A discrete-time overdetermined problem for the heat equation

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Abstract

In this paper, we consider a parabolic counterpart of Serrin's overdetermined problem, in which the overdetermined condition (constant flux condition) is imposed only on a discrete infinite set of time values. We show that, under suitable regularity assumptions on the domain, such a discrete-time overdetermined problem admits a solution if and only if the domain is a ball. Remarkably, depending on the temporal scale, the same overdetermined condition captures either geometric or spectral information, yet both mechanisms lead to the same rigidity conclusion. We study both the case in which the constant flux condition is imposed on the boundary and the case in which the constant flux condition is imposed on an interior surface. We remark that the methods employed in our analysis do not depend on the location of the overdetermined surface but only on whether the sequence of time instants accumulates away from zero. Finally, we will show how this problem generalizes to complete Riemannian manifolds.

Key words. overdetermined problem, discrete-time overdetermined condition, heat equation, symmetry, ball, eigenfunction expansion, Serrin's problem, heat content asymptotics, Aleksandrov's soap bubble theorem

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1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$. We consider the following initial/boundary-value problem for the heat equation:

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times (0, \infty), \\ u = 1 & \text{on } \Omega \times \{0\}, \\ u = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{1.1}$$

In what follows, it will be convenient to think of u as a function of time with values in a function space $X(\Omega)$, instead of a real-valued function of several variables. In such cases, we will employ the notation $t \mapsto u(t) \in X(\Omega)$, where $X(\Omega)$ is a function space of real-valued functions defined on Ω (which may depend on the context) and $u(t)(x) := u(x,t)$ for $x \in \Omega$ and $t > 0$.

This paper addresses the problem of *characterizing Euclidean balls by the behavior of the heat flux* $\partial_\nu u(t)$ of the solution to (1.1). This can be seen as a natural extension of Serrin's overdetermined problem to a parabolic setting. In his celebrated paper [34], J. Serrin studied the following elliptic overdetermined problem:

$$-\Delta U = 1 \quad \text{in } \Omega, \quad U = 0 \quad \text{on } \partial\Omega, \quad \partial_\nu U = c \quad \text{on } \partial\Omega, \tag{1.2}$$

where ∂_ν denotes the exterior normal derivative, and c is a real constant. He showed that if Ω is a bounded domain of class C^2, then problem (1.2) admits a solution $U \in C^2(\overline{\Omega})$ if and only if Ω is a ball (and the solution U is radial). Serrin's original proof relies on a newly introduced method, now referred to as the *method of moving planes*, which refines Aleksandrov's *reflection principle* (see [1]). In the following years, many mathematicians worked on improving Serrin's result in multiple directions: by providing alternative methods of proof (see [37] for a new strategy using a P-function and the maximum principle), by extending it to the manifold setting (see [21] for positive results in the hyperbolic space, and in the sphere for domains included in a hemisphere; and [11] for counterexamples to symmetry in the sphere under no additional assumption), by weakening the regularity assumptions on the domain ([36] showed that C^1 regularity for the boundary is enough to obtain symmetry if the overdetermination is considered in some suitable generalized sense, [28] extended Serrin's result to domains with a cusp, and finally [13] showed that Serrin's result holds true among bounded domains of finite perimeter that satisfy some density bounds), and by extending the result to nonlinear operators (see [16, 14, 10] for an extension to the p-Laplacian and [12, 15, 9] for the ∞-Laplacian; further developments also include equations driven by fully nonlinear non-divergence-form operators, such as k-Hessian equations [4].) For a more general overview and a complete list of references, we suggest the surveys [27, 23].

The parabolic counterpart of Serrin's problem has also been studied from several perspectives. A first important contribution in this direction is [2], in which the authors considered symmetry for a class of possibly degenerate parabolic equations, thus providing an early parabolic analogue of the Aleksandrov–Serrin theory. For the heat equation, a more specific line of research was later developed in [24] through the study of *stationary isothermic surfaces*, that is, spatial level surfaces of the temperature which do not depend on time. The results proved in [24], together with subsequent extensions provided in [25], show that such overdetermined parabolic information has strong geometric consequences and, in the bounded case, characterizes balls. This point of view is also closely related to the *constant flow property*, introduced in the Riemannian setting in [3, 32] and further investigated for two-phase conductors in [5], and later for general multi-phase conductors in [6].

In the pursuit of extending Serrin's problem to the parabolic setting, one might first consider the following naïve question:
"For what bounded domain $\Omega \subset \mathbb{R}^N$ *does the solution* u *of* (1.1) *also satisfy the following overdetermined condition at all times* $t > 0$ *for some function* $b(\cdot)$*?"*

$$\partial_\nu u(t) = b(t) \quad \text{on } \partial\Omega. \tag{1.3}$$

In other words, such a problem aims to characterize all bounded domains satisfying the "*constant flow property*" at the boundary (see [32], where such terminology was first introduced in the context of the heat flow on Riemannian manifolds, and [5], where the authors characterized the geometry of two-phase heat conductors with the constant flow property). Although maybe not obvious at first glance, requiring the overdetermined condition (1.3) to hold for all $t > 0$ is a very restrictive assumption. Indeed, if u satisfies both (1.1) and (1.3) for all $t > 0$, then one can readily check that the function

$$U(x) := \int_0^\infty u(x,t)\, dt$$

satisfies Serrin's overdetermined problem (1.2) for $c := \int_0^\infty b(t)\, dt$. In other words, if (1.3) holds *for all* $t > 0$, then Ω must be a ball. Analogous reduction techniques used to turn a parabolic overdetermined problem into a more manageable elliptic one via time integrals or integral transforms in the time variable are presented in [29, 30, 31, 5, 6].

The previous observation shows that prescribing the overdetermined condition (1.3) for every $t > 0$ is too restrictive if one wishes to obtain genuinely parabolic phenomena, since the problem reduces to the classical elliptic setting. It is therefore natural to ask whether a weaker requirement in time still carries geometric information about the domain.

