpha^2)}\langle J(\mathbf{D}),\nu\rangle_\varepsilon - \frac{\varepsilon^2\alpha^2}{\varepsilon(1+\varepsilon^2\alpha^2)}\langle J(\mathbf{D}),\nu\rangle_\varepsilon + \frac{\varepsilon\mathbf{D}\alpha}{1+\varepsilon^2\alpha^2}\\ &= \frac{1}{\varepsilon}\langle \mathbf{D}, J(\nu)\rangle + \frac{\varepsilon\mathbf{D}\alpha}{1+\varepsilon^2\alpha^2}. \end{aligned}$$

□

A first trivial consequence of Lemma B.4 is the behavior of the mean curvature.

**Corollary B.5.** *Let $p \in S \setminus S_0$. Then*

$$H^\varepsilon = \frac{1}{\sqrt{1+\varepsilon^2\alpha^2}} H^{\mathcal{H}} + \frac{\varepsilon^2 \mathcal{S}\alpha}{(1+\varepsilon^2\alpha^2)^{\frac{3}{2}}}. \tag{B.13}$$

*In particular,*

$$\mathbf{A}_1 \cdots \mathbf{A}_k H^\varepsilon \xrightarrow[\varepsilon \to 0]{} \mathbf{A}_1 \cdots \mathbf{A}_k H^{\mathcal{H}} \text{ locally uniformly on } S \setminus S_0 \text{ for any } k \in \mathbb{N},\ \mathbf{A}_1, \dots, \mathbf{A}_k \in \Gamma(TS). \tag{B.14}$$

In the next result, the approximation occurs both in the second fundamental form and in its entries.

**Lemma B.6.** *Let $p \in S \setminus S_0$. Let $f, g \in C^\infty(S)$. Then*

$$\begin{aligned} h^\varepsilon\left(\nabla^{\varepsilon,S} f, \nabla^{\varepsilon,S} g\right) = & \frac{\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S} f, \nabla^{\mathcal{H},S} g\right)}{\sqrt{1+\varepsilon^2\alpha^2}} + \frac{J(\nu) f \mathcal{S} g + J(\nu) g \mathcal{S} f}{\sqrt{1+\varepsilon^2\alpha^2}} \\ & + \frac{\varepsilon^2 \left\langle \nabla^{\mathcal{H},S}\alpha, \mathcal{S} g \nabla^{\mathcal{H},S} f + \mathcal{S} f \nabla^{\mathcal{H},S} g\right\rangle}{(1+\varepsilon^2\alpha^2)^{\frac{3}{2}}} + \frac{\varepsilon^4 \mathcal{S}\alpha \mathcal{S} f \mathcal{S} g}{(1+\varepsilon^2\alpha^2)^{\frac{5}{2}}}. \end{aligned} \tag{B.15}$$

*In particular, if $(f^\varepsilon)_\varepsilon$, $(g^\varepsilon)_\varepsilon$ converge smoothly to $f$ and $g$ as in (B.14), then*

$$h^\varepsilon\left(\nabla^{\varepsilon,S} f^\varepsilon, \nabla^{\varepsilon,S} g^\varepsilon\right) \xrightarrow[\varepsilon \to 0]{} \tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S} f, \nabla^{\mathcal{H},S} g\right) + J(\nu) f\, \mathcal{S} g + J(\nu) g\, \mathcal{S} f \text{ locally uniformly on } S \setminus S_0. \tag{B.16}$$

*Proof.* Fix $p \in S \setminus S_0$. Let $\mathbf{E}_1, \dots, \mathbf{E}_{n-1}, \mathbf{E}_{n+1}, \dots, \mathbf{E}_{2n-1}$ be a local orthonormal frame of $\mathcal{H}'TS$. Set $\mathbf{E}_n = J(\nu)$ and $\mathbf{E}_{2n} = \mathcal{S}^\varepsilon$. In this way, $\mathbf{E}_1, \dots, \mathbf{E}_{2n}$ is a local orthonormal frame of $TS$. Then

$$\begin{aligned} h^\varepsilon\left(\nabla^{\varepsilon,S} f, \nabla^{\varepsilon,S} g\right) &= \sum_{i,j=1}^{2n} h^\varepsilon(\mathbf{E}_i, \mathbf{E}_j) \mathbf{E}_i f \mathbf{E}_i g \\ &= \sum_{i,j=1}^{2n-1} h^\varepsilon(\mathbf{E}_i, \mathbf{E}_j) \mathbf{E}_i f \mathbf{E}_i g + \sum_{i=1}^{2n-1} h^\varepsilon(\mathbf{E}_i, \mathcal{S}^\varepsilon)\left(\mathbf{E}_i f \mathcal{S}^\varepsilon g + \mathbf{E}_i g \mathcal{S}^\varepsilon f\right) + h^\varepsilon(\mathcal{S}^\varepsilon, \mathcal{S}^\varepsilon) \mathcal{S}^\varepsilon f \mathcal{S}^\varepsilon g \\ &\overset{(B.11)}{=} \frac{\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S} f, \nabla^{\mathcal{H},S} g\right)}{\sqrt{1+\varepsilon^2\alpha^2}} + \frac{\varepsilon^2}{(1+\varepsilon^2\alpha^2)^{\frac{3}{2}}} \sum_{\substack{i=1 \\ i \neq n}}^{2n-1} \mathbf{E}_i \alpha \left(\mathbf{E}_i f \mathcal{S} g + \mathbf{E}_i g \mathcal{S} f\right) \\ &\quad + \frac{1}{\sqrt{1+\varepsilon^2\alpha^2}} \left(1 + \frac{\varepsilon^2 J(\nu)\alpha}{1+\varepsilon^2\alpha^2}\right) \left(J(\nu) f \mathcal{S} g + J(\nu) g \mathcal{S} f\right) + \frac{\varepsilon^4 \mathcal{S}\alpha \mathcal{S} f \mathcal{S} g}{(1+\varepsilon^2\alpha^2)^{\frac{5}{2}}} \\ &= \frac{\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S} f, \nabla^{\mathcal{H},S} g\right)}{\sqrt{1+\varepsilon^2\alpha^2}} + \frac{J(\nu) f \mathcal{S} g + J(\nu) g \mathcal{S} f}{\sqrt{1+\varepsilon^2\alpha^2}} + \frac{\varepsilon^2 \left\langle \nabla^{\mathcal{H},S}\alpha, \mathcal{S} g \nabla^{\mathcal{H},S} f + \mathcal{S} f \nabla^{\mathcal{H},S} g\right\rangle}{(1+\varepsilon^2\alpha^2)^{\frac{3}{2}}} + \frac{\varepsilon^4 \mathcal{S}\alpha \mathcal{S} f \mathcal{S} g}{(1+\varepsilon^2\alpha^2)^{\frac{5}{2}}}. \end{aligned}$$

In particular, (B.16) follows by (B.15). ☐

The next convergence result has been achieved, through a different approach, in [37].

