Homogenization effects on non-local functionals

Enrico Micalizio*

Politecnico di Torino, Dipartimento di Scienze Matematiche (DISMA)
Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Abstract

We study the homogenization of a class of non-local functionals featuring a rapidly oscillating periodic weight. By means of two-scale convergence, we explicitly evaluate the Γ-limit for constant target functions, revealing how the interplay between periodicity and non-locality forces the minimizing sequences to develop highly oscillating microstructures. As a natural consequence, we establish that the effective macroscopic functional fails to admit a standard double-integral representation.

Keywords: homogenization; non-local functionals; Γ-convergence; two-scale convergence; optimal microstructures; integral representation
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1 Introduction

The macroscopic behavior of materials and physical systems is often linked to energy minimization. However, as pointed out in [4, 11, 14], competing interactions at the microscopic level prevent the formation of homogeneous ground states and instead induce the formation of patterns and microstructures, leading to non-trivial and often unexpected macroscopic properties. This complex phenomenology naturally leads to the study of long-range interactions, whose features are effectively captured by the mathematical framework of non-local functionals. Furthermore, this class of functionals has recently attracted significant attention within the Calculus of Variations. Indeed, beyond the aforementioned long-range interactions, non-local energies – typically formulated as double integrals – naturally arise in models involving peridynamics [12, 16] and image processing [10, 13]. In this context, recent works by Braides and Dal Maso have investigated some fundamental properties of functionals taking the form

$$F_k(u) = \int_{\Omega\times\Omega} f_k(u(x) - u(y))\, \mathrm{d}\mu_k(x, y) + \int_{\Omega} g_k(x, \nabla u(x))\, \mathrm{d}x,$$

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[*] **Correspondence:** Email: enrico.micalizio@studenti.polito.it

defined on $W_0^{1,p}(\Omega)$ with $p > 1$, where Ω is a bounded open set in $\mathbb{R}^d$ for $d \geq 1$. In [5], the authors provide suitable conditions on the measures μ_k guaranteeing the stability of the Γ-limit, while in [7] it is shown that the Γ-limit retains a decoupled integral form, highlighting how the effective local part arises from the interaction between local and non-local terms. The integral representability of the limit, however, is a delicate issue: while positive results can be achieved for $p = 2$ [6], non-representability counterexamples emerge for $p \neq 2$ [6] or when the local term is dropped [3].

A natural further step in this investigation is to understand how these non-local energies behave when the interaction itself oscillates at a microscopic scale $\varepsilon > 0$. When dealing with materials exhibiting a heterogeneous yet periodic microstructure, homogenization provides a powerful tool to derive the effective macroscopic behavior. We refer to [2] for an in-depth analysis via Young measures, and to [8, 9] for the homogenization of convolution-type energies.

Motivated by these results, in this paper we study the asymptotic behavior of the sequence of non-local functionals $F_\varepsilon \colon L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$F_\varepsilon(u) = \int_{\Omega\times\Omega} a\left(\frac{x-y}{\varepsilon}\right) f(u(x) - u(y))\, \mathrm{d}x\mathrm{d}y. \tag{1}$$

Here, f is the same triple-well potential considered in [3], given by

$$f(z) = \begin{cases} 0 & \text{if } z \in \{-1, 1\}, \\ 1 & \text{if } z = 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{2}$$

while the periodic weight $a \in L^\infty(\mathbb{R})$ is assumed to be strictly positive. To tackle this problem, we apply the theory of two-scale convergence introduced by Nguetseng and Allaire [1, 15], which allows us to effectively capture the microscopic oscillations during the limit process. By explicitly characterizing the Γ-limit for constant target functions, which identify the ground state energy of the system, we prove that energy minimization strictly enforces the emergence of highly oscillating step-like microstructures.

The paper is organized as follows. In Section 2, after reformulating the problem using characteristic functions, we compute the two-scale limit of the functionals, highlighting the structural properties of the minimizing sequences. In Section 3, we solve the cell problem, showing how the minimizing profiles strongly depend on the structure of the kernel a. While proving that the minimum of the energy is attained by constant functions, we illustrate how interactions associated with the zero-cost wells of the potential f are energetically favored. Finally, in Section 4 we establish that the Γ-limit of the sequence of functionals defined in (1) does not admit a standard non-local double integral representation.

