

Widely degenerate anisotropic diffusion: local boundedness and semicontinuity

Pasquale Ambrosio, Simone Ciani, Giovanni Cupini

October 20, 2026

Abstract

We investigate the regularity of local weak solutions to evolution equations of the form

$$\partial_t u = \sum_{i=1}^n \partial_{x_i} \left[a_i(x, t) (|\partial_{x_i} u| - \delta_i)_+^{p_i-1} \frac{\partial_{x_i} u}{|\partial_{x_i} u|} \right] \quad \text{in } \Omega_T = \Omega \times (0, T),$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 2$, the coefficients a_i are measurable and bounded, $p_i > 1$ and $\delta_i \geq 0$ are fixed parameters. Under suitable assumptions on the exponents p_i , we first show that the local boundedness of weak solutions follows from their membership in an appropriate non-homogeneous parabolic De Giorgi class. We then establish the existence of semicontinuous representatives for local weak sub(super)-solutions. Our analysis extends analogous results available for less degenerate operators and generalizes the local boundedness results obtained in [7] to fully anisotropic, widely degenerate parabolic PDEs with non-smooth coefficients depending additionally on the space-time variables (x, t) , whose growth is governed by a family of exponents p_i rather than by a single exponent.

Mathematics Subject Classification: 35B45, 35B65, 35K10, 35K65, 35K92.

Keywords: Degenerate parabolic equations; anisotropic equations; local boundedness; semicontinuity.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and let $T \in (0, \infty)$. We are concerned with local regularity properties of weak sub(super)-solutions to the following parabolic equation:

$$\partial_t u = \sum_{i=1}^n \partial_{x_i} \left[a_i(x, t) (|\partial_{x_i} u| - \delta_i)_+^{p_i-1} \frac{\partial_{x_i} u}{|\partial_{x_i} u|} \right] \quad \text{in } \Omega_T = \Omega \times (0, T), \quad (1.1)$$

where $p_1, \dots, p_n > 1$, $\delta_1, \dots, \delta_n$ are non-negative real numbers, and $(\cdot)_+$ stands for the positive part. Here a_i , $i \in \{1, \dots, n\}$, are measurable coefficients satisfying

$$\Lambda^{-1} \leq a_i(x, t) \leq \Lambda \quad \text{a.e. in } \Omega_T, \quad (1.2)$$

for some constant $\Lambda \geq 1$.

Evolution equations of a form similar to (1.1) have been studied since the 1960s, especially

by the Soviet school; see, for instance, the paper [74] by Vishik. The above equation, in the special case $p_i = p$, $\delta_i = 0$ and $a_i = 1$ for all $i \in \{1, \dots, n\}$, also appears explicitly in the monographs [61], [68, Example 4.A, Chap. III] and [76, Example 30.8], among others. More recently, equations of the type (1.1), with $p_i = p \geq 2$ and $a_i = 1$ for all $i \in \{1, \dots, n\}$, have been considered in the works [7] and [3]: in the former, we have established the local boundedness of local weak solutions, while in the latter we have proved that such solutions are locally Lipschitz continuous in the spatial variable (see also [23] for the case in which all δ_i are equal to zero). The equation considered in this paper combines, in a unified way, two of the most studied classes of nonlinear equations in recent years: the *widely degenerate equations* and the *anisotropic* ones.

Widely degenerate equations have attracted growing attention in the literature; see, for instance, [1, 4, 10, 13, 24, 27, 37, 38, 39, 57, 58, 64, 65, 66, 70] in the elliptic setting and [2, 5, 6, 8, 11, 14, 54, 71] in the parabolic context. In these latter works, the model equation under consideration is the following:

$$\partial_t u - \operatorname{div} \left((|Du| - \lambda)_+^{p-1} \frac{Du}{|Du|} \right) = f, \quad \text{with } \lambda > 0. \quad (1.3)$$

According to the now-standard terminology, this equation is widely degenerate, in the sense that the diffusion part is uniformly elliptic only outside a ball with radius λ , while it behaves asymptotically, that is, for large values of $|Du|$, like the parabolic p -Laplace operator. Therefore, equations of the form (1.3) fall within the class of *asymptotically regular parabolic problems* (for a comprehensive overview of this topic, see [2, 11, 14] and the references therein). As already pointed out in [5, 14], no more than Lipschitz regularity can be expected for solutions to equation (1.3). In fact, when $f = 0$, any time-independent λ -Lipschitz function solves (1.3), and even more, it is a solution of the associated stationary equation.

Returning to our setting, equation (1.1) exhibits a similar type of degeneracy as (1.3), since the ellipticity of the corresponding operator in divergence form breaks down on the set

$$\bigcup_{i=1}^n \{|\partial_{x_i} u| \leq \delta_i\}.$$

In particular, equation (1.1) displays a threshold-type behavior (possibly with different thresholds δ_i), with regions where the diffusion coefficients may vanish identically, thereby resulting in a loss of the parabolic structure on sets of possibly positive measure.

Anisotropic equations and functionals were introduced in the 1980s by Giaquinta [55] and Marcellini [63] and have since been extensively studied under many different aspects; see, e.g., [9, 12, 18, 19, 21, 29, 33, 35, 40, 41, 42, 43, 51, 52, 53, 59, 67, 69] in the elliptic setting and [15, 16, 17, 30, 31, 32, 34, 44, 72] in the parabolic framework. The term *anisotropic* refers to the fact that the diffusion is of power type with exponents that may vary with the spatial directions. While a fairly well-developed regularity theory is available for equations with differentiable coefficients (see, for instance, [19] and [50]), the situation is much less understood in the case of rough coefficients. In this latter setting, although the L^∞ -estimates and the Critical Mass Lemma, two key tools necessary to prove regularity (see (4.21), (4.34) and Lemma 5.2 later on), can still be adapted, the same does not apply for the so-called shrinking lemma (see [46, Lemma 7.2 Chap. III and Lemma 5.1 Chap. IV]). We recall that the shrinking lemma, originally introduced by De Giorgi [45], is another fundamental ingredient needed to demonstrate regularity. Roughly speaking, it ensures that, upon restriction to smaller domains, if we denote by μ the infimum of the solution, the measure of the set where the solution takes

values between μ and $\mu + \varepsilon$ tends to zero as $\varepsilon \rightarrow 0$ (see [48] for further details). A partial breakthrough in this direction was obtained by Liskevich and Skrypnik [62], who succeeded in proving regularity in a very special case by replacing the shrinking lemma with the positivity expansion method introduced in [47] in the parabolic framework. However, despite these recent advances, the regularity theory remains highly fragmented. For example, Harnack's estimates for the elliptic operators are currently available, to our knowledge, only for operators with constant coefficients and with exponents $p_i > 2$ satisfying a suitable parabolic condition (see [34] for more details).

As for widely degenerate anisotropic problems, while in the elliptic setting they have already been studied from several viewpoints (see, for instance, [20, 22, 25, 26, 28] and the references therein), in the parabolic context only a few regularity results are currently available (we refer again to [3, 7]), and many aspects still remain to be investigated.

Before describing the aims and structure of this paper, we also mention the work [36], where the authors establish several qualitative properties of weak solutions to doubly nonlinear anisotropic evolution equations, whose prototype is

$$\partial_t(|u|^{\alpha-1}u) - \sum_{i=1}^n \partial_{x_i}(|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u) = 0, \quad \text{with } \alpha > 0. \quad (1.4)$$

Among other things, they show that certain regularity properties, such as local boundedness and the existence of semicontinuous representatives for weak solutions, are actually embodied in suitable energy estimates rather than in the mere class of solutions to the equation. However, their framework does not cover our widely degenerate operator. This is because, from the energetic viewpoint, equation (1.1) exhibits features that are not captured by (1.4). Roughly speaking, equation (1.1) behaves as if a lower-order term on the right-hand side of (1.4) were capable of suppressing the diffusion when the relevant energy is below a certain threshold; see, for instance, the bound for the energies $\mathcal{E}_j$ in the proof of Lemma 5.2 below.

In this paper, we not only extend the aforementioned regularity properties to local weak sub(super)-solutions of (1.1), but we also generalize the local boundedness results obtained in [7] to fully anisotropic and widely degenerate operators, with non-smooth coefficients depending also on the space-time variables (x, t) , and whose growth is prescribed by a whole family of exponents $p_i > 1$, $i \in \{1, \dots, n\}$, rather than by a single exponent $p \geq 2$.

1.1 Plan of the paper

Concerning local weak sub(super)-solutions to (1.1), we carry out a detailed analysis of

1. the weak formulation of the notion of sub(super)-solution via Steklov averages;
2. energy estimates for sub(super)-solutions;
3. local boundedness and the existence of semicontinuous representatives (for the whole energy classes defined at the end of Section 3).

In what follows, we introduce each one of these aspects. After presenting the definition of weak sub(super)-solution, in Subsection 2.1 we give a weak formulation of sub(super)-solution based on the usual Steklov averaging technique. This is then employed to show that local weak sub(super)-solutions to (1.1) are elements of special energy classes.

Energy classes

In this paper, we follow an approach dating back to De Giorgi (see [45]) and subsequently widely developed in the literature: we show that certain regularity properties are, in general, encoded in suitable energy estimates rather than in the mere class of local weak sub(super)-solutions to (1.1). More precisely, here we define two classes of functions $\mathcal{DG}^+(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$ and $\mathcal{DG}^-(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$ (see the end of Section 3), which we believe to be of fundamental importance in the study of the local behavior of local weak sub(super)-solutions. The inclusion of the latter in the two aforementioned classes, explicitly observed in Remark 3.5 below, allows us to state results of broader scope and application within the regularity theory for widely degenerate and anisotropic operators. Indeed, the results obtained here can be formulated directly for elements of the (larger) classes $\mathcal{DG}^\pm(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$, rather than merely in terms of local weak sub(super)-solutions to (1.1).

Local boundedness and semicontinuity

Regarding the boundedness of local weak sub(super)-solutions to (1.1), we are interested in the parameter range

$$1 < p_i < \bar{p} \left(1 + \frac{2}{n}\right), \quad \bar{p} < n, \quad (1.5)$$

where

$$\bar{p} := \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} \right)^{-1}.$$

More generally, we prove that, under certain conditions involving the exponents p_i , the functions in the energy class $\mathcal{DG}^+(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$ (resp. $\mathcal{DG}^-(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$) are locally essentially bounded from above (resp. below) in Ω_T . Hence, by the inclusion of local weak solutions of (1.1) in the classes $\mathcal{DG}^\pm(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$, we obtain the local boundedness for weak solutions.

In our proofs, we need to distinguish between the cases $\bar{p} > \frac{2n}{n+2}$ and $\bar{p} \leq \frac{2n}{n+2}$. In the latter case, we require the extra integrability condition

$$u \in L_{loc}^m(\Omega_T), \quad \text{for some } m > \frac{n}{\bar{p}}(2 - \bar{p}), \quad (1.6)$$

similarly to subcritical p -Laplacian equations (see, for instance, [46]). We note that the two ranges for $\bar{p}$ are identical to those appearing in [75], and also the extra integrability condition (1.6) coincides with the integrability assumption used in [75].

Furthermore, we show that local weak sub(super)-solutions to (1.1) admit an upper (lower) semicontinuous representative in Ω_T . In fact, the same result holds more generally for elements of the classes $\mathcal{DG}^+(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$ and $\mathcal{DG}^-(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$, respectively, under appropriate assumptions on the exponents p_i , and additionally assuming that condition (1.6) holds in the case $\bar{p} \leq 2n/(n+2)$. To prove this, we adapt to our anisotropic setting the very general method developed in [60]. There, it is shown that the existence of lower or upper semicontinuous representatives is a consequence of some kind of measure-theoretical maximum principle, referred to in the literature as a Critical Mass Lemma, or a De Giorgi-type Lemma. In establishing this result, we follow the approach of [30]; however, the novelty here lies in the consideration of an alternative formulation embodying the non-homogeneity of the energy classes $\mathcal{DG}^\pm$. Therefore, differently from [30], the Critical Mass Lemma proved here holds under an additional condition involving the exponents p_i , the degeneracy thresholds δ_i , and the parameters M, ρ defining the

intrinsic cylinders used in the proof (see Lemma 5.2 below).

1.2 Structure of the paper

The paper is organized as follows. Section 2 introduces the notion of local weak sub(super)-solution and the associated function spaces, and collects the necessary preliminary material, including standard notation, the main embedding results and Steklov averages. In Section 3 we derive the main energy estimates and, accordingly, we define the functional classes $\mathcal{DG}^+(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathcal{C})$ and $\mathcal{DG}^-(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathcal{C})$. Next, in Section 4, we study the local boundedness from above or below of functions belonging to the classes $\mathcal{DG}^\pm(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathcal{C})$. Lastly, in Section 5 we establish a Critical Mass Lemma, which is then used to prove the existence of semicontinuous representatives for local weak sub(super)-solutions to (1.1). The final Appendix provides details on the construction of suitable cut-off functions employed in our proofs.

2 Notation and preliminaries

In this paper we shall denote by C or c a general positive constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. The norm we use on $\mathbb{R}^k$, $k \in \mathbb{N}$, will be the standard Euclidean one and it will be denoted by $|\cdot|$. In particular, for the vectors $\xi, \eta \in \mathbb{R}^k$, we write $\langle \xi, \eta \rangle$ for the usual inner product and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm.

For $\rho > 0$ and $x_0 \in \mathbb{R}^n$, we define the following n -dimensional hyper-rectangles:

$$K_\rho := \prod_{i=1}^n (-\rho^{\frac{1}{p_i}}, \rho^{\frac{1}{p_i}}),$$

$$K_\rho(x_0) := \prod_{i=1}^n (x_{0,i} - \rho^{\frac{1}{p_i}}, x_{0,i} + \rho^{\frac{1}{p_i}}).$$

Moreover, for $\rho > 0$ we define the space-time cylinder

$$Q_\rho := K_\rho \times (-\rho, 0),$$

and if $(x_0, t_0) \in \mathbb{R}^{n+1}$, we let $Q_\rho(x_0, t_0)$ denote the cylinder with vertex at (x_0, t_0) congruent to Q_ρ , i.e.,

$$Q_\rho(x_0, t_0) := K_\rho(x_0) \times (t_0 - \rho, t_0).$$

For a general cylinder $Q = B \times (t_0, t_1)$, where $B \subset \mathbb{R}^n$ and $t_0 < t_1$, we denote by

$$\partial_{\text{par}} Q := (\overline{B} \times \{t_0\}) \cup (\partial B \times (t_0, t_1))$$

the usual *parabolic boundary* of Q , which is nothing but its standard topological boundary without the upper cap $\overline{B} \times \{t_1\}$.

If $E \subseteq \mathbb{R}^k$ is a Lebesgue-measurable set, then we will denote by $|E|$ its k -dimensional Lebesgue measure. When $0 < |E| < \infty$, the mean value of a function $v \in L^1(E)$ is defined by

$$\oint_E v(y) dy := \frac{1}{|E|} \int_E v(y) dy.$$

Given a vector of real numbers $\mathbf{p} = (p_1, \dots, p_n)$ with $p_i > 1$ for every $i \in \{1, \dots, n\}$, we define the following spaces

$$W_0^{1,\mathbf{p}}(\Omega) := \{v \in W_0^{1,1}(\Omega) : \partial_{x_i} v \in L^{p_i}(\Omega), \text{ for all } i = 1, \dots, n\},$$

$$W_{loc}^{1,\mathbf{p}}(\Omega) := \{v \in W_{loc}^{1,1}(\Omega) : \partial_{x_i} v \in L_{loc}^{p_i}(\Omega), \text{ for all } i = 1, \dots, n\},$$

$$L^{\mathbf{p}}(0, T; W_0^{1,\mathbf{p}}(\Omega)) := \{v \in L^1(0, T; W_0^{1,1}(\Omega)) : \partial_{x_i} v \in L^{p_i}(\Omega_T), \text{ for all } i = 1, \dots, n\},$$

$$L_{loc}^{\mathbf{p}}(0, T; W_0^{1,\mathbf{p}}(\Omega)) := \{v \in L_{loc}^1(0, T; W_0^{1,1}(\Omega)) : \partial_{x_i} v \in L_{loc}^{p_i}(0, T; L_{loc}^{p_i}(\Omega)), \text{ for all } i = 1, \dots, n\},$$

$$L_{loc}^{\mathbf{p}}(0, T; W_{loc}^{1,\mathbf{p}}(\Omega)) := \{v \in L_{loc}^1(0, T; W_{loc}^{1,1}(\Omega)) : \partial_{x_i} v \in L_{loc}^{p_i}(\Omega_T), \text{ for all } i = 1, \dots, n\}.$$

Throughout this paper, we let

$$\mathcal{P} := \max\{p_1, \dots, p_n\} \quad \text{and} \quad P := \max\{2, \mathcal{P}\}.$$

Now let $F : \Omega_T \times \mathbb{R}^n \rightarrow \mathbb{R}$ be the function defined by

$$F(x, t, \xi) := \sum_{i=1}^n \frac{a_i(x, t)}{p_i} (|\xi_i| - \delta_i)_+^{p_i}. \quad (2.1)$$

In this work, we define a local weak sub(super)-solution of (1.1) as follows.

Definition 2.1. A function $u \in C_{loc}^0(0, T; L_{loc}^2(\Omega)) \cap L_{loc}^{\mathbf{p}}(0, T; W_{loc}^{1,\mathbf{p}}(\Omega)) \cap L_{loc}^P(\Omega_T)$ is a *local weak sub(super)-solution* of (1.1) in Ω_T if

$$\iint_{\Omega_T} (u \partial_t \varphi - \langle D_\xi F(x, t, Du), D\varphi \rangle) dx dt \geq (\leq) 0 \quad (2.2)$$

for all test functions $\varphi \in C_0^\infty(\Omega_T)$, $\varphi \geq 0$.

Definition 2.2. A function $u \in C_{loc}^0(0, T; L_{loc}^2(\Omega)) \cap L_{loc}^{\mathbf{p}}(0, T; W_{loc}^{1,\mathbf{p}}(\Omega)) \cap L_{loc}^P(\Omega_T)$ is a *local weak solution* of (1.1) in Ω_T if

$$\iint_{\Omega_T} (u \partial_t \varphi - \langle D_\xi F(x, t, Du), D\varphi \rangle) dx dt = 0$$

for all test functions $\varphi \in C_0^\infty(\Omega_T)$.