**Lemma B.7.** *Let $p \in S \setminus S_0$. Then*

$$|h^\varepsilon|^2 + \operatorname{Ric}^\varepsilon(\mathbf{N}^\varepsilon, \mathbf{N}^\varepsilon) = \frac{|\tilde{h}^{\mathcal{H}}|^2 + 4J(\nu)\alpha + (2n+2)\alpha^2}{1+\varepsilon^2\alpha^2} + \frac{2\varepsilon^2 \left|\nabla^{\mathcal{H},S}\alpha\right|^2}{(1+\varepsilon^2\alpha^2)^2} + \frac{\varepsilon^4 (\mathcal{S}\alpha)^2}{(1+\varepsilon^2\alpha^2)^3}$$

*In particular,*

$$|h^\varepsilon|^2 + \operatorname{Ric}^\varepsilon(\mathbf{N}^\varepsilon, \mathbf{N}^\varepsilon) \xrightarrow[\varepsilon \to 0]{} |\tilde{h}^{\mathcal{H}}|^2 + 4J(\nu)\alpha + (2n+2)\alpha^2 \qquad \text{locally uniformly on } S \setminus S_0. \tag{B.17}$$

*Proof.* Let $\mathbf{E}_1, \dots, \mathbf{E}_{2n-1}$ be as in the proof of Lemma B.6. First, by (B.6) and (B.8),

$$\operatorname{Ric}^\varepsilon(\mathbf{N}^\varepsilon, \mathbf{N}^\varepsilon) = -\frac{2}{\varepsilon^2} + \frac{(2n+2)\alpha^2}{1+\varepsilon^2\alpha^2}.$$

Moreover,

$$
\begin{aligned}
|h^\varepsilon|^2 &= \sum_{i,j=1}^{2n-1} h^\varepsilon(\mathbf{E}_i,\mathbf{E}_j)^2 + 2h^\varepsilon(J(\nu),\mathcal{S}^\varepsilon)^2 + 2\sum_{\substack{i=1\\ i\neq n}}^{2n-1} h^\varepsilon(\mathbf{E}_i,\mathcal{S}^\varepsilon)^2 + h^\varepsilon(\mathcal{S}^\varepsilon,\mathcal{S}^\varepsilon)^2\\
&\overset{(B.11)}{=} \frac{|\tilde h^{\mathcal{H}}|^2}{1+\varepsilon^2\alpha^2} + \frac{2}{\varepsilon^2} + \frac{4J(\nu)\alpha}{1+\varepsilon^2\alpha^2} + \frac{2\varepsilon^2\left(J(\nu)\alpha\right)^2}{(1+\varepsilon^2\alpha^2)^2} + \frac{2\varepsilon^2}{(1+\varepsilon^2\alpha^2)^2}\sum_{\substack{i=1\\ i\neq n}}^{2n-1}(\mathbf{E}_i\alpha)^2 + \frac{\varepsilon^4\left(\mathcal{S}\alpha\right)^2}{(1+\varepsilon^2\alpha^2)^3}\\
&= \frac{2}{\varepsilon^2} + \frac{|\tilde h^{\mathcal{H}}|^2 + 4J(\nu)\alpha}{1+\varepsilon^2\alpha^2} + \frac{2\varepsilon^2\left|\nabla^{\mathcal{H},S}\alpha\right|^2}{(1+\varepsilon^2\alpha^2)^2} + \frac{\varepsilon^4\left(\mathcal{S}\alpha\right)^2}{(1+\varepsilon^2\alpha^2)^3}.
\end{aligned}
$$

The thesis follows combining the above computations. ☐

Lemma B.4 allows to compare the Laplace-Beltrami operator with $\hat\Delta^{\mathcal{H},S}$. We need the following lemma.

**Lemma B.8.** *Let $p\in S\setminus S_0$. Then*

$$
\nabla^\varepsilon_{\mathcal{S}^\varepsilon}\mathcal{S}^\varepsilon = -2\alpha J(\nu) - \frac{\varepsilon^2\alpha}{1+\varepsilon^2\alpha^2}\nabla^{\mathcal{H},S}\alpha - \frac{\varepsilon^2\mathcal{S}\alpha}{(1+\varepsilon^2\alpha^2)^{\frac32}}\mathbf{N}^\varepsilon. \tag{B.18}
$$

*Proof.* First,

$$
\langle\nabla^\varepsilon_{\mathcal{S}^\varepsilon}\mathcal{S}^\varepsilon,\mathbf{N}^\varepsilon\rangle_\varepsilon = -h^\varepsilon(\mathcal{S}^\varepsilon,\mathcal{S}^\varepsilon) \overset{(B.11)}{=} -\frac{\varepsilon^2\mathcal{S}\alpha}{(1+\varepsilon^2\alpha^2)^{\frac32}}.
$$

Moreover, $\langle\nabla^\varepsilon_{\mathcal{S}^\varepsilon}\mathcal{S}^\varepsilon,\mathcal{S}^\varepsilon\rangle_\varepsilon = 0$. Finally, fix $\mathbf{E}\in\Gamma(\mathcal{H}TS)$. Then

$$
\begin{aligned}
&\left\langle\nabla^\varepsilon_{\mathcal{S}^\varepsilon}\mathcal{S}^\varepsilon,\mathbf{E}\right\rangle_\varepsilon = \frac{1}{\sqrt{1+\varepsilon^2\alpha^2}}\left\langle\nabla^\varepsilon_{-\varepsilon\alpha\nu+\varepsilon T}\left(\frac{-\varepsilon\alpha\nu+\varepsilon T}{\sqrt{1+\varepsilon^2\alpha^2}}\right),\mathbf{E}\right\rangle_\varepsilon\\
&= \frac{1}{1+\varepsilon^2\alpha^2}\left\langle\nabla^\varepsilon_{-\varepsilon\alpha\nu+\varepsilon T}\left(-\varepsilon\alpha\nu\right),\mathbf{E}\right\rangle_\varepsilon + \frac{1}{1+\varepsilon^2\alpha^2}\left\langle\nabla^\varepsilon_{-\varepsilon\alpha\nu+\varepsilon T}\varepsilon T,\mathbf{E}\right\rangle_\varepsilon\\
&\overset{(B.3)}{=} \frac{\varepsilon^2\alpha^2}{1+\varepsilon^2\alpha^2}\langle\nabla^\varepsilon_\nu\nu,\mathbf{E}\rangle_\varepsilon - \frac{\varepsilon\alpha}{1+\varepsilon^2\alpha^2}\langle\nabla^\varepsilon_{\varepsilon T}\nu,\mathbf{E}\rangle_\varepsilon - \frac{\alpha}{1+\varepsilon^2\alpha^2}\langle\mathbf{E},J(\nu)\rangle\\
&\overset{(B.1),(2.6)}{=} -\frac{2\varepsilon^2\alpha^3}{1+\varepsilon^2\alpha^2}\langle\mathbf{E},J(\nu)\rangle - \frac{\varepsilon^2\alpha}{1+\varepsilon^2\alpha^2}\sum_{i=1}^{2n}T(\nu^i)E^i - \frac{\alpha}{1+\varepsilon^2\alpha^2}\sum_{i=1}^{n}\left(\nu^iY_i-\nu^{n+i}X_i\right) - \frac{\alpha}{1+\varepsilon^2\alpha^2}\langle\mathbf{E},J(\nu)\rangle\\
&\overset{(2.5)}{=} -2\alpha\langle\mathbf{E},J(\nu)\rangle - \frac{\varepsilon^2\alpha\mathbf{E}\alpha}{1+\varepsilon^2\alpha^2}.
\end{aligned}
$$