In what follows, we consider the following overdetermined boundary condition at infinitely many times.

$$\begin{gathered}\text{There exist sequences } (t_n)_n \subset (0,\infty) \text{ and } (b_n)_n \subset \mathbb{R} \text{ such that} \\ \partial_\nu u(t_n) \equiv b_n \quad \text{on } \partial\Omega \quad \text{for all } n \in \mathbb{N},\end{gathered} \tag{1.4}$$

where the equality is understood in the sense of *weak conormal derivatives* (see Definition 2.1).

At first glance, imposing the condition only at a sequence of times, as in (1.4), might appear as a purely technical relaxation. However, this is not the case. Indeed, unlike condition (1.3), the discrete-time overdetermination does not allow for a direct reduction to an elliptic problem via time-integration. As a consequence, the problem retains an intrinsically parabolic nature, and its analysis requires a different set of tools, combining spectral information and short-time asymptotics. In this sense, the newly introduced condition (1.4) should be regarded not merely as a weakening of (1.3), but rather as a genuinely different regime where the temporal structure of the heat flow plays a crucial role.

At first glance, imposing the condition only at a sequence of times, as in (1.4), might appear as a purely technical relaxation. However, this is not the case. The novelty of our approach lies in the fact that the discrete-time overdetermination prevents a direct reduction to an elliptic problem via time-integration. Consequently, the problem remains genuinely parabolic and exhibits two distinct rigidity mechanisms depending on the accumulation point $t^\star \in [0, \infty]$ of the sequence $(t_n)_n$:

- **A spectral rigidity mechanism** when $t^\star \in (0, \infty]$, where the overdetermination encodes the long-time behavior of the heat flow and the spectral properties of the Dirichlet Laplacian;

- **A short-time geometric rigidity mechanism** when $t^\star = 0$, governed by the local geometry of the boundary via heat-content asymptotics.

This dichotomy highlights an intrinsically parabolic feature: the same boundary condition, observed at different temporal scales, triggers two independent phenomena that both ultimately enforce spherical symmetry.

Clearly, if Ω is a ball, then the solution to (1.1) satisfies the overdetermined condition (1.4) for all times $t > 0$, thus in particular, for any sequence $(t_n)_n$. In fact, we will show that satisfying (1.4) is sufficient to characterize the ball among domains with a suitable regularity assumption.

We now present our main results, which show that both regimes $t^\star \in (0, \infty]$ and $t^\star = 0$ yield the same rigidity conclusion.

Theorem I. *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ $(N \geq 2)$. Suppose that the solution to* (1.1) *satisfies the overdetermined condition* (1.4) *and the sequence $(t_n)_n$ has an accumulation point $t^\star \in (0, \infty]$. Then, Ω is a ball.*

Theorem II. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *(*$N \geq 2$*) whose boundary is of class* C^∞*. Suppose that the solution to* (1.1) *satisfies the overdetermined condition* (1.4) *and the sequence* $(t_n)_n$ *has* $t^\star = 0$ *as an accumulation point. Then,* Ω *is a ball.*

Remark 1.1 (On the difference in the regularity assumptions of Theorems I-II)**.** *As previously stated, the two cases* $t^\star \in (0, \infty]$ *and* $t^\star = 0$ *translate to essentially different information on the behavior of the solution* u*. Indeed, while case* (*i*) *determines the long-time behavior of the heat flow, thus giving us information on the eigenfunctions of the Dirichlet Laplacian on* Ω *(see the proof of Theorem I given in section 3), case* (*ii*) *determines the short-time behavior of the heat flow, thus giving us information on the curvature of* $\partial\Omega$ *(see the proof of Theorem II given in section 4). This essential difference is reflected in the distinct regularity assumptions imposed in Theorems I-II.*

In addition to Theorems I-II, an analogous symmetry result can be obtained by imposing the overdetermination on a surface inside Ω. Similarly to condition (1.4), we can consider the following overdetermined condition, imposed on the boundary of a *subdomain* ω of Ω for countably infinitely many instants of time.

$$\begin{gathered}\text{There exist a Lipschitz subdomain } \emptyset \neq \omega \subset \overline{\omega} \subset \Omega \text{ and} \\ \text{sequences } (t_n)_n,\ (\tau_n)_n \subset (0,\infty), \quad (a_n)_n,\ (b_n)_n \subset \mathbb{R} \text{ such that} \\ u(\tau_n) \equiv a_n, \quad \partial_\nu u(t_n) \equiv b_n \quad \text{on } \partial\omega \quad \text{for all } n \in \mathbb{N},\end{gathered} \tag{1.5}$$

where the equalities above have to be understood in the sense of traces (for $u(\tau_n)$) and in the sense of weak conormal derivatives (for $\partial_\nu u(t_n)$).

Theorem III. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *(*$N \geq 2$*) whose boundary* $\partial\Omega$ *is made entirely of regular points for the Dirichlet Laplacian (see [18, Chapter 8]). Suppose that the solution to* (1.1) *satisfies the overdetermined condition* (1.5) *for some sequences* $(t_n)_n$ *and* $(\tau_n)_n$*. Assume that* $(t_n)_n$ *and* $(\tau_n)_n$ *have accumulation points* $t^\star, \tau^\star \in (0, \infty]$ *respectively. Then,* ω *and* Ω *are concentric balls.*

Remark 1.2 (Why the overdetermined surface in Theorem III must be the boundary of a subdomain)**.** *In the setting of Theorem III, we assumed that the* overdetermined surface $\partial\omega$ *is the boundary of a subdomain* $\omega \subset \overline{\omega} \subset \Omega$*. We remark that this is not a technical assumption. Indeed, just requiring condition* (1.5) *to simply hold on a closed manifold* $\Gamma \subset \Omega$ *is not enough to conclude that* $\partial\Omega$ *and* Γ *are concentric spheres. A simple counterexample is given by the following annular configuration:*

$$\Omega := B_R \setminus \overline{B_r}, \quad \Gamma := \partial B_\rho, \quad (0 < r < \rho < R).$$

Plan of the paper. This paper is organized as follows. In section 2, we present some preliminary results on weak conormal derivatives and the eigenvalue expansion of the solution to the heat equation. In section 3, we show Theorem I by turning the discrete-time overdetermination (1.4) into an overdetermined condition on the Dirichlet eigenfunctions of the Laplacian and then conclude by making use of Serrin's theorem on rough domains [13]. In section 4, we show Theorem 4 by turning the discrete-time overdetermination (1.4) into a condition on the mean curvature of $\partial\Omega$, and then conclude by the classical Aleksandrov's soap bubble theorem [1]. In section 5, we give a proof of Theorem III along the lines of Theorem I. Finally, in section 6, we discuss how the above results generalize to the case of complete Riemannian manifolds such as space forms.