Given a real number $\ell \geq 1$, ℓ^* is its Sobolev exponent, i.e.

$$\ell^* := \begin{cases} \frac{n\ell}{n-\ell} & \text{if } \ell < n, \\ \text{any value in } (\ell, \infty) & \text{if } \ell \geq n, \end{cases} \quad (2.3)$$

and ℓ' is the conjugate exponent of ℓ , i.e. $\ell' = \frac{\ell}{\ell-1}$ if $\ell > 1$, while $\ell' = \infty$ if $\ell = 1$. Moreover, we denote by $\bar{p}$ the harmonic average of $\mathbf{p} = (p_1, \dots, p_n)$, i.e.

$$\bar{p} := \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} \right)^{-1}.$$

Next, we recall some useful anisotropic Sobolev embeddings.

Lemma 2.3 (Sobolev-Troisi embedding, [49]). *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a rectangular domain, $\bar{p} < n$ and $\alpha_i > 0$, $i \in \{1, \dots, n\}$. If we define*

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n), \quad \tilde{\alpha} = \sum_{i=1}^n \alpha_i, \quad p_{\alpha}^* = \bar{p}^* \frac{\tilde{\alpha}}{n},$$

then there exists a constant $C = C(n, \mathbf{p}, \boldsymbol{\alpha}) > 0$ such that

$$\|v\|_{L^{p_{\alpha}^*}(\mathcal{K})} \leq C \prod_{i=1}^n \|\partial_{x_i} |v|^{\alpha_i}\|_{L^{p_i}(\mathcal{K})}^{\frac{1}{\tilde{\alpha}}}$$

for all $v \in W_0^{1,\mathbf{p}}(\mathcal{K})$.

As a corollary, one gets the usual Troisi inequality (see [73]).

Remark 2.4. Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a rectangular domain. Then there exists a constant $C = C(n, \mathbf{p}) > 0$ such that

$$\|v\|_{L^{\bar{p}^*}(\mathcal{K})} \leq C \prod_{i=1}^n \|\partial_{x_i} v\|_{L^{p_i}(\mathcal{K})}^{\frac{1}{n}} \quad (2.4)$$

for all $v \in W_0^{1,\mathbf{p}}(\mathcal{K})$. Moreover, taking both sides of (2.4) to the exponent $\bar{p}$ and using Young's inequality on the right-hand side with exponents $\frac{n p_i}{\bar{p}}$, whose inverses add up to 1 due to the definition of $\bar{p}$, we obtain the following useful inequality:

$$\left(\int_{\mathcal{K}} |v|^{\bar{p}^*} dx \right)^{\frac{\bar{p}}{\bar{p}^*}} \leq C \sum_{i=1}^n \int_{\mathcal{K}} |\partial_{x_i} v|^{p_i} dx, \quad v \in W_0^{1,\mathbf{p}}(\mathcal{K}). \quad (2.5)$$

The next lemma plays a crucial role in the De Giorgi-type iterations; we refer to [56, Lemma 7.1] for a proof.

Lemma 2.5. *Let $\mu > 0$ and let $\{Y_j\}_{j \in \mathbb{N}_0}$ be a sequence of non-negative real numbers, satisfying the recursive inequalities*

$$Y_{j+1} \leq C b^j Y_j^{1+\mu}$$

where $C > 0$ and $b > 1$. If $Y_0 \leq C^{-\frac{1}{\mu}} b^{-\frac{1}{\mu^2}}$, then

$$\lim_{j \rightarrow \infty} Y_j = 0.$$

2.1 Steklov averages

In this section, we recall the definition and some elementary properties of Steklov averages. Let us denote a domain in space-time by $Q' := \Omega' \times I$, where $\Omega' \subseteq \Omega$ is a bounded domain and $I := (t_0, t_1) \subseteq (0, T)$. For every $h \in (0, t_1 - t_0)$ and $v \in L^1(Q', \mathbb{R}^k)$, where $k \in \mathbb{N}$, the *Steklov average* $[v]_h(\cdot, t)$ is defined by

$$[v]_h(x, t) := \begin{cases} \frac{1}{h} \int_t^{t+h} v(x, s) ds & \text{if } t \in (t_0, t_1 - h], \\ 0 & \text{if } t \in (t_1 - h, t_1), \end{cases}$$

for $x \in \Omega'$. This definition implies, for almost every $(x, t) \in \Omega' \times (t_0, t_1 - h)$,

$$\frac{\partial[v]_h}{\partial t}(x, t) = \frac{v(x, t+h) - v(x, t)}{h}.$$

The proof of the following result is straightforward from the theory of Lebesgue spaces (see [46, Lemma 3.2, Chapter I]).

Lemma 2.6. *Let $q, r \geq 1$ and $v \in L^r(t_0, t_1; L^q(\Omega'))$. Then, as $h \rightarrow 0$, $[v]_h$ converges to v in $L^r(t_0, t_1 - \varepsilon; L^q(\Omega'))$ for every $\varepsilon \in (0, t_1 - t_0)$. If $v \in C^0(t_0, t_1; L^q(\Omega'))$, then as $h \rightarrow 0$, $[v]_h(\cdot, t)$ converges to $v(\cdot, t)$ in $L^q(\Omega')$ for every $t \in (t_0, t_1 - \varepsilon)$, $\forall \varepsilon \in (0, t_1 - t_0)$.*

A very useful formulation, equivalent to (2.2), is the one involving Steklov averages. Assume that $u \in C_{loc}^0(0, T; L_{loc}^2(\Omega)) \cap L_{loc}^{\mathbf{P}}(0, T; W_{loc}^{1, \mathbf{P}}(\Omega)) \cap L_{loc}^P(\Omega_T)$ is a local weak sub(super)-solution of (1.1) in Ω_T and let $h \in (0, T)$. Then, the Steklov average $[u]_h$ satisfies

$$\int_{\mathcal{K} \times \{\tau\}} \left(\frac{\partial[u]_h}{\partial t} \cdot \varphi + \langle [D_\xi F(x, t, Du)]_h, D\varphi \rangle \right) dx \leq (\geq) 0 \quad (2.6)$$

for every compact subset $\mathcal{K}$ of Ω , for all $\tau \in (0, T - h]$ and all test functions

$$\varphi \in C_{loc}^0(0, T; L^2(\mathcal{K})) \cap L_{loc}^{\mathbf{P}}(0, T; W_0^{1, \mathbf{P}}(\mathcal{K})), \quad \varphi \geq 0.$$

3 Energy classes

In this section, we establish some energy estimates that will be useful for proving our results. Throughout the sequel, $(x_0, t_0) \in \Omega_T$ and $\rho > 0$ are such that $Q_\rho(x_0, t_0) \Subset \Omega_T$, while $\zeta \in C^\infty(\mathbb{R}^{n+1})$ is a smooth cut-off function of the form

$$\zeta(x, t) := \psi(t) \prod_{i=1}^n \zeta_i^{p_i}(x_i), \quad (3.1)$$

where $\psi \in C^\infty(\mathbb{R}; [0, 1])$ is a non-decreasing map such that

$$\psi \equiv 0 \quad \text{on} \quad \left(-\infty, t_0 - \rho + \frac{\epsilon}{2}\right], \quad \psi \equiv 1 \quad \text{on} \quad [t_0 - \rho + \epsilon, \infty), \quad \text{with } \epsilon \in (0, \rho), \quad (3.2)$$

while the functions ζ_i satisfy

$$\zeta_i \in C_0^\infty(\mathbb{R}; [0, 1]), \quad \text{supp } \zeta_i \subset (x_{0,i} - \rho^{\frac{1}{p_i}}, x_{0,i} + \rho^{\frac{1}{p_i}}) \quad \text{for every } i \in \{1, \dots, n\}. \quad (3.3)$$

As an immediate consequence of these conditions, we have that

$$\zeta \equiv 0 \quad \text{on the parabolic boundary of } Q_\rho(x_0, t_0).$$

Proposition 3.1. *Let $n \geq 2$ and $p_1, \dots, p_n > 1$. Assume that (1.2) holds and that u is a local weak subsolution of (1.1) in the sense of Definition 2.1. Then there exists a positive constant $\mathcal{C}$, depending only on Λ and $\max\{p_1, \dots, p_n\}$, such that for every $k \in \mathbb{R}$ we have*

$$\sup_{t_0 - \rho < \tau < t_0} \int_{K_\rho(x_0)} (u - k)_+^2 \zeta(x, \tau) dx + \sum_{i=1}^n \iint_{Q_\rho(x_0, t_0)} (|\partial_{x_i} u| - \delta_i)_+^{p_i} \zeta \mathbb{1}_{\{u > k\}} dx dt$$

$$\leq \mathfrak{C} \iint_{Q_\rho(x_0, t_0)} (u - k)_+^2 \partial_t \zeta \, dx \, dt + \mathfrak{C} \sum_{i=1}^n \iint_{Q_\rho(x_0, t_0)} (u - k)_+^{p_i} |\partial_{x_i} \zeta^{\frac{1}{p_i}}|^{p_i} \, dx \, dt. \quad (3.4)$$

Similarly, if u is a local weak supersolution of (1.1), then for every $k \in \mathbb{R}$ we have

$$\begin{aligned} & \sup_{t_0 - \rho < \tau < t_0} \int_{K_\rho(x_0)} (u - k)_-^2 \zeta(x, \tau) \, dx + \sum_{i=1}^n \iint_{Q_\rho(x_0, t_0)} (|\partial_{x_i} u| - \delta_i)_+^{p_i} \zeta \mathbb{1}_{\{u < k\}} \, dx \, dt \\ & \leq \mathfrak{C} \iint_{Q_\rho(x_0, t_0)} (u - k)_-^2 \partial_t \zeta \, dx \, dt + \mathfrak{C} \sum_{i=1}^n \iint_{Q_\rho(x_0, t_0)} (u - k)_-^{p_i} |\partial_{x_i} \zeta^{\frac{1}{p_i}}|^{p_i} \, dx \, dt. \end{aligned} \quad (3.5)$$

Proof. Let u be a local weak subsolution of (1.1). After a translation, we may assume that $(x_0, t_0) = (0, 0)$. Hence, it suffices to prove (3.4) for the cylinder Q_ρ . In (2.6) we take the test functions

$$\varphi = ([u]_h - k)_+ \zeta$$

and integrate with respect to time over $(-\rho, \tau)$, with $\tau \in (-\rho, 0)$. We thus obtain

$$\int_{-\rho}^{\tau} \int_{K_\rho} \frac{\partial [u]_h}{\partial t} ([u]_h - k)_+ \zeta \, dx \, dt + \iint_{Q^\tau} \langle [A(x, t, Du)]_h, D[(u - k)_+ \zeta] \rangle \, dx \, dt \leq 0, \quad (3.6)$$

where, for convenience of notation, we set

$$Q^\tau := K_\rho \times (-\rho, \tau) \quad \text{and} \quad A(x, t, \eta) := D_\xi F(x, t, \eta), \quad \eta \in \mathbb{R}^n,$$

with $F : \Omega_T \times \mathbb{R}^n \rightarrow \mathbb{R}$ denoting the function defined in (2.1). The first term in (3.6) can be rewritten as

$$\int_{-\rho}^{\tau} \int_{K_\rho} \frac{\partial [u]_h}{\partial t} ([u]_h - k)_+ \zeta \, dx \, dt = \frac{1}{2} \int_{-\rho}^{\tau} \int_{K_\rho} \frac{\partial ([u]_h - k)_+^2}{\partial t} \zeta \, dx \, dt.$$

Therefore, integrating by parts, using that $\zeta \equiv 0$ on $\partial_{\text{par}} Q_\rho$ and letting $h \rightarrow 0$, by Lemma 2.6 we have

$$\begin{aligned} & \int_{-\rho}^{\tau} \int_{K_\rho} \frac{\partial [u]_h}{\partial t} ([u]_h - k)_+ \zeta \, dx \, dt \longrightarrow \\ & \frac{1}{2} \int_{K_\rho} (u - k)_+^2 \zeta(x, \tau) \, dx - \frac{1}{2} \iint_{Q^\tau} (u - k)_+^2 \partial_t \zeta \, dx \, dt. \end{aligned} \quad (3.7)$$

Now observe that $|A_i(x, t, Du)| \leq \Lambda |\partial_{x_i} u|^{p_i-1}$ for any $i \in \{1, \dots, n\}$. Then, taking the limit as $h \rightarrow 0$ in the second term of (3.6), we can apply Lemma 2.6 again. Thus we get

$$\begin{aligned} & \iint_{Q^\tau} \langle [A(x, t, Du)]_h, D[(u - k)_+ \zeta] \rangle \, dx \, dt \longrightarrow \\ & \iint_{Q^\tau} \langle A(x, t, Du), D(u - k)_+ \rangle \zeta \, dx \, dt + \iint_{Q^\tau} \langle A(x, t, Du), D\zeta \rangle (u - k)_+ \, dx \, dt. \end{aligned} \quad (3.8)$$

From (3.6)–(3.8), we then obtain

$$\frac{1}{2} \int_{K_\rho} (u - k)_+^2 \zeta(x, \tau) \, dx + \iint_{Q^\tau} \langle A(x, t, Du), D(u - k)_+ \rangle \zeta \, dx \, dt$$

$$\leq \frac{1}{2} \iint_{Q^\tau} (u - k)_+^2 \partial_t \zeta \, dx \, dt - \iint_{Q^\tau} \langle A(x, t, Du), D\zeta \rangle (u - k)_+ \, dx \, dt. \quad (3.9)$$

We now estimate

$$\begin{aligned} \iint_{Q^\tau} \langle A(x, t, Du), D(u - k)_+ \rangle \zeta \, dx \, dt &= \sum_{i=1}^n \iint_{Q^\tau \cap \{u > k\}} a_i(x, t) (|\partial_{x_i} u| - \delta_i)_+^{p_i-1} |\partial_{x_i} u| \zeta \, dx \, dt \\ &\geq \Lambda^{-1} \sum_{i=1}^n \iint_{Q^\tau \cap \{u > k\}} (|\partial_{x_i} u| - \delta_i)_+^{p_i} \zeta \, dx \, dt. \end{aligned}$$

To deal with the last term in (3.9), we set, for each $i \in \{1, \dots, n\}$,

$$\hat{\zeta}_i := \prod_{j \neq i} \zeta_j^{p_j}(x_j).$$

Using the definition of $\hat{\zeta}_i$, we get

$$\psi \hat{\zeta}_i \zeta_i^{p_i} = \zeta \quad \text{and} \quad \psi \hat{\zeta}_i |\partial_{x_i} \zeta_i|^{p_i} = |\partial_{x_i} \zeta^{\frac{1}{p_i}}|^{p_i}.$$

Therefore, applying Young's inequality with $\varepsilon_i > 0$, we have

$$\begin{aligned} & - \iint_{Q^\tau} \langle A(x, t, Du), D\zeta \rangle (u - k)_+ \, dx \, dt \\ & \leq \sum_{i=1}^n p_i \iint_{Q^\tau} a_i(x, t) (|\partial_{x_i} u| - \delta_i)_+^{p_i-1} \psi \hat{\zeta}_i \zeta_i^{p_i-1} |\partial_{x_i} \zeta_i| (u - k)_+ \, dx \, dt \\ & \leq \Lambda \sum_{i=1}^n \varepsilon_i (p_i - 1) \iint_{Q^\tau \cap \{u > k\}} (|\partial_{x_i} u| - \delta_i)_+^{p_i} \zeta \, dx \, dt + \Lambda \sum_{i=1}^n \frac{1}{\varepsilon_i^{p_i-1}} \iint_{Q^\tau} (u - k)_+^{p_i} |\partial_{x_i} \zeta^{\frac{1}{p_i}}|^{p_i} \, dx \, dt. \end{aligned}$$

Choosing $\varepsilon_i = [2(p_i - 1)\Lambda^2]^{-1}$ for every $i \in \{1, \dots, n\}$, and collecting the three previous estimates, we obtain

$$\begin{aligned} & \int_{K_\rho} (u - k)_+^2 \zeta(x, \tau) \, dx + \sum_{i=1}^n \iint_{Q^\tau \cap \{u > k\}} (|\partial_{x_i} u| - \delta_i)_+^{p_i} \zeta \, dx \, dt \\ & \leq \mathcal{C} \iint_{Q^\tau} (u - k)_+^2 \partial_t \zeta \, dx \, dt + \mathcal{C} \sum_{i=1}^n \iint_{Q^\tau} (u - k)_+^{p_i} |\partial_{x_i} \zeta^{\frac{1}{p_i}}|^{p_i} \, dx \, dt, \end{aligned}$$

where $\mathcal{C}$ is a positive constant depending only on Λ and $\max\{p_1, \dots, p_n\}$. Recalling that $\tau \in (-\rho, 0)$ is arbitrary, from the above inequality we get

$$\begin{aligned} & \sup_{-\rho < \tau < 0} \int_{K_\rho} (u - k)_+^2 \zeta(x, \tau) \, dx + \sum_{i=1}^n \iint_{Q_\rho} (|\partial_{x_i} u| - \delta_i)_+^{p_i} \zeta \mathbb{1}_{\{u > k\}} \, dx \, dt \\ & \leq \mathcal{C} \iint_{Q_\rho} (u - k)_+^2 \partial_t \zeta \, dx \, dt + \mathcal{C} \sum_{i=1}^n \iint_{Q_\rho} (u - k)_+^{p_i} |\partial_{x_i} \zeta^{\frac{1}{p_i}}|^{p_i} \, dx \, dt. \end{aligned}$$

We have thus proved estimate (3.4).

Now observe that if u is a local weak supersolution of (1.1), then $-u$ is a local weak subsolution. Hence, applying (3.4) with $-u$ and $-k$ in place of u and k , respectively, and using the identity $(-v)_+ = v_-$, we immediately obtain inequality (3.5). $\square$

Definition 3.2 (De Giorgi classes $\mathcal{DG}^+(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$). A measurable function $u : \Omega_T \rightarrow \mathbb{R}$ belongs to the class $\mathcal{DG}^+(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$ if

$$u \in C_{loc}^0(0, T; L_{loc}^2(\Omega)) \cap L_{loc}^{\mathbf{p}}(0, T; W_{loc}^{1, \mathbf{p}}(\Omega)) \cap L_{loc}^P(\Omega_T)$$

and for all $k \in \mathbb{R}$ and all ζ as above, the function u satisfies inequality (3.4).

Definition 3.3 (De Giorgi classes $\mathcal{DG}^-(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$). A measurable function $u : \Omega_T \rightarrow \mathbb{R}$ belongs to the class $\mathcal{DG}^-(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$ if

$$u \in C_{loc}^0(0, T; L_{loc}^2(\Omega)) \cap L_{loc}^{\mathbf{p}}(0, T; W_{loc}^{1, \mathbf{p}}(\Omega)) \cap L_{loc}^P(\Omega_T)$$

and for all $k \in \mathbb{R}$ and all ζ as above, the function u satisfies inequality (3.5).