The thesis follows by the above computations. ☐

**Proposition B.9.** *Let $\mathbf{A}\in\Gamma(\mathcal{H}TS)$. Let $f\in C^\infty(S)$. Let $p\in S\setminus S_0$. Then*

$$
\operatorname{div}^{\varepsilon,S}\left(\mathbf{A}+f\mathcal{S}\right) = \operatorname{div}^{\mathcal{H},S}\mathbf{A} + 2\alpha\langle\mathbf{A},J(\nu)\rangle + \mathcal{S}f - fH^{\mathcal{H}}\alpha + \frac{\varepsilon^2\alpha}{1+\varepsilon^2\alpha^2}\left(\mathbf{A}\alpha+f\mathcal{S}\alpha\right). \tag{B.19}
$$

*In particular, if $\varphi\in\mathbf{C}^\infty(S)$ and $p\in S\setminus S_0$,*

$$
\Delta^{\varepsilon,S}\varphi = \hat\Delta^{\mathcal{H},S}\varphi + \frac{\varepsilon^2\mathcal{S}\mathcal{S}\varphi}{1+\varepsilon^2\alpha^2} - \frac{\varepsilon^4\alpha\mathcal{S}\alpha\mathcal{S}\varphi}{(1+\varepsilon^2\alpha^2)^2} - \frac{\varepsilon^2H^{\mathcal{H}}\alpha\mathcal{S}\varphi}{1+\varepsilon^2\alpha^2} + \frac{\varepsilon^2\alpha}{1+\varepsilon^2\alpha^2}\nabla^{\mathcal{H},S}\alpha. \tag{B.20}
$$

*Therefore, if $(\varphi^\varepsilon)_\varepsilon$ converges to $\varphi$ smoothly as in (B.14), then*

$$
\mathcal{J}^\varepsilon\varphi^\varepsilon \xrightarrow[\varepsilon\to0]{} \mathcal{J}^{\mathcal{H}}\varphi, \qquad \textit{locally uniformly on } S\setminus S_0. \tag{B.21}
$$

*Proof.* Let $\mathbf{E}_1,\dots,\mathbf{E}_{2n-1}$ be as in the proof of Lemma B.6. First,

$$
\operatorname{div}^{\varepsilon,S}\mathbf{A} \overset{(B.4)}{=} \sum_{i=1}^{2n-1}\langle\nabla_{\mathbf{E}_i}\mathbf{A},\mathbf{E}_i\rangle - \langle\nabla^\varepsilon_{\mathcal{S}^\varepsilon}\mathcal{S}^\varepsilon,\mathbf{A}\rangle_\varepsilon \overset{(B.18)}{=} \operatorname{div}^{\mathcal{H},S}\mathbf{A} + 2\alpha\langle\mathbf{A},J(\nu)\rangle + \frac{\varepsilon^2\alpha}{1+\varepsilon^2\alpha^2}\mathbf{A}\alpha.
$$

Moreover,

$$\begin{aligned}
\operatorname{div}^{\varepsilon,S}(f\mathcal{S}) &= \mathcal{S}f + f\sum_{i=1}^{2n-1}\left\langle \nabla^{\varepsilon}_{\mathbf{E}_i}\mathcal{S},\mathbf{E}_i\right\rangle_{\varepsilon} + f\left\langle \nabla^{\varepsilon}_{\mathcal{S}^{\varepsilon}}\mathcal{S},\mathcal{S}^{\varepsilon}\right\rangle_{\varepsilon}\\
&= \mathcal{S}f - f\alpha\sum_{i=1}^{2n-1}\left\langle \nabla^{\varepsilon}_{\mathbf{E}_i}\nu,\mathbf{E}_i\right\rangle_{\varepsilon} + f\sum_{i=1}^{2n-1}\left\langle \nabla^{\varepsilon}_{\mathbf{E}_i}T,\mathbf{E}_i\right\rangle_{\varepsilon} + f\left\langle \nabla^{\varepsilon}_{\mathcal{S}^{\varepsilon}}\left(\frac{\sqrt{1+\varepsilon^2\alpha^2}}{\varepsilon}\mathcal{S}^{\varepsilon}\right),\mathcal{S}^{\varepsilon}\right\rangle_{\varepsilon}\\
&\overset{(B.3),(B.4)}{=} \mathcal{S}f - f\alpha\sum_{i=1}^{2n-1}\left\langle \nabla_{\mathbf{E}_i}\nu,\mathbf{E}_i\right\rangle + \frac{f}{\varepsilon^2}\sum_{i=1}^{2n-1}\left\langle J(\mathbf{E}_i),\mathbf{E}_i\right\rangle + \frac{f}{\sqrt{1+\varepsilon^2\alpha^2}}\mathcal{S}\left(\sqrt{1+\varepsilon^2\alpha^2}\right)\\
&= \mathcal{S}f - fH^{\mathcal{H}}\alpha + \frac{\varepsilon^2 f\alpha\mathcal{S}\alpha}{1+\varepsilon^2\alpha^2}.
\end{aligned}$$

To prove (B.20) it suffices to apply (B.19), noticing that

$$\nabla^{\varepsilon,S}\varphi = \nabla^{\mathcal{H},S}\varphi + \frac{\varepsilon^2\mathcal{S}\varphi}{1+\varepsilon^2\alpha^2}\mathcal{S}.$$

Finally, (B.21) follows by (B.17) and (B.20). ☐

The above results and the divergence theorem yield the following integration-by-parts formula (cf. [15]).