2 Preliminaries

In this section, we present some preliminary technical results concerning *weak conormal derivatives* and the *eigenvalue expansion* of the solution u of (1.1).

Definition 2.1 (Weak conormal derivative)**.** *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ ($N \geq 2$). Let $\varphi \in H^1(\Omega)$ be such that $\Delta\varphi \in L^2(\Omega)$. Then, we can define its conormal derivative*

$$\partial_\nu \varphi \in H^{-1/2}(\partial\Omega) := \left(H^{1/2}(\partial\Omega)\right)^*$$

via the following duality pairing:

$${}_{H^{-1/2}}\langle \partial_\nu\varphi, \psi\rangle_{H^{1/2}} := \int_\Omega \Delta\varphi\widetilde{\psi} + \int_\Omega \nabla\varphi \cdot \nabla\widetilde{\psi}. \tag{2.1}$$

Here, $\widetilde{\psi}$ denotes the harmonic extension of ψ, that is, the unique function $\widetilde{\psi} \in H^1(\Omega)$ satisfying

$$\widetilde{\psi}\Big|_{\partial\Omega} = \psi \quad \textit{and} \quad \int_\Omega \nabla\widetilde{\psi}\cdot\nabla w = 0 \quad \textit{for all } w \in H^1_0(\Omega). \tag{2.2}$$

Finally, for any given constant $b \in \mathbb{R}$, the notation $\partial_\nu\varphi \equiv b$ will be used as a shorthand for the following:

$${}_{H^{-1/2}}\langle \partial_\nu\varphi, \psi\rangle_{H^{1/2}} = b\int_{\partial\Omega}\psi \quad \textit{for all } \psi \in H^{1/2}(\partial\Omega).$$

Remark 2.2 (On the regularity assumptions in Definition 2.1)**.** *In Definition 2.1 we consider a function $\varphi \in H^1(\Omega)$ with $\Delta\varphi \in L^2(\Omega)$. One might wonder whether such a φ automatically belongs to $H^2(\Omega)$. Unfortunately, in general, this is not the case. Indeed, peculiar Lipschitz domains Ω are known to exist where the solution to the Laplace equation* *with data in* $L^2(\Omega)$ *and Dirichlet* 0 *boundary condition fails to belong to* $H^2(\Omega)$ *(see [20, Theorem A, 2]). As a result, the gradient* $\nabla\varphi$ *only belongs to* $L^2(\Omega)$ *in general, and so it does not necessarily admit a trace in the classical sense.*

Remark 2.3. *Notice that the pairing* (2.1) *coincides with a formal application of integration by parts. Also, notice that the second term in the right-hand side of* (2.1) *vanishes if* φ *has constant trace on* $\partial\Omega$ *(thus, in particular, when* $\varphi \in H_0^1(\Omega)$*) by* (2.2)

Remark 2.4. *The pairing* (2.1) *actually defines a bounded linear mapping from* $H^{1/2}(\partial\Omega)$ *to* $\mathbb{R}$*, that is,* $\partial_\nu\varphi \in H^{-1/2}(\partial\Omega)$*, as claimed. Linearity is obvious, while boundedness is a consequence of the following estimate (see [26, Lemma 4.3]):*

$$\|\partial_\nu\varphi\|_{H^{-1/2}(\partial\Omega)} \le C_1\|\varphi\|_{H^1(\Omega)} + C_2\|\Delta\varphi\|_{L^2(\Omega)}\,, \tag{2.3}$$

for some $C_1, C_2 > 0$ *independent of* φ*.*

Functions with a constant weak conormal derivative can actually be characterized without explicitly mentioning the constant at play. As the following result shows, this can be done by making use of the subspace of zero-average functions:

$$H_*^{1/2}(\partial\Omega) := \left\{\psi \in H^{1/2}(\partial\Omega) \;\middle|\; \int_{\partial\Omega}\psi = 0\right\}. \tag{2.4}$$

Lemma 2.5. *Let* Ω *and* φ *satisfy the assumptions of Definition 2.1. Then, the following are equivalent:*

(*a*) $\partial_\nu\varphi \equiv b$ *for some* $b \in \mathbb{R}$ *in the sense of Definition 2.1.*

(*b*) ${}_{H^{-1/2}}\langle\partial_\nu\varphi, \psi\rangle_{H^{1/2}} = 0$ *holds true for all* $\psi \in H_*^{1/2}(\partial\Omega)$*.*

Proof. The implication $(a) \implies (b)$ follows readily from Definition 2.1. In what follows, we will show $(b) \implies (a)$. To this end, assume (b) and take an arbitrary $\psi \in H^{1/2}(\partial\Omega)$. Set

$$\overline{\psi} := \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\psi, \quad \psi_* := \psi - \overline{\psi} \in H_*^{1/2}(\partial\Omega).$$

We have

$$\begin{aligned} {}_{H^{-1/2}}\langle\partial_\nu\varphi, \psi\rangle_{H^{1/2}} &= \underbrace{{}_{H^{-1/2}}\langle\partial_\nu, \psi_*\rangle_{H^{1/2}}}_{=0} + {}_{H^{-1/2}}\left\langle\partial_\nu\varphi, \overline{\psi}\right\rangle_{H^{1/2}} \\ &= \overline{\psi}\;{}_{H^{-1/2}}\langle\partial_\nu\varphi, 1\rangle_{H^{1/2}} = \overline{\psi}\int_\Omega \Delta\varphi \cdot 1 = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\psi\int_\Omega\Delta\varphi, \end{aligned}$$

which is nothing but (a) with $b = \frac{1}{|\partial\Omega|}\int_\Omega \Delta\varphi$. □

