A representation result for Γ-limit of supremal functionals

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Abstract

In this paper we prove a representation result for the weak$^*L^\infty$ Γ-limit of a sequence of supremal functionals $F_n(u) := \operatorname*{ess\,sup}_{x\in\Omega} f_n(x, u(x))$ where $\Omega \subset \mathbb{R}^n$ and $u \in L^\infty(\Omega, \mathbb{R}^N)$. This Γ-limit is still a supremal functional and we give an explicit formula to obtain it. The basic tools we use are the definition of level convexity and the related notion of duality introduced by Volle in [13].

1 Introduction

In the last decade people become interested in the so-called problems in L^∞, where instead of the classical integral functional a supremal one is considered, i.e.

$$F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x), Du(x)) \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ and $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$. From an application point of view this approach is very important because in severable models not only the L^∞ approach arises naturally but cannot be ignored if one wants a realistic solution. For example, in designing a support structure, one can be interested in minimizing the maximum pointwise stress more that the average stress; or in trying to steer a trajectory to a target the aim can be to minimize the maximum distance to the target. With these motivations, the calculus of variation in L^∞ has been developed (see [5], [6], [7], [4]).

In this paper we consider a sequence of supremal functionals

$$F_n(u) := \operatorname*{ess\,sup}_{x\in\Omega} f_n(x, u(x)) \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ and $u \in L^\infty(\Omega, \mathbb{R}^N)$. Our goal is to give a representation result for the Γ-limit of this sequence with respect to the weak* topology of L^∞. Indeed, our main theorem (see Theorem 3.12) gives an explicit formula to compute it.

In [2] and [3], Attouch and Aze give a general representation formula for strong and weak Γ-limits of lower semicontinuous and convex functions defined on a Banach space X through the

Γ-limits of their polar functions, defined on the dual space X' by the Fenchel transform $f \mapsto f^*$. This result has an important application in the case of Γ-limit of integral functionals

$$F_n(u) := \int_\Omega f_n(x, u'(x))dx \tag{1.3}$$

with $\Omega \subset \mathbb{R}$, $u : \Omega \to \mathbb{R}^N$ and f_n convex in the u' variable. In (1.2) and in (1.1), the natural hypotheses on f and f_n in order to obtain the weak* lower semicontinuity of F in $W^{1,\infty}(\Omega, \mathbb{R}^N)$ and of F_n in $L^\infty(\Omega, \mathbb{R}^N)$ is not the convexity of f in the gradient variable and of f_n in the u variable. The property of lower semicontinuity for a functional expressed by a supremum requires a different notion of convexity: the level convexity. Indeed, in 1997, Barron and Liu, (see [6]), proved a relaxation theorem with respect to the weak ${}^*W^{1,\infty}$ topology for the supremal functional (1.1) when $n = 1$. Their approach was to consider the calculus of variation problem as a particular optimal control problem and to apply the theory of viscosity solutions for the Hamilton-Jacobi equation previously developed. Thus they require f to be continuous in all variables. They showed that the relaxed functional of (1.1) is a supremal functional represented by the greatest level convex function minorant f with respect to the u' variable. Moreover they showed that the greatest level convex minorant the functional (1.1) is the same as its lower semicontinuous regularization. To determine the greatest level convex minorant two different definitions of level convex duality were given.

A more direct approach to the calculus of variation problem has been started in [1]. In this work the authors characterized completely the class of functionals on L^∞ which can be represented by a supremum. Following that, in [14], Prinari obtained a relaxation theorem for a functional $F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x))$, where $\Omega \subset \mathbb{R}^n$ and $u \in L^\infty(\Omega, \mathbb{R}^N)$, without any continuity assumptions on f. Also in this case, the relaxed functional with respect to the weak* topology of $L^\infty(\Omega, \mathbb{R}^N)$ is a supremal functional represented by the level convex envelope of f in the u-variable. Moreover, in [14], it is proved that the Γ-limit of a sequence of supremal functional is still a supremal functional represented by a level convex function.
The previous results imply that the notion of duality has to be changed if one wants to give a representation formula for the Γ-limit of (1.2) as in [2] and [3]. We use a duality introduced by Volle in [13]. If X is a Banach space and X' is its dual space, in order to define this duality we introduce two operators: one operates on a function $f : X \to \overline{\mathbb{R}}$ giving a function $f^c : X' \times \mathbb{R} \to \overline{\mathbb{R}}$, while the other one operates on a function $\phi : X' \times \mathbb{R} \to \overline{\mathbb{R}}$ giving a function $\phi^\gamma : X \to \overline{\mathbb{R}}$. The lower semicontinuous and level convex functions result to be the invariant functions with respect to the compositions of these two operators: $f \mapsto f^{c\gamma}$. The level convex duality so defined allows us to compute the Γ-limit of (1.2) by computing the Γ-limit G of the sequence $(F_n^c)_n$ and by applying to G the operator $\phi \mapsto \phi^\gamma$. In this paper we study the assumption one has to require on the sequence $(f_n)_n$ in order to calculate the weak ${}^*L^\infty$ Γ-limit. To obtain this we derive also some general results about the level convex duality and the relation with Γ-limit of sequences of functionals. In Section 2, after recalling the basic definitions we use, we prove a first representation result for the weak* Γ-limit of sequences of functionals defined on an abstract Banach space. Moreover a necessary and sufficient condition is derived in the class of lower semicontinuous and level convex functionals. We start Section 3 by precisely describing the supremal functionals we are going to work with. We then prove that the duality defined in Section 2 transforms a supremal functional into an integral one and vice versa, and we use this and the general result we obtained in Section 2 to derive our full representation result. The latest gives us a way to compute explicitly the weak ${}^*L^\infty$ Γ-limit of the sequence (1.2). Two explicit examples are computed in Section 4. This section and the work are ended by the description of

two counterexamples on a behavior which is natural in the integral case. (See Theorem 3.7 in [2], and Theorem 3.2.4 in [3].)
We observe that if $n = 1$ this result applies to the case when $F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, u'(x))$, while the same representations results (relaxation and Γ-convergence) for the functional $F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, Du(x))$, with $\Omega \in \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}^N$, are still open.

2 Duality: basic properties and Γ-limit.

In this section we want to describe the conjugate operators and the behavior of Γ-limits of sequences of functions with respect to this duality. This section is divided in two parts, in the first one we recall the definitions of conjugate functions given in [13] and their basic properties, while the second one is devoted to the proof of some general Γ-convergence results.

We begin giving the basic definitions. Let X be a Hausdorff locally convex topological space with topological dual Y. We will denote the duality product between X and Y by $\langle \cdot, \cdot \rangle$.

Definition 2.1 *A function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is called* level convex *if for each $t \in \mathbb{R}$ the set $\{\varphi \le t\} := \{x \in X \,:\, \varphi(x) \le t\}$ is convex. Moreover*

(i) Given $\varphi : X \to \mathbb{R}\cup\{+\infty\}$, we set $\varphi^c(y, \lambda) := \sup\{\langle x, y\rangle :\ \varphi(x) < \lambda\}$ for any $(y, \lambda) \in Y \times \mathbb{R}$.

(ii) Given $\varphi : X \to \mathbb{R}\cup\{+\infty\}$, we set $\varphi^{\bar{c}}(y, \lambda) := \sup\{\langle x, y\rangle :\ \varphi(x) \le \lambda\}$ for any $(y, \lambda) \in Y \times \mathbb{R}$.

(iii) Given $\psi : Y \times \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$, we set $\psi^\gamma(x) := \sup_{y\in Y} \sup\{\lambda :\ \psi(y, \lambda) < \langle x, y\rangle\}$ for any $y \in Y$.

We refer to φ^c, $\varphi^{\bar{c}}$, ψ^γ as the *conjugate functions* of φ, φ, ψ and to $\varphi^{c\gamma}$, $\psi^{\gamma c}$ as the *biconjugate functions* of φ, ψ. The following proposition specifies some useful properties. (See Proposition 3.2 and Theorem 3.4 of [13] for the proof of (a) and (c), and Proposition 2.1 in [6] for (b).)

Proposition 2.2

(a) *Given $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, for any $\lambda \in \mathbb{R}$ $\varphi^c(\cdot, \lambda)$ is convex, proper or identically $-\infty$ and for any $y \in Y$, $\varphi^c(y, \cdot)$ is lower semicontinuous and nondecreasing.*

(b) *Given $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, for any $\lambda \in \mathbb{R}$ $\varphi^{\bar{c}}(\cdot, \lambda)$ is convex, proper or identically $-\infty$ and for any $y \in Y$, $\varphi^{\bar{c}}(y, \cdot)$ is upper semicontinuous and nondecreasing.*

(c) *Given $\psi : Y \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, ψ^γ is lower semicontinuous and level convex.*

(d) *$\varphi^c \le \psi$ if and only if $\varphi \ge \psi^\gamma$.*

The next theorems describe the class of functions on X (on $Y \times \mathbb{R}$) which are invariant with respect to the operator $\varphi \mapsto \varphi^{c\gamma}$ ($\psi \mapsto \psi^{\gamma c}$). In the sequel we will call *regular* these functions.

Theorem 2.3 (Theorem 3.4 in [13].) *Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$. Then $\varphi = \varphi^{c\gamma}$ if and only if φ is lower semicontinuous and level convex.*

Theorem 2.4 (Theorem 3.5 in [13].) *Let $\psi : Y \times \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$. Then $\psi = \psi^{\gamma c}$ if and only if*

(i) *for every $y \in Y$, $\psi(y, \cdot)$ is lower semicontinuous and nondecreasing;*

(ii) *for every $\lambda \in \mathbb{R}$ $\psi(\cdot, \lambda)$ is lower semicontinuous, sublinear, proper, or identically $-\infty$.*

Remark 2.5 From [13], pag. 333, we have that the following equalities hold:

$$\varphi^{c\gamma c} = \varphi^{c}, \;\; \psi^{\gamma} = \psi^{\gamma c\gamma}.$$

In the next propositions we describe some properties of the relation between the operators $\varphi \mapsto \varphi^c$ and $\varphi \mapsto \varphi^{\bar{c}}$ that will be very useful in what follows.

Proposition 2.6 *Let $\varphi : X \to \mathbb{R}\cup\{+\infty\}$. If $(y,\lambda) \in Y\times\mathbb{R}$ are such that $\varphi^c(y,\lambda) \in \mathbb{R}\cup\{-\infty\}$ then*

$$\varphi^c(y,\lambda) = \bigvee_{r<\lambda,\, r\in\mathbf{Q}} \varphi^{\bar{c}}(y,r). \tag{2.1}$$

Moreover,

$$\varphi^{c\gamma} = \varphi^{\bar{c}\gamma}. \tag{2.2}$$

Proof. The inequality $\varphi^c(y,\lambda) \geq \bigvee_{r<\lambda,\, r\in\mathbf{Q}} \varphi^{\bar{c}}(y,r)$ follows from the definitions of φ^c and $\varphi^{\bar{c}}$. For the opposite inequality, without loss of generality, we may assume $\varphi^c(y,\lambda) > -\infty$. By hypothesis $\varphi^c(y,\lambda) < +\infty$. So, fixed $\varepsilon > 0$, there exists $x_\varepsilon \in X$ such that $\varphi(x_\varepsilon) < \lambda$ and $\varphi^c(y,\lambda) \leq \langle x_\varepsilon, y\rangle + \varepsilon$. In particular, there exists $r \in \mathbf{Q}$ such that $\varphi(x_\varepsilon) \leq r < \lambda$, thus $\varphi^{\bar{c}}(y,r) \geq \langle x_\varepsilon, y\rangle \geq \varphi^c(y,\lambda) - \varepsilon$, which implies $\bigvee_{r<\lambda,\, r\in\mathbf{Q}} \varphi^{\bar{c}}(y,r) \geq \varphi^c(y,\lambda) - \varepsilon$. Since ε is arbitrary, (2.1) follows .
Since $\varphi^{\bar{c}} \geq \varphi^c$, by properties of the operator $\psi \mapsto \psi^\gamma$, we obtain

$$\varphi^{\bar{c}\gamma} \leq \varphi^{c\gamma}.$$

For the opposite inequality, let $x \in X$ and, without loss of generality, assume $\varphi^{c\gamma} > -\infty$.
If $\varphi^{c\gamma} < +\infty$, for every $\varepsilon > 0$ there exists $\lambda_\varepsilon \in \mathbb{R}$ and $y_\varepsilon \in Y$ such that $\varphi^c(y_\varepsilon, \lambda_\varepsilon) < \langle y_\varepsilon, x\rangle$ and $\varphi^{c\gamma}(x) < \lambda_\varepsilon + \varepsilon$. So, for every $\lambda < \lambda_\varepsilon$, $\varphi^{\bar{c}}(y_\varepsilon, \lambda) \leq \varphi^c(y_\varepsilon, \lambda_\varepsilon) < \langle y_\varepsilon, x\rangle$, which implies for every $\lambda < \lambda_\varepsilon$, $\varphi^{\bar{c}\gamma}(x) > \lambda$. In particular $\varphi^{\bar{c}\gamma}(x) \geq \lambda_\varepsilon > \varphi^{c\gamma}(x) - \varepsilon$ and since ε is arbitrary, we obtain (2.2).
If $\varphi^{c\gamma}(x) = +\infty$, then there exist two sequences $(\lambda_n) \subset \mathbb{N}$ and $(y_n) \subset Y$ such that $\lambda_n \to \infty$ and $\varphi^c(p_n, \lambda_n) < \langle p_n, x\rangle$ for all $n \in \mathbb{N}$. By (2.1), for each $n \in \mathbb{N}$ there exists $\rho_n \in \mathbb{R}$, $\rho_n < \lambda_n$ such that $\rho_n \to \infty$ and $\varphi^{\bar{c}}(p_n, \rho_n) \leq \varphi^c(p_n, \lambda_n) < \langle p_n, x\rangle$. So $\varphi^{\bar{c}\gamma}(x) = +\infty$, which implies again (2.2), and concludes the proof. $\square$

Proposition 2.7 *Let Y be a separable Banach space and let X its topological dual space. Let $\varphi : X \to \mathbb{R}\cup\{+\infty\}$ and assume that $\varphi^c : Y\times\mathbb{R} \to \mathbb{R}\cup\{-\infty\}$ and $\varphi^{\bar{c}} : Y\times\mathbb{R} \to \mathbb{R}\cup\{-\infty\}$. Then there exists a countable $\Lambda \subset \mathbb{R}$ such that*

$$\varphi^c(y,\lambda) = \varphi^{\bar{c}}(y,\lambda)$$

for every $y \in Y$ and for every $\lambda \in \mathbb{R}\setminus\Lambda$ (in particular it holds for every (y,λ) which is a continuity point for the function φ^c (or for $\varphi^{\bar{c}}$)).

Proof. Setting $\bar{\lambda} := \inf_X \varphi$, we have

$$\lambda > \bar{\lambda} \;\Rightarrow\; \varphi^c(\cdot,\lambda), \varphi^{\bar{c}}(\cdot,\lambda) \in \mathbb{R} \quad \text{and} \quad \lambda < \bar{\lambda} \;\Rightarrow\; \varphi^c(\cdot,\lambda) = \varphi^{\bar{c}}(\cdot,\lambda) = -\infty.$$

Define $\Lambda_0 := \{\lambda \in \mathbb{R} : \lambda > \bar{\lambda}\}$. If $\lambda \in \Lambda_0$, then $\varphi^c(\cdot, \lambda), \varphi^{\bar{c}}(\cdot, \lambda)$ are real convex functions, so they are continuous. Let $\mathcal{D} \subset Y$, $\mathcal{D} = (y_n)$ dense in Y. Since $\varphi^c(y_n, \cdot)$ and $\varphi^{\bar{c}}(y_n, \cdot)$ are nondecreasing for every $n \in \mathbb{N}$, there exists a countable $\Lambda_n \subset \Lambda_0$ such that $\varphi^c(y_n, \cdot)$ and $\varphi^{\bar{c}}(y_n, \cdot)$ are continuous on $\Lambda_0 \setminus \Lambda_n$. Thus, for every $n \in \mathbb{N}$, $\varphi^c(y_n, \cdot)$ and $\varphi^{\bar{c}}(y_n, \cdot)$ are continuous on $\bar{\Lambda} := \Lambda_0 \setminus \left(\bigcup_{n=1}^{\infty} \Lambda_n \right)$. Let $y \in \mathcal{D}$, $\lambda \in \bar{\Lambda}$ and $\lambda_n \to \lambda$. Then we have

$$\varphi^{\bar{c}}(y, \lambda) = \lim_{n\to\infty} \varphi^{\bar{c}}(y, \lambda_n) \leq \lim_{n\to\infty} \varphi^c(y, \lambda_n + \frac{1}{n}) = \varphi^c(y, \lambda). \tag{2.3}$$

Since $\mathcal{D}$ is dense in Y and $\varphi^c(\cdot, \lambda), \varphi^{\bar{c}}(\cdot, \lambda)$ are continuous for every $\lambda \in \bar{\Lambda}$, (2.3) gives us

$$\varphi^c(y, \lambda) = \varphi^{\bar{c}}(y, \lambda) \quad \forall y \in Y, \, \forall \lambda \in \bar{\Lambda}.$$

It is sufficient to choose $\Lambda := \{\bar{\lambda}\} \cup \left(\bigcup_{n=1}^{\infty} \Lambda_n \right)$ to conclude. $\square$

Notation. In the sequel, when X is a Banach space, we will denote its norm by $\| \cdot \|_X$.

Remark 2.8 Observe that in the previous theorem, if $\lim\limits_{\|x\|_X \to \infty} \varphi(x) = +\infty$ (i.e. for every $\lambda \in \mathbb{R}$ $E_\lambda := \{x \in X : \varphi(x) \leq \lambda\}$ is bounded in X), then $\varphi^{\bar{c}} : Y \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ and $\varphi^c : Y \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$.
In fact, since $\varphi^c(y, \lambda) \leq \varphi^{\bar{c}}(y, \lambda)$, it is sufficient to prove the property for $\varphi^{\bar{c}}(y, \lambda)$. If $\varphi^{\bar{c}}(y, \lambda) > -\infty$, then the level set $E_\lambda \neq \emptyset$ and, by coercivity condition, there exists K_λ such that $\| x \|_X \leq K_\lambda$ for every $x \in E_\lambda$. This implies $|\varphi^{\bar{c}}(y, \lambda)| \leq K_\lambda \| y \|_Y$.
Moreover, it holds a vice versa: if, for every $\lambda \in \mathbb{R}$, there exists $K_\lambda > 0$ such that $\varphi^c(y, \lambda) \leq K_\lambda \| y \|_Y$, then φ is coercive. In fact, fix $\lambda \in \mathbb{R}$, $\varepsilon > 0$ and let $x \in E_\lambda$. Then for every $y \in Y$, $\| y \|_Y = 1$, we have $K_{\lambda+\varepsilon} \geq \varphi^c(y, \lambda + \varepsilon) \geq \langle x, y \rangle$, which implies $K_{\lambda+\varepsilon} \geq \| x \|_X$. The same argument shows that, if $\phi : Y \times \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is such that for every $\lambda \in \mathbb{R}$, there exists K_λ such that $\phi(y, \lambda) \leq K_\lambda \| y \|_Y$, then ϕ^γ is coercive. In fact, fix $\lambda \in \mathbb{R}$ and $\varepsilon > 0$. Let $x \in G_\lambda := \{x \in X : \phi^\gamma(x) \leq \lambda\}$. Then for every $y \in Y$, $\| y \|_Y = 1$ we have $K_{\lambda+\varepsilon} \geq \phi(y, \lambda + \varepsilon) \geq \phi^{\gamma c}(y, \lambda + \varepsilon) \geq \langle x, y \rangle$, which implies $K_{\lambda+\varepsilon} \geq \| x \|_X$.

For the sake of completeness we begin our description of the Γ-convergence results giving the characterization of Γ-limit we will use in the sequel. (For the general definition and the properties of Γ-limits see [10].)

First, let us recall a sequential characterization of Γ-liminf, Γ-limsup and Γ-limit in a topological space X satisfying the first axiom of countability.

Proposition 2.9 (Proposition 8.1 in [10].) *Let $\varphi_n : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a sequence of functionals. The functional $\varphi'(x) = \Gamma(X) \liminf\limits_{n\to\infty} \varphi_n(x)$ [respectively $\varphi''(x) = \Gamma(X) \limsup\limits_{n\to\infty} \varphi_n(x)$] is characterized by the following properties:*

(a) *For every $x \in X$ and for every sequence (x_n) converging to x in X it is*

$$\varphi'(x) \leq \liminf_{n\to\infty} \varphi_n(x_n). \quad [\textit{respectively } \varphi''(x) \leq \limsup_{n\to\infty} \varphi_n(x_n)]$$

(b) *For every $x \in X$ there exists a sequence (x_n) converging to x in X such that*

$$\varphi'(x) = \liminf_{n\to\infty} \varphi_n(x_n). \quad [\textit{respectively } \varphi''(x) = \limsup_{n\to\infty} \varphi_n(x_n)]$$

The functional $\varphi(x) = \Gamma(X) \lim_{n\to\infty} \varphi_n(x)$ *is characterized by the following properties:*

(c) *For every* $x \in X$ *and for every sequence* (x_n) *converging to* x *in* X *it is*

$$\varphi(x) \le \liminf_{n\to\infty} \varphi_n(x_n).$$

(d) *For every* $x \in X$ *there exists a sequence* (x_n) *converging to* x *in* X *such that*

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x_n).$$

Since our result will apply to $X = L^\infty(\Omega)$ endowed with its weak* topology, we need a sequential characterization of Γ- limit when X is the dual of a separable Banach space Y. To obtain this characterization and our general result we need to assume the following hypothesis.

(Ho) Let Y be a separable Banach space and X its dual space endowed with its weak* topology. Let $\varphi_n : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ for every $n \in \mathbb{N}$. There exists $\Phi : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ such that:

$$\lim_{\|x\|_X \to +\infty} \Phi(x) = +\infty,$$

and $\varphi_n \ge \Phi$ for every $n \in \mathbb{N}$.

In the following Γ-limits we will denote the space Y endowed with the strong topology by sY and the space X endowed by the weak* topology by w^*X.

Proposition 2.10 *Assume* **(Ho)**. *Then* (φ_n) Γ*-converges to* φ *in the weak* topology of* X *if and only if*

(i) *for every* $x \in X$ *and for every sequence* (x_n) *converging weakly* to* $x \in X$, *it is*

$$\varphi(x) \le \liminf_{n\to\infty} \varphi_n(x_n);$$

(ii) *for every* $x \in X$ *there exists a sequence* (x_n) *converging weakly* to* $x \in X$ *such that*

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x_n).$$

The proof is analogous to the one of Proposition 8.1 in [10].

Moreover, the following compactness theorem will be useful (analogous to Corollary 8.12 in [10]).

Proposition 2.11 *Assume* **(Ho)**. *Then there exists a subsequence of* (φ_n) *which* Γ*-converges in the weak* topology of X.*

Let us now prove two results on the behavior of the Γ-limit of a sequence of functionals with respect to the duality previously studied.

Theorem 2.12 *Assume* **(Ho)**. *If* (φ_n) Γ*-converges to a level convex function* φ *with respect to the weak* topology of* X, *then*

$$\varphi(x) = \left(\Gamma(sY) \liminf_{n\to\infty} \varphi_n^c\right)^\gamma (x) = \left(\Gamma(sY) \limsup_{n\to\infty} \varphi_n^{\bar{c}}\right)^\gamma (x). \tag{2.4}$$

Proof. We first observe that to obtain (2.4) it is sufficient to prove that for every $y \in Y$ and for every $\lambda \in \mathbb{R}$ the following inequalities hold:

$$\varphi^c(y,\lambda) \leq \Gamma(sY) \liminf_{n\to\infty} \varphi_n^c(y,\lambda), \tag{2.5}$$

$$\Gamma(sY) \limsup_{n\to\infty} \varphi_n^{\bar{c}}(y,\lambda) \leq \varphi^{\bar{c}}(y,\lambda). \tag{2.6}$$

Indeed, inequalities (2.5), (2.6) and the fact that we always have $\varphi_n^c \leq \varphi_n^{\bar{c}}$, imply that for every $y \in Y$ and for every $\lambda \in \mathbb{R}$ we have

$$\varphi^c(y,\lambda) \leq \Gamma(sY) \liminf_{n\to\infty} \varphi_n^c(y,\lambda) \leq \Gamma(sY) \limsup_{n\to\infty} \varphi_n^{\bar{c}}(y,\lambda) \leq \varphi^{\bar{c}}(y,\lambda) \tag{2.7}$$

and so, by definition of the operator $\varphi \mapsto \varphi^\gamma$, for each $x \in X$, we have

$$\varphi^{c\gamma}(x) \geq (\Gamma(sY) \liminf_{n\to\infty} \varphi_n^c(y,\lambda))^\gamma(x) \geq (\Gamma(sY) \limsup_{n\to\infty} \varphi_n^{\bar{c}}(y,\lambda))^\gamma(x) \geq \varphi^{\bar{c}\gamma}(x). \tag{2.8}$$

Now, by (2.2) we have $\varphi^{c\gamma}(x) = \varphi^{\bar{c}\gamma}(x)$. By hypotheses, φ is a level convex and, since it is a Γ-limit, φ is a w^*X l.s.c. function. So, applying Theorem 2.3, it follows $\varphi(x) = \varphi^{c\gamma}(x)(= \varphi^{\bar{c}\gamma}(x))$. This fact and (2.8) imply (2.4).
In order to prove (2.5), fix $\lambda \in \mathbb{R}$ and $y \in Y$. Without loss of generality, we can suppose $\varphi^c(y,\lambda) \in \mathbb{R}$. Let (y_n) be a sequence converging strongly to y in Y. Now, by definition of $\varphi^c(y,\lambda)$, for every $\varepsilon > 0$ there exists $x^\varepsilon \in X$ such that

$$\varphi^c(y,\lambda) - \varepsilon \leq \langle x^\varepsilon, y\rangle \qquad \text{and} \qquad \varphi(x^\varepsilon) < \lambda. \tag{2.9}$$

Moreover since $\varphi(x) = \Gamma(w^*X) \lim_{n\to\infty} \varphi_n(x)$, by Proposition 2.10, there exists a sequence (x_n^ε) which tends to x^ε weakly* in X and $\varphi(x^\varepsilon) = \lim_{n\to\infty} \varphi_n(x_n^\varepsilon)$. By (2.9), this implies that there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $\varphi_n(x_n^\varepsilon) < \lambda$. Thus, by definition of φ_n^c,

$$\varphi_n^c(y_n,\lambda) \geq \langle y_n, x_n^\varepsilon\rangle.$$

Using also (2.9) we obtain

$$\liminf_{n\to\infty} \varphi_n^c(y_n,\lambda) \geq \liminf_{n\to\infty} \langle y_n, x_n^\varepsilon\rangle = \langle y, x^\varepsilon\rangle \geq \varphi^c(y,\lambda) - \varepsilon.$$

Letting $\varepsilon \to 0$ we have

$$\varphi^c(y,\lambda) \leq \liminf_{n\to\infty} \varphi_n^c(y_n,\lambda). \tag{2.10}$$

Thanks to the sequential characterization of Γ-liminf seen in Proposition 2.9, (2.5) follows. To prove (2.6), by the sequential characterization of Γ-limsup of Proposition 2.9, it will be sufficient to prove that, for every $y \in Y$ and for every $\lambda \in \mathbb{R}$,

$$\varphi^{\bar{c}}(p,\lambda) \geq \lim_{n\to\infty} \varphi_n^{\bar{c}}(y,\lambda). \tag{2.11}$$

Fix $\lambda \in \mathbb{R}$ and $y \in Y$. We observe that, if $\varphi^{\bar{c}}(y,\lambda) = -\infty$, then $\varphi(x) > \lambda$ for every $x \in L^\infty$ and so $\min\{\varphi(x) \ : \ x \in X\} > \lambda$. By properties of Γ-convergence (see Theorem 7.4 in [10]), there exists $n_0 \in \mathbb{N}$ such that $\inf\{\varphi_n(x) \ : \ x \in X\} > \lambda$ for every $n \geq n_0$. Thus $\varphi_n^{\bar{c}}(\cdot,\lambda) = -\infty$ for every $n \geq n_0$ and, in this case, (2.6) is satisfied. Now, without loss of generality, we can assume $\varphi^{\bar{c}}(y,\lambda) \in \mathbb{R}$. By definition of $\varphi_n^{\bar{c}}(y,\lambda)$, for every $n \in \mathbb{N}$ there exists a sequence $(x_k^n)_k$ such that

$$\varphi_n^{\bar{c}}(y,\lambda) = \lim_{k\to\infty} \langle x_n^k, y\rangle \qquad \text{and} \qquad \varphi(x_k^n) \leq \lambda. \tag{2.12}$$

Thus, by hypothesis of coercivity, the sequence $((x_k^n)_k)_n$ is bounded in X which is the dual of the separable space Y. Then there exists a subsequence (which we still denote by $(x_k^n)_k)_n)$ converging weakly* to a function $x \in X$. So

$$\langle x, y \rangle = \lim_{n\to\infty} \lim_{k\to\infty} \langle x_n^k, y \rangle. \tag{2.13}$$

Moreover, since $\varphi(x) = \Gamma(w^*X) \lim_{n\to\infty} \varphi_n(x)$, we have (also by (2.12))

$$\varphi(x) \le \lim_{n\to\infty} \lim_{k\to\infty} f(x_k^n) \le \lambda$$

and this implies, by definition of $\varphi^{\bar{c}}(y,\lambda)$, that

$$\varphi^{\bar{c}}(y,\lambda) \ge \langle x, y \rangle = \lim_{n\to\infty} \varphi_n^{\bar{c}}(y,\lambda),$$

and this completes the proof of (2.11).

□

Remark 2.13 We can observe that, by Proposition 2.11, up to a subsequence, the Γ-limit φ exists. Moreover, if $(\varphi_n)_n$ are level convex, then φ is level convex too. In fact, fixed $x, w \in X$ and $\lambda \in (0,1)$, by Proposition 2.10 (ii), there exist $(x_n) \subset X$ weak* converging to x and $(w_n) \subset X$ weak* converging to w such that $\varphi(x) = \lim_{n\to\infty} \varphi_n(x_n)$ and $\varphi(w) = \lim_{n\to\infty} \varphi_n(w_n)$. So, by Proposition 2.10 (i),

$$\begin{aligned}\varphi(\lambda x + (1-\lambda)w) &\le \lim_{n\to\infty} \varphi_n(\lambda x_n + (1-\lambda)w_n) \le \\ &\le \lim_{n\to\infty} \varphi_n(x_n) \vee \lim_{n\to\infty} \varphi_n(x_n) = \\ &= \varphi(x) \vee \varphi(w).\end{aligned}$$

One of the main differences with the theory of Γ-convergence with respect to the classical Legendre duality is that we do not have an equality in (2.7). (See Theorem 3.7 in [2], and Theorem 3.2.4 in [3].) This is due to the different structure of the problem and we will give some examples in Section 5. On the other hand, we will prove that if we consider the Γ-limit with respect to both the y and λ variables, we can obtain equality. More precisely we have the following theorem.

Theorem 2.14 *Assume* **(Ho)**.*Then*

$$\left(\Gamma(w^*X) \lim_{n\to\infty} \varphi_n\right)^c (y,\lambda) = \Gamma(sY \times \mathbb{R}) \lim_{n\to\infty} \varphi_n^c(y,\lambda), \tag{2.14}$$

and

$$\left(\Gamma(w^*X) \lim_{n\to\infty} \varphi_n\right)^c (y,\lambda) = \Gamma(sY \times \mathbb{R}) \lim_{n\to\infty} \varphi_n^{\bar{c}}(y,\lambda). \tag{2.15}$$

Proof. Let us denote $\varphi(x) := \Gamma(w^*X) \lim_{n\to\infty} \varphi_n(x)$ and fix $(y,\lambda) \in Y \times \mathbb{R}$. We have, by repeating the proof of (2.5) and (2.6),

$$\varphi^c(y,\lambda) \le \Gamma(sY \times \mathbb{R}) \liminf_{n\to\infty} \varphi_n^c(y,\lambda), \tag{2.16}$$

and

$$\Gamma(sY \times \mathbb{R}) \limsup_{n\to\infty} \varphi_n^{\bar{c}}(y,\lambda) \le \varphi^{\bar{c}}(y,\lambda). \tag{2.17}$$

Thus

$$\begin{aligned}
\varphi^c(y,\lambda) &\le \Gamma(sY\times\mathbb{R})\liminf_{n\to\infty}\varphi_n^c(y,\lambda)\le\\
&\le \Gamma(sY\times\mathbb{R})\limsup_{n\to\infty}\varphi_n^c(y,\lambda)\le\\
&\le \Gamma(sY\times\mathbb{R})\limsup_{n\to\infty}\varphi_n^{\bar c}(y,\lambda)\le\\
&\le \varphi^{\bar c}(y,\lambda). \qquad (2.18)
\end{aligned}$$

Fix now $\bar\lambda\in\mathbb{R}$, since (2.18) holds for every $\lambda<\bar\lambda$, we have,

$$\bigvee_{\lambda<\bar\lambda}\varphi^c(y,\lambda)\le\bigvee_{\lambda<\bar\lambda}\Gamma(sY\times\mathbb{R})\liminf_{n\to\infty}\varphi_n^c(y,\lambda)\le$$

$$\le\bigvee_{\lambda<\bar\lambda}\Gamma(sY\times\mathbb{R})\limsup_{n\to\infty}\varphi_n^c(y,\lambda)\le\bigvee_{\lambda<\bar\lambda}\varphi^{\bar c}(y,\lambda).$$

Applying (2.1) in Proposition 2.6 and recalling that φ^c, the Γ-limsup and the Γ-liminf are lower semicontinuous functionals w.r.t. λ we have

$$\varphi^c(y,\bar\lambda)\le\Gamma(sY\times\mathbb{R})\liminf_{n\to\infty}\varphi_n^c(y,\bar\lambda)\le\Gamma(sY\times\mathbb{R})\limsup_{n\to\infty}\varphi_n^c(y,\bar\lambda)\le\varphi^c(y,\bar\lambda),$$

which, since $\bar\lambda$ is arbitrary, implies (2.14). Analogously, (2.16) and (2.17) give

$$\begin{aligned}
\varphi^c(y,\lambda)&\le\Gamma(sY\times\mathbb{R})\liminf_{n\to\infty}\varphi_n^c(y,\lambda)\le\\
&\le\Gamma(sY\times\mathbb{R})\liminf_{n\to\infty}\varphi_n^{\bar c}(y,\lambda)\le\\
&\le\Gamma(sY\times\mathbb{R})\limsup_{n\to\infty}\varphi_n^{\bar c}(y,\lambda)\le\varphi^{\bar c}(y,\lambda).
\end{aligned}$$

Again fix $\bar\lambda\in\mathbb{R}$, we obtain

$$\bigvee_{\lambda<\bar\lambda}\varphi^c(y,\lambda)\le\bigvee_{\lambda<\bar\lambda}\Gamma(sY\times\mathbb{R})\liminf_{n\to\infty}\varphi_n^{\bar c}(y,\lambda)\le$$

$$\le\bigvee_{\lambda<\bar\lambda}\Gamma(sY\times\mathbb{R})\limsup_{n\to\infty}\varphi_n^{\bar c}(y,\lambda)\le\bigvee_{\lambda<\bar\lambda}\varphi^{\bar c}(y,\lambda),$$

which implies (again by (2.1) and the lower semicontinuity w.r.t. λ of φ^c, the Γ -limsup and the Γ-liminf)

$$\begin{aligned}
\varphi^c(y,\bar\lambda)&\le\Gamma(sY\times\mathbb{R})\liminf_{n\to\infty}\varphi_n^{\bar c}(y,\bar\lambda)\le\\
&\le\Gamma(sY\times\mathbb{R})\limsup_{n\to\infty}\varphi_n^{\bar c}(y,\bar\lambda)\le\varphi^c(y,\bar\lambda),
\end{aligned}$$

and this proves (2.15). □

In Section 5, some counterexamples will show that the previous result is false if we consider $(\Gamma(w^*X)\lim_{n\to\infty}\varphi_n)^{\bar c}$ in (2.14) or (2.15). We end this Section by showing two theorems which precise how the operators $\varphi\mapsto\varphi^c$ and $\phi\mapsto\phi^\gamma$ behave with respect to Γ-convergence in the class of the functions φ (ϕ) which are regular, i.e. $\varphi=\varphi^{c\gamma}$ ($\phi=\phi^{\gamma c}$), whose characterization is given by Theorem 2.3 (Theorem 2.4). The following results are the analogous of Theorem 3.11 of [2].

Theorem 2.15 *Let Y be a separable Banach space and X its dual space endowed with its weak* topology. Let $\varphi_n,\ \varphi : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be regular functions satisfying* **(Ho)**. *Then*

$$\varphi = \Gamma(w^*X) \lim_{n\to\infty} \varphi_n \quad \textit{if and only if} \quad \varphi^c = \Gamma(sY \times \mathbb{R}) \lim_{n\to\infty} \varphi_n^c. \tag{2.19}$$

Proof. The necessary condition is assured by Theorem 2.14. In order to prove the sufficient condition, let us observe that, by Remark 2.13, there exist a subsequence $(\varphi_{n_k})_k$ and a regular function $g : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$g = \Gamma(w^*X) \lim_{k\to\infty} \varphi_{n_k}.$$

By necessary condition

$$g^c = \Gamma(sY \times \mathbb{R}) \lim_{k\to\infty} \varphi_{n_k}^c = \varphi^c.$$

So, also by the regularity of φ, $g = g^{c\gamma} = \varphi^{c\gamma} = \varphi$. Thus

$$\varphi = \Gamma(w^*X) \lim_{n\to\infty} \varphi_{n_k}$$

for every subsequence $(\varphi_{n_k})_k$ and the thesis follows. □

Theorem 2.16 *Let Y be a separable Banach space and X its dual space endowed with its weak* topology. Let $\phi_n, \phi : Y \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be regular functions such that for every $\lambda \in \mathbb{R}$ there exists $K_\lambda > 0$ such that*

$$\phi_n(y, \lambda) \leq K_\lambda \parallel y \parallel_Y \tag{2.20}$$

for every $y \in Y$ and for every $n \in \mathbb{N}$. Then

$$\phi = \Gamma(sY \times \mathbb{R}) \lim_{n\to\infty} \phi_n \quad \textit{if and only if} \quad \phi^\gamma = \Gamma(w^*X) \lim_{n\to\infty} \phi_n^\gamma. \tag{2.21}$$

Proof. Thanks to hypothesis (2.20), following the proof of the vice versa in Remark 2.8, we obtain that ϕ_n^γ satisfies the following property:
For every $\lambda \in \mathbb{R}$ there exists $K_\lambda \in \mathbb{R}$ such that if $\inf_{n\in\mathbb{N}} \phi_n^\gamma(x) \leq \lambda$, then $\parallel x \parallel_X \leq K_\lambda$.
This implies that ϕ_n^γ satisfies **(Ho)**. In fact, it is sufficient to define $\Phi(x) = \inf\{\mu \in \mathbb{R} \ : \ x \in H_\lambda \ \forall \lambda > \mu\}$ where $H_\lambda = \{x \in X \ : \ \inf_{n\in\mathbb{N}} \phi_n^\gamma(x) \leq \lambda\}$. If $n \in \mathbb{N}$ and $\phi_n^\gamma(x) \leq \mu$, then $x \in H_\lambda$ for every $\lambda > \mu$, so $\Phi(x) \leq \mu$, which implies $\Phi(x) \leq \phi_n^\gamma(x)$. Moreover, Φ is coercive: if $\Phi(x) \leq \mu$, then $\parallel x \parallel_X \leq K_\lambda$.
Thus the sufficient condition follows from Theorem 2.15. Indeed, if $\phi^\gamma = \Gamma(w^*X) \lim_{n\to\infty} \phi_n^\gamma$, then $\phi^{\gamma c} = \Gamma(sY \times \mathbb{R}) \lim_{n\to\infty} \phi_n^{\gamma c}$ and, by regularity of ϕ, ϕ_n, $\phi = \Gamma(sY \times \mathbb{R}) \lim_{n\to\infty} \phi_n$.
To prove the necessary condition, let us observe that, since ϕ_n^γ is a sequence of regular function, by Remark 2.13, there exist a subsequence $(\phi_{n_k}^\gamma)_k$ and a regular function $g : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$g = \Gamma(w^*X) \lim_{k\to\infty} \phi_{n_k}^\gamma$$

where the regularity of g follows from the regularity of the sequence (ϕ_n^γ). By Theorem 2.15,

$$g^c = \Gamma(sY \times \mathbb{R}) \lim_{k\to\infty} \phi_{n_k}^{\gamma c} = \Gamma(sY \times \mathbb{R}) \lim_{k\to\infty} \phi_{n_k} = \phi.$$

This implies $g = g^{c\gamma} = \phi^\gamma$. Thus

$$\phi^\gamma = \Gamma(w^*X) \lim_{n\to\infty} \phi_n^\gamma$$

for every subsequence $(\phi_{n_k}^\gamma)_k$ and the thesis follows.

□

3 Representation results

In this section we prove our representation results. Let us describe first the supremal functional we are going to work with. Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote for brevity by L^p the classical Lebesgue space $L^p(\Omega; \mathbb{R}^N)$ $(1 \leq p \leq +\infty)$, by $\mathcal{L}(A)$ the Lebesgue measure of a subset A of $\mathbb{R}^N$ and by $\mathcal{B}_N$ the Borel σ-field of $\mathbb{R}^N$.
In the sequel we will work with *supremal functional* on L^∞:

$$F(u) = \operatorname*{ess\,sup}_{x \in \Omega} f(x, u(x)),$$

and *integral functional* on L^1,

$$F(u) = \int_\Omega f(x, u(x))dx.$$

The function f, which represents the functional, will be said *supremand* (respectively *integrand*) if represents a supremal (respectively integral) functional. We will give the following definitions.

Definition 3.1 *A function $f : \Omega \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty, -\infty\}$ is said to be*

(a) a supremand *(respectively* integrand*) if f is $\mathcal{B}_n \otimes \mathcal{B}_N$-measurable;*

(b) a normal supremand *(respectively* normal integrand*) if f is $\mathcal{B}_n \otimes \mathcal{B}_N$-measurable and $f(x, \cdot)$ is lower semicontinuous (l.c.s. for short) on $\mathbb{R}^N$ for a. e. $x \in \Omega$;*

(c) a level convex normal supremand *if f is a normal supremand such that, for a. e. $x \in \Omega$, $f(x, \cdot)$ is level convex, i.e. for every $t \in \mathbb{R}$ the level set $\{s \in \mathbb{R}^N \ : \ f(x, s) \leq t\}$ is convex;*

(d) a convex normal integrand *if f is a normal integrand such that, for a. e. $x \in \Omega$, $f(x, \cdot)$ is convex.*

Our aim is now to represent the conjugate of a supremal functional F given by a normal supremand f through the corresponding conjugate function. In all the paper we will write $f^c(x, p, \lambda)$ $(f^{\bar{c}}(x, p, \lambda))$ to indicate the conjugate of the function $f(x, \cdot)$ at x fixed. See Definition 2.1. In the sequel we will assume that f fulfills the following condition:

(Hc) $f : \Omega \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is a normal supremand and there exists a measurable function $\phi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ such that

$$\lim_{|s| \to \infty} \phi(s) = +\infty \tag{3.1}$$

and

$$f(x, s) \geq \phi(s) \tag{3.2}$$

for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}^N$.

Remark 3.2 Note that, by Remark 2.8 the coercivity assumption **(Hc)** implies that for each $(x, p, \lambda) \in \Omega \times \mathbb{R}^N \times \mathbb{R}$ we have $f^c(x, p, \lambda) \in \mathbb{R} \cup \{-\infty\}$ and $f^{\bar{c}}(x, p, \lambda) \in \mathbb{R} \cup \{-\infty\}$. Thus, assuming **(Hc)**, we will always be in position to apply Proposition 2.6 and Proposition 2.7.

The result is that the conjugate of a supremal functional is an integral functional. To obtain this, we need the following measurability result for f^c and $f^{\bar{c}}$.

Proposition 3.3 *Let $f : \Omega \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a normal supremand which satisfies* **(Hc)**. *Then,*

(a) *For every* $\lambda \in \mathbb{R}$ *there exists a measurable set* $\Omega_\lambda \subset \Omega$ *such that*

$$f^c(\cdot,\cdot,\lambda) = -\infty \text{ on } \Omega \setminus \Omega_\lambda$$

and $f^c(\cdot,\cdot,\lambda)$ *is a convex normal integrand on* $\Omega_\lambda \times \mathbb{R}^N$.

(b) *For every* $\lambda \in \mathbb{R}$ *there exists a measurable set* $\Omega'_\lambda \subset \Omega$ *such that*

$$f^{\bar{c}}(\cdot,\cdot,\lambda) = -\infty \text{ on } \Omega \setminus \Omega'_\lambda$$

and $f^{\bar{c}}(\cdot,\cdot,\lambda)$ *is a convex normal integrand on* $\Omega'_\lambda \times \mathbb{R}^N$.

Proof. Fix $\lambda \in \mathbb{R}$. To prove (a) we begin to observe that, by Proposition 2.6 of [14], $f^c(\cdot,w,\lambda)$ is $\mathcal{B}_n$-measurable for every $w \in \mathbb{R}^N$ and, by Proposition 2.2 (a), $f^c(x,\cdot,\lambda)$ is convex for every $x \in \Omega$. So it is sufficient to define $\Omega_\lambda := \{x \in \Omega \; : \text{ there exists } z \in \mathbb{R}^N \text{such that} f(x,z) < \lambda\}$. It is a measurable set and, by Remark 3.2, $f^c(x,\cdot,\lambda)$ is a finite convex function for every $x \in \Omega_\lambda$. Moreover, by Proposition 2.2, $f^c(\cdot,\cdot,\lambda)$ is identically $=-\infty$ on $\Omega \setminus \Omega_\lambda$. Since $f^c(x,\cdot,\lambda)$ is a continuous function for every $x \in \Omega_\lambda$, by Proposition 1.1 in [11], pag. 234, $f(\cdot,\cdot,\lambda)$ is a normal integrand on $\Omega_\lambda \times \mathbb{R}^N$. In the same way we can prove (b). □

The representation results are the following.

Theorem 3.4 *Let us consider the supremal functional*

$$F(u) := \operatorname*{ess\,sup}_{x\in\Omega} f(x,u(x))$$

for every $u \in L^\infty(\Omega)$, *and assume that* f *fulfills hypothesis* **(Hc)**. *Then*

$$F^{\bar{c}}(p,\lambda) = \int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx \tag{3.3}$$

for every $\lambda \in \mathbb{R}$ *and for every* $p \in L^1$.

Corollary 3.5 *Under the assumptions of Theorem 3.4, for every* $p \in L^1$ *and* $\lambda \in \mathbb{R}$, *if* $F^c(p,\lambda) > -\infty$ *then*

$$F^c(p,\lambda) = \int_\Omega f^c(x,p(x),\lambda)dx.$$

We first need a Lemma.

Lemma 3.6 *Under the assumptions of Theorem 3.4 for every* $p \in L^1$ *and* $\lambda \in \mathbb{R}$, *we have*

(a) $F^{\bar{c}}(p,\lambda) \in \mathbb{R} \cup \{-\infty\}$, $F^c(p,\lambda) \in \mathbb{R} \cup \{-\infty\}$.

(b) $\int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx \in \mathbb{R} \cup \{-\infty\}$, $\int_\Omega f^c(x,p(x),\lambda)dx \in \mathbb{R} \cup \{-\infty\}$.

(c) $F^{\bar{c}}(p,\lambda) = -\infty$ *if and only if* $\int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx = -\infty$.

(d) *If* $\int_\Omega f^c(x,p(x),\lambda)dx = -\infty$ *then* $F^c(p,\lambda) = -\infty$. *Moreover*

$$\sup\{\lambda \in \mathbb{R} : \int_\Omega f^c(x,\cdot,\lambda)dx = -\infty\} = \sup\{\lambda \in \mathbb{R} : \; F^c(\cdot,\lambda) = -\infty\} = \inf_{u\in L^\infty} F(u). \tag{3.4}$$

Remark 3.7 Let us observe that in general the implication

$$F^c(p,\lambda) = -\infty \Rightarrow \int_\Omega f^c(x,p(x),\lambda)dx = -\infty$$

is false. For example, let $\Omega = (-1,1)$ and let $f(x,s) : \Omega\times\mathbb{R} \to \mathbb{R}$ be defined by $f(x,s) := s(x+s)$. Observe that $f(x,s) \geq s^2 - |s|$ for every $x \in (-1,1)$ and so f satisfies hypothesis **(Hc)**. Let us compute f^c.
The equation $f(x,s) < \lambda$ admits a solution s if and only if $x^2 + 4\lambda > 0$. Let us consider three cases.
If $\lambda > 0$, then $f^c(x,\cdot,\lambda) \in \mathbb{R}$ and $\int_\Omega f^c(x,p(x),\lambda)dx \in \mathbb{R}$ for every $p \in L^1$. In fact, by coercivity, $|f^c(x,p,\lambda)| \leq k_\lambda |p|$ for every $p \in \mathbb{R}$ and for every $x \in (-1,1)$.
If $\lambda = 0$, then $f^c(x,\cdot,\lambda) \in \mathbb{R}$ for every $x \neq 0$ and $f^c(0,\cdot,\lambda) = -\infty$. Since $|f^c(x,p,0)| \leq |p|$ for every $p \in \mathbb{R}$ and for every $x \neq 0$, again $\int_\Omega f^c(x,p(x),0)dx \in \mathbb{R}$ for every $p \in L^1$.
If $\lambda < 0$, then $x^2 + 4\lambda > 0$ if and only if $|x| > 2\sqrt{-\lambda}$. So, if $2\sqrt{-\lambda} \geq 1$ (i.e. $\lambda \leq -\frac{1}{4}$), then $f^c(x,\cdot,\lambda) = -\infty$ for every $x \in (-1,1)$ and so $\int_\Omega f^c(x,p(x),\lambda)dx = -\infty$. If $\lambda > -\frac{1}{4}$, $f^c(x,\cdot,\lambda) = -\infty$ for every $x \in (-1,1)$ such that $|x| \leq 2\sqrt{-\lambda}$, which implies $\int_\Omega f^c(x,p(x),\lambda)dx = -\infty$.
Now, from (3.4), if $\lambda < 0$, then $\lambda \leq \inf_{L^\infty} F$ so $\inf_{L^\infty} F \geq 0$. Since $F(0) = 0$, we have that $\min_{L^\infty} F = 0$ which implies $F^c(\cdot,0) = -\infty < \int_\Omega f^c(x,\cdot,0)dx$.

Proof of Lemma 3.6. To prove (a), if $u \in L^\infty$ is such that $F(u) \leq \lambda$, then by (3.2), $\operatorname{ess\,sup}_{x\in\Omega} \phi(u(x)) \leq \lambda$ and so, by (3.1), there exists $M = M(\lambda)$ such that $||u||_{L^\infty(\Omega)} \leq M$. So, it is sufficient to apply to F the Remark 2.8.
To prove (b), since $\int_\Omega f^c(x,p(x),\lambda)dx \leq \int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx$, it is sufficient to show the conclusion for $\int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx$. Set $\Omega_\infty := \{x \in \Omega \;\; f^{\bar{c}}(x,p(x),\lambda) = -\infty\}$. If $x_0 \in \Omega \setminus \Omega_\infty$, then the level set $E_\lambda(x_0) := \{z \in \mathbb{R}^N : \; f(x_0,z) \leq \lambda\} \neq \emptyset$. If $z \in E_\lambda(x_0)$, then, by (3.2), $\phi(z) \leq \lambda$ and, by (3.1), there exists $M = M(\lambda)$ such that $|z| \leq M$. By definition of $f^{\bar{c}}$, this implies $f^{\bar{c}}(x_0,p(x_0),\lambda) \leq M|p(x_0)|$ for every $x_0 \in \Omega \setminus \Omega_\infty$. In particular $\int_{\Omega\setminus\Omega_\infty} f^{\bar{c}}(x,p(x),\lambda) < +\infty$. Then $\int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx = -\infty$ if $|\Omega_\infty| > 0$ and $\int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx \in \mathbb{R}$ otherwise.
In order to prove the necessary condition in (c), by contradiction, we suppose that $F^{\bar{c}}(p,\lambda) = -\infty$ and $\int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx \in \mathbb{R}$. Then there exists $N \subset \Omega$ such that $\mathcal{L}(N) = 0$ and for every $x \in \Omega \setminus N$, $f^{\bar{c}}(x,p(x),\lambda) \in \mathbb{R}$. Then for every $x \in \Omega \setminus N$, there exists $z(x) \in \mathbb{R}^N$ such that $f(x,z(x)) \leq \lambda$. By (3.2), $\phi(z(x)) \leq \lambda$ and so, by (3.1), there exists $M = M(\lambda)$ such that $|z(x)| \leq M$ for every $x \in \Omega \setminus N$. We want to select a function $u \in L^\infty$ such that $f(x,u(x)) \leq \lambda$. Set $B_M(0) := \{z \in \mathbb{R}^N \; : \; |z| \leq M\}$ and $h : \Omega \times B_M(0) \to \bar{\mathbb{R}}$ defined by $h(x,z) = (f(x,z)-\lambda)\vee 0$: it is a nonnegative measurable function, l.s.c. with respect to z and $h(x,z(x)) = 0$ for every $x \in \Omega \setminus N$. Thanks to Theorem 1.2 pag. 236 in [11], there exists $u : \Omega \to B_M(0)$ measurable such that $h(x,u(x)) = 0$ for every $x \in \Omega \setminus N$. In particular $u \in L^\infty$ and $F(u) \leq \lambda$. By definition (2.1)(ii) and by (a), it follows $F^{\bar{c}}(p,\lambda) \in \mathbb{R}$, in contradiction with the hypothesis.
To prove the sufficient condition in (c), by contradiction, we suppose that $F^{\bar{c}}(p,\lambda) \in \mathbb{R}$ and $\int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx = -\infty$. Then the level set $\{u \in L^\infty : \; F(u) \leq \lambda\} \neq \emptyset$. Let $u \in L^\infty$ such that $F(u) \leq \lambda$. Then there exists $N \subset \Omega$ such that $\mathcal{L}(N) = 0$ and $f(x,u(x)) \leq \lambda$ for every $x \in \Omega \setminus N$. By definition of $f^{\bar{c}}$, this implies $f^{\bar{c}}(x,p(x),\lambda) \geq p(x) \cdot u(x)$ for every $x \in \Omega \setminus N$. So $\int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx \geq \int_\Omega p(x) \cdot u(x)dx > -\infty$, in contradiction with the hypothesis.

With the same proof we obtain the implication

$$\int_\Omega f^c(x,p(x),\lambda)dx = -\infty \Rightarrow F^c(p,\lambda) = -\infty.$$

So

$$\{\lambda \in \mathbb{R} : \int_\Omega f^c(x,\cdot,\lambda)dx = -\infty\} \subseteq \{\lambda \in \mathbb{R} : F^c(\cdot,\lambda) = -\infty\} \subseteq$$
$$\subseteq \{\lambda \in \mathbb{R} : \lambda \leq \inf_{u\in L^\infty} F(u)\}.$$

In particular

$$\sup\{\lambda \in \mathbb{R} : \int_\Omega f^c(x,\cdot,\lambda)dx = -\infty\} \leq \inf_{u\in L^\infty} F(u).$$

For the opposite inequality, it is sufficient to observe that if

$$\sup\{\lambda \in \mathbb{R} : \int_\Omega f^c(x,\cdot,\lambda)dx = -\infty\} < \lambda_0 < \inf_{u\in L^\infty} F(u),$$

then $F^{\bar{c}}(\cdot,\lambda_0) = -\infty$ and so, by property (c) of Lemma 3.6, $\int_\Omega f^{\bar{c}}(x,\cdot,\lambda_0)dx = -\infty$ which implies $\int_\Omega f^c(x,\cdot,\lambda_0)dx = -\infty$, in contradiction with the assumption on λ_0. $\square$

Proof of Theorem 3.4. Let $\lambda \in \mathbb{R}$ and fix $p \in L^1$. Thanks to Lemma 3.6, we can assume, without loss of generality, that $F^{\bar{c}}(p,\lambda) \in \mathbb{R}$ and $\int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx \in \mathbb{R}$.
Then, by definition of $F^{\bar{c}}$, for every $\varepsilon > 0$ there exists $u_\varepsilon \in L^\infty$ such that $F(u_\varepsilon) \leq \lambda$ and $F^{\bar{c}}(p,\lambda) \leq \int_\Omega u_\varepsilon(x)\cdot p(x)dx + \varepsilon$. So there exists $N \subset \Omega$ such that $\mathcal{L}(N) = 0$ and $f(x,u_\varepsilon(x)) \leq \lambda$ for every $x \in \Omega \setminus N$. By definition of $f^{\bar{c}}$, this implies $f^{\bar{c}}(x,p(x),\lambda) \geq p(x)\cdot u_\varepsilon(x)$ for every $x \in \Omega \setminus N$ and so

$$\int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx \geq \int_\Omega p(x)\cdot u_\varepsilon(x)dx \geq F^{\bar{c}}(p,\lambda) - \varepsilon.$$

By the arbitrariness of ε, we obtain

$$\int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx \geq F^{\bar{c}}(p,\lambda).$$

To prove the opposite inequality, since $\int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx \in \mathbb{R}$, there exists $N \subset \Omega$ such that $\mathcal{L}(N) = 0$ and $f^{\bar{c}}(x,p(x),\lambda) \in \mathbb{R}$ for every $x \in N$. Now fix $\varepsilon > 0$. Then, for every $x \in \Omega \setminus N$, there exists $z_\varepsilon(x) \in \mathbb{R}^n$ such that $f(x,z_\varepsilon(x)) \leq \lambda$ and $f^{\bar{c}}(x,p(x),\lambda) \leq p(x)\cdot z_\varepsilon(x) + \varepsilon$. Moreover $|z_\varepsilon(x)| \leq M$. We want to select a function $u \in L^\infty$ such that $f^{\bar{c}}(x,p(x),\lambda) \leq p(x)\cdot u_\varepsilon(x) + \varepsilon$. Define $h(x,z) = (f(x,z) - \lambda) \vee 0 + (f^{\bar{c}}(x,p(x),\lambda) - p(x)\cdot z - \varepsilon) \vee 0$. Thanks to Proposition 2.6 of [14] h is a nonnegative measurable function, l.s.c. with respect to z and $h(x,z_\varepsilon(x)) = 0$ for every $x \in \Omega \setminus N$. Thanks to Theorem 2.1 pag. 236 in [11], there exists $u_\varepsilon \in L^\infty(\Omega)$ such that $h(x,u_\varepsilon(x)) = 0$ for every $x \in \Omega \setminus N$. In particular $f(x,u_\varepsilon(x)) \leq \lambda$ and $f^{\bar{c}}(x,p(x),\lambda) \leq p(x)\cdot u_\varepsilon(x) + \varepsilon$ for every $x \in \Omega \setminus N$, which implies

$$F^{\bar{c}}(p,\lambda) \geq \int_\Omega u_\varepsilon(x)\cdot p(x)dx \geq \int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx - \varepsilon.$$

Since ε is arbitrary , we obtain

$$F^{\bar{c}}(p,\lambda) \geq \int_\Omega f^{\bar{c}}(x,p(x),\lambda)dx,$$

and this completes the proof. $\square$

Proof of Corollary 3.5. Fix $p \in L^1$ and $\lambda \in \mathbb{R}$ such that $F^c(p,\lambda) > -\infty$. Then, thanks to Lemma 3.6 (a), $F^c(p,\lambda) \in \mathbb{R}$. So the level set $\{u \in L^\infty : \ F(u) \leq \lambda\} \neq \emptyset$. If $u \in L^\infty$ is such that $F(u) < \lambda$, then $f^{\bar{c}}(x,p(x),r) \geq p(x)\cdot u(x)$ for a.e. $x \in \Omega$ and for every $r \in \mathbf{Q}$ such that $F(u) < r < \lambda$. So, we can apply Beppo Levi Theorem to the nondecreasing sequence $\{f^{\bar{c}}(\cdot,\cdot,r)\}_{r\in\mathbf{Q},\ r<\lambda}$ and by using twice (2.1) in Proposition 2.6 we obtain

$$\begin{aligned} F^c(p,\lambda) &= \bigvee_{r<\lambda, r\in\mathbf{Q}} F^{\bar{c}}(p,r) = \lim_{r\to\lambda^-}\int_\Omega f^{\bar{c}}(x,p(x),r)dx \\ &= \int_\Omega \lim_{r\to\lambda^-} f^{\bar{c}}(x,p(x),r)dx = \int_\Omega f^c(x,p(x),\lambda)dx. \end{aligned}$$

$\square$

We will now prove that the conjugate of an integral functional can be represented by a supremal functional provided that the integrand fulfills a set of hypotheses that we resume as follows.

(Hγ) $g : \Omega\times\mathbb{R}^N\times\mathbb{R} \to \mathbb{R}\cup\{-\infty\}$ is such that:

- for each $(x,w)\in\Omega\times\mathbb{R}^N$, $g(x,w,\cdot)$ is nondecreasing;
- for each $\lambda\in\mathbb{R}$, there exists a measurable set $\Omega_\lambda\subset\Omega$, such that $g(\cdot,\cdot,\lambda)$ is a convex normal integrand on $\Omega_\lambda\times\mathbb{R}^N$ and identically $-\infty$ on $\Omega\setminus\Omega_\lambda$;
- for each $\lambda\in\mathbb{R}$ there exists $K(\lambda)>0$ such that

  $$|g(x,p,\lambda)| \leq K(\lambda)|p|$$

  for every $p\in\mathbb{R}^N$ and for every $x\in\Omega_\lambda$. Moreover $K(\lambda)$ is a nondecreasing function of λ.

In all the paper we will write $g^\gamma(x,u)$ to indicate the conjugate of the function $g(x,\cdot,\cdot)$ at x fixed. (See also the Definition 2.1.) We need the following measurability property. (For the proof just follow the proof of Theorem 2.7 of [14].)

Proposition 3.8 *If a function g fulfills hypothesis* **(Hγ)** *then g^γ is a normal supremand.*

Remark 3.9 Note that, in Proposition 3.8, if g fulfills **(Hγ)** then g^γ fulfills **(Hc)**. Furthermore by Proposition 3.3 if a function f fulfills **(Hc)** then f^c and $f^{\bar{c}}$ fulfill **(Hγ)**. See also Remark 2.8 for the same property in the general case.

The representation result is the following.

Theorem 3.10 *Assume that g fulfills hypothesis* **(Hγ)**. *Fix $p\in L^1$ and $\lambda\in\mathbb{R}$ If we define*

$$G(p,\lambda) := \begin{cases} \int_\Omega g(x,p(x),\lambda)dx & \text{if } \ \mathcal{L}(\Omega\setminus\Omega_\lambda)=0 \\ -\infty & \text{otherwise} \end{cases}$$

then

$$G^\gamma(u) = \operatorname*{ess\,sup}_{x\in\Omega} g^\gamma(x,u(x)) \tag{3.5}$$

for every $u\in L^\infty$.

Proof. Let us define $H(u) := \operatorname{ess\,sup}_{x\in\Omega} g^\gamma(x,u(x))$. Observe that by Proposition 2.2 (c) and Remark 4.4 in [1] the functional H is weakly* lower semicontinuous in L^∞ and level convex. Thus Theorem 2.3 ensure us that $H = H^{c\gamma}$. We first show that

$$H(u) = \left(\int_\Omega g^{\gamma c}(x,p(x),\lambda)dx\right)^\gamma(u). \tag{3.6}$$

Indeed, by Corollary 3.5 and by Theorem 3.4 ,

$$\begin{aligned}
H^{c\gamma}(u) &= \sup_{p\in L^1}\sup\left\{\lambda\in\mathbb{R}:\ H^c(p,\lambda) < \int_\Omega p(x)\cdot u(x)dx\right\} = \\
&= \sup\{\lambda\in\mathbb{R}:\ H^c(\cdot,\lambda) = -\infty\}\bigvee \\
&\bigvee \sup_{p\in L^1}\sup\left\{\lambda\in\mathbb{R}:\ -\infty < H^c(p,\lambda) < \int_\Omega p(x)\cdot u(x)dx\right\} = \\
&= \sup\{\lambda\in\mathbb{R}:\ \int_\Omega g^{\gamma c}(x,\cdot,\lambda)dx = -\infty\}\bigvee \\
&\bigvee \sup_{p\in L^1}\sup\left\{\lambda\in\mathbb{R}:\ -\infty < \int_\Omega g^{\gamma c}(x,\cdot,\lambda)dx < \int_\Omega p(x)\cdot u(x)dx\right\} = \\
&= \sup_{p\in L^1}\sup\left\{\lambda\in\mathbb{R}:\ \int_\Omega g^{\gamma c}(x,p,\lambda)dx < \int_\Omega p(x)\cdot u(x)dx\right\} = \left(\int_\Omega g^{\gamma c}(x,\cdot,\cdot)dx\right)^\gamma.
\end{aligned}$$

For each p in L^1, let us denote by $\overline{G}(p,\cdot)$ the lower semicontinuous envelope of G w.r.t. λ. Since G is monotone in λ, we have

$$\overline{G}(p,\lambda) = \sup_{r<\lambda} G(p,r) = \lim_{r\to\lambda^-} G(p,r).$$

Analogously, if for every $(x,p)\in\Omega\times\mathbb{R}^N$ we denote by $\overline{g}(x,p,\cdot)$ the lower semicontinuous envelope of g w.r.t. λ, then we have

$$\overline{g}(x,p,\lambda) = \sup_{r<\lambda} g(x,p,r) = \lim_{r\to\lambda^-} g(x,p,r).$$

By applying Lebesgue Theorem, we have

$$\overline{G}(p,\lambda) := \begin{cases} \int_\Omega \overline{g}(x,p(x),\lambda)dx & \text{if } \mathcal{L}(\Omega\setminus\Omega_\lambda) = 0 \\ -\infty & \text{otherwise.}\end{cases} \tag{3.7}$$

For every $(x,p,\lambda)\in\Omega\times\mathbb{R}^N\times\mathbb{R}$, we have

$$\overline{g}(x,p,\lambda) = g^{\gamma c}(x,p,\lambda). \tag{3.8}$$

In fact, if $x\notin\Omega_\lambda$, it is trivial. If $x\in\Omega_\lambda$, since $\overline{g}$ satisfies Proposition 2.4, we have $\overline{g}(x,p,\lambda) = \overline{g}^{\gamma c}(x,p,\lambda) \le g^{\gamma c}(x,p,\lambda)$, while the opposite inequality follows from the lower semicontinuity of $g^{\gamma c}(x,p,\cdot)$. Analogously $\overline{G} = G^{\gamma c}$ and, from Remark 2.5, we have

$$\overline{G}^\gamma = G^\gamma. \tag{3.9}$$

By (3.6), (3.8), (3.7) and (3.9), we obtain

$$H(u) = \left(\int_\Omega g^{\gamma c}(x,\cdot,\cdot)dx\right)^\gamma(u) =$$

$$= \left(\int_\Omega \overline{g}(x,\cdot,\cdot)dx\right)^\gamma (u) = \overline{G}^\gamma(u) = G^\gamma(u)$$

which concludes the proof. □

Our aim is finally to study the $\Gamma(w^*L^\infty)$-limit of a sequence of supremal functionals $F_n(u) = \operatorname{ess\,sup}_{x\in\Omega} f_n(x,u(x))$ defined on L^∞. We will always consider a sequence of supremands (f_n) that fulfills the following uniform coercivity assumption.

(Hcn) For each $n \in \mathbb{N}$, $f_n : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a normal supremand. Moreover, there exists a function $\alpha : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ such that

$$\lim_{|w|\to\infty} \alpha(|w|) = +\infty$$

and

$$f_n(x,w) \geq \alpha(|w|) \quad \forall x \in \Omega,\ \forall w \in \mathbb{R}^N,\ \forall n \in \mathbb{N}.$$

Indeed, under hypothesis **(Hcn)**, by Theorem 2.11, there exists a subsequence $(F_{n_k})_k$ of $(F_n)_n$ such that F_{n_k} Γ- converges to F with respect to the weak* topology of L^∞. Moreover, by Theorem 3.6 in [14], there exists a level convex normal supremand f such that the representation formula $F(u) = \operatorname{ess\,sup}_{x\in\Omega} f(x,u(x))$ holds for each $u \in L^\infty$. So, up to a subsequence, hypothesis **(Hcn)** ensures us that we can always consider $F(u) := \Gamma(w^*L^\infty) \lim_{n\to\infty} F_n(u)$. Moreover, F is a level convex functional, so just rewriting Theorem 2.12, we have the following first representation result.

Proposition 3.11 *Let $\Omega \subset \mathbb{R}^N$ be an open set. For every $n \in \mathbb{N}$ let*

$$F_n(u) := \operatorname{ess\,sup}_{x\in\Omega} f_n(x,u(x)) \qquad \text{with } u \in L^\infty. \tag{3.10}$$

Assume **(Hcn)**. *If*

$$F(u) = \Gamma(w^*L^\infty) \lim_{n\to\infty} F_n(u),$$

then

$$F(u) = (\Gamma(sL^1) \liminf_{n\to\infty} F_n^c(p,\lambda))^\gamma(u) = (\Gamma(sL^1) \limsup_{n\to\infty} F_n^{\bar{c}}(p,\lambda))^\gamma(u). \tag{3.11}$$

From the equality (3.11), it becomes clear that to specify further our representation result we have to be able to calculate $\Gamma(sL^1) \liminf_{n\to\infty} F_n^c(p,\lambda)$ and $\Gamma(sL^1) \limsup_{n\to\infty} F_n^{\bar{c}}(p,\lambda)$. Since those functionals admit an integral representation it is natural to assume the following. (Our aim is to apply the classical result in [12].)

(H) Let

$$\Lambda = \{\lambda \in \mathbb{R} \text{ s.t. } \exists n_0 \in \mathbb{N} \text{ s.t. } F_n^c(\cdot,\lambda) \text{ is proper } \forall n \geq n_0\}.$$

There exists a function $f : \Omega \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{n\to\infty} f_n^c(\cdot,w,\lambda) = f(\cdot,w,\lambda) \text{ weakly}^* L^\infty \quad \forall w \in \mathbb{R}^N, \forall \lambda \in \Lambda,$$

and

$$f(\cdot,\cdot,\lambda) = -\infty \quad \forall \lambda \in \mathbb{R} \setminus \Lambda.$$

The complete representation result is the following.

Theorem 3.12 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let us consider, for each $n \in \mathbb{N}$, the supremal functional*

$$F_n(u) = \operatorname*{ess\,sup}_{x\in\Omega} f_n(x, u(x)) \qquad \text{with } u \in L^\infty. \tag{3.12}$$

Assume **(Hcn)** *and* **(H)**. *If*

$$F(u) = \Gamma(w^*L^\infty) \lim_{n\to\infty} F_n(u)$$

then

$$F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f^\gamma(x, u(x)). \tag{3.13}$$

Proof. By Corollary 3.5, for every $p \in L^1(\Omega)$, for every $\lambda \in \Lambda$ and for every $n \geq n_0$, we have

$$F_n^c(p, \lambda) = \int_\Omega f_n^c(x, p(x), \lambda)dx.$$

By definition $f_n^c(x, \cdot, \lambda)$ is a convex function. Moreover, by **(Hcn)**, for every $\lambda \in \Lambda$ there exists a constant $K(\lambda)$ such that $|f_n^c(x, p(x), \lambda)| \leq K(\lambda)|p(x)|$, for a.e. $x \in \Omega, \forall n \in \mathbb{N}$. Thus Theorem 3.3 of [12] applies and we obtain

$$\Gamma(sL^1) \lim_{n\to\infty} F_n^c(p, \lambda) = \Gamma(sL^1) \lim_{n\to\infty} \int_\Omega f_n^c(x, p(x), \lambda)dx = \int_\Omega f(x, p(x), \lambda)dx.$$

for every $\lambda \in \Lambda$. If $\lambda \notin \Lambda$, then there exists a subsequence $(F_{n_k}^c)_k$ of F_n^c such that $F_{n_k}^c(\cdot, \lambda) = -\infty$, for all $k \in \mathbb{N}$. By **(H)** we have again

$$\Gamma(sL^1) \lim_{n\to\infty} F_n^c(p, \lambda) = \int_\Omega f(x, p(x), \lambda)dx = -\infty.$$

Thus applying (3.11) in Proposition 3.11 we get

$$F(u) = (\Gamma(sL^1) \lim_{n\to\infty} F_n^c(p, \lambda))^\gamma(u) = \left(\int_\Omega f(x, p(x), \lambda)dx\right)^\gamma(u).$$

Observe now that by **(H)**, f fulfills hypothesis **(Hγ)**, then Theorem 3.10 applies and (3.13) follows. □

Remark 3.13 In the previous theorem we assume that the open set Ω is bounded in order to apply Theorem 3.3 in [12].

Remark 3.14 By (3.11) it is clear that instead of calculating the $\Gamma(sL^1) \liminf_{n\to\infty} F_n^c(p, \lambda)$ one can calculate the $\Gamma(sL^1) \limsup_{n\to\infty} F_n^{\bar{c}}(p, \lambda)$. Indeed, assuming instead of **(H)**,

(H1) Let

$$\Lambda = \{\lambda \in \mathbb{R} \text{ s.t. } \exists n_0 \in \mathbb{N} \text{ s.t. } F_n^{\bar{c}}(\cdot, \lambda) \text{ is proper } \forall n \geq n_0\}.$$

There exists a function $g : \Omega \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{n\to\infty} f_n^{\bar{c}}(\cdot, w, \lambda) = g(\cdot, w, \lambda) \text{ weakly}^* L^\infty \quad \forall w \in \mathbb{R}^N, \forall \lambda \in \Lambda,$$

and

$$g(\cdot, \cdot, \lambda) = -\infty \quad \forall \lambda \in \mathbb{R} \setminus \Lambda,$$

one can repeat the same proof and obtain the following representation

$$F(u) = \operatorname*{ess\,sup}_{x\in\Omega} g^{\gamma}(x, u(x)).$$

Remark 3.15 Of course one can use Theorem 2.14 instead of Theorem 2.12 in order to represent F. We choose not to use it because the classical result does not apply directly, but it is sufficient to calculate $\Gamma(sL^1 \times \mathbb{R}) \liminf_{n\to\infty} F_n^{\bar{c}}(p,\lambda)$ or $\Gamma(sL^1 \times \mathbb{R}) \limsup_{n\to\infty} F_n^{\bar{c}}(p,\lambda)$ and proceed as in the proof of Theorem 3.12.

Our last result is a vice versa for Theorem 3.12.

Theorem 3.16 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. For every $n \in \mathbb{N}$ let*

$$F_n(u) := \operatorname*{ess\,sup}_{x\in\Omega} f_n(x, u(x)) \quad \text{with } u \in L^\infty. \tag{3.14}$$

Let $\alpha : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ such that the coercivity assumption **(Hcn)** *holds. Suppose that there exists a level convex normal supremand $f : \Omega \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ such that*

$$f(x,w) \geq \alpha(|w|) \quad \forall x \in \Omega,\ \forall w \in \mathbb{R}^N, \tag{3.15}$$

and

$$\Gamma(w^* L^\infty) \lim_{n\to\infty} F_n(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x)) (:= F(u)).$$

Let $\bar{\lambda} = \min\{F(u) : \ u \in L^\infty\}$. Then there exists a countable $\bar{\Lambda} \subset \mathbb{R}$ such that, for every $\lambda > \bar{\lambda}$ and $\lambda \notin \bar{\Lambda}$

$$\lim_{n\to\infty} f_n^c(\cdot, w, \lambda) = f^c(\cdot, w, \lambda) \ \text{weakly}^* L^\infty \quad \forall w \in \mathbb{R}^N,$$

and

$$\lim_{n\to\infty} f_n^{\bar{c}}(\cdot, w, \lambda) = f^{\bar{c}}(\cdot, w, \lambda) \ \text{weakly}^* L^\infty \quad \forall w \in \mathbb{R}^N.$$

Proof. By Proposition 2.7, there exists a countable $\bar{\Lambda} \subset \mathbb{R}$ such that $F^c(p,\lambda) = F^{\bar{c}}(p,\lambda)$ for every $p \in L^1$ and for every $\lambda \in \mathbb{R} \setminus \bar{\Lambda}$. In particular, by (2.5) and (2.6), we obtain

$$F^c(p,\lambda) = \Gamma(sL^1) \lim_{n\to\infty} F_n^c(p,\lambda) = \Gamma(sL^1) \lim_{n\to\infty} F_n^{\bar{c}}(p,\lambda) = F^{\bar{c}}(p,\lambda)$$

for every $\lambda \in \mathbb{R} \setminus \bar{\Lambda}$. Let us observe that if $\lambda > \bar{\lambda}$ and $\lambda \notin \bar{\Lambda}$, $F^c(p,\lambda) = F^{\bar{c}}(p,\lambda) \in \mathbb{R}$ for every $p \in L^1$. Moreover, by (2.10) and Remark 2.8, there exists $n_0 \in \mathbb{N}$ such that $F_n^c(p,\lambda) \in \mathbb{R}$ for every $n \geq n_0$. It follows that also $F_n^{\bar{c}}(p,\lambda) \in \mathbb{R}$ for every $n \geq n_0$. Thus, Theorem 3.4 and Corollary 3.5 apply and we have

$$F_n^{\bar{c}}(p,\lambda) = \int_\Omega f_n^{\bar{c}}(x, p(x), \lambda)dx, \quad F_n^c(p,\lambda) = \int_\Omega f_n^c(x, p(x), \lambda)dx, \quad \forall n \geq n_0, \quad \forall p \in L^1,$$

and

$$F^{\bar{c}}(p,\lambda) = \int_\Omega f^{\bar{c}}(x, p(x), \lambda)dx, \quad F^c(p,\lambda) = \int_\Omega f^c(x, p(x), \lambda)dx \quad \forall p \in L^1.$$

So, thanks also to assumptions **(Hcn)** and (3.15), we can apply Theorem 3.3 in [12] and obtain that

$$\lim_{n\to\infty} f_n^c(\cdot, w, \lambda) = f^c(\cdot, w, \lambda) \ \text{weakly}^* L^\infty \quad \forall w \in \mathbb{R}^N$$

and

$$\lim_{n\to\infty} f_n^{\bar{c}}(\cdot, w, \lambda) = f^{\bar{c}}(\cdot, w, \lambda) \ \text{weakly}^* L^\infty \quad \forall w \in \mathbb{R}^N.$$

□

4 Examples and Counterexamples

We give now two examples of application for our main representation result, i.e. Theorem 3.12.

Example 1. Let us consider $\Omega \subset \mathbb{R}^N$ an open bounded set, and $a_n : \Omega \to]0, +\infty[$ a sequence such that $a_n \in L^\infty$ for each $n \in \mathbb{N}$. We consider the sequence of supremand $f_n(x, u) := a_n(x)|u|$ and we want to calculate the $\Gamma(w^* L^\infty)$-limit of $F_n(u) := \operatorname*{ess\,sup}_{x\in\Omega} f_n(x, u(x))$. Since $a_n(\cdot)$ is strictly positive for each $n \in \mathbb{N}$ assumption (**Hcn**) follows. By calculation we have, fix $x \in \Omega$,

$$f_n^c(x, \cdot, \lambda) = \begin{cases} \dfrac{\lambda|\cdot|}{a_n(x)} & \text{if } \lambda > 0 \\ -\infty & \text{if } \lambda \leq 0. \end{cases}$$

So, in order to have **(H)** we assume that there exists a function $a : \Omega \to]0, +\infty[$ such that

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a} \quad \text{weakly}^* L^\infty.$$

Thus, Theorem 3.12 applies and we obtain

$$\Gamma(w^* L^\infty) \lim_{n\to\infty} F_n(u) = \operatorname*{ess\,sup}_{x\in\Omega} a(x)|u(x)|.$$

Example 2. Let us consider a bounded open subset Ω of $\mathbb{R}^N$. Fix $x \in \Omega$ let $A_n(x)$, $(\forall n \in \mathbb{N})$, be a nonempty closed subset of $\mathbb{R}$ and suppose that there exists a bounded subset of $\mathbb{R}$, $A(x)$, such that $A_n(x) \subseteq A(x)$ for every $n \in \mathbb{N}$. Thus

$$f_n(x, u) := 1_{A_n(x)}(u) = \begin{cases} +\infty & \text{if} \quad u \notin A_n(x) \\ 0 & \text{if} \quad u \in A_n(x) \end{cases}$$

is a sequence that fulfills **(Hcn)**.
In order to apply Theorem 3.12 to $F_n(u) := \operatorname*{ess\,sup}_{x\in\Omega} f_n(x, u(x))$, we have to calculate $f_n^c(x, p, \lambda)$. One gets

$$f_n^c(x, p, \lambda) = \begin{cases} p\,\bar{a}_n(x) & \text{if} \quad \lambda > 0, \quad p > 0 \\ p\,\underline{a}_n(x) & \text{if} \quad \lambda > 0, \quad p \leq 0 \\ -\infty & \text{if} \quad \lambda \leq 0, \quad \forall p \end{cases}$$

where $\bar{a}_n(x) = \max_{u\in A_n(x)} u$ and $\underline{a}_n(x) = \min_{u\in A_n(x)} u$. Assuming that $\lim_{n\to\infty} \bar{a}_n(x) = \bar{a}(x)$ weakly* in L^∞ and $\lim_{n\to\infty} \underline{a}_n(x) = \underline{a}(x)$ weakly* in L^∞, assumption **(H)** is fulfilled with

$$f(x, p, \lambda) = \begin{cases} p\,\bar{a}(x) & \text{if} \quad \lambda > 0, \quad p > 0 \\ p\,\underline{a}(x) & \text{if} \quad \lambda > 0, \quad p \leq 0 \\ -\infty & \text{if} \quad \lambda \leq 0, \quad \forall p. \end{cases}$$

Theorem 3.12 applies and a calculation yields to

$$\Gamma(w^* L^\infty) \lim_{n\to\infty} F_n(u) = \operatorname*{ess\,sup}_{x\in\Omega} 1_{[\underline{a}(x), \bar{a}(x)]}(u(x)).$$

As we have announced in Section 2 we want to discuss now inequality (2.7). Since our main aim was to study the representation results we will discuss it in the contest of supremal functional. We have seen in the proof of Theorem 3.12 that in order to characterize the $\Gamma(w^*L^\infty)$-limit of it was sufficient to consider the $\Gamma(sL^1)$-limit of the first conjugate with respect to the p variable for almost every λ. (See also Theorem 3.16). More precisely the key point is in the following inequality (analogous of (2.7)). For each $\lambda \in \mathbb{R}$ we have

$$F^c(p,\lambda) \le \Gamma(sL^1)\liminf_{n\to\infty} F^c_n(p,\lambda) \le \Gamma(sL^1)\limsup_{n\to\infty} F^{\bar{c}}_n(p,\lambda) \le F^{\bar{c}}(p,\lambda). \tag{4.1}$$

We want to stress now that in general none of this is an equality. First, we remark that in general to produce a function $\varphi : X \to \mathbb{R}\cup\{+\infty\}$ such that $\varphi^c(p,\lambda) < \varphi^{\bar{c}}(p,\lambda)$ it is sufficient to consider a φ with $\varphi(x) := k \quad \forall k \in M$ with $\mathcal{L}(M) > 0$ and to choose $\lambda = k$. So, it is easily proved that one can have $F^c(p,\lambda) < F^{\bar{c}}(p,\lambda)$ or $F^c_n(p,\lambda) < F^{\bar{c}}_n(p,\lambda)$. Therefore we will focus now on the two Γ-limit inequalities. We give two examples.

Counterexample 1.

Let Ω be a bounded subset of $\mathbb{R}$ and consider the sequence of functionals $F_n(u) = \operatorname*{ess\,sup}_{x\in\Omega} f_n(u(x))$, where $f_n : \mathbb{R} \to \mathbb{R}$ is defined as follows.

$$f_n(u) = \begin{cases} |u|-1 & |u| > 2 \\ 1-\frac{1}{n} & 1 \le |u| \le 2 \\ (\frac{2}{3}-\frac{1}{n})|u| + \frac{1}{3} & |u| < 1. \end{cases}$$

One can directly prove that $\Gamma(w^*L^\infty)\lim_{n\to\infty} F_n(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(u(x))$ with

$$f(u) = \begin{cases} |u|-1 & |u| > 2 \\ 1 & 1 \le |u| \le 2 \\ \frac{2}{3}|u| + \frac{1}{3} & |u| < 1. \end{cases}$$

Thus, by calculation, one get

$$F^c(p,\lambda) = \begin{cases} (\int_\Omega |p(x)|dx)\,(\lambda+1) & \lambda > 1 \\ (\int_\Omega |p(x)|dx)\,\frac{3}{2}(\lambda-\frac{1}{3}) & \frac{1}{3} < \lambda \le 1 \\ -\infty & \lambda \le \frac{1}{3}. \end{cases}$$

Let us now compute $\Gamma(sL^1)\liminf_{n\to\infty} F^c_n(p,\lambda)$. We have

$$f^c_n(p,\lambda) = \begin{cases} |p|(\lambda+1) & \lambda > 1 \\ |p|2 & 1-\frac{1}{n} < \lambda \le 1-\frac{1}{n} \\ |p|\frac{3n}{2n-3}(\lambda-\frac{1}{3}) & \frac{1}{3} < \lambda \le 1-\frac{1}{n} \\ -\infty & \lambda \le \frac{1}{3} \end{cases}$$

then, for each $(p,\lambda) \in \mathbb{R}^N \times \mathbb{R}$,

$$\lim_{n\to\infty} f^c_n(p,\lambda) = \begin{cases} |p|(\lambda+1) & \lambda > 1 \\ |p|2 & \lambda = 1 \\ |p|\frac{3}{2}(\lambda-\frac{1}{3}) & \frac{1}{3} < \lambda < 1 \\ -\infty & \lambda \le \frac{1}{3}. \end{cases}$$

Thus, applying Theorem 3.3 in [12], or simply calculating,

$$\Gamma(sL^1)\lim_{n\to\infty} F_n^c(p,\lambda) = \begin{cases} (\int_\Omega |p(x)|dx)(\lambda+1) & \lambda > 1 \\ (\int_\Omega |p(x)|dx)2 & \lambda = 1 \\ (\int_\Omega |p(x)|dx)\frac{3}{2}(\lambda - \frac{1}{3}) & \frac{1}{3} < \lambda < 1 \\ -\infty & \lambda \leq \frac{1}{3}. \end{cases}$$

For $\lambda = 1$ we have

$$F^c(p,1) = \int_\Omega |p(x)|dx < 2\int_\Omega |p(x)|dx = \Gamma(sL^1)\liminf_{n\to\infty} F_n^c(p,1),$$

which prove that the first inequality in (4.1) can be strict.

Counterexample 2

Let Ω be a bounded subset of $\mathbb{R}$ and consider the sequence of functionals $F_n(u) = \operatorname{ess\,sup}_{x\in\Omega} f_n(u(x))$, where $f_n : \mathbb{R} \to \mathbb{R}$ is defined as follows.

$$f_n(u) = \begin{cases} |u| + \frac{1}{n} & |u| > 1 \\ \frac{1}{n}|u| + 1 & |u| \leq 1. \end{cases}$$

We have $\Gamma(w^*L^\infty)\lim_{n\to\infty} F_n(u) = \operatorname{ess\,sup}_{x\in\Omega} f(u(x))$ with

$$f(u) = \begin{cases} |u| & |u| > 1 \\ 1 & |u| \leq 1. \end{cases}$$

Thus

$$F^{\bar{c}}(p,\lambda) = \begin{cases} (\int_\Omega |p(x)|dx)\ \lambda & \lambda \geq 1 \\ -\infty & \lambda < 1. \end{cases}$$

Let us now calculate $\Gamma(sL^1)\limsup_{n\to\infty} F_n^{\bar{c}}(p,\lambda)$. First,

$$f_n^{\bar{c}}(p,\lambda) = \begin{cases} |p|(\lambda - \frac{1}{n}) & \lambda \geq 1 + \frac{1}{n} \\ |p|n(\lambda - 1) & 1 \leq \lambda < 1 + \frac{1}{n} \\ -\infty & \lambda < 1, \end{cases}$$

so, applying Theorem 3.3 in [12] or simply calculating,

$$\Gamma(sL^1)\lim_{n\to\infty} F_n^{\bar{c}}(p,\lambda) = \begin{cases} (\int_\Omega |p(x)|dx)\lambda & \lambda \geq 1 \\ 0 & \lambda = 1 \\ -\infty & \lambda < 1. \end{cases}$$

For $\lambda = 1$ and $p \neq 0$ we have

$$\Gamma(sL^1)\limsup_{n\to\infty} F_n^{\bar{c}}(p,1) = 0 < \int_\Omega |p(x)|dx = F^{\bar{c}}(p,1)$$

which prove that the third inequality in (4.1) can be strict. Note that this prove also that there is no hope to obtain $F^{\bar{c}}(p,\lambda) = \Gamma(sL^1 \times \mathbb{R})\lim_{n\to\infty} F_n^{\bar{c}}(p,\lambda)$ or
$F^{\bar{c}}(p,\lambda) = \Gamma(sL^1 \times \mathbb{R})\lim_{n\to\infty} F_n^c(p,\lambda)$.

Remark 4.1 Let us remark that in both examples all functions f_n are positive, coercive, lower semicontinuous and level convex. The strictness of the inequalities is not then due to any lack of regularity but is a characteristic of the problem.

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