**Proposition B.10.** *Let* $\varphi \in C^{\infty}(S)$*. Let* $\mathbf{A} \in \Gamma(\mathcal{H}TS)$*. If* $\operatorname{supp}(\varphi\mathbf{A}) \subseteq S \setminus S_0$*, then*

$$\int_S \varphi \operatorname{div}^{\mathcal{H},S}\mathbf{A}\, d\sigma^{\mathcal{H}} + 2\int_S \varphi\alpha\left\langle \mathbf{A}, J(\nu)\right\rangle\, d\sigma^{\mathcal{H}} = -\int_S \mathbf{A}\varphi\, d\sigma^{\mathcal{H}}. \tag{B.22}$$

*In addition, if* $\psi \in C^{\infty}(S)$ *and* $\operatorname{supp}(\varphi\psi) \subseteq S \setminus S_0$*, then*

$$\int_S \varphi\,\mathcal{S}\psi\, d\sigma^{\mathcal{H}} = \int_S \varphi\,\psi H^{\mathcal{H}}\alpha\, d\sigma^{\mathcal{H}} - \int_S \mathcal{S}\varphi\,\psi\, d\sigma^{\mathcal{H}}. \tag{B.23}$$

Finally, we show the convergence of the cubic curvature term appearing in (A.43).

**Lemma B.11.** *Let* $p \in S \setminus S_0$*. Then*

$$\begin{aligned}
\operatorname{trace}\left((h^{\varepsilon})^3\right) &= \frac{3\tilde{h}^{\mathcal{H}}(J(\nu),J(\nu))}{\varepsilon^2\left(1+\varepsilon^2\alpha^2\right)^{\frac{1}{2}}} + \frac{1}{\left(1+\varepsilon^2\alpha^2\right)^{\frac{3}{2}}}\left(\operatorname{trace}\left(\left(\tilde{h}^{\mathcal{H}}\right)^3\right) + 6\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\alpha, J(\nu)\right) + 3\mathcal{S}\alpha\right)\\
&\quad + \frac{3\varepsilon^2}{\left(1+\varepsilon^2\alpha^2\right)^{\frac{5}{2}}}\left(\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\alpha,\nabla^{\mathcal{H},S}\alpha\right) + 2J(\nu)\alpha\mathcal{S}\alpha\right) + \frac{3\varepsilon^4}{\left(1+\varepsilon^2\alpha^2\right)^{\frac{7}{2}}}\mathcal{S}\alpha\left|\nabla^{\mathcal{H},S}\alpha\right|^2 + \frac{\varepsilon^6}{\left(1+\varepsilon^2\alpha^2\right)^{\frac{9}{2}}}(\mathcal{S}\alpha)^3
\end{aligned}$$

*Proof.* Let $\mathbf{E}_1,\dots,\mathbf{E}_{2n-1}$ be as in the proof of Lemma B.6. Set $h^{\varepsilon}_{ij} = h^{\varepsilon}(\mathbf{E}_i,\mathbf{E}_j)$ for $i,j = 1,\dots,2n$. Then

$$\operatorname{trace}\left((h^{\varepsilon})^3\right) = \underbrace{\sum_{i,j,k=1}^{2n-1} h^{\varepsilon}_{ij}h^{\varepsilon}_{jk}h^{\varepsilon}_{ki}}_{\mathrm{I}} + \underbrace{3\sum_{i,j=1}^{2n-1} h^{\varepsilon}_{ij}h^{\varepsilon}_{i,2n}h^{\varepsilon}_{j,2n}}_{\mathrm{II}} + \underbrace{3h^{\varepsilon}_{2n,2n}\sum_{i=1}^{2n-1}\left(h^{\varepsilon}_{i,2n}\right)^2}_{\mathrm{III}} + \underbrace{\left(h^{\varepsilon}_{2n,2n}\right)^3}_{\mathrm{IV}}.$$

First,

$$\mathrm{I} \overset{(B.11)}{=} \frac{1}{\left(1+\varepsilon^2\alpha^2\right)^{\frac{3}{2}}}\operatorname{trace}\left(\left(\tilde{h}^{\mathcal{H}}\right)^3\right).$$

Moreover,

$$
\begin{aligned}
\mathrm{II} &= 3 \sum_{\substack{i,j=1\\ i,j\neq n}}^{2n-1} h^{\varepsilon}_{ij} h^{\varepsilon}_{i,2n} h^{\varepsilon}_{j,2n} + 6 h^{\varepsilon}_{n,2n} \sum_{\substack{i=1\\ i\neq n}}^{2n-1} h^{\varepsilon}_{i,n} h^{\varepsilon}_{i,2n} + 3 h^{\varepsilon}_{n,n} \left(h^{\varepsilon}_{n,2n}\right)^2 \\
&\overset{(B.11)}{=} \frac{3\varepsilon^2}{(1+\varepsilon^2\alpha^2)^{\frac52}} \sum_{\substack{i,j=1\\ i,j\neq n}}^{2n-1} \tilde{h}^{\mathcal{H}}(\mathbf{E}_i,\mathbf{E}_j)\mathbf{E}_i\alpha\mathbf{E}_j\alpha + 6\left(1+\frac{\varepsilon^2 J(\nu)\alpha}{1+\varepsilon^2\alpha^2}\right)\frac{1}{(1+\varepsilon^2\alpha^2)^{\frac32}} \sum_{\substack{i=1\\ i\neq n}}^{2n-1} \tilde{h}^{\mathcal{H}}(\mathbf{E}_i,J(\nu))\mathbf{E}_i\alpha \\
&\quad + \frac{3}{\varepsilon^2(1+\varepsilon^2\alpha^2)^{\frac12}} \tilde{h}^{\mathcal{H}}(J(\nu),J(\nu)) \left(1+\frac{\varepsilon^2 J(\nu)\alpha}{1+\varepsilon^2\alpha^2}\right)^2 \\
&= \frac{3\varepsilon^2}{(1+\varepsilon^2\alpha^2)^{\frac52}} \tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\alpha,\nabla^{\mathcal{H},S}\alpha\right) + \frac{6}{(1+\varepsilon^2\alpha^2)^{\frac32}} \tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\alpha,J(\nu)\right) + \frac{3}{\varepsilon^2(1+\varepsilon^2\alpha^2)^{\frac12}} \tilde{h}^{\mathcal{H}}(J(\nu),J(\nu)).
\end{aligned}
$$

In addition,

$$
\begin{aligned}
\mathrm{III} &= 3 h^{\varepsilon}_{2n,2n} \sum_{\substack{i=1\\ i\neq n}}^{2n-1} \left(h^{\varepsilon}_{i,2n}\right)^2 + 3 h^{\varepsilon}_{2n,2n} \left(h^{\varepsilon}_{n,2n}\right)^2 \\
&\overset{(B.11)}{=} \frac{3\varepsilon^4}{(1+\varepsilon^2\alpha^2)^{\frac72}} \mathcal{S}\alpha \sum_{\substack{i=1\\ i\neq n}}^{2n-1} (\mathbf{E}_i\alpha)^2 + \frac{3}{(1+\varepsilon^2\alpha^2)^{\frac32}} \mathcal{S}\alpha \left(1+\frac{\varepsilon^2 J(\nu)\alpha}{1+\varepsilon^2\alpha^2}\right)^2 \\
&= \frac{3\varepsilon^4}{(1+\varepsilon^2\alpha^2)^{\frac72}} \mathcal{S}\alpha \left|\nabla^{\mathcal{H},S}\alpha\right|^2 + \frac{6\varepsilon^2}{(1+\varepsilon^2\alpha^2)^{\frac52}} \mathcal{S}\alpha J(\nu)\alpha + \frac{3}{(1+\varepsilon^2\alpha^2)^{\frac32}} \mathcal{S}\alpha.
\end{aligned}
$$