Let $D(\Delta_D)$ denote the "domain of the Dirichlet Laplacian", defined as

$$D(\Delta_D) := \left\{ v \in H_0^1(\Omega) \;\middle|\; \Delta v \in L^2(\Omega) \text{ (in the sense of distributions)} \right\}.$$

We recall that, for a function $v \in H_0^1(\Omega)$, the expression "$\Delta v \in L^2(\Omega)$ in the sense of distributions" means that there exists some $f \in L^2(\Omega)$ such that

$$\int_\Omega \nabla v \cdot \nabla w = -\int_\Omega f w \quad \text{for all } w \in H_0^1(\Omega). \tag{2.5}$$

Moreover, in such a case, we will write $\Delta v = f$. We remark that the space $D(\Delta_D)$ becomes a Banach space when endowed with the following graph norm:

$$\|v\|_{D(\Delta_D)} := \|v\|_{H_0^1(\Omega)} + \|\Delta v\|_{L^2(\Omega)}.$$

Lemma 2.6. *Let $\Phi \in D(\Delta_D)$ satisfy $-\Delta\Phi = 1 \in L^2(\Omega)$. If, in addition, its weak conormal derivative $\partial_\nu\Phi$ satisfies ${}_{H^{-1/2}}\langle \partial_\nu \Phi, \psi\rangle_{H^{1/2}} = 0$ for all $\psi \in H_*^{1/2}(\partial\Omega)$, then there exists some $b \in \mathbb{R}$ such that Φ is also a weak solution to the following boundary value problem:*

$$-\Delta\Phi = 1 \quad \text{in } \Omega, \qquad \partial_\nu \Phi = b \quad \text{on } \partial\Omega,$$

that is, $\Phi \in H_0^1(\Omega)$ satisfies

$$\int_\Omega \nabla\Phi \cdot \nabla w = \int_\Omega w + b\int_{\partial\Omega} w \quad \text{for all } w \in H^1(\Omega). \tag{2.6}$$

Proof. By Lemma 2.5, there exists some $b \in \mathbb{R}$ such that $\partial_\nu\Phi \equiv b$ in the sense of weak conormal derivatives. Take an arbitrary $w \in H^1(\Omega)$. We can decompose it as

$$w = w_0 + \widetilde{w},$$

where $\widetilde{w} \in H^1(\Omega)$ is the harmonic extension of $w\big|_{\partial\Omega}$. In other words, $\widetilde{w}\big|_{\partial\Omega} = w\big|_{\partial\Omega}$ and

$$\int_\Omega \nabla\widetilde{w} \cdot \nabla v = 0 \quad \text{for all } v \in H_0^1(\Omega). \tag{2.7}$$

Also, notice that $w_0 \in H_0^1(\Omega)$ by construction.

Using $\Phi \in H_0^1(\Omega)$ as a test function in (2.7) yields

$$\int_\Omega \nabla\widetilde{w} \cdot \nabla\Phi = 0. \tag{2.8}$$

On the other hand, integrating $-\Delta\Phi = 1$ against w_0 yields

$$\int_\Omega w_0 = -\int_\Omega \Delta\Phi w_0 = \int_\Omega \nabla\Phi \cdot \nabla w_0. \tag{2.9}$$

Now, combining (2.8), (2.9), and the fact that $\partial_\nu \Phi \equiv b$ in the sense of conormal derivatives yields

$$\int_\Omega \nabla\Phi \cdot \nabla w = \int_\Omega \nabla\Phi \cdot \nabla w_0 + \underbrace{\int_\Omega \nabla\Phi \cdot \nabla \widetilde{w}}_{=0} = \int_\Omega w_0 = \int_\Omega w - \int_\Omega \widetilde{w}$$

$$= \int_\Omega w + \int_\Omega \Delta\Phi \widetilde{w} = \int_\Omega w + {}_{H^{-1/2}}\Big\langle \partial_\nu \Phi, w\big|_{\partial\Omega} \Big\rangle_{H^{1/2}} = \int_\Omega w + b \int_{\partial\Omega} w.$$

Since $w \in H^1(\Omega)$ was arbitrary, this concludes the proof. $\square$

2.1 On the eigenfunction expansion of the solution

Consider the following eigenvalue problem:

$$\text{Find } \phi \text{ such that } \int_\Omega \nabla\phi \cdot \nabla w = \lambda \int_\Omega \phi w \quad \text{for all } w \in H_0^1(\Omega). \tag{2.10}$$

It is known that, without any additional smoothness assumptions on Ω, problem (2.10) admits an increasing sequence of positive eigenvalues, which will be counted *with multiplicity* as follows:

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \lambda_{k+1} \le \cdots \nearrow \infty.$$

Also, let $(\phi_k)_{k\in\mathbb{N}} \subset H_0^1(\Omega)$ denote an orthonormal family (in $L^2(\Omega)$) of eigenfunctions of (2.10). We remark that sometimes it will be more convenient to index the eigenvalues of (2.10) by counting them *without multiplicity*, as

$$0 < \Lambda_1 < \Lambda_2 < \Lambda_3 < \cdots < \Lambda_k < \Lambda_{k+1} < \cdots \nearrow \infty.$$

In this case, Λ_k will be defined inductively by

$$\Lambda_1 := \lambda_1, \quad \Lambda_{k+1} := \min\left\{\lambda_j \;\middle|\; j \in \mathbb{N}, \quad \lambda_j > \Lambda_k\right\} \quad \text{for } k \ge 1.$$

Also, for $k \in \mathbb{N}$, V_k will denote the eigenspace corresponding to Λ_k:

$$V_k := \left\{\phi \in H_0^1(\Omega) \;\middle|\; -\Delta\phi = \Lambda_k \phi\right\}.$$

Lemma 2.7. *Let $r \in \mathbb{R}$ and $\tau > 0$. Then the following series converge:*

$$\sum_{k=1}^\infty \lambda_k^r e^{-\lambda_k \tau}, \quad \sum_{k=1}^\infty \Lambda_k^r e^{-\Lambda_k \tau}. \tag{2.11}$$

Proof. First, we recall the following well-known lower bound for λ_k (see [22, Corollary 1]):

$$\lambda_k \geq C(N, |\Omega|) k^{2/N}, \tag{2.12}$$

where $C(N, |\Omega|)$ is an explicit positive constant depending only on N and $|\Omega|$.