2 Two-scale Limit and Homogenization

Without loss of generality, we can choose $\Omega = (0, 1)$ for simplicity.

Lemma 2.1. *Let $u \in L^1(\Omega)$ and let $\{u_\varepsilon\}$ be a sequence weakly converging to u in $L^1(\Omega)$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Then, there exists a bounded sequence $\{z_\varepsilon\} \subset \mathbb{R}$ such that, up to subsequences, $z_\varepsilon \to z \in \mathbb{R}$ and*

$$u_\varepsilon(x) \in \{z_\varepsilon, z_\varepsilon + 1\} \quad \textit{for almost every } x \in \Omega.$$

Furthermore, the limit z satisfies

$$z \leq \operatorname{ess-inf} u \qquad \textit{and} \qquad \operatorname{ess-sup} u \leq z + 1. \tag{3}$$

Proof. The strict positivity of a implies that the only way to achieve $F_\varepsilon(u_\varepsilon) < +\infty$ is for $f(u_\varepsilon(x) - u_\varepsilon(y))$ to take values in $\{0, 1\}$ for almost every $(x, y) \in \Omega \times \Omega$. Therefore, fixing a suitable $y \in \Omega$, there exists z_ε such that $u_\varepsilon(x) \in \{z_\varepsilon, z_\varepsilon - 1, z_\varepsilon + 1\}$ for almost every $x \in \Omega$. If both values $z_\varepsilon - 1$ and $z_\varepsilon + 1$ were taken on sets of positive measure, there we would have $F_\varepsilon(u_\varepsilon) = +\infty$; hence we can restrict the values to $u_\varepsilon(x) \in \{z_\varepsilon, z_\varepsilon + 1\}$. Since the sequence is weakly convergent, $\{z_\varepsilon\}$ is bounded, and we can assume $z_\varepsilon \to z$ up to subsequences. Given that $z_\varepsilon \leq u_\varepsilon \leq z_\varepsilon + 1$, integrating over any measurable subset $A \subset \Omega$ and passing to the limit yields

$$|A|z \leq \int_A u(x)\,\mathrm{d}x \leq |A|(z + 1).$$

Since this holds for every measurable subset A, inequality (3) follows almost everywhere, which in turn implies that $u \in L^\infty(\Omega)$. ☐

Following Lemma 2.1, we define the characteristic functions

$$\chi_\varepsilon(x) := \begin{cases} 1 & \text{if } u_\varepsilon(x) = z_\varepsilon + 1 \\ 0 & \text{if } u_\varepsilon(x) = z_\varepsilon, \end{cases} \tag{4}$$

so that

$$f(u_\varepsilon(x) - u_\varepsilon(y)) = \chi_\varepsilon(x)\chi_\varepsilon(y) + (1 - \chi_\varepsilon(x))(1 - \chi_\varepsilon(y))$$

and the functional rewrites as

$$F_\varepsilon(u_\varepsilon) = \int_{\Omega \times \Omega} a\left(\frac{x - y}{\varepsilon}\right) \Big[\chi_\varepsilon(x)\chi_\varepsilon(y) + (1 - \chi_\varepsilon(x))(1 - \chi_\varepsilon(y))\Big]\,\mathrm{d}x\mathrm{d}y. \tag{5}$$

We pass to the limit by using a two-scale convergence approach.

Let $Y := (0, 1)$ be the unit cell. Note that since $\chi_\varepsilon \in \{0, 1\}$ almost everywhere and Ω has finite measure, the sequence $\{\chi_\varepsilon\}$ is uniformly bounded not only in $L^\infty(\Omega)$ but also in $L^2(\Omega)$. This allows us to apply classical two-scale convergence theory. By [1, Theorem 1.2], we can extract a subsequence that two-scale converges to a limit in $L^2(\Omega \times Y)$, namely there exists a function $\phi \in L^2(\Omega \times Y)$ such that $\chi_\varepsilon \xrightharpoonup{2} \phi$.

Lemma 2.2. *If $\chi_\varepsilon \xrightharpoonup{2} \phi$, then $w_\varepsilon(x, y) := \chi_\varepsilon(x)\chi_\varepsilon(y) \xrightharpoonup{2} \phi(x, \sigma)\phi(y, \tau)$.*

Proof. We test the bounded sequence $\{w_\varepsilon\} \subset L^2(\Omega \times \Omega)$ against functions in $\mathcal{D}(\Omega \times \Omega; C^\infty_\#(Y \times Y))$, which is the space of smooth, compactly supported functions on $\Omega \times \Omega$, taking values in the space of smooth, $(Y \times Y)$-periodic functions. By invoking the Stone–Weierstrass Theorem, we may restrict our analysis to test functions of the form $\Psi(x, y, \sigma, \tau) = \psi_1(x, \sigma)\psi_2(y, \tau)$, with $\psi_1, \psi_2 \in \mathcal{D}(\Omega; C^\infty_\#(Y))$. We obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\Omega\times\Omega} w_\varepsilon(x, y)\Psi\left(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \mathrm{d}x\mathrm{d}y \\
&= \lim_{\varepsilon\to 0} \left(\int_\Omega \chi_\varepsilon(x)\psi_1\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x\right) \left(\int_\Omega \chi_\varepsilon(y)\psi_2\left(y, \frac{y}{\varepsilon}\right) \mathrm{d}y\right) \\
&= \int_{\Omega\times\Omega} \int_Y \int_Y \phi(x, \sigma)\phi(y, \tau)\psi_1(x, \sigma)\psi_2(y, \tau) \,\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where we used the Fubini–Tonelli Theorem; the thesis follows by density. □

We define the functional $F\colon L^\infty(Y; [0, 1]) \to \mathbb{R}$ by

$$F(\varphi) := \int_Y \int_Y a(\sigma - \tau)\Big[\varphi(\sigma)\varphi(\tau) + (1 - \varphi(\sigma))(1 - \varphi(\tau))\Big] \,\mathrm{d}\sigma\mathrm{d}\tau. \tag{6}$$

Proposition 2.3. *Let $u \in L^1(\Omega)$ and let $\{u_\varepsilon\}$ be a sequence weakly converging to u in $L^1(\Omega)$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Let χ_ε be the characteristic functions associated with u_ε defined by (4), and suppose they admit a two-scale limit $\phi \in L^2(\Omega \times Y)$. Upon defining $\varphi(\cdot) := \int_\Omega \phi(\xi, \cdot) \,\mathrm{d}\xi$, we have*

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = F(\varphi). \tag{7}$$

Proof. While the two-scale convergence theorem [1, Theorem 1.2] typically requires test functions to be in $L^2(\Omega; C_\#(Y))$, a standard density argument of C^0 in L^1 allows us to use $a \in L^\infty_\#(Y)$ as an admissible test function. Lemma 2.2 along with the Fubini–Tonelli Theorem yields

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_{\Omega\times\Omega} a\left(\frac{x - y}{\varepsilon}\right) \chi_\varepsilon(x)\chi_\varepsilon(y) \,\mathrm{d}x\mathrm{d}y &= \int_{\Omega\times\Omega} \int_Y \int_Y a(\sigma - \tau)\phi(x, \sigma)\phi(y, \tau) \,\mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}x\mathrm{d}y \\
&= \int_Y \int_Y a(\sigma - \tau)\varphi(\sigma)\varphi(\tau) \,\mathrm{d}\sigma\mathrm{d}\tau.
\end{aligned}$$

Defining $v_\varepsilon(x, y) := (1 - \chi_\varepsilon(x))(1 - \chi_\varepsilon(y))$, Lemma 2.2 similarly ensures $v_\varepsilon(x, y) \xrightarrow{2} (1 - \phi(x, \sigma))(1 - \phi(y, \tau))$, and the conclusion follows by linearity. □

Corollary 2.4. *Let $u \in L^1(\Omega)$ and let $\{u_\varepsilon\}$ be a sequence weakly converging to u in $L^1(\Omega)$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$, and let $z \in \mathbb{R}$ be the limit of the associated sequence $\{z_\varepsilon\}$ given by Lemma 2.1. By letting $t := \int_Y \varphi(\sigma) \,\mathrm{d}\sigma$, the limit function u satisfies the relation*

$$\int_\Omega u(x) \,\mathrm{d}x = z + t. \tag{8}$$

Proof. From [1, Proposition 1.6], the two-scale convergence $\chi_\varepsilon \xrightharpoonup{2} \phi$ implies the weak convergence $\chi_\varepsilon(\cdot) \rightharpoonup \int_Y \phi(\cdot,\sigma)\,\mathrm{d}\sigma$ in $L^2(\Omega)$, and thus in $L^1(\Omega)$. By integrating over Ω, we get

$$\int_\Omega \chi_\varepsilon(x)\,\mathrm{d}x \to \int_\Omega \left(\int_Y \phi(x,\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}x = \int_Y \varphi(\sigma)\,\mathrm{d}\sigma = t.$$

By Lemma 2.1, we can write $u_\varepsilon(x) = z_\varepsilon + \chi_\varepsilon(x)$. Integrating and passing to the limit yields the conclusion. □

3 Minimization of the Cell Problem

To explicitly investigate the optimal microstructures governing the homogenized functional, we specialize our analysis to a symmetric, non-trivial piece-wise constant periodic weight. For fixed parameters $\beta > \alpha > 0$ and a given $\lambda \in (0,1)$, we consider a symmetric profile for the kernel a_λ, centered around $t = 1/2$:

$$a_\lambda(t) = \begin{cases} \alpha & \text{if } t \in [0,\frac{\lambda}{2}) \cup [1-\frac{\lambda}{2},1) \\ \beta & \text{if } t \in [\frac{\lambda}{2}, 1-\frac{\lambda}{2}), \end{cases}$$

extended periodically to $\mathbb{R}$. Without loss of generality, we can restrict ourselves to the case $0 < \lambda \le 1/2$. The average of the weight is denoted by $\overline{a}_\lambda = \lambda\alpha + (1-\lambda)\beta$.

The sequence of non-local functionals associated with this weight is

$$F_\varepsilon^\lambda(u) = \int_{\Omega\times\Omega} a_\lambda\left(\frac{x-y}{\varepsilon}\right) f(u(x)-u(y))\,\mathrm{d}x\mathrm{d}y. \tag{9}$$

As proved in Proposition 2.3, we have that

$$\lim_{\varepsilon\to 0} F_\varepsilon^\lambda(u_\varepsilon) = \int_Y \int_Y a_\lambda(\sigma-\tau)\Big[\varphi(\sigma)\varphi(\tau) + (1-\varphi(\sigma))(1-\varphi(\tau))\Big]\,\mathrm{d}\sigma\mathrm{d}\tau =: F^\lambda(\varphi), \tag{10}$$

which we aim to minimize over all admissible microscopic profiles φ subject to the volume fraction constraint $\int_Y \varphi(\sigma)\,\mathrm{d}\sigma =: t \in [0,1]$. Thus, we define the cell problem energy:

$$\gamma_\lambda(t) := \min\left\{F^\lambda(\varphi) : \varphi \in L^\infty(Y;[0,1]) \text{ with } \int_Y \varphi(\sigma)\,\mathrm{d}\sigma = t\right\}. \tag{11}$$

Proposition 3.1. *The optimal profile minimizing the cell problem (11) is given by the characteristic function $\varphi_t(x) = \chi_{[0,t/2)\cup(1-t/2,1)}(x)$ modulo 1, and the explicit minimum energy is*

$$\gamma_\lambda(t) = \begin{cases} 2\alpha t^2 - 2\overline{a}_\lambda t + \overline{a}_\lambda & \text{if } t \in [0,\frac{\lambda}{2}], \\ 2\beta(t^2-t) - \frac{\alpha-\beta}{2}\lambda^2 + \overline{a}_\lambda & \text{if } t \in [\frac{\lambda}{2}, 1-\frac{\lambda}{2}], \\ 2\alpha(1-t)^2 + 2\overline{a}_\lambda t - \overline{a}_\lambda & \text{if } t \in [1-\frac{\lambda}{2}, 1]. \end{cases} \tag{12}$$

Proof. Let us expand the integrand defining $F^{\lambda}(\varphi)$:

$$a_{\lambda}(\sigma-\tau)\big[\varphi(\sigma)\varphi(\tau)+(1-\varphi(\sigma))(1-\varphi(\tau))\big]=a_{\lambda}(\sigma-\tau)\big[2\varphi(\sigma)\varphi(\tau)-\varphi(\sigma)-\varphi(\tau)+1\big].$$

Integrating this over $Y\times Y$ and noting that $\int_Y a_{\lambda}(\sigma-\tau)\mathrm{d}\tau=\overline{a}_{\lambda}$ for any fixed σ, the functional simplifies to

$$F^{\lambda}(\varphi)=2J^{\lambda}(\varphi)-2\overline{a}_{\lambda}t+\overline{a}_{\lambda},$$

where $J^{\lambda}(\varphi):=\int_Y\int_Y a_{\lambda}(\sigma-\tau)\varphi(\sigma)\varphi(\tau)\,\mathrm{d}\sigma\mathrm{d}\tau$.

Since t and $\overline{a}_{\lambda}$ are fixed, minimizing F^{λ} is equivalent to minimizing J^{λ}. We can rewrite the weight as $a_{\lambda}(x)=\alpha+(\beta-\alpha)\chi_{(\lambda/2,1-\lambda/2)}(x)$. Consequently, the functional J^{λ} reads

$$\begin{aligned}J^{\lambda}(\varphi)&=\int_Y\int_Y\big[\alpha+(\beta-\alpha)\chi_{(\lambda/2,1-\lambda/2)}(\sigma-\tau)\big]\varphi(\sigma)\varphi(\tau)\,\mathrm{d}\sigma\mathrm{d}\tau\\&=\alpha t^2+(\beta-\alpha)\int_Y\int_Y\chi_{(\lambda/2,1-\lambda/2)}(\sigma-\tau)\varphi(\sigma)\varphi(\tau)\,\mathrm{d}\sigma\mathrm{d}\tau\\&=\alpha t^2+(\beta-\alpha)\left[t^2-\int_Y\int_Y\chi_{(0,\frac{\lambda}{2})\cup(1-\frac{\lambda}{2},1)}(\sigma-\tau)\varphi(\sigma)\varphi(\tau)\,\mathrm{d}\sigma\mathrm{d}\tau\right],\end{aligned}$$

so that minimizing it is, in turn, equivalent to maximizing (recall that $\beta-\alpha>0$) the term

$$K^{\lambda}(\varphi):=\int_Y\int_Y\chi_{(-\lambda/2,\lambda/2)}(\sigma-\tau)\varphi(\sigma)\varphi(\tau)\,\mathrm{d}\sigma\mathrm{d}\tau,$$

where the intervals are understood modulo 1. Since the kernel $\chi_{(-\lambda/2,\lambda/2)}$ is symmetrically decreasing for any $\lambda\in(0,1/2]$, the Riesz–Sobolev rearrangement inequality implies that K^{λ} is maximized when φ is also a symmetrically decreasing indicator function. Hence, the optimal profile is

$$\varphi_t(x):=\chi_{(-t/2,t/2)}(x)\equiv\chi_{[0,t/2)\cup(1-t/2,1)}(x)\quad\text{modulo }1,$$

which satisfies the constraint $\int_Y\varphi(\sigma)\,\mathrm{d}\sigma=t$.

Computing the integral $J^{\lambda}(\varphi_t)=\int_{-t/2}^{t/2}\int_{-t/2}^{t/2}a_{\lambda}(\sigma-\tau)\,\mathrm{d}\sigma\mathrm{d}\tau$ for the three regimes $t<\lambda/2$, $t\in[\lambda/2,1-\lambda/2)$, and $t\geq 1-\lambda/2$ yields the polynomial expressions in (12), whose continuity with respect to t is immediate to verify. ☐

Remark 3.2. It is worth noting that the assumption $\alpha<\beta$ drives the symmetric rearrangement to concentrate the mass of the optimal profile φ_t around the boundary of the unit cell (i.e., around $t=0$), precisely where the low-cost kernel α is distributed.

If, conversely, one considers $\alpha>\beta$, the energetic penalty is inverted. To minimize the intersection with the expensive α-phase, the optimal profile concentrates its mass symmetrically within the central low-cost plateau, yielding

$$\varphi_t(x)=\chi_{(1/2-t/2,1/2+t/2)}(x).$$

However, the minimum energy $\gamma_{\lambda}(t)$ has exactly the same shape as in (12), except that α and β are swapped, as well as λ and $1-\lambda$. In the sequel, it may be useful to take this into consideration.

We define the final homogenized functional evaluated on a generic function $u \in L^1(\Omega)$ as:

$$\overline{F}^{\lambda}(u) \coloneqq \min\{\gamma_\lambda(t) : t \in I_u \coloneqq [\iota(u), \varsigma(u)]\}, \tag{13}$$

where $\iota(u) \coloneqq \int_\Omega u(x)\,\mathrm{d}x - \text{ess-inf}\, u$ and $\varsigma(u) \coloneqq \int_\Omega u(x)\,\mathrm{d}x - \text{ess-sup}\, u + 1$. Note that if the essential oscillation of u is strictly greater than 1 (i.e., $\text{ess-sup}\, u - \text{ess-inf}\, u > 1$), the interval I_u is empty and $\overline{F}^{\lambda}(u) = \min_{t\in\emptyset}\{\gamma_\lambda(t)\} = +\infty$, which is consistent with the conditions established by Lemma 2.1.

Theorem 3.3. *Let $u \equiv c \in \mathbb{R}$ be a constant function. Then, the Γ-limit of the sequence $\{F_\varepsilon^\lambda\}$ exists and evaluates exactly to the minimum energy of the cell problem:*

$$\Big(\Gamma - \lim_{\varepsilon\to 0} F_\varepsilon^\lambda\Big)(c) = \overline{F}^{\lambda}(c) = \frac{(1-(1-\lambda)^2)\alpha + (1-\lambda)^2\beta}{2}. \tag{14}$$

Proof. Liminf inequality. From Proposition 2.3 and formulas (11) and (13), we deduce that for any $u_\varepsilon \rightharpoonup c$ in $L^1(\Omega)$ with bounded energy, it holds

$$\liminf_{\varepsilon\to 0} F_\varepsilon^\lambda(u_\varepsilon) = F^\lambda(\varphi) \geq \gamma_\lambda(t) \geq \overline{F}^{\lambda}(c).$$

Limsup inequality. For the constant function $u \equiv c$, we have that $I_c = [0,1]$. The optimal parameter $\overline{t}$ that minimizes $\gamma_\lambda(t)$ over I_c imposes the optimal shift to be $\overline{z} = \int_\Omega u(x)\,\mathrm{d}x - \overline{t}$, see (8). Minimizing $\gamma_\lambda(t)$ in (12) over $[0,1]$ yields that the minimizer is $\overline{t} = 1/2$; consequently the optimal shift is $\overline{z} = c - 1/2$.

Recalling the definition of φ_t as the minimizer of the problem defining $\gamma_\lambda(t)$ (see (11) and Proposition 3.1), we construct the recovery sequence as

$$u_\varepsilon^*(x) \coloneqq \overline{z} + \varphi_{\overline{t}}\left(\frac{x}{\varepsilon}\right) = c - \frac{1}{2} + \varphi_{1/2}\left(\frac{x}{\varepsilon}\right),$$

which weakly converges in $L^1(\Omega)$ to $c - 1/2 + \int_0^1 \varphi_{1/2}(\sigma)\,\mathrm{d}\sigma = c$. Plugging u_ε^* into F_ε and passing to the limit as $\varepsilon \to 0$ yields precisely the integral defining $F^\lambda(\varphi_{1/2})$. Since $\varphi_{1/2}$ is the optimal profile for $t = 1/2$, this integral evaluates exactly to the cell-problem minimum $\gamma_\lambda(1/2)$. Furthermore, since $t = 1/2$ is the global minimizer of $\gamma_\lambda(t)$ over the interval $[0,1]$, we obtain that $F^\lambda(\varphi_{1/2}) = \overline{F}^{\lambda}(c)$. Thus, we have proved that

$$\lim_{\varepsilon\to 0} F_\varepsilon^\lambda(u_\varepsilon^*) = \overline{F}^{\lambda}(c) = F^\lambda(\varphi_{1/2}) = \gamma_\lambda(1/2) = \frac{(1-(1-\lambda)^2)\alpha + (1-\lambda)^2\beta}{2}.$$

□

Remark 3.4. It is worth noting how the macroscopic energy forces the interactions into the zero-cost wells of the potential f, rejecting a flat microscopic profile (which would cost the full average weight $\overline{a}_\lambda$, strictly greater than the expression in (14)) in favor of a highly oscillating recovery sequence to achieve the constant macroscopic state $u \equiv c$. Therefore, taking the limit to the homogeneous cases, if $\lambda \to 1$ ($a(t) \equiv \alpha$), the high oscillations of the optimal profile between $c - \frac{1}{2}$ and $c + \frac{1}{2}$ halve the macroscopic energy: $\overline{F}^{1}(c) = \frac{\alpha}{2}$. Similarly, $\overline{F}^{0}(c) = \frac{\beta}{2}$ for $\lambda \to 0$.

4 Consequences on Integral Representation

The explicit evaluation of the Γ-limit on optimal microstructures allows us to deduce a strong property of the homogenized macroscopic energy: the loss of the classical double-integral form. To prove this, the following lemma regarding the strong convergence of characteristic functions will be useful.

Lemma 4.1. *Let $\{A_n\}$ be a sequence of measurable subsets of Ω and let $A \subset \Omega$. If the sequence of characteristic functions $\{\chi_{A_n}\}$ weakly converges to the characteristic function χ_A in $L^1(\Omega)$ as $n \to \infty$, then the convergence is indeed strong in $L^1(\Omega)$.*

Proof. The weak convergence in L^1 implies that for any test function $v \in L^\infty(\Omega)$, we have $\int_\Omega \chi_{A_n} v \to \int_\Omega \chi_A v$. Choosing $v \equiv 1$, we get $\int_\Omega \chi_{A_n} \to \int_\Omega \chi_A$, which means $|A_n| \to |A|$. Choosing $v = \chi_A$, we get $\int_\Omega \chi_{A_n}\chi_A \to \int_\Omega \chi_A \chi_A = \int_\Omega \chi_A$, which means that $|A_n \cap A| \to |A|$. We can now conclude by expanding the L^1 norm of the difference:

$$\lim_{n\to\infty} \|\chi_{A_n} - \chi_A\|_{L^1(\Omega)} = \lim_{n\to\infty} (|A_n| + |A| - 2|A_n \cap A|) = |A| + |A| - 2|A| = 0. \qquad \square$$

Theorem 4.2. *There exists no function g such that the Γ-limit of $\{F_\varepsilon^\lambda\}$ can be represented as an integral functional of the form*

$$\Big(\Gamma - \lim_{\varepsilon\to 0} F_\varepsilon^\lambda\Big)(u) = \int_{\Omega\times\Omega} g(u(x) - u(y))\, \mathrm{d}x\mathrm{d}y, \tag{15}$$

for every $u \in L^1(\Omega)$.

Proof. We argue by contradiction, similarly to [3], and suppose that such a representation exists. We already explicitly computed the exact Γ-limit for constant functions in Theorem 3.3. Now, let $s \in (0,1)$ and define the target step function u_s as

$$u_s(x) := \begin{cases} 1 & \text{if } x \in (0,s], \\ 0 & \text{if } x \in (s,1). \end{cases}$$

We claim that for this specific profile, the Γ-limit evaluates to

$$\Big(\Gamma - \lim_{\varepsilon\to 0} F_\varepsilon^\lambda\Big)(u_s) = \overline{a}_\lambda \int_{\Omega\times\Omega} \Big[u_s(x)u_s(y) + (1-u_s(x))(1-u_s(y))\Big]\, \mathrm{d}x\mathrm{d}y =: T^\lambda(u_s). \tag{16}$$

The value $T^\lambda(u_s)$ can be computed explicitly, noting that the integrand is 1 when x and y are on the same side of the jump s, and 0 otherwise, yielding $T^\lambda(u_s) = \overline{a}_\lambda(s^2 + (1-s)^2)$.

Liminf inequality. Let $\{u_\varepsilon\}$ be an arbitrary sequence weakly converging to u_s in $L^1(\Omega)$. As a consequence of Lemma 2.1, see (4), we can write $u_\varepsilon = z_\varepsilon + \chi_\varepsilon$. Since $u_s \in \{0,1\}$ is itself a characteristic function, the weak convergence forces $z_\varepsilon \to 0$ and $\chi_\varepsilon \rightharpoonup u_s$ in $L^1(\Omega)$. By Lemma 4.1, the convergence of $\{\chi_\varepsilon\}$ to u_s is indeed strong in $L^1(\Omega)$. This allows us to pass to the limit in the term $F_\varepsilon^\lambda(u_\varepsilon)$, decoupling the strong convergence of $\chi_\varepsilon(x)\chi_\varepsilon(y)$ from the weak convergence of $a_\lambda\left(\frac{x-y}{\varepsilon}\right) \rightharpoonup \overline{a}_\lambda$, achieving the expression in (16).

Limsup inequality. We choose the constant recovery sequence $u_\varepsilon^* \equiv u_s$. Applying the Riemann–Lebesgue Lemma to the oscillating weight $a_\lambda\left(\frac{x-y}{\varepsilon}\right)$, we immediately recover the explicit value of $T^\lambda(u_s)$.

We now match these two exact evaluations with the hypothetical formula (15), yielding a counterexample analogous to [3]. For the constant function $u \equiv c$, since $u(x) - u(y) = 0$, formula (15) combined with Theorem 3.3 implies

$$g(0) = \int_{\Omega\times\Omega} g(0)\,\mathrm{d}x\mathrm{d}y = \overline{F}^\lambda(c) = \frac{(1-(1-\lambda)^2)\alpha + (1-\lambda)^2\beta}{2}.$$

Now, evaluating (15) on the step function u_s and equating it to (16), we obtain

$$T^\lambda(u_s) = \int_{\Omega\times\Omega} g(u_s(x) - u_s(y))\,\mathrm{d}x\mathrm{d}y = (s^2 + (1-s)^2)g(0) + 2s(1-s)g(1).$$

By substituting the known values for $T^\lambda(u_s)$ and $g(0)$, we obtain

$$\overline{a}_\lambda(s^2 + (1-s)^2) = (s^2 + (1-s)^2)\frac{(1-(1-\lambda)^2)\alpha + (1-\lambda)^2\beta}{2} + 2s(1-s)g(1),$$

and solving for $g(1)$ gives

$$g(1) = \frac{(s^2 + (1-s)^2)}{2s(1-s)}\frac{\lambda^2\alpha + (1-\lambda^2)\beta}{2}.$$

Consequently, the value of $g(1)$ depends explicitly on the arbitrary jump point $s \in (0,1)$, which is a contradiction. ☐

Remark 4.3. Our main results rely on a function f that takes the value $+\infty$ almost everywhere. However, as pointed out in [3], an extension to finite functions occurs by considering, for a fixed constant $M > 0$, the everywhere finite integrand

$$f_M(z) = \begin{cases} 0 & \text{if } z \in \{-1, 1\} \\ 1 & \text{if } z = 0 \\ M & \text{otherwise,} \end{cases}$$

which is such that $f_M \nearrow f$. By choosing a sufficiently large M, one can easily show that any deviation from increments in $\{-1, 0, 1\}$ becomes strictly sub-optimal. Consequently, the optimal profiles for f_M rigidly coincide with those of the infinite function, yielding the same macroscopic energy on our target functions.

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