Finally,

$$
\mathrm{IV} \overset{(B.11)}{=} \frac{\varepsilon^6}{(1+\varepsilon^2\alpha^2)^{\frac92}} (\mathcal{S}\alpha)^3 .
$$

The thesis follows combining the above computations. ☐

**Lemma B.12.** *Let $p \in S \setminus S_0$. Then*

$$
\langle h^{\varepsilon}, \mathcal{C}^{\varepsilon}(\mathbf{N}^{\varepsilon},\mathbf{N}^{\varepsilon})\rangle_{\varepsilon} = \frac{3\tilde{h}^{\mathcal{H}}(J(\nu),J(\nu))}{\varepsilon^2(1+\varepsilon^2\alpha^2)^{\frac32}} - \frac{1}{(1+\varepsilon^2\alpha^2)^{\frac32}}\Big(\mathcal{S}\alpha + H^{\mathcal{H}}\alpha^2\Big).
$$

*Proof.* Let $\mathbf{E}_1,\dots,\mathbf{E}_{2n-1}$ be as in the proof of Lemma B.6. By (B.5) and (B.8), if $\mathbf{B},\mathbf{D} \in \Gamma(TS)$,

$$
\mathcal{C}^{\varepsilon}(\mathbf{N}^{\varepsilon},\mathbf{N}^{\varepsilon})(\mathbf{B},\mathbf{D}) = \frac{3}{\varepsilon^2}\langle \mathbf{N}^{\varepsilon}, J(\mathbf{B})\rangle_{\varepsilon} \langle \mathbf{N}^{\varepsilon}, J(\mathbf{D})\rangle_{\varepsilon} - \frac{B^{2n+1,\varepsilon}D^{2n+1,\varepsilon}}{\varepsilon^2} - \frac{\alpha^2}{1+\varepsilon^2\alpha^2}\langle \mathbf{B},\mathbf{D}\rangle_{\varepsilon}. \tag{B.24}
$$

Therefore,

$$
\begin{aligned}
\langle h^{\varepsilon}, \mathcal{C}^{\varepsilon}(\mathbf{N}^{\varepsilon},\mathbf{N}^{\varepsilon})\rangle_{\varepsilon} &= \sum_{i,j=1}^{2n} h^{\varepsilon}(\mathbf{E}_i,\mathbf{E}_j)\mathcal{C}^{\varepsilon}(\mathbf{N}^{\varepsilon},\mathbf{N}^{\varepsilon})(\mathbf{E}_i,\mathbf{E}_j) \\
&\overset{(B.24)}{=} \frac{3}{\varepsilon^2} h^{\varepsilon}(J(\nu),J(\nu)) \langle \mathbf{N}^{\varepsilon},\nu\rangle_{\varepsilon}^2 - \frac{1}{\varepsilon^2} h^{\varepsilon}(\mathcal{S}^{\varepsilon},\mathcal{S}^{\varepsilon}) \langle \mathcal{S}^{\varepsilon},\varepsilon T\rangle_{\varepsilon}^2 - \frac{\alpha^2}{1+\varepsilon^2\alpha^2} H^{\varepsilon} \\
&\overset{(B.8),(B.9),(B.11),(B.13)}{=} \frac{3\tilde{h}^{\mathcal{H}}(J(\nu),J(\nu))}{\varepsilon^2(1+\varepsilon^2\alpha^2)^{\frac32}} - \frac{\mathcal{S}\alpha}{(1+\varepsilon^2\alpha^2)^{\frac52}} - \frac{H^{\mathcal{H}}\alpha^2}{(1+\varepsilon^2\alpha^2)^{\frac32}} - - \frac{\varepsilon^2\alpha^2\mathcal{S}\alpha}{(1+\varepsilon^2\alpha^2)^{\frac52}} \\
&= \frac{3\tilde{h}^{\mathcal{H}}(J(\nu),J(\nu))}{\varepsilon^2(1+\varepsilon^2\alpha^2)^{\frac32}} - \frac{\mathcal{S}\alpha}{(1+\varepsilon^2\alpha^2)^{\frac32}} - \frac{H^{\mathcal{H}}\alpha^2}{(1+\varepsilon^2\alpha^2)^{\frac32}}.
\end{aligned}
$$

☐

**Theorem B.13.** *Let $\varepsilon > 0$. Let $p \in S \setminus S_0$. Then*

$$\begin{aligned}
&2\,\mathrm{trace}\left((h^\varepsilon)^3\right) - 2\left\langle h^\varepsilon, \mathcal{C}^\varepsilon(\mathbf{N}^\varepsilon, \mathbf{N}^\varepsilon)\right\rangle_\varepsilon - \left(\nabla^\varepsilon_{\mathbf{N}^\varepsilon} \mathrm{Ric}^\varepsilon\right)(\mathbf{N}^\varepsilon, \mathbf{N}^\varepsilon) \\
&\quad = \frac{1}{(1+\varepsilon^2\alpha^2)^{\frac{3}{2}}}\left(2\,\mathrm{trace}\left(\left(\tilde{h}^{\mathcal{H}}\right)^3\right) + 12\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\alpha, J(\nu)\right) + 8\mathcal{S}\alpha + 6\tilde{h}^{\mathcal{H}}(J(\nu),J(\nu))\alpha^2 + 2H^{\mathcal{H}}\alpha^2\right) \\
&\qquad + \frac{6\varepsilon^2}{(1+\varepsilon^2\alpha^2)^{\frac{5}{2}}}\left(\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\alpha, \nabla^{\mathcal{H},S}\alpha\right) + 2J(\nu)\alpha\mathcal{S}\alpha\right) + \frac{6\varepsilon^4}{(1+\varepsilon^2\alpha^2)^{\frac{7}{2}}}\mathcal{S}\alpha\left|\nabla^{\mathcal{H},S}\alpha\right|^2 + \frac{2\varepsilon^6}{(1+\varepsilon^2\alpha^2)^{\frac{9}{2}}}(\mathcal{S}\alpha)^3.
\end{aligned}$$