Notice that, for k large enough, we get

$$k^2 \Lambda_k^r e^{-\Lambda_k \tau} \leq k^2 \lambda_k^r e^{-\lambda_k \tau} \leq C(N, |\Omega|)^{-N} \lambda_k^{N+r} e^{-\lambda_k \tau},$$

where we have made use of (2.12) in the last inequality. Now, since $\lambda_k \to \infty$ when $k \to \infty$, the above implies

$$\lambda_k^r e^{-\lambda_k \tau} = o(1/k^2), \quad \Lambda_k^r e^{-\Lambda_k \tau} = o(1/k^2) \quad \text{as } k \to \infty,$$

whence the series in (2.11) converge, as claimed. □

The following immediate corollary of Lemma 2.7 will turn out to be useful in studying overdetermined conditions in case $t^\star \in (0, \infty)$.

Corollary 2.8. *Let $(\gamma_k)_k \subset \mathbb{C}$ be a sequence satisfying $|\gamma_k| \leq D\Lambda_k$ for some positive constant D independent of k. Then, the series*

$$f(z) := \sum_{k=1}^{\infty} \gamma_k e^{-\Lambda_k z}$$

defines a holomorphic function in $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$.

Proof. Since each term in the series is a holomorphic function, it will suffice to show that the series converges uniformly on any compact subset $K \subset \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$.

To this end, set $\tau := \min_{z \in K} \operatorname{Re} z > 0$ and notice that

$$\left| \gamma_k e^{-\Lambda_k z} \right| \leq D \Lambda_k e^{-\Lambda_k \tau},$$

whence, Lemma 2.7 implies the desired convergence. As a result, f is holomorphic in K, being the uniform limit of a sequence of holomorphic functions. We conclude by the arbitrariness of $K \subset \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. □

In what follows, we will show how to express the solution to (1.1) as an infinite series through an eigenfunction expansion. We will focus on the convergence properties of the series expansion.

For each $k \in \mathbb{N}$, let $\alpha_k := \int_\Omega \phi_k$ and consider the following series:

$$u(t) := \sum_{k=1}^{\infty} \alpha_k e^{-\lambda_k t} \phi_k. \tag{2.13}$$

Sometimes it will be more convenient to express (2.13) by gathering together the elements belonging to the same eigenspace, as follows:

$$u(t) := \sum_{k=1}^{\infty} e^{-\Lambda_k t} \Phi_k, \quad \text{where } \Phi_k := \sum_{\substack{j \in \mathbb{N} \\ \lambda_j = \Lambda_k}} \alpha_j \phi_j. \tag{2.14}$$

Proposition 2.9. *The series* (2.13)–(2.14) *converge, to the same function, in the norms of* $C^0\big([0,\infty), L^2(\Omega)\big)$ *and* $C^1\big([\tau,\infty), D(\Delta_D)\big)$ *for all* $\tau > 0$.

Proof. First, we will show convergence in the $C^0\left([0,\infty), L^2(\Omega)\right)$-norm. To this end, we will show that both (2.13)–(2.14) are Cauchy sequences. For natural numbers $m \geq n$, the orthonormality of the family $\{\phi_k\}_k$ yields the following

$$\begin{aligned} \left\| \sum_{k=n}^{m} \alpha_k e^{-\lambda_k t} \phi_k \right\|_{C^0\left([0,\infty), L^2(\Omega)\right)} &= \max_{t \in [0,\infty)} \left\| \sum_{k=n}^{m} \alpha_k e^{-\lambda_k t} \phi_k \right\|_{L^2(\Omega)} \\ = \max_{t \in [0,\infty)} \left(\sum_{k=n}^{m} |\alpha_k|^2 e^{-2\lambda_k t} \|\phi_k\|^2_{L^2(\Omega)} \right)^{1/2} &= \left(\sum_{n}^{m} |\alpha_k|^2 \right)^{1/2}. \end{aligned}$$

Since the above converges to 0 as $m, n \to \infty$ by the definition of α_k, we conclude that (2.13) converges in the $C^0\left([0,\infty), L^2(\Omega)\right)$-norm as claimed.

Now, to show convergence in the $C^1\left([\tau,\infty), D(\Delta_D)\right)$-norm, it will be enough to show that (2.13) is uniformly absolutely convergent in the same norm. We have

$$\begin{aligned} \left\| \alpha_k e^{-\lambda_k t} \phi_k \right\|_{C^1\left([\tau,\infty), D(\Delta_D)\right)} &= \max_{t \in [\tau,\infty)} |\alpha_k| e^{-\lambda_k t} \|\phi_k\|_{D(\Delta_D)} + \max_{t \in [\tau,\infty)} \lambda_k |\alpha_k| e^{-\lambda_k t} \|\phi_k\|_{D(\Delta_D)} \\ = (\lambda_k + 1)|\alpha_k| e^{-\lambda_k \tau} \|\phi_k\|_{D(\Delta_D)} = (\lambda_k + 1)|\alpha_k| e^{-\lambda_k \tau} (\sqrt{\lambda_k} + \lambda_k) &\leq (\sqrt{\lambda_k} + \lambda_k)(\lambda_k + 1)|\Omega|^{1/2} e^{-\lambda_k \tau}. \end{aligned}$$

The desired convergence then follows from Lemma 2.7.