Definition 3.4 (De Giorgi classes $\mathcal{DG}(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$). A measurable function $u : \Omega_T \rightarrow \mathbb{R}$ belongs to the class $\mathcal{DG}(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$ if

$$u \in C_{loc}^0(0, T; L_{loc}^2(\Omega)) \cap L_{loc}^{\mathbf{p}}(0, T; W_{loc}^{1, \mathbf{p}}(\Omega)) \cap L_{loc}^P(\Omega_T)$$

and for all $k \in \mathbb{R}$ and all ζ as above, the function u satisfies both inequalities (3.4) and (3.5).

Remark 3.5. By Proposition 3.1, under assumption (1.2), local weak subsolutions (resp. supersolutions) to (1.1) in the sense of Definition 2.1 belong to $\mathcal{DG}^+(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$ (resp. $\mathcal{DG}^-(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$). In particular, local weak solutions to (1.1) in the sense of Definition 2.2 belong to $\mathcal{DG}(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$.

Starting from the above definitions, in the following sections we prove local boundedness and the existence of semicontinuous representatives for functions in $\mathcal{DG}^\pm(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$, under suitable assumptions on the exponents p_i appearing in (1.1).

4 Local boundedness

In this section we prove the local boundedness of local weak solutions to equation (1.1). Our proofs are based on De Giorgi-type iterations combining the energy estimates established in Proposition 3.1 with the Sobolev embedding from Remark 2.4. Throughout the section, we will use the space-time cylinders defined in Section 2, which turn out to be convenient in our setting.

As mentioned in the introduction, there are two ranges for $\bar{p}$ that require somewhat different arguments, and in the range corresponding to small values of $\bar{p}$ we also require some extra integrability of the local weak solutions. For clarity, the two cases have been treated in separate subsections.

4.1 The case $\bar{p} > \frac{2n}{n+2}$

In this section we focus on the case in which $\bar{p}$ satisfies the following lower bound:

$$\bar{p} > \frac{2n}{n+2}. \quad (4.1)$$

The proof of local boundedness in Theorem 4.1 below is somewhat different in the cases $\mathcal{P} \geq 2$ and $\mathcal{P} < 2$, where $\mathcal{P} := \max\{p_1, \dots, p_n\}$. In fact, in the former case the condition (4.1) is not explicitly used in the argument. However, in this particular case, (4.1) must necessarily be true due to the upper bound for the parameters p_i in (1.5).

Theorem 4.1. *Let $u \in \mathcal{DG}^+(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathcal{C})$, where $\mathbf{p} = (p_1, \dots, p_n)$ satisfies (1.5), and assume that (4.1) holds. Then u is locally bounded from above and, for every cylinder $Q_\rho(x_0, t_0) \Subset \Omega_T$ and every $\sigma \in (0, 1)$, we have the explicit bound*

$$\operatorname{ess\,sup}_{Q_{\sigma\rho}(x_0, t_0)} u \leq \max \left\{ 1, \rho^{1/P}, C \left[(1 - \sigma)^{-\frac{\mathcal{P}(n+\bar{p})}{\bar{p}}} \iint_{Q_\rho(x_0, t_0)} u_+^P dx dt \right]^{\frac{\bar{p}}{\bar{p}(n+2) - nP}} \right\}, \quad (4.2)$$

where $\mathcal{P} := \max \{p_1, \dots, p_n\}$, $P := \max \{2, \mathcal{P}\}$ and C is a positive constant depending only on n , Λ , $\mathbf{p}$ and $\max \{\delta_1, \dots, \delta_n\}$. Similarly, if $u \in \mathcal{DG}^-(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathcal{C})$ under assumptions (1.5) and (4.1), then u is locally bounded from below and, for every $Q_\rho(x_0, t_0) \Subset \Omega_T$ and every $\sigma \in (0, 1)$, we have

$$\operatorname{ess\,inf}_{Q_{\sigma\rho}(x_0, t_0)} u \geq - \max \left\{ 1, \rho^{1/P}, C \left[(1 - \sigma)^{-\frac{\mathcal{P}(n+\bar{p})}{\bar{p}}} \iint_{Q_\rho(x_0, t_0)} u_-^P dx dt \right]^{\frac{\bar{p}}{\bar{p}(n+2) - nP}} \right\}. \quad (4.3)$$

Remark 4.2. In view of Remark 3.5 and Theorem 4.1, under assumptions (1.2), (1.5) and (4.1), local weak subsolutions to (1.1) are locally bounded from above in Ω_T , whereas local weak supersolutions are locally bounded from below.

Proof of Theorem 4.1. Let us first assume that $u \in \mathcal{DG}^+(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathcal{C})$. After a translation, we may suppose that (x_0, t_0) coincides with the origin. For a fixed $\sigma \in (0, 1)$, we consider the sequence

$$\rho_j := \sigma\rho + \frac{(1 - \sigma)}{2^j} \rho, \quad j \in \mathbb{N}_0, \quad (4.4)$$

and the corresponding cylinders $\mathcal{Q}_j := Q_{\rho_j}$. From the definitions it follows that $\mathcal{Q}_0 = Q_\rho$. We also consider the family of boxes

$$\tilde{\mathcal{Q}}_j := Q_{\tilde{\rho}_j},$$

where, for $j \in \mathbb{N}_0$,

$$\tilde{\rho}_j := \frac{\rho_j + \rho_{j+1}}{2} = \sigma\rho + \frac{3(1 - \sigma)}{2^{j+2}} \rho. \quad (4.5)$$

For these boxes, we have the inclusions

$$\mathcal{Q}_{j+1} \subset \tilde{\mathcal{Q}}_j \subset \mathcal{Q}_j, \quad j \in \mathbb{N}_0.$$

We now introduce the sequence of increasing levels

$$k_j := k - \frac{k}{2^j}, \quad (4.6)$$

where $k \geq 1$ is a number to be chosen later. To move forward, we choose cut-off functions $\tilde{\zeta}_{i,j} \in C_0^\infty((-\rho_j^{\frac{1}{p_i}}, \rho_j^{\frac{1}{p_i}}); [0, 1])$ such that

$$\tilde{\zeta}_{i,j} \equiv 1 \quad \text{on} \quad (-\rho_{j+1}^{\frac{1}{p_i}}, \rho_{j+1}^{\frac{1}{p_i}}) \quad \text{and} \quad |\tilde{\zeta}'_{i,j}|^{p_i} \leq \frac{c 2^{j\mathcal{P}}}{(1 - \sigma)^{\mathcal{P}} \rho},$$

where $c = c(\mathcal{P}) > 0$. Furthermore, we define

$$\eta_j(x) := \prod_{i=1}^n \tilde{\zeta}_{i,j}^{p_i}(x_i).$$

Similarly, we take $\psi_j \in C^\infty(\mathbb{R}; [0, 1])$ such that

$$\psi_j \equiv 0 \quad \text{on } (-\infty, -\tilde{\rho}_j], \quad \psi_j \equiv 1 \quad \text{on } [-\rho_{j+1}, \infty) \quad \text{and} \quad 0 \leq \psi'_j \leq \frac{c 2^j}{(1-\sigma)\rho}.$$

It follows that the smooth cut-off function

$$\tilde{\zeta}_j(x, t) := \psi_j(t) \eta_j(x)$$

satisfies the following properties:

$$\begin{cases} 0 \leq \tilde{\zeta}_j \leq 1, & \tilde{\zeta}_j \equiv 1 \quad \text{in } \mathcal{Q}_{j+1}, & \tilde{\zeta}_j \equiv 0 \quad \text{on } \partial_{\text{par}} \mathcal{Q}_j, \\ 0 \leq \partial_t \tilde{\zeta}_j \leq \frac{c 2^j}{(1-\sigma)\rho}, & |\partial_{x_i} \tilde{\zeta}_j^{\frac{1}{p_i}}|^{p_i} \leq \frac{c 2^{j\mathcal{P}}}{(1-\sigma)^{\mathcal{P}}\rho} \quad \text{for every } i \in \{1, \dots, n\}. \end{cases} \quad (4.7)$$

Details on the explicit construction of such a cut-off function are provided in the [Appendix](#). Now we rewrite estimate (3.4) with ρ_j , k_{j+1} and $\tilde{\zeta}_j$ in place of ρ , k and ζ , respectively. We thus obtain

$$\begin{aligned} & \sup_{-\rho_j < \tau < 0} \int_{K_{\rho_j}} (u - k_{j+1})_+^2 \tilde{\zeta}_j(x, \tau) dx + \sum_{i=1}^n \iint_{\mathcal{Q}_j \cap \{u > k_{j+1}\}} (|\partial_{x_i} u| - \delta_i)_+^{p_i} \tilde{\zeta}_j dx dt \\ & \leq \frac{C_0 2^j}{(1-\sigma)\rho} \iint_{\mathcal{Q}_j} (u - k_{j+1})_+^2 dx dt + \frac{C_0 2^{j\mathcal{P}}}{(1-\sigma)^{\mathcal{P}}\rho} \sum_{i=1}^n \iint_{\mathcal{Q}_j} (u - k_{j+1})_+^{p_i} dx dt, \end{aligned} \quad (4.8)$$

where C_0 is a positive constant depending only on Λ and $\mathcal{P}$. In addition, we set

$$q := \bar{p} \left(1 + \frac{2}{n}\right).$$

In the case $\mathcal{P} \geq 2$, the upper bound of the exponents p_i in (1.5) ensures that $\frac{q}{P} > 1$, where $P := \max\{2, \mathcal{P}\}$. In the case $\mathcal{P} < 2$, we can use the lower bound (4.1) to deduce that $\frac{q}{P} > 1$.

At this point of the proof, we define the sequence

$$Y_j := \iint_{\mathcal{Q}_j} (u - k_j)_+^P dx dt, \quad j \in \mathbb{N}_0. \quad (4.9)$$

Note that Y_j is finite for every $j \in \mathbb{N}_0$, since $u \in L_{loc}^P(\Omega_T)$. We may use Hölder's inequality to estimate

$$Y_{j+1} \leq \left[\iint_{\mathcal{Q}_{j+1}} (u - k_{j+1})_+^q dx dt \right]^{\frac{P}{q}} |A_{j+1}|^{1 - \frac{P}{q}}, \quad (4.10)$$

where we have set

$$A_{j+1} := \{(x, t) \in \mathcal{Q}_j : u(x, t) > k_{j+1}\}. \quad (4.11)$$

In the following calculation, we estimate the integral in (4.10) by applying Hölder's inequality with $H := n/(n - \bar{p})$ in the integral over space. Then, we estimate one of the integrals in space by its supremum for all times in the interval $(-\rho_{j+1}, 0)$, and introduce the functions η_j and ψ_j while expanding the set of integration, which allows us to use (2.5). All in all, the calculation takes the form

$$\iint_{\mathcal{Q}_{j+1}} (u - k_{j+1})_+^q dx dt \leq \int_{-\rho_{j+1}}^0 \left[\int_{K_{\rho_{j+1}}} (u - k_{j+1})_+^{\frac{2\bar{p}}{n} H'} dx \right]^{\frac{1}{H'}} \left[\int_{K_{\rho_{j+1}}} (u - k_{j+1})_+^{\bar{p}H} dx \right]^{\frac{1}{H}} dt$$

$$\begin{aligned}
&= \int_{-\rho_{j+1}}^0 \left[\int_{K_{\rho_{j+1}}} (u - k_{j+1})_+^2 dx \right]^{\frac{\bar{p}}{n}} \left[\int_{K_{\rho_{j+1}}} (u - k_{j+1})_+^{\bar{p}^*} dx \right]^{\frac{\bar{p}}{\bar{p}^*}} dt \\
&\leq \left[\sup_{-\rho_{j+1} < \tau < 0} \int_{K_{\rho_{j+1}}} (u(x, \tau) - k_{j+1})_+^2 dx \right]^{\frac{\bar{p}}{n}} \int_{-\rho_{j+1}}^0 \left[\int_{K_{\rho_{j+1}}} (u - k_{j+1})_+^{\bar{p}^*} dx \right]^{\frac{\bar{p}}{\bar{p}^*}} dt \\
&\leq \left[\sup_{-\rho_j < \tau < 0} \int_{K_{\rho_j}} (u - k_{j+1})_+^2 \tilde{\zeta}_j(x, \tau) dx \right]^{\frac{\bar{p}}{n}} \int_{-\rho_j}^0 \psi_j(t) \left[\int_{K_{\rho_j}} [(u - k_{j+1})_+ \eta_j]^{\bar{p}^*} dx \right]^{\frac{\bar{p}}{\bar{p}^*}} dt \\
&\leq c_0 \left[\sup_{-\rho_j < \tau < 0} \int_{K_{\rho_j}} (u - k_{j+1})_+^2 \tilde{\zeta}_j(x, \tau) dx \right]^{\frac{\bar{p}}{n}} \left[\sum_{i=1}^n \iint_{Q_j} |\partial_{x_i} [(u - k_{j+1})_+ \eta_j]|^{p_i} \psi_j dx dt \right], \tag{4.12}
\end{aligned}$$

where $c_0 = c_0(n, \mathbf{p}) > 0$. Now we estimate the last term in (4.12). Setting

$$\delta := \max \{ \delta_i : i = 1, \dots, n \},$$

applying the definition of A_{j+1} in (4.11), and using the bounds for the functions η_j , ψ_j and their derivatives, we obtain

$$\begin{aligned}
&\sum_{i=1}^n \iint_{Q_j} |\partial_{x_i} [(u - k_{j+1})_+ \eta_j]|^{p_i} \psi_j dx dt = \sum_{i=1}^n \iint_{A_{j+1}} |(\partial_{x_i} u) \eta_j + (u - k_{j+1})_+ \partial_{x_i} \eta_j|^{p_i} \psi_j dx dt \\
&\leq c \left[\sum_{i=1}^n \iint_{A_{j+1}} |\partial_{x_i} u|^{p_i} \tilde{\zeta}_j dx dt + \sum_{i=1}^n \iint_{A_{j+1}} (u - k_{j+1})_+^{p_i} |\partial_{x_i} \eta_j|^{p_i} \psi_j dx dt \right] \\
&\leq c \left[\sum_{i=1}^n \iint_{A_{j+1}} [(|\partial_{x_i} u| - \delta_i)_+ + \delta_i]^{p_i} \tilde{\zeta}_j dx dt + \frac{2^{j\mathcal{P}}}{(1 - \sigma)^\mathcal{P} \rho} \sum_{i=1}^n \iint_{Q_j} (u - k_{j+1})_+^{p_i} dx dt \right] \\
&\leq c \left[\sum_{i=1}^n \iint_{A_{j+1}} (|\partial_{x_i} u| - \delta_i)_+^{p_i} \tilde{\zeta}_j dx dt + n(\delta + 1)^\mathcal{P} |A_{j+1}| + \frac{2^{j\mathcal{P}}}{(1 - \sigma)^\mathcal{P} \rho} \sum_{i=1}^n \iint_{Q_j} (u - k_{j+1})_+^{p_i} dx dt \right], \tag{4.13}
\end{aligned}$$

where $c = c(\mathcal{P}) > 0$. Combining (4.12) and (4.13), majorizing the supremum in (4.12) and the final term in (4.13) by means of the energy estimate (4.8), and using that $\frac{1}{1-\sigma} > 1$, we ultimately get

$$\begin{aligned}
&\iint_{Q_{j+1}} (u - k_{j+1})_+^q dx dt \\
&\leq \frac{C_1 2^{j \frac{\mathcal{P}(n+\bar{p})}{n}}}{(1 - \sigma)^{\frac{\mathcal{P}(n+\bar{p})}{n}}} \left[\frac{1}{\rho} \iint_{Q_j} (u - k_{j+1})_+^2 dx dt + \frac{1}{\rho} \sum_{i=1}^n \iint_{Q_j} (u - k_{j+1})_+^{p_i} dx dt + |A_{j+1}| \right]^{\frac{n+\bar{p}}{n}}, \tag{4.14}
\end{aligned}$$

for a positive constant C_1 depending only on n , Λ , $\mathbf{p}$ and δ . Invoking again the definition of A_{j+1} , recalling that $k_j < k$, $k \geq 1$ and $p_i \leq \mathcal{P} \leq P$, and applying Young's inequality, we can handle the last sum as follows

$$\sum_{i=1}^n \iint_{Q_j} (u - k_{j+1})_+^{p_i} dx dt \leq \sum_{i=1}^n \iint_{A_{j+1}} u^{p_i} dx dt \leq \sum_{i=1}^n \iint_{A_{j+1}} [(u - k_j)_+ + k_j]^{p_i} dx dt$$

$$\begin{aligned}
&\leq 2^{\mathcal{P}-1} \left[\sum_{i=1}^n \iint_{A_{j+1}} (u - k_j)_+^{p_i} dx dt + n k^P |A_{j+1}| \right] \\
&\leq 2^{\mathcal{P}-1} n \left[\iint_{A_{j+1}} (u - k_j)_+^P dx dt + (1 + k^P) |A_{j+1}| \right] \\
&\leq 2^{\mathcal{P}} n [Y_j + k^P |A_{j+1}|], \tag{4.15}
\end{aligned}$$

where, in the last line, we have used the definition of Y_j in (4.9). Similarly, since $P \geq 2$, the first integral on the right-hand side of (4.14) can also be treated using Young's inequality:

$$\iint_{Q_j} (u - k_{j+1})_+^2 dx dt \leq \iint_{A_{j+1}} (u - k_{j+1})_+^P dx dt + |A_{j+1}| \leq Y_j + |A_{j+1}|,$$

where, in the last inequality, we have used the fact that $k_j < k_{j+1}$. Collecting the last three estimates, redefining the constant C_1 and recalling that $k \geq 1$, we obtain

$$\begin{aligned}
\iint_{Q_{j+1}} (u - k_{j+1})_+^q dx dt &\leq \frac{C_1 2^j \frac{\mathcal{P}(n+\bar{p})}{n}}{(1-\sigma)^{\frac{\mathcal{P}(n+\bar{p})}{n}}} \left[\frac{Y_j}{\rho} + \left(1 + \frac{1+k^P}{\rho}\right) |A_{j+1}| \right]^{\frac{n+\bar{p}}{n}} \\
&\leq \frac{C_1 2^j \frac{\mathcal{P}(n+\bar{p})}{n}}{(1-\sigma)^{\frac{\mathcal{P}(n+\bar{p})}{n}}} \left[\frac{Y_j}{\rho} + \left(1 + \frac{2k^P}{\rho}\right) |A_{j+1}| \right]^{\frac{n+\bar{p}}{n}}.
\end{aligned}$$