*In particular,*

$$\begin{aligned}
&2\,\mathrm{trace}\left((h^\varepsilon)^3\right) - 2\left\langle h^\varepsilon, \mathcal{C}^\varepsilon(\mathbf{N}^\varepsilon, \mathbf{N}^\varepsilon)\right\rangle_\varepsilon - \left(\nabla^\varepsilon_{\mathbf{N}^\varepsilon} \mathrm{Ric}^\varepsilon\right)(\mathbf{N}^\varepsilon, \mathbf{N}^\varepsilon) \\
&\quad \xrightarrow[\varepsilon\to 0]{} 2\,\mathrm{trace}\left(\left(\tilde{h}^{\mathcal{H}}\right)^3\right) + 12\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\alpha, J(\nu)\right) + 8\mathcal{S}\alpha + 6\tilde{h}^{\mathcal{H}}(J(\nu),J(\nu))\alpha^2 + 2H^{\mathcal{H}}\alpha^2
\end{aligned} \tag{B.25}$$

*locally uniformly on $S \setminus S_0$.*

*Proof.* Notice that

$$\frac{6\tilde{h}^{\mathcal{H}}(J(\nu),J(\nu))}{\varepsilon^2\,(1+\varepsilon^2\alpha^2)^{\frac{1}{2}}} - \frac{6\tilde{h}^{\mathcal{H}}(J(\nu),J(\nu))}{\varepsilon^2\,(1+\varepsilon^2\alpha^2)^{\frac{3}{2}}} = \frac{6\tilde{h}^{\mathcal{H}}(J(\nu),J(\nu))\alpha^2}{(1+\varepsilon^2\alpha^2)^{\frac{3}{2}}}.$$

The thesis follows by (B.7), Lemma B.11 and Lemma B.12. □

B.4. **Proof of Theorem 4.2.** The proof of Theorem 4.2 follows by applying the results of Appendix B.3.

*Proof of Theorem 4.2.* Fix $\varepsilon > 0$. Set $\mathscr{H}^\varepsilon_f(S) := \int_S f(H^\varepsilon)\, d\sigma^\varepsilon$. Then

$$\begin{aligned}
\mathscr{H}^\varepsilon_f(S) &\overset{\text{(B.10),(B.13)}}{=} \int_S f\left(\frac{H^{\mathcal{H}}}{\sqrt{1+\varepsilon^2\alpha^2}} + \frac{\varepsilon^2\mathcal{S}\alpha}{(1+\varepsilon^2\alpha^2)^{\frac{3}{2}}}\right)\frac{\sqrt{1+\varepsilon^2\alpha^2}}{\varepsilon\sqrt{1+\alpha^2}}\, d\sigma^1, \\
\mathscr{H}^{\mathcal{H}}_f(S) &\overset{\text{(2.8)}}{=} \int_S f\left(H^{\mathcal{H}}\right)\frac{1}{\sqrt{1+\alpha^2}}\, d\sigma^1.
\end{aligned} \tag{B.26}$$

Therefore, by (B.26) and since $\Phi$ is supported on $S \setminus S_0$, we infer that

$$\delta\mathscr{H}^{\mathcal{H}}(S)[\Phi] = \lim_{\varepsilon\to 0}\varepsilon\delta\mathscr{H}^\varepsilon(S)[\Phi], \qquad \delta^2\mathscr{H}^{\mathcal{H}}(S)[\Phi] = \lim_{\varepsilon\to 0}\varepsilon\delta^2\mathscr{H}^\varepsilon(S)[\Phi]. \tag{B.27}$$

Set $\varphi^\varepsilon = \langle \mathbf{X}, \mathbf{N}^\varepsilon\rangle_\varepsilon$. Then, by (A.42),

$$\varepsilon\delta\mathscr{H}^\varepsilon(S)[\Phi] = \int_S \left(f'(H^\varepsilon)\left(-\Delta^{\varepsilon,S}\varphi^\varepsilon - \varphi^\varepsilon\left(|h^\varepsilon|^2 + \mathrm{Ric}^\varepsilon(\mathbf{N}^\varepsilon,\mathbf{N}^\varepsilon)\right)\right) + \varphi^\varepsilon f(H^\varepsilon)H^\varepsilon\right)\frac{\sqrt{1+\varepsilon^2\alpha^2}}{\sqrt{1+\alpha^2}}\, d\sigma^1.$$

Notice that

$$\varphi^\varepsilon = \langle \mathbf{X}, \mathbf{N}^\varepsilon\rangle = \frac{\langle \mathbf{X}, \nu\rangle}{\sqrt{1+\varepsilon^2\alpha^2}} + \frac{\varepsilon\alpha\left\langle \mathbf{X}, \varepsilon T\right\rangle_\varepsilon}{\sqrt{1+\varepsilon^2\alpha^2}} = \frac{\langle \mathbf{X}, \nu\rangle}{\sqrt{1+\varepsilon^2\alpha^2}} + \frac{\alpha\left\langle \mathbf{X}, T\right\rangle_1}{\sqrt{1+\varepsilon^2\alpha^2}} = \frac{\varphi}{\sqrt{1+\varepsilon^2\alpha^2}}.$$

Therefore, (B.20) implies that

$$\varphi^\varepsilon \xrightarrow[\varepsilon\to 0]{} \varphi, \qquad \Delta^{\varepsilon,S}\varphi^\varepsilon \xrightarrow[\varepsilon\to 0]{} \hat{\Delta}^{\mathcal{H},S}\varphi \qquad \text{uniformly on } S.$$

Then, by (B.27) and recalling (B.14) and (B.17), (4.4) follows. Next, we prove (4.5). Set

$$\mathbf{X}^{T,\varepsilon} = \mathbf{X} - \varphi^\varepsilon\mathbf{N}^\varepsilon = \mathbf{X} - \frac{\varphi}{1+\varepsilon^2\alpha^2}\left(\nu + \varepsilon^2\alpha T\right),$$

and denote by $\mathbf{Z}^\varepsilon$ the acceleration of $\Phi$ with respect to $\nabla^\varepsilon$. By (A.43),

$$\begin{aligned}
\delta^2\mathscr{H}_f^\varepsilon(S)[\Phi] &= \delta\mathscr{H}_f^\varepsilon(S)\left[\langle \mathbf{Z}|_S,\mathbf{N}^\varepsilon\rangle_\varepsilon - 2\mathbf{X}^{T,\varepsilon}\varphi^\varepsilon + h^\varepsilon\left(\mathbf{X}^{T,\varepsilon},\mathbf{X}^{T,\varepsilon}\right)\right] + \int_S \varphi^\varepsilon\,\mathcal{J}^\varepsilon\left(f''(H^\varepsilon)\mathcal{J}^\varepsilon\varphi^\varepsilon\right)\,d\sigma^\varepsilon\\
&\quad+\int_S 2\varphi^\varepsilon\operatorname{div}^{\varepsilon,S}\left(f'(H^\varepsilon)A^\varepsilon\left(\nabla^{\varepsilon,S}\varphi^\varepsilon\right)\right)\,d\sigma^\varepsilon - \int_S\varphi^\varepsilon\left(f'(H^\varepsilon)H^\varepsilon+f(H^\varepsilon)\right)\Delta^{\varepsilon,S}\varphi^\varepsilon\,d\sigma^\varepsilon\\
&\quad-\int_S 4\varphi^\varepsilon f''(H^\varepsilon)h^\varepsilon\left(\nabla^{\varepsilon,S}H^\varepsilon,\nabla^{\varepsilon,S}\varphi^\varepsilon\right)\,d\sigma^\varepsilon + \int_S\varphi^\varepsilon\left(f''(H^\varepsilon)H^\varepsilon-2f'(H^\varepsilon)\right)\left\langle\nabla^{\varepsilon,S}H^\varepsilon,\nabla^{\varepsilon,S}\varphi^\varepsilon\right\rangle_\varepsilon\,d\sigma^\varepsilon\\
&\quad+\int_S(\varphi^\varepsilon)^2 f'(H^\varepsilon)\Big(2\operatorname{trace}\left((h^\varepsilon)^3\right)-2\left\langle h^\varepsilon,\mathcal{C}^\varepsilon(\mathbf{N}^\varepsilon,\mathbf{N}^\varepsilon)\right\rangle-\left(\nabla^\varepsilon_{\mathbf{N}^\varepsilon}\operatorname{Ric}^\varepsilon\right)(\mathbf{N}^\varepsilon,\mathbf{N}^\varepsilon)\Big)\,d\sigma^\varepsilon\\
&\quad+\int_S(\varphi^\varepsilon)^2\Big(f(H^\varepsilon)\,(H^\varepsilon)^2-\left(2f'(H^\varepsilon)H^\varepsilon+f(H^\varepsilon)\right)\left(|h^\varepsilon|^2+\operatorname{Ric}^\varepsilon(\mathbf{N}^\varepsilon,\mathbf{N}^\varepsilon)\right)\Big)\,d\sigma^\varepsilon.
\end{aligned}$$

Denote the terms on the right-hand side by $\mathrm{I}^\varepsilon,\dots,\mathrm{VIII}^\varepsilon$. First,

$$\langle\mathbf{Z}^\varepsilon,\mathbf{N}^\varepsilon\rangle_\varepsilon=\frac{\langle\mathbf{X}',\nu+\alpha T\rangle_1}{\sqrt{1+\varepsilon^2\alpha^2}}+\langle\nabla^\varepsilon_{\mathbf{X}}\mathbf{X},\mathbf{N}^\varepsilon\rangle_\varepsilon\overset{(B.2)}{=}\frac{\langle\mathbf{Z},\nu+\alpha T\rangle_1}{\sqrt{1+\varepsilon^2\alpha^2}}+\frac{2\langle\mathbf{X},\varepsilon T\rangle_\varepsilon}{\varepsilon}\langle J(\mathbf{X}),\mathbf{N}^\varepsilon\rangle_\varepsilon\,.$$

Moreover, $\mathbf{X}-\varphi\nu\in\Gamma(TS)$, and

$$-2\mathbf{X}^{T,\varepsilon}\varphi^\varepsilon=-2\left(\mathbf{X}-\frac{\varphi}{1+\varepsilon^2\alpha^2}\left(\nu+\varepsilon^2\alpha T\right)\right)\left(\frac{\varphi}{\sqrt{1+\varepsilon^2\alpha^2}}\right)\xrightarrow[\varepsilon\to0]{}-2\left(\mathbf{X}-\varphi\nu\right)\varphi\qquad\text{uniformly on } S.$$

In addition,

$$h^\varepsilon\left(\mathbf{X}^{T,\varepsilon},\mathbf{X}^{T,\varepsilon}\right)=-\left\langle\nabla^\varepsilon_{\mathbf{X}^{T,\varepsilon}}\mathbf{X}^{T,\varepsilon},\mathbf{N}^\varepsilon\right\rangle_\varepsilon\overset{(B.2)}{=}-\left\langle\nabla_{\mathbf{X}^{T,\varepsilon}}\mathbf{X}^{T,\varepsilon},\mathbf{N}^\varepsilon\right\rangle_\varepsilon-\frac{2\left\langle\mathbf{X}^{T,\varepsilon},\varepsilon T\right\rangle_\varepsilon}{\varepsilon}\langle J(\mathbf{X}),\mathbf{N}^\varepsilon\rangle_\varepsilon\,.$$

Therefore

$$\begin{aligned}
\langle\mathbf{Z}^\varepsilon,\mathbf{N}^\varepsilon\rangle_\varepsilon+h^\varepsilon\left(\mathbf{X}^{T,\varepsilon},\mathbf{X}^{T,\varepsilon}\right)&=\frac{\langle\mathbf{Z},\nu+\alpha T\rangle_1}{\sqrt{1+\varepsilon^2\alpha^2}}-\left\langle\nabla_{\mathbf{X}^{T,\varepsilon}}\mathbf{X}^{T,\varepsilon},\mathbf{N}^\varepsilon\right\rangle_\varepsilon+\frac{2\langle J(\mathbf{X}),\mathbf{N}^\varepsilon\rangle_\varepsilon}{\varepsilon}\left\langle\mathbf{X}-\mathbf{X}^{T,\varepsilon},\varepsilon T\right\rangle_\varepsilon\\
&=\frac{\langle\mathbf{Z},\nu+\alpha T\rangle_1}{\sqrt{1+\varepsilon^2\alpha^2}}-\left\langle\nabla_{\mathbf{X}^{T,\varepsilon}}\mathbf{X}^{T,\varepsilon},\mathbf{N}^\varepsilon\right\rangle_\varepsilon+\frac{2\alpha\varphi_\epsilon\langle J(\mathbf{X}),\mathbf{N}^\varepsilon\rangle_\varepsilon}{\sqrt{1+\varepsilon^2\alpha^2}}\\
&\xrightarrow[\varepsilon\to0]{}\langle\mathbf{Z},\nu+\alpha T\rangle_1-\Big\langle\nabla_{\mathbf{X}-\varphi\nu}\left(\mathbf{X}-\varphi\nu\right),\nu+\alpha T\Big\rangle_1-2\alpha\varphi\left\langle\mathbf{X},J(\nu)\right\rangle
\end{aligned}$$

uniformly on $S$. By the first part of the proof, we conclude that

$$\varepsilon\mathrm{I}^\varepsilon\xrightarrow[\varepsilon\to0]{}\delta\mathscr{H}_f^{\mathcal{H}}(S)\left[\langle\mathbf{Z},\nu+\alpha T\rangle_1-2\left(\mathbf{X}-\varphi\nu\right)\varphi-\Big\langle\nabla_{\mathbf{X}-\varphi\nu}\left(\mathbf{X}-\varphi\nu\right),\nu+\alpha T\Big\rangle_1-2\alpha\varphi\left\langle\mathbf{X},J(\nu)\right\rangle\right].$$

Moreover, by (B.21),

$$\varepsilon\mathrm{II}^\varepsilon=\varepsilon\int_S f''(H^\varepsilon)\mathcal{J}^\varepsilon\varphi^\varepsilon\,\mathcal{J}^\varepsilon\varphi^\varepsilon\,d\sigma^\varepsilon\xrightarrow[\varepsilon\to0]{}\int_S f''(H^{\mathcal{H}})\mathcal{J}^{\mathcal{H}}\varphi\,\mathcal{J}^{\mathcal{H}}\varphi\,d\sigma^{\mathcal{H}}=\int_S\varphi\,\mathcal{J}^{\mathcal{H}}\left(f''(H^{\mathcal{H}})\mathcal{J}^{\mathcal{H}}\varphi\right)\,d\sigma^{\mathcal{H}}.$$

Next,

$$\begin{aligned}
\varepsilon\mathrm{III}^\varepsilon&=-\varepsilon\int_S 2f'(H^\varepsilon)h^\varepsilon\left(\nabla^{\varepsilon,S}\varphi^\varepsilon,\nabla^{\varepsilon,S}\varphi^\varepsilon\right)\,d\sigma^\varepsilon\\
&\xrightarrow[\varepsilon\to0]{(B.16)}-\int_S 2f'(H^{\mathcal{H}})\Big(\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\varphi,\nabla^{\mathcal{H},S}\varphi\right)+2J(\nu)\varphi\mathcal{S}\varphi\Big)\,d\sigma^{\mathcal{H}}\\
&\overset{(2.7)}{=}-\int_S 2\left\langle f'(H^{\mathcal{H}})A^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\varphi\right),\nabla^{\mathcal{H},S}\varphi\right\rangle\,d\sigma^{\mathcal{H}}-\int_S 4f'(H^{\mathcal{H}})J(\nu)\varphi\mathcal{S}\varphi\,d\sigma^{\mathcal{H}}\\
&\overset{(B.22),(B.23)}{=}\int_S 2\varphi\operatorname{div}^{\mathcal{H},S}\left\langle f'(H^{\mathcal{H}})A^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\varphi\right)\right)\,d\sigma^{\mathcal{H}}+\int_S 4\varphi\alpha f'(H^{\mathcal{H}})h^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\varphi,J(\nu)\right)\,d\sigma^{\mathcal{H}}\\
&\quad+\int_S 4\varphi J(\nu)\varphi f''(H^{\mathcal{H}})\mathcal{S}H^{\mathcal{H}}\,d\sigma^{\mathcal{H}}+\int_S 4\varphi f'(H^{\mathcal{H}})\mathcal{S}J(\nu)\varphi,d\sigma^{\mathcal{H}}-\int_S 4\varphi\alpha f'(H^{\mathcal{H}})H^{\mathcal{H}}J(\nu)\varphi\,d\sigma^{\mathcal{H}}\\
&\overset{(2.7)}{=}\int_S 2\varphi\operatorname{div}^{\mathcal{H},S}\left\langle f'(H^{\mathcal{H}})A^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\varphi\right)\right)\,d\sigma^{\mathcal{H}}+\int_S 4\varphi\alpha f'(H^{\mathcal{H}})\Big(\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\varphi,J(\nu)\right)-H^{\mathcal{H}}J(\nu)\varphi\Big)\,d\sigma^{\mathcal{H}}\\
&\quad+\int_S 4\varphi J(\nu)\varphi f''(H^{\mathcal{H}})\mathcal{S}H^{\mathcal{H}}\,d\sigma^{\mathcal{H}}+\int_S 4\varphi f'(H^{\mathcal{H}})\mathcal{S}J(\nu)\varphi,d\sigma^{\mathcal{H}}.
\end{aligned}$$

Moreover,

$$\varepsilon \mathrm{IV}^{\varepsilon} \xrightarrow[\varepsilon\to 0]{} -\int_S \varphi \left(f'(H^{\mathcal{H}})H^{\mathcal{H}} + f(H^{\mathcal{H}})\right) \hat{\Delta}^{\mathcal{H},S}\varphi \, d\sigma^{\mathcal{H}}.$$

In addition,

$$\varepsilon \mathrm{V}^{\varepsilon} \xrightarrow[\varepsilon\to 0]{(B.16)} -\int_S 4\varphi f''(H^{\mathcal{H}})\Big(\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}H^{\mathcal{H}}, \nabla^{\mathcal{H},S}\varphi\right) + J(\nu)\varphi\mathcal{S}H^{\mathcal{H}} + \mathcal{S}\varphi J(\nu)H^{\mathcal{H}}\Big)\, d\sigma^{\mathcal{H}}.$$

Moreover,

$$\varepsilon \mathrm{VI}^{\varepsilon} \xrightarrow[\varepsilon\to 0]{} \int_S \varphi \left(f''(H^{\mathcal{H}})H^{\mathcal{H}} - 2f'(H^{\mathcal{H}})\right) \left\langle \nabla^{\mathcal{H},S}H^{\mathcal{H}}, \nabla^{\mathcal{H},S}\varphi\right\rangle \, d\sigma^{\mathcal{H}}.$$

Furthermore, by (B.25),

$$\varepsilon \mathrm{VII}^{\varepsilon} \xrightarrow[\varepsilon\to 0]{} \int_S \varphi^2 f'(H^{\mathcal{H}})\Big(2\,\mathrm{trace}\left(\left(\tilde{h}^{\mathcal{H}}\right)^3\right) + 12\tilde{h}^{\mathcal{H}}\left(\nabla^{\mathcal{H},S}\alpha, J(\nu)\right) + 8\mathcal{S}\alpha + 6\tilde{h}^{\mathcal{H}}(J(\nu),J(\nu))\alpha^2 + 2H^{\mathcal{H}}\alpha^2\Big)\, d\sigma^{\mathcal{H}}.$$

Finally, by (B.17),

$$\varepsilon \mathrm{VIII}^{\varepsilon} \xrightarrow[\varepsilon\to 0]{} \int_S \varphi^2 \Big(f(H^{\mathcal{H}})\left(H^{\mathcal{H}}\right)^2 - \left(2f'(H^{\mathcal{H}})H^{\mathcal{H}} + f(H^{\mathcal{H}})\right)\left(|\tilde{h}^{\mathcal{H}}|^2 + 4J(\nu)\alpha + (2n+2)\alpha^2\right)\Big)\, d\sigma^{\mathcal{H}}.$$

The thesis follows combining the above computations. □

## DECLARATIONS

**Conflict of interest.** The authors have no financial interests or conflicts of interest related to the subject matter.

**Data availability statement.** Data sharing not applicable as no datasets were generated or analyzed during the current study.

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