Finally, since the sequence of partial sums of (2.14) is nothing but a subsequence of those of (2.13), (2.14) converges to the same function as (2.13) in the norms of $C^0\left([0,\infty), L^2(\Omega)\right)$ and $C^1\left([\tau,\infty), D(\Delta_D)\right)$ as well. This concludes the proof. □

Corollary 2.10. *Let* $u = u(t)$ *be the function defined via the eigenfunction expansions* (2.13)–(2.14). *Then, the differential operators* ∂_t, Δ, *and* ∂_ν *act on* u *term-wise. In other* *words, for all* $t > 0$*, we have:*

$$\partial_t u(t) = \Delta u(t) = \sum_{k=1}^{\infty} -\lambda_k \alpha_k e^{-\lambda_k t} \phi_k = \sum_{k=1}^{\infty} -\Lambda_k e^{-\Lambda_k t} \Phi_k, \tag{2.15}$$

$$\partial_\nu u(t) = \sum_{k=1}^{\infty} \alpha_k e^{-\lambda_k t} \partial_\nu \phi_k = \sum_{k=1}^{\infty} e^{-\Lambda_k t} \partial_\nu \Phi_k, \tag{2.16}$$

where convergence holds in $C^0\big([\tau,\infty), D(\Delta_D)\big) \cap C^1\big([\tau,\infty), L^2(\Omega)\big)$ *for* (2.15) *and in* $C^1\big([\tau,\infty), H^{-1/2}(\partial\Omega)\big)$ *for* (2.16) *(both, for any given* $\tau > 0$*). In particular,* u *solves* (1.1)*.*

Proof. The claim follows by combining Proposition 2.9 with the continuity of the following operators:

$$\begin{aligned}
\partial_t &: C^1\left([\tau,\infty), D(\Delta_D)\right) \longrightarrow C^0\left([\tau,\infty), D(\Delta_D)\right), \\
\Delta &: C^1\left([\tau,\infty), D(\Delta_D)\right) \longrightarrow C^1\Big([\tau,\infty), L^2(\Omega)\Big), \\
\partial_\nu &: C^1\left([\tau,\infty), D(\Delta_D)\right) \longrightarrow C^1\Big([\tau,\infty), H^{-1/2}(\partial\Omega)\Big).
\end{aligned}$$

□

3 Proof of Theorem I

In this section, we give a proof of Theorem I.

Proof of Theorem I. Let u be the solution to (1.1) and assume that u satisfies (1.4) for a sequence $(t_n)_n$ with an accumulation point $t^\star \in (0,\infty]$. By the unique solvability of (1.1), Corollary 2.10 implies that

$$u(t) := \sum_{k=1}^{\infty} e^{-\Lambda_k t} \Phi_k,$$

where Φ_k are defined according to (2.14). Take now an arbitrary function $\psi \in H_*^{1/2}(\Omega)$. Following Definition 2.1, overdetermined condition (1.4) then yields:

$$0 = {}_{H^{-1/2}}\big\langle \partial_\nu u(t_n), \psi \big\rangle_{H^{1/2}} = \sum_{k=1}^{\infty} \gamma_k e^{-\Lambda_k t_n}, \quad \text{where } \gamma_k := {}_{H^{-1/2}}\langle \partial_\nu \Phi_k, \psi \rangle_{H^{1/2}}. \tag{3.1}$$

Here we made use of (2.16) of Corollary 2.10 to justify the interchange between the duality pairing and the series in the above.

In what follows, we will give an upper bound for $|\gamma_k|$. First, (2.3) (combined with the Poincaré inequality) yields:

$$\begin{aligned}|\gamma_k| = \left|{}_{H^{-1/2}}\langle \partial_\nu \Phi_k, \psi\rangle_{H^{1/2}}\right| &\le \|\partial_\nu \Phi_k\|_{H^{-1/2}(\partial\Omega)} \|\psi\|_{H^{1/2}(\partial\Omega)} \\ &\le \Big\{C_1' \|\Phi_k\|_{H_0^1(\Omega)} + C_2 \|\Delta \Phi_k\|_{L^2(\Omega)}\Big\} \|\psi\|_{H^{1/2}(\partial\Omega)} \\ &\le \Big(C_1' \Lambda_k^{1/2} + C_2 \Lambda_k\Big) \|\Phi_k\|_{L^2(\Omega)} \|\psi\|_{H^{1/2}(\partial\Omega)}\,.\end{aligned}$$

Since, Φ_k is nothing but the orthogonal projection of the constant function $1 \in L^2(\Omega)$ onto the eigenspace V_k, the bound $\|\Phi_k\|_{L^2(\Omega)} \le \|1\|_{L^2(\Omega)} = |\Omega|^{1/2}$ holds for all k. Finally, since $\Lambda_k \to \infty$ as $k \to \infty$, there exists a constant $D > 0$ independent of k (but that might depend on ψ) such that

$$|\gamma_k| \le D\Lambda_k \quad \text{for all } k \in \mathbb{N}. \tag{3.2}$$

Now, we will deal with cases $t^\star = \infty$ and $t^\star \in (0,\infty)$ separately.

Case $t^\star = \infty$

Without loss of generality, assume that $(t_n)_n$ satisfies $\lim_{n\to\infty} t_n = \infty$. Then (3.1) yields:

$$0 = 0 \cdot e^{\Lambda_1 t_n} = \gamma_1 + \sum_{k=2}^{\infty} \gamma_k e^{(\Lambda_1 - \Lambda_k)t_n}. \tag{3.3}$$

In what follows, we will take the limit of (3.3) as $n \to \infty$.