Without loss of generality, we can now assume that $k \geq \max\{1, \rho^{1/P}\}$. It then follows from the previous inequality that

$$\iint_{Q_{j+1}} (u - k_{j+1})_+^q dx dt \leq \frac{C_1 2^j \frac{\mathcal{P}(n+\bar{p})}{n}}{(1-\sigma)^{\frac{\mathcal{P}(n+\bar{p})}{n}} \rho^{\frac{n+\bar{p}}{n}}} (Y_j + k^P |A_{j+1}|)^{\frac{n+\bar{p}}{n}}, \tag{4.16}$$

up to a different constant $C_1 = C_1(n, \Lambda, \mathbf{p}, \delta) > 0$. At this point, we note that

$$|A_{j+1}| = \iint_{A_{j+1}} \frac{(k_{j+1} - k_j)^P}{(k_{j+1} - k_j)^P} dx dt \leq \iint_{A_{j+1}} \frac{(u - k_j)^P}{(k_{j+1} - k_j)^P} dx dt \leq \frac{2^{(j+1)P}}{k^P} Y_j. \tag{4.17}$$

Combining this estimate with (4.16) and using that $1 \leq \mathcal{P} \leq P$, we get

$$\begin{aligned}
\iint_{Q_{j+1}} (u - k_{j+1})_+^q dx dt &\leq \frac{C_1 2^j \frac{\mathcal{P}(n+\bar{p})}{n}}{(1-\sigma)^{\frac{\mathcal{P}(n+\bar{p})}{n}} \rho^{\frac{n+\bar{p}}{n}}} (1 + 2^{(j+1)P})^{\frac{n+\bar{p}}{n}} Y_j^{1+\frac{\bar{p}}{n}} \\
&\leq \frac{C_1 2^j \frac{\mathcal{P}(n+\bar{p})}{n} 2^{(j+2)\frac{P(n+\bar{p})}{n}}}{(1-\sigma)^{\frac{\mathcal{P}(n+\bar{p})}{n}} \rho^{\frac{n+\bar{p}}{n}}} Y_j^{1+\frac{\bar{p}}{n}} \\
&\leq \frac{C_1 4^j \frac{P(n+\bar{p})}{n}}{(1-\sigma)^{\frac{\mathcal{P}(n+\bar{p})}{n}} \rho^{\frac{n+\bar{p}}{n}}} Y_j^{1+\frac{\bar{p}}{n}}. \tag{4.18}
\end{aligned}$$

Putting together (4.10), (4.17) and (4.18), and recalling that $q := \bar{p}(1 + \frac{2}{n})$, we end up with

$$Y_{j+1} \leq \frac{C_2 4^j \frac{P^2(n+\bar{p})}{nq}}{(1-\sigma)^{\frac{\mathcal{P}(n+\bar{p})P}{nq}} \rho^{\frac{P(n+\bar{p})}{nq}}} \frac{2^{(j+1)P(1-\frac{P}{q})}}{k^{P-\frac{P^2}{q}}} Y_j^{1+\frac{P\bar{p}}{nq}}$$

$$\begin{aligned}
&\leq \frac{C_3 4^{jP} \left[\frac{P(n+\bar{p})}{nq} + 1 - \frac{P}{q} \right]}{(1-\sigma)^{\frac{\mathcal{P}(n+\bar{p})P}{nq}} \rho^{\frac{P(n+\bar{p})}{nq}}} k^{\frac{P^2}{q} - P} Y_j^{1 + \frac{P\bar{p}}{nq}} \\
&= \frac{C_3 b^j}{[(1-\sigma)^{\mathcal{P}} \rho]^{\frac{P(n+\bar{p})}{\bar{p}(n+2)}}} \mathcal{A}_k Y_j^{1 + \frac{P}{n+2}}, \tag{4.19}
\end{aligned}$$

where C_2 and C_3 are positive constants depending only n , Λ , $\mathbf{p}$ and δ ,

$$b := 4^{P + \frac{P^2}{n+2}},$$

$$\mathcal{A}_k := k^{\frac{P^2 n}{\bar{p}(n+2)} - P}.$$

We are thus in a position to apply the result in Lemma 2.5 with the choices

$$C = C_3 \mathcal{A}_k [(1-\sigma)^{\mathcal{P}} \rho]^{-\frac{P(n+\bar{p})}{\bar{p}(n+2)}}, \quad \mu = \frac{P}{n+2},$$

where C_3 is the constant from (4.19), provided that

$$Y_0 \leq C^{-\frac{1}{\mu}} b^{-\frac{1}{\mu^2}} = C_3 [(1-\sigma)^{\mathcal{P}} \rho]^{\frac{n+\bar{p}}{\bar{p}}} k^{n+2 - \frac{nP}{\bar{p}}}. \tag{4.20}$$

Since

$$Y_0 := \iint_{Q_\rho} u_+^P dx dt \quad \text{and} \quad \bar{p}(n+2) - nP = (q-P)n > 0,$$

and since $|Q_\rho| = 2^n \rho^{\frac{n+\bar{p}}{\bar{p}}}$, we see that (4.20) is satisfied provided that

$$k \geq C_4 \left[(1-\sigma)^{-\frac{\mathcal{P}(n+\bar{p})}{\bar{p}}} \iint_{Q_\rho} u_+^P dx dt \right]^{\frac{\bar{p}}{\bar{p}(n+2) - nP}} =: \mathcal{B},$$

for a positive constant C_4 depending only n , Λ , $\mathbf{p}$ and $\max\{\delta_1, \dots, \delta_n\}$. Therefore, choosing $k = \max\{1, \rho^{1/P}, \mathcal{B}\}$, by Lemma 2.5 we have

$$\iint_{Q_{\sigma\rho}} (u - k)_+^P dx dt \leq Y_j \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

This implies that the integrand above is zero almost everywhere in $Q_{\sigma\rho}$, and hence

$$\operatorname{ess\,sup}_{Q_{\sigma\rho}} u \leq \max \left\{ 1, \rho^{1/P}, C_4 \left[(1-\sigma)^{-\frac{\mathcal{P}(n+\bar{p})}{\bar{p}}} \iint_{Q_\rho} u_+^P dx dt \right]^{\frac{\bar{p}}{\bar{p}(n+2) - nP}} \right\}.$$

We have thus proved estimate (4.2).

Finally, if $u \in \mathcal{DG}^-(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathcal{C})$, the previous arguments, with minor modifications, yield the lower bound (4.3). We leave the details to the reader. $\square$

As an easy consequence of Definition 3.4, Remark 3.5 and Theorem 4.1, we obtain the following local boundedness result.

Corollary 4.3. *Let $\mathbf{p} = (p_1, \dots, p_n)$ satisfy (1.5). Assume that (4.1) holds and that $u \in \mathcal{DG}(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$. Then u is locally bounded and, for every cylinder $Q_\rho(x_0, t_0) \Subset \Omega_T$ and every $\sigma \in (0, 1)$, we have*

$$\operatorname{ess\,sup}_{Q_{\sigma\rho}(x_0, t_0)} |u| \leq \max \left\{ 1, \rho^{1/P}, C \left[(1 - \sigma)^{-\frac{\mathcal{P}(n+\bar{p})}{\bar{p}}} \iint_{Q_\rho(x_0, t_0)} |u|^P dx dt \right]^{\frac{\bar{p}}{\bar{p}(n+2) - nP}} \right\}, \quad (4.21)$$

where $\mathcal{P} := \max\{p_1, \dots, p_n\}$, $P := \max\{2, \mathcal{P}\}$ and C is a positive constant depending only on n , Λ , $\mathbf{p}$ and $\max\{\delta_1, \dots, \delta_n\}$. In particular, if u is a local weak solution of (1.1) under assumptions (1.2), (1.5) and (4.1), then u belongs to $L_{loc}^\infty(\Omega_T)$ and satisfies the estimate (4.21).

By iterating the estimate of the previous corollary, we can actually get a better bound for the essential supremum. We first note that, by the upper bound for the exponents p_i in (1.5) and inequality (4.1), we have

$$P > P - \frac{n}{\bar{p}} \left[\bar{p} \left(1 + \frac{2}{n} \right) - P \right] =: \omega(n, \mathbf{p}).$$

Moreover, from the definitions of $\omega(n, \mathbf{p})$ and P , we see that

$$\omega(n, \mathbf{p}) = P - 2 + n \left(\frac{P}{\bar{p}} - 1 \right) \geq 0.$$

Now we are in a position to prove the following result.

Theorem 4.4. *Under the assumptions of Corollary 4.3, for every cylinder $Q_{2\rho}(x_0, t_0) \Subset \Omega_T$ and every $\nu \in (\omega(n, \mathbf{p}), P]$ we have*

$$\operatorname{ess\,sup}_{Q_\rho(x_0, t_0)} |u| \leq C \left[\left(\iint_{Q_{2\rho}(x_0, t_0)} |u|^\nu dx dt \right)^{\frac{1}{\nu - \omega(n, \mathbf{p})}} + 1 + \rho^{1/P} \right], \quad (4.22)$$

for a positive constant C depending only on n , Λ , $\mathbf{p}$, ν and $\max\{\delta_1, \dots, \delta_n\}$.

Proof. Let $\nu \in (\omega(n, \mathbf{p}), P]$. Observe that, in the case $\nu = P$, estimate (4.22) is a straightforward consequence of Corollary 4.3. Indeed, by choosing $\sigma = \frac{1}{2}$ and replacing ρ with 2ρ in (4.21), one immediately obtains the bound (4.22). Therefore, from now on we assume that $\nu < P$. Let $Q_{2\rho}(x_0, t_0)$ be compactly contained in Ω_T and define

$$\rho_j := 2\rho - \frac{\rho}{2^j}, \quad Q_j := Q_{\rho_j}(x_0, t_0), \quad j \in \mathbb{N}_0.$$

Using (4.21) with $\sigma = \frac{\rho_j}{\rho_{j+1}}$ and noting that $\rho_{j+1} < 2\rho$, we obtain

$$\begin{aligned} M_j := \operatorname{ess\,sup}_{Q_j} |u| &\leq C \left(2^{j \frac{\mathcal{P}(n+\bar{p})}{\bar{p}}} \iint_{Q_{j+1}} |u|^P dx dt \right)^{\frac{\bar{p}}{\bar{p}(n+2) - nP}} + 1 + \rho_{j+1}^{1/P} \\ &\leq C M_{j+1}^{\frac{\bar{p}(P-\nu)}{\bar{p}(n+2) - nP}} \left(2^{j \frac{\mathcal{P}(n+\bar{p})}{\bar{p}}} \iint_{Q_{j+1}} |u|^\nu dx dt \right)^{\frac{\bar{p}}{\bar{p}(n+2) - nP}} + 1 + (2\rho)^{1/P}, \end{aligned}$$

where C is a positive constant depending only on n , Λ , $\mathbf{p}$ and $\max\{\delta_1, \dots, \delta_n\}$. Since $\omega(n, \mathbf{p}) < \nu < P$, we see that the exponent of M_{j+1} in the last expression belongs to the interval $(0, 1)$. Thus we can use Young's inequality with $\varepsilon > 0$ to obtain

$$M_j \leq \varepsilon M_{j+1} + c(\varepsilon) b^j \left(\iint_{Q_{2\rho}(x_0, t_0)} |u|^\nu dx dt \right)^{\frac{1}{\nu - \omega(n, \mathbf{p})}} + 1 + (2\rho)^{1/P},$$

where we have also used the fact that Q_{j+1} is contained in $Q_{2\rho}(x_0, t_0)$. Here we emphasize that the positive constant $c(\varepsilon)$ depends on ε , in addition to n , Λ , $\mathbf{p}$, ν and $\max\{\delta_1, \dots, \delta_n\}$. The constant $b > 1$ depends on n , $\mathbf{p}$ and ν . By iterating the last estimate, we end up with

$$M_0 \leq \varepsilon^N M_N + c(\varepsilon) \left(\sum_{\ell=0}^{N-1} (\varepsilon b)^\ell \right) \left(\iint_{Q_{2\rho}(x_0, t_0)} |u|^\nu dx dt \right)^{\frac{1}{\nu - \omega(n, \mathbf{p})}} + [1 + (2\rho)^{1/P}] \sum_{\ell=0}^{N-1} \varepsilon^\ell, \quad (4.23)$$

for any $N \in \mathbb{N}$. Choose ε so small that $\varepsilon b < 1$. Then the sums in the previous inequality converge as $N \rightarrow \infty$. Since the sequence $\{M_N\}$ is bounded due to the local boundedness of u , the first term on the right-hand side of (4.23) tends to zero in the limit as $N \rightarrow \infty$, which yields the bound (4.22). $\square$

4.2 The case $\bar{p} \leq \frac{2n}{n+2}$

We now turn our attention to the range

$$\bar{p} \leq \frac{2n}{n+2}, \quad (4.24)$$

and recall that we also require (1.6) in this case. In the previous section, the condition $u \in L_{loc}^P(\Omega_T)$, combined with (4.1), implies (1.6), so this assumption was not needed. The argument consists of two parts. First, we establish the local boundedness from above (resp. from below) for functions $u \in \mathcal{DG}^+(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$ (resp. $u \in \mathcal{DG}^-(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$) without an explicit bound. Then, we follow the approach of DiBenedetto in [46, Chapter V, Section 10] to derive an explicit estimate for the essential supremum of u_+ and u_- , respectively.

Theorem 4.5. *Let $u \in \mathcal{DG}^+(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$, where $\mathbf{p} = (p_1, \dots, p_n)$ satisfies (1.5). Suppose that (4.24) holds and that u satisfies the extra integrability condition (1.6). Then u is locally bounded from above and, if $Q_{2\rho}(x_0, t_0)$ is compactly contained in Ω_T , we have the following explicit bound:*

$$\operatorname{ess\,sup}_{Q_\rho(x_0, t_0)} u_+ \leq C \left[\left(\iint_{Q_{2\rho}(x_0, t_0)} u_+^m dx dt \right)^{\frac{1}{m - \frac{n}{\bar{p}}(2 - \bar{p})}} + 1 + \sqrt{\rho} \right], \quad (4.25)$$

where C is a positive constant depending only on n , Λ , $\mathbf{p}$, m and $\max\{\delta_1, \dots, \delta_n\}$. Similarly, if $u \in \mathcal{DG}^-(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$ under assumptions (1.5), (1.6) and (4.24), then u is locally bounded from below and, if $Q_{2\rho}(x_0, t_0) \Subset \Omega_T$, we have

$$\operatorname{ess\,sup}_{Q_\rho(x_0, t_0)} u_- \leq C \left[\left(\iint_{Q_{2\rho}(x_0, t_0)} u_-^m dx dt \right)^{\frac{1}{m - \frac{n}{\bar{p}}(2 - \bar{p})}} + 1 + \sqrt{\rho} \right]. \quad (4.26)$$

Remark 4.6. In view of Remark 3.5 and Theorem 4.5, under assumptions (1.2), (1.5), (1.6) and (4.24), local weak subsolutions u to (1.1) are locally bounded from above in Ω_T , whereas local weak supersolutions are locally bounded from below.

Proof of Theorem 4.5. After a translation, we may assume again that $(x_0, t_0) = (0, 0)$. By (1.6) and (4.24) we have

$$m > 2n \left(\frac{1}{\bar{p}} - \frac{1}{2} \right) \geq 2. \quad (4.27)$$

For the sake of readability, we divide the proof into two steps.

Step 1: local boundedness. Let us first assume that $u \in \mathcal{DG}^+(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathcal{C})$. We choose σ , δ , the cylinders $\mathcal{Q}_j$, and the functions $\tilde{\zeta}_{i,j}$ and ψ_j as in the proof of Theorem 4.1, and we show that u is essentially bounded from above in $\mathcal{Q}_{\sigma\rho}$. To this end, we define

$$\mathfrak{q} := \frac{m - 2\bar{p}(\frac{1}{n} + \frac{1}{2})}{m - 2} \geq 1, \quad \vartheta := \frac{\bar{p}(\frac{1}{n} + \frac{1}{2})}{\mathfrak{q}} \in (0, 1].$$

The lower bound for $\mathfrak{q}$ follows from (4.24), and from this we deduce the range of ϑ . Moreover, we set

$$Y_j := \iint_{\mathcal{Q}_j} (u - k_j)_+^2 dx dt, \quad k_j := 2k - \frac{k}{2^j}, \quad j \in \mathbb{N}_0.$$

We require $k \geq 1$ as before. If strict inequality holds in (4.24), then $\mathfrak{q} > 1$ and we have by Hölder's inequality

$$\begin{aligned} Y_{j+1} &= \iint_{\mathcal{Q}_{j+1}} (u - k_{j+1})_+^{2\vartheta} (u - k_{j+1})_+^{2(1-\vartheta)} dx dt \\ &\leq \left(\iint_{\mathcal{Q}_{j+1}} (u - k_{j+1})_+^{2\vartheta\mathfrak{q}} dx dt \right)^{\frac{1}{\mathfrak{q}}} \left(\iint_{\mathcal{Q}_{j+1}} (u - k_{j+1})_+^{2(1-\vartheta)\mathfrak{q}'} dx dt \right)^{\frac{\mathfrak{q}-1}{\mathfrak{q}}} \\ &= \left(\iint_{\mathcal{Q}_{j+1}} (u - k_{j+1})_+^{2\bar{p}(\frac{1}{n} + \frac{1}{2})} dx dt \right)^{\frac{1}{\mathfrak{q}}} \left(\iint_{\mathcal{Q}_{j+1}} (u - k_{j+1})_+^m dx dt \right)^{\frac{\mathfrak{q}-1}{\mathfrak{q}}} \\ &\leq \left(\iint_{\mathcal{Q}_{j+1}} (u - k_{j+1})_+^{2\bar{p}(\frac{1}{n} + \frac{1}{2})} dx dt \right)^{\frac{1}{\mathfrak{q}}} \left(\iint_{\mathcal{Q}_0} u_+^m dx dt \right)^{\frac{\mathfrak{q}-1}{\mathfrak{q}}}. \end{aligned} \quad (4.28)$$

If, instead, (4.24) holds with equality, then $\mathfrak{q} = 1$ and the previous estimate is valid without applying Hölder's inequality, provided that we adopt the standard convention $0^0 = 1$ if

$$I := \iint_{\mathcal{Q}_0} u_+^m dx dt = 0.$$

We note that inequality (4.14) is valid for the first integral in the last line of (4.28), despite the different definition of k_j . The upper bound for the exponents p_i in (1.5), combined with (4.24), implies that $p_i < 2$ for every $i \in \{1, \dots, n\}$; hence $P = 2$. Therefore, arguing as in (4.15), but this time using that $k_j < 2k$, we have

$$\sum_{i=1}^n \iint_{\mathcal{Q}_j} (u - k_{j+1})_+^{p_i} dx dt \leq 2^{j+1} n [Y_j + (1 + k^2) |A_{j+1}|],$$

where A_{j+1} is defined as in (4.11). Combining the above estimate with (4.14) and recalling that $k \geq 1$, we obtain

$$\iint_{Q_{j+1}} (u - k_{j+1})_+^{2\bar{p}(\frac{1}{n} + \frac{1}{2})} dx dt \leq \frac{C_1 2^j \frac{\mathcal{P}(n+\bar{p})}{n}}{(1-\sigma)^{\frac{\mathcal{P}(n+\bar{p})}{n}}} \left[\frac{Y_j}{\rho} + \left(1 + \frac{2k^2}{\rho}\right) |A_{j+1}| \right]^{\frac{n+\bar{p}}{n}},$$

where C_1 is a positive constant depending only on n , Λ , $\mathbf{p}$ and δ . Without loss of generality, we can now assume that $k \geq \max\{1, \sqrt{\rho}\}$. It then follows from the previous inequality that

$$\iint_{Q_{j+1}} (u - k_{j+1})_+^{2\bar{p}(\frac{1}{n} + \frac{1}{2})} dx dt \leq \frac{C_1 2^j \frac{\mathcal{P}(n+\bar{p})}{n}}{[(1-\sigma)^{\mathcal{P}} \rho]^{\frac{n+\bar{p}}{n}}} [Y_j + k^2 |A_{j+1}|]^{\frac{n+\bar{p}}{n}},$$

and reasoning as in (4.17) we have

$$|A_{j+1}| \leq \frac{4^{j+1}}{k^2} Y_j.$$

At this point, noting that in the current parameter range one has $\mathcal{P} < 2$, we can combine the two previous estimates with (4.28) to conclude that

$$Y_{j+1} \leq \frac{C_2 b^j}{[(1-\sigma)^{\mathcal{P}} \rho]^{\frac{n+\bar{p}}{n\mathbf{q}}}} I^{\frac{\mathbf{q}-1}{\mathbf{q}}} Y_j^{\frac{n+\bar{p}}{n\mathbf{q}}}, \quad (4.29)$$

where I denotes the integral over Q_0 appearing in the last line of (4.28), $b > 1$ is a constant depending only on n , $\mathbf{p}$ and m , while C_2 is a positive constant depending only on n , Λ , $\mathbf{p}$, δ and m . Now we consider the following four cases:

- (i) $I = 0$ and $\mathbf{q} > 1$;
- (ii) $I = 0$ and $\mathbf{q} = 1$;
- (iii) $I > 0$ and $\mathbf{q} = 1$;
- (iv) $I > 0$ and $\mathbf{q} > 1$.

In case (i), from the recursive inequality (4.29) we deduce

$$\iint_{Q_{\sigma\rho}} (u - 2k)_+^2 dx dt \leq Y_{j+1} = 0$$

for every $j \in \mathbb{N}_0$, which implies that $u \leq 2k$ a.e. in $Q_{\sigma\rho}$.

In cases (ii) and (iii), inequality (4.29) reduces to

$$Y_{j+1} \leq \frac{C_2 b^j}{[(1-\sigma)^{\mathcal{P}} \rho]^{\frac{n+\bar{p}}{n}}} Y_j^{1 + \frac{\bar{p}}{n}}.$$

Thus, the sequence $\{Y_j\}$ satisfies the recursive estimate of Lemma 2.5 with

$$C = C_2 [(1-\sigma)^{\mathcal{P}} \rho]^{-\frac{n+\bar{p}}{n}}, \quad \mu = \frac{\bar{p}}{n}.$$

By Lemma 2.5, the sequence $\{Y_j\}$ converges to zero provided that

$$\iint_{Q_\rho} (u - k)_+^2 dx dt = Y_0 \leq C^{-\frac{1}{\mu}} b^{-\frac{1}{\mu^2}} = C_2 [(1-\sigma)^{\mathcal{P}} \rho]^{\frac{n+\bar{p}}{\bar{p}}}.$$

The integral on the left-hand side vanishes in the limit as $k \rightarrow \infty$ by the dominated convergence theorem and the expression on the right-hand side is independent of k , so the estimate must indeed hold for large $k \geq \max\{1, \sqrt{\rho}\}$. Thus, we have

$$\iint_{Q_{\sigma\rho}} (u - 2k)_+^2 dx dt \leq Y_j \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

which implies that $u \leq 2k$ a.e. in $Q_{\sigma\rho}$ for some sufficiently large k .

It remains to analyze case (iv). In this case, (4.24) holds with strict inequality and, by the definition of $\mathfrak{q}$ and the lower bound for m in (4.27), we see that

$$\begin{aligned} \frac{n + \bar{p}}{n\mathfrak{q}} &= \left(\frac{n + \bar{p}}{n} \right) \frac{m - 2}{m - 2\bar{p}(\frac{1}{n} + \frac{1}{2})} = \left(\frac{n + \bar{p}}{n} \right) \left(1 + \frac{2\bar{p}(\frac{1}{n} + \frac{1}{2}) - 2}{m - 2\bar{p}(\frac{1}{n} + \frac{1}{2})} \right) \\ &> \left(\frac{n + \bar{p}}{n} \right) \left(1 + \frac{\bar{p}(\frac{1}{n} + \frac{1}{2}) - 1}{n(\frac{1}{\bar{p}} - \frac{1}{2}) - \bar{p}(\frac{1}{n} + \frac{1}{2})} \right) = 1. \end{aligned}$$

Therefore, the sequence $\{Y_j\}$ satisfies the recursive estimate of Lemma 2.5 with

$$C = C_2 [(1 - \sigma)^{\mathcal{P}} \rho]^{-\frac{n + \bar{p}}{n\mathfrak{q}}} I^{\frac{\mathfrak{q}-1}{\mathfrak{q}}}, \quad \mu = \frac{n + \bar{p}}{n\mathfrak{q}} - 1 > 0.$$

By Lemma 2.5, the sequence $\{Y_j\}$ converges to zero provided that

$$\iint_{Q_\rho} (u - k)_+^2 dx dt = Y_0 \leq C^{-\frac{1}{\mu}} b^{-\frac{1}{\mu^2}} = C_2 [(1 - \sigma)^{\mathcal{P}} \rho]^{\frac{n + \bar{p}}{n + \bar{p} - n\mathfrak{q}}} I^{-\frac{n(\mathfrak{q}-1)}{n + \bar{p} - n\mathfrak{q}}}.$$

Once more, the integral on the left-hand side vanishes in the limit as $k \rightarrow \infty$ and the expression on the right-hand side is independent of k ; hence the estimate must hold for large $k \geq \max\{1, \sqrt{\rho}\}$. Reasoning as before, we conclude again that $u \leq 2k$ a.e. in $Q_{\sigma\rho}$ for some sufficiently large k .

If $u \in \mathcal{DG}^-(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$, the same arguments, with minor modifications, give $u \geq -2k$ a.e. in $Q_{\sigma\rho}$ for some k large enough. We leave the details to the reader.

Step 2: explicit L^∞ bounds. Let us again assume that $u \in \mathcal{DG}^+(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$. Knowing that u is locally essentially bounded from above, we now proceed to establish the explicit bound in (4.25). By combining (4.24) and (4.27), we can introduce a positive parameter as follows:

$$\nu := m - 2\bar{p} \left(\frac{1}{n} + \frac{1}{2} \right) > 0.$$

Let k_j be defined again as in (4.6), and set

$$Z_j := \iint_{\Omega_j} (u - k_j)_+^m dx dt, \quad j \in \mathbb{N}_0.$$

Then we have

$$Z_{j+1} \leq \|u_+\|_{L^\infty(\Omega_0)}^\nu \iint_{\Omega_{j+1}} (u - k_{j+1})_+^{2\bar{p}(\frac{1}{n} + \frac{1}{2})} dx dt.$$

Using (4.14) to estimate the last integral, we conclude that

$$Z_{j+1} \leq C_1 \frac{\|u_+\|_{L^\infty(\Omega_0)}^\nu 2^{j \frac{\mathcal{P}(n + \bar{p})}{n}}}{(1 - \sigma)^{\frac{\mathcal{P}(n + \bar{p})}{n}}}$$

$$\times \left[\frac{1}{\rho} \iint_{\mathcal{Q}_j} (u - k_{j+1})_+^2 dx dt + \frac{1}{\rho} \sum_{i=1}^n \iint_{\mathcal{Q}_j} (u - k_{j+1})_+^{p_i} dx dt + |A_{j+1}| \right]^{\frac{n+\bar{p}}{n}}.$$

Exploiting the definitions of A_{j+1} and k_j , applying Young's inequality with exponents $\frac{2}{p_i}$, using the fact that $k \geq 1$ and $m > 2$, and arguing as in (4.17), we end up with

$$\begin{aligned} & \iint_{\mathcal{Q}_j} (u - k_{j+1})_+^2 dx dt + \sum_{i=1}^n \iint_{\mathcal{Q}_j} (u - k_{j+1})_+^{p_i} dx dt \\ & \leq \iint_{A_{j+1}} (u - k_j)_+^2 dx dt + \sum_{i=1}^n \iint_{A_{j+1}} (u - k_j)_+^{p_i} dx dt \\ & \leq (n+1) \left[\iint_{A_{j+1}} (u - k_j)_+^2 dx dt + |A_{j+1}| \right] \\ & \leq (n+1) \left(1 + \frac{4^{j+1}}{k^2} \right) \iint_{A_{j+1}} \frac{(u - k_j)_+^m}{(u - k_j)_+^{m-2}} dx dt \\ & \leq (n+1) (1 + 4^{j+1}) \iint_{A_{j+1}} \frac{(u - k_j)_+^m}{(k_{j+1} - k_j)^{m-2}} dx dt \\ & \leq 2(n+1) 4^{j+1} \frac{2^{(j+1)(m-2)}}{k^{m-2}} Z_j = 2^{m+1}(n+1) \frac{2^{mj}}{k^{m-2}} Z_j. \end{aligned}$$

Combining the two previous estimates and recalling that $\mathcal{Q}_0 = Q_\rho$, we obtain

$$Z_{j+1} \leq C_2 \frac{\|u_+\|_{L^\infty(Q_\rho)}^\nu 2^j \frac{\mathcal{P}(n+\bar{p})}{n}}{(1-\sigma) \frac{\mathcal{P}(n+\bar{p})}{n}} \left[\frac{2^{mj} Z_j}{\rho k^{m-2}} + |A_{j+1}| \right]^{\frac{n+\bar{p}}{n}}, \quad (4.30)$$

where $C_2 = C_2(n, \Lambda, \mathbf{p}, \delta, m) > 0$. We now use again the definitions of A_{j+1} and k_j , together with $k \geq \max\{1, \sqrt{\rho}\}$, to deduce that

$$\begin{aligned} |A_{j+1}| &= \iint_{A_{j+1}} \frac{(u - k_j)_+^m}{(u - k_j)_+^m} dx dt \leq \iint_{A_{j+1}} \frac{(u - k_j)_+^m}{(k_{j+1} - k_j)^m} dx dt \\ &= \frac{2^{m(j+1)}}{k^m} \iint_{A_{j+1}} (u - k_j)_+^m dx dt \leq 2^m \frac{2^{mj} Z_j}{\rho k^{m-2}}. \end{aligned} \quad (4.31)$$

Joining (4.30) and (4.31), we find

$$Z_{j+1} \leq C_2 \frac{\|u_+\|_{L^\infty(Q_\rho)}^\nu k^{\frac{n+\bar{p}}{n}(2-m)} b^j}{[(1-\sigma)^\mathcal{P} \rho]^{\frac{n+\bar{p}}{n}}} Z_j^{1+\frac{\bar{p}}{n}},$$

where $b > 1$ is a constant depending only on n , $\mathbf{p}$ and m .

Henceforth, we assume that $\|u_+\|_{L^\infty(Q_\rho)} > 0$. In this case, from Lemma 2.5 we see that $\{Z_j\}$ converges to zero provided that

$$\iint_{Q_\rho} u_+^m dx dt = Z_0 \leq C_2 \|u_+\|_{L^\infty(Q_\rho)}^{-\frac{n\nu}{\bar{p}}} k^{\frac{n+\bar{p}}{\bar{p}}(m-2)} [(1-\sigma)^\mathcal{P} \rho]^{\frac{n+\bar{p}}{\bar{p}}},$$

which holds if

$$k \geq C_2 \|u_+\|_{L^\infty(Q_\rho)}^\zeta (1-\sigma)^{-\frac{\mathcal{P}}{m-2}} \left(\iint_{Q_\rho} u_+^m dx dt \right)^{\frac{\bar{p}}{(n+\bar{p})(m-2)}}, \quad (4.32)$$

where

$$\begin{aligned}\zeta &= \frac{n\nu}{(n+\bar{p})(m-2)} = \frac{n}{(n+\bar{p})(m-2)} \left[m - 2\bar{p} \left(\frac{1}{n} + \frac{1}{2} \right) \right] \\ &= \frac{n}{n+\bar{p}} \left[1 + \frac{2-2\bar{p} \left(\frac{1}{n} + \frac{1}{2} \right)}{m-2} \right].\end{aligned}$$

If (4.24) holds with equality, then the numerator in the last fractional term vanishes and $\zeta = n/(n+\bar{p}) < 1$. If, instead, (4.24) holds with a strict inequality, then also the last inequality in (4.27) is strict and we may estimate

$$\zeta < \frac{n}{n+\bar{p}} \left[1 + \frac{2-2\bar{p} \left(\frac{1}{n} + \frac{1}{2} \right)}{2n \left(\frac{1}{\bar{p}} - \frac{1}{2} \right) - 2} \right] = 1.$$

Thus, we have confirmed that in any case $\zeta \in (0, 1)$. In addition, we have shown that if (4.32) and $k \geq \max \{1, \sqrt{\rho}\}$ hold, then

$$\iint_{Q_{\sigma\rho}} (u-k)_+^m dx dt \leq Z_j \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

which means that $u \leq k$ almost everywhere in $Q_{\sigma\rho}$. Therefore,

$$\|u_+\|_{L^\infty(Q_{\sigma\rho})} \leq \frac{C_2 \|u_+\|_{L^\infty(Q_\rho)}^\zeta}{(1-\sigma)^{\frac{\mathfrak{p}}{m-2}}} \left(\iint_{Q_\rho} u_+^m dx dt \right)^{\frac{\bar{p}}{(n+\bar{p})(m-2)}} + 1 + \sqrt{\rho}.$$

Since $\zeta \in (0, 1)$, we may use Young's inequality with $\varepsilon > 0$ on the right-hand side to conclude that

$$\|u_+\|_{L^\infty(Q_{\sigma\rho})} \leq \varepsilon \|u_+\|_{L^\infty(Q_\rho)} + \frac{c(\varepsilon)}{(1-\sigma)^{\frac{\mathfrak{p}}{(m-2)(1-\zeta)}}} \left(\iint_{Q_\rho} u_+^m dx dt \right)^{\frac{1}{m-\frac{\mathfrak{p}}{2-\bar{p}}}} + 1 + \sqrt{\rho},$$

where we may choose ε arbitrarily small. Here we emphasize that the positive constant $c(\varepsilon)$ depends on ε , in addition to $n, \Lambda, \mathbf{p}, m$ and $\max \{\delta_1, \dots, \delta_n\}$. Now an iteration similar to that performed in the proof of Theorem 4.4 leads to the following estimate:

$$\operatorname{ess\,sup}_{Q_\rho} u_+ \leq C \left[\left(\iint_{Q_{2\rho}} u_+^m dx dt \right)^{\frac{1}{m-\frac{\mathfrak{p}}{2-\bar{p}}}} + 1 + \sqrt{\rho} \right], \quad (4.33)$$

where C is a positive constant depending only on $n, \Lambda, \mathbf{p}, m$ and $\max \{\delta_1, \dots, \delta_n\}$. Note that estimate (4.33) holds trivially in the case $\|u_+\|_{L^\infty(Q_\rho)} = 0$.

Finally, if $u \in \mathcal{DG}^-(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathcal{C})$, analogous arguments yield the bound (4.26). In this regard, we leave the straightforward details to the reader. $\square$

We conclude this section with the following corollary on local boundedness, which immediately follows from Definition 3.4, Remark 3.5 and Theorem 4.5.

Corollary 4.7. *Let $u \in \mathcal{DG}(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$, where $\mathbf{p} = (p_1, \dots, p_n)$ satisfies (1.5). Suppose that (4.24) holds and that u satisfies the extra integrability condition (1.6). Then u is locally bounded and, for every cylinder $Q_{2\rho}(x_0, t_0) \in \Omega_T$, we have*

$$\operatorname{ess\,sup}_{Q_\rho(x_0, t_0)} |u| \leq C \left[\left(\iint_{Q_{2\rho}(x_0, t_0)} |u|^m dx dt \right)^{\frac{1}{m - \frac{n}{\bar{p}}(2 - \bar{p})}} + 1 + \sqrt{\rho} \right], \quad (4.34)$$

where C is a positive constant depending only on $n, \Lambda, \mathbf{p}, m$ and $\max\{\delta_1, \dots, \delta_n\}$. In particular, if u is a local weak solution of (1.1) under assumptions (1.2), (1.5), (1.6) and (4.24), then u belongs to $L^\infty_{\text{loc}}(\Omega_T)$ and satisfies the estimate (4.34).

5 Critical mass lemma and semicontinuity

In this section, we prove a measure-theoretical maximum principle for local weak super(sub)-solutions to (1.1) that are locally bounded from below (from above), commonly referred to as De Giorgi-type Lemma, or Critical Mass Lemma. As a consequence of this result, local weak super(sub)-solutions to (1.1) admit a lower (upper) semicontinuous representative, provided they are locally bounded from below (from above).

Actually, in view of what has been established in Sections 3 and 4, analogous results hold more generally for functions $u \in \mathcal{DG}^\pm(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$, under suitable assumptions on the parameters p_i , and additionally assuming that condition (1.6) holds in the case $\bar{p} \leq 2n/(n+2)$. However, for the sake of simplicity, here we restrict ourselves to stating the aforementioned results only for local weak super(sub)-solutions to (1.1). The corresponding statements for functions in $\mathcal{DG}^\pm(\mathbf{p}, \{\delta_i\}, \Omega_T, \mathbb{C})$ can be easily deduced by the reader.

In the sequel, we will use the cubes and cylinders defined below, which turn out to be particularly convenient in our anisotropic setting. For any $(y, s) \in \Omega_T$ and any $M, \rho > 0$, the *intrinsic cubes* $\mathcal{K}_\rho(M)$, $[y + \mathcal{K}_\rho(M)]$ and the *backward intrinsic cylinders* $\mathcal{Q}_\rho^-(M)$, $(y, s) + \mathcal{Q}_\rho^-(M)$ are defined respectively by

$$\mathcal{K}_\rho(M) := \prod_{i=1}^n \left\{ |x_i| < M^{\frac{p_i-2}{p_i}} \rho^{\frac{\bar{p}}{p_i}} \right\}, \quad [y + \mathcal{K}_\rho(M)] := \prod_{i=1}^n \left\{ |x_i - y_i| < M^{\frac{p_i-2}{p_i}} \rho^{\frac{\bar{p}}{p_i}} \right\}$$

and

$$\mathcal{Q}_\rho^-(M) := \mathcal{K}_\rho(M) \times (-\rho^{\bar{p}}, 0], \quad (y, s) + \mathcal{Q}_\rho^-(M) := [y + \mathcal{K}_\rho(M)] \times (s - \rho^{\bar{p}}, s].$$

Throughout this section, $(y, s) \in \Omega_T$ and $M, \rho > 0$ are such that $(y, s) + \mathcal{Q}_\rho^-(M) \Subset \Omega_T$, while $\eta \in C^\infty(\mathbb{R}^{n+1})$ is a smooth cut-off function of the form

$$\eta(x, t) := \Psi(t) \prod_{i=1}^n \eta_i^{p_i}(x_i),$$

where $\Psi \in C^\infty(\mathbb{R}; [0, 1])$ is a non-decreasing map such that

$$\Psi \equiv 0 \quad \text{on} \quad \left(-\infty, s - \rho^{\bar{p}} + \frac{\epsilon}{2}\right], \quad \Psi \equiv 1 \quad \text{on} \quad [s - \rho^{\bar{p}} + \epsilon, \infty), \quad \text{with } \epsilon \in (0, \rho^{\bar{p}}),$$

while the functions η_i satisfy

$$\eta_i \in C_0^\infty(\mathbb{R}; [0, 1]), \quad \operatorname{supp} \eta_i \subset \left(y_i - M^{\frac{p_i-2}{p_i}} \rho^{\frac{\bar{p}}{p_i}}, y_i + M^{\frac{p_i-2}{p_i}} \rho^{\frac{\bar{p}}{p_i}}\right) \quad \text{for every } i \in \{1, \dots, n\}.$$

As an immediate consequence of these conditions, we have that

$$\eta \equiv 0 \quad \text{on the parabolic boundary of } (y, s) + \mathcal{Q}_\rho^-(M).$$

The proof of the following two energy estimates is entirely analogous to that of Proposition 3.1, and is therefore omitted.

Proposition 5.1. *Let $n \geq 2$ and $p_1, \dots, p_n > 1$. Assume that (1.2) holds and that u is a local weak supersolution to (1.1) in the sense of Definition 2.1. Then there exists a positive constant C , depending only on Λ and $\max\{p_1, \dots, p_n\}$, such that for every $k \in \mathbb{R}$ we have*

$$\begin{aligned} & \sup_{\tau \in (s-\rho^{\bar{p}}, s]} \int_{[y + \mathcal{K}_\rho(M)]} (u - k)_-^2 \eta(x, \tau) dx + \sum_{i=1}^n \iint_{(y, s) + \mathcal{Q}_\rho^-(M)} (|\partial_{x_i} u| - \delta_i)_+^{p_i} \eta \mathbf{1}_{\{u < k\}} dx dt \\ & \leq C \iint_{(y, s) + \mathcal{Q}_\rho^-(M)} (u - k)_-^2 \partial_t \eta dx dt + C \sum_{i=1}^n \iint_{(y, s) + \mathcal{Q}_\rho^-(M)} (u - k)_-^{p_i} |\partial_{x_i} \eta^{\frac{1}{p_i}}|^{p_i} dx dt. \end{aligned} \quad (5.1)$$

Similarly, if u is a local weak subsolution to (1.1), then for every $k \in \mathbb{R}$ we have

$$\begin{aligned} & \sup_{\tau \in (s-\rho^{\bar{p}}, s]} \int_{[y + \mathcal{K}_\rho(M)]} (u - k)_+^2 \eta(x, \tau) dx + \sum_{i=1}^n \iint_{(y, s) + \mathcal{Q}_\rho^-(M)} (|\partial_{x_i} u| - \delta_i)_+^{p_i} \eta \mathbf{1}_{\{u > k\}} dx dt \\ & \leq C \iint_{(y, s) + \mathcal{Q}_\rho^-(M)} (u - k)_+^2 \partial_t \eta dx dt + C \sum_{i=1}^n \iint_{(y, s) + \mathcal{Q}_\rho^-(M)} (u - k)_+^{p_i} |\partial_{x_i} \eta^{\frac{1}{p_i}}|^{p_i} dx dt. \end{aligned} \quad (5.2)$$

We are now in a position to prove the Critical Mass Lemma mentioned above. We recall that local weak subsolutions (resp. supersolutions) to (1.1) are locally bounded from above (resp. below) in Ω_T , provided that additional conditions are satisfied (see Remarks 4.2 and 4.6). Let us fix a cylinder $(y, s) + \mathcal{Q}_{2\rho}^-(M) \Subset \Omega_T$, being $(y, s) \in \Omega_T$ and $M, \rho > 0$ appropriate. Let μ^+ , μ^- be such that

$$\mu^- \leq \operatorname{ess\,inf}_{(y, s) + \mathcal{Q}_{2\rho}^-(M)} u \leq \operatorname{ess\,sup}_{(y, s) + \mathcal{Q}_{2\rho}^-(M)} u \leq \mu^+.$$

We also fix $a \in (0, 1)$ and define

$$\Gamma = \Gamma(\delta_1, \dots, \delta_n, \mathbf{p}) := \sum_{i=1}^n \delta_i^{p_i}. \quad (5.3)$$

Lemma 5.2 (De Giorgi-type/Critical Mass). *Assume that (1.2) holds and that u is a local weak supersolution to (1.1) in the sense of Definition 2.1, locally bounded from below. Let ρ , M , $\mu^\pm$, a and Γ be as above. Then there exists $\nu^- \in (0, 1)$, depending on the data $\{n, \Lambda, \mathbf{p}\}$ and on the parameter a , but not on the radius ρ nor on M , such that if*

$$|\{u \leq \mu^- + M\} \cap [(y, s) + \mathcal{Q}_{2\rho}^-(M)]| \leq \nu^- |\mathcal{Q}_{2\rho}^-(M)|$$

and $M^2 > \Gamma \rho^{\bar{p}}$, then

$$u \geq \mu^- + aM \quad \text{a.e. in } [(y, s) + \mathcal{Q}_\rho^-(M)]. \quad (5.4)$$

Likewise, if u is a local weak subsolution to (1.1) which is locally bounded from above, then there exists $\nu^+ \in (0, 1)$, depending on the data $\{n, \Lambda, \mathbf{p}\}$ and on the parameter a , but not on the radius ρ nor on M , such that if

$$|\{u \geq \mu^+ - M\} \cap [(y, s) + \mathcal{Q}_{2\rho}^-(M)]| \leq \nu^+ |\mathcal{Q}_{2\rho}^-(M)|$$

and $M^2 > \Gamma \rho^{\bar{p}}$, then

$$u \leq \mu^+ - aM \quad \text{a.e. in } [(y, s) + \mathcal{Q}_\rho^-(M)]. \quad (5.5)$$

Proof. We prove (5.4), since the proof of (5.5) is analogous. Without loss of generality, we assume $(y, s) = (0, 0)$, just to simplify the notation. Let us set, for any $j, \ell \in \mathbb{N}_0$,

$$\rho_j := \rho + \frac{\rho}{2^j}, \quad \mathbb{K}_j := \prod_{i=1}^n \left\{ |x_i| < M^{\frac{p_i-2}{p_i}} \rho^{\frac{\bar{p}}{p_i}} \left(1 + \frac{1}{2^{j+\ell}} \right) \right\}, \quad \mathcal{Q}_j := \mathbb{K}_j \times (-\rho_j^{\bar{p}}, 0].$$

Since $\mathbb{K}_0 \rightarrow \overline{\mathcal{K}_\rho(M)}$ as $\ell \rightarrow \infty$, we fix ℓ such that $\mathbb{K}_0 \subset \mathcal{K}_{2\rho}(M)$. Notice that ℓ can be chosen in such a way that it depends only on n and $\mathbf{p}$. We now introduce the sequence of levels

$$k_j := \mu^- + aM + \frac{(1-a)M}{2^j},$$

and note that

$$(u - k_j)_- \leq \left[aM + \frac{(1-a)M}{2^j} \right] \mathbf{1}_{\{u < k_j\}} \leq M \mathbf{1}_{\{u < k_j\}} \quad \text{a.e. in } \mathcal{Q}_{2\rho}^-(M). \quad (5.6)$$

To move forward, we choose cut-off functions $\eta_{i,j} \in C_0^\infty(\mathbb{R}; [0, 1])$ such that

$$\eta_{i,j} \equiv 1 \quad \text{in } \pi_i(\mathbb{K}_{j+1}), \quad \eta_{i,j} \equiv 0 \quad \text{in } \mathbb{R} \setminus \pi_i(\mathbb{K}_j) \quad \text{and} \quad |\eta'_{i,j}| \leq \frac{c 2^j}{M^{\frac{p_i-2}{p_i}} \rho^{\frac{\bar{p}}{p_i}}},$$

where $c = c(n, \mathbf{p}) > 0$ and $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $\pi_i(x) = x_i$, $i \in \{1, \dots, n\}$, denotes the canonical projection onto the i -th spatial coordinate. Similarly, we take $\Psi_j \in C^\infty(\mathbb{R}; [0, 1])$ such that

$$\Psi_j \equiv 0 \quad \text{on } \left(-\infty, -\frac{\rho_j^{\bar{p}} + \rho_{j+1}^{\bar{p}}}{2} \right], \quad \Psi_j \equiv 1 \quad \text{on } [-\rho_{j+1}^{\bar{p}}, \infty) \quad \text{and} \quad 0 \leq \Psi'_j \leq \frac{c 2^{(j+1)\bar{p}}}{\rho^{\bar{p}}}.$$

It follows that the smooth cut-off function

$$\eta_j(x, t) := \Psi_j(t) \prod_{i=1}^n \eta_{i,j}^{p_i}(x_i)$$

satisfies the following properties:

$$\begin{cases} 0 \leq \eta_j \leq 1, & \eta_j \equiv 1 \quad \text{in } \mathcal{Q}_{j+1}, & \eta_j \equiv 0 \quad \text{on } \partial_{\text{par}} \mathcal{Q}_j, \\ 0 \leq \partial_t \eta_j \leq \frac{c 2^{(j+1)\bar{p}}}{\rho^{\bar{p}}}, & |\partial_{x_i} \eta_j^{\frac{1}{p_i}}|^{p_i} \leq \frac{c 2^{j\mathcal{P}}}{M^{p_i-2} \rho^{\bar{p}}} \quad \text{for every } i \in \{1, \dots, n\}. \end{cases} \quad (5.7)$$

The construction of this cut-off function is similar to that of $\tilde{\zeta}_j$ in the proof of Theorem 4.1 and is obtained by suitably modifying the calculations carried out in the [Appendix](#). Now we rewrite estimate (5.1) with ρ_j , $\mathbb{K}_j$, $\mathcal{Q}_j$, k_j and η_j in place of ρ , $\mathcal{K}_\rho(M)$, $\mathcal{Q}_\rho^-(M)$, k and η , respectively. We thus obtain

$$\sup_{\tau \in (-\rho_j^{\bar{p}}, 0]} \int_{\mathbb{K}_j} (u - k_j)_-^2 \eta_j(x, \tau) dx + \sum_{i=1}^n \iint_{\mathcal{Q}_j \cap \{u < k_j\}} (|\partial_{x_i} u| - \delta_i)_+^{p_i} \eta_j dx dt$$

$$\leq \frac{C 2^{j\bar{p}}}{\rho^{\bar{p}}} \iint_{\mathcal{Q}_j} (u - k_j)_-^2 dx dt + \sum_{i=1}^n \frac{C 2^{j\mathcal{P}}}{M^{p_i-2} \rho^{\bar{p}}} \iint_{\mathcal{Q}_j} (u - k_j)_-^{p_i} dx dt,$$

where C is a positive constant depending only on n , Λ and $\mathbf{p}$. At this stage, we set

$$\mathcal{A}_j := \{(x, t) \in \mathcal{Q}_j : u(x, t) < k_j\}.$$

Using the properties of η_j in (5.7) and the definition of Γ in (5.3), we get

$$\begin{aligned} & \sum_{i=1}^n \iint_{\mathcal{Q}_j} |\partial_{x_i} [\eta_j (u - k_j)_-]|^{p_i} dx dt \\ & \leq 2^{\mathcal{P}-1} \left[\sum_{i=1}^n \iint_{\mathcal{Q}_j} (u - k_j)_-^{p_i} |\partial_{x_i} \eta_j|^{p_i} dx dt + \sum_{i=1}^n \iint_{\mathcal{A}_j} \eta_j |\partial_{x_i} u|^{p_i} dx dt \right] \\ & \leq 2^{2\mathcal{P}-2} \left[\sum_{i=1}^n \frac{C 2^{j\mathcal{P}}}{M^{p_i-2} \rho^{\bar{p}}} \iint_{\mathcal{Q}_j} (u - k_j)_-^{p_i} dx dt + \sum_{i=1}^n \iint_{\mathcal{A}_j} (|\partial_{x_i} u| - \delta_i)_+^{p_i} \eta_j dx dt + |\mathcal{A}_j| \Gamma \right]. \end{aligned}$$

Putting together the two previous estimates, applying (5.6) and recalling that $M^2 > \Gamma \rho^{\bar{p}}$, we deduce that

$$\begin{aligned} \mathcal{E}_j &:= \sup_{\tau \in (-\rho_j^{\bar{p}}, 0]} \int_{\mathbb{K}_j} (u - k_j)_-^2 \eta_j(x, \tau) dx + \sum_{i=1}^n \iint_{\mathcal{Q}_j} |\partial_{x_i} [\eta_j (u - k_j)_-]|^{p_i} dx dt \\ &\leq \frac{C 2^{j\bar{p}}}{\rho^{\bar{p}}} \iint_{\mathcal{Q}_j} (u - k_j)_-^2 dx dt + \sum_{i=1}^n \frac{C 2^{j\mathcal{P}}}{M^{p_i-2} \rho^{\bar{p}}} \iint_{\mathcal{Q}_j} (u - k_j)_-^{p_i} dx dt + C |\mathcal{A}_j| \Gamma \\ &\leq C 2^{j\mathcal{P}} |\mathcal{A}_j| \left(\frac{M^2}{\rho^{\bar{p}}} + \Gamma \right) \leq \frac{C 2^{j\mathcal{P}+1} |\mathcal{A}_j| M^2}{\rho^{\bar{p}}}. \end{aligned}$$

Now we combine these estimates of the energies $\mathcal{E}_j$ with the following embedding inequality (see [49])

$$\begin{aligned} & \iint_{\mathcal{Q}_j} |\eta_j (u - k_j)_-|^{\bar{p}(\frac{n+2}{n})} dx dt \\ & \leq c \left[\sup_{\tau \in (-\rho_j^{\bar{p}}, 0]} \int_{\mathbb{K}_j} (u - k_j)_-^2 \eta_j^2(x, \tau) dx + \sum_{i=1}^n \iint_{\mathcal{Q}_j} |\partial_{x_i} [\eta_j (u - k_j)_-]|^{p_i} dx dt \right]^{\frac{n+\bar{p}}{n}} \leq c \mathcal{E}_j^{\frac{n+\bar{p}}{n}}. \end{aligned}$$

Observing that $(u - k_j)_- \geq (k_j - k_{j+1}) = (1 - a)M/2^{j+1}$ in $\mathcal{A}_{j+1}$, and recalling that $\eta_j = 1$ in $\mathcal{Q}_{j+1}$, we get the chain

$$\begin{aligned} \left[\frac{(1-a)M}{2^{j+1}} \right]^{\bar{p}(\frac{n+2}{n})} |\mathcal{A}_{j+1}| &\leq \iint_{\mathcal{A}_{j+1}} (u - k_j)_-^{\bar{p}(\frac{n+2}{n})} dx dt \\ &\leq \iint_{\mathcal{Q}_j} |\eta_j (u - k_j)_-|^{\bar{p}(\frac{n+2}{n})} dx dt \leq c \mathcal{E}_j^{\frac{n+\bar{p}}{n}} \\ &\leq \frac{C 2^{bj} M^{\frac{2(n+\bar{p})}{n}}}{\rho^{\frac{\bar{p}(n+\bar{p})}{n}}} |\mathcal{A}_j|^{1+\frac{\bar{p}}{n}}, \end{aligned} \tag{5.8}$$

where $b = b(n, \mathbf{p}) > 1$. At this point, we set

$$Y_j := \frac{|\mathcal{A}_j|}{|\mathcal{Q}_j|}$$

and observe that $|\mathcal{Q}_j|^{\frac{\bar{p}}{n}} \leq c(n, \mathbf{p}) M^{\bar{p}-2} \rho^{\frac{\bar{p}}{n}(n+\bar{p})}$. Dividing both sides of (5.8) by $|\mathcal{Q}_{j+1}|$, and noticing that $|\mathcal{Q}_{j+1}| \geq 2^{-(n+\bar{p})} |\mathcal{Q}_j|$, we obtain

$$\begin{aligned} Y_{j+1} &\leq \frac{C 2^{bj} M^{2-\bar{p}}}{(1-a)^{\frac{\bar{p}(n+2)}{n}} \rho^{\frac{\bar{p}(n+\bar{p})}{n}}} \frac{|\mathcal{A}_j|^{1+\frac{\bar{p}}{n}}}{|\mathcal{Q}_j|} \\ &\leq \frac{C 2^{bj}}{(1-a)^{\frac{\bar{p}(n+2)}{n}}} Y_j^{1+\frac{\bar{p}}{n}} =: \gamma 2^{bj} Y_j^{1+\frac{\bar{p}}{n}}, \end{aligned} \quad (5.9)$$

where γ is a positive constant depending only on $n, \Lambda, \mathbf{p}$ and a . In the previous estimate, we may assume that $C > 1$; hence, $\gamma > 1$, since $a \in (0, 1)$. According to Lemma 2.5, the estimate (5.9) yields $Y_\infty := \lim_{j \rightarrow \infty} Y_j = 0$ provided that

$$Y_0 := \frac{|\{u < \mu^- + M\} \cap \mathcal{Q}_0|}{|\mathcal{Q}_0|} \leq \gamma^{-\frac{n}{\bar{p}}} 2^{-\frac{bn^2}{\bar{p}^2}} =: \nu^*. \quad (5.10)$$

Then $Y_\infty = 0$ implies (5.4). It remains to prove that (5.10) holds.

We set $\nu^- := \nu^*/\gamma^*$, where $\nu^* \in (0, 1)$ is defined in (5.10) and γ^* is such that $|\mathcal{Q}_{2\rho}^-(M)| \leq \gamma^* |\mathcal{Q}_0|$. Therefore, $\gamma^* \geq 1$ and

$$|\{u < \mu^- + M\} \cap \mathcal{Q}_0| \leq |\{u \leq \mu^- + M\} \cap \mathcal{Q}_{2\rho}^-(M)| \leq \nu^- |\mathcal{Q}_{2\rho}^-(M)| \leq \nu^* |\mathcal{Q}_0|,$$

which ensures (5.10).

In order to prove (5.5), we may proceed as above, considering the energy estimates (5.2) in the same iterative geometry, but this time using the truncations $(u - k_j)_+$ with

$$k_j = \mu^+ - aM - \frac{(1-a)M}{2^j}.$$

This concludes the proof. $\square$

At this point, we are ready to state the following result.

Theorem 5.3. *Assume that (1.2) holds and that u is a local weak supersolution to (1.1) in the sense of Definition 2.1, locally bounded from below. Then u has a lower semicontinuous representative in Ω_T .*

Likewise, if u is a local weak subsolution to (1.1) which is locally bounded from above, then u has an upper semicontinuous representative in Ω_T .

Proof. We proceed in a way reminiscent of [30] and [60]. Let u be a local weak supersolution to (1.1) in the sense of Definition 2.1, locally bounded from below. Set $\mathcal{Q}_\rho := \mathcal{Q}_\rho^-(1)$ for all $\rho > 0$, and consider the lower semicontinuous regularization of u , defined by

$$u_*(x, t) := \lim_{\rho \rightarrow 0^+} \operatorname{ess\,inf}_{(x, t) + \mathcal{Q}_\rho} u \quad \text{for } (x, t) \in \Omega_T.$$

We observe that this function is well defined at every point of Ω_T , since u is locally bounded from below and $(x, t) + \mathcal{Q}_\rho \subset \Omega_T$ for small values of ρ . It is well known that u_* is lower

semicontinuous. Therefore, showing that $u = u_*$ almost everywhere in Ω_T yields the desired conclusion. In order to prove this equality, we also define the set

$$\mathcal{L} := \left\{ (x, t) \in \Omega_T : |u(x, t)| < \infty, \lim_{\rho \rightarrow 0^+} \iint_{(x, t) + \mathcal{Q}_\rho} |u(x, t) - u(y, s)| dy ds = 0 \right\}.$$

This set is well defined, since $u \in L^1_{loc}(\Omega_T)$. Moreover,

$$|\mathcal{L}| = |\Omega_T| \quad (5.11)$$

(see [30, Section 4]). Taking (5.11) into account, it is sufficient to prove $u = u_*$ in $\mathcal{L}$. For all $(x, t) \in \mathcal{L}$, we have

$$u_*(x, t) = \lim_{\rho \rightarrow 0^+} \operatorname{ess\,inf}_{(x, t) + \mathcal{Q}_\rho} u \leq \lim_{\rho \rightarrow 0^+} \iint_{(x, t) + \mathcal{Q}_\rho} u(y, s) dy ds = u(x, t).$$

To show the reverse inequality, we argue by contradiction. Let us take $(x_0, t_0) \in \mathcal{L}$ and suppose that $u_*(x_0, t_0) < u(x_0, t_0)$. Let $r, b > 0$ be sufficiently small so that $(x_0, t_0) + \mathcal{Q}_r \subset \Omega_T$ and

$$\operatorname{ess\,inf}_{(x_0, t_0) + \mathcal{Q}_r} u =: \mu_- \leq u_*(x_0, t_0) < \mu_- + b < u(x_0, t_0).$$

This choice is possible by the very definition of u_* , since $u_*(x_0, t_0)$ is close to μ_- for small values of r . Let us introduce $a \in (0, 1)$ such that

$$\mu_- + ab > u_*(x_0, t_0), \quad \text{i.e.,} \quad \frac{u_*(x_0, t_0) - \mu_-}{b} < a < 1.$$

Now we set

$$\rho^* := \inf \left\{ \left(\frac{b^2}{\Gamma} \right)^{\frac{1}{\bar{p}}}, \frac{r}{2}, \min_{1 \leq i \leq n} \left\{ b^{\frac{2-p_i}{\bar{p}}} \frac{r}{2} \right\} \right\},$$

where Γ is defined in (5.3). If $\Gamma = 0$ (which occurs only when $\delta_i = 0$ for all $i \in \{1, \dots, n\}$), the quantity $(b^2/\Gamma)^{1/\bar{p}}$ is understood as $+\infty$. With the above choice of ρ^* , for every $\rho \in (0, \rho^*)$ one has

$$b^2 > \Gamma \rho^{\bar{p}} \quad \text{and} \quad \mathcal{Q}_{2\rho}^-(b) \subset \mathcal{Q}_r.$$

Therefore, for every $\rho \in (0, \rho^*)$ we have

$$\mu_- \leq \operatorname{ess\,inf}_{(x_0, t_0) + \mathcal{Q}_{2\rho}^-(b)} u.$$

At this point, we prove that there exists $\nu_a^- > 0$, depending only on $a, n, \Lambda, \mathbf{p}$, such that for some $\tilde{\rho} \in (0, \rho^*)$ one has

$$|\{u \leq \mu_- + b\} \cap [(x_0, t_0) + \mathcal{Q}_{2\tilde{\rho}}^-(b)]| \leq \nu_a^- |\mathcal{Q}_{2\tilde{\rho}}^-(b)|. \quad (5.12)$$

If this were not the case, then for every $\rho \in (0, \rho^*)$ we would have

$$\begin{aligned} \iint_{(x_0, t_0) + \mathcal{Q}_{2\rho}^-(b)} |u(x_0, t_0) - u(x, t)| dx dt &\geq \iint_{\{u \leq \mu_- + b\} \cap [(x_0, t_0) + \mathcal{Q}_{2\rho}^-(b)]} [u(x_0, t_0) - (\mu_- + b)] dx dt \\ &\geq \nu_a^- [u(x_0, t_0) - (\mu_- + b)] |\mathcal{Q}_{2\rho}^-(b)|, \end{aligned}$$

which, after dividing by $|\mathcal{Q}_{2\rho}^-(b)|$, yields

$$\iint_{(x_0, t_0) + \mathcal{Q}_{2\rho}^-(b)} |u(x_0, t_0) - u(x, t)| dx dt \geq \nu_a^- [u(x_0, t_0) - (\mu_- + b)] > 0, \quad (5.13)$$

for all $\rho \in (0, \rho^*)$. Setting

$$\kappa = \kappa(n, \mathbf{p}, b) := \max \left\{ 2, \max_{1 \leq i \leq n} \left\{ 2b^{\frac{p_i - 2}{p}} \right\} \right\},$$

it is easy to verify that $\mathcal{Q}_{2\rho}^-(b) \subseteq \mathcal{Q}_{\kappa\rho}$, and hence

$$\iint_{(x_0, t_0) + \mathcal{Q}_{2\rho}^-(b)} |u(x_0, t_0) - u| dx dt \leq \frac{|\mathcal{Q}_{\kappa\rho}|}{|\mathcal{Q}_{2\rho}^-(b)|} \iint_{(x_0, t_0) + \mathcal{Q}_{\kappa\rho}} |u(x_0, t_0) - u| dx dt. \quad (5.14)$$

Observe that

$$\frac{|\mathcal{Q}_{\kappa\rho}|}{|\mathcal{Q}_{2\rho}^-(b)|} = \max \left\{ 1, \max_{1 \leq i \leq n} \left\{ b^{\frac{(p_i - 2)(n + \bar{p})}{\bar{p}}} \right\} \right\} \prod_{j=1}^n b^{\frac{2 - p_j}{p_j}},$$

i.e., this ratio depends only on $n, \mathbf{p}, b$, but not on the radius ρ . Then, from (5.14) and the fact that $(x_0, t_0) \in \mathcal{L}$, it follows that

$$\lim_{\rho \rightarrow 0^+} \iint_{(x_0, t_0) + \mathcal{Q}_{2\rho}^-(b)} |u(x_0, t_0) - u(x, t)| dx dt = 0,$$

which contradicts (5.13). Therefore, condition (5.12) holds for some $\tilde{\rho} \in (0, \rho^*)$, and we may apply Lemma 5.2 to deduce that

$$u(x, t) \geq \mu_- + ab > u_*(x_0, t_0), \quad \text{for a.e. } (x, t) \in [(x_0, t_0) + \mathcal{Q}_{\tilde{\rho}}^-(b)].$$

This however contradicts the definition of $u_*(x_0, t_0)$, since

$$u_*(x_0, t_0) = \lim_{\rho \rightarrow 0^+} \operatorname{ess\,inf}_{(x_0, t_0) + \mathcal{Q}_\rho} u \geq \operatorname{ess\,inf}_{(x_0, t_0) + \mathcal{Q}_{\tilde{\rho}}^-(b)} u > u_*(x_0, t_0).$$

As a result, we must have $u_*(x, t) \geq u(x, t)$ for all $(x, t) \in \mathcal{L}$.

Finally, the second part of the statement follows from the observation that if u is a local weak subsolution to (1.1), then $-u$ is a local weak supersolution, together with the fact that the additive inverse of a lower semicontinuous function is upper semicontinuous. $\square$

Remark 5.4. Theorem 5.3 implies that the corresponding semicontinuous representatives of u actually satisfy the pointwise bound (5.4) or (5.5) for every point in the intrinsic cylinder $(y, s) + \mathcal{Q}_\rho^-(M)$.

Remark 5.5. Let u be a local weak solution to (1.1) in the sense of Definition 2.2, and assume that u is locally bounded in Ω_T . By Theorem 5.3, u admits a lower semicontinuous representative u_* and an upper semicontinuous representative $\tilde{u}$. However, this result does not, in general, imply that u is continuous on Ω_T . Indeed, one cannot exclude that

$$u_*(x, t) < \tilde{u}(x, t)$$

on a nonempty set of Lebesgue measure zero. Therefore, continuity of u cannot be deduced solely from the existence of these semicontinuous representatives. In this regard, we aim to address the continuity of local weak solutions in the case $p_1 = \dots = p_n$ in future work.

A Appendix

In what follows, we first show how to explicitly construct a cut-off function ζ of the form (3.1), satisfying (3.2) and (3.3) with $(x_0, t_0) = (0, 0)$. Then, we construct a smooth cut-off function $\tilde{\zeta}_j$ having the properties stated in (4.7).

Construction of the cut-off function ζ . Let $\chi \in C^\infty(\mathbb{R})$ be a non-decreasing function such that

$$0 \leq \chi \leq 1, \quad \chi \equiv 0 \quad \text{on } (-\infty, 0], \quad \chi \equiv 1 \quad \text{on } [1, \infty).$$

For example, one can take

$$\chi(t) = \frac{Z(t)}{Z(t) + Z(1-t)}, \quad \text{with } Z(t) := \begin{cases} e^{-\frac{1}{t}} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

As an immediate consequence of Lagrange's mean value theorem, we have that $\|\chi'\|_\infty \geq 1$. For $i \in \{1, \dots, n\}$ and $\rho \in (0, \infty)$, we choose three positive numbers a_i, b_i and ϵ such that

$$0 < b_i < a_i < \rho^{\frac{1}{p_i}} \quad \text{and} \quad \epsilon \in (0, \rho).$$

Furthermore, we define

$$\psi(t) := \chi\left(\frac{2t + 2\rho - \epsilon}{\epsilon}\right), \quad t \in \mathbb{R},$$

and

$$\zeta_i(s) := \chi\left(\frac{s + a_i}{a_i - b_i}\right) \chi\left(\frac{a_i - s}{a_i - b_i}\right), \quad s \in \mathbb{R}.$$

One can easily check that $\psi \in C^\infty(\mathbb{R}; [0, 1])$ is a non-decreasing map such that

$$\psi \equiv 0 \quad \text{on } \left(-\infty, -\rho + \frac{\epsilon}{2}\right], \quad \psi \equiv 1 \quad \text{on } [-\rho + \epsilon, \infty),$$

while the functions ζ_i satisfy

$$\zeta_i \equiv 1 \quad \text{on } [-b_i, b_i], \quad \zeta_i \equiv 0 \quad \text{on } \mathbb{R} \setminus (-a_i, a_i).$$

Therefore,

$$\zeta_i \in C_0^\infty(\mathbb{R}; [0, 1]) \quad \text{and} \quad \text{supp } \zeta_i \subset (-\rho^{\frac{1}{p_i}}, \rho^{\frac{1}{p_i}}) \quad \text{for every } i \in \{1, \dots, n\}.$$

Now we consider the smooth cut-off function

$$\zeta(x, t) := \psi(t) \prod_{i=1}^n \zeta_i^{p_i}(x_i), \quad (x, t) \in \mathbb{R}^{n+1}.$$

By the properties of ψ and $\zeta_1, \dots, \zeta_n$, we immediately deduce that

$$0 \leq \zeta \leq 1, \quad \zeta \equiv 0 \quad \text{on } \partial_{\text{par}} Q_\rho, \quad \zeta \equiv 1 \quad \text{in } \prod_{i=1}^n (-b_i, b_i) \times (-\rho + \epsilon, 0).$$

Construction of the cut-off function $\tilde{\zeta}_j$. For $i \in \{1, \dots, n\}$ and a fixed $j \in \mathbb{N}_0$, we set

$$a_{i,j} := (\tilde{\rho}_j)^{\frac{1}{p_i}}, \quad b_{i,j} := \rho_{j+1}^{\frac{1}{p_i}},$$

where ρ_{j+1} and $\tilde{\rho}_j$ are defined in (4.4) and (4.5), respectively. Thus we have $0 < b_{i,j} < a_{i,j} < \rho_j^{\frac{1}{p_i}}$. Moreover, we define

$$\psi_j(t) := \chi\left(\frac{t + \tilde{\rho}_j}{\tilde{\rho}_j - \rho_{j+1}}\right), \quad t \in \mathbb{R}, \quad (\text{A.1})$$

and

$$\tilde{\zeta}_{i,j}(s) := \chi\left(\frac{s + a_{i,j}}{a_{i,j} - b_{i,j}}\right) \chi\left(\frac{a_{i,j} - s}{a_{i,j} - b_{i,j}}\right), \quad s \in \mathbb{R}. \quad (\text{A.2})$$

Then, $\psi_j \in C^\infty(\mathbb{R}; [0, 1])$ is a non-decreasing map such that

$$\psi_j \equiv 0 \quad \text{on } (-\infty, -\tilde{\rho}_j], \quad \psi_j \equiv 1 \quad \text{on } [-\rho_{j+1}, \infty),$$

while the functions $\tilde{\zeta}_{i,j}$ satisfy

$$\tilde{\zeta}_{i,j} \equiv 1 \quad \text{on } [-b_{i,j}, b_{i,j}], \quad \tilde{\zeta}_{i,j} \equiv 0 \quad \text{on } \mathbb{R} \setminus (-a_{i,j}, a_{i,j}).$$

Therefore,

$$\tilde{\zeta}_{i,j} \in C_0^\infty(\mathbb{R}; [0, 1]) \quad \text{and} \quad \text{supp } \tilde{\zeta}_{i,j} \subset (-\rho_j^{\frac{1}{p_i}}, \rho_j^{\frac{1}{p_i}}) \quad \text{for every } i \in \{1, \dots, n\}.$$

Now we consider the smooth cut-off function

$$\tilde{\zeta}_j(x, t) := \psi_j(t) \prod_{i=1}^n \tilde{\zeta}_{i,j}^{p_i}(x_i), \quad (x, t) \in \mathbb{R}^{n+1}.$$

By the properties of ψ_j and $\tilde{\zeta}_{1,j}, \dots, \tilde{\zeta}_{n,j}$, we immediately have that

$$0 \leq \tilde{\zeta}_j \leq 1, \quad \tilde{\zeta}_j \equiv 1 \quad \text{in } \mathcal{Q}_{j+1}, \quad \tilde{\zeta}_j \equiv 0 \quad \text{on } \partial_{\text{par}} \mathcal{Q}_j,$$

where the cylinders $\mathcal{Q}_j$ and $\mathcal{Q}_{j+1}$ are defined as in the proof of Theorem 4.1. Furthermore, by (A.1) and the definitions of ρ_{j+1} , $\tilde{\rho}_j$, we have

$$\psi_j'(t) = \frac{1}{\tilde{\rho}_j - \rho_{j+1}} \chi'\left(\frac{t + \tilde{\rho}_j}{\tilde{\rho}_j - \rho_{j+1}}\right) = \frac{2^{j+2}}{(1 - \sigma)\rho} \chi'\left(\frac{t + \tilde{\rho}_j}{\tilde{\rho}_j - \rho_{j+1}}\right),$$

and therefore

$$0 \leq \partial_t \tilde{\zeta}_j(x, t) = \psi_j'(t) \prod_{i=1}^n \tilde{\zeta}_{i,j}^{p_i}(x_i) \leq \frac{2^{j+2} \|\chi'\|_\infty}{(1 - \sigma)\rho} \quad \text{for every } (x, t) \in \mathbb{R}^{n+1}. \quad (\text{A.3})$$

Now, using (A.2), the definitions of $a_{i,j}$ and $b_{i,j}$, and the fact that $0 \leq \chi \leq 1$, we obtain, for every $s \in \mathbb{R}$,

$$\begin{aligned} |\tilde{\zeta}_{i,j}'(s)| &= \frac{1}{a_{i,j} - b_{i,j}} \left| \chi'\left(\frac{s + a_{i,j}}{a_{i,j} - b_{i,j}}\right) \chi\left(\frac{a_{i,j} - s}{a_{i,j} - b_{i,j}}\right) - \chi\left(\frac{s + a_{i,j}}{a_{i,j} - b_{i,j}}\right) \chi'\left(\frac{a_{i,j} - s}{a_{i,j} - b_{i,j}}\right) \right| \\ &\leq \frac{2 \|\chi'\|_\infty}{(\tilde{\rho}_j)^{\frac{1}{p_i}} - \rho_{j+1}^{\frac{1}{p_i}}}. \end{aligned}$$

Then, for every $(x, t) \in \mathbb{R}^{n+1}$, we get

$$|\partial_{x_i} \tilde{\zeta}_j^{\frac{1}{p_i}}(x, t)|^{p_i} = \psi_j(t) |\tilde{\zeta}_{i,j}'(x_i)|^{p_i} \prod_{\ell \neq i} \tilde{\zeta}_{\ell,j}^{p_\ell}(x_\ell) \leq \left[\frac{2 \|\chi'\|_\infty}{(\tilde{\rho}_j)^{\frac{1}{p_i}} - \rho_{j+1}^{\frac{1}{p_i}}} \right]^{p_i}. \quad (\text{A.4})$$

At this point, we consider the function $g_i : [0, \infty) \rightarrow [0, \infty)$ defined by

$$g_i(s) := s^{\frac{1}{p_i}}, \quad i \in \{1, \dots, n\}.$$

By Lagrange's mean value theorem, there exists some $\kappa_{i,j} \in (\rho_{j+1}, \tilde{\rho}_j)$ such that

$$(\tilde{\rho}_j)^{\frac{1}{p_i}} - \rho_{j+1}^{\frac{1}{p_i}} = g_i(\tilde{\rho}_j) - g_i(\rho_{j+1}) = \frac{\tilde{\rho}_j - \rho_{j+1}}{p_i} \kappa_{i,j}^{\frac{1}{p_i} - 1}, \quad (\text{A.5})$$

Joining (A.4) and (A.5), using the definitions of ρ_{j+1} and $\tilde{\rho}_j$, together with the inequalities $\kappa_{i,j} < \tilde{\rho}_j < \rho$ and $p_i > 1$, and recalling that $\|\chi'\|_\infty \geq 1$, $\sigma \in (0, 1)$ and $\mathcal{P} := \max\{p_1, \dots, p_n\}$, we obtain

$$|\partial_{x_i} \tilde{\zeta}_j^{\frac{1}{p_i}}|^{p_i} \leq \frac{(2 p_i \|\chi'\|_\infty)^{p_i} \kappa_{i,j}^{p_i-1}}{(\tilde{\rho}_j - \rho_{j+1})^{p_i}} \leq \frac{(2^{j+3} \mathcal{P} \|\chi'\|_\infty)^{\mathcal{P}} \rho^{p_i-1}}{(1-\sigma)^{p_i} \rho^{p_i}} \leq \frac{(2^{j+3} \mathcal{P} \|\chi'\|_\infty)^{\mathcal{P}}}{(1-\sigma)^{\mathcal{P}} \rho}. \quad (\text{A.6})$$

Finally, taking $c = (8 \mathcal{P} \|\chi'\|_\infty)^{\mathcal{P}}$, from (A.3) and (A.6) we get

$$0 \leq \partial_t \tilde{\zeta}_j \leq \frac{c 2^j}{(1-\sigma) \rho} \quad \text{and} \quad |\partial_{x_i} \tilde{\zeta}_j^{\frac{1}{p_i}}|^{p_i} \leq \frac{c 2^{j\mathcal{P}}}{(1-\sigma)^{\mathcal{P}} \rho} \quad \text{for every } i \in \{1, \dots, n\}.$$

Acknowledgments. We would like to thank the reviewer for his/her valuable comments, which helped to improve our paper. This work has been partially supported by the Wallenberg AI, Autonomous Systems and Software Program (WASP) funded by the Knut and Alice Wallenberg Foundation. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). P. Ambrosio and G. Cupini have been partially supported through the INdAM–GNAMPA 2026 Project “Esistenza e regolarità per soluzioni di equazioni ellittiche e paraboliche anisotrope” (CUP E53C25002010001). S. Ciani has been partially supported through the INdAM–GNAMPA 2026 Project “Classificazione delle soluzioni di equazioni evolutive non locali” (CUP E53C25002010001).

Declarations. On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability. This manuscript has no associated data.

References

- [1] P. Ambrosio, *Besov regularity for a class of singular or degenerate elliptic equations*, J. Math. Anal. Appl. **505**(2), 125636 (2022).
- [2] P. Ambrosio, *Fractional Sobolev regularity for solutions to a strongly degenerate parabolic equation*, Forum Math. **35**(6), 1485-1497 (2023).
- [3] P. Ambrosio, *Gradient bounds for a widely degenerate orthotropic parabolic equation*, Nonlinear Differ. Equ. Appl. **33**, 105 (2026).
- [4] P. Ambrosio, *Gradient regularity for strongly singular or degenerate elliptic and parabolic equations*, Bruno Pini Mathematical Analysis Seminar **16**(1), 68-101 (2025).

- [5] P. Ambrosio, *Sharp Sobolev regularity for widely degenerate parabolic equations*, Calc. Var. **64**, 32 (2025).
- [6] P. Ambrosio, F. Bäuerlein, *Gradient bounds for strongly singular or degenerate parabolic systems*, J. Differ. Equ. **401**, 492-549 (2024).
- [7] P. Ambrosio, S. Ciani, *Local boundedness for weak solutions to strongly degenerate orthotropic parabolic equations*, Ricerche mat **75**, 561-579 (2026).
- [8] P. Ambrosio, S. Cuomo, M. De Rosa, *A physics-informed deep learning approach for solving strongly degenerate parabolic problems*, Engineering with Computers **41**, 4013-4029 (2025).
- [9] P. Ambrosio, G. Cupini, E. Mascolo, *Regularity of vectorial minimizers for non-uniformly elliptic anisotropic integrals*, Nonlinear Anal. **261**, Paper No. 113897, 18 (2025).
- [10] P. Ambrosio, A.G. Grimaldi, A. Passarelli di Napoli, *On the second-order regularity of solutions to widely singular or degenerate elliptic equations*, Annali di Matematica **205**, 327-364 (2026).
- [11] P. Ambrosio, A. Passarelli di Napoli, *Regularity results for a class of widely degenerate parabolic equations*, Adv. Calc. Var. **17**(3), 805-829 (2023).
- [12] S. Biagi, G. Cupini, E. Mascolo, *Local boundedness for solutions of a class of non-uniformly elliptic anisotropic problems*, Nonlinear Anal. **262**, Paper No. 113915, (2026).
- [13] V. Bögelein, F. Duzaar, R. Giova, A. Passarelli di Napoli, *Higher regularity in congested traffic dynamics*, Math. Ann. **385**, 1-56 (2023).
- [14] V. Bögelein, F. Duzaar, R. Giova, A. Passarelli di Napoli, *Gradient regularity for a class of widely degenerate parabolic systems*, SIAM J. Math. Anal. **56**(4), 5017-5078 (2024).
- [15] V. Bögelein, F. Duzaar, P. Marcellini, *Parabolic equations with p, q -growth*, J. Math. Pures Appl. **100** (4), 535-563 (2013).
- [16] V. Bögelein, F. Duzaar, P. Marcellini, S. Signoriello, *Parabolic equations and the bounded slope condition*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **34**(2), 355-379 (2017).
- [17] V. Bögelein, F. Duzaar, G. Mingione, *The regularity of general parabolic systems with degenerate diffusion*, Mem. Amer. Math. Soc. **221** (1041) (2013).
- [18] P. Bousquet, L. Brasco, *C^1 regularity of orthotropic p -harmonic functions in the plane*, Anal. PDE **11**, no. 4, 813-854 (2018).
- [19] P. Bousquet, L. Brasco, *Lipschitz regularity for orthotropic functionals with nonstandard growth conditions*, Rev. Mat. Iberoamericana **36**, 1989-2032 (2020).
- [20] P. Bousquet, L. Brasco, V. Julin, *Lipschitz regularity for local minimizers of some widely degenerate problems*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **16** (2016), no. 4, 1235-1274.
- [21] P. Bousquet, L. Brasco, C. Leone, *Singular orthotropic functionals with nonstandard growth conditions*, Rev. Mat. Iberoam. **40**, no. 2, 753-802 (2024).
- [22] P. Bousquet, L. Brasco, C. Leone, A. Verde, *On the Lipschitz character of orthotropic p -harmonic functions*, Calc. Var. **57**, 88 (2018).
- [23] P. Bousquet, L. Brasco, C. Leone, A. Verde, *Gradient estimates for an orthotropic nonlinear diffusion equation*, Adv. Calc. Var. **16** (3), 705-730 (2023).
- [24] L. Brasco, *Global L^∞ gradient estimates for solutions to a certain degenerate elliptic equation*, Nonlinear Anal. **74**, 516-531 (2011).
- [25] L. Brasco, G. Carlier, *Congested traffic equilibria and degenerate anisotropic PDEs*, Dyn. Games Appl. **3** (4), 508-522 (2013).

- [26] L. Brasco, G. Carlier, *On certain anisotropic elliptic equations arising in congested optimal transport: local gradient bounds*, Adv. Calc. Var. **7**, 379-407 (2014).
- [27] L. Brasco, G. Carlier, F. Santambrogio, *Congested traffic dynamics, weak flows and very degenerate elliptic equations* [corrected version of mr2584740], J. Math. Pures Appl. (9) **93**(6), 652-671 (2010).
- [28] L. Brasco, C. Leone, G. Pisante, A. Verde, *Sobolev and Lipschitz regularity for local minimizers of widely degenerate anisotropic functionals*, Nonlinear Anal. **153**, 169-199 (2017).
- [29] A. Cianchi, *Local boundedness of minimizers of anisotropic functionals*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **17**, no. 2, 147-168 (2000).
- [30] S. Ciani, U. Guarnotta, V. Vespri, *On a particular scaling for the prototype anisotropic p -Laplacian*. In: Recent Advances in Mathematical Analysis. Trends in Mathematics, pp. 289-308. Birkhäuser, Basel (2023).
- [31] S. Ciani, E. Henriques, M.O. Savchenko, I.I. Skrypnik, *Parabolic De Giorgi classes with doubly nonlinear, nonstandard growth: local boundedness under exact integrability assumptions*, J. Differ. Equ. **462**, 114235 (2026).
- [32] S. Ciani, E. Henriques, I.I. Skrypnik, *The impact of intrinsic scaling on the rate of extinction for anisotropic non-Newtonian fast diffusion*, Nonlinear Anal. **242**, 113497 (2024).
- [33] S. Ciani, E. Henriques, I.I. Skrypnik, *On the continuity of solutions to anisotropic elliptic operators in the limiting case*, Bull. London Math. Soc., **57**: 1548-1567 (2025).
- [34] S. Ciani, S. Mosconi, V. Vespri, *Parabolic Harnack estimates for anisotropic slow diffusion*, JAMA **149**, 611-642 (2023).
- [35] S. Ciani, I.I. Skrypnik, V. Vespri, *On the local behavior of local weak solutions to some singular anisotropic elliptic equations*, Adv. Nonlinear Anal. **12**(1), 237-265 (2022).
- [36] S. Ciani, V. Vespri, M. Vestberg, *Boundedness, ultracontractive bounds and optimal evolution of the support for doubly nonlinear anisotropic diffusion*, Annali di Matematica **205**, 181-231 (2026).
- [37] A. Clop, R. Giova, F. Hatami, A. Passarelli di Napoli, *Very degenerate elliptic equations under almost critical Sobolev regularity*, Forum Math. **32** (6), 1515-1537 (2020).
- [38] M. Colombo, A. Figalli, *Regularity results for very degenerate elliptic equations*, J. Math. Pures Appl. **101**(1), 94-117 (2014).
- [39] G. Cupini, F. Giannetti, R. Giova, A. Passarelli di Napoli, *Regularity results for vectorial minimizers of a class of degenerate convex integrals*, J. Differ. Equ. **265**(9), 4375-4416 (2018).
- [40] G. Cupini, P. Marcellini, E. Mascolo, *Regularity under sharp anisotropic general growth conditions*, Discrete Contin. Dyn. Syst. Ser. B **11**, no. 1, 66-86 (2009).
- [41] G. Cupini, P. Marcellini, E. Mascolo, *Local boundedness of solutions to quasilinear elliptic systems*, Manuscripta Math. **137**, no. 3-4, 287-315 (2012).
- [42] G. Cupini, P. Marcellini, E. Mascolo, *Local boundedness of minimizers with limit growth conditions*, J. Optim. Theory Appl. **166**, no. 1, 1-22 (2015).
- [43] G. Cupini, P. Marcellini, E. Mascolo, *Regularity of minimizers under limit growth conditions*, Nonlinear Anal. **153**, 294-310 (2017).
- [44] C. De Filippis, *Gradient bounds for solutions to irregular parabolic equations with (p, q) -growth*, Calc. Var. **59**(5), 171 (2020).
- [45] E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat., P.I III Ser. **3**, 25-43 (1957).
- [46] E. DiBenedetto, *Degenerate Parabolic Equations*, Universitext, Springer, New York, 1993.

- [47] E. DiBenedetto, U. Gianazza, V. Vespri, *Harnack estimates for quasi-linear degenerate parabolic differential equations*, Acta Math. **200**, 181-209 (2008).
- [48] E. DiBenedetto, J.M. Urbano, V. Vespri, *Current issues on singular and degenerate evolution equations*, in: Handbook of Differential Equations, vol. 1, pp. 169-286. North-Holland, Amsterdam (2004).
- [49] F.G. Düzgün, S. Mosconi, V. Vespri, *Anisotropic Sobolev embeddings and the speed of propagation for parabolic equations*, J. Evol. Equ. **19**, 845-882 (2019).
- [50] M. Eleuteri, P. Marcellini, E. Mascolo, *Regularity for scalar integrals without structure conditions*, Adv. Calc. Var. **13**, 279-300 (2020).
- [51] F. Feo, A. Passarelli di Napoli, M.R. Posteraro, *Local boundedness for minimizers of anisotropic functionals with monomial weights*, J. Optim. Theory Appl. **201**, 1313-1332 (2024).
- [52] N. Fusco, C. Sbordone, *Local boundedness of minimizers in a limit case*, Manuscripta Math. **69**, 19-25 (1990).
- [53] N. Fusco, C. Sbordone, *Some remarks on the regularity of minima of anisotropic integrals*, Comm. Partial Differential Equations **18**, no. 1-2, 153-167 (1993).
- [54] A. Gentile, A. Passarelli di Napoli, *Higher regularity for weak solutions to degenerate parabolic problems*, Calc. Var. **62**, 225 (2023).
- [55] M. Giaquinta, *Growth conditions and regularity. A counterexample*, Manuscr. Math. **59**, 245-248 (1987).
- [56] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific Publishing Co., 2003.
- [57] A.G. Grimaldi, *Higher regularity for minimizers of very degenerate convex integrals*, Nonlinear Anal. **242**, 113520 (2024).
- [58] A.G. Grimaldi, S. Russo, *Weak comparison principle for widely degenerate elliptic equations*, J Elliptic Parabol Equ (2026). DOI: <https://doi.org/10.1007/s41808-026-00480-6>.
- [59] A.G. Grimaldi, S. Russo, *Regularity for minimizers of degenerate, non-autonomous, orthotropic integral functionals*, Ricerche mat **75**, 1017-1053 (2026).
- [60] N. Liao, *Regularity of weak supersolutions to elliptic and parabolic equations: lower semicontinuity and pointwise behavior*, J. Math. Pures Appl. **147**, 179-204 (2021).
- [61] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Gauthier-Villars, Paris, 1969.
- [62] V. Liskevich, I.I. Skrypnik, *Hölder continuity of solutions to an anisotropic elliptic equation*, Nonlinear Anal. **71**, 1699-1708 (2009).
- [63] P. Marcellini, *Un esempio di soluzione discontinua d'un problema variazionale nel caso scalare*, Ist. Mat. "U. Dini" No. 11, Firenze (1987).
- [64] L. Mons, *Higher regularity for minimizers of very degenerate integral functionals*, J. Math. Anal. Appl. **518**(2), 126717 (2023).
- [65] M. Piccirillo, *Regularity for a strongly degenerate equation with explicit u -dependence*, preprint (2025), [arXiv: 2511.00976](https://arxiv.org/abs/2511.00976).
- [66] S. Russo, *On widely degenerate p -Laplace equations with symmetric data*, Rev Mat Complut **38**, 939-963 (2025).
- [67] S. Russo, *Higher differentiability of minimizers for non-autonomous orthotropic functionals*, Nonlinear Analysis: Real World Applications, **87**, 104450 (2026).

- [68] R.E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Math. Surveys Monogr. **49**, American Mathematical Society, Providence, 1997.
- [69] B. Stroffolini, *Global boundedness of solutions of anisotropic variational problems*, Boll. Un. Mat. Ital. **A 7** (5) (1991) 345-352, (3).
- [70] M. Strunk, *Gradient regularity for widely degenerate elliptic partial differential equations*, Partial Differ. Equ. Appl. **6**, 39 (2025).
- [71] M. Strunk, *Gradient regularity for widely degenerate parabolic equations*, preprint (2025), [arXiv: 2510.07999](https://arxiv.org/abs/2510.07999).
- [72] A.S. Tersenov, A.S. Tersenov, *Existence of Lipschitz continuous solutions to the Cauchy-Dirichlet problem for anisotropic parabolic equations*, J. Funct. Anal. **272**, no. 10, 3965-3986 (2017).
- [73] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche Mat. **18**, 3-24 (1969).
- [74] M.I. Vishik, *Solubility of boundary-value problems for quasi-linear parabolic equations of higher orders*, Mat. Sb. (N. S.) **59(101)** (1962), 289-325.
- [75] M. Yu, X. Lian, *Boundedness of solutions of parabolic equations with anisotropic growth conditions*, Can. J. Math. **49**, 798-809 (1997).
- [76] E. Zeidler, *Nonlinear Functional Analysis and its Applications. II/A – Linear Monotone Operators*, Springer, New York, 1989.

Pasquale Ambrosio

Department of Mathematics, Uppsala University
 P.O. Box 524, 751 20 Uppsala, Sweden.
E-mail address: pasquale.ambrosio@math.uu.se

Simone Ciani

Dipartimento di Matematica, Università di Bologna
 Piazza di Porta S. Donato 5, 40126 Bologna, Italy.
E-mail address: simone.ciani3@unibo.it

Giovanni Cupini

Dipartimento di Matematica, Università di Bologna
 Piazza di Porta S. Donato 5, 40126 Bologna, Italy.
E-mail address: giovanni.cupini@unibo.it