RIGIDITY FOR THE PÓLYA-SZEGÖ INEQUALITY UNDER CIRCULAR REARRANGEMENT

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Abstract. A Pólya-Szegö inequality for the circular rearrangement is proven, under general assumptions. In addition, sufficient conditions are given, under which all the extremals of the inequality are symmetric.

1. Introduction

Rearrangement inequalities are a fundamental tool in the Calculus of Variations, since they allow to prove symmetry of solutions of variational problems and PDEs. For instance, the radial decreasing rearrangement and the Steiner rearrangement are by now standard tools that are commonly employed in various contexts. However, these techniques cannot be used in situations in which the functions involved can change sign, or when they are defined in domains that are not simply connected. To tackle these problems, in this paper we consider the circular rearrangement of functions, in which level sets are rearranged using circular arcs. Our main contributions are the following:

(1) We establish a Pólya-Szegö inequality for the circular rearrangement in any dimension and under general assumptions, both on the integrand and on the class of functions under consideration, see Theorem 1.12 and Theorem 1.13 (see also Theorem 4.3 for a more general statement).

  (1a) To the best of our knowledge, the Pólya-Szegö inequalities currently available in the literature for the circular symmetrization only deal with nonnegative functions [25, 29, 34, 36]. Instead, we consider also the case of functions that can change sign.

  (1b) Our results can be applied in any dimension, in situations in which one expects minimizers to have cylindrical symmetry (or, more precisely, to be 2*-sectionally foliated Schwarz symmetric*, see [18, Definition 1.2]). In [25, 34, 36], the authors consider the spherical symmetrization, in which level sets are rearranged using spherical caps. This coincides with the circular symmetrization only in dimension 2. Even in the 2-dimensional case, we extend the known results, by considering a general class of integrands, and functions that can change sign.

  (1c) Our proof of the Pólya-Szegö inequality is *not* obtained via approximation. We give a direct proof, and this allows us to study in detail the equality cases.

(2) We give sufficient conditions under which rigidity holds, that is, under which all extremals are symmetric, see Theorem 1.18. Such conditions are given in terms of the notion of essential connectedness (see Definition 2.2 and [8, 9]).

(3) The results we prove in this paper will be instrumental in the study of a general Pólya-Szegö inequality under spherical symmetrization. In this case, when trying to give a direct proof of the inequality one encounters a major technical difficulty, due to the existence of functions that are the spherical counterpart of what Almgren and Lieb call *Coarea irregular functions* [1, Definition 1.2.6]. This implies that an identity of the type (3.7) cannot be obtained for the spherical rearrangement, see Remark 3.16 and Remark 3.18. This will be the object of further study [10].

We now describe how the rest of the Introduction is arranged. We start by recalling the classical Pólya–Szegö inequality (Section 1.1), and then we explain how the circular symmetrization operates on sets (Section 1.2) and on functions (Section 1.3). In Section 1.4 we state the Pólya–Szegö inequality under general assumptions (see Theorem 1.12 and Theorem 1.13), which is our first main result. In Section 1.5 we motivate the assumptions of Theorem 1.12, giving some counterexamples. Section 1.6 contains our second main result, which gives sufficient conditions for rigidity (Theorem 1.18), while in Section 1.7 we exhibit examples in which the assumptions of Theorem 1.18 are not satisfied and rigidity fails.

1.1. **Pólya-Szegö inequality for the Schwarz rearrangement.** In its most well-known version, the Pólya-Szegö inequality states that if $u : \mathbb{R}^n \to \mathbb{R}$ is any nonnegative smooth function with compact support and $1 \le p < \infty$, then

$$\|\nabla u^*\|_{L^p(\mathbb{R}^n)} \le \|\nabla u\|_{L^p(\mathbb{R}^n)}, \tag{1.1}$$

where $u^* : \mathbb{R}^n \to \mathbb{R}$ denotes the radial decreasing (also known as Schwarz) rearrangement of u, see [30]. More in general, (1.1) is satisfied if u is nonnegative with compact support and belongs to the Sobolev space $W^{1,p}(\mathbb{R}^n)$. We say that rigidity holds for the previous inequality if the following implication is true:

$$u \text{ satisfies equality in (1.1)} \quad \Longrightarrow \quad \exists\, c \in \mathbb{R}^n : \quad u(\cdot) = u^*(\cdot + c)\ \mathcal{L}^n\text{-a.e. in } \mathbb{R}^n.$$

where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure. Let us now set $M = \operatorname{ess\,sup} u^*$, and $C^* := \{\nabla u^* = 0\} \cap (u^*)^{-1}((0, M))$. In the seminal paper [6], Brothers and Ziemer showed that if

$$\mathcal{L}^n(C^*) = 0 \tag{1.2}$$

then rigidity holds, see [6, Theorem 1.1]. Very recently, the first author has proved that condition (1.2) is also necessary for rigidity [7]. Inequality (1.1) and its rigidity have also been studied for functions of bounded variation [14], and for the Steiner rearrangement of codimension 1 [15] and of any codimension [12]. When rigidity holds, the stability of the inequality can also be investigated, see [3] for the cases of Steiner and Schwarz rearrangements.

Inequality (1.1) has been used in a number of applications, as for instance the characterization of the extremals of Sobolev inequality [35], and the proof of a sharp quantitative version of the same inequality [16].

There are, however, several variational problems in which these techniques cannot be applied. This happens, for instance, if the domain of the functions under consideration is not simply connected (e.g. annular domains in the plane), or in which the functions can change sign and have nodal domains.

As an example of a situation in which Steiner and Schwarz symmetrization, or the moving planes method [32] cannot be applied, we mention [23], where the authors use the spherical symmetrization to show the symmetry of the extremals of Morrey's inequality (see also [22, 24], where several other properties of the extremals are proved). We observe that in [23], using additional symmetry properties of the extremals, the authors are able to apply the rearrangement to non negative functions. One of the goals of this paper is to show that the circular rearrangement can also be used with functions that change sign, see Definition 1.2 and Remark 1.16 for more details.

1.2. **Circular symmetrization.** To the best of our knowledge, the circular symmetrization of sets and the associated rearrangement for functions were firstly introduced by Pólya in [29]. Let $N \in \mathbb{N}$ with $N \ge 2$, and let us label the points of $\mathbb{R}^N$ as (x, z), with $x \in \mathbb{R}^2$ and $z \in \mathbb{R}^{N-2}$. Moreover, for every $r > 0$ we set $D(r) = \{x \in \mathbb{R}^2 : |x| < r\}$ and $\partial D(r) = \{x \in \mathbb{R}^2 : |x| = r\}$.

Let now $E \subset \mathbb{R}^N$ be a Lebesgue measurable set. For every $(r, z) \in (0, \infty) \times \mathbb{R}^{N-2}$, we define the slice $E_{(r,z)}$ of E at (r, z) as the subset of $\mathbb{R}^2$ given by

$$E_{(r,z)} := \{x \in \partial D(r) \text{ such that } (x, z) \in E\}.$$

We introduce the circular projection $\Pi_{N-1}(E)$ of E and its annular part $\Pi^{\mathrm{a}}_{N-1}(E)$ as

$$\Pi_{N-1}(E) := \{(r, z) \in (0, \infty) \times \mathbb{R}^{N-2} : \mathcal{H}^1(E_{(r,z)}) \neq 0\}, \tag{1.3}$$

and

$$\Pi^{\mathrm{a}}_{N-1}(E) := \left\{(r, z) \in \Pi_{N-1}(E) : \mathcal{H}^1(E_{(r,z)}) = 2\pi r\right\}, \tag{1.4}$$

respectively, where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure in $\mathbb{R}^N$.

The circular symmetrization (with respect to the half-hyperplane $\{x_2 = 0\} \cap \{x_1 > 0\}$) of a Lebesgue measurable set E is the Lebesgue measurable set E^s such that, for every $(r, z) \in (0, \infty) \times \mathbb{R}^{N-2}$, the slice $E^s_{(r,z)}$ is a connected arc centred at the point $(r, 0)$ having the same $\mathcal{H}^1$-measure as $E_{(r,z)}$. More precisely, setting $\mathbb{R}^2_0 = \mathbb{R}^2 \setminus \{(0, 0)\}$ and $\hat{x} = x/|x|$, we define E^s as (see Figure 1.1)

$$E^s := \{(x, z) \in \mathbb{R}^2_0 \times \mathbb{R}^{N-2} : 2|x| \arccos(\hat{x} \cdot e_1) < \mathcal{H}^1(E_{(r,z)})\},$$

where $e_1 = (1, 0)$. Let us observe that, if E is open, then the set E^s defined above is open, see Remark 2.5 and Proposition 2.6. Moreover, the circular symmetrization preserves the N-dimensional Lebesgue measure and does not increase the perimeter.

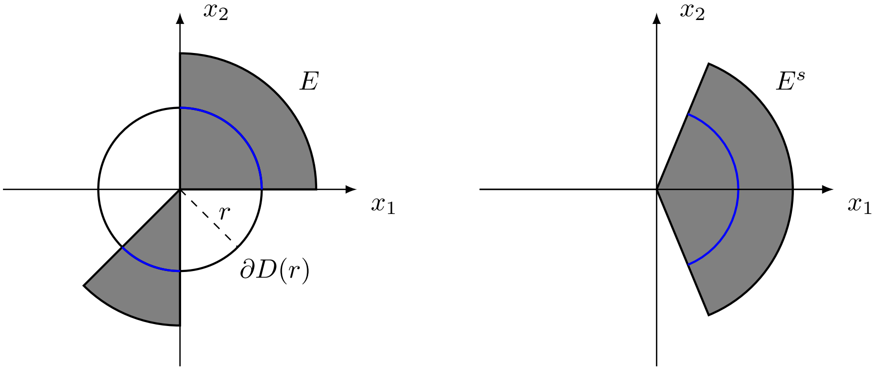

FIGURE 1.1. An example of circular symmetrization when $N = 2$. For a given value of $r > 0$, the slices E_r and E^s_r are highlighted in blue.

In order to give a precise statement of this fact, let us denote by $P(E)$ the distributional perimeter of E and, for $A \subset \mathbb{R}^N$ Borel, let $P(E, A)$ stand for the distributional perimeter of E in A (see Section 2). Moreover, let $\Phi_N : (0, \infty) \times \mathbb{R}^{N-2} \times \mathbb{S}^1 \to \mathbb{R}^2_0 \times \mathbb{R}^{N-2}$ be the diffeomorphism given by

$$\Phi_N(r, z, \omega) := (r\omega, z) \quad \text{for every } (r, z, \omega) \in (0, \infty) \times \mathbb{R}^{N-2} \times \mathbb{S}^1. \tag{1.5}$$

We then have the following result, see [11, Theorem 1.1], and [28, Theorem 1.3].

Theorem 1.1. *Let $E \subset \mathbb{R}^N$ be a set of finite perimeter in $\mathbb{R}^N$ with $\mathcal{L}^N(E) < \infty$. Then, E^s is a set of finite perimeter in $\mathbb{R}^N$ with $\mathcal{L}^N(E^s) = \mathcal{L}^N(E)$. Moreover,*

$$P(E^s; \Phi_N(B \times \mathbb{S}^1)) \leq P(E; \Phi_N(B \times \mathbb{S}^1)) \quad \text{for every Borel set } B \subset (0, \infty) \times \mathbb{R}^{N-2}. \tag{1.6}$$

Choosing $B = (0, \infty) \times \mathbb{R}^{N-2}$ in (1.6), we obtain that $P(E^s) \leq P(E)$, see Figure 1.2. However, more in general inequality (1.6) holds locally, as shown in Figure 1.3.

Let us mention that some variants of the circular symmetrization can be used when one wants to preserve the barycenter or some symmetry properties of a set, see for instance [5, 20].

1.3. **State of the art.** Let $n \in \mathbb{N}$, $n \geq 2$, and let us denote points of $\mathbb{R}^n$ as (x, y), with $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^{n-2}$. Let $\Omega \subset \mathbb{R}^n$ be open, and let $u : \Omega \to \mathbb{R}$ be a Lebesgue measurable function. The circular rearrangement u^s of u is the Lebesgue measurable function $u^s : \Omega^s \to \mathbb{R}$ such that

$$\{u^s > t\} = \{u > t\}^s, \quad \text{for every } t \in \mathbb{R}, \tag{1.7}$$

where Ω^s and $\{u > t\}^s$ are the circular rearrangements (in $\mathbb{R}^n$) of the sets Ω and $\{u > t\}$, respectively, see Figure 1.4.

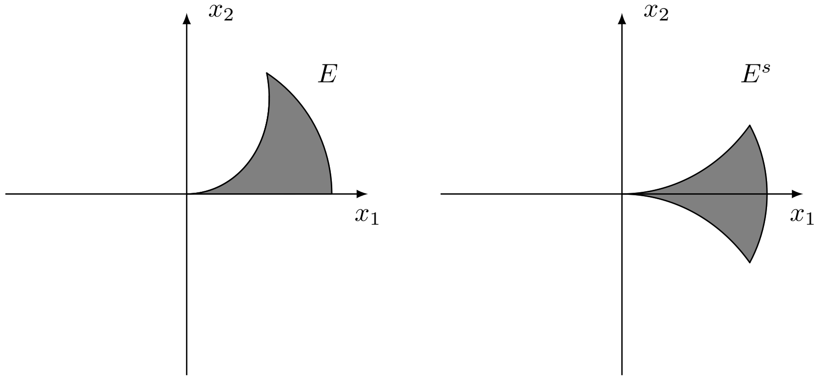

FIGURE 1.2. Perimeter inequality for $N = 2$. The perimeter of the set E^s (right), is less than or equal to the perimeter of the set E (left).

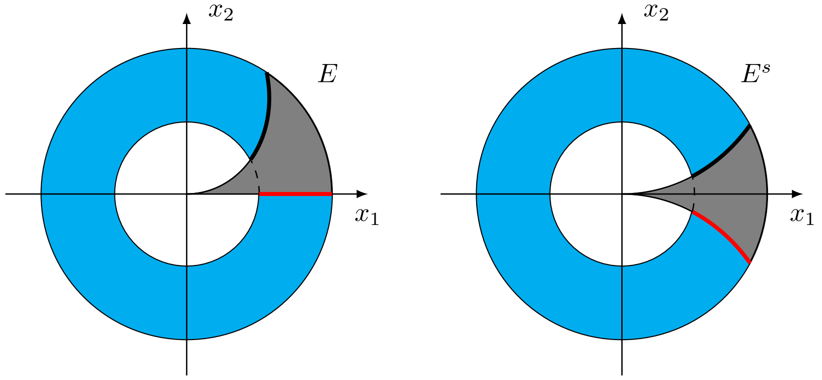

FIGURE 1.3. The same sets considered in Figure 1.2. If one chooses B as an open interval, the corresponding set $\Phi_2(B \times \mathbb{S}^1)$ is the open annulus highlighted in blue. Then, inequality (1.6) states that the **sum** of the lenghts of the black and red arcs in the right, is less than or equal to the **sum** of the lenghts of the two corresponding arcs in the figure in the left. Note that this **does not mean** that the length of each arc decreases after symmetrization. In fact, the length of the red arc in the right is larger than the length of the corresponding red arc in the left.

Pólya firstly observed that, if $n = 2$, Ω is bounded, and u is smooth with $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, then

$$\int_{\Omega^s} |\nabla u^s(x)|^2 \, dx \le \int_\Omega |\nabla u(x)|^2 \, dx. \tag{1.8}$$

The proof provided by Pólya in [29] is quite elegant and goes along the following lines. For every $\varepsilon > 0$, one can consider the subgraph of εu:

$$\Sigma^{\varepsilon u} := \{(x,t) \in \Omega \times \mathbb{R} : t < \varepsilon u(x)\}.$$

Then, if $(\Sigma^{\varepsilon u})^s$ denotes the circular symmetrization of $\Sigma^{\varepsilon u}$ in $\mathbb{R}^3$, we have

$$(\Sigma^{\varepsilon u})^s = \Sigma^{\varepsilon u^s}.$$

Since the circular symmetrization does not increase the perimeter, and since $\mathcal{L}^2(\Omega^s) = \mathcal{L}^2(\Omega)$, one has

$$\int_{\Omega^s} \left[\sqrt{1 + \varepsilon^2 |\nabla u^s(x)|^2} - 1\right] dx \le \int_\Omega \left[\sqrt{1 + \varepsilon^2 |\nabla u(x)|^2} - 1\right] dx. \tag{1.9}$$

Dividing the previous inequality by ε^2 and taking the limit as $\varepsilon \to 0^+$, we obtain (1.8).

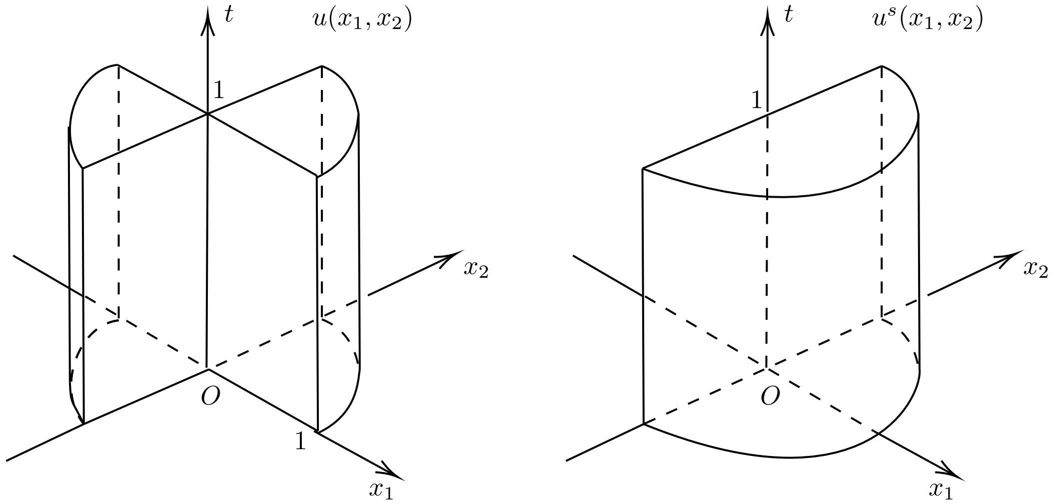

FIGURE 1.4. In this case, $n = 2$ and $\Omega = D(1) = \{|x| < 1\}$. In the left, the graph of the measurable function $u : D(1) \to \mathbb{R}$ defined as $u(x_1, x_2) = 1$ if $x_1 x_2 > 0$, and $u(x_1, x_2) = 0$ elsewhere in $D(1)$. In the right, the graph of the function u^s. We have $u^s(x_1, x_2) = 1$ if $x_1 > 0$, and $u^s(x_1, x_2) = 0$ elsewhere in $D(1)$.

In 1985, Kawohl showed that if $n = 2$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^2$ is bounded and $u \in W_0^{1,p}(\Omega)$ with $u \geq 0$, then $u^s \in W_0^{1,p}(\Omega^s)$ and

$$\int_{\Omega^s} F(|x|, u^s)|\nabla u^s|^p \, dx \leq \int_{\Omega} F(|x|, u)|\nabla u|^p \, dx,$$

whenever $F : [0, \infty) \times \mathbb{R} \to [0, \infty)$ is continuous [25, Corollary 2.35, ii)]. Under the additional assumptions that $\partial\Omega$ is piecewise analytic and u is analytic, he also proved that

$$\int_{\Omega^s} F(|x|, u^s) \, G(|\nabla u^s|) \, dx \leq \int_{\Omega} F(|x|, u) \, G(|\nabla u|) \, dx, \tag{1.10}$$

whenever $G : [0, \infty) \to \mathbb{R}$ is nondecreasing and convex.

A more recent result is due to Smets and Willem. If $n = 2$, $1 < p < \infty$, $u \in W_0^{1,p}(\Omega)$ and $u \geq 0$, they showed that

$$\int_{\Omega^s} F(|x|)|\nabla u^s|^p \, dx \leq \int_{\Omega} F(|x|)|\nabla u|^p \, dx, \tag{1.11}$$

whenever F is nonnegative, measurable and bounded [34, Theorem 2.8]. More precisely, the authors prove (1.11) when u^s is interpreted as the foliated Schwarz symmetrization of u, in which level sets are rearranged via spherical symmetrization (see [34, Definition 2.4]), in any dimension. We observe that, while in the case $n = 2$ this coincides with the circular symmetrization, for $n \geq 3$ the two symmetrizations differ.

1.4. **Pólya–Szegö inequality for the circular rearrangement under general assumptions.** The first result we show in this paper is that a Pólya-Szegö inequality holds for the circular symmetrization under very general assumptions. First of all, let us introduce the class of functions under consideration. In the following, when $A, B \subset \mathbb{R}^n$ we write $A \subset\subset B$ if A is compactly contained in B.

Definition 1.2. *Let $n \in \mathbb{N}$ with $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be open, let $p \in [1, +\infty)$, let $u : \Omega \to \mathbb{R}$, and let u_0 be given by*

$$u_0(x, y) := \begin{cases} u(x, y) & \text{if } (x, y) \in \Omega, \\ 0 & \text{if } (x, y) \in \Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1) \setminus \Omega. \end{cases} \tag{1.12}$$

We say that $u \in W_{0,\tau}^{1,p}(\Omega)$ if the following conditions are satisfied:

(a$_p$) *$u_0 \in W^{1,p}(\Phi_n(A \times \mathbb{S}^1))$ for every open set $A \subset (0, \infty) \times \mathbb{R}^{n-2}$ with $A \subset\subset \Pi_{n-1}(\Omega)$,*

(b) $u \geq 0 \quad \mathcal{L}^n$-*a.e. in* $\Omega \setminus \Phi_n(\Pi^{\mathrm{a}}_{n-1}(\Omega) \times \mathbb{S}^1)$,

where Φ_n, $\Pi_{n-1}(\Omega)$, *and* $\Pi^{\mathrm{a}}_{n-1}(\Omega)$ *are defined by* (1.3), (1.4), *and* (1.5), *respectively.*

Let now $u : \Omega \to \mathbb{R}$ be Lebesgue measurable. We define the distribution of u as the function $\mu : \Pi_{n-1}(\Omega) \times \mathbb{R} \to [0, \infty)$ given by

$$\mu(r, y, t) = \mathcal{H}^1(\{u_0 > t\}_{(r,y)}), \quad \text{for every } (r, y, t) \in \Pi_{n-1}(\Omega) \times \mathbb{R}, \tag{1.13}$$

where u_0 is given by (1.12).

Remark 1.3. *We observe that, if* (1.13) *holds for some Lebesgue measurable function* $u : \Omega \to \mathbb{R}$, *then*

$$0 \leq \mu(r, y, t) \leq 2\pi r \quad \text{for every } (r, y, t) \in \Pi_{n-1}(\Omega) \times \mathbb{R},$$

and

$$t \longmapsto \mu(r, y, t) \quad \text{is non-increasing and right-continuous,} \quad \text{for every } (r, y) \in \Pi_{n-1}(\Omega).$$

Moreover, since $u(x, y)$ *is finite for every* $(x, y) \in \Omega$, *we have*

$$\lim_{t \to +\infty} \mu(r, y, t) = 0 \quad \text{and} \quad \lim_{t \to -\infty} \mu(r, y, t) = 2\pi r, \quad \text{for every } (r, y) \in \Pi_{n-1}(\Omega).$$

Remark 1.4. *Let us note that, by Hölder inequality,* $W^{1,p}_{0,\tau}(\Omega) \subset W^{1,1}_{0,\tau}(\Omega)$ *for every* $p \in [1, +\infty)$, *see also Remark 3.4.*

We stress the fact that if $u \in W^{1,p}_{0,\tau}(\Omega)$, this does not imply that $u = 0$ on all of $\partial\Omega$, as explained in the next remark.

Remark 1.5. *If* $u \in W^{1,p}_0(\Omega)$, *then condition* (a_p) *is satisfied. However, the opposite implication is false. Indeed, let* $n = 2$, *let* $\Omega = D(1) \cap \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0\}$, *let* $u(x_1, x_2) = x_1 x_2$, *and let* $p \in [1, +\infty)$. *Then* u *satisfies* (a_p), *but* $u \notin W^{1,p}_0(\Omega)$. *This example also shows that* $W^{1,p}_{0,\tau}(\Omega) \not\subset W^{1,p}_0(\Omega)$.

Remark 1.6. *Note that* $W^{1,p}_{0,\tau}(\Omega)$ *is not a vector space. Indeed, condition (b) is not closed under scalar multiplication. For the same reason, unless* $\Pi^{\mathrm{a}}_{n-1}(\Omega) = \Pi_{n-1}(\Omega)$, *we have* $W^{1,p}_0(\Omega) \not\subset W^{1,p}_{0,\tau}(\Omega)$. *However,*

$$u \geq 0 \quad \text{and} \quad u \in W^{1,p}_0(\Omega) \quad \Longrightarrow \quad u \in W^{1,p}_{0,\tau}(\Omega).$$

Definition 1.7. *Let* $n \in \mathbb{N}$ *with* $n \geq 2$, *let* $\Omega \subset \mathbb{R}^n$ *be open and let* $\mu : \Pi_{n-1}(\Omega) \times \mathbb{R} \to [0, \infty)$ *be a Lebesgue measurable function. We say that* μ *is an admissible distribution if* (1.13) *holds for (a representative of) some* $u \in W^{1,1}_{0,\tau}(\Omega)$. *The set of all admissible distributions is denoted by* $\mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$. *We say that a function* $u \in W^{1,1}_{0,\tau}(\Omega)$ *is* μ*-distributed if there exists a representative of* u *such that* (1.13) *holds.*

It turns out that if $\mu \in \mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$, then $\mu \in BV(A \times (-d, +\infty))$ for every open set $A \subset\subset \Pi_{n-1}(\Omega)$ and for every $d > 0$, see Proposition 3.24.

The circular rearrangement of a function only depends on its distribution. For this reason, whenever $\mu \in \mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$ (or, more in general, whenever (1.13) holds for some Lebesgue measurable function $u : \Omega \to \mathbb{R}$), we define $v_\mu : \Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1) \to \overline{\mathbb{R}}$ as

$$v_\mu(x, y) := \inf \{t \in \mathbb{R} : \mu(|x|, y, t) \leq 2|x| \arccos(\hat{x} \cdot e_1)\}. \tag{1.14}$$

Thanks to Remark 1.3, $v_\mu(x, y)$ is finite except possibly in the hylperplane $\{x_2 = 0\}$, which is $\mathcal{H}^n$-negligible. Note that, even if u is defined only in Ω, the function v_μ is defined in the (larger) set $\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1)$, see Remark 1.16 for further comments. We also point out that the notion of distribution is well defined, as clarified by the next remarks.

Remark 1.8. *Let* $u_1, u_2 \in L^1_{\mathrm{loc}}(\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1))$, *and let* μ_{u_1} *and* μ_{u_2} *be the distributions of* u_1 *and* u_2, *respectively. If* $u_1 = u_2$ $\mathcal{L}^n$*-a.e. in* $\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1))$, *then one can see that* $\mu_{u_1} = \mu_{u_2}$ $\mathcal{L}^n$*-a.e. in* $\Pi_{n-1}(\Omega) \times \mathbb{R}$ *(see Proposition 3.10).*

Remark 1.9. *By definition, v_μ is μ-distributed. Moreover, if $\mu_1, \mu_2 \in \mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$ with $\mu_1 = \mu_2$ $\mathcal{L}^n$-a.e. in $\Pi_{n-1}(\Omega) \times \mathbb{R}$, one can see that $v_{\mu_1} = v_{\mu_2}$ $\mathcal{L}^n$-a.e. in $\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1)$ (see Proposition 3.21). Also, if $u \in W^{1,1}_{0,\tau}(\Omega)$ is a μ-distributed function, then $v_\mu|_{\Omega^s} \in W^{1,1}_{0,\tau}(\Omega^s)$ (see Proposition 3.31), and $v_\mu = u^s$ $\mathcal{L}^n$-a.e. in Ω^s, where u^s is such that* (1.7) *holds.*

Let us now focus on the general assumptions on the integrand. If $u \in W^{1,p}_{0,\tau}(\Omega)$, for every $(x, y) \in \Omega$ such that the gradient $\nabla u(x, y)$ at (x, y) is defined, we write $\nabla u(x, y) = (\nabla_x u(x, y), \nabla_y u(x, y))$, where $\nabla_x u(x, y) \in \mathbb{R}^2$ and $\nabla_y u(x, y) \in \mathbb{R}^{n-2}$, with

$$\nabla_x u(x, y) = (\partial_{x_1} u(x, y), \partial_{x_2} u(x, y)), \quad \nabla_y u(x, y) = (\partial_{y_1} u(x, y), \dots, \partial_{y_{n-2}} u(x, y)).$$

Moreover, if $x = (x_1, x_2) \in \mathbb{R}^2_0$, we can further decompose $\nabla_x u(x, y)$ into radial and tangential components as

$$\nabla_x u(x, y) = (\hat{x} \cdot \nabla_x u(x, y))\hat{x} + \nabla_{x_\parallel} u(x, y) x_\parallel, \qquad \nabla_{x_\parallel} u(x, y) = \nabla_x u(x, y) \cdot x_\parallel,$$

where

$$\hat{x} = \left(\frac{x_1}{|x|}, \frac{x_2}{|x|}\right) \qquad \text{and} \qquad x_\parallel = \left(-\frac{x_2}{|x|}, \frac{x_1}{|x|}\right). \tag{1.15}$$

Definition 1.10. *Let $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \to [0, \infty)$. We say that $f \in \mathscr{F}$ if the following assumptions are satisfied:*

(f1) *f is convex;*

(f2) *$f(\eta, -\tau, \zeta) = f(\eta, \tau, \zeta)$ for every $(\eta, \tau, \zeta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$.*

We say that $f \in \mathscr{F}'$ if, in addition,

(f1') *f is **strictly** convex.*

Remark 1.11. *Note that the classical Dirichlet functional can be written as*

$$\int_\Omega |\nabla u|^2 \, dx\, dy = \int_\Omega f(\hat{x} \cdot \nabla_x u, \nabla_{x_\parallel} u, \nabla_y u) \, dx\, dy,$$

with $f \in \mathscr{F}'$ given by $f(\eta, \tau, \zeta) = \eta^2 + \tau^2 + |\zeta|^2$.

We can now state our first result, which shows that a general version of the Pólya–Szegö inequality holds locally. We recall that the set $\mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$ of admissible distributions has been introduced in Definition 1.7.

Theorem 1.12. *Let $n \in \mathbb{N}$ with $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be open, and let $\mu \in \mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$. Let $a \in L^\infty((0, \infty) \times \mathbb{R}^{n-2} \times \mathbb{R})$ with $a \geq 0$ $\mathcal{H}^n$-a.e., and let $f \in \mathscr{F}$. Then, for every μ-distributed function $u \in W^{1,1}_{0,\tau}(\Omega)$ we have*

$$\begin{aligned} &\int_{\Phi_n(B \times \mathbb{S}^1)} a(|x|, y, v_\mu) f(\hat{x} \cdot \nabla_x v_\mu, \nabla_{x_\parallel} v_\mu, \nabla_y v_\mu) \, dxdy \\ &\leq \int_{\Phi_n(B \times \mathbb{S}^1)} a(|x|, y, u_0) f(\hat{x} \cdot \nabla_x u_0, \nabla_{x_\parallel} u_0, \nabla_y u_0) \, dxdy, \end{aligned} \tag{1.16}$$

for every Borel set $B \subset \Pi_{n-1}(\Omega)$, where u_0 and v_μ are given by (1.12) *and* (1.14), *respectively.*

A consequence of the previous result is the following.

Theorem 1.13. *Let $n \in \mathbb{N}$ with $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be open, and let $\mu \in \mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$. Let $a \in L^\infty((0, \infty) \times \mathbb{R}^{n-2} \times \mathbb{R})$ with $a \geq 0$ $\mathcal{H}^n$-a.e., and let $f \in \mathscr{F}$. Then, for every μ-distributed function $u \in W^{1,1}_{0,\tau}(\Omega)$ we have*

$$\begin{aligned} &\int_{\Omega^s} a(|x|, y, v_\mu) f(\hat{x} \cdot \nabla_x v_\mu, \nabla_{x_\parallel} v_\mu, \nabla_y v_\mu) \, dxdy \\ &\leq \int_\Omega a(|x|, y, u) f(\hat{x} \cdot \nabla_x u, \nabla_{x_\parallel} u, \nabla_y u) \, dxdy, \end{aligned} \tag{1.17}$$

where v_μ is given by (1.14).

Remark 1.14. *In the statements above Ω does not need to be bounded. Moreover,* (1.16) *and* (1.17) *are inequalities between extended real numbers in* $[0,\infty]$.

We can actually prove a more general inequality than (1.16), see Theorem 4.3. From these results, in particular, it follows that if $p \in [1,+\infty)$ and $u \in W^{1,p}_{0,\tau}(\Omega)$, then $v_\mu \,|_{\Omega^s} \in W^{1,p}_{0,\tau}(\Omega^s)$, see Corollary 4.4.

1.5. **Comments on the assumptions of Theorem 1.12.**

In the next example we show why we need condition (a$_p$) of Definition 1.2.

Example 1.15. *Let $n=2$, let $\Omega = D(1) \cap \{x=(x_1,x_2)\in\mathbb{R}^2 : x_1,x_2>0\}$, and let $u:\Omega\to\mathbb{R}$ be given by*

$$u(x_1,x_2) = x_1 \qquad \text{for every } (x_1,x_2)\in\Omega.$$

Then, $\Omega^s = D(1)\cap\{x=(x_1,x_2)\in\mathbb{R}^2_0 : |x_2|<x_1\}$ and

$$v_\mu(x_1,x_2) = \frac{x_1^2-x_2^2}{|x|} \qquad \text{for every } (x_1,x_2)\in\Omega^s,$$

where μ and v_μ are given by (1.13) *and* (1.14)*, respectively. Thus, recalling that $\mathcal{L}^2(\Omega^s)=\mathcal{L}^2(\Omega)$, we have*

$$\int_{\Omega^s}|\nabla v_\mu|^2\,dx = \int_{\Omega^s}\left(1+12\frac{x_1^2x_2^2}{|x|^4}\right)dx > \mathcal{L}^2(\Omega) = \int_\Omega |\nabla u|^2\,dx.$$

Note that in the previous example $u\notin W^{1,p}_{0,\tau}(\Omega)$, since condition (a$_p$) of Definition 1.2 is not satisfied. Let us explain why this causes inequality (1.8) to fail. First of all, we observe that in this case the perimeter inequality (1.6) (with $N=3$ and $B=(0,1)\times\mathbb{R}$), applied to the sets $E=\Sigma^u$ and $E^s=\Sigma^{v_\mu}$, still holds. However, the boundary conditions of u are such that $u\neq 0$ on $I=\{(x_1,0):0<x_1<1\}$. For this reason, the perimeter of Σ^u in the open cylinder $\Phi_3(B\times\mathbb{S}^1)=(D(1)\setminus\{(0,0)\})\times\mathbb{R}$ has a non trivial contribution coming from $\partial\Sigma^u\cap(I\times\mathbb{R})$.

Then, if we try to reproduce Pólya's proof in this case, inequality (1.9) has an additional term appearing in the right-hand side, and this does not allow us to conclude. Note that one cannot simply disregard this additional term since, as shown in Figure 1.3, the perimeter inequality does not hold if one cherry-picks subsets of $\partial\Sigma^u$. Thus, some boundary conditions (which are encoded in condition (a$_p$)) are needed if one wants to avoid this situation.

Remark 1.16. *We observe that, although the function u is defined in Ω, the rearranged function v_μ is defined in the (larger) set $\Phi_n(\Pi_{n-1}(\Omega)\times\mathbb{S}^1)$. A natural question is whether it is possible to just define v_μ in Ω^s, without the need of considering the set $\Phi_n(\Pi_{n-1}(\Omega)\times\mathbb{S}^1)$. In this case, the idea would be to substitute the function μ given in* (1.13) *with the function*

$$\mu'(r,y,t) := \mathcal{H}^1(\{u>t\}_{(r,y)}).$$

Note that $\mu'\le\mu$, since $\{u>t\}\subset\{u_0>t\}$, and u_0 is an extension of u to the whole $\Phi_n(\Pi_{n-1}(\Omega)\times\mathbb{S}^1)$. Then, one could define a rearranged function $w_{\mu'}$ using the following variant of (1.14)*:*

$$w_{\mu'}(x,y) := \inf\{t\in\mathbb{R} : \mu'(|x|,y,t)\le 2|x|\arccos(\hat{x}\cdot e_1)\} \quad \forall\,(x,y)\in\Omega^s \text{ with } \hat{x}\cdot e_1 > -1.$$

One can check that, if $u\ge 0$ and condition (a$_p$) is satisfied, this definition is equivalent to the one given in (1.14)*. That is, $w_{\mu'}$ coincides with the restriction of the function v_μ given in* (1.14) *to the set Ω^s. However, if we are dealing with functions that change sign, the situation can be very different, as explained in the following example.*

In the next example we show why we need to impose condition (b) of Definition 1.2, and why an approach like the one described in Remark 1.16 does not work with functions that can change sign.

Example 1.17. *Let $n=2$, and let $\phi:(0,+\infty)\times[-\pi,\pi)\to\mathbb{R}^2_0$ be the polar change of coordinates, given by*

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta).$$

Let $a\in(0,\pi/4)$ and let $\Omega=\phi(O)$, where

$$O = \left\{(r,\theta) : 1<r<2,\ |\theta|<\frac{\pi}{4}+a(r-1)\right\}.$$

Note that, by construction, $\Omega^s = \Omega$. Consider now the function $u : \Omega \to \mathbb{R}$ given by $u = U \circ \phi^{-1}$, where $U : O \to \mathbb{R}$ is the piecewise affine function given by

$$U(r,\theta) = \begin{cases} |\theta| - \frac{\pi}{4} & \text{if } |\theta| \le \frac{\pi}{4}, \\ 0 & \text{otherwise in } O. \end{cases}$$

Let now μ' and $w_{\mu'}$ be given by Remark 1.16. We have $w_{\mu'} = V \circ \phi^{-1}$, where $V : O \to \mathbb{R}$ is the following piecewise affine function:

$$V(r,\theta) = \begin{cases} 0 & \text{if } |\theta| \le a(r-1), \\ -|\theta| + a(r-1) & \text{if } a(r-1) < |\theta| < a(r-1) + \frac{\pi}{4}. \end{cases}$$

In this case, u satisfies condition (a_p) for every $p \in [1, +\infty)$. However, a direct calculation shows that

$$\int_{\Omega^s} |\nabla w_{\mu'}|^2 \, dx = \frac{\pi}{2} \log(2) + \frac{3}{4}\pi a^2 \quad \textit{and} \quad \int_{\Omega} |\nabla u|^2 \, dx = \frac{\pi}{2} \log(2).$$

In the example above, we have

$$u = 0 \quad \text{on } \partial\Omega \cap \{1 < |x| < 2\},$$

so that u satisfies condition (a_p). Note, however, that $u \notin W^{1,p}_{0,\tau}(\Omega)$, since condition (b) of Definition 1.2 is not satisfied. In polar coordinates, u is independent of r and has global minimum $-\pi/4$, which is attained in the segment $\{(x_1, 0) : 1 < x_1 < 2\}$. Instead, the function $w_{\mu'}$ satisfies

$$w_{\mu'} = -\frac{\pi}{4} \quad \text{on } \partial\Omega \cap \{1 < |x| < 2\},$$

Thus, the construction of the function $w_{\mu'}$ shifts the minimum value $-\pi/4$ towards (a subset of) the boundary of Ω. This means that, depending on the shape of Ω, we can force the function $w_{\mu'}$ to also depend on r, therefore creating additional gradient. In fact, we have

$$|\nabla u(x)|^2 = \frac{1}{|x|^2} \qquad \text{for every } x \in \{\nabla u \neq 0\},$$

and

$$|\nabla w_{\mu'}(x)|^2 = \frac{1}{|x|^2} + a^2 \qquad \text{for every } x \in \{\nabla w_{\mu'} \neq 0\},$$

and this eventually causes the failure of Pólya-Szegö inequality.

1.6. **Rigidity.** Once a Pólya–Szegö inequality is established, we want to understand if the equality

$$\begin{aligned} &\int_{\Phi_n(\Pi_{n-1}(\Omega)\times\mathbb{S}^1)} a(|x|, y, v_\mu) f(\hat{x} \cdot \nabla_x v_\mu, \nabla_{x\|} v_\mu, \nabla_y v_\mu) \, dxdy \\ &= \int_{\Phi_n(\Pi_{n-1}(\Omega)\times\mathbb{S}^1)} a(|x|, y, u_0) f(\hat{x} \cdot \nabla_x u_0, \nabla_{x\|} u_0, \nabla_y u_0) \, dxdy \end{aligned} \tag{1.18}$$

implies that (up to orthogonal transformations) $u = v_\mu$. This question makes sense if the integrals under consideration are finite, so we will require that

$$\int_{\Phi_n(\Pi_{n-1}(\Omega)\times\mathbb{S}^1)} a(|x|, y, v_\mu) f(\hat{x} \cdot \nabla_x v_\mu, \nabla_{x\|} v_\mu . \nabla_y v_\mu) \, dxdy < \infty, \tag{1.19}$$

When (1.19) is satisfied, we define the set of extremals of (1.16) as

$$\mathcal{E}(\mu, \Omega) := \{u \in W^{1,1}_{0,\tau}(\Omega) : u \text{ is } \mu\text{-distributed and (1.18) holds}\}.$$

First of all, note that the following inclusion is always satisfied:

$$\mathcal{E}(\mu, \Omega) \supset \{u \in W^{1,1}_{0,\tau}(\Omega) : \; u(x, y) = v_\mu(Rx, y) \text{ for } \mathcal{H}^n\text{-a.e. } (x, y) \in \Omega, \text{ for some } R \in O(2)\},$$

where $O(2)$ is the set of orthogonal transformations of $\mathbb{R}^2$. We will say that *rigidity holds* for (1.16) if also the opposite inclusion is true, that is if

$$\mathcal{E}(\mu, \Omega) = \{u \in W^{1,1}_{0,\tau}(\Omega) : u(x, y) = v_\mu(Rx, y) \text{ for } \mathcal{H}^n\text{-a.e. } (x, y) \in \Omega, \text{ for some } R \in O(2)\}. \tag{$\mathcal{R}$}$$

As observed by Kawohl, in [30, pag. 186] Pólya and Szegö dismiss the study of rigidity (for the Steiner rearrangement) as "hopeless". Indeed, Pólya's proof of (1.8) given above does not provide any information about the functions u satisfying equality. Moreover, the proof of (1.11) by Smets and Willem (see [34]) is obtained by combining polarization and approximation arguments and, also in this case, this does not allow to study rigidity.

To the best of our knowledge, the problem of rigidity for the Pólya–Szegö inequality in the context of circular symmetrization was firstly considered by Kawohl. He showed that if in (1.10) one further assumes that Ω is an annulus, F is continuous and positive, G is increasing and strictly convex, and u is analytic, then rigidity holds [25, Corollary 2.35, iii)]. The proof relies on the smoothness of u, and uses the implicit function theorem.

Our goal is to show that rigidity holds also in higher dimensions, under much milder assumptions. To this aim, if $\mu \in \mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$, we define the function $\alpha_\mu : \Pi_{n-1}(\Omega) \times \mathbb{R} \to [0, \pi]$ as

$$\alpha_\mu(r, y, t) := \frac{\mu(r, y, t)}{2r} \qquad \text{for every } (r, y, t) \in \Pi_{n-1}(\Omega) \times \mathbb{R}. \tag{1.20}$$

For every $(r, y, t) \in \Pi_{n-1}(\Omega) \times \mathbb{R}$, the number $\alpha_\mu(r, y, t)$ gives half of the angle corresponding to a connected arc of $\partial D(r)$ with length $\mu(r, y, t)$. Thanks to Proposition 3.24, whenever $\mu \in \mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$, we have that $\alpha_\mu \in BV(A \times (-d, +\infty))$ for every open set $A \subset\subset \Pi_{n-1}(\Omega)$ and for every $d > 0$. We will make the following assumption:

$$\{0 < \alpha_\mu < \pi\} \text{ is not } \mathcal{H}^n\text{-equivalent to the empty set.} \tag{1.21}$$

This is because, if $\{0 < \alpha_\mu < \pi\}$ is empty, then rigidity trivially holds. Indeed, in this case **every** μ-distributed function is $\mathcal{H}^n$-equivalent to v_μ.

We can now state our second main result, which gives a sufficient condition for rigidity. This is written in terms of the notion of essential connectedness, see Section 2 and also [8, 9]. Roughly speaking, one can create a counterexample to rigidity whenever the set $\{0 < \alpha_\mu < \pi\}$ is "split into two pieces" by points where $\alpha_\mu = 0$, $\alpha_\mu = \pi$, or by points where the singular part of the distributional derivative $D\alpha_\mu$ is concentrated. Here, $\alpha^\wedge$ and $\alpha_\mu^\vee$ are the approximate liminf and the approximate limsup of α_μ, respectively, while $S_{\alpha_\mu} = \{\alpha^\wedge < \alpha_\mu^\vee\}$ is the singular set of α_μ, see again Section 2.

Theorem 1.18. *Let $n \in \mathbb{N}$ with $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be open, and let $\mu \in \mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$. Let $a \in L^\infty((0, \infty) \times \mathbb{R}^{n-2} \times \mathbb{R})$ with $a > 0$ $\mathcal{H}^n$-a.e., let $f \in \mathscr{F}'$, and suppose that (1.19) and (1.21) hold. Assume, in addition, that the Cantor part $D^c\alpha_\mu$ of $D\alpha_\mu$ is concentrated on a Borel set $K \subset \Pi_{n-1}(\Omega) \times \mathbb{R}$ such that*

$$\{\alpha_\mu^\wedge = 0\} \cup \{\alpha_\mu^\vee = \pi\} \cup S_{\alpha_\mu} \cup K \textit{ does not essentially disconnect } \{0 < \alpha_\mu < \pi\}.$$

Then, $(\mathcal{R})$ holds.

1.7. **Comments on the assumptions of Theorem 1.18.** First of all let us observe that, without assuming strict convexity of the integrand f (i.e. without assuming $f \in \mathscr{F}'$), one cannot expect rigidity to hold.

Example 1.19. *Let $n = 2$, $a \equiv 1$, and $f(\eta, \tau) = |\tau|$. Let $\Omega = \{x \in \mathbb{R}^2 : 1 < |x| < 2\}$, and let $\phi : (0, +\infty) \times [-\pi, \pi) \to \mathbb{R}^2_0$ be as in Example 1.17. Let $O = \{(r, \theta) : 1 < r < 2\}$, and let $u = W \circ \phi^{-1}$, where $W : O \to \mathbb{R}$ is defined as*

$$W(r, \theta) = \begin{cases} 2\theta + \frac{\pi}{2} & \text{if } \theta \in \left[-\frac{\pi}{4}, 0\right], \\ -\frac{2}{3}\theta + \frac{\pi}{2} & \text{if } \theta \in \left[0, \frac{3}{4}\pi\right], \\ 0 & \text{otherwise in } O. \end{cases}$$

Then, we have $\Omega^s = \Omega$ and $v_\mu = T \circ \phi^{-1}$, where $T : O \to \mathbb{R}$ is given by

$$T(r, \theta) = \begin{cases} -|\theta| + \frac{\pi}{2} & \text{if } |\theta| \leq \frac{\pi}{2}, \\ 0 & \text{otherwise in } D. \end{cases}$$

In this case,

$$\int_\Omega |\nabla_{x_\parallel} v_\mu| \, dx = \int_\Omega |\nabla_{x_\parallel} u| \, dx = \pi,$$

so that rigidity fails.

Remark 1.20. *More in general, let $a \equiv 1$ and $f(\eta,\tau,\zeta) = |\tau|$, and suppose that Ω is bounded. Then, any smooth function $u \in W^{1,1}_{0,\tau}(\Omega) \cap W^{1,1}(\Omega)$ with the property that*

$$\{u_0 > t\}_{(r,y)} \text{ is a (possibly empty) connected arc} \quad \forall\,(r,y,t) \in \Pi_{n-1}(\Omega)\times\mathbb{R}, \tag{1.22}$$

belongs to $\mathcal{E}(\mu,\Omega)$, and therefore rigidity fails.

Indeed, using property ii) of Proposition 3.12), we have

$$(\partial\Sigma^{u_0})_{(r,y,t)} = \partial\left((\Sigma^{u_0})_{(r,y,t)}\right) = \partial(\{u_0 > t\}_{(r,y)}), \quad \text{for } \mathcal{L}^n\text{-a.e. } (r,y,t) \in \Pi_{n-1}(\Omega)\times\mathbb{R}.$$

Then, by Coarea Formula (see Proposition 3.6) and Remark 3.14 we have

$$\begin{aligned}
\int_\Omega |\nabla_{x\|} u|\,dx\,dy &= \int_{\partial\Sigma^{u_0}\cap(\Omega\times\mathbb{R})} \frac{|\nabla_{x\|} u_0|}{\sqrt{1+|\nabla u_0|^2}}\,d\mathcal{H}^n(x,y,t)\\
&= \int_{(\Pi_{n-1}(\Omega)\times\mathbb{R})\cap\{\nabla_{x\|}u_0\neq 0\}} \left(\int_{(\partial\Sigma^{u_0})_{(r,y,t)}} d\mathcal{H}^0(x)\right) dr\,dy\,dt\\
&= \int_{\Pi_{n-1}(\Omega)\times\mathbb{R}} \left(\int_{(\partial\Sigma^{u_0})_{(r,y,t)}} d\mathcal{H}^0(x)\right) dr\,dy\,dt\\
&= \int_{\Pi_{n-1}(\Omega)\times\mathbb{R}} \left(\int_{(\partial\Sigma^{v_\mu})_{(r,y,t)}} d\mathcal{H}^0(x)\right) dr\,dy\,dt\\
&= \int_{(\Pi_{n-1}(\Omega)\times\mathbb{R})\cap\{\nabla_{x\|}v_\mu\neq 0\}} \left(\int_{(\partial\Sigma^{v_\mu})_{(r,y,t)}} d\mathcal{H}^0(x)\right) dr\,dy\,dt\\
&= \int_\Omega |\nabla_{x\|} v_\mu|\,dx\,dy,
\end{aligned}$$

where we used the fact that, by definition, v_μ satisfies property (1.22). *In the context of Schwarz rearrangement, a similar phenomenon had already been observed by Brothers and Ziemer* [6].

Let us show that even asking $f \in \mathscr{F}'$ is not enough to guarantee rigidity.

Example 1.21. *Let $n = 2$, let $\Omega = D(5)$, let $a \equiv 1$, and let $f \in \mathscr{F}'$ be given by*

$$f(\eta,\tau) = \eta^2 + \tau^2.$$

Let $x' = (2,0)$, $x'' = (4,0)$, $x''' = (0,2)$, and let

$$u(x) = 2\max\{0, 1-|x|, 1-|x-x''|, 1-|x-x'''|\} \quad \text{for every } x \in \Omega.$$

In this case $\Omega^s = \Omega$ and the circular rearrangement of u is given by

$$v_\mu(x) = 2\max\{0, 1-|x|, 1-|x-x'|, 1-|x-x''|\} \quad \text{for every } x \in \Omega,$$

see Figure 1.5.

One can check that $u \in \mathcal{E}(\mu,\Omega)$, since

$$\int_\Omega |\nabla u(x)|^2\,dx = \int_\Omega |\nabla v_\mu(x)|^2\,dx = 12\pi.$$

Note that u cannot be written as the composition of the symmetric function v_μ with an orthogonal transformation, so rigidity is violated.

Remark 1.22. *Let us clarify why rigidity is violated in Example 1.21. Observing that in this case $\Pi_1(\Omega) = (0,5)$, it will be convenient to visualize the behaviour of the function $\alpha_\mu : (0,5)\times\mathbb{R} \to [0,\pi]$, see Figure 1.6. We note that, up to removing the singleton $\{(3,0)\}$ (which is $\mathcal{H}^1$-negligible), the set $\{0 < \alpha_\mu < \pi\}$ is disconnected. For this reason, it is possible to rotate only the part of the graph of v_μ lying above the annulus $\{1 < |x| < 3\}$, and to obtain functions for which the value of the integral is the same.*

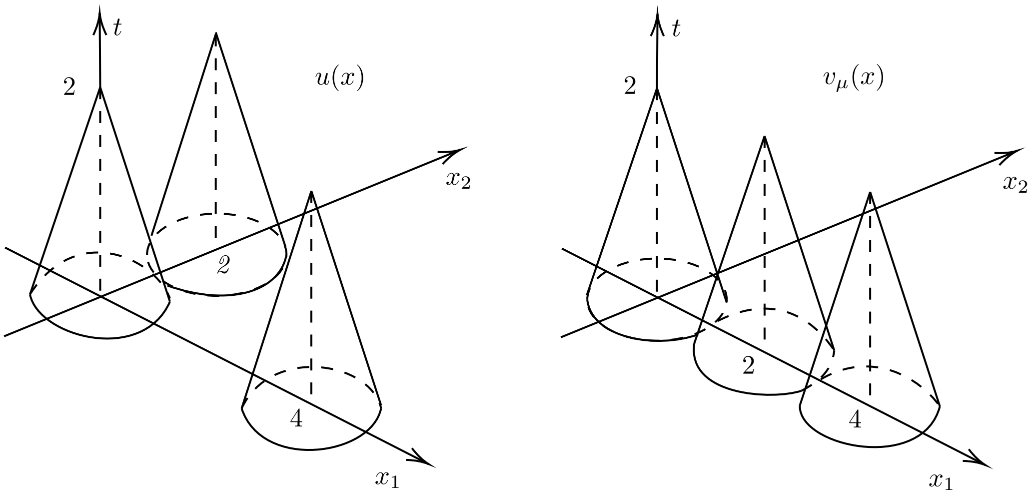

FIGURE 1.5. The function u given in Example 1.21 and its circular rearrangement v_μ. In this case, rigidity fails. This is because the set $\{0 < \alpha_\mu < \pi\}$ is essentially disconnected, see Figure 1.6.

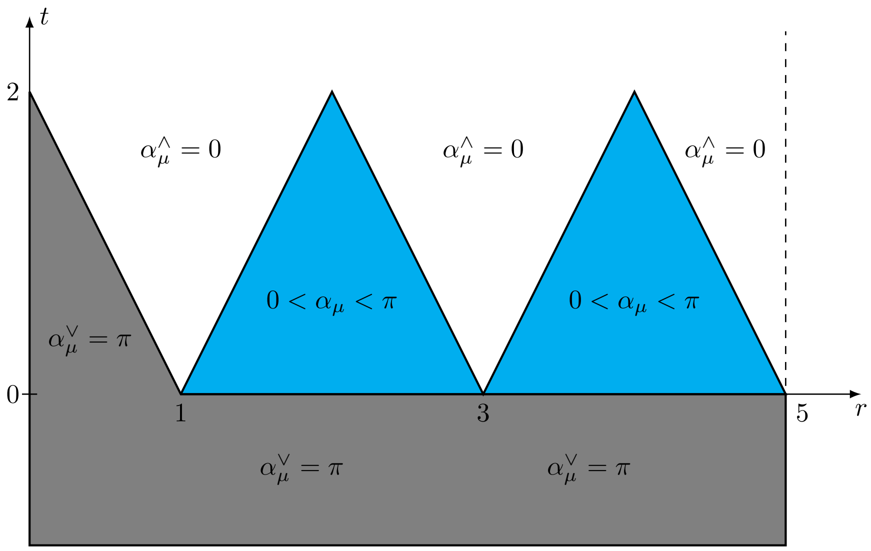

FIGURE 1.6. The values of $\alpha_\mu(r,t)$ when $(r,t) \in (0,5)\times\mathbb{R}$, in Example 1.21. Note that the set $\{0 < \alpha_\mu < \pi\}$ is essentially disconnected, since the singleton $\{(3,0)\}$ has $\mathcal{H}^1$-measure zero.

Example 1.23. *Let us modify Example 1.21, by removing one of the cones in the graph of u. More precisely, let $n = 2$, and let Ω, a, f, and x', x''' be as in Example 1.21. Let now $u : \Omega \to \mathbb{R}$ be given by*

$$u(x) = 2\max\{0, 1 - |x|, 1 - |x - x'''|\} \quad \textit{for every } x \in \Omega.$$

Denoting by μ the distribution of u (note that this is not the same as the distribution μ in Example 1.21), the circular rearrangement of u is now given by

$$v_\mu(x) = 2\max\{0, 1 - |x|, 1 - |x - x'|\} \quad \textit{for every } x \in \Omega,$$

see Figure 1.7.

Then,

$$\int_\Omega |\nabla u(x)|^2\, dx = \int_\Omega |\nabla v_\mu(x)|^2\, dx = 8\pi,$$

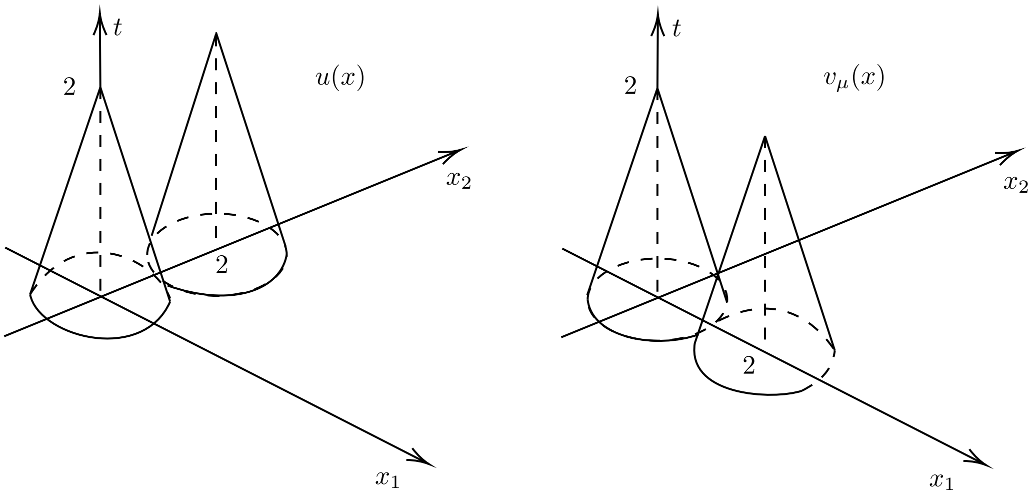

FIGURE 1.7. The function u given in Example 1.23 and its circular rearrangement v_μ. In this case rigidity holds, since every extremal is obtained as the composition of v_μ with a rotation. Note that now the set $\{0 < \alpha_\mu < \pi\}$ is essentially connected, see Figure 1.8.

so that $u \in \mathcal{E}(\mu, \Omega)$. However, rigidity is not violated, since $u(x) = v_\mu(Rx)$ for every $x \in \Omega$, where R is the clockwise rotation of $\pi/2$ around the origin. Note that in this case the set $\{0 < \alpha_\mu < \pi\}$ is connected, see Figure 1.8.

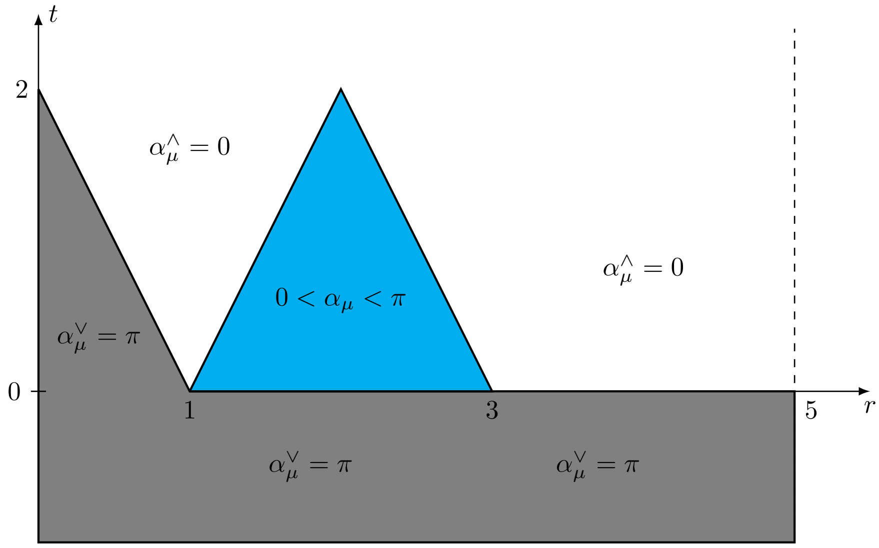

FIGURE 1.8. The values of the function $\alpha_\mu(r,t)$ given in Example 1.23. In this case, the set $\{0 < \alpha_\mu < \pi\}$ is essentially connected.

The next example shows that even assuming $f \in \mathscr{F}'$ and $\{0 < \alpha_\mu < \pi\}$ (essentially) connected might not be enough to guarantee rigidity. This is because also sets where the singular part of $D\alpha_\mu$ is concentrated play an important role.

Example 1.24. *Let $n = 2$, let $\Omega = D(5)$, $0 < \delta \ll 1$, $\tilde{x} = (2 + \delta, 0) \in \mathbb{R}^2$, and set:*

$$v_\mu(x) = \max\{v_1(x), v_2(x), v_3(x)\},$$

where

$$v_1(x) = 2\max\{0, 1-|x|\}, \quad v_2(x) = \frac{3}{4}\max\{0, 2-|x|\}\varphi(\hat{x}\cdot e_1) \quad v_3(x) = 2\max\{0, 1-|x-\tilde{x}|\},$$

and where $\varphi \in C_c^\infty((-1,1])$ *is a non decreasing function with* $0 \le \varphi \le 1$ *and* $\varphi \equiv 1$ *in* $[0,1]$*, see Figure 1.9.*

Let now $\gamma \in (0, \pi/4)$*, and let* $R : \mathbb{R}^2 \to \mathbb{R}^2$ *be a counterclockwise rotation of an angle* γ*. Then, setting (see Figure 1.9)*

$$u(x) = \max\{v_1(x), v_2(x), v_3(Rx)\},$$

one can check that

$$\int_\Omega |\nabla u|^2\,dx = \int_\Omega |\nabla v_\mu|^2\,dx,$$

so that rigidity is violated. Note that in this case the singular set S_{α_μ} *essentially disconnects* $\{0 < \alpha_\mu < \pi\}$*, see Figure 1.10.*

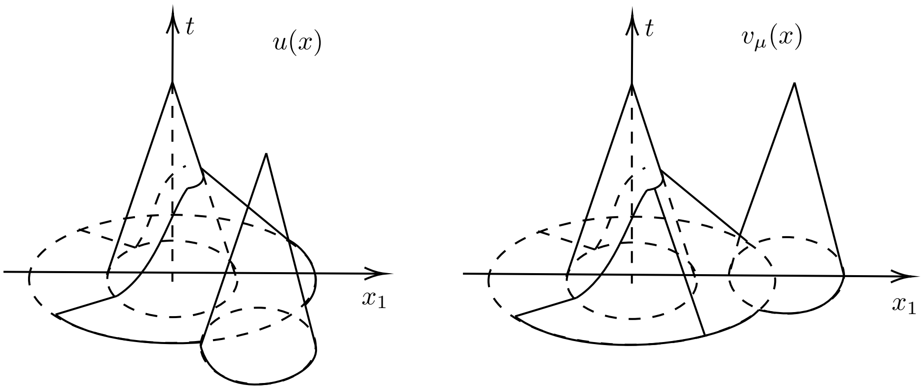

FIGURE 1.9. The function u given in Example 1.24 and its circular rearrangement v_μ, showing that rigidity fails. In this case, the singular set S_{α_μ} essentially disconnects $\{0 < \alpha_\mu < \pi\}$, see Figure 1.10.

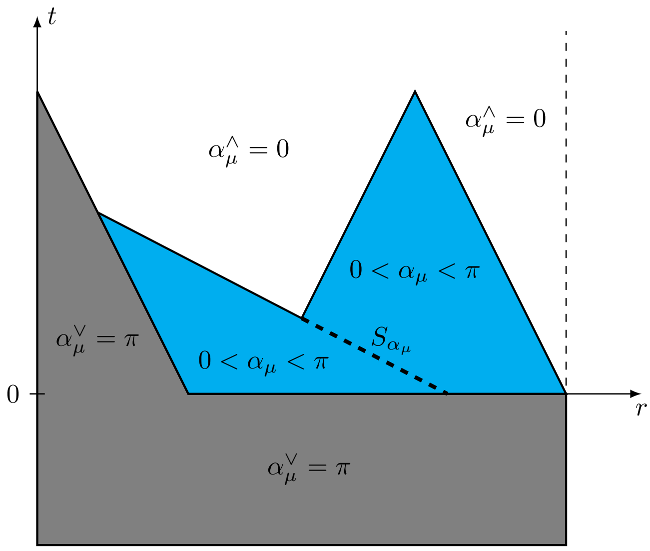

FIGURE 1.10. The values of the function $\alpha_\mu(r,t)$ given in Example 1.24. The singular set S_{α_μ} (see bold dashed line) essentially disconnects $\{0 < \alpha_\mu < \pi\}$.

The rest of the paper is divided as follows. In Section 2 we introduce some notions and we recall some basic results of Geometric Measure Theory, while in Section 3 we discuss the circular symmetrization of subgraphs. In Section 4 we prove a general version of the Pólya–Szegö inequality (see Theorem 4.3), and we show that this implies Theorem 1.12 and Theorem 1.13. Finally, Section 5 contains the proof of Theorem 1.18.

2. Preliminaries and notation

In this section we introduce the notation and the background needed to prove the main results of the paper. For more details we direct the reader to the monographs [2, 21, 26, 33].

2.1. **Basic notation.** Let $N \in \mathbb{N}$ with $N \geq 2$. We decompose $\mathbb{R}^N$ as $\mathbb{R}^2 \times \mathbb{R}^{N-2}$, and we set $\mathbb{R}^2_0 = \mathbb{R}^2 \setminus (0,0)$. Depending on the context, $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^N$. Moreover, for every $x \in \mathbb{R}^2_0$ we set $\hat{x} := x/|x|$. If $r > 0$ and $w = (x,z) \in \mathbb{R}^N$ with $x \in \mathbb{R}^2$ and $z \in \mathbb{R}^{N-2}$, we will denote the open ball of $\mathbb{R}^N$ of radius r and centre w by $B^N_r(w)$, or simply B^N_r when $w = 0$. For every $r > 0$, we set $D(r) = \{x \in \mathbb{R}^2 : |x| < r\}$ and $\partial D(r) = \{x \in \mathbb{R}^2_0 : |x| = r\}$, while for the unit circle of $\mathbb{R}^2$ we use the standard notation $\mathbb{S}^1 = \{x \in \mathbb{R}^2_0 : |x| = 1\}$. In $\mathbb{S}^1$, we consider the topology induced by the arclength distance $d_{\mathbb{S}^1} : \mathbb{S}^1 \times \mathbb{S}^1 \to [0,\pi]$, given by

$$d_{\mathbb{S}^1}(\omega', \omega'') := \arccos(\omega' \cdot \omega''), \qquad \text{for every } \omega', \omega'' \in \mathbb{S}^1.$$

This topology coincides with the topology induced in $\mathbb{S}^1$ by the Euclidean topology of $\mathbb{R}^2$. If $R \subset \mathbb{S}^1$, we set

$$d_{\mathbb{S}^1}(\omega, R) := \inf\{d_{\mathbb{S}^1}(\omega, \sigma) : \ \sigma \in R\}, \qquad \text{for every } \omega \in \mathbb{S}^1.$$

In particular, with the usual convention $\inf \emptyset = +\infty$, we have that $d_{\mathbb{S}^1}(\omega, \emptyset) = +\infty$. Let now $\beta \in (0, \pi)$ and let $x \in \mathbb{S}^1$. We denote by $\mathbf{B}_\beta(x)$ be the (relatively) open connected arc of $\mathbb{S}^1$ centred at x and with length 2β, i.e.:

$$\mathbf{B}_\beta(x) := \{\sigma \in \mathbb{S}^1 : d_{\mathbb{S}^1}(\sigma, x) < \beta\}.$$

For every $k \in \mathbb{N}$ with $1 \leq k \leq N$, we denote the Hausdorff k-dimensional measure in $\mathbb{R}^N$ by $\mathcal{H}^k$, while $\mathcal{L}^N$ stands for the N-dimensional Lebesgue measure. If $A, B \subset \mathbb{R}^N$, we write $A \subset_{\mathcal{H}^k} B$ when $\mathcal{H}^k(A \setminus B) = 0$ and $A =_{\mathcal{H}^k} B$ when $\mathcal{H}^k(A \Delta B) = 0$, where $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of A and B.

Let $E \subset \mathbb{R}^N$ be a Lebesgue measurable set. We denote by χ_E its characteristic function. Moreover, the upper and lower N-dimensional densities of E at w are defined as

$$\theta^*(E, w) := \limsup_{\rho \to 0^+} \frac{\mathcal{H}^N(E \cap B^N_\rho(w))}{\omega_N \, \rho^N}, \qquad \theta_*(E, w) := \liminf_{\rho \to 0^+} \frac{\mathcal{H}^N(E \cap B^N_\rho(w))}{\omega_N \, \rho^N},$$

respectively, where ω_N is the N-dimensional Lebesgue measure of the unit ball of $\mathbb{R}^N$. The functions $w \mapsto \theta^*(E, w)$ and $w \mapsto \theta_*(E, w)$ are Borel measurable, and they agree $\mathcal{L}^N$-a.e. on $\mathbb{R}^N$.

Therefore, the N-dimensional density of E at w

$$\theta(E, w) := \lim_{\rho \to 0^+} \frac{\mathcal{H}^N(E \cap B^N_\rho(w))}{\omega_N \, \rho^N},$$

is defined for $\mathcal{L}^N$-a.e. $w \in \mathbb{R}^N$, and $w \mapsto \theta(E, w)$ is a Borel function on $\mathbb{R}^N$. Given $t \in [0,1]$, we define the set of points of density t of E as

$$E^{(t)} := \{w \in \mathbb{R}^N : \theta(E, w) = t\}.$$

The set $\partial^e E := \mathbb{R}^N \setminus (E^{(0)} \cup E^{(1)})$ is called the *essential boundary* of E.

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a Lebesgue measurable function. If $M \in \mathbb{R}$, we set

$$(f \vee M)(w) = \max\{f(w), M\} \quad \text{and} \quad (f \wedge M)(w) = \min\{f(w), M\}.$$

When $M = 0$, we will also use the notation $f_+ = f \vee 0$ and $f_- = (-f) \vee 0$. We define the *approximate upper limit* $f^\vee(w)$ and the *approximate lower limit* $f^\wedge(w)$ of f at $w \in \mathbb{R}^N$ as

$$f^\vee(w) = \inf\left\{t \in \mathbb{R} : w \in \{f > t\}^{(0)}\right\}, \tag{2.1}$$

$$f^\wedge(w) = \sup\left\{t \in \mathbb{R} : w \in \{f > t\}^{(1)}\right\}. \tag{2.2}$$

We observe that $f^\vee$ and $f^\wedge$ are Borel functions that are defined at *every* point of $\mathbb{R}^N$, with values in $\mathbb{R}\cup\{\pm\infty\}$. Moreover, if $f_1:\mathbb{R}^N\to\mathbb{R}$ and $f_2:\mathbb{R}^N\to\mathbb{R}$ are measurable functions satisfying $f_1=f_2$ $\mathcal{L}^N$-a.e. on $\mathbb{R}^N$, then $f_1^\vee=f_2^\vee$ and $f_1^\wedge=f_2^\wedge$ *everywhere* on $\mathbb{R}^N$. We define the *approximate discontinuity* set S_f of f as $S_f:=\{f^\wedge<f^\vee\}$. Note that, by the above considerations, it follows that $\mathcal{L}^N(S_f)=0$. Although $f^\wedge$ and $f^\vee$ may take infinite values on S_f, the difference $f^\vee(z)-f^\wedge(z)$ is well defined in $\mathbb{R}\cup\{\pm\infty\}$ for every $w\in S_f$. Then, we can define the *approximate jump* $[f]$ of f as the Borel function $[f]:\mathbb{R}^N\to[0,\infty]$ given by

$$[f](w):=\begin{cases} f^\vee(w)-f^\wedge(w) & \text{if } w\in S_f\,,\\ 0 & \text{if } w\in\mathbb{R}^N\setminus S_f\,.\end{cases}$$

For $w\in\mathbb{R}^N$ and $\nu\in\mathbb{S}^{N-1}$, we will denote by $H^+_{w,\nu}$ and $H^-_{w,\nu}$ the closed half-spaces whose boundaries are orthogonal to ν:

$$H^+_{w,\nu}:=\Big\{w'\in\mathbb{R}^N:\,(w'-w)\cdot\nu\geq 0\Big\},\quad H^-_{w,\nu}:=\Big\{w'\in\mathbb{R}^N:\,(w'-w)\cdot\nu\leq 0\Big\}.$$

Let $A\subset\mathbb{R}^N$ be a Lebesgue measurable set. We say that $t\in\mathbb{R}\cup\{\pm\infty\}$ is the approximate limit of f at w with respect to A, and write $t=\operatorname{ap}\lim(f,A,w)$, if

$$\theta\Big(\{|f-t|>\varepsilon\}\cap A;w\Big)=0\,,\qquad\forall\varepsilon>0\,,\qquad(t\in\mathbb{R})\,,\tag{2.3}$$

$$\theta\Big(\{f<M\}\cap A;w\Big)=0\,,\qquad\forall M>0\,,\qquad(t=+\infty)\,,\tag{2.4}$$

$$\theta\Big(\{f>-M\}\cap A;w\Big)=0\,,\qquad\forall M>0\,,\qquad(t=-\infty)\,.\tag{2.5}$$

We say that $w\in S_f$ is a *jump point* of f if there exists $\nu\in\mathbb{S}^{N-1}$ such that

$$f^\vee(w)=\operatorname{ap}\lim(f,H^+_{w,\nu},x)>f^\wedge(w)=\operatorname{ap}\lim(f,H^-_{w,\nu},x)\,.$$

If this is the case, we say that $\nu_f(w):=\nu$ is the approximate jump direction of f at w. Denoting by J_f the set of approximate jump points of f, we have that $J_f\subset S_f$ and $\nu_f:J_f\to\mathbb{S}^{N-1}$ is a Borel function.

Let $w\in S_f^c$. We say that f is *approximately differentiable* at w if $\tilde f(w):=f^\wedge(w)=f^\vee(w)\in\mathbb{R}$ and there exists $\xi\in\mathbb{R}^N$ such that

$$\operatorname{ap}\lim(g_\xi,\mathbb{R}^N,w)=0,$$

where $g_\xi(w')=(f(w')-\tilde f(w)-\xi\cdot(w'-w))/|w'-w|$ for $w'\in\mathbb{R}^N\setminus\{w\}$. If this is the case, then ξ is uniquely determined, we set $\xi=\nabla f(w)$, and call $\nabla f(w)$ the *approximate differential* of f at w. In components, we write $\nabla f(w)=(\nabla_x f(w),\nabla_z f(w))$, where $\nabla_x f(w)=(\partial_{x_1}f(w),\partial_{x_2}f(w))\in\mathbb{R}^2$ and $\nabla_z f(w)=(\partial_{z_1}f(w),\dots,\partial_{z_{N-2}}f(w))\in\mathbb{R}^{N-2}$. Moreover, we denote the set of points of approximate differentiability of f as

$$\mathcal{D}_f:=\big\{w\in\mathbb{R}^N:\,f\text{ is approximately differentiable at }w\big\}\,.$$

2.2. **Essential connectedness.** We now introduce the concept of essential connectedness, which will be used to study the rigidity of the Pólya–Szegö inequality.

Definition 2.1. *Let $G\subset\mathbb{R}^N$ be a Borel set, and let $G_+,G_-\subset\mathbb{R}^N$ be Borel sets. We say that $\{G_+,G_-\}$ is a non-trivial Borel partition of G modulo $\mathcal{H}^N$ if*

$$\mathcal{H}^N(G_+\cap G_-)=0\,,\qquad\mathcal{H}^N(G\Delta(G_+\cup G_-))=0\,,\qquad\mathcal{H}^N(G_+)\,\mathcal{H}^N(G_-)>0\,.$$

Definition 2.2. *Let K and G be Borel sets in $\mathbb{R}^N$. We say that K essentially disconnects G if there exists a non-trivial Borel partition $\{G_+,G_-\}$ of G modulo $\mathcal{H}^N$ such that*

$$\mathcal{H}^{N-1}\Big(\Big(G^{(1)}\cap\partial^{\mathrm{e}}G_+\cap\partial^{\mathrm{e}}G_-\Big)\setminus K\Big)=0\,.\tag{2.6}$$

Instead, we say that K does not essentially disconnect G if, for every non-trivial Borel partition $\{G_+,G_-\}$ of G modulo $\mathcal{H}^N$,

$$\mathcal{H}^{N-1}\Big(\Big(G^{(1)}\cap\partial^{\mathrm{e}}G_+\cap\partial^{\mathrm{e}}G_-\Big)\setminus K\Big)>0\,.\tag{2.7}$$

Finally, we say that G is essentially connected if $\emptyset$ does not essentially disconnect G.

Remark 2.3. *If $\mathcal{H}^N(G\Delta G')=0$ and $\mathcal{H}^{N-1}(K\Delta K')=0$, then K essentially disconnects G if and only if K' essentially disconnects G'.*

2.3. BV **functions.** Let $\Omega\subset\mathbb{R}^N$ be open. We denote by $C^0(\Omega)$ ($C^0_c(\Omega)$) the space of continuous functions (with compact support) in Ω, while $C^0_b(\Omega)$ stands for the space of bounded continuous functions in Ω. Moreover, $C^1_c(\Omega;\mathbb{R}^N)$ is the space of $\mathbb{R}^N$-valued C^1 functions with compact support in Ω. Let $f\in L^1(\Omega)$. We say that f is a function of bounded variation in Ω, and we write $f\in BV(\Omega)$, if

$$\sup\left\{\int_\Omega f(w)\,\mathrm{div}\,\varphi(w)\,dw:\ \varphi\in C^1_c(\Omega;\mathbb{R}^N)\,,|\varphi|\le 1\right\}<\infty. \tag{2.8}$$

More in general, we say that $f\in BV_{\rm loc}(\Omega)$ if $f\in BV(\Omega')$ for every open set $\Omega'\subset\subset\Omega$. If $f\in BV_{\rm loc}(\Omega)$, then the distributional derivative Df of f is an $\mathbb{R}^N$-valued Radon measure. In this case, when needed we will write $Df=(D_xf,D_zf)$, where D_xf is an $\mathbb{R}^2$-valued Radon measure, and D_zf is an $\mathbb{R}^{N-2}$-valued Radon measure.

Let now $f\in BV(\Omega)$. Then, the total variation of Df is finite in Ω, and its value $|Df|(\Omega)$ coincides with the left–hand side of (2.8). One can write the Radon–Nikodým decomposition of Df with respect to $\mathcal{L}^N$ as $Df=D^af+D^sf$, where $D^af\ll\mathcal{L}^N$, and where D^sf and D^af are mutually singular.

It turns out that for $\mathcal{L}^N$-a.e. $w\in\Omega$ the function f is approximately differentiable, and its approximate differential is the density of D^af with respect to $\mathcal{L}^N$. Thus, we have $D^af=\nabla f d\mathcal{L}^N$ and $\nabla f\in L^1(\Omega;\mathbb{R}^N)$. Moreover, the singular set S_f and the jump set J_f of f are countably $(N-1)$-rectifiable, with $\mathcal{H}^{N-1}(S_f\setminus J_f)=0$. In addition, for $\mathcal{H}^{N-1}$-a.e. $w\in J_f$ there exists a vector $\nu_f(w)\in\mathbb{S}^{N-1}$ such that we can further decompose the singular part of Df as $D^sf=D^jf+D^cf$, where D^jf and D^cf are mutually singular, $D^jf=[f](w)\nu_f(w)\,d\mathcal{H}^{N-1}\llcorner J_f$ is called the jump part of Df, and $D^cf=D^sf-D^jf$ is called the Cantor part of Df, and is concentrated on a set $K\subset\Omega$ such that $\mathcal{H}^N(K)=0$ and $\mathcal{H}^{N-1}(K)=+\infty$. If $f:\Omega\to\mathbb{R}^m$, with $m\in\mathbb{N}$ and $m\ge 2$, we say that $f\in BV(\Omega;\mathbb{R}^m)$ if and only if $f_i\in BV(\Omega)$ for $i=1,\dots,m$, where with $f=(f_1,\dots,f_m)$. Accordingly, we say that $f\in BV_{\rm loc}(\Omega;\mathbb{R}^m)$ if $f\in BV(\Omega';\mathbb{R}^m)$ for every open set Ω' compactly contained in Ω. If $f\in BV_{\rm loc}(\Omega;\mathbb{R}^m)$ the distributional derivative Df of f is an $(m\times N)$-valued Radon measure.

If $a\in\mathbb{R}^m$, and $b\in\mathbb{R}^N$ we denote the tensor product between a and b as $a\otimes b$. This is the $m\times N$ matrix whose components are given by

$$(a\otimes b)_{i,j}=a_ib_j\quad i=1,\dots,m,\ j=1,\dots,N.$$

2.4. **Sets of finite perimeter.** Let $E\subset\mathbb{R}^N$ be a Lebesgue measurable set, and let $O\subset\mathbb{R}^N$ be open. We say that E is a set of finite perimeter in O if

$$P(E;O):=\sup\left\{\int_E \mathrm{div}\,\varphi(w)\,dw:\ \varphi\in C^1_c(O;\mathbb{R}^N),\ |\varphi|\le 1\right\}<\infty, \tag{2.9}$$

and in this case we say that $P(E;O)$ is the perimeter of E in O. Note that, if $\mathcal{L}^N(E)<\infty$, then E is a set of finite perimeter in O if and only if $\chi_E\in BV(O)$. In the special case $O=\mathbb{R}^N$ we set $P(E):=P(E;\mathbb{R}^N)$ and when $P(E)<\infty$ we say that E is a set of finite perimeter. We say that E is a set of locally finite perimeter in O if $\chi_E\in BV_{\rm loc}(O)$.

Let E be a set of locally finite perimeter in O. We define the *reduced boundary* ∂^*E of E as the set of those points $w\in O$ such that

$$\nu^E(w):=\lim_{\rho\to 0^+}\frac{D\chi_E(B^N_\rho(w))}{|D\chi_E|(B^N_\rho(w))},$$

exists and belongs to $\mathbb{S}^{N-1}$. We call $\nu^E:\partial^*E\to\mathbb{S}^{N-1}$ the *measure-theoretic inner unit normal* to E. It turns out that ν^E is a Borel function, and that the distributional derivative $D\chi_E$ of χ_E is given by

$$D\chi_E=\nu^E d\mathcal{H}^{N-1}\llcorner\partial^*E.$$

Therefore, if $A\subset O$ is a Borel set, we can define the perimeter of E in A as

$$P(E;A):=|D\chi_E|(A)=\mathcal{H}^{N-1}(\partial^*E\cap A).$$

It turns out that

$$(\partial^* E \cap O) \subset (E^{(1/2)} \cap O) \subset (\partial^{\mathrm{e}} E \cap O)\,.$$

Moreover, *Federer's theorem* holds true (see [2, Theorem 3.61] and [26, Theorem 16.2]):

$$\mathcal{H}^{N-1}((\partial^{\mathrm{e}} E \cap O) \setminus (\partial^* E \cap O)) = 0.$$

2.5. **Sets of finite perimeter in $\mathbb{S}^1$.** We now briefly introduce sets of finite perimeter in the unit circle $\mathbb{S}^1$. Then, if $r > 0$ one can argue in a similar way to define sets of finite perimeter on $\partial D(r)$. We direct the reader to [33, Chapter 6] for more details (see also [4]).

We denote by $\Lambda_1(\mathbb{R}^2)$ and $\Lambda^1(\mathbb{R}^2)$ the linear spaces of 1-vectors and 1-covectors in $\mathbb{R}^2$, respectively, while $\mathcal{D}^1(\mathbb{R}^2)$ stands for the set of smooth 1-forms with compact support in $\mathbb{R}^2$.

A 1-*dimensional current* in $\mathbb{R}^2$ is a continuous linear functional on $\mathcal{D}^1(\mathbb{R}^2)$. Instead, 0-dimensional currents in $\mathbb{R}^2$ are simply defined as the usual distributions in $\mathbb{R}^2$. The family of 1-dimensional (0-dimensional) currents in $\mathbb{R}^2$ is denoted by $\mathcal{D}_1(\mathbb{R}^2)$ ($\mathcal{D}_0(\mathbb{R}^2)$).

We say that $T \in \mathcal{D}_1(\mathbb{R}^2)$ is an *integer multiplicity rectifiable* 1-*current* if it can be represented as

$$T(\omega) = \int_M \langle \omega(x), \tau(x) \rangle\, \theta(x)\, d\mathcal{H}^1(x) \qquad \text{for every } \omega \in \mathcal{D}^1(\mathbb{R}^2),$$

where M is an $\mathcal{H}^1$-measurable countably 1-rectifiable subset of $\mathbb{R}^2$, θ is an $\mathcal{H}^1$-measurable positive integer-valued function, $\tau : M \to \Lambda_1(\mathbb{R}^2)$ is an $\mathcal{H}^1$-measurable function such that $\tau(x)$ is a unit vector belonging to the approximate tangent space of M at x for $\mathcal{H}^1$-a.e. $x \in M$, and $\langle \cdot, \cdot \rangle$ denotes the usual pairing between $\Lambda^1(\mathbb{R}^2)$ and $\Lambda_1(\mathbb{R}^2)$. In the special case when

$$T(\omega) = \int_M \langle \omega(x), \tau(x) \rangle\, d\mathcal{H}^1(x) \qquad \text{for every } \omega \in \mathcal{D}^1(\mathbb{R}^2),$$

we write $T = [\![M]\!]$. The boundary ∂T of T is then defined as the 0-dimensional current in $\mathbb{R}^2$ such that

$$\partial T(\omega) = T(d\omega) \qquad \text{for every } \omega \in C^0_c(\mathbb{R}^2),$$

while the mass $\mathbf{M}(T)$ of T is given by

$$\mathbf{M}(T) := \sup \left\{ T(\omega) : \omega \in \mathcal{D}^1(\mathbb{R}^2),\ |\omega| \le 1 \right\}.$$

More in general, for any open set $U \subset \mathbb{R}^2$, we set

$$\mathbf{M}_U(T) := \sup \left\{ T(\omega) : \omega \in \mathcal{D}^1(\mathbb{R}^2),\ |\omega| \le 1,\ \operatorname{supp} \omega \subset U \right\}.$$

Let $A \subset \mathbb{S}^1$ be an $\mathcal{H}^1$-measurable set. We will say that A is a set of finite perimeter on $\mathbb{S}^1$ if there exists $Q \in \mathcal{D}_0(\mathbb{R}^2)$ with $\operatorname{supp} Q \subset \mathbb{S}^1$ and

$$Q = \partial [\![A]\!],$$

with the property that $\mathbf{M}_U(Q) < \infty$ for every $U \subset\subset \mathbb{R}^2$. By the Riesz representation theorem it follows that there exists a Radon measure μ_Q such that

$$\int_A \operatorname{div}_{\|} \varphi(x)\, d\mathcal{H}^1(x) = \int_{\mathbb{S}^1} \varphi(x) \cdot d\mu_Q(x),$$

for every smooth vector field $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$ such that $\varphi(x) \cdot \hat{x} = 0$ for every $x \in \mathbb{S}^1$, where $\operatorname{div}_{\|}\varphi$ stands for the tangential divergence of φ on $\mathbb{S}^1$. If $A \subset \mathbb{S}^1$ is a set of finite perimeter on the sphere, the reduced boundary $\partial^* A$ of A is the set of points $x \in \mathbb{S}^1$ such that the limit

$$\nu^A(x) := \lim_{\rho \to 0^+} \frac{\mu_Q(\mathbf{B}_\rho(x))}{|\mu_Q|(\mathbf{B}_\rho(x))}$$

exists, $\nu^A(x) \in T_x\mathbb{S}^1$, and $|\nu^A(x)| = 1$. The De Giorgi structure theorem holds true also for sets of finite perimeter on the sphere. In particular, $\partial^* A$ is finite, $\mu_Q = \nu^A \mathcal{H}^0 \llcorner \partial^* A$, and

$$\int_A \operatorname{div}_{\|} \varphi(x)\, d\mathcal{H}^1(x) = \int_{\partial^* A} \varphi(x) \cdot \nu^A(x)\, d\mathcal{H}^0(x), \tag{2.10}$$

for every smooth vector field $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$ such that $\varphi(x) \cdot \hat{x} = 0$ for every $x \in \mathbb{S}^1$.

The isoperimetric inequality on the circle states that, if $A \subset \mathbb{S}^1$ is a set of finite perimeter on $\mathbb{S}^1$ with $\mathcal{H}^1(A) = \mathcal{H}^1(\mathbf{B}_\beta(e_1))$, then (see [31])

$$\mathcal{H}^0(\partial^* \mathbf{B}_\beta(e_1)) \le \mathcal{H}^0(\partial^* A), \tag{2.11}$$

and

$$\mathcal{H}^0(\partial^*\mathbf{B}_\beta(e_1)) = \mathcal{H}^0(\partial^* A) \quad \Longleftrightarrow \quad A =_{\mathcal{H}^1} \mathbf{B}_\beta(p) \text{ for some } p \in \mathbb{S}^1. \tag{2.12}$$

2.6. **Circular rearrangement of sets.** If $E \subset \mathbb{R}^N$ is a Lebesgue measurable set and $(r, z) \in (0, +\infty) \times \mathbb{R}^{N-2}$, we define the slice $E_{(r,z)}$ as the subset of $\mathbb{R}^2_0$ given by

$$E_{(r,z)} := \{x \in \partial D(r) : (x, z) \in E\}.$$

We now give the definition of circular symmetrization of a set in $\mathbb{R}^N$.

Definition 2.4. *Let $E \subset \mathbb{R}^N$ be a Lebesgue measurable set. The circular rearrangement of E is the set E^s defined as*

$$E^s := \{(x, z) \in \Phi_N(\Pi_{N-1}(E) \times \mathbb{S}^1) : d_{\mathbb{S}^1}(\hat{x} \cdot e_1) < \beta(|x|, z)\} \tag{2.13}$$

where

$$\beta(r, z) = \frac{\mathcal{H}^1(E_{(r,z)})}{2r}, \qquad \text{for every } (r, z) \in \Pi_{N-1}(E). \tag{2.14}$$

Remark 2.5. *Note that with the definition given above the set E^s does not include points of the half-hyperplane $H = \{\hat{x} \cdot e_1 = -1\}$, even when the original set E does include some of these points. This is not affecting our results, since H is an $\mathcal{H}^n$-negligible set. The advantage of definition* (2.13) *is clarified by the next result.*

Proposition 2.6. *Let $E \subset \mathbb{R}^N$ be a Lebesgue measurable set, and let $\beta : \Pi_{N-1}(E) \to [0, \pi]$ be given by* (2.14)*. Then,*

(i) *E^s is open $\Longleftrightarrow$ β is lower semicontinuous;*

(ii) *E is open $\Longrightarrow$ E^s is open.*

Proof. We divide the proof into steps.

Step 1. E open $\Longrightarrow$ β lower semicontinuous. Let E be open and assume, by contradiction, that β is not lower semicontinuous. Then, there exist $(r, z) \in \Pi_{N-1}(E)$ and a sequence $\{(r_h, z_h)\}_{h\in\mathbb{N}} \subset \Pi_{N-1}(E)$ with $(r_h, z_h) \to (r, z)$ such that

$$\beta(r, z) > \liminf_{h\to+\infty} \beta(r_h, z_h).$$

Passing to a suitable (not relabelled) subsequence, we can assume that the liminf is a limit and so we have

$$\beta(r, z) > \lim_{h\to+\infty} \beta(r_h, z_h) \quad \Longleftrightarrow \quad \mathcal{H}^1(C) > \lim_{h\to+\infty} \mathcal{H}^1(S_h),$$

where the sets $C, S_h \subset \mathbb{S}^1$ are defined as:

$$C := \frac{1}{r} E_{(r,z)} \quad \text{ and } \quad S_h := \frac{1}{r_h} E_{(r_h,z_h)}, \quad \text{for every } h \in \mathbb{N}.$$

Then, there exist $\delta > 0$ and $j \in \mathbb{N}$ such that

$$\mathcal{H}^1(C) > \mathcal{H}^1(S_h) + 2\delta, \qquad \text{for every } h > j,$$

and so

$$\mathcal{H}^1(C \setminus S_h) \geq \mathcal{H}^1(C) - \mathcal{H}^1(S_h) > 2\delta, \qquad \text{for every } h > j. \tag{2.15}$$

Since E is open and $C = \frac{1}{r}E_{(r,y)}$, we have that C is (nonempty and) relatively open in $\mathbb{S}^1$. Equivalently, C is open in the topology induced in $\mathbb{S}^1$ by $d_{\mathbb{S}^1}$. For every $k \in \mathbb{N}$, we define the set $C^k \subset \mathbb{S}^1$ as

$$C^k := \left\{\omega \in \mathbb{S}^1 : d_{\mathbb{S}^1}(\omega, \mathbb{S}^1 \setminus C) \geq \frac{1}{k}\right\}.$$

For $k \in \mathbb{N}$ sufficiently large, C^k is nonempty and compact in $\mathbb{S}^1$. We also observe that, with the usual convention $d_{\mathbb{S}^1}(\omega, \emptyset) = +\infty$, if $C = \mathbb{S}^1$ we have $C^k = \mathbb{S}^1$. Moreover, $C^{k_1} \subset C^{k_2} \subset C$ whenever $k_1 < k_2$, and

$$C = \bigcup_{k\in\mathbb{N}} C^k.$$

Therefore, there exists $k_\delta \in \mathbb{N}$ such that

$$\mathcal{H}^1(C \setminus C^{k_\delta}) < \delta.$$

Last inequality, together with (2.15), implies that

$$\mathcal{H}^1(C^{k_\delta} \setminus S_h) > \delta, \qquad \text{for every } h > j.$$

Then, for every $h > j$ there exists $\omega_h \in C^{k_\delta} \setminus S_h$. By compactness of C^{k_δ}, there exists $\overline{\omega} \in C^{k_\delta}$ such that, up to subsequences,

$$\omega_h \to \overline{\omega}.$$

Since $\overline{\omega} \in C^{k_\delta} \subset C$, by definition of C we have that $(r\overline{\omega}, z) \in E$. Thus, recalling the definition of S_h, the sequence $\{(r_h\omega_h, z_h)\}_{h>j}$ is such that

$$(r_h\omega_h, z_h) \notin E \qquad \text{for every } h > j,$$

and

$$(r_h\omega_h, z_h) \to (r\overline{\omega}, z) \in E.$$

But this is impossible, since E is open.

Step 2. β lower semicontinuous $\Longrightarrow$ E^s open. Let β be lower semicontinuous and suppose, by contradiction, that E^s is not open. Then, there exist $(x, z) \in E^s$ and a sequence $\{(x_h, z_h)\}_h \subset \Phi_N(\Pi_{N-1}(E) \times \mathbb{S}^1)$ such that $(x_h, z_h) \to (x, z)$ and $(x_h, z_h) \notin E^s$ for all $h \in \mathbb{N}$. Then, we have

$$\beta(|x|, z) > d_{\mathbb{S}^1}(\hat{x}, e_1) \quad \text{ and } \quad d_{\mathbb{S}^1}(\hat{x}_h, e_1) \geq \beta(|x_h|, z_h) \quad \forall\, h \in \mathbb{N}.$$

Using that fact that β is lower semicontinuous, we obtain

$$\beta(|x|, z) > d_{\mathbb{S}^1}(\hat{x}, e_1) = \lim_{h\to+\infty} d_{\mathbb{S}^1}(\hat{x}_h, e_1) \geq \liminf_{h\to\infty} \beta(|x_h|, z_h) \geq \beta(|x|, z),$$

which is impossible.

Step 3. We conclude. Property (i) follows by combining Step 1 (applied to the set E^s) and Step 2. Let us now show (ii). If E is open, thanks to Step 1, the function β is lower semicontinuous and then, by Step 2, E^s is open. □

3. Circular rearrangement of subgraphs

In this section and in the rest of the paper, we assume $n \in \mathbb{N}$ with $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ open. We decompose $\mathbb{R}^{n+1}$ as $\mathbb{R}^{n+1} = \mathbb{R}^2 \times \mathbb{R}^{n-2} \times \mathbb{R}$, and we label points of $\mathbb{R}^{n+1}$ as (x, y, t), with $x \in \mathbb{R}^2$, $y \in \mathbb{R}^{n-2}$ and $t \in \mathbb{R}$. If $E \subset \mathbb{R}^{n+1}$ is a set of locally finite perimeter, for every $(x, y, t) \in \partial^* E \cap (\mathbb{R}^2_0 \times \mathbb{R}^{n-2} \times \mathbb{R})$ we decompose the measure theoretic inner unit normal to E at (x, y, t) as $\nu^E(x, y, t) = (\nu^E_x(x, y, t), \nu^E_y, (x, y, t), \nu^E_t(x, y, t))$, where $\nu^E_x(x, y, t) \in \mathbb{R}^2$, $\nu^E_y(x, y, t) \in \mathbb{R}^{n-2}$, and $\nu^E_t(x, y, t) \in \mathbb{R}$. Moreover, we further decompose $\nu^E_x(x, y, t)$ as

$$\nu^E_x(x, y, t) = (\hat{x} \cdot \nu^E_x(x, y, t))\hat{x} + \nu^E_{x\parallel}(x, y, t)x_\parallel,$$

where $\hat{x}$ and $x_\parallel$ are defined in (1.15), and $\nu^E_{x\parallel}(x, y, t) = \nu^E_x(x, y, t) \cdot x_\parallel$.

3.1. **Subgraphs.** Let now $u : \Omega \to \mathbb{R}$ be a Lebesgue measurable function. We denote by $\Sigma^u \subset \mathbb{R}^{n+1}$ the subgraph of u, defined as

$$\Sigma^u = \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^{n-2} \times \mathbb{R} : \, (x, y) \in \Omega \text{ and } u(x, y) > t\}$$

The following result follows from [21, Chapter 4, Section 1.5, Theorem 1–Theorem 5] (see also [9, Proposition 3.4], and [27, Theorem 5.2]).

Proposition 3.1. *Let $U \subset \mathbb{R}^n$ be open and bounded, and let $u \in L^1(U)$. Then, $u \in BV(U)$ if and only if Σ^u is a set of finite perimeter in $U \times \mathbb{R}$. In this case, we have:*

$$\partial^*\Sigma^u \cap ((S_u)^c \times \mathbb{R}) =_{\mathcal{H}^n} \{(x, y, t) \in (S_u)^c \times \mathbb{R} : u^\wedge(x, y) = u^\vee(x, y) = t\},$$

with

$$\nu^{\Sigma^u}(x, y, u(x, y)) = \left(\frac{\nabla_x u(x, y)}{\sqrt{1 + |\nabla u(x, y)|^2}}, \frac{\nabla_y u(x, y)}{\sqrt{1 + |\nabla u(x, y)|^2}}, \frac{-1}{\sqrt{1 + |\nabla u(x, y)|^2}} \right) \tag{3.1}$$

for $\mathcal{H}^n$-a.e. $(x, y) \in \mathcal{D}_u$, and

$$\nu^{\Sigma^u}(x, y, u(x, y)) = \left(\frac{dD^c u}{d|D^c u|}(x, y), 0 \right) \qquad \text{for } |D^c u|\text{-a.e. } (x, y) \in U.$$

Moreover,

$$\partial^*\Sigma^u \cap (S_u \times \mathbb{R}) =_{\mathcal{H}^n} \{(x,y,t) \in S_u \times \mathbb{R} : u^\wedge(x,y) < t < u^\vee(x,y)\},$$

and

$$\nu^{\Sigma^u}(x,y,t) = (\nu_u(x,y), 0)\,,$$

for $\mathcal{H}^{n-1}$-a.e. $(x,y) \in S_u$ and for every $t \in (u^\wedge(x,y), u^\vee(x,y))$. Finally, if $B \subset U$ is a Borel set, the perimeter of Σ^u in $B \times \mathbb{R}$ is given by

$$P(\Sigma^u; B \times \mathbb{R}) = \int_B \sqrt{1 + |\nabla u|^2}\, dx\, dy + |D^s u|(B). \tag{3.2}$$

We will also need the following statement.

Proposition 3.2. *Let $U \subset \mathbb{R}^n$ be open and bounded, let $u \in BV(U)$ and let $A \subset U$ be a Borel set. Then, the following are equivalent:*

(i) $\mathcal{H}^n\Big(\{(x,y,t) \in \partial^*\Sigma^u : \nu_t^{\Sigma^u}(x,y,t) = 0\} \cap (A \times \mathbb{R})\Big) = 0$*;*

(ii) $P(\Sigma^u; B \times \mathbb{R}) = 0$ *for every Borel set $B \subset A$ with $\mathcal{H}^n(B) = 0$.*

Moreover, if A is open, (i) and (ii) are also equivalent to:

(iii) $u \in W^{1,1}(A)$.

Proof. The equivalence between (i) and (ii) follows directly from [13, Lemma 4.1]. Let us now assume that A is open, and let us show that (ii) is equivalent to (iii). Thanks to formula (3.2), we have that for every Borel set $B \subset A$ with $\mathcal{H}^n(B) = 0$

$$P(\Sigma^u; B \times \mathbb{R}) = \int_B \sqrt{1 + |\nabla u|^2}\, dx\, dy + |D^s u|(B) = |D^s u|(B).$$

Therefore,

$$\text{(ii)} \iff |D^s u|(B) = 0 \text{ for every Borel set } B \subset A \text{ with } \mathcal{H}^n(B) = 0 \iff u \in W^{1,1}(A).$$

□

We now introduce a class of functions that extends Definition 1.2.

Definition 3.3. *Let $n \in \mathbb{N}$ with $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be open, let $u : \Omega \to \mathbb{R}$, and let u_0 be given by (1.12). We say that $u \in BV_{0,\tau}(\Omega)$ if the following conditions are satisfied:*

(a) $u_0 \in BV(\Phi_n(A \times \mathbb{S}^1))$ *for every open set $A \subset (0,\infty) \times \mathbb{R}^{n-2}$ with $A \subset\subset \Pi_{n-1}(\Omega)$,*

(b) $u \geq 0 \quad \mathcal{L}^n$*-a.e. in* $\Omega \setminus \Phi_n(\Pi^{\mathrm{a}}_{n-1}(\Omega) \times \mathbb{S}^1)$,

where Φ_n, $\Pi_{n-1}(\Omega)$, and $\Pi^{\mathrm{a}}_{n-1}(\Omega)$ are defined by (1.3), (1.4), and (1.5), respectively.

Remark 3.4. *We have $W^{1,p}_{0,\tau}(\Omega) \subset W^{1,1}_{0,\tau}(\Omega) \subset BV_{0,\tau}(\Omega)$ for every $p \in [1, +\infty)$, where the first inclusion follows from the Hölder inequality on compact sets.*

Remark 3.5. *Thanks to Proposition 3.1, if $u \in BV_{0,\tau}(\Omega)$ then Σ^{u_0} is a set of finite perimeter in $\Phi_n(A \times \mathbb{S}^1) \times \mathbb{R} = \Phi_{n+1}((A \times \mathbb{R}) \times \mathbb{S}^1)$, for every open set $A \subset\subset \Pi_{n-1}(\Omega)$.*

Adapting [11, Proposition 6.1] to the case of subgraphs of functions belonging to $BV_{0,\tau}(\Omega)$, we obtain the following.

Proposition 3.6 (Coarea Formula)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, let $u \in BV_{0,\tau}(\Omega)$, and let $g : \mathbb{R}^{n+1} \to [0,\infty]$ be a Borel function. Then,*

$$\begin{aligned}\int_{\partial^*\Sigma^{u_0}} & g(x,y,t) |\nu_{x_\parallel}^{\Sigma^{u_0}}(x,y,t)|\, d\mathcal{H}^n(x,y,t) \\ &= \int_{(0,\infty)\times\mathbb{R}^{n-2}\times\mathbb{R}} \left(\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} g(x,y,t)\, d\mathcal{H}^0(x) \right) dr\, dy\, dt.\end{aligned}$$

Remark 3.7. *From Proposition 3.1 it follows that when $u \in BV_{0,\tau}(\Omega)$, the subgraph Σ^{u_0} is a set of locally finite perimeter in $\Phi_{n+1}((\Pi_{n-1}(\Omega) \times \mathbb{R}) \times \mathbb{S}^1)$. Moreover, if $u \in W^{1,1}_{0,\tau}(\Omega)$*

$$\nu_{x_\parallel}^{\Sigma^{u_0}}(x,y,u_0(x,y)) = \frac{\nabla_{x_\parallel} u_0(x,y)}{\sqrt{1 + |\nabla u_0(x,y)|^2}}, \qquad \text{for } \mathcal{H}^n\text{-a.e. } (x,y) \in \mathcal{D}_{u_0} \cap \Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1).$$

Remark 3.8. *Combining Proposition 3.6 and Remark 3.7 we obtain that if $u \in W^{1,1}_{0,\tau}(\Omega)$, then*

$$\mathcal{H}^n\left(\partial^*\Sigma^{u_0} \cap \Phi_{n+1}(R \times \mathbb{S}^1) \cap (\{\nabla_{x\|} u_0 \neq 0\} \times \mathbb{R})\right) = 0, \tag{3.3}$$

for every Borel set $R \subset \Pi_{n-1}(\Omega) \times \mathbb{R}$ with $\mathcal{H}^n(R) = 0$. Indeed,

$$\begin{aligned} &\mathcal{H}^n\left(\partial^*\Sigma^{u_0} \cap \Phi_{n+1}(R \times \mathbb{S}^1) \cap (\{\nabla_{x\|} u_0 \neq 0\} \times \mathbb{R})\right) \\ &= \int_R \left(\int_{(\partial^*\Sigma^{u_0} \cap (\{\nabla_{x\|} u_0 \neq 0\} \times \mathbb{R}))_{(r,y,t)}} \frac{\chi_{\{\nabla_{x\|} u_0 \neq 0\}}(x,y)}{|\nabla_{x\|} u_0(x,y)|} \sqrt{1 + |\nabla u_0|^2}\, d\mathcal{H}^0(x) \right) dr\, dy\, dt = 0, \end{aligned}$$

where in the last equality we used the fact that $\mathcal{H}^n(R) = 0$.

Proposition 3.9. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u : \Omega \to \mathbb{R}$ be a Lebesgue measurable function such that $u_0 \in L^1_{\mathrm{loc}}(\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1))$, where u_0 is given by* (1.12)*. Let μ, v_μ, and α_μ be given by* (1.13)*,* (1.14) *and* (1.20)*, respectively. Then, $\mu \in L^1(A \times (-d, +\infty))$ for every open set $A \subset\subset \Pi_{n-1}(\Omega)$ and for every $d > 0$.*

Proof. Let $A \subset\subset \Pi_{n-1}(\Omega)$ be open and let $d > 0$. We have

$$\begin{aligned} \|\mu\|_{L^1(A \times (0,+\infty))} &= \int_0^\infty \left(\int_A \mu(r,y,t)\, drdy \right) dt = \int_0^\infty \left(\int_A \int_{\partial D(r)} \chi_{\{u_0 > t\}}(x,y)\, d\mathcal{H}^1(x)\, drdy \right) dt \\ &= \int_0^\infty \left(\int_{\Phi_n(A \times \mathbb{S}^1)} \chi_{\{u_0 > t\}}(x,y)\, dxdy \right) dt = \|(u_0)_+\|_{L^1(\Phi_n(A \times \mathbb{S}^1))} < +\infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\mu\|_{L^1(A \times (-d,0))} &= \int_{-d}^0 \left(\int_A \mu(r,y,t)\, drdy \right) dt = \int_{-d}^0 \left(\int_A \int_{\partial D(r)} \chi_{\{u_0 > t\}}(x,y)\, d\mathcal{H}^1(x)\, drdy \right) dt \\ &= \int_{-d}^0 \left(\int_{\Phi_n(A \times \mathbb{S}^1)} \chi_{\{u_0 > t\}}(x,y)\, dxdy \right) dt \leq \mathcal{H}^n\left(\Phi_n(A \times \mathbb{S}^1)\right) d < +\infty. \end{aligned}$$

Therefore, $\mu \in L^1(A \times (-d, +\infty))$. □

Let us now show that the function μ is well defined.

Proposition 3.10. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u_1, u_2 : \Omega \to \mathbb{R}$ be Lebesgue measurable functions such that $u_{1,0}, u_{2,0} \in L^1_{\mathrm{loc}}(\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1))$, where $u_{1,0}$ and $u_{2,0}$ are given by* (1.12) *with u_1 and u_2 in place of u, respectively. Let μ_1 and μ_2 be defined by* (1.13) *with u_1 and u_2 in place of u, respectively. Then, the following are equivalent:*

(i) *$u_1 = u_2$ $\mathcal{H}^n$-a.e. in Ω;*

(ii) *$\Sigma^{u_{1,0}} =_{\mathcal{H}^{n+1}} \Sigma^{u_{2,0}}$ in $\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1)) \times \mathbb{R}$.*

Moreover, both (i) *and* (ii) *imply*

(iii) *$\mu_1 = \mu_2$ $\mathcal{H}^n$-a.e. in $\Pi_{n-1}(\Omega) \times \mathbb{R}$.*

Proof. We start by showing that for every open set $A \subset\subset \Pi_{n-1}(\Omega)$

$$\begin{aligned} &\|u_1 - u_2\|_{L^1(\Omega \cap \Phi_n(A \times \mathbb{S}^1))} \\ &= \mathcal{H}^{n+1}\Big((\Sigma^{u_{1,0}} \Delta \Sigma^{u_{2,0}}) \cap (\Phi_{n+1}((A \times \mathbb{R}) \times \mathbb{S}^1)) \Big) \\ &= \int_{A \times \mathbb{R}} \mathcal{H}^1\Big((\partial D(r) \cap \{u_{1,0}(\cdot, y) > t\}) \Delta (\partial D(r) \cap \{u_{2,0}(\cdot, y) > t\}) \Big)\, dr\, dy\, dt. \end{aligned} \tag{3.4}$$

Indeed, let $A \subset\subset \Pi_{n-1}(\Omega)$ be open. Then,

$$
\begin{aligned}
&\|u_1 - u_2\|_{L^1(\Omega\cap\Phi_n(A\times\mathbb{S}^1))} = \|u_{1,0} - u_{2,0}\|_{L^1(\Phi_n(A\times\mathbb{S}^1))} \\
&= \int_{\Phi_n(A\times\mathbb{S}^1)} |u_{1,0}(x,y) - u_{2,0}(x,y)|\, dx\, dy \\
&= \int_{\Phi_n(A\times\mathbb{S}^1)} \left(\int_{\mathbb{R}} (\chi_{[u_{1,0}(x,y),u_{2,0}(x,y))}(t) + \chi_{[u_{2,0}(x,y),u_{1,0}(x,y))}(t))\, dt \right) dx\, dy \\
&= \int_{\Phi_n(A\times\mathbb{S}^1)} \left(\int_{\mathbb{R}} (\chi_{\{u_{2,0}>t\}\setminus\{u_{1,0}>t\}}(x,y) + \chi_{\{u_{1,0}>t\}\setminus\{u_{2,0}>t\}}(x,y))\, dt \right) dx\, dy \\
&= \int_{\Phi_n(A\times\mathbb{S}^1)} \left(\int_{\mathbb{R}} (\chi_{\Sigma^{u_{2,0}}\setminus\Sigma^{u_{1,0}}}(x,y,t) + \chi_{\Sigma^{u_{1,0}}\setminus\Sigma^{u_{2,0}}}(x,y,t))\, dt \right) dx\, dy \\
&= \mathcal{H}^{n+1}\Big((\Sigma^{u_{1,0}} \Delta \Sigma^{u_{2,0}}) \cap (\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)) \Big).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\mathcal{H}^{n+1}\Big((\Sigma^{u_{1,0}} \Delta \Sigma^{u_{2,0}}) \cap (\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)) \Big) \\
&= \int_{\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)} \chi_{\{u_{1,0}>t\}\Delta\{u_{2,0}>t\}}(x,y)\, dt\, dx\, dy \\
&= \int_{A\times\mathbb{R}} \left(\int_{\partial D(r)} \chi_{\{u_{1,0}>t\}\Delta\{u_{2,0}>t\}}(x,y)\, d\mathcal{H}^1(x) \right) dr\, dy\, dt \\
&= \int_{A\times\mathbb{R}} \mathcal{H}^1\Big((\partial D(r) \cap \{u_{1,0}(\cdot,y) > t\}) \Delta (\partial D(r) \cap \{u_{2,0}(\cdot,y) > t\}) \Big)\, dr\, dy\, dt,
\end{aligned}
$$

and this shows (3.4). Since $A \subset\subset \Pi_{n-1}(\Omega)$ was arbitrary, from the first equality in (3.4), it follows that (i) $\iff$ (ii).

Suppose now that (ii) holds. Then, since $A \subset\subset \Pi_{n-1}(\Omega)$ was arbitrary, from the second equality in (3.4) we obtain that

$$
\partial D(r) \cap \{u_{1,0}(\cdot,y) > t\} =_{\mathcal{H}^1} \partial D(r) \cap \{u_{2,0}(\cdot,y) > t\}, \qquad \text{for } \mathcal{H}^n\text{-a.e. } (r,y,t) \in \Pi_{n-1}(\Omega)\times\mathbb{R}.
$$

As a consequence,

$$
\mu_{u_1}(r,y,t) = \mathcal{H}^1(\partial D(r) \cap \{u_{1,0}(\cdot,y) > t\}) = \mathcal{H}^1(\partial D(r) \cap \{u_{2,0}(\cdot,y) > t\}) = \mu_{u_2}(r,y,t),
$$

for $\mathcal{H}^n$-a.e. $(r,y,t) \in \Pi_{n-1}(\Omega)\times\mathbb{R}$, and this shows (iii). ☐

The next result will be used to prove the Pólya–Szegö inequality.

Proposition 3.11. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u \in BV_{0,\tau}(\Omega)$, and let μ and α_μ be given by* (1.13) *and* (1.20)*, respectively. Then,*

$$
\begin{aligned}
(r,y,t) \in \{\alpha_\mu = 0\}^{(1)} \quad &\Longrightarrow \quad (x,y,t) \in (\Sigma^{u_0})^{(0)} \quad \text{for every } x \in \partial D(r); \\
(r,y,t) \in \{\alpha_\mu = \pi\}^{(1)} \quad &\Longrightarrow \quad (x,y,t) \in (\Sigma^{u_0})^{(1)} \quad \text{for every } x \in \partial D(r).
\end{aligned} \tag{3.5}
$$

Proof. We will only prove the first implication, since the second one is analogous. In the following, we will use the fact that

$$
(x',y',t') \in B^{n+1}_\rho(x,y,t) \quad \Longrightarrow \quad (|x'|,y',t') \in B^n_\rho(|x|,y,t) \quad \text{for every } \rho > 0. \tag{3.6}
$$

Let $(r,y,t) \in \{\alpha_\mu = 0\}^{(1)}$, and let $x \in \partial D(r)$. Let now $\rho \in (0,|x|)$. Thanks to (3.6), we have

$$
\begin{aligned}
&\mathcal{H}^{n+1}\left(\Sigma^{u_0} \cap B^{n+1}_\rho(x,y,t)\right) = \int_{\mathbb{R}^{n+1}} \chi_{\Sigma^{u_0}\cap B^{n+1}_\rho(x,y,t)}(x',y',t')\,dx'\,dy'\,dt' \\
&= \int_{\Pi_{n-1}(\Omega)\times\mathbb{R}} \left(\int_{\partial D(r')} \chi_{\Sigma^{u_0}\cap B^{n+1}_\rho(x,y,t)}(x',y',t') \chi_{B^2_\rho(x)}(x')\,d\mathcal{H}^1(x') \right) dr'\,dy'\,dt' \\
&\le \int_{\Pi_{n-1}(\Omega)\times\mathbb{R}} \chi_{B^n_\rho(|x|,y,t)}(r',y',t') \left(\int_{\partial D(r')} \chi_{\Sigma^{u_0}}(x',y',t') \chi_{B^2_\rho(x)}(x')\,d\mathcal{H}^1(x') \right) dr'\,dy'\,dt' \\
&\le \int_{\Pi_{n-1}(\Omega)\times\mathbb{R}} \chi_{\{\alpha_\mu>0\}\cap B^n_\rho(|x|,y,t)}(r',y',t') \left(\int_{\partial D(r')} \chi_{B^2_\rho(x)}(x')\,d\mathcal{H}^1(x') \right) dr'\,dy'\,dt' \\
&= \int_{\Pi_{n-1}(\Omega)\times\mathbb{R}} \mathcal{H}^1(\partial D(r') \cap B^2_\rho(x)) \chi_{\{\alpha_\mu>0\}\cap B^n_\rho(|x|,y,t)}(r',y',t')\,dr'\,dy'\,dt'.
\end{aligned}
$$

Note now that in the last integral only values of $r' \in (|x|-\rho, |x|+\rho)$ give a contribution. Therefore, for ρ sufficiently small, we have

$$\mathcal{H}^1(\partial D(r') \cap B^2_\rho(x)) \le C\rho,$$

for some $C > 0$ independent of ρ. Thus,

$$0 \le \limsup_{\rho\to 0^+} \frac{\mathcal{H}^{n+1}\left(\Sigma^{u_0} \cap B^{n+1}_\rho(x,y,t)\right)}{\mathcal{H}^{n+1}\left(B^{n+1}_\rho(r\omega,y,t)\right)} \le \frac{C}{\omega_{n+1}} \limsup_{\rho\to 0^+} \frac{\mathcal{H}^n(\{\alpha_\mu > 0\} \cap B^n_\rho(|x|,y,t))}{\rho^n} = 0,$$

where we used the fact that $(|x|,y,t) \in \{\alpha_\mu = 0\}^{(1)} = \{\alpha_\mu > 0\}^{(0)}$. □

We now state a useful result, which will be used extensively in the following, and can be obtained by combining together [11, Theorem 6.2] and Proposition 3.1 (see also [37]).

Proposition 3.12 (Vol'pert)**.** *Let $n \in \mathbb{N}$ with $n \ge 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u \in BV_{0,\tau}(\Omega)$, and let μ and α_μ be given by (1.13) and (1.20), respectively. Let $A \subset\subset \Pi_{n-1}(\Omega)$ be open. Then, there exists a Borel set $G_{u_0} \subset \{0 < \alpha_\mu < \pi\} \cap (A\times\mathbb{R})$ with $\mathcal{H}^n((\{0 < \alpha_\mu < \pi\} \cap (A\times\mathbb{R})) \setminus G_{u_0}) = 0$ such that the following properties hold. For every $(r,y,t) \in G_{u_0}$:*

(i) *$(\Sigma^{u_0})_{(r,y,t)}$ is a set of finite perimeter in $\partial D(r)$;*

(ii) *$\partial^*\left((\Sigma^{u_0})_{(r,y,t)}\right) = (\partial^*\Sigma^{u_0})_{(r,y,t)}$;*

(iii) *for **every** $\omega \in \mathbb{S}^1$ such that $(r\omega,y,t) \in (\partial^*\Sigma^{u_0})_{(r,y,t)} \cap \partial^*((\Sigma^{u_0})_{(r,y,t)})$:*

(iiia) $|\nu^{\Sigma^{u_0}}_{x_\parallel}(r\omega,y,t)| = \dfrac{|\nabla_{x_\parallel} u_0(r\omega,y)|}{\sqrt{1+|\nabla u_0(r\omega,y)|^2}} > 0$*;*

(iiib) $\nu^{(\Sigma^{u_0})_{(r,y,t)}}(r\omega,y,t) = \dfrac{\nu^{\Sigma^{u_0}}_{x_\parallel}(r\omega,y,t)}{|\nu^{\Sigma^{u_0}}_{x_\parallel}(r\omega,y,t)|} = \dfrac{\nabla_{x_\parallel} u_0(r\omega,y)}{|\nabla_{x_\parallel} u_0(r\omega,y)|}$*;*

Remark 3.13. *Note that the set G_{u_0} in general depends on A.*

Proof. The proposition is a direct consequence of the results contained in [21, Section 2.5]. □

Remark 3.14. *The previous proposition implies that*

$$\mathcal{H}^n(B_{\Sigma^{u_0}}) = 0, \tag{3.7}$$

where

$$B_{\Sigma^{u_0}} := \{0 < \alpha_\mu < \pi\} \cap \{(r,y,t) \in A\times\mathbb{R} : \exists\,\omega \in \mathbb{S}^1 : (r\omega,y,t) \in \partial^*\Sigma^{u_0} \text{ and } \nu^{\Sigma^{u_0}}_{x_\parallel}(r\omega,y,t) = 0\}.$$

Remark 3.15. *From Remark 3.14 it follows that if $(x,y) \in \mathcal{D}_{u_0} \cap \Phi_n(A\times\mathbb{S}^1)$ is such that $(x,y,u(x,y)) \in \partial^*\Sigma^{u_0}$ and*

$$\nabla_{x_\parallel} u_0(x,y) = 0,$$

then $(|x|,y,u_0(x,y)) \notin G_{u_0}$.

Remark 3.16. *If one considers the spherical symmetrization of u, in which the level sets of u are rearranged using spheres of $\mathbb{R}^n$ (rather than circles of $\mathbb{R}^2$), a result similar to Proposition 3.12 holds. However, in that case the analogue of identity* (3.7) *is false in general. This is due to the fact that for the spherical symmetrization property (iii) of Proposition 3.12 is satisfied in each slice only* ***up to an $\mathcal{H}^{n-2}$-negligible set****.*

Let us now show a consequence of (3.7).

Proposition 3.17. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u \in BV_{0,\tau}(\Omega)$, and let $B \subset\subset \Pi_{n-1}(\Omega) \times \mathbb{R}$ be a Borel set. Then the following statements are equivalent:*

(i) $\mathcal{H}^n \left(\left\{ (x, y, t) \in \partial^* \Sigma^{u_0} \cap \Phi_{n+1}(B \times \mathbb{S}^1) : \nu^{\Sigma^{u_0}}_{x\parallel}(x, y, t) = 0 \right\} \right) = 0$*;*

(ii) $P(\Sigma^{u_0}; \Phi_{n+1}(B' \times \mathbb{S}^1)) = 0$ *for every Borel set $B' \subset B$, such that $\mathcal{H}^n(B') = 0$.*

Proof. We show the two implications.

Step 1: (i) $\Longrightarrow$ (ii). Suppose (i) is satisfied, and let $B' \subset B$ be a Borel set such that $\mathcal{H}^n(B') = 0$. Then, thanks to Proposition 3.6,

$$\begin{aligned}
&P(\Sigma^{u_0}; \Phi_{n+1}(B' \times \mathbb{S}^1)) \\
&= \int_{\partial^*\Sigma^{u_0} \cap \Phi_{n+1}(B' \times \mathbb{S}^1) \cap \{\nu^{\Sigma^{u_0}}_{x\parallel} = 0\}} 1 \, d\mathcal{H}^n(x, y, t) + \int_{\partial^*\Sigma^{u_0} \cap \Phi_{n+1}(B' \times \mathbb{S}^1) \cap \{\nu^{\Sigma^{u_0}}_{x\parallel} \neq 0\}} 1 \, d\mathcal{H}^n(x, y, t) \\
&\leq \mathcal{H}^n \left(\left\{ (x, y, t) \in \partial^*\Sigma^{u_0} \cap \Phi_{n+1}(B \times \mathbb{S}^1) : \nu^{\Sigma^{u_0}}_{x\parallel}(x, y, t) = 0 \right\} \right) \\
&\quad + \int_{B'} \int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{1}{|\nu^{\Sigma^{u_0}}_{x\parallel}(x, y, t)|} \, d\mathcal{H}^0(x) \, dr \, dy \, dt = 0,
\end{aligned}$$

where we used (i) and the fact that $\mathcal{H}^n(B') = 0$.

Step 2: (ii) $\Longrightarrow$ (i). Suppose (ii) is satisfied. Then, since

$$\begin{aligned}
&\mathcal{H}^n \left(\left\{ (x, y, t) \in \partial^*\Sigma^{u_0} \cap \Phi_{n+1}((B \cap \{\alpha_\mu = 0\}) \times \mathbb{S}^1) \right\} \right) \\
&\overset{(ii)}{=} \mathcal{H}^n \left(\left\{ (x, y, t) \in \partial^*\Sigma^{u_0} \cap \Phi_{n+1}((B \cap \{\alpha_\mu = 0\}^{(1)}) \times \mathbb{S}^1) \right\} \right) \overset{(3.5)}{=} 0.
\end{aligned}$$

Similarly, we have

$$\mathcal{H}^n \left(\left\{ (x, y, t) \in \partial^*\Sigma^{u_0} \cap \Phi_{n+1}((B \cap \{\alpha_\mu = \pi\}) \times \mathbb{S}^1) \right\} \right) = 0.$$

Let now $A \subset\subset \Pi_{n-1}(\Omega)$ be an open set such that $B \subset A \times \mathbb{R}$, and let $B_{\Sigma^{u_0}}$ be the set defined in Remark 3.14. Then, thanks to (3.7), we have

$$\begin{aligned}
&\mathcal{H}^n \left(\left\{ (x, y, t) \in \partial^*\Sigma^{u_0} \cap \Phi_{n+1}(B \times \mathbb{S}^1) : \nu^{\Sigma^{u_0}}_{x\parallel}(x, y, t) = 0 \right\} \right) \\
&= \mathcal{H}^n \left(\left\{ (x, y, t) \in \partial^*\Sigma^{u_0} \cap \Phi_{n+1}((B \cap \{0 < \alpha_\mu < \pi\}) \times \mathbb{S}^1) : \nu^{\Sigma^{u_0}}_{x\parallel}(x, y, t) = 0 \right\} \right) \\
&\quad = \mathcal{H}^n \left(\left\{ (x, y, t) \in \partial^*\Sigma^{u_0} \cap \Phi_{n+1}((B \cap (A \times \mathbb{R}) \cap \{0 < \alpha_\mu < \pi\}) \times \mathbb{S}^1) : \nu^{\Sigma^{u_0}}_{x\parallel}(x, y, t) = 0 \right\} \right) \\
&\quad = P(\Sigma^{u_0}; \Phi_{n+1}(B \cap B_{\Sigma^{u_0}} \times \mathbb{S}^1)) = 0,
\end{aligned}$$

where we used (ii) with $B' = B \cap B_{\Sigma^{u_0}}$. □

Remark 3.18. *One cannot expect a result analogous to Proposition 3.17 to hold for the spherical rearrangement of u. Indeed, to prove the implication (ii) $\Longrightarrow$ (i) above we have used identity* (3.7)*, which fails for the spherical symmetrization (see Remark 3.16).*

3.2. **Further properties of μ, v_μ, and Σ^{v_μ}.** In this section we show some important properties of the distribution μ, and how these affect the function v_μ and the set Σ^{v_μ}. We start by showing that the circular rearrangement preserves the L^p norm, provided condition (b) of Definition 1.2 is satisfied.

Proposition 3.19. *Let $n \in \mathbb{N}$ with $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be open, and let $p \in [1, \infty)$. Let $u : \Omega \to \mathbb{R}$ be a Lebesgue measurable function such that $u_0 \in L^p_{\rm loc}(\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1))$, where u_0 is given by* (1.12)*. Let μ and v_μ be defined by* (1.13) *and* (1.14)*, respectively. Then, for every open set $A \subset\subset \Pi_{n-1}(\Omega)$ we have $v_\mu \in L^p(\Phi_n(A \times \mathbb{S}^1))$, with $\|v_\mu\|_{L^p(\Phi_n(A\times\mathbb{S}^1))} = \|u_0\|_{L^p(\Phi_n(A\times\mathbb{S}^1))}$.*

Proof. Let $t \geq 0$. We have

$$\begin{aligned}
&\mathcal{H}^n(\{(v_\mu)_+ > t\} \cap \Phi_n(A \times \mathbb{S}^1))) = \mathcal{H}^n(\{v_\mu > t\} \cap \Phi_n(A \times \mathbb{S}^1))) \\
&= \int_A \mathcal{H}^1(\{v_\mu(\cdot, y) > t\} \cap \partial D(r))\, dr\, dy = \int_A \mu(r, y, t)\, dr\, dy \\
&= \mathcal{H}^n(\{(u_0)_+ > t\} \cap \Phi_n(A \times \mathbb{S}^1))),
\end{aligned}$$

where the last equality follows by applying backward the previous equalities to the function u. Similarly,

$$\begin{aligned}
&\mathcal{H}^n(\{(v_\mu)_- \geq t\} \cap \Phi_n(A \times \mathbb{S}^1)) = \mathcal{H}^n(\{v_\mu \leq -t\} \cap \Phi_n(A \times \mathbb{S}^1)) \\
&= \int_A \mathcal{H}^1(\{v_\mu(\cdot, y) \leq -t\} \cap \partial D(r))\, dr\, dy = \int_A \left[2\pi r - \mathcal{H}^1(\{v_\mu(\cdot, y) > -t\} \cap \partial D(r))\right] dr\, dy \\
&= \int_A [2\pi r - \mu(r, y, -t)]\, dr\, dy = \mathcal{H}^n(\{(u_0)_- \geq t\} \cap \Phi_n(A \times \mathbb{S}^1))).
\end{aligned}$$

Thanks to the Layer-cake formula, we then obtain

$$\begin{aligned}
&\|v_\mu\|_{L^p(\Phi_n(A\times\mathbb{S}^1))} = \|(v_\mu)_+\|_{L^p(\Phi_n(A\times\mathbb{S}^1))} + \|(v_\mu)_-\|_{L^p(\Phi_n(A\times\mathbb{S}^1))} \\
&= \|(u_0)_+\|_{L^p(\Phi_n(A\times\mathbb{S}^1))} + \|(u_0)_-\|_{L^p(\Phi_n(A\times\mathbb{S}^1))} = \|u_0\|_{L^p(\Phi_n(A\times\mathbb{S}^1))}.
\end{aligned}$$

□

In the next proposition we show the connection between the circular rearrangement of a function and the circular symmetrization of its subgraph.

Proposition 3.20. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u : \Omega \to \mathbb{R}$ be a Lebesgue measurable function such that $u_0 \in L^1_{\rm loc}(\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1))$, where u_0 is given by* (1.12)*. Let μ, v_μ, and α_μ be given by* (1.13)*,* (1.14) *and* (1.20)*, respectively, and let $F_\mu \subset \mathbb{R}^{n+1}$ be defined as*

$$F_\mu := \{(x, y, t) \in \Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1) \times \mathbb{R} : d_{\mathbb{S}^1}(\hat{x}, e_1) < \alpha_\mu(|x|, y, t)\}. \tag{3.8}$$

Then,

$$\Sigma^{v_\mu} =_{\mathcal{H}^{n+1}} F_\mu. \tag{3.9}$$

Proof. Let $(x, y, t) \in \Phi_n(\Pi_{n-1}(\Omega)\times\mathbb{S}^1)\times\mathbb{R}$. Since the hyperplane $\{(x_1, x_2, y_1, \ldots, y_{n-2}, t) \in \mathbb{R}^{n+1} : x_2 = 0\}$ is $\mathcal{H}^{n+1}$-negligible, we can assume $0 < d_{\mathbb{S}^1}(\hat{x}, e_1) < \pi$. By definition of v_μ, we have

$$v_\mu(x, y) = \inf A(x, y) \quad \text{where} \quad A(x, y) := \{s \in \mathbb{R} : \alpha_\mu(|x|, y, s) \leq d_{\mathbb{S}^1}(\hat{x}, e_1)\}.$$

Thanks to Remark 1.3, the function $s \mapsto \mu(|x|, y, s)$ is non-increasing and right-continuous, and so is $s \mapsto \alpha_\mu(|x|, y, s)$. Therefore, since $0 < d_{\mathbb{S}^1}(\hat{x}, e_1) < \pi$, we have that $A(x, y)$ is a closed half-line which is unbounded from above and bounded from below, and

$$A(x, y) = [v_\mu(x, y), +\infty) \quad \text{and} \quad \alpha_\mu(|x|, y, v_\mu(x, y)) \leq d_{\mathbb{S}^1}(\hat{x} \cdot e_1).$$

Thus,

$$v_\mu(x, y) > t \iff t \notin A(x, y) \iff \alpha_\mu(|x|, y, t) > d_{\mathbb{S}^1}(\hat{x} \cdot e_1).$$

From this, it follows that

$$(x, y, t) \in \Sigma^{v_\mu} \iff v_\mu(x, y) > t \iff \alpha_\mu(|x|, y, t) > d_{\mathbb{S}^1}(\hat{x} \cdot e_1) \iff (x, y, t) \in F_\mu.$$

□

The next result should be compared with Proposition 3.10.

Proposition 3.21. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u_1, u_2 : \Omega \to \mathbb{R}$ be Lebesgue measurable functions such that $u_{1,0}, u_{2,0} \in L^1_{\rm loc}(\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1))$, where $u_{1,0}$ and $u_{2,0}$ are given by* (1.12) *with u_1 and u_2 in place of u, respectively. Let μ_1 and μ_2 be defined by* (1.13) *with u_1 and u_2 in place of u, respectively. Let now v_{μ_1}, v_{μ_2} and F_{μ_1}, F_{μ_2} be defined by* (1.14) *and by* (3.8)*, with μ_1 and μ_2 in place of μ, respectively. Then, the following are equivalent:*

(a) $\mu_1 = \mu_2$ $\mathcal{H}^n$-*a.e. in* $\Pi_{n-1}(\Omega) \times \mathbb{R}$;

(b) $v_{\mu_1} = v_{\mu_2}$ $\mathcal{H}^n$-*a.e. in* $\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1)$;

(c) $\Sigma^{v_{\mu_1}} =_{\mathcal{H}^{n+1}} \Sigma^{v_{\mu_2}}$ *in* $\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1)) \times \mathbb{R}$;

(d) $F_{\mu_1} =_{\mathcal{H}^{n+1}} F_{\mu_2}$ *in* $\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1)) \times \mathbb{R}$.

Proof. Let $A \subset\subset \Pi_{n-1}(\Omega)$ be open. Applying (3.4) to the two functions v_{μ_1} and v_{μ_2}, we obtain

$$\begin{aligned}
&\|v_{\mu_1} - v_{\mu_2}\|_{L^1(\Phi_n(A\times\mathbb{S}^1))} \\
&= \mathcal{H}^{n+1}\Big((\Sigma^{v_{\mu_1}} \Delta \Sigma^{v_{\mu_2}}) \cap (\Phi_{n+1}((A \times \mathbb{R}) \times \mathbb{S}^1)))\Big) \\
&= \int_{A\times\mathbb{R}} \mathcal{H}^1\Big((\partial D(r) \cap \{v_{\mu_1}(\cdot, y) > t\}) \Delta (\partial D(r) \cap \{v_{\mu_2}(\cdot, y) > t\})\Big)\, dr\, dy\, dt \\
&= \int_{A\times\mathbb{R}} \mathcal{H}^1(\mathbf{B}_{\alpha_{\mu_1}}(re_1) \Delta \mathbf{B}_{\alpha_{\mu_2}}(re_1))\, dr\, dy\, dt = \|\mu_1 - \mu_2\|_{L^1(A\times\mathbb{R})}.
\end{aligned} \tag{3.10}$$

This shows that $\mu_1 - \mu_2 \in L^1(A \times \mathbb{R})$ (even though, thanks to Proposition 3.9, in general we only have $\mu_1, \mu_2 \in L^1(A \times (-d, +\infty))$ for every $d > 0$). Since $A \subset\subset \Pi_{n-1}(\Omega)$ is arbitrary, from (3.10) it follows that (a), (b), and (c) are equivalent. Finally, from (3.9) we conclude that (c) is equivalent to (d). □

Proposition 3.22. *Let $n \in \mathbb{N}$ with $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be open. Let $u : \Omega \to \mathbb{R}$ be a measurable function satisfying property (b) of Definition 1.2, such that $u_0 \in L^1_{\mathrm{loc}}(\Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1))$, where u_0 is given by (1.12). Let μ and v_μ be defined by (1.13) and (1.14), respectively. Then,*

$$v_\mu(x, y) = 0 \quad \textit{for } \mathcal{H}^n\textit{-a.e. } (x, y) \in \Phi_n(\Pi_{n-1}(\Omega) \times \mathbb{S}^1) \setminus \Omega^s. \tag{3.11}$$

As a consequence,

$$(v_\mu|_{\Omega^s})_0 = v_\mu, \tag{3.12}$$

where $(v_\mu|_{\Omega^s})_0$ is given by (1.12) with v_μ in place of u.

Proof. Note that, by definition of $\Pi^{\mathrm{a}}_{n-1}(\Omega)$ (see (1.4)), we have that

$$\mathcal{L}^n(\Phi_n(\Pi^{\mathrm{a}}_{n-1}(\Omega) \times \mathbb{S}^1) \setminus \Omega^s) = 0.$$

Therefore, to show (3.11) it will be enough to prove that

$$v_\mu(x, y) = 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } (x, y) \in \Phi_n((\Pi_{n-1}(\Omega) \setminus \Pi^{\mathrm{a}}_{n-1}(\Omega)) \times \mathbb{S}^1) \setminus \Omega^s.$$

Let now $(x, y) \in \Phi_n((\Pi_{n-1}(\Omega) \setminus \Pi^{\mathrm{a}}_{n-1}(\Omega)) \times \mathbb{S}^1) \setminus \Omega^s$. Since the hyperplane $\{(x, y) \in \mathbb{R}^n : x_2 = 0\}$ is $\mathcal{H}^n$-negligible, we can assume that $0 < d_{\mathbb{S}^1}(\hat{x}, e_1) < \pi$. Then, we have $(|x|, y) \notin \Pi^{\mathrm{a}}_{n-1}(\Omega)$ and

$$2\pi|x| > 2|x| d_{\mathbb{S}^1}(\hat{x}, e_1) \geq \mathcal{H}^1((\Omega)_{(|x|,y)}).$$

Thanks to Proposition 3.10 and Proposition 3.21, if we modify u on a set of $\mathcal{H}^n$-measure zero, this will only affect v_μ on a $\mathcal{H}^n$-negligible set. Therefore, thanks to property (b) of Definition 1.2, we can assume that

$$u \geq 0 \quad \text{everywhere in } \Omega \setminus \Phi_n(\Pi^{\mathrm{a}}_{n-1}(\Omega) \times \mathbb{S}^1). \tag{3.13}$$

Then, since $u_0 = 0$ outside of Ω, for every $t > 0$ we have

$$\begin{aligned}
\mu(|x|, y, t) &= \mathcal{H}^1(\{u_0(\cdot, y) > t\} \cap \partial D(|x|)) \\
&= \mathcal{H}^1(\{u(\cdot, y) > t\} \cap \Omega \cap \partial D(|x|)) \\
&\leq \mathcal{H}^1((\Omega)_{(|x|,y)}) \leq 2|x| d_{\mathbb{S}^1}(\hat{x}, e_1).
\end{aligned}$$

On the other hand, thanks to (3.13), for every $t < 0$ we have

$$\mu(|x|, y, t) = \mathcal{H}^1(\{u_0(\cdot, y) > t\} \cap \partial D(|x|)) = 2\pi|x| > 2|x| d_{\mathbb{S}^1}(\hat{x}, e_1).$$

Thus, recalling (1.14),

$$v_\mu(x, y) = \inf\{t \in \mathbb{R} : \mu(|x|, y, t) \leq 2|x| d_{\mathbb{S}^1}(\hat{x}, e_1)\} = 0.$$

□

The next result gives a refinement of Proposition 3.11 in the special case in which one considers the function v_μ, and will be used in the proof of The Pólya–Szegö inequality.

Proposition 3.23. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u \in BV_{0,\tau}(\Omega)$, and let μ, v_μ, and α_μ be given by (1.13), (1.14), and (1.20), respectively. Then,*

$$(r, y, t) \in \{\alpha_\mu^\vee = 0\} \quad \Longrightarrow \quad (r\omega, y, t) \in (\Sigma^{v_\mu})^{(0)} \text{ for every } \omega \in \mathbb{S}^1 \setminus \{e_1\};$$
$$(r, y, t) \in \{\alpha_\mu^\wedge = \pi\} \quad \Longrightarrow \quad (r\omega, y, t) \in (\Sigma^{v_\mu})^{(1)} \text{ for every } \omega \in \mathbb{S}^1 \setminus \{-e_1\}.$$

Moreover,

$$\mathcal{H}^n\left(\partial^* \Sigma^{v_\mu} \cap \left(\Phi_{n+1}\left(\{\alpha_\mu^\vee = 0\} \times \mathbb{S}^1\right)\right)\right) = 0, \tag{3.14}$$

and

$$\mathcal{H}^n\left(\partial^* \Sigma^{v_\mu} \cap \left(\Phi_{n+1}\left(\{\alpha_\mu^\wedge = \pi\} \times \mathbb{S}^1\right)\right)\right) = 0. \tag{3.15}$$

As a consequence, for every Borel set $B \subset \Pi_{n-1}(\Omega) \times \mathbb{R}$ we have

$$\begin{aligned} &\mathcal{H}^n\left(\partial^* \Sigma^{v_\mu} \cap \left(\Phi_{n+1}\left(B \times \mathbb{S}^1\right)\right)\right) \\ &= \mathcal{H}^n\left(\partial^* \Sigma^{v_\mu} \cap \left(\Phi_{n+1}\left((B \cap \{\alpha_\mu^\vee > 0\} \cap \{\alpha_\mu^\wedge < \pi\}) \times \mathbb{S}^1\right)\right)\right). \end{aligned} \tag{3.16}$$

Proof. We will just show the first implication and (3.14), since the second implcation and (3.15) can be proved in a similar way. We proceed by steps.

Step 1: We show that

$$(r, y, t) \in \{\alpha_\mu^\vee = 0\} \quad \Longrightarrow \quad (r\omega, y, t) \in (\Sigma^{v_\mu})^{(0)} \text{ for every } \omega \in \mathbb{S}^1 \setminus \{e_1\}.$$

Let $(r, y, t) \in \{\alpha_\mu^\vee = 0\}$, and let $\omega \in \mathbb{S}^1 \setminus \{e_1\}$. We set

$$\delta := \arccos(\omega \cdot e_1) > 0. \tag{3.17}$$

Note that

$$(x', y', t') \in \Sigma^{v_\mu} \cap B_\rho^{n+1}(r\omega, y, t) \quad \Longrightarrow \quad (|x'|, y', t') \in \{\alpha_\mu > \delta/2\} \quad \text{if } 0 < \rho \ll 1. \tag{3.18}$$

Indeed, if $(x', y', t') \in \Sigma^{v_\mu} \cap B_\rho^{n+1}(r\omega, y, t)$, we have $(x', y', t') = (r\omega, y, t) + \rho(\overline{x}, \overline{y}, \overline{t})$, for some $(\overline{x}, \overline{y}, \overline{t}) \in B_1^{n+1}$. Therefore, thanks to (3.17), for ρ sufficiently small

$$\alpha_\mu(|x'|, y', t') > \arccos\left(\frac{x'}{|x'|} \cdot e_1\right) = \arccos\left(\frac{(r\omega + \rho\overline{x}) \cdot e_1}{|r\omega + \rho\overline{x}|}\right) > \frac{\delta}{2},$$

so that (3.18) holds. Then, using the same argument of the proof of Proposition 3.11, we obtain that for $0 < \rho \ll 1$

$$\begin{aligned} &\mathcal{H}^{n+1}\left(\Sigma^{v_\mu} \cap B_\rho^{n+1}(r\omega, y, t)\right) = \int_{\mathbb{R}^{n+1}} \chi_{\Sigma^{v_\mu} \cap B_\rho^{n+1}(r\omega, y, t)}(x', y', t')\, dx'\, dy'\, dt' \\ &= \int_{\Pi_{n-1}(\Omega) \times \mathbb{R}} \left(\int_{\partial D(r')} \chi_{B_\rho^2(r\omega)}(x') \chi_{\Sigma^{v_\mu} \cap B_\rho^{n+1}(r\omega, y, t)}(x', y', t')\, d\mathcal{H}^1(x')\right) dr'\, dy'\, dt' \\ &\leq \int_{\Pi_{n-1}(\Omega) \times \mathbb{R}} \left(\int_{\partial D(r')} \chi_{B_\rho^2(r\omega)}(x') \chi_{\{\alpha_\mu > \delta/2\} \cap B_\rho^n(r, y, t)}(r', y', t')\, d\mathcal{H}^1(x')\right) dr'\, dy'\, dt' \\ &\leq \int_{\Pi_{n-1}(\Omega) \times \mathbb{R}} \mathcal{H}^1\left(\partial D(r') \cap B_\rho^2(r\omega)\right) \chi_{\{\alpha_\mu > \delta/2\} \cap B_\rho^n(r, y, t)}(r', y', t')\, d\mathcal{H}^1(x')\, dr'\, dy'\, dt' \\ &\leq C\rho \mathcal{H}^n(\{\alpha_\mu > \delta/2\} \cap B_\rho^n(r, y, t)), \end{aligned}$$

for some $C > 0$ independent of ρ. Therefore,

$$0 \leq \limsup_{\rho \to 0^+} \frac{\mathcal{H}^{n+1}\left(\Sigma^{v_\mu} \cap B_\rho^{n+1}(r\omega, y, t)\right)}{\mathcal{H}^{n+1}\left(B_\rho^{n+1}(r\omega, y, t)\right)} \leq \frac{C}{\omega_{n+1}} \limsup_{\rho \to 0^+} \frac{\mathcal{H}^n(\{\alpha_\mu > \delta/2\} \cap B_\rho^n(r, y, t))}{\rho^n} = 0,$$

where we used the fact that $\alpha^\vee(r, y, t) = 0$.

Step 2: We show (3.14). Thanks to (3.5) and Step 1, if $(x, y, t) \in \partial^* \Sigma^{v_\mu}$ and $(|x|, y, t) \in \{\alpha_\mu^\vee = 0\}$, with $x = (x_1, x_2)$, then

$$x_1 = |x|,\ x_2 = 0, \quad \text{and} \quad (x_1, y, t) \in (\Pi_{n-1}(\Omega) \times \mathbb{R}) \cap (\{\alpha_\mu^\vee = 0\} \setminus \{\alpha_\mu = 0\}^{(1)}).$$

Therefore,

$$\mathcal{H}^n\left(\partial^*\Sigma^{v_\mu}\cap\left(\Phi_{n+1}\left(\{\alpha_\mu^\vee=0\}\times\mathbb{S}^1\right)\right)\right)\leq\int_{(\Pi_{n-1}(\Omega)\times\mathbb{R})\cap(\{\alpha_\mu^\vee=0\}\setminus\{\alpha_\mu=0\}^{(1)})}1\,d\mathcal{H}^n(x_1,y,t)$$
$$\leq\mathcal{H}^n(\{\alpha_\mu^\vee=0\}\setminus\{\alpha_\mu=0\}^{(1)})=\mathcal{H}^n(\{\alpha_\mu^\vee=0\}\setminus\{\alpha_\mu^\vee=0\}^{(1)})=0.$$

□

In the next proposition we give the explicit expression of the distributional derivatives of μ and ξ_μ, see [11, Lemma 6.7] and [28, Lemma 3.6] for related results.

Proposition 3.24. *Let $n\in\mathbb{N}$ with $n\geq 2$, and let $\Omega\subset\mathbb{R}^n$ be open. Let $u\in BV_{0,\tau}(\Omega)$, and let μ and α_μ be given by (1.13) and (1.20), respectively, and let $\xi_\mu:\Pi_{n-1}(\Omega)\times\mathbb{R}\to[0,2\pi]$ be given by*

$$\xi_\mu(r,y,t):=\frac{\mu(r,y,t)}{r}=2\alpha_\mu(r,y,t),\qquad \textit{for every } (r,y,t)\in\Pi_{n-1}(\Omega)\times\mathbb{R}. \tag{3.19}$$

Let $A\subset\subset\Pi_{n-1}(\Omega)$ be open and let $d>0$. Then, Σ^{u_0} is a set of finite perimeter in $\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)$, and $\mu,\xi_\mu,\alpha_\mu\in BV(A\times(-d,+\infty))$. Moreover, for every Borel set $B\subset A\times(-d,+\infty)$ and for every bounded Borel function $\varphi:B\to\mathbb{R}$ we have

$$\int_B\varphi(r,y,t)\,dD_y\mu(r,y,t)=\int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}(B\times\mathbb{S}^1))}\varphi(|x|,y,t)\,\nu_y^{\Sigma^{u_0}}(x,y,t)\,d\mathcal{H}^n(x,y,t), \tag{3.20}$$

$$\int_B\varphi(r,y,t)\,dD_t\mu(r,y,t)=\int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}(B\times\mathbb{S}^1))}\varphi(|x|,y,t)\,\nu_t^{\Sigma^{u_0}}(x,y,t)\,d\mathcal{H}^n(x,y,t), \tag{3.21}$$

$$\int_B\varphi(r,y,t)r\,dD_r\xi_\mu(r,y,t)=\int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}(B\times\mathbb{S}^1))}\varphi(|x|,y,t)\,\hat{x}\cdot\nu_x^{\Sigma^{u_0}}(x,y,t)\,d\mathcal{H}^n(x,y,t). \tag{3.22}$$

Remark 3.25. *Applying the previous result to the μ-distributed function v_μ, thanks to Proposition 3.23, we obtain that*

$$D_y\mu=D_y\mu\llcorner\{\alpha^\vee>0\}\cap\{\alpha^\wedge<\pi\};$$
$$D_t\mu=D_t\mu\llcorner\{\alpha^\vee>0\}\cap\{\alpha^\wedge<\pi\};$$
$$rD_r\xi_\mu=rD_r\xi_\mu\llcorner\{\alpha^\vee>0\}\cap\{\alpha^\wedge<\pi\}.$$

Proof of Proposition 3.24. By definition of $BV_{0,\tau}(\Omega)$ we have that $u_0\in BV(\Phi_n(A\times\mathbb{S}^1))$ and so, thanks to Proposition 3.1, Σ^{u_0} is a set of finite perimeter in $\Phi_n(A\times\mathbb{S}^1)\times\mathbb{R}=\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)$. Moreover, $\mu\in L^1(A\times(-d,+\infty))$ thanks to Proposition 3.9. We now divide the remaining part of the proof in several steps.

Step 1: We show that formulas (3.20) and (3.21) hold in the special case $B=A\times(-d,+\infty)$ and $\varphi\in C_c^1(A\times(-d,+\infty))$. Concerning (3.21), we have

$$\begin{aligned}&\int_{A\times(-d,+\infty)}\varphi(r,y,t)\,dD_t\mu(r,y,t)=-\int_{A\times(-d,+\infty)}\mu(r,y,t)\frac{\partial\varphi}{\partial t}(r,y,t)\,drdydt\\&=-\int_{\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)}\chi_{\Sigma^{u_0}}(x,y,t)\frac{\partial\varphi}{\partial t}(|x|,y,t)\,dx\,dy\,dt\\&=\int_{\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)}\varphi(|x|,y,t)\,dD_t\chi_{\Sigma^{u_0}}(x,y,t)\\&=\int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}((A\times(-d,+\infty))\times\mathbb{S}^1))}\varphi(|x|,y,t)\,\nu_t^{\Sigma^{u_0}}(x,y,t)\,d\mathcal{H}^n(x,y,t).\end{aligned}$$

One can argue in a similar way for identity (3.20).

Step 2: We show that for every $\varphi\in C_c^1(A\times(-d,+\infty))$

$$\begin{aligned}\int_{A\times(-d,+\infty)}\varphi(r,y,t)\,dD_r\mu(r,y,t)&=\int_{A\times(-d,+\infty)}\varphi(r,y,t)\xi_\mu(r,y,t)\,dr\,dy\,dt\\&+\int_{\partial^*\Sigma^{u_0}\cap\Phi_{n+1}((A\times(-d,+\infty))\times\mathbb{S}^1)}\varphi(|x|,y,t)\,\hat{x}\cdot\,\nu_x^{\Sigma^{u_0}}(x,y,t)\,d\mathcal{H}^n(x,y,t).\end{aligned} \tag{3.23}$$

Let $\varphi \in C^1_c(A\times(-d,+\infty))$. Then, a direct calculation gives

$$\mathrm{div}_{(x,y,t)}(\varphi(|x|,y,t)\hat{x}) = \frac{\partial\varphi}{\partial r}(|x|,y,t) + \frac{1}{|x|}\varphi(|x|,y,t), \qquad \forall\,(x,y,t)\in\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1),$$

where $\mathrm{div}_{(x,y,t)}$ denotes the divergence of a vector field in $\mathbb{R}^{n+1}$ with respect to the variables (x,y,t). Therefore,

$$\begin{aligned}
&\int_{A\times(-d,+\infty)}\varphi(r,y,t)\,dD_r\mu(r,y,t) = -\int_{A\times(-d,+\infty)}\mu(r,y,t)\frac{\partial\varphi}{\partial r}(r,y,t)\,dr\,dy\,dt\\
&= -\int_{A\times\mathbb{R}}\mu(r,y,t)\frac{\partial\varphi}{\partial r}(r,y,t)\,dr\,dy\,dt = -\int_{\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)}\chi_{\Sigma^{u_0}}(x,y,t)\frac{\partial\varphi}{\partial r}(|x|,y,t)\,dx\,dy\,dt\\
&= -\int_{\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)}\chi_{\Sigma^{u_0}}(x,y,t)\left(\mathrm{div}_{(x,y,t)}\Big(\varphi(|x|,y,t)\hat{x}\Big) - \frac{1}{|x|}\varphi(|x|,y,t)\right)\,dx\,dy\,dt\\
&= \int_{\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)}\varphi(|x|,y,t)\,\hat{x}\cdot\,dD\chi_{\Sigma^{u_0}}(x,y,t) + \int_{\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)}\chi_{\Sigma^{u_0}}(x,y,t)\frac{1}{|x|}\varphi(|x|,y,t)\,dx\,dy\,dt\\
&= \int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}((A\times(-d,+\infty))\times\mathbb{S}^1))}\varphi(|x|,y,t)\,\hat{x}\cdot\,\nu_x^{\Sigma^{u_0}}(x,y,t)\,d\mathcal{H}^n(x,y,t)\\
&+\int_{A\times(-d,+\infty)}\xi_\mu(r,y,t)\varphi(r,y,t)\,dr\,dy\,dt,
\end{aligned}$$

which gives (3.23).

Step 3: We show that $D_y\mu$, $D_t\mu$ and $D_r\mu$ are bounded Radon measures on $A\times(-d,+\infty)$. From Step 2, we know that (3.21) holds with $B = A\times(-d,+\infty)$ and $\varphi\in C^1_c(A\times(-d,+\infty))$. Taking the supremum over all $\varphi\in C^1_c(A\times(-d,+\infty))$ with $|\varphi|\le 1$ we obtain that $D_t\mu$ is a bounded Radon measure on $A\times(-d,+\infty)$, with

$$|D_t\mu|(A\times(-d,+\infty)) \le \mathcal{H}^n\left(\partial^*\Sigma^{u_0}\cap\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)\right) < \infty.$$

One can argue in a similar way for $D_y\mu$.

We are left to show that $D_r\mu$ is a bounded Radon measure on $A\times(-d,+\infty)$. Let $c>0$ be such that $r\ge c$ for every $(r,y)\in A$. Thanks to (3.23), for every $\varphi\in C^1_c(A\times(-d,+\infty))$ with $|\varphi|\le 1$

$$\begin{aligned}
\int_{A\times(-d,+\infty)}\varphi(r,y,t)\,dD_r\mu(r,y,t) &= \int_{A\times(-d,+\infty)}\varphi(r,y,t)\frac{\mu(r,y,t)}{r}\,dr\,dy\,dt\\
&+\int_{\partial^*\Sigma^{u_0}\cap\Phi_{n+1}((A\times(-d,+\infty))\times\mathbb{S}^1)}\varphi(|x|,y,t)\,\hat{x}\cdot\,\nu_x^{\Sigma^{u_0}}(x,y,t)\,d\mathcal{H}^n(x,y,t)\\
&\le \frac{1}{c}\|\mu\|_{L^1(A\times(-d,+\infty))} + \mathcal{H}^n\left(\partial^*\Sigma^{u_0}\cap\Phi_{n+1}((A\times(-d,+\infty))\times\mathbb{S}^1)\right) < \infty.
\end{aligned}$$

Taking the supremum over all $\varphi\in C^1_c(A\times(-d,+\infty))$ with $|\varphi|\le 1$ we conclude.

Step 4: We show that $\xi_\mu,\alpha_\mu\in BV(A\times(-d,+\infty))$. From Step 3, we have $\mu\in BV(A\times(-d,+\infty))$. Observe now that also the map $(r,y,t)\mapsto 1/r$ belongs to $BV(A\times(-d,+\infty))$. Therefore, from [2, Example 3.97] it follows that both $\xi_\mu = \mu/r$ and $\alpha_\mu = \mu/(2r)$ belong to $BV(A\times(-d,+\infty))$.

Step 5: We show formulas (3.20) and (3.21). We will only give the proof of (3.21), because the proof of (3.20) is similar. Using the fact that every function in $C^0_b(A\times(-d,+\infty))$ can be approximated uniformly on compact subsets of $A\times(-d,+\infty)$ by functions in $C^1_c(A\times(-d,+\infty))$, and the fact that $D_t\mu$ is a bounded Radon measure on $A\times(-d,+\infty)$, we have that (3.21) holds for every $\varphi\in C^0_b(A\times(-d,+\infty))$.

Let now $\varphi : A\times(-d,+\infty)\to\mathbb{R}$ be a bounded Borel function, and let λ be the bounded Radon measure on $A\times(-d,+\infty)$ defined by

$$\lambda(B) := |D_t\mu|(B) + \mathcal{H}^n\left(\partial^*\Sigma^{u_0}\cap\left(\Phi_{n+1}(B\times\mathbb{S}^1)\right)\right), \qquad \text{for every Borel set } B\subset A\times(-d,+\infty).$$

By Lusin Theorem, for every $h \in \mathbb{N}$ there exists $\varphi_h \in C^0_b(A\times(-d,+\infty))$ with $\|\varphi_h\|_{L^\infty(A\times(-d,+\infty))} \leq \|\varphi\|_{L^\infty(A\times(-d,+\infty))}$ such that

$$\lambda\left(\{(r,y,t) \in A \times (-d,+\infty) : \varphi(r,y,t) \neq \varphi_h(r,y,t)\}\right) < \frac{1}{h}.$$

For each $h \in \mathbb{N}$ we can apply (3.21) to φ_h, so that

$$\int_{A\times(-d,+\infty)} \varphi_h(r,y,t)\, dD_t\mu(r,y,t) = \int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}((A\times(-d,+\infty))\times\mathbb{S}^1))} \varphi_h(|x|,y,t)\, \nu_t^{\Sigma^{u_0}}(x,y,t)\, d\mathcal{H}^n(x,y,t).$$

Using this identity, we have

$$\begin{aligned}
&\left|\int_{A\times(-d,+\infty)} \varphi(r,y,t)\, dD_t\mu(r,y,t) - \int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}((A\times(-d,+\infty))\times\mathbb{S}^1))} \varphi(|x|,y,t)\, \nu_t^{\Sigma^{u_0}}(x,y,t)\, d\mathcal{H}^n(x,y,t)\right| \\
&= \left|\int_{A\times(-d,+\infty)} (\varphi(r,y,t) - \varphi_h(r,y,t))\, dD_t\mu(r,y,t) + \int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}((A\times(-d,+\infty))\times\mathbb{S}^1))} (\varphi_h(|x|,y,t) - \varphi(|x|,y,t))\, \nu_t^{\Sigma^{u_0}}(x,y,t)\, d\mathcal{H}^n(x,y,t)\right| \\
&\leq \int_{A\times(-d,+\infty)} |\varphi(r,y,t) - \varphi_h(r,y,t)|\, d\,|D_t\mu|\,(r,y,t) \\
&+ \int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}((A\times(-d,+\infty))\times\mathbb{S}^1))} |\varphi_h(|x|,y,t) - \varphi(|x|,y,t)|\, d\mathcal{H}^n(x,y,t) \leq \frac{2}{h}\|\varphi\|_{L^\infty(A\times(-d,+\infty))}.
\end{aligned}$$

Passing to the limit as $h \to \infty$ we obtain (3.21).

Step 6: We show (3.22). Arguing as in Step 5 it follows that (3.23) holds whenever φ is a bounded Borel function on $A \times (-d,+\infty)$. Observe now that by [2, Example 3.97] we have

$$D_r\mu = D_r(r\xi_\mu) = rD_r\xi_\mu + \xi_\mu\, dr\, dy\, dt.$$

Comparing last identity with (3.23) we conclude. ☐

Before stating the next result, we recall that $\nabla\mu = (\partial_r\mu, \nabla_y\mu, \partial_t\mu)$ and $\nabla\xi = (\partial_r\xi, \nabla_y\xi, \partial_t\xi)$ denote the absolutely continuous parts of the Radon measures $D\mu$ and $D\xi$.

Proposition 3.26. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u \in BV_{0,\tau}(\Omega)$, and let μ, α_μ and ξ_μ be given by* (1.13), (1.20), *and* (3.19), *respectively. Then, $\nabla\mu = (\partial_r\mu, \nabla_y\mu, \partial_t\mu)$ and $\nabla\xi = (\partial_r\xi, \nabla_y\xi, \partial_t\xi)$ are concentrated on $\{0 < \alpha_\mu < \pi\}$. Moreover,*

$$\partial_t\mu(r,y,t) = -\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{1}{|\nabla_{x\|}u_0(x,y)|}\, d\mathcal{H}^0(x) \tag{3.24}$$

$$\nabla_y\mu(r,y,t) = \int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{\nabla_y u_0(x,y)}{|\nabla_{x\|}u_0(x,y)|}\, d\mathcal{H}^0(x), \tag{3.25}$$

and

$$r\partial_r\xi_\mu(r,y,t) = \int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{\hat{x}\cdot\nabla_x u_0(x,y)}{|\nabla_{x\|}u_0(x,y)|}\, d\mathcal{H}^0(x), \tag{3.26}$$

$$r\nabla_y\xi_\mu(r,y,t) = \int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{\nabla_y u_0(x,y)}{|\nabla_{x\|}u_0(x,y)|}\, d\mathcal{H}^0(x),$$

$$r\partial_t\xi_\mu(r,y,t) = -\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{1}{|\nabla_{x\|}u_0(x,y)|}\, d\mathcal{H}^0(x),$$

for $\mathcal{H}^n$-a.e. $(r, y, t) \in \{0 < \alpha_\mu < \pi\}$.

Proof. We will only show the statement for $\partial_t \mu$, since the other identities can be proven in a similar way. Let $A \subset\subset \Pi_{n-1}(\Omega)$ be open and let $d > 0$. Let now G_{u_0} be given by Proposition 3.12 for the set $A \times (-d, +\infty)$, and let $q \in C_c^0(A \times (-d, +\infty))$. Then, thanks to Remark 3.25

$$\begin{aligned}
&\int_{(A\times(-d,+\infty))} q(r,y,t)\, dD_t\mu(r,y,t) \\
&= \int_{(A\times(-d,+\infty))\cap\{\alpha^\vee>0\}\cap\{\alpha^\wedge<\pi\}} q(r,y,t)\, dD_t\mu(r,y,t) \\
&= \int_{(A\times(-d,+\infty))\cap\{\alpha^\vee>0\}\cap\{\alpha^\wedge<\pi\}\cap G_{u_0}} q(r,y,t)\, dD_t\mu(r,y,t) \\
&+ \int_{\Big((A\times(-d,+\infty))\cap\{\alpha^\vee>0\}\cap\{\alpha^\wedge<\pi\}\Big)\setminus G_{u_0}} q(r,y,t)\, dD_t\mu(r,y,t).
\end{aligned}$$

Note now that the last integral does not contain any contribution coming from the absolutely continuous part of $D_t\mu$, since the region of integration is $\mathcal{H}^n$-negligible:

$$\mathcal{H}^n\left(\Big((A\times(-d,+\infty))\cap\{\alpha^\vee>0\}\cap\{\alpha^\wedge<\pi\}\Big)\setminus G_{u_0}\right)=0.$$

For the remaining integral, setting $R := (A \times (-d, +\infty)) \cap \{\alpha^\vee > 0\} \cap \{\alpha^\wedge < \pi\} \cap G_{u_0}$, thanks to Proposition 3.1, (3.21), and Remark 3.15, we have

$$\begin{aligned}
&\int_R q(r,y,t)\, dD_t\mu(r,y,t) = \int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}(R\times\mathbb{S}^1))} q(|x|,y,t)\nu_t^{\Sigma^{u_0}}(x,y,t)\, d\mathcal{H}^n(x,y,t) \\
&= -\int_{\partial^*\Sigma^{u_0}\cap\{\nabla_{x\parallel}u_0\neq 0\}\cap(\Phi_{n+1}(R\times\mathbb{S}^1))} \frac{q(|x|,y,t)}{\sqrt{1+|\nabla u_0(x,y)|^2}}\, d\mathcal{H}^n(x,y,t) \\
&= -\int_{(A\times(-d,+\infty))\cap\{\alpha^\vee>0\}\cap\{\alpha^\wedge<\pi\}\cap G_{u_0}} q(r,y,t)\left(\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{1}{|\nabla_{x\parallel}u_0(x,y)|}\, d\mathcal{H}^0(x)\right) dr\, dy\, dt \\
&= -\int_{(A\times(-d,+\infty))\cap\{0<\alpha_\mu<\pi\}\cap G_{u_0}} q(r,y,t)\left(\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{1}{|\nabla_{x\parallel}u_0(x,y)|}\, d\mathcal{H}^0(x)\right) dr\, dy\, dt,
\end{aligned}$$

where we also used Proposition 3.6 and identity (iiia) of Proposition 3.12. Since q is arbitrary, this shows that the density $\partial_t\mu$ of the absolutely continuous part of the measure $D_t\mu$ satisfies

$$\partial_t\mu(r,y,t) = -\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{1}{|\nabla_{x\parallel}u_0(x,y)|}\, d\mathcal{H}^0(x), \tag{3.27}$$

for $\mathcal{H}^n$-a.e. $(r, y, t) \in A \times (-d, +\infty) \cap \{0 < \alpha_\mu < \pi\} \cap G_{u_0}$. Recalling now that

$$\mathcal{H}^n\Big((A\times(-d,+\infty))\cap\{0<\alpha_\mu<\pi\}\setminus G_{u_0}\Big)=0,$$

we obtain that (3.27) is satisfied for $\mathcal{H}^n$-a.e. $(r, y, t) \in (A \times (-d, +\infty)) \cap \{0 < \alpha_\mu < \pi\}$. Since A and d are arbitrary, the conclusion follows. □

The next result shows a perimeter inequality under circular symmetrization of subgraphs.

Proposition 3.27. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u \in BV_{0,\tau}(\Omega)$, let μ be given by (1.13), and let v_μ be defined by (1.14). Then, $v_\mu|_{\Omega^s} \in BV_{0,\tau}(\Omega^s)$. Moreover, Σ^{v_μ} is a set of locally finite perimeter in $\Phi_{n+1}((\Pi_{n-1}(\Omega) \times \mathbb{R}) \times \mathbb{S}^1)$ and*

$$P(\Sigma^{v_\mu};\Phi_{n+1}(B\times\mathbb{S}^1)) \leq P(\Sigma^{u_0};\Phi_{n+1}(B\times\mathbb{S}^1)), \tag{3.28}$$

for every Borel set $B \subset \Pi_{n-1}(\Omega) \times \mathbb{R}$.

Remark 3.28. *Thanks to Remark 3.5, we have that Σ^{v_μ} and Σ^{u_0} are sets of finite perimeter in $\Phi_{n+1}((A \times \mathbb{R}) \times \mathbb{S}^1)$, for every open set $A \subset\subset \Pi_{n-1}(\Omega)$. Therefore, (3.28) should be interpreted as an inequality between extended real numbers.*

Proof. Let $A \subset\subset \Pi_{n-1}(\Omega)$ be open. We will show that $v_\mu \in BV(\Phi_n(A \times \mathbb{S}^1))$. By assumption, we already know that $u_0 \in BV(\Phi_n(A \times \mathbb{S}^1))$. Thanks to Proposition 3.1, Σ^{u_0} is a set of finite perimeter in $\Phi_n(A \times \mathbb{S}^1) \times \mathbb{R}$. Let now $(\Sigma^{u_0})^s$ and F_μ be defined by (2.13) and (3.8), respectively. We have

$$(\Sigma^{u_0})^s = F_\mu =_{\mathcal{H}^{n+1}} \Sigma^{v_\mu},$$

where the last equality follows from (3.9). We would like to apply Theorem 1.1 to the sets Σ^{v_μ} and Σ^{u_0}, but this is not possible, since they do not have finite (n-dimensional) Lebesgue measure. Therefore, for each $d > 0$ if we set

$$\Sigma^{u_0}_d = \Sigma^{u_0} \cap \{(x, y, t) \in \mathbb{R}^{n+1} : t > -d\} \quad \text{and} \quad \Sigma^{v_\mu}_d = \Sigma^{v_\mu} \cap \{(x, y, t) \in \mathbb{R}^{n+1} : t > -d\},$$

so that

$$(\Sigma^{u_0}_d)^s =_{\mathcal{H}^{n+1}} \Sigma^{v_\mu}_d.$$

From Proposition 3.9 it follows that $\mu \in L^1(A \times (-d, +\infty))$, so that $\mathcal{L}^{n+1}(\Sigma^{u_0}_d) = \mathcal{L}^{n+1}(\Sigma^{v_\mu}_d) < +\infty$. Thanks to [26, Proposition 2.16], there exists a sequence $d_k \to +\infty$ such that for each $k \in \mathbb{N}$

$$\mathcal{H}^n(\partial^*\Sigma^{u_0}_{d_k} \cap \{(x, y, t) \in \mathbb{R}^{n+1} : t = -d_k\}) = \mathcal{H}^n(\partial^*\Sigma^{v_\mu}_{d_k} \cap \{(x, y, t) \in \mathbb{R}^{n+1} : t = -d_k\}) = 0. \quad (3.29)$$

If we set $\tilde{A} := A \times \mathbb{R}$, we have $\Phi_n(A \times \mathbb{S}^1) \times \mathbb{R} = \Phi_{n+1}(\tilde{A} \times \mathbb{S}^1)$. Then, thanks to Theorem 1.1, $\Sigma^{v_\mu}_{d_k}$ is a set of finite perimeter in $\Phi_{n+1}(\tilde{A} \times \mathbb{S}^1)$ and

$$P(\Sigma^{v_\mu}_{d_k}; \Phi_{n+1}(\tilde{A} \times \mathbb{S}^1)) \leq P(\Sigma^{u_0}_{d_k}; \Phi_{n+1}(\tilde{A} \times \mathbb{S}^1)). \quad (3.30)$$

Thanks to [26, formula (16.10)] and recalling (3.29), we have

$$\begin{aligned} P(\Sigma^{u_0}_{d_k}; \Phi_{n+1}(\tilde{A} \times \mathbb{S}^1)) &= P(\Sigma^{u_0}; \Phi_{n+1}(\tilde{A} \times \mathbb{S}^1) \cap \{t > -d_k\}) + P(\{t > -d_k\}; (\Sigma^{u_0})^{(1)}) \\ &= P(\Sigma^{u_0}; \Phi_{n+1}(\tilde{A} \times \mathbb{S}^1) \cap \{t > -d_k\}) + \mathcal{L}^n(\{u_0 > -d_k\} \cap \Phi_n(A \times \mathbb{S}^1)) \\ &= P(\Sigma^{u_0}; \Phi_{n+1}(\tilde{A} \times \mathbb{S}^1) \cap \{t > -d_k\}) + \mathcal{L}^n(\{v_\mu > -d_k\} \cap \Phi_n(A \times \mathbb{S}^1)). \end{aligned}$$

Similarly, we have

$$P(\Sigma^{v_\mu}_{d_k}; \Phi_{n+1}(\tilde{A} \times \mathbb{S}^1)) = P(\Sigma^{v_\mu}; \Phi_{n+1}(\tilde{A} \times \mathbb{S}^1) \cap \{t > -d_k\}) + \mathcal{L}^n(\{v_\mu > -d_k\} \cap \Phi_n(A \times \mathbb{S}^1)).$$

Therefore, thanks to (3.30),

$$P(\Sigma^{v_\mu}; \Phi_{n+1}(\tilde{A} \times \mathbb{S}^1) \cap \{t > -d_k\}) \leq P(\Sigma^{u_0}; \Phi_{n+1}(\tilde{A} \times \mathbb{S}^1) \cap \{t > -d_k\}), \quad \forall\, k \in \mathbb{N}.$$

Passing to the limit as $k \to \infty$, we have

$$P(\Sigma^{v_\mu}; \Phi_{n+1}(\tilde{A} \times \mathbb{S}^1)) \leq P(\Sigma^{u_0}; \Phi_{n+1}(\tilde{A} \times \mathbb{S}^1)). \quad (3.31)$$

Therefore, Σ^{v_μ} is a set of finite perimeter in $\Phi_{n+1}(\tilde{A} \times \mathbb{S}^1)$. Thanks to Proposition 3.19, we have $v_\mu \in L^1(\Phi_n(A \times \mathbb{S}^1))$. Then, from Proposition 3.1 it follows that $v_\mu \in BV(\Phi_n(A \times \mathbb{S}^1))$. From (3.31), since $A \subset\subset \Pi_{n-1}(\Omega)$ was arbitrary, inequality (3.28) follows. Finally, thanks to (3.12) and using again the arbitrariness of A, we conclude that $v_\mu|_{\Omega^s} \in BV_{0,\tau}(\Omega^s)$. $\square$

Remark 3.29. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u \in BV_{0,\tau}(\Omega)$, and let μ be given by* (1.13)*. Then, thanks to* [28, Proposition 1.2] *applied to Σ^{v_μ}, we have that for every $(r, y, t) \in \Pi_{n-1}(\Omega) \times \mathbb{R}$ such that $(\partial^*\Sigma^{v_\mu})_{(r,y,t)} \neq \emptyset$, the functions*

$$x \mapsto \hat{x} \cdot \nu^{\Sigma^{v_\mu}}_x(x, y, t), \quad x \mapsto |\nu^{\Sigma^{v_\mu}}_{x\parallel}(x, y, t)|, \quad x \mapsto \nu^{\Sigma^{v_\mu}}_y(x, y, t), \quad x \mapsto \nu^{\Sigma^{v_\mu}}_t(x, y, t),$$

are constant in $(\partial^*\Sigma^{v_\mu})_{(r,y,t)}$. Thanks to (3.1)*, this means that*

$$x \mapsto \hat{x} \cdot \nabla_x v_\mu(x, y, t), \quad x \mapsto |\nabla_{x\parallel} v_\mu(x, y, t)|, \quad x \mapsto \nabla_y v_\mu(x, y, t), \quad x \mapsto |\nabla v_\mu|(x, y, t),$$

are constant in $(\partial^*\Sigma^{v_\mu})_{(r,y,t)}$.

Remark 3.30. *Suppose that $u \in BV_{0,\tau}(\Omega)$ and let μ be given by* (1.13)*. Thanks to Proposition 3.27, we also have $v_\mu\,|_{\Omega^s} \in BV_{0,\tau}(\Omega^s)$. Then, applying Proposition 3.26 to v_μ and taking into account Remark 3.29, we have*

$$\partial_t \mu(r,y,t) = -\frac{2}{|\nabla_{x\|} v_\mu(x,y)|}, \tag{3.32}$$

$$\nabla_y \mu(r,y,t) = 2\frac{\nabla_y v_\mu(x,y)}{|\nabla_{x\|} v_\mu(x,y)|}, \tag{3.33}$$

$$r\partial_r \xi(r,y,t) = 2\frac{\widehat{x}\cdot\nabla_x v_\mu(x,y)}{|\nabla_{x\|} v_\mu(x,y)|}, \tag{3.34}$$

for $\mathcal{H}^n$-a.e. $(r,y,t) \in \Pi_{n-1}(\Omega)\times\mathbb{R}$.

We are now ready to show that the circular rearrangement of a function in $W^{1,1}_{0,\tau}(\Omega)$ also belongs to $W^{1,1}_{0,\tau}(\Omega)$.

Proposition 3.31. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u \in W^{1,1}_{0,\tau}(\Omega)$, let μ be given by* (1.13)*, and let v_μ be defined by* (1.14)*. Then, $v_\mu\,|_{\Omega^s} \in W^{1,1}_{0,\tau}(\Omega^s)$.*

Proof. Let $A \subset\subset \Pi_{n-1}(\Omega)$ be open. Thanks to Proposition 3.27, $v_\mu \in BV(\Phi_n(A\times\mathbb{S}^1))$, and Σ^{u_0} and Σ^{v_μ} are sets of finite perimeter in $\Phi_n(A\times\mathbb{S}^1)\times\mathbb{R}$. Let us first show that

$$\mathcal{H}^n(B) = 0, \qquad \text{where} \qquad B := \{(x,y,t) \in \partial^*\Sigma^{v_\mu} : \nu_t^{\Sigma^{v_\mu}}(x,y,t) = 0\} \cap (\Phi_n(A\times\mathbb{S}^1)\times\mathbb{R}). \tag{3.35}$$

First of all note that, by definition of Φ_n and Φ_{n+1}, we have

$$B := \{(x,y,t) \in \partial^*\Sigma^{v_\mu} : \nu_t^{\Sigma^{v_\mu}}(x,y,t) = 0\} \cap (\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)).$$

Let now $S : \mathbb{R}^2_0 \times \mathbb{R}^{n-2}\times\mathbb{R} \to (0,+\infty)\times\mathbb{R}^{n-2}\times\mathbb{R}$ be defined as $S(x,y,t) := (|x|,y,t)$. Thanks to Remark 3.29, we have

$$B = \partial^*\Sigma^{v_\mu} \cap (\Phi_{n+1}(S(B)\times\mathbb{S}^1)).$$

Since S is continuous, B is Borel, and $D_t\mu$ is a finite Radon measure on $(0,+\infty)\times\mathbb{R}^{n-2}\times\mathbb{R}$, the set $S(B)$ is $D_t\mu$-measurable. Moreover, since $D_t\mu$ is a finite Radon measure, formula (3.21) holds also for bounded $D_t\mu$-measurable functions. In particular, we have

$$\begin{aligned}&\int_{\partial^*\Sigma^{v_\mu}\cap(\Phi_{n+1}(S(B)\times\mathbb{S}^1))} \nu_t^{\Sigma^{v_\mu}}(x,y,t)\,d\mathcal{H}^n(x,y,t)\\ &= D_t\mu(S(B)) = \int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}(S(B)\times\mathbb{S}^1))} \nu_t^{\Sigma^{u_0}}(x,y,t)\,d\mathcal{H}^n(x,y,t).\end{aligned} \tag{3.36}$$

Now, since $\nu_t^{\Sigma^{v_\mu}}(x,y,t) = 0$ for every $(x,y,t)\in B$, thanks to (3.36)

$$\begin{aligned}0 &= \int_B \nu_t^{\Sigma^{v_\mu}}(x,y,t)\,d\mathcal{H}^n(x,y,t) = \int_{\partial^*\Sigma^{v_\mu}\cap(\Phi_{n+1}(S(B)\times\mathbb{S}^1))} \nu_t^{\Sigma^{v_\mu}}(x,y,t)\,d\mathcal{H}^n(x,y,t)\\ &= \int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}(S(B)\times\mathbb{S}^1))} \nu_t^{\Sigma^{u_0}}(x,y,t)\,d\mathcal{H}^n(x,y,t)\\ &= -\int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}(S(B)\times\mathbb{S}^1))} \frac{1}{\sqrt{1+|\nabla u_0(x,y)|^2}}\,d\mathcal{H}^n(x,y,t),\end{aligned}$$

where the last equality follows from Proposition 3.1 and from the fact that $u_0 \in W^{1,1}(\Phi_n(A\times\mathbb{S}^1))$. From the above chain of equalities we infer that

$$\mathcal{H}^n\left(\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}(S(B)\times\mathbb{S}^1))\right) = 0.$$

Then, thanks to (3.28),

$$\begin{aligned}0 &= \mathcal{H}^n\left(\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}(S(B)\times\mathbb{S}^1))\right) = P(\Sigma^{u_0};\Phi_{n+1}(S(B)\times\mathbb{S}^1))\\ &\geq P(\Sigma^{v_\mu};\Phi_{n+1}(S(B)\times\mathbb{S}^1)) = \mathcal{H}^n\left(\partial^*\Sigma^{v_\mu}\cap(\Phi_{n+1}(S(B)\times\mathbb{S}^1))\right) = \mathcal{H}^n(B),\end{aligned}$$

which shows (3.35).

Once (3.35) is satisfied, thanks to Proposition 3.2 it follows that $v_\mu \in W^{1,1}(\Phi_n(A \times \mathbb{S}^1))$. Using the arbitrariness of A and thanks to (3.12), we have $v_\mu|_{\Omega^s} \in W^{1,1}_{0,\tau}(\Omega^s)$. $\square$

The next proposition will be useful to understand rigidity of the Pólya–Szegö inequality.

Proposition 3.32. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u \in W^{1,1}_{0,\tau}(\Omega)$, and let μ be given by* (1.13). *Let v_μ be defined by* (1.14) *and let $B \subset\subset \Pi_{n-1}(\Omega) \times \mathbb{R}$ be a Borel set. Then,*

$$\mathcal{H}^n\left(\left\{(x,y,t) \in \partial^*\Sigma^{u_0} \cap \Phi_{n+1}(B \times \mathbb{S}^1) : \nu^{\Sigma^{u_0}}_{x\parallel}(x,y,t) = 0\right\}\right) = 0 \tag{3.37}$$

if and only if

$$\mathcal{H}^n\left(\left\{(x,y,t) \in \partial^*\Sigma^{v_\mu} \cap \Phi_{n+1}(B \times \mathbb{S}^1) : \nu^{\Sigma^{v_\mu}}_{x\parallel}(x,y,t) = 0\right\}\right) = 0. \tag{3.38}$$

Proof. Applying formula (3.21) with $\varphi = \chi_{B'}$ first to Σ^{u_0} and then to Σ^{v_μ}, taking into account Proposition 3.1 and Proposition 3.31, we have

$$\begin{aligned}
&-\int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}(B'\times\mathbb{S}^1))} \frac{1}{\sqrt{1+|\nabla u_0(x,y)|^2}}\, d\mathcal{H}^n(x,y,t)\\
&= D_t\mu(B')\\
&= -\int_{\partial^*\Sigma^{v_\mu}\cap(\Phi_{n+1}(B'\times\mathbb{S}^1))} \frac{1}{\sqrt{1+|\nabla v_\mu(x,y)|^2}}\, d\mathcal{H}^n(x,y,t),
\end{aligned} \tag{3.39}$$

for every Borel set $B' \subset B$. Then, thanks to Proposition 3.17, we have

$$\begin{aligned}
(3.37) \quad &\Longleftrightarrow \quad P(\Sigma^{u_0}; \Phi_{n+1}(B' \times \mathbb{S}^1)) = 0 \quad \forall \text{ Borel set } B' \subset B \text{ with } \mathcal{H}^n(B') = 0\\
&\Longleftrightarrow \quad \mathcal{H}^n(\partial^*\Sigma^{u_0} \cap \Phi_{n+1}(B' \times \mathbb{S}^1)) = 0 \quad \forall \text{ Borel set } B' \subset B \text{ with } \mathcal{H}^n(B') = 0\\
&\overset{(3.39)}{\Longleftrightarrow} \quad D_t\mu(B') = 0 \quad \forall \text{ Borel set } B' \subset B \text{ with } \mathcal{H}^n(B') = 0\\
&\overset{(3.39)}{\Longleftrightarrow} \quad \mathcal{H}^n(\partial^*\Sigma^{v_\mu} \cap \Phi_{n+1}(B' \times \mathbb{S}^1)) = 0 \quad \forall \text{ Borel set } B' \subset B \text{ with } \mathcal{H}^n(B') = 0\\
&\Longleftrightarrow \quad P(\Sigma^{v_\mu}; \Phi_{n+1}(B' \times \mathbb{S}^1)) = 0 \quad \forall \text{ Borel set } B' \subset B \text{ with } \mathcal{H}^n(B') = 0\\
&\Longleftrightarrow \quad (3.38).
\end{aligned}$$

$\square$

4. Proof of the Pólya–Szegö inequality

In this section we prove a general version of the Pólya–Szegö inequality (see Theorem 4.3), and then we show how this implies Theorem 1.12 and Theorem 1.13. We start with some preliminary results that concerning functions f belonging to $\mathscr{F}$ and $\mathscr{F}'$.

Lemma 4.1. *Let $\mathscr{F}$ and $\mathscr{F}'$ be given by Definition 1.10, and let $f \in \mathscr{F}$. Then,*

$$\tau \longmapsto f(\eta,\tau,\zeta) \text{ is increasing in } [0,\infty) \qquad \text{for every } (\eta,\zeta) \in \mathbb{R} \times \mathbb{R}^{n-2}.$$

If, in addition, $f \in \mathscr{F}'$, then

$$\tau \longmapsto f(\eta,\tau,\zeta) \text{ is } \textbf{strictly} \text{ increasing in } [0,\infty) \qquad \text{for every } (\eta,\zeta) \in \mathbb{R} \times \mathbb{R}^{n-2}.$$

Proof. Suppose that $f \in \mathscr{F}$, let $(\eta,\zeta) \in \mathbb{R} \times \mathbb{R}^{n-2}$ be fixed, and let $g : \mathbb{R} \to [0,+\infty)$ be defined as $g(\tau) := f(\eta,\tau,\zeta)$. We want to show that g is increasing. First of all, note that since f is convex we have that g is convex. Let now $0 \leq \tau_1 < \tau_2$. Then, we can write $\tau_1 = \lambda(-\tau_2) + (1-\lambda)\tau_2$, where $\lambda := \frac{\tau_2-\tau_1}{2\tau_2} \in (0,1/2)$. Therefore, by convexity of g

$$g(\tau_1) = g(\lambda(-\tau_2) + (1-\lambda)\tau_2) \leq \lambda g(-\tau_2) + (1-\lambda)g(\tau_2) = g(\tau_2),$$

which shows the first part of the statement. If, in addition, $f \in \mathscr{F}'$, then the function g is strictly convex and so the inequality above is strict. $\square$

If $f \in \mathscr{F}$, we recall that the Legendre transform f^* of f is defined as:

$$f^*(w) := \sup_{(\eta,\zeta,\tau)\in\mathbb{R}^n} ((\eta,\zeta,\tau)\cdot w - f(\eta,\zeta,\tau)), \qquad \text{for every } w \in \mathbb{R}^n.$$

Since f is convex and finite everywhere, f^* is convex with $-\infty < f^*(w) \le +\infty$ for every $w \in \mathbb{R}^n$. Moreover, for every $(\eta,\zeta,\tau) \in \mathbb{R}^n$ we have

$$\begin{aligned} f(\eta,\zeta,\tau) = (f^*)^*(\eta,\zeta,\tau) &= \sup_{w\in\mathbb{R}^n} (w\cdot(\eta,\zeta,\tau) - f^*(w)) \\ &= \sup_{h\in\mathbb{N}} (w_h\cdot(\eta,\zeta,\tau) - f^*(w_h)), \end{aligned} \tag{4.1}$$

where $\{w_h\}_{h\in\mathbb{N}}$ is a countable dense subset of the set $\{w \in \mathbb{R}^n : f^*(w) < +\infty\}$.

The next result is a variant of [2, Lemma 2.35], and will be used in the proof of Theorem 4.3.

Lemma 4.2. *Let $n \in \mathbb{N}$ with $n \ge 2$, and let $\Omega \subset \mathbb{R}^n$ be open. Let $u \in BV_{0,\tau}(\Omega)$, let $A \subset\subset \Pi_{n-1}(\Omega)$ be open, and let $B \subset A\times\mathbb{R}$ be a Borel set. For every $h \in \mathbb{N}$, let $\varphi_h : \Phi_{n+1}(B\times\mathbb{S}^1) \to \mathbb{R}$ be a Borel function such that*

$$x \longmapsto \varphi_h(x,y,t) \text{ is constant in } (\partial^*\Sigma^{u_0})_{(r,y,t)} \quad \text{for every } (r,y,t) \in B. \tag{4.2}$$

Suppose, in addition, that there exists $\tilde{h} \in \mathbb{N}$ such that

$$\int_{\partial^*\Sigma^{u_0}\cap\Phi_{n+1}(B\times\mathbb{S}^1)} |\varphi_{\tilde{h}}(x,y,t)|\, d\mathcal{H}^n(x,y,t) < +\infty.$$

Then,

$$\begin{aligned} &\int_{\partial^*\Sigma^{u_0}\cap\Phi_{n+1}(B\times\mathbb{S}^1)} \sup_{h\in\mathbb{N}} \varphi_h(x,y,t)\, d\mathcal{H}^n(x,y,t) \\ &= \sup_H \left\{ \sum_{h\in H} \int_{\partial^*\Sigma^{u_0}\cap\Phi_{n+1}(B_h\times\mathbb{S}^1)} \varphi_h(x,y,t)\, d\mathcal{H}^n(x,y,t) \right\}, \end{aligned}$$

where the supremum in the right hand side ranges over all finite sets $H \subset \mathbb{N}$ and over all pairwise disjoint Borel partitions $\{B_h\}_{h\in H}$ of B.

Proof. In order to prove the lemma, we first need to introduce some tools. We define the map $S : \mathbb{R}^2_0 \times \mathbb{R}^{n-2}\times\mathbb{R} \to (0,+\infty)\times\mathbb{R}^{n-2}\times\mathbb{R}$ as

$$S(x,y,t) := (|x|,y,t), \qquad \text{for every } (x,y,t) \in \mathbb{R}^2_0\times\mathbb{R}^{n-2}\times\mathbb{R}. \tag{4.3}$$

Let λ be the finite Radon measure on $\mathbb{R}^2_0\times\mathbb{R}^{n-2}\times\mathbb{R}$ given by

$$\lambda(R) := \mathcal{H}^n\left(R\cap\partial^*\Sigma^{u_0}\cap\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)\right), \quad \forall \text{ Borel set } R \subset \mathbb{R}^2_0\times\mathbb{R}^{n-2}\times\mathbb{R},$$

and let $\sigma := S_\#\lambda$, where $S_\#\lambda$ denotes the push–forward measure of λ through S. We observe that both λ and σ are finite Borel measures. Finally, we set for every $h \in \{1,\dots,k\}$,

$$A_h := \left\{(x,y,t) \in \partial^*\Sigma^{u_0}\cap\Phi_{n+1}(B\times\mathbb{S}^1) : \varphi_h(x,y,t) = \max_{1\le i\le k}\varphi_i(x,y,t)\right\}. \tag{4.4}$$

We now divide the proof into steps.

Step 1: We show that the statement follows if we prove that for every $k \in \mathbb{N}$ with $k \ge \tilde{h}$, we have

$$\begin{aligned} &\int_{\partial^*\Sigma^{u_0}\cap\Phi_{n+1}(B\times\mathbb{S}^1)} \max_{1\le h\le k} \varphi_h(x,y,t)\, d\mathcal{H}^n(x,y,t) \\ &= \sup\left\{ \sum_{h=1}^k \int_{\partial^*\Sigma^{u_0}\cap\Phi_{n+1}(B_h\times\mathbb{S}^1)} \varphi_h(x,y,t)\, d\mathcal{H}^n(x,y,t) \right\}, \end{aligned} \tag{4.5}$$

where the last supremum ranges over all the k-ples $B_1,\dots,B_k$ of pairwise disjoint Borel partitions of B.

To this aim, for every $k \ge \tilde{h}$, let us define the function $M_k : \Phi_{n+1}(B\times\mathbb{S}^1) \to \mathbb{R}$ as

$$M_k(x,y,t) := \max_{1\le h\le k} \varphi_h(x,y,t).$$

Then, we have $M_k \geq \varphi_{\tilde{h}}$ and $(M_k)_- \leq (\varphi_{\tilde{h}})_-$. Moreover, by the assumption on $\varphi_{\tilde{h}}$ it follows that

$$\int_{\partial^* \Sigma^{u_0} \cap \Phi_{n+1}(B \times \mathbb{S}^1)} (M_k)_-(x,y,t)\, d\mathcal{H}^n(x,y,t) \leq \int_{\partial^* \Sigma^{u_0} \cap \Phi_{n+1}(B \times \mathbb{S}^1)} (\varphi_{\tilde{h}})_-(x,y,t)\, d\mathcal{H}^n(x,y,t) < \infty.$$

Thus, the integral

$$\int_{\partial^* \Sigma^{u_0} \cap \Phi_{n+1}(B \times \mathbb{S}^1)} M_k(x,y,t)\, d\mathcal{H}^n(x,y,t),$$

is well defined, and its value belongs to $(-\infty, +\infty]$.

We also have that $M_k - \varphi_{\tilde{h}} \geq 0$, and $M_k - \varphi_{\tilde{h}} \uparrow \sup_{h \in \mathbb{N}} \varphi_h - \varphi_{\tilde{h}}$ as $k \to \infty$. Therefore, by the Monotone Convergence Theorem applied to $M_k - \varphi_{\tilde{h}}$, we have

$$\begin{aligned}
&\lim_{k \to +\infty} \int_{\partial^* \Sigma^{u_0} \cap \Phi_{n+1}(B \times \mathbb{S}^1)} \max_{1 \leq h \leq k} \varphi_h(x,y,t)\, d\mathcal{H}^n(x,y,t) \\
&= \int_{\partial^* \Sigma^{u_0} \cap \Phi_{n+1}(B \times \mathbb{S}^1)} \sup_{h \in \mathbb{N}} \varphi_h(x,y,t)\, d\mathcal{H}^n(x,y,t),
\end{aligned}$$

so the claim follows.

Step 2: For every $k \in \mathbb{N}$ with $k \geq \tilde{h}$, we construct a pairwise disjoint Borel partition $\{C_1, \ldots, C_k\}$ of B such that for every $h = 1, \ldots, k$

$$\varphi_h(x,y,t) = \max_{1 \leq i \leq k} \varphi_i(x,y,t), \qquad \text{for } \mathcal{H}^n\text{-a.e. } (x,y,t) \in \partial^* \Sigma^{u_0} \cap \Phi_{n+1}(C_h \times \mathbb{S}^1).$$

Let $k \in \mathbb{N}$ with $k \geq \tilde{h}$ be fixed. We divide this step into sub-steps.

Step 2a: We show that for every $h = 1, \ldots, k$ the set $S(A_h)$ is σ-measurable, where S and A_h are defined in (4.3) and (4.4), respectively.

Let $h \in \{1, \ldots, k\}$ be fixed. Since A_h is Borel and S is continuous, $S(A_h)$ is an analytic subset of $(0,\infty) \times \mathbb{R}^{n-2} \times \mathbb{R}$ (see [17, Section 8.2]). Note that, in general, there is no guarantee that $S(A_h)$ is Borel (see [17, Corollary 8.2.17]). However, since σ is a finite Borel measure on $(0,\infty) \times \mathbb{R}^{n-2} \times \mathbb{R}$, we have that $S(A_h)$ is σ–measurable, see [17, Theorem 8.4.1].

Step 2b: We show that for every $h = 1, \ldots, k$

$$A_h = \partial^* \Sigma^{u_0} \cap \Phi_{n+1}(S(A_h) \times \mathbb{S}^1). \tag{4.6}$$

The inclusion $A_h \subset \partial^* \Sigma^{u_0} \cap \Phi_{n+1}(S(A_h) \times \mathbb{S}^1)$ follows by the definition of S, so we only need to prove the opposite inclusion. Let $(x', y, t) \in \partial^* \Sigma^{u_0} \cap \Phi_{n+1}(S(A_h) \times \mathbb{S}^1)$. In particular, this implies that $(A_h)_{(|x'|,y,t)} \neq \emptyset$, and so there exists $x \in (A_h)_{(|x'|,y,t)}$ such that $(x,y,t) \in A_h$. Then, by definition of A_h,

$$\varphi_h(x,y,t) = \max_{1 \leq i \leq k} \varphi_i(x,y,t).$$

On the other hand, since $x, x' \in (\partial^* \Sigma^{u_0})_{(|x'|,y,t)}$, by (4.2) we have $\varphi_i(x',y,t) = \varphi_i(x,y,t)$ for every $i = 1, \ldots, k$. Therefore,

$$\varphi_h(x',y,t) = \varphi_h(x,y,t) = \max_{1 \leq i \leq k} \varphi_i(x,y,t) = \max_{1 \leq i \leq k} \varphi_i(x',y,t),$$

so that $(x',y,t) \in A_h$, and this shows (4.6).

Step 2c: We conclude the proof of Step 2. For each $h = 1, \ldots, k$, we can find a Borel set R_h such that $\sigma(S(A_h) \Delta R_h) = 0$. Note now that, by definition of σ,

$$\begin{aligned}
0 = \sigma(R_h \setminus S(A_h)) &= \lambda(S^{-1}(R_h \setminus S(A_h))) = \lambda(\Phi_{n+1}((R_h \setminus S(A_h)) \times \mathbb{S}^1) \\
&= \mathcal{H}^n \left(\partial^* \Sigma^{u_0} \cap \Phi_{n+1}((R_h \setminus S(A_h)) \times \mathbb{S}^1) \right).
\end{aligned}$$

Combining the above identity with (4.4) and (4.6), we obtain that for every $h = 1, \ldots, k$

$$\varphi_h(x,y,t) = \max_{1 \leq i \leq k} \varphi_i(x,y,t), \qquad \text{for } \mathcal{H}^n\text{-a.e. } (x,y,t) \in \partial^* \Sigma^{u_0} \cap \Phi_{n+1}(R_h \times \mathbb{S}^1). \tag{4.7}$$

Therefore, the Borel sets $R_1, \ldots, R_h$ are such that (4.7) holds, and

$$\bigcup_{i=1}^{h} R_i = B' := \left\{ (r,y,t) \in B : (\partial^* \Sigma^{u_0})_{(r,y,t)} \neq \emptyset \right\}.$$

We now set $C_1 := R_1$,

$$C_h := R_h \setminus \Big(\bigcup_{1\le i<h} R_i \Big), \quad h = 2, \dots, k-1, \qquad\qquad R_k := B \setminus \Big(\bigcup_{1\le i<k} R_i \Big),$$

and the claim follows.

Step 3: We conclude. Thanks to Step 1 and Step 2,

$$\begin{aligned}
&\int_{\partial^*\Sigma^{u_0}\cap\Phi_{n+1}(B\times\mathbb{S}^1)} \max_{1\le h\le k} \varphi_h(x,y,t)\, d\mathcal{H}^n(x,y,t) \\
&= \sum_{h=1}^k \int_{\partial^*\Sigma^{u_0}\cap\Phi_{n+1}(C_h\times\mathbb{S}^1)} \varphi_h(x,y,t)\, d\mathcal{H}^n(x,y,t) \\
&= \sup \left\{ \sum_{h=1}^k \int_{\partial^*\Sigma^{u_0}\cap\Phi_{n+1}(B_h\times\mathbb{S}^1)} \varphi_h(x,y,t)\, d\mathcal{H}^n(x,y,t) \right\},
\end{aligned}$$

where the last supremum ranges over all the k-ples $B_1, \dots, B_k$ of pairwise disjoint Borel partitions of B. This shows (4.5) and, in turn, the lemma.

□

We can now state a general Pólya-Szegö inequality under circular rearrangement, which is the main result of this section.

Theorem 4.3. *Let $n \in \mathbb{N}$ with $n \ge 2$, let $\Omega \subset \mathbb{R}^n$ be open, and let $\mu \in \mathcal{A}(\Pi_{n-1}(\Omega)\times\mathbb{R})$. Let $a \in L^\infty((0,\infty)\times\mathbb{R}^{n-2}\times\mathbb{R})$ with $a \ge 0$ $\mathcal{H}^n$-a.e., and let $f \in \mathscr{F}$. Then, for every μ-distributed function $u \in W^{1,1}_{0,\tau}(\Omega)$ we have*

$$\begin{aligned}
&\int_{\partial^*\Sigma^{v_\mu}\cap(\Phi_{n+1}(B\times\mathbb{S}^1))} \frac{a(|x|,y,v_\mu) f(\hat{x}\cdot\nabla_x v_\mu, \nabla_{x\|} v_\mu, \nabla_y v_\mu)}{\sqrt{1+|\nabla v_\mu|^2}}\, d\mathcal{H}^n(x,y,t) \\
&\le \int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}(B\times\mathbb{S}^1))} \frac{a(|x|,y,u_0) f(\hat{x}\cdot\nabla_x u_0, \nabla_{x\|} u_0, \nabla_y u_0)}{\sqrt{1+|\nabla u_0|^2}}\, d\mathcal{H}^n(x,y,t),
\end{aligned} \tag{4.8}$$

for every Borel set $B \subset \Pi_{n-1}(\Omega)\times\mathbb{R}$.

Before discussing the proof, we show some consequences of this result.

4.1. **Consequences of Theorem 4.3.** We start by showing that Theorem 4.3 implies Theorem 1.12.

Proof of Theorem 1.12. The result follows immediately from (4.8), choosing Borel sets of the type $B = \tilde{B}\times\mathbb{R}$ with $\tilde{B} \subset \Pi_{n-1}(\Omega)$. □

The next result is a consequence of Theorem 1.12.

Corollary 4.4. *Let $n \in \mathbb{N}$ with $n \ge 2$, let $\Omega \subset \mathbb{R}^n$ be open, and let $p \in [1,\infty)$. Let $u \in W^{1,p}_{0,\tau}(\Omega)$, let μ be given by (1.13), and let v_μ be defined by (1.14). Then, $v_\mu\,|_{\Omega^s} \in W^{1,p}_{0,\tau}(\Omega^s)$.*

Proof. The case $p = 1$ was already considered in Proposition 3.31, so let us assume $p > 1$, and let $u \in W^{1,p}_{0,\tau}(\Omega)$. By Remark 3.4 it follows that $u \in W^{1,1}_{0,\tau}(\Omega)$ and so, by Proposition 3.31, we have $v_\mu\,|_{\Omega^s} \in W^{1,1}_{0,\tau}(\Omega^s)$.

Let now $A \subset\subset \Pi_{n-1}(\Omega)$ be open. Fom Proposition 3.19 it follows that $v_\mu \in L^p(\Phi_n(A\times\mathbb{S}^1))$. Moreover, Applying Theorem 1.12 with $a \equiv 1$, $B = A$ and

$$f(\eta,\zeta,\tau) = \left(\eta^2 + \zeta^2 + |\tau|^2\right)^{p/2},$$

we obtain that $v_\mu \in W^{1,p}(\Phi_n(A\times\mathbb{S}^1))$. Since A was arbitrary, thanks to (3.12) we conclude. □

Let us now show that Theorem 1.12 implies Theorem 1.13.

Proof of Theorem 1.13. Let $A \subset\subset \Pi_{n-1}(\Omega)$ be a Borel set. From Theorem 1.12 it follows that

$$\begin{aligned}&\int_{\Phi_n(A\times\mathbb{S}^1)} a(|x|,y,v_\mu)f(\hat{x}\cdot\nabla_x v_\mu,\nabla_{x\parallel}v_\mu,\nabla_y v_\mu)\,dxdy\\&\qquad\le\int_{\Phi_n(A\times\mathbb{S}^1)} a(|x|,y,u_0)f(\hat{x}\cdot\nabla_x u_0,\nabla_{x\parallel}u_0,\nabla_y u_0)\,dxdy,\end{aligned}\tag{4.9}$$

where this is an inequality between extended nonnegative real numbers.

Note now that, thanks to (3.11), we have $v_\mu = 0$ $\mathcal{H}^n$-a.e. in $\Phi_n(\Pi_{n-1}(\Omega)\times\mathbb{S}^1)\setminus\Omega^s$. Then (see, for instance, [19, Section 4.2.2, Theorem 4, part iv]) it follows that $\nabla v_\mu = 0$ $\mathcal{H}^n$-a.e. in $\Phi_n(\Pi_{n-1}(\Omega)\times\mathbb{S}^1)\setminus\Omega^s$. For the same reason, since $u_0 = 0$ $\mathcal{H}^n$-a.e. in $\Phi_n(\Pi_{n-1}(\Omega)\times\mathbb{S}^1)\setminus\Omega$, we have $\nabla u_0 = 0$ $\mathcal{H}^n$-a.e. in $\Phi_n(\Pi_{n-1}(\Omega)\times\mathbb{S}^1)\setminus\Omega$. Therefore,

$$\begin{aligned}&\int_{\Phi_n(A\times\mathbb{S}^1)\setminus\Omega^s} a(|x|,y,v_\mu)f(\hat{x}\cdot\nabla_x v_\mu,\nabla_{x\parallel}v_\mu,\nabla_y v_\mu)\,dxdy\\&\quad= f(0,0,0)\int_{\Phi_n(A\times\mathbb{S}^1)\setminus\Omega^s} a(|x|,y,0)\,dxdy\\&\quad= f(0,0,0)\int_A a(r,y,0)\left(\int_{\partial D(r)\setminus(\Omega^s)_{(r,y)}} d\mathcal{H}^1(x)\right)dr\,dy\\&\quad= f(0,0,0)\int_A a(r,y,0)\Big(2\pi r-\mathcal{H}^1\left((\Omega^s)_{(r,y)}\right)\Big)\,dr\,dy\\&\quad= f(0,0,0)\int_A a(r,y,0)\Big(2\pi r-\mathcal{H}^1\left((\Omega)_{(r,y)}\right)\Big)\,dr\,dy\\&\quad= f(0,0,0)\int_{\Phi_n(A\times\mathbb{S}^1)\setminus\Omega} a(|x|,y,0)\,dxdy\\&\quad=\int_{\Phi_n(A\times\mathbb{S}^1)\setminus\Omega} a(|x|,y,u_0)f(\hat{x}\cdot\nabla_x u_0,\nabla_{x\parallel}u_0,\nabla_y u_0)\,dxdy,\end{aligned}\tag{4.10}$$

where we used the fact that by definition of Ω^s we have

$$\mathcal{H}^1\left((\Omega^s)_{(r,y)}\right)=\mathcal{H}^1\left((\Omega)_{(r,y)}\right)\qquad\text{for every }(r,y)\in A.$$

Note that, since $A \subset\subset \Pi_{n-1}(\Omega)$ and a is bounded, all the integrals appearing in (4.10) are finite. Therefore, we can combine (4.9) and (4.10), obtaining

$$\begin{aligned}&\int_{\Phi_n(A\times\mathbb{S}^1)\cap\Omega^s} a(|x|,y,v_\mu)f(\hat{x}\cdot\nabla_x v_\mu,\nabla_{x\parallel}v_\mu,\nabla_y v_\mu)\,dxdy\\&\le\int_{\Phi_n(A\times\mathbb{S}^1)\cap\Omega} a(|x|,y,u_0)f(\hat{x}\cdot\nabla_x u_0,\nabla_{x\parallel}u_0,\nabla_y u_0)\,dxdy,\end{aligned}$$

for every Borel set $A \subset\subset \Pi_{n-1}(\Omega)$. By considering a sequence $\{A_j\}_{j\in\mathbb{N}}$ of Borel sets with $A_j \subset\subset \Pi_{n-1}(\Omega)$ and $A_j \nearrow \Pi_{n-1}(\Omega)$, we conclude. □

We can now give the proof of Theorem 4.3.

Proof of Theorem 4.3. First of all, note that it is not restrictive to assume

$$B\subset\{\alpha_\mu^\vee>0\}\cap\{\alpha_\mu^\wedge<\pi\}.\tag{4.11}$$

Indeed, if not, one can split B as the disjoint union $B = B_1 \cup B_2$, where

$$B_1 = B\cap\{\alpha_\mu^\vee>0\}\cap\{\alpha_\mu^\wedge<\pi\},\quad B_2 = B\setminus(\{\alpha_\mu^\vee>0\}\cap\{\alpha_\mu^\wedge<\pi\}),$$

and observe that, thanks to (3.16), inequality (4.8) is trivially satisfied in B_2. Let's then assume that (4.11) is satisfied. We can also suppose $B \subset A\times\mathbb{R}$ for some open set $A \subset\subset \Pi_{n-1}(\Omega)$, since the general case can be obtained by approximation.

By assumption, $u_0 \in W^{1,1}(\Phi_n(A\times\mathbb{S}^1))$ and thanks to Proposition 3.31 $v_\mu \in W^{1,1}(\Phi_n(A\times\mathbb{S}^1))$. Let G_{v_μ} and G_{u_0} denote the sets given by Proposition 3.12 (note that in general both G_{v_μ} and

G_{u_0} will depend on A). Then, setting $B_{v_\mu,u_0} := B \cap G_{v_\mu} \cap G_{u_0}$ we have

$$\begin{aligned}
&\int_{\partial^*\Sigma^{v_\mu}\cap(\Phi_{n+1}(B\times\mathbb{S}^1))} \frac{a(|x|,y,v_\mu)f(\hat{x}\cdot\nabla_x v_\mu, \nabla_{x\|}v_\mu, \nabla_y v_\mu)}{\sqrt{1+|\nabla v_\mu|^2}}\, d\mathcal{H}^n(x,y,t) \\
&= \int_{\partial^*\Sigma^{v_\mu}\cap(\Phi_{n+1}(B\times\mathbb{S}^1))} \frac{a(|x|,y,t)f(\hat{x}\cdot\nabla_x v_\mu, \nabla_{x\|}v_\mu, \nabla_y v_\mu)}{\sqrt{1+|\nabla v_\mu|^2}}\, d\mathcal{H}^n(x,y,t) = I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= \int_{\partial^*\Sigma^{v_\mu}\cap\left(\Phi_{n+1}(B_{v_\mu,u_0}\times\mathbb{S}^1)\right)} \frac{a(|x|,y,t)f(\hat{x}\cdot\nabla_x v_\mu, \nabla_{x\|}v_\mu, \nabla_y v_\mu)}{\sqrt{1+|\nabla v_\mu|^2}}\, d\mathcal{H}^n(x,y,t), \\
I_2 &:= \int_{\partial^*\Sigma^{v_\mu}\cap\left(\Phi_{n+1}\left((B\setminus B_{v_\mu,u_0})\times\mathbb{S}^1\right)\right)} \frac{a(|x|,y,t)f(\hat{x}\cdot\nabla_x v_\mu, \nabla_{x\|}v_\mu, \nabla_y v_\mu)}{\sqrt{1+|\nabla v_\mu|^2}}\, d\mathcal{H}^n(x,y,t),
\end{aligned}$$

and where we used the fact that, since $v_\mu \in W^{1,1}(\Phi_n(A\times\mathbb{S}^1))$, thanks to Proposition 3.1 we have $v_\mu^\wedge(x,y) = v_\mu^\vee(x,y) = t$ for $\mathcal{H}^n$-a.e. $(x,y,t) \in \partial^*\Sigma^{v_\mu}$. We divide the proof into two steps.

Step 1: We show that

$$I_1 \leq \int_{\partial^*\Sigma^{u_0}\cap\left(\Phi_{n+1}(B_{v_\mu,u_0}\times\mathbb{S}^1)\right)} \frac{a(|x|,y,u_0)f(\hat{x}\cdot\nabla_x u_0, \nabla_{x\|}u_0, \nabla_y u_0)}{\sqrt{1+|\nabla u_0|^2}}\, d\mathcal{H}^n(x,y,t). \tag{4.12}$$

Taking into account property (f2) of Definition 1.10, we have

$$\begin{aligned}
I_1 &= \int_{\partial^*\Sigma^{v_\mu}\cap\left(\Phi_{n+1}(B_{v_\mu,u_0}\times\mathbb{S}^1)\cap(\{\nabla_{x\|}v_\mu\neq 0\}\times\mathbb{R})\right)} \frac{a(|x|,y,t)f(\hat{x}\cdot\nabla_x v_\mu, \nabla_{x\|}v_\mu, \nabla_y v_\mu)}{\sqrt{1+|\nabla v_\mu|^2}}\, d\mathcal{H}^n(x,y,t) \\
&= \int_{B_{v_\mu,u_0}} a(r,y,t) \int_{\partial^*((\Sigma^{v_\mu})_{(r,y,t)})} \frac{f(\hat{x}\cdot\nabla_x v_\mu, |\nabla_{x\|}v_\mu|, \nabla_y v_\mu)}{|\nabla_{x\|}v_\mu|}\, d\mathcal{H}^0(x)\, dr\, dy\, dt \\
&= \int_{B_{v_\mu,u_0}} a(r,y,t)\,(-\partial_t\mu(r,y,t)) f\left(\frac{r\partial_r\xi_\mu(r,y,t)}{-\partial_t\mu(r,y,t)}, \frac{2}{-\partial_t\mu(r,y,t)}, \frac{\nabla_y\mu(r,y,t)}{-\partial_t\mu(r,y,t)}\right) dr\, dy\, dt,
\end{aligned} \tag{4.13}$$

where the first equality is due to Remark 3.15, the second one to Proposition 3.6, Remark 3.7 and Proposition 3.12, and in the last one we used Remark 3.29, Remark 3.30, and the fact that $\mathcal{H}^0\left(\partial^*((\Sigma^{v_\mu})_{(r,y,t)})\right) = 2$ for $\mathcal{H}^n$-a.e. $(r,y,t) \in B_{v_\mu,u_0}$.

Now we would like to compare the integral above with an integral involving the function u_0. To this aim, we first observe that, thanks to (2.11),

$$2 \leq \mathcal{H}^0\left(\partial^*((\Sigma^{u_0})_{(r,y,t)})\right), \qquad \text{for } \mathcal{H}^n\text{-a.e. } (r,y,t) \in B_{v_\mu,u_0}. \tag{4.14}$$

Let us now introduce, for every $(r,y,t) \in B_{v_\mu,u_0}$, the probability measure $\lambda^t_{r,y}$ on $(\partial^*\Sigma^{u_0})_{(r,y,t)}$ given by

$$\lambda^t_{r,y} := \frac{\dfrac{1}{|\nabla_{x\|}u_0(x,y)|}\,\mathcal{H}^0 \llcorner (\partial^*\Sigma^{u_0})_{(r,y,t)}}{\displaystyle\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{1}{|\nabla_{x\|}u_0(x,y)|}\, d\mathcal{H}^0(x)}.$$

In the following, to ease the notation, in the integrals below we drop the integration set $(\partial^*\Sigma^{u_0})_{(r,y,t)}$. Using Lemma 4.1 and (4.14), together with Proposition 3.26, for $\mathcal{H}^n$-a.e. $(r,y,t) \in B_{v_\mu,u_0}$ one has

$$a(r,y,t)\,(-\partial_t\mu(r,y,t))f\left(\frac{r\partial_r\xi_\mu(r,y,t)}{-\partial_t\mu(r,y,t)},\frac{2}{-\partial_t\mu(r,y,t)},\frac{\nabla_y\mu(r,y,t)}{-\partial_t\mu(r,y,t)}\right)$$

$$\begin{aligned}
&= a(r,y,t)\left(\int\frac{d\mathcal{H}^0(x)}{|\nabla_{x\parallel}u_0|}\right)f\left(\frac{\left(\int\frac{\hat{x}\cdot\nabla_x u_0}{|\nabla_{x\parallel}u_0|}\,d\mathcal{H}^0(x),2,\int\frac{\nabla_y u_0}{|\nabla_{x\parallel}u_0|}\,d\mathcal{H}^0(x)\right)}{\int\frac{d\mathcal{H}^0(x)}{|\nabla_{x\parallel}u_0|}}\right)\\
&\le a(r,y,t)\left(\int\frac{d\mathcal{H}^0(x)}{|\nabla_{x\parallel}u_0|}\right)f\left(\frac{\left(\int\frac{\hat{x}\cdot\nabla_x u_0}{|\nabla_{x\parallel}u_0|}\,d\mathcal{H}^0(x),\int d\mathcal{H}^0(x),\int\frac{\nabla_y u_0}{|\nabla_{x\parallel}u_0|}\,d\mathcal{H}^0(x)\right)}{\int\frac{d\mathcal{H}^0(x)}{|\nabla_{x\parallel}u_0|}}\right)\\
&= a(r,y,t)\left(\int\frac{d\mathcal{H}^0(x)}{|\nabla_{x\parallel}u_0|}\right)f\left(\int\hat{x}\cdot\nabla_x u_0\,d\lambda^t_{r,y}(x),\int|\nabla_{x\parallel}u_0|\,d\lambda^t_{r,y}(x),\int\nabla_y u_0\,d\lambda^t_{r,y}(x)\right).
\end{aligned} \tag{4.15}$$

Thanks to properties (f1) of Definition 1.10, we can apply Jensen's inequality. Therefore, using also property (f2) of Definition 1.10, we obtain

$$\begin{aligned}
&a(r,y,t)\,(-\partial_t\mu(r,y,t))f\left(\frac{r\partial_r\xi(r,y,t)}{-\partial_t\mu(r,y,t)},\frac{2}{-\partial_t\mu(r,y,t)},\frac{\partial_y\mu(r,y,t)}{-\partial_t\mu(r,y,t)}\right)\\
&\le a(r,y,t)\left(\int\frac{d\mathcal{H}^0(x)}{|\nabla_{x\parallel}u_0|}\right)\int f(\hat{x}\cdot\nabla_x u_0,|\nabla_{x\parallel}u_0|,\nabla_y u_0)\,d\lambda^t_{r,y}(x)\\
&= a(r,y,t)\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}}\frac{f(\hat{x}\cdot\nabla_x u_0,\nabla_{x\parallel}u_0,\nabla_y u_0)}{|\nabla_{x\parallel}u_0|}\,d\mathcal{H}^0(x),
\end{aligned} \tag{4.16}$$

for $\mathcal{H}^n$-a.e. $(r,y,t)\in B_{v_\mu,u_0}$. From this and (4.13), using again Proposition 3.6, it follows that

$$\begin{aligned}
I_1 &\le \int_{B_{v_\mu,u_0}} a(r,y,t)\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}}\frac{f(\hat{x}\cdot\nabla_x u_0,\nabla_{x\parallel}u_0,\nabla_y u_0)}{|\nabla_{x\parallel}u_0|}\,d\mathcal{H}^0(x)\,dr\,dy\,dt\\
&= \int_{\partial^*\Sigma^{u_0}\cap\left(\Phi_{n+1}(B_{v_\mu,u_0}\times\mathbb{S}^1)\right)}\frac{a(|x|,y,u_0)f(\hat{x}\cdot\nabla_x u_0,\nabla_{x\parallel}u_0,\nabla_y u_0)}{\sqrt{1+|\nabla u_0|^2}}\,d\mathcal{H}^n(x,y,t),
\end{aligned}$$

thus showing (4.12).

Step 2: We conclude, showing that

$$I_2 \le \int_{\partial^*\Sigma^{u_0}\cap\left(\Phi_{n+1}\left((B\setminus B_{v_\mu,u_0})\times\mathbb{S}^1\right)\right)}\frac{a(|x|,y,u_0)\,f(\hat{x}\cdot\nabla_x u_0(x,y),\nabla_{x\parallel}u_0(x,y),\nabla_y u_0(x,y))}{\sqrt{1+|\nabla u_0(x,y)|^2}}\,d\mathcal{H}^n(x,y,t). \tag{4.17}$$

Recalling the definition of B_{v_μ,u_0} and the properties of the sets G_{v_μ} and G_{u_0}, from (4.11) it follows that $\mathcal{H}^n(B\setminus B_{v_\mu,u_0})=0$. Therefore, applying (3.3) to v_μ we obtain

$$\begin{aligned}
I_2 &= \int_{\partial^*\Sigma^{v_\mu}\cap\left(\Phi_{n+1}\left((B\setminus B_{v_\mu,u_0})\times\mathbb{S}^1\right)\right)}\frac{a(|x|,y,t)f(\hat{x}\cdot\nabla_x v_\mu,\nabla_{x\parallel}v_\mu,\nabla_y v_\mu)}{\sqrt{1+|\nabla v_\mu|^2}}\,d\mathcal{H}^n(x,y,t)\\
&= \int_{\partial^*\Sigma^{v_\mu}\cap\left(\Phi_{n+1}\left((B\setminus B_{v_\mu,u_0})\times\mathbb{S}^1\right)\right)}\frac{a(|x|,y,t)f(\hat{x}\cdot\nabla_x v_\mu,0,\nabla_y v_\mu)}{\sqrt{1+|\nabla v_\mu|^2}}\,d\mathcal{H}^n(x,y,t)\\
&= \int_{\partial^*\Sigma^{v_\mu}\cap\left(\Phi_{n+1}\left((B\setminus B_{v_\mu,u_0})\times\mathbb{S}^1\right)\right)}\sup_{h\in\mathbb{N}}\varphi_h(x,y,t)\,d\mathcal{H}^n(x,y,t),
\end{aligned}$$

where for every $h\in\mathbb{N}$ we set

$$\varphi_h(x,y,t) := a(|x|,y,t)\frac{w_h\cdot(\hat{x}\cdot\nabla_x v_\mu(x,y),0,\nabla_y v_\mu(x,y)) - f^*(w_h)}{\sqrt{1+|\nabla v_\mu(x,y)|^2}},$$

and we used (4.1), where $\{w_h\}_{h\in\mathbb{N}}$ is a countable dense subset of $\{w \in \mathbb{R}^n : f^*(w) < +\infty\}$. Note now that

$$|\varphi_1(x,y,t)| \le a(|x|,y,t)\frac{|w_1||\nabla v_\mu(x,y)| + |f^*(w_1)|}{\sqrt{1+|\nabla v_\mu(x,y)|^2}} \le C(|w_1| + |f^*(w_1)|),$$

where $C > 0$ is such that $|a| \le C$. Then,

$$\begin{aligned}
&\int_{\partial^*\Sigma^{v_\mu}\cap\left(\Phi_{n+1}\left((B\setminus B_{v_\mu,u_0})\times\mathbb{S}^1\right)\right)} |\varphi_1(x,y,t)|\, d\mathcal{H}^n(x,y,t)\\
&\quad\le C(|w_1| + |f^*(w_1)|)\mathcal{H}^n\left(\partial^*\Sigma^{v_\mu}\cap\left(\Phi_{n+1}(B\times\mathbb{S}^1)\right)\right)\\
&\quad\le \mathcal{H}^n\left(\partial^*\Sigma^{v_\mu}\cap\left(\Phi_{n+1}((A\times\mathbb{R})\times\mathbb{S}^1)\right)\right) < \infty.
\end{aligned}$$

Thanks to Remark 3.29, we can apply Lemma 4.2. Therefore, we have

$$I_2 = \sup_H \left\{ \sum_{h\in H} \int_{\partial^*\Sigma^{v_\mu}\cap\Phi_{n+1}(B_h\times\mathbb{S}^1)} \varphi_h(x,y,t)\, d\mathcal{H}^n(x,y,t) \right\},$$

where the supremum ranges over all finite sets $H \subset \mathbb{N}$ and over all partitions $\{B_h\}_{h\in H}$ of $B\setminus B_{u,v_\mu}$ composed of pairwise disjoint Borel sets. Let now $H \subset \mathbb{N}$ be a finite set, and let $h \in H$. Thanks to Proposition 3.24, applied first to v_μ and then to u, and recalling (3.1), we have

$$\begin{aligned}
&\int_{\partial^*\Sigma^{v_\mu}\cap\Phi_{n+1}(B_h\times\mathbb{S}^1)} \varphi_h(x,y,t)\, d\mathcal{H}^n(x,y,t)\\
&= \int_{\partial^*\Sigma^{v_\mu}\cap\Phi_{n+1}(B_h\times\mathbb{S}^1)} a(|x|,y,t)\frac{w_h\cdot(\hat{x}\cdot\nabla_x v_\mu, 0, \nabla_y v_\mu) - f^*(w_h)}{\sqrt{1+|\nabla v_\mu|^2}}\, d\mathcal{H}^n(x,y,t)\\
&= \int_{\partial^*\Sigma^{v_\mu}\cap\Phi_{n+1}(B_h\times\mathbb{S}^1)} a(|x|,y,t)\Big(w_h\cdot\Big(\hat{x}\cdot\nu_x^{\Sigma^{v_\mu}}(x,y,t), 0, \nu_y^{\Sigma^{v_\mu}}(x,y,t)\Big)\\
&\qquad + f^*(w_h)\,\nu_t^{\Sigma^{v_\mu}}(x,y,t)\Big)\, d\mathcal{H}^n(x,y,t)\\
&= w_h\cdot\left(\int_{B_h} a(r,y,t)\, r\, dD_r\xi_\mu(r,y,t), 0, \int_{B_h} a(r,y,t)\, dD_y\mu(r,y,t)\right)\\
&\qquad + \int_{B_h} a(r,y,t) f^*(w_h)\, dD_t\mu(r,y,t)\\
&= \int_{\partial^*\Sigma^{u_0}\cap\Phi_{n+1}(B_h\times\mathbb{S}^1)} a(|x|,y,t)\frac{w_h\cdot(\hat{x}\cdot\nabla_x u_0, 0, \nabla_y u_0) - f^*(w_h)}{\sqrt{1+|\nabla u_0|^2}}\, d\mathcal{H}^n(x,y,t)\\
&\le \int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}(B_h\times\mathbb{S}^1))} \frac{a(|x|,y,t) f(\hat{x}\cdot\nabla_x u_0, 0, \nabla_y u_0)}{\sqrt{1+|\nabla u_0|^2}}\, d\mathcal{H}^n(x,y,t),
\end{aligned}$$

where the last inequality follows from (4.1). From this, it follows that

$$\begin{aligned}
&\sum_{h\in H}\int_{\partial^*\Sigma^{v_\mu}\cap\Phi_{n+1}(B_h\times\mathbb{S}^1)} \varphi_h(x,y,t)\, d\mathcal{H}^n(x,y,t)\\
&\le \sum_{h\in H}\int_{\partial^*\Sigma^{u_0}\cap(\Phi_{n+1}(B_h\times\mathbb{S}^1))} \frac{a(|x|,y,t) f(\hat{x}\cdot\nabla_x u_0, 0, \nabla_y u_0)}{\sqrt{1+|\nabla u_0|^2}}\, d\mathcal{H}^n(x,y,t)\\
&= \int_{\partial^*\Sigma^{u_0}\cap\left(\Phi_{n+1}\left((B\setminus B_{v_\mu,u_0})\times\mathbb{S}^1\right)\right)} \frac{a(|x|,y,t) f(\hat{x}\cdot\nabla_x u_0, 0, \nabla_y u_0)}{\sqrt{1+|\nabla u_0|^2}}\, d\mathcal{H}^n(x,y,t)\\
&= \int_{\partial^*\Sigma^{u_0}\cap\left(\Phi_{n+1}\left((B\setminus B_{v_\mu,u_0})\times\mathbb{S}^1\right)\right)} \frac{a(|x|,y,t) f(\hat{x}\cdot\nabla_x u_0, \nabla_{x_\parallel} u_0, \nabla_y u_0)}{\sqrt{1+|\nabla u_0|^2}}\, d\mathcal{H}^n(x,y,t),
\end{aligned}$$

where in the last equality we applied (3.3) to u_0 with $R = B\setminus B_{v_\mu,u_0}$. Taking the supremum with respect to H we obtain (4.17) which, combined with (4.12), allows us to conclude. □

5. Sufficient conditions for rigidity

In this section we give the proof of Theorem 1.18. We start by considering inequality (4.8) with $B = \Pi_{n-1}(\Omega) \times \mathbb{R}$:

$$\begin{aligned} &\int_{\partial^* \Sigma^{v_\mu} \cap (\Phi_{n+1}((\Pi_{n-1}(\Omega)\times\mathbb{R})\times\mathbb{S}^1))} \frac{a(|x|, y, v_\mu) f(\hat{x} \cdot \nabla_x v_\mu, \nabla_{x\parallel} v_\mu, \nabla_y v_\mu)}{\sqrt{1+|\nabla v_\mu|^2}} \, d\mathcal{H}^n(x,y,t) \\ &\le \int_{\partial^* \Sigma^{u_0} \cap (\Phi_{n+1}((\Pi_{n-1}(\Omega)\times\mathbb{R})\times\mathbb{S}^1))} \frac{a(|x|, y, u_0) f(\hat{x} \cdot \nabla_x u_0, \nabla_{x\parallel} u_0, \nabla_y u_0)}{\sqrt{1+|\nabla u_0|^2}} \, d\mathcal{H}^n(x,y,t), \end{aligned} \tag{5.1}$$

under the assumption that

$$\int_{\partial^* \Sigma^{v_\mu} \cap (\Phi_{n+1}((\Pi_{n-1}(\Omega)\times\mathbb{R})\times\mathbb{S}^1))} \frac{a(|x|, y, v_\mu) f(\hat{x} \cdot \nabla_x v_\mu, \nabla_{x\parallel} v_\mu, \nabla_y v_\mu)}{\sqrt{1+|\nabla v_\mu|^2}} \, d\mathcal{H}^n(x,y,t) < \infty. \tag{5.2}$$

First of all, we define the set of extremals in (5.1).

Definition 5.1. *Let $n \in \mathbb{N}$ with $n \ge 2$, let $\Omega \subset \mathbb{R}^n$ be open, and let $\mu \in \mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$. Let $a \in L^\infty((0,\infty) \times \mathbb{R}^{n-2} \times \mathbb{R})$ with $a > 0$ $\mathcal{H}^n$-a.e., let $f \in \mathscr{F}'$, and suppose that* (5.2) *is satisfied. Then, we set*

$$\mathcal{E}(\mu, \Omega) := \{u \in W^{1,1}_{0,\tau}(\Omega) : u \text{ is } \mu\text{-distributed and equality holds in } (5.1)\}.$$

Remark 5.2. *Note that, under the assumption* (5.2)*, using* (4.10) *we have that* (5.1) *is equivalent to* (1.18) *and to equality in* (1.17)*.*

We start with some necessary conditions for a function u to belong to $\mathcal{E}(\mu, \Omega)$.

Proposition 5.3. *Let $n \in \mathbb{N}$ with $n \ge 2$, let $\Omega \subset \mathbb{R}^n$ be open, and let $\mu \in \mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$. Let $a \in L^\infty((0,\infty) \times \mathbb{R}^{n-2} \times \mathbb{R})$ with $a > 0$ $\mathcal{H}^n$-a.e., let $f \in \mathscr{F}'$, and suppose that* (5.2) *is satisfied. Let $u \in \mathcal{E}(\mu, \Omega)$. Then, for $\mathcal{H}^n$-a.e. $(r,y,t) \in \{0 < \alpha_\mu < \pi\}$*

$$(\Sigma^{u_0})_{(r,y,t)} =_{\mathcal{H}^1} \mathbf{B}_{\alpha_\mu(r,y,t)}(p(r,y,t)) \quad \text{for some } p(r,y,t) \in \mathbb{S}^1, \tag{5.3}$$

and the functions

$$x \longmapsto \hat{x} \cdot \nabla u_0(x,y), \qquad x \longmapsto |\nabla_{x\parallel} u_0(x,y)|, \qquad x \longmapsto \nabla_y u_0(x,y),$$

are constant on $\partial^*((\Sigma^{u_0})_{(r,y,t)})$.

Proof. Since $u \in \mathcal{E}(\mu, \Omega)$, equality in (5.1) holds. Then, thanks to Theorem 4.3 equality holds in (4.8) for every Borel set $B \subset \Pi_{n-1}(\Omega) \times \mathbb{R}$. In particular, equality holds in (4.15) for $\mathcal{H}^n$-a.e. $(r,y,t) \in \{0 < \alpha_\mu < \pi\}$. Since $a > 0$ $\mathcal{H}^n$-a.e. and since, thanks to Lemma 4.1, $\tau \longmapsto f(\eta, \tau, \zeta)$ is strictly increasing in $[0,\infty)$, we conclude that equality holds in (4.14) for $\mathcal{H}^n$-a.e. $(r,y,t) \in \{0 < \alpha_\mu < \pi\}$. Therefore, thanks to (2.12) we obtain (5.3).

Observe now that for every Borel set $B \subset \Pi_{n-1}(\Omega) \times \mathbb{R}$ we also have equality in Jensen's inequality (4.16). Since $a > 0$ $\mathcal{H}^n$-a.e. and f is strictly convex, the remaining part of the statement follows. ☐

If $u \in W^{1,1}_{0,\tau}(\Omega)$ satisfies (5.3), we can define the direction function $d_{u_0} : \Pi_{n-1}(\Omega) \times \mathbb{R} \to \mathbb{S}^1$ given by

$$d_{u_0}(r,y,t) := \begin{cases} \dfrac{1}{2r \sin \alpha_\mu(r,y,t)} \displaystyle\int_{(\Sigma^{u_0})_{(r,y,t)}} \hat{x} \, d\mathcal{H}^1(x) & \text{if } (r,y,t) \in \{0 < \alpha_\mu < \pi\}, \\ e_1 & \text{otherwise in } \Pi_{n-1}(\Omega) \times \mathbb{R}. \end{cases} \tag{5.4}$$

Remark 5.4. *If $u \in W^{1,1}_{0,\tau}(\Omega)$ satisfies* (5.3) *and d_{u_0} is defined by* (5.4)*, then a direct calculation shows that*

$$\Sigma^{u_0} \cap \{0 < \alpha_\mu < \pi\} =_{\mathcal{H}^{n+1}} \{(x,y,t) \in \mathbb{R}^2_0 \times \mathbb{R}^{n-2} \times \mathbb{R} : d_{\mathbb{S}^1}(\hat{x}, d_{u_0}(|x|,y,t)) < \alpha_\mu(|x|,y,t)\}.$$

We now give a regularity result for d_u.

Theorem 5.5. *Let $n \in \mathbb{N}$ with $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be open, and let $\mu \in \mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$. Let $u \in W^{1,1}_{0,\tau}(\Omega)$ be a μ-distributed function satisfying (5.3), let $A \subset\subset \Pi_{n-1}(\Omega)$ be open, let $d > 0$, and let $\delta \in (0, \pi/2)$. Then, $d^\delta_{u_0} \in BV(A \times (-d, +\infty); \mathbb{R}^2)$, where*

$$d^\delta_{u_0}(r, y, t) := \frac{1}{2r \sin \alpha^\delta_\mu(r, y, t)} \int_{(\Sigma^{u_0})_{(r,y,t)}} \hat{x}\, d\mathcal{H}^1(x), \qquad \alpha^\delta_\mu := (\alpha_\mu \vee \delta) \wedge (\pi - \delta).$$

Moreover, the following coarea formula holds. Using the notation $d^\delta_{u_0} = ((d^\delta_{u_0})_1, (d^\delta_{u_0})_2)$, for every $j \in \{1, 2\}$ we have

$$\begin{aligned} &\int_{\mathbb{R}} \mathcal{H}^{n-1}(B \cap \partial^e \{(d^\delta_{u_0})_j > s\})\, ds \\ &= \int_B |\nabla (d^\delta_{u_0})_j| dr\, dy\, dt + \int_{B \cap S_{(d^\delta_{u_0})_j}} [(d^\delta_{u_0})_j]\, d\mathcal{H}^{n-1} + |D^c (d^\delta_{u_0})_j|(B), \end{aligned} \tag{5.5}$$

for every Borel set $B \subset A \times (-d, +\infty)$.

Proof. Let $c > 0$ be such that $r \geq c$ for every $(r, y, t) \in A \times (-d, +\infty)$. To ease the notation, we define the function $m_{u_0} : \Pi_{n-1}(\Omega) \times \mathbb{R} \to \mathbb{R}^2$ as

$$m_{u_0}(r, y, t) := \frac{1}{r} \int_{(\Sigma^{u_0})_{(r,y,t)}} \hat{x}\, d\mathcal{H}^1(x), \quad \text{for every } (r, y, t) \in \Pi_{n-1}(\Omega) \times \mathbb{R}. \tag{5.6}$$

Step 1: We show that $m_u \in BV(A \times (-d, +\infty); \mathbb{R}^2)$. We have

$$\begin{aligned} \int_{A \times (-d,+\infty)} |m_{u_0}(r, y, t)|\, dr\, dy\, dt &\leq \frac{1}{c} \int_{A \times (-d,+\infty)} \mathcal{H}^1((\Sigma^{u_0})_{(r,y,t)})\, dr\, dy\, dt \\ &= \frac{1}{c} \|\mu\|_{L^1(A \times (-d,+\infty))} < \infty, \end{aligned}$$

where the last inequality follows from Proposition 3.9. This shows that $m_{u_0} \in L^1(A \times (-d, +\infty); \mathbb{R}^2)$.

Let now $j \in \{1, 2\}$ and, denoting by $(m_{u_0})_j$ the j-th component of m_{u_0}, let us show that $D(m_{u_0})_j$ is a bounded Radon measure on $A \times (-d, +\infty)$. Let $\psi \in C^1_c(A \times (-d, +\infty))$. Then,

$$\begin{aligned} &\int_{A \times (-d,+\infty)} \frac{\partial \psi}{\partial r}(r, y, t)(m_{u_0})_j(r, y, t)\, dr\, dy\, dt \\ &= \int_{A \times (-d,+\infty)} \frac{\partial \psi}{\partial r}(r, y, t) \left(\frac{1}{r} \int_{(\Sigma^{u_0})_{(r,y,t)}} \hat{x}_j\, d\mathcal{H}^1(x) \right) dr\, dy\, dt \\ &= \int_{A \times (-d,+\infty)} \frac{\partial \psi}{\partial r}(r, y, t) \left(\frac{1}{r} \int_{\partial D(r)} \frac{x_j}{r} \chi_{\Sigma^{u_0}}(x, y, t)\, d\mathcal{H}^1(x) \right) dr\, dy\, dt \\ &= \int_{\Phi_{n+1}((A \times (-d,+\infty)) \times \mathbb{S}^1)} \frac{x_j}{|x|^2} \frac{\partial \psi}{\partial r}(|x|, y, t) \chi_{\Sigma^{u_0}}(x, y, t)\, dx\, dy\, dt. \end{aligned}$$

A direct calculation shows that

$$\operatorname{div}_x \left(\frac{x_j}{|x|^2} \psi(|x|, y, t) \hat{x} \right) = \frac{x_j}{|x|^2} \frac{\partial \psi}{\partial r}(|x|, y, t),$$

where div_x denotes the divergence with respect to the variables (x_1, x_2). Therefore,

$$\begin{aligned} &\int_{A \times (-d,+\infty)} \frac{\partial \psi}{\partial r}(r, y, t)(m_{u_0})_j(r, y, t)\, dr\, dy\, dt \\ &= \int_{\Phi_{n+1}((A \times (-d,+\infty)) \times \mathbb{S}^1))} \operatorname{div}_x \left(\frac{x_j}{|x|^2} \psi(|x|, y, t) \hat{x} \right) \chi_{\Sigma^{u_0}}(x, y, t)\, dx\, dy\, dt \\ &= -\int_{\Phi_{n+1}((A \times (-d,+\infty)) \times \mathbb{S}^1)} \frac{x_j}{|x|^2} \psi(|x|, y, t)\, \hat{x} \cdot dD_x \chi_{\Sigma^{u_0}}(x, y, t) \\ &= -\int_{\partial^* \Sigma^{u_0} \cap \Phi_{n+1}((A \times (-d,+\infty)) \times \mathbb{S}^1)} \psi(|x|, y, t) \frac{x_j}{|x|^2} \frac{\hat{x} \cdot \nabla_x u_0(x, y)}{\sqrt{1 + |\nabla u_0(x, y)|^2}} d\mathcal{H}^n(x, y, t). \end{aligned} \tag{5.7}$$

Taking the supremum over all $\psi \in C^1_c(A \times (-d, +\infty))$ with $|\psi| \leq 1$ we obtain that $D_r(m_{u_0})_j$ is a bounded Radon measure over $A \times (-d, +\infty)$ with

$$|D_r(m_{u_0})_j|(A \times (-d, +\infty)) \leq \frac{1}{c}\mathcal{H}^n(\partial^*\Sigma^{u_0} \cap \Phi_{n+1}((A \times (-d, +\infty)) \times \mathbb{S}^1)) < \infty.$$

From (5.7), by approximation we obtain that

$$D_r(m_{u_0})_j(B) = \int_{\partial^*\Sigma^{u_0} \cap \Phi_{n+1}(B \times \mathbb{S}^1)} \frac{x_j}{|x|^2} \frac{\hat{x} \cdot \nabla_x u_0(x, y)}{\sqrt{1 + |\nabla u_0(x, y)|^2}} d\mathcal{H}^n(x, y, t), \tag{5.8}$$

for every Borel set $B \subset A \times (-d, +\infty)$. In a similar way, one can show that $D_y(m_{u_0})_j$ and $D_t(m_{u_0})_j$ are bounded Radon measures on $A \times (-d, +\infty)$, with

$$\begin{aligned} D_y(m_{u_0})_j(B) &= \int_{\partial^*\Sigma^{u_0} \cap \Phi_{n+1}(B \times \mathbb{S}^1)} \frac{x_j}{|x|^2} \frac{\nabla_y u_0(x, y)}{\sqrt{1 + |\nabla u_0(x, y)|^2}} d\mathcal{H}^n(x, y, t), \\ D_t(m_{u_0})_j(B) &= -\int_{\partial^*\Sigma^{u_0} \cap \Phi_{n+1}(B \times \mathbb{S}^1)} \frac{x_j}{|x|^2} \frac{1}{\sqrt{1 + |\nabla u_0(x, y)|^2}} d\mathcal{H}^n(x, y, t), \end{aligned} \tag{5.9}$$

for every Borel set $B \subset A \times (-d, +\infty)$.

Step 2: We conclude. First of all, observe that the function $f_\delta : [0, \pi] \to \mathbb{R}$ defined as

$$f_\delta(s) := \frac{1}{\sin\Big((s \vee \delta) \wedge (\pi - \delta)\Big)}, \tag{5.10}$$

is Lipschitz continuous. Therefore, by the chain rule in BV [2, Theorem 3.69] we have that

$$f_\delta(\alpha_\mu) := \frac{1}{2\sin\Big((\alpha_\mu \vee \delta) \wedge (\pi - \delta)\Big)} \in BV(A \times (-d, +\infty)).$$

We can now write $d^\delta_{u_0}$ as

$$d^\delta_{u_0}(r, y, t) = f_\delta(\alpha_\mu(r, y, t))m_{u_0}(r, y, t).$$

Thanks to Step 1, using again the chain rule in BV (see, in particular, [2, Example 3.97]) we have that $d^\delta_{u_0} \in BV(A \times (-d, +\infty); \mathbb{R}^2)$. Finally, let $j \in \{1, 2\}$. Thanks to Step 2 and [2, Theorem 3.40], formula (5.5) follows. ☐

We can now give further necessary conditions for a function u to belong to the set $\mathcal{E}(\mu, \Omega)$ of extremals.

Proposition 5.6. *Let $n \in \mathbb{N}$ with $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be open, and let $\mu \in \mathcal{A}(\Pi_{n-1}(\Omega) \times \mathbb{R})$ be such that (1.21) holds. Let $a \in L^\infty((0, \infty) \times \mathbb{R}^{n-2} \times \mathbb{R})$ with $a > 0$ $\mathcal{H}^n$-a.e., and let $f \in \mathscr{F}'$. Suppose that (5.2) is satisfied, and let $u \in \mathcal{E}(\mu, \Omega)$. Then, the function d_{u_0} defined in (5.4) is $\mathcal{H}^n$-a.e. approximately differentiable in $\{0 < \alpha_\mu < \pi\}$ and*

$$\nabla d_{u_0}(r, y, t) = 0 \qquad \textit{for } \mathcal{H}^n\textit{-a.e. } (r, y, t) \textit{ in } \{0 < \alpha_\mu < \pi\}. \tag{5.11}$$

Moreover, denoting the j-th component of d_{u_0} by $(d_{u_0})_j$, for every $j = 1, 2$ we have

$$S_{(d_{u_0})_j} \cap \{0 < \alpha^\wedge_\mu \leq \alpha^\vee_\mu < \pi\} \subset_{\mathcal{H}^{n-1}} S_{\alpha_\mu} \cap \{0 < \alpha^\wedge_\mu \leq \alpha^\vee_\mu < \pi\} \quad \textit{and} \quad |D^c(d_{u_0})_j|^+ \ll |D^c\alpha_\mu|,$$

where $|D^c(d_{u_0})_j|^+$ is the Borel measure given by

$$|D^c(d_{u_0})_j|^+(B) := \lim_{\delta \to 0^+} |D^c(d^\delta_{u_0})_j|(B), \qquad \textit{for every Borel set } B \subset \Pi_{n-1}(\Omega) \times \mathbb{R}.$$

Finally, if W is a Borel subset of $\{0 < \alpha^\wedge_\mu \leq \alpha^\vee_\mu < \pi\}$ and K is a concentration set for $D^c\alpha_\mu$, then for every $j = 1, 2$

$$\int_{-1}^{1} \mathcal{H}^{n-1}(W \cap \partial^e\{(d_{u_0})_j > s\})\, ds = \int_{W \cap S_{(d_{u_0})_j} \cap S_{\alpha_\mu}} [(d_{u_0})_j]\, d\mathcal{H}^{n-1} + |D^c(d_{u_0})_j|^+(W \cap K).$$

Proof. Thanks to Proposition 3.12 and Proposition 5.3, for $\mathcal{H}^n$-a.e. $(r, y, t) \in \{0 < \alpha_\mu < \pi\}$, the set $(\partial^*\Sigma^{u_0})_{(r,y,t)}$ is composed of just two points, and the vector function

$$x \longmapsto \frac{(\hat{x} \cdot \nabla_x u_0(x, y), \nabla_y u_0(x, y), -1)}{|\nabla_{x_\parallel} u_0(x, y)|}$$

is constant on $(\partial^*\Sigma^{u_0})_{(r,y,t)}$. Therefore, for $\mathcal{H}^n$-a.e. $(r,y,t) \in \{0 < \alpha_\mu < \pi\}$ we can define the vector $c_{u_0}(r,y,t) \in \mathbb{R}^n$ given by

$$c_{u_0}(r,y,t) := \frac{(\hat{x} \cdot \nabla_x u_0(x,y), \nabla_y u_0(x,y), -1)}{|\nabla_{x\parallel} u_0(x,y)|} = r\nabla\alpha_\mu(r,y,t), \tag{5.12}$$

where $x \in (\partial^*\Sigma^{u_0})_{(r,y,t)}$, and the last equality follows from Proposition 3.26. We now divide the proof into several steps.

Step 1: We show that the density of the absolutely continuous part of Dm_{u_0} is given by

$$\nabla m_{u_0}(r,y,t) = \frac{1}{r^2}\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{x \otimes (\hat{x} \cdot \nabla_x u_0(x,y), \nabla_y u_0(x,y), -1)}{|\nabla_{x\parallel} u_0(x,y)|}\, d\mathcal{H}^0(x),$$

for $\mathcal{H}^n$-a.e. (r,y,t) in $\{0 < \alpha_\mu < \pi\}$, where m_{u_0} is defined by (5.6).

Let $A \subset\subset \Pi_{n-1}(\Omega)$ be open, let $d > 0$, let G_{u_0} be given by Proposition 3.12. Thanks to (5.8) and (5.9), we have that

$$Dm_{u_0}(B) = \int_{\partial^*\Sigma^{u_0} \cap \Phi_{n+1}(B\times\mathbb{S}^1)} \frac{x \otimes (\hat{x} \cdot \nabla_x u_0(x,y), \nabla_y u_0(x,y), -1)}{|x|^2\sqrt{1+|\nabla u_0(x,y)|^2}} d\mathcal{H}^n(x,y,t), \tag{5.13}$$

for every Borel set $B \subset A \times (-d, +\infty)$. Since $\mathcal{H}^n((\{0 < \alpha_\mu < \pi\} \cap (A \times (-d, +\infty))) \setminus G_{u_0}) = 0$, we have

$$\begin{aligned}
D^a m_{u_0}(B) &= D^a m_{u_0}(B \cap G_{u_0}) \\
&= \int_{\partial^*\Sigma^{u_0} \cap \Phi_{n+1}((B\cap G_{u_0})\times\mathbb{S}^1)} \frac{x \otimes (\hat{x} \cdot \nabla_x u_0(x,y), \nabla_y u_0(x,y), -1)}{|x|^2\sqrt{1+|\nabla u_0(x,y)|^2}} d\mathcal{H}^n(x,y,t) \\
&= \int_{B\cap G_{u_0}} \frac{1}{r^2}\left(\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{x \otimes (\hat{x} \cdot \nabla_x u_0(x,y), \nabla_y u_0(x,y), -1)}{|\nabla_{x\parallel} u_0(x,y)|}\, d\mathcal{H}^0(x)\right) dr\, dy\, dt \\
&= \int_{B} \frac{1}{r^2}\left(\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{x \otimes (\hat{x} \cdot \nabla_x u_0(x,y), \nabla_y u_0(x,y), -1)}{|\nabla_{x\parallel} u_0(x,y)|}\, d\mathcal{H}^0(x)\right) dr\, dy\, dt.
\end{aligned}$$

Since A and d are arbitrary, the conclusion follows.

Step 2: We show that for $\mathcal{H}^n$-a.e. (r,y,t) in $\{0 < \alpha_\mu < \pi\}$

$$\nabla m_{u_0}(r,y,t) = \frac{\cos\alpha_\mu(r,y,t)}{r\sin\alpha_\mu(r,y,t)}\Big(m_{u_0}(r,y,t) \otimes c_{u_0}(r,y,t)\Big). \tag{5.14}$$

where m_{u_0} is defined by (5.6) and c_{u_0} is given by (5.12). Indeed, since $(\partial^*\Sigma^{u_0})_{(r,y,t)}$ is composed by two points that are symmetric with respect to $d_{u_0}(r,y,t)$, using Step 1 we obtain that

$$\begin{aligned}
&\nabla m_{u_0}(r,y,t) = \frac{1}{r^2}\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} \frac{x \otimes (\hat{x} \cdot \nabla_x u_0(x,y), \nabla_y u_0(x,y), -1)}{|\nabla_{x\parallel} u_0(x,y)|}\, d\mathcal{H}^0(x) \\
&= \frac{1}{r^2}\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} (x \otimes c_{u_0}(r,y,t))\, d\mathcal{H}^0(x) = \frac{1}{r^2}\left(\int_{(\partial^*\Sigma^{u_0})_{(r,y,t)}} x\, d\mathcal{H}^0(x)\right) \otimes c(r,y,t) \\
&= \frac{1}{r^2}\left(2r\cos\alpha_\mu(r,y,t)\right)\Big(d_{u_0}(r,y,t) \otimes c_{u_0}(r,y,t)\Big) = \frac{2\cos\alpha_\mu(r,y,t)}{r}\Big(d_{u_0}(r,y,t) \otimes c_{u_0}(r,y,t)\Big) \\
&= \frac{\cos\alpha_\mu(r,y,t)}{r\sin\alpha_\mu(r,y,t)}\Big(m_{u_0}(r,y,t) \otimes c_{u_0}(r,y,t)\Big).
\end{aligned}$$

Step 3: We show (5.11). Let $\delta \in (0, \pi/2)$. Note that $d^\delta_{u_0} = d_{u_0}$ $\mathcal{H}^n$-a.e. in $\{\delta < \alpha_\mu < \pi - \delta\}^{(1)}$. Then, thanks to Theorem 5.5 we can write

$$\nabla d_{u_0} = \nabla d^\delta_{u_0} = \frac{1}{2}\nabla\left(\frac{m_{u_0}}{\sin\alpha^\delta_\mu}\right) = \frac{1}{2}\nabla\left(\frac{m_{u_0}}{\sin\alpha_\mu}\right), \quad \mathcal{H}^n\text{-a.e. in } \{\delta < \alpha_\mu < \pi - \delta\}^{(1)}.$$

Thanks to the chain rule in BV (see [2, Example 3.97]) and using Step 2, we have that $\mathcal{H}^n$-a.e. in $\{\delta < \alpha_\mu < \pi - \delta\}^{(1)}$

$$\begin{aligned}\nabla d_{u_0} &= \frac{1}{2}\frac{\nabla m_{u_0}}{\sin\alpha_\mu} - \frac{1}{2}\frac{\cos\alpha_\mu}{(\sin\alpha_\mu)^2}(m_{u_0}\otimes\nabla\alpha_\mu)\\ &= \frac{\cos\alpha_\mu}{2r\sin^2\alpha_\mu}(m_{u_0}\otimes c) - \frac{1}{2}\frac{\cos\alpha_\mu}{(\sin\alpha_\mu)^2}(m_{u_0}\otimes\nabla\alpha_\mu) = 0,\end{aligned}$$

where we used (5.12) and (5.14). This shows that

$$\nabla d_{u_0} = 0, \qquad \mathcal{H}^n\text{-a.e. in } \{\delta < \alpha_\mu < \pi - \delta\}^{(1)}.$$

Note now that the set $\{\delta < \alpha_\mu < \pi - \delta\}^{(1)}$ is decreasing in δ and that, thanks to [9, (2.3) and (2.4)], we have

$$\bigcup_{\delta\in(0,\pi/2)} \{\delta < \alpha_\mu < \pi - \delta\}^{(1)} = \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}. \tag{5.15}$$

Therefore, we obtain that

$$\nabla d_{u_0} = 0, \qquad \mathcal{H}^n\text{-a.e. in } \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}.$$

Since $\{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\} =_{\mathcal{H}^n} \{0 < \alpha_\mu < \pi\}$, the conclusion follows.

Step 4: We show that

$$|D(m_{u_0})_j| \llcorner \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\} \ll |D\mu| \llcorner \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}. \tag{5.16}$$

Let $A \subset\subset \Pi_{n-1}(\Omega)$ be open, and let $d > 0$.

Let now $W \subset \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}$ with $W \subset A \times (-d, +\infty)$ be such that $|D\mu|(W) = 0$. Then, thanks to (3.21),

$$0 = D_t\mu(W) = -\int_{\partial^*\Sigma^{u_0}\cap\Phi_{n+1}(W\times\mathbb{S}^1)} \frac{1}{\sqrt{1+|\nabla u_0(x,y)|^2}}\, d\mathcal{H}^n(x,y,t),$$

which implies

$$\mathcal{H}^n\left(\partial^*\Sigma^{u_0}\cap\Phi_{n+1}(W\times\mathbb{S}^1)\right) = 0. \tag{5.17}$$

On the other hand, from (5.13)

$$\begin{aligned}|D_t(m_{u_0})_j(W)| &= \left|\int_{\partial^*\Sigma^{u_0}\cap\Phi_{n+1}(W\times\mathbb{S}^1)} \frac{x_j}{|x|^2}\frac{1}{\sqrt{1+|\nabla u_0(x,y)|^2}} d\mathcal{H}^n(x,y,t)\right|\\ &\le \frac{1}{c}\mathcal{H}^n\left(\partial^*\Sigma^{u_0}\cap\Phi_{n+1}(W\times\mathbb{S}^1)\right) = 0,\end{aligned}$$

where $c > 0$ is such that $r \ge c$ for every $(r,y,t) \in W$, and where we used (5.17). Arguing in a similar way, one can show that

$$D_y(m_{u_0})_j(W) = 0, \qquad \text{and} \qquad D_r(m_{u_0})_j(W) = 0.$$

Since A and d are arbitrary, the conclusion follows.

Step 5: We show that for every $j = 1, 2$

$$S_{(d_{u_0})_j} \cap \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\} \subset_{\mathcal{H}^{n-1}} S_{\alpha_\mu} \cap \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}. \tag{5.18}$$

Indeed, let $j \in \{1,2\}$, and let $\delta \in (0,\pi/2)$. Recall that $d_{u_0}^\delta$ is defined as

$$d_{u_0}^\delta(r,y,t) = f_\delta(\alpha_\mu(r,y,t))m_{u_0}(r,y,t),$$

where f_δ is the Lipschitz function given by (5.10). Then, recalling the definition of d_{u_0}, we have

$$S_{(d_{u_0})_j} \cap \{\delta < \alpha_\mu < \pi - \delta\}^{(1)} = S_{(d_{u_0}^\delta)_j} \cap \{\delta < \alpha_\mu < \pi - \delta\}^{(1)}. \tag{5.19}$$

On the other hand, thanks to the chain rule in BV

$$S_{(d_{u_0}^\delta)_j} \cap \{\delta < \alpha_\mu < \pi - \delta\}^{(1)} \subset_{\mathcal{H}^{n-1}} (S_{\alpha_\mu} \cup S_{(m_{u_0})_j}) \cap \{\delta < \alpha_\mu < \pi - \delta\}^{(1)}. \tag{5.20}$$

Observe now that, thanks to (5.16), and taking into account that $S_\mu = S_{\alpha_\mu}$, we have

$$S_{(m_{u_0})_j} \cap \{\delta < \alpha_\mu < \pi - \delta\}^{(1)} \subset_{\mathcal{H}^{n-1}} S_{\alpha_\mu} \cap \{\delta < \alpha_\mu < \pi - \delta\}^{(1)}. \tag{5.21}$$

Combining (5.19), (5.20), and (5.21), and recalling (5.15), we conclude.

Step 6: We show that for every $j = 1, 2$

$$|D^c(d_{u_0})_j|^+ \llcorner \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\} \ll |D^c\alpha_\mu| \llcorner \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}. \tag{5.22}$$

Let $\delta \in (0, \pi/2)$. Since $d^\delta_{u_0} = d_{u_0}$ $\mathcal{H}^n$-a.e. in $\{\delta < \alpha_\mu < \pi - \delta\}^{(1)}$, thanks to [8, Lemma 2.2], we have that

$$D^c(d_{u_0})_j \llcorner \{\delta < \alpha_\mu < \pi - \delta\}^{(1)} = D^c(d^\delta_{u_0})_j \llcorner \{\delta < \alpha_\mu < \pi - \delta\}^{(1)}.$$

On the other hand, by definition of $d^\delta_{u_0}$ and using again the chain rule in BV ([2, Theorem 3.96 and Example 3.97])

$$\begin{aligned} D^c\left((d^\delta_{u_0})_j\right) &= D^c(f_\delta(\alpha_\mu)(m_{u_0})_j) = D^c(f_\delta(\alpha_\mu))(m_{u_0})_j + f_\delta(\alpha_\mu) D^c\left((m_{u_0})_j\right) \\ &= (m_{u_0})_j(\nabla f_\delta(\alpha_\mu)) D^c\alpha_\mu + f_\delta(\alpha_\mu) D^c\left((m_{u_0})_j\right). \end{aligned} \tag{5.23}$$

From (5.16) we know that

$$\begin{aligned} |D^c\left((m_{u_0})_j\right)| \llcorner \{\delta < \alpha_\mu < \pi - \delta\}^{(1)} &\ll |D^c\mu| \llcorner \{\delta < \alpha_\mu < \pi - \delta\}^{(1)} \\ &\ll |D^c\alpha_\mu| \llcorner \{\delta < \alpha_\mu < \pi - \delta\}^{(1)}. \end{aligned} \tag{5.24}$$

Combining (5.23) and (5.24) we then obtain

$$|D^c\left((d^\delta_{u_0})_j\right)| \llcorner \{\delta < \alpha_\mu < \pi - \delta\}^{(1)} \ll |D^c\alpha_\mu| \llcorner \{\delta < \alpha_\mu < \pi - \delta\}^{(1)}.$$

Thanks to (5.15), this implies that

$$|D^c\left((d^\delta_{u_0})_j\right)| \llcorner \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\} \ll |D^c\alpha_\mu| \llcorner \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}.$$

Let now $W \subset \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}$ be a Borel set with $|D^c\alpha_\mu|(W) = 0$. From what we have just proved it follows that

$$|D^c\left((d^\delta_{u_0})_j\right)|(W) = 0, \qquad \text{for every } \delta \in (0, \pi/2).$$

Then,

$$|D^c(d_{u_0})_j|^+(W) = \lim_{\delta \to 0^+} |D^c\left((d^\delta_{u_0})_j\right)|(W) = 0,$$

which shows the claim.

Step 7: We conclude. Let $A \subset\subset \Pi_{n-1}(\Omega)$ be open, let $d > 0$, let $\delta \in (0, \pi/2)$, and let $j \in \{1, 2\}$. Since $(d^\delta_{u_0})_j = (d_{u_0})_j$ $\mathcal{H}^n$-a.e. on $\{\delta < \alpha_\mu < \pi - \delta\}^{(1)}$, thanks to (5.5) we have that for every Borel set $W \subset (A \times (-d, +\infty)) \cap \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}$

$$\begin{aligned} &\int_{\mathbb{R}} \mathcal{H}^{n-1}(W \cap \{\delta < \alpha_\mu < \pi - \delta\}^{(1)} \cap \partial^{\mathrm{e}}\{(d_{u_0})_j > s\})\, ds \\ &= \int_{W \cap \{\delta < \alpha_\mu < \pi - \delta\}^{(1)}} |\nabla(d_{u_0})_j|\, dr\, dy\, dt + \int_{W \cap \{\delta < \alpha_\mu < \pi - \delta\}^{(1)} \cap S_{(d_{u_0})_j} \cap S_{\alpha_\mu}} [(d_{u_0})_j]\, d\mathcal{H}^{n-1} \\ &+ |D^c(d_{u_0})_j|(W \cap \{\delta < \alpha_\mu < \pi - \delta\}^{(1)}), \end{aligned}$$

where we also used (5.18). Thanks to (5.15), passing to the limit as $\delta \to 0^+$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \mathcal{H}^{n-1}(W \cap \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\} \cap \partial^{\mathrm{e}}\{(d_{u_0})_j > s\})\, ds \\ &= \int_{W \cap \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}} |\nabla(d_{u_0})_j|\, dr\, dy\, dt + \int_{W \cap \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\} \cap S_{(d_{u_0})_j} \cap S_{\alpha_\mu}} [(d_{u_0})_j]\, d\mathcal{H}^{n-1} \\ &+ |D^c(d_{u_0})_j|^+(W \cap \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}). \end{aligned}$$

Let now K be a concentration set for $D^c\alpha_\mu$. Thanks to (5.22), we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \mathcal{H}^{n-1}(W \cap \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\} \cap \partial^{\mathrm{e}}\{(d_{u_0})_j > s\})\, ds \\ &= \int_{W \cap \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}} |\nabla(d_{u_0})_j|\, dr\, dy\, dt + \int_{W \cap \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\} \cap S_{(d_{u_0})_j} \cap S_{\alpha_\mu}} [(d_{u_0})_j]\, d\mathcal{H}^{n-1} \\ &+ |D^c(d_{u_0})_j|^+(W \cap K \cap \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}). \end{aligned}$$

Since A and d are arbitrary, the conclusion follows. □

We are now ready to give the proof of Theorem 1.18.

Proof of Theorem 1.18. Let $u \in \mathcal{E}(\mu, \Omega)$. We want to show that the function d_{u_0} defined in (5.4) is $\mathcal{H}^n$-a.e. constant on $\{0 < \alpha_\mu < \pi\}$. Let $j \in \{1, 2\}$ be fixed. Thanks to Proposition 5.6, for every Borel subset W of $\{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}$ we have

$$\begin{aligned} &\int_{-1}^{1} \mathcal{H}^{n-1}(W \cap \partial^{\mathrm{e}}\{(d_{u_0})_j > s\})\, ds \\ &= \int_{W \cap S_{(d_{u_0})_j} \cap S_{\alpha_\mu}} [(d_{u_0})_j]\, d\mathcal{H}^{n-1} + |D^c (d_{u_0})_j|^+ (W \cap K). \end{aligned} \tag{5.25}$$

Suppose now, by contradiction, that $(d_{u_0})_j$ is not $\mathcal{H}^n$-a.e. constant on $\{0 < \alpha_\mu < \pi\}$. Then, there exists a Lebesgue measurable set $I \subset (-1, 1)$ such that $\mathcal{H}^1(I) > 0$ and, for every $s \in I$, the Borel sets

$$W_+^s = \{(d_{u_0})_j > s\} \cap \{0 < \alpha_\mu < \pi\} \quad \text{and} \quad W_-^s = \{(d_{u_0})_j \le s\} \cap \{0 < \alpha_\mu < \pi\}$$

define a non trivial Borel partition $\{W_+^s, W_-^s\}$ of $\{0 < \alpha_\mu < \pi\}$.

By assumption, $\{\alpha_\mu^\wedge = 0\} \cup \{\alpha_\mu^\vee = \pi\} \cup S_{\alpha_\mu} \cup K$ does not essentially disconnect $\{0 < \alpha_\mu < \pi\}$. Then, for every $s \in I$ we have

$$\mathcal{H}^{n-1}\Big(\{0 < \alpha_\mu < \pi\}^{(1)} \cap \partial^{\mathrm{e}} W_+^s \cap \partial^{\mathrm{e}} W_-^s \setminus \big(\{\alpha_\mu^\wedge = 0\} \cup \{\alpha_\mu^\vee = \pi\} \cup S_{\alpha_\mu} \cup K\big)\Big) > 0. \tag{5.26}$$

Note now that

$$\begin{aligned} &\{0 < \alpha_\mu < \pi\}^{(1)} \cap \partial^{\mathrm{e}} W_+^s \cap \partial^{\mathrm{e}} W_-^s = \{0 < \alpha_\mu < \pi\}^{(1)} \cap \partial^{\mathrm{e}} W_-^s \\ &= \{0 < \alpha_\mu < \pi\}^{(1)} \cap \partial^{\mathrm{e}} W_+^s = \{0 < \alpha_\mu < \pi\}^{(1)} \cap \partial^{\mathrm{e}}\{(d_{u_0})_j > s\}. \end{aligned}$$

Thanks to last equality and (5.26)

$$\mathcal{H}^{n-1}\Big(\{0 < \alpha_\mu < \pi\}^{(1)} \cap \partial^{\mathrm{e}}\{(d_{u_0})_j > s\} \setminus \big(\{\alpha_\mu^\wedge = 0\} \cup \{\alpha_\mu^\vee = \pi\} \cup S_{\alpha_\mu} \cup K\big)\Big) > 0, \tag{5.27}$$

for every $s \in I$. Let us now consider equality (5.25) with W given by

$$W = \{0 < \alpha_\mu < \pi\}^{(1)} \setminus \big(\{\alpha_\mu^\wedge = 0\} \cup \{\alpha_\mu^\vee = \pi\} \cup S_{\alpha_\mu} \cup K\big)$$

(note that $W \subset \{0 < \alpha_\mu^\wedge \le \alpha_\mu^\vee < \pi\}$). We obtain

$$\begin{aligned} 0 &\overset{(5.27)}{<} \int_{-1}^{1} \mathcal{H}^{n-1}\Big(\{0 < \alpha_\mu < \pi\}^{(1)} \cap \partial^{\mathrm{e}}\{(d_{u_0})_j > s\} \setminus \big(\{\alpha_\mu^\wedge = 0\} \cup \{\alpha_\mu^\vee = \pi\} \cup S_{\alpha_\mu} \cup K\big)\Big)\, ds \\ &= \int_{W \cap S_{(d_{u_0})_j} \cap S_{\alpha_\mu}} [(d_{u_0})_j]\, d\mathcal{H}^{n-1} + |D^c (d_{u_0})_j|^+ (W \cap K) = 0, \end{aligned}$$

where we used the fact that with our choice of W we have $W \cap S_{\alpha_\mu} = W \cap K = \emptyset$ and that, thanks to Proposition 5.6, $|D^c (d_{u_0})_j|^+ \ll |D^c \alpha_\mu|$.

This gives a contradiction, and thus shows that $(d_{u_0})_j$ is constant. □

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