To this end, first notice that, since $\Lambda_1 < \Lambda_k$ for all $k \ge 2$ and $n \in \mathbb{N}$, we have:

$$\left|\gamma_k e^{(\Lambda_1-\Lambda_k)t_n}\right| \le |\gamma_k| e^{(\Lambda_1-\Lambda_k)\tau}, \quad \text{where } \tau := \min_{n\in\mathbb{N}} t_n > 0. \tag{3.4}$$

As a result, the estimate (3.2) combined with Lemma 2.7 readily implies that the series in (3.3) is dominated by a summable series uniformly in n. Thus, by the dominated convergence theorem (with respect to the counting measure on $\mathbb{N}$), we can take the limit of (3.3) as $n \to \infty$ to obtain

$$\gamma_1 = 0. \tag{3.5}$$

The same process can be repeated inductively to (3.1), leading to

$$\gamma_k := {}_{H^{-1/2}}\langle \partial_\nu \Phi_k, \psi\rangle_{H^{1/2}} = 0 \quad \text{for all } k \in \mathbb{N} \text{ and } \psi \in H_*^{1/2}(\partial\Omega). \tag{3.6}$$

Consider now the following auxiliary function:

$$\Phi := \sum_{k=1}^{\infty} \lambda_k^{-1} \alpha_k \phi_k = \sum_{k=1}^{\infty} \Lambda_k^{-1} \Phi_k. \tag{3.7}$$

By construction, (3.7) converges in the $D(\Delta_D)$-norm. As a result, we have

$$\begin{aligned} -\Delta\Phi &= \sum_{k=1}^{\infty} \Phi_k = 1, \\ {}_{H^{-1/2}}\langle \partial_\nu \Phi, \psi \rangle_{H^{1/2}} &= \sum_{k=1}^{\infty} \Lambda_k^{-1}\, {}_{H^{-1/2}}\langle \partial_\nu \Phi_k, \psi \rangle_{H^{1/2}} = 0 \quad \text{for } \psi \in H_*^{1/2}(\partial\Omega). \end{aligned}$$

Thus, by Lemma 2.6, Φ is a weak solution to the following overdetermined problem:

$$-\Delta\Phi = 1 \quad \text{in } \Omega, \qquad \Phi = 0 \quad \text{on } \partial\Omega, \qquad \partial_\nu \Phi = b \quad \text{on } \partial\Omega$$

for some constant $b \in \mathbb{R}$. By [13], Ω must be a ball, as claimed.

Case $t^\star \in (0, \infty)$

Without loss of generality, assume that $(t_n)_n$ converges to some $t^\star \in (0, \infty)$. Combining (3.2) and Corollary 2.8 yields that the function

$$f(z) := \sum_{k=1}^{\infty} \gamma_k e^{-\Lambda_k z}$$

is holomorphic in $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. On the other hand, by (3.1), $(t_n)_n$ is a sequence of zeros for f with $t^\star$ as an accumulation point. Thus, by the identity theorem, $f \equiv 0$ in the whole $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. In particular, $f(t) = 0$ for all $t > 0$. Since $\psi \in H_*^{1/2}(\partial\Omega)$ was arbitrary, this means that the function u satisfies overdetermined condition (1.4) at all times $t > 0$, thus in particular, along a sequence $(t_n)_n$ satisfying $\lim_{n\to\infty} t_n = \infty$. The problem is thus reduced to the case $t^\star = \infty$, completing the proof. $\square$

Remark 3.1. *Under stronger regularity assumptions, say, $\partial\Omega \in C^{2,\alpha}$ ($0 < \alpha < 1$), then a shorter proof is available. Indeed, directly from (3.5), one concludes that Φ_1 satisfies*

$$-\Delta\Phi_1 = \Lambda_1 \Phi_1 \quad \textit{in } \Omega, \qquad \Phi_1 = 0 \quad \textit{on } \partial\Omega, \qquad \partial_\nu \Phi_1 = b \quad \textit{on } \partial\Omega$$

for some constant $b \in \mathbb{R}$. Moreover, since $\Phi_1 > 0$ in Ω by Krein–Rutman's theorem, the desired conclusion readily follows from [34, Theorem 2].

We remark that, unfortunately, the shortcut above is not available in our setting, as, to the best of our knowledge, no analogue of [34, Theorem 2] is (yet) known to hold for Lipschitz domains.

4 Proof of Theorem II

In this section, we give a proof of Theorem II.

Proof of Theorem II. By [35, Theorem 1.2 and Theorem 1.3 (1): (a),(b)] or [19, Theorem 2.3.3: 1., 2., 3.], for every $\psi \in C^\infty(\Omega)$ with $\Delta\psi = 0$ we have

$$f(t) := \int_\Omega u(t^2)\,\psi = \int_\Omega \psi - \frac{2t}{\sqrt{\pi}} \int_{\partial\Omega} \psi + \frac{t^2}{2} \int_{\partial\Omega} H\,\psi + o(t^2) \qquad \text{as } t \to 0^+, \tag{4.1}$$

where H is the mean curvature of $\partial\Omega$, that is, the sum of the principal curvatures of $\partial\Omega$ defined with respect to the inward unit normal. In particular, if $\partial\Omega$ is a sphere of radius R, then $H = \frac{N-1}{R}$ (see [19, page 6] or [35]).

On the other hand, for $t > 0$,

$$\begin{aligned} f'(t) &= 2t \int_\Omega u_t(t^2)\,\psi = 2t \int_\Omega \Delta u(t^2)\,\psi \\ &= 2t \int_{\partial\Omega} \partial_\nu u(t^2)\,\psi, \end{aligned}$$

where the last equality follows by integration by parts using $\Delta\psi = 0$.

That is, if, in addition

$$\int_{\partial\Omega} \psi = 0 \quad \text{and} \quad t = \sqrt{t_n},$$

we get

$$f'(\sqrt{t_n}) = 2\sqrt{t_n}\ {}_{H^{-1/2}}\langle \partial_\nu u(t_n), \psi \rangle_{H^{1/2}} = 0 \qquad \forall n \in \mathbb{N},$$

and hence

$$f''(0) = 0.$$

On the other hand, differentiating (4.1) gives

$$f''(0) = \int_{\partial\Omega} H\,\psi.$$

Since ψ is arbitrary (subject to $\Delta\psi = 0$ and $\int_{\partial\Omega} \psi = 0$), we conclude that H is constant on $\partial\Omega$, whence Ω is a ball by Aleksandrov's soap bubble theorem (see [1]). ☐

5 Proof of Theorem III

In this section, we are going to show Theorem III. The proof will follow that of Theorem I, with some adequate modifications.

Proof of Theorem III. Let u be the solution to (1.1) and assume that u satisfies (1.5) with respect to sequences $(t_n)_n$ and $(\tau_n)_n$ with accumulation points $t^\star, \tau^\star \in (0,\infty]$ respectively. By the unique solvability of (1.1), Corollary 2.10 implies that

$$u(t) := \sum_{k=1}^{\infty} e^{-\Lambda_k t}\Phi_k, \tag{5.1}$$

where Φ_k are defined according to (2.14).

Take an arbitrary function $\psi \in H^{1/2}_*(\partial\omega)$. Then, (1.5) yields:

$$0 = a_n \int_{\partial\omega} \psi = \Big(u(\tau_n)\big|_{\partial\omega}, \psi\Big)_{L^2(\partial\omega)} = \sum_{k=1}^{\infty} \Big(\Phi_k\big|_{\partial\omega}, \psi\Big)_{L^2(\partial\omega)} e^{-\Lambda_k \tau_n},$$

$$0 = b_n \int_{\partial\omega} \psi = {}_{H^{-1/2}}\Big\langle \partial_\nu u(t_n)\big|_{\partial\omega}, \psi\Big\rangle_{H^{1/2}} = \sum_{k=1}^{\infty} {}_{H^{-1/2}}\Big\langle \partial_\nu \Phi_k\big|_{\partial\omega}, \psi\Big\rangle_{H^{1/2}} e^{-\Lambda_k t_n}.$$

Thus, along the same line as proof of Theorem I, in each case (recall that, by assumption, we have either $t^\star \in (0,\infty)$ or $t^\star = \infty$, and either $\tau^\star \in (0,\infty)$ or $\tau^\star = \infty$), one can show inductively that for all $k \in \mathbb{N}$ and $\psi \in H^{1/2}_*(\partial\omega)$ the following holds:

$$\Big(\Phi_k\big|_{\partial\omega}, \psi\Big)_{L^2(\partial\omega)} = 0 \quad \text{and} \quad {}_{H^{-1/2}}\Big\langle \partial_\nu \Phi_k\big|_{\partial\omega}, \psi\Big\rangle_{H^{1/2}} = 0. \tag{5.2}$$

Let now $\Phi \in H^1_0(\Omega)$ denote the following auxiliary function:

$$\Phi := \sum_{k=1}^{\infty} \lambda_k^{-1}\alpha_k\phi_k = \sum_{k=1}^{\infty} \Lambda_k^{-1}\Phi_k. \tag{5.3}$$

By construction, (5.3) converges in the $D(\Delta_D)$-norm. As a result, term-wise application of the Laplacian shows that Φ satisfies the following boundary value problem:

$$-\Delta\Phi = 1 \quad \text{in } \Omega, \quad \Phi = 0 \quad \text{on } \partial\Omega. \tag{5.4}$$

Furthermore, by (5.2), the following hold for all $\psi \in H^{1/2}_*(\partial\omega)$:

$$\Big(\Phi\big|_{\partial\omega}, \psi\Big)_{L^2(\omega)} = \sum_{k=1}^{\infty} \Lambda_k^{-1} \Big(\Phi_k\big|_{\partial\omega}, \psi\Big)_{L^2(\omega)} = 0, \tag{5.5}$$

$${}_{H^{-1/2}}\Big\langle \partial_\nu \Phi\big|_{\partial\omega}, \psi\Big\rangle_{H^{1/2}} = \sum_{k=1}^{\infty} \Lambda_k^{-1} {}_{H^{-1/2}}\Big\langle \partial_\nu \Phi_k\big|_{\partial\omega}, \psi\Big\rangle_{H^{1/2}} = 0. \tag{5.6}$$

In particular, (5.5) implies that $\Phi\big|_{\partial\omega} \equiv a$ for some $a \in \mathbb{R}$. In turn, by combining (5.6) with Lemma 2.6, we obtain that the function $U \in H^1_0(\omega)$ given by $U := \Phi\big|_{\omega} - a$ is a weak solution to the following overdetermined problem:

$$-\Delta U = 1 \quad \text{in } \omega, \quad U = 0 \quad \text{on } \partial\omega, \quad \partial_\nu U = b \quad \text{on } \partial\omega$$

for some constant $b \in \mathbb{R}$. By [13], ω must be a ball and U is radial in ω.

In particular, also Φ is also radial inside ω, being $\Phi = U + a$. Moreover, notice that Φ is real analytic inside the whole Ω, being a solution to (5.4). This implies that Φ is radial in Ω. Finally, recall that $\Phi \in C^0(\overline{\Omega})$, since $\partial\Omega$ is made entirely of regular points for the Dirichlet Laplacian. Now, since $\Phi = 0$ on $\partial\Omega$ because of the boundary condition, and $\Phi > 0$ in Ω by the maximum principle, we conclude that $\partial\Omega$ is a sphere concentric with ω. In other words, ω and Ω are concentric balls, as claimed. ☐

6 Possible extensions to complete Riemannian manifolds

In this section, we discuss which parts of the previous sections extend to the Riemannian setting, and which conclusions have to be modified. Throughout, (M, g) is a complete Riemannian manifold of dimension $N \geq 2$, $\Omega \subset \overline{\Omega} \subset M$ is a connected domain with C^∞ boundary, and $u = u(x, t)$ denotes the solution of the Dirichlet heat problem

$$\begin{cases} (\partial_t + \Delta_g)u = 0 & \text{in } (0, \infty) \times \Omega, \\ u = 1 & \text{on } \Omega \times \{0\}, \\ u = 0 & \text{on } (0, \infty) \times \partial\Omega. \end{cases} \tag{6.1}$$

where Δ_g denotes the Laplace–Beltrami operator on (M, g). We write ν for the *inward* unit normal along $\partial\Omega$ (the choice of the normal only affects the sign of $\partial_\nu u$, and therefore is immaterial for all the rigidity statements below).

By standard spectral theory for the Dirichlet Laplace–Beltrami operator on compact manifolds with boundary, there exists an orthonormal basis $(\phi_j)_{j \geq 1} \subset C^\infty(\overline{\Omega})$ of $L^2(\Omega)$ and a nondecreasing sequence of positive eigenvalues $\lambda_j \to \infty$ such that

$$\Delta_g \phi_j = \lambda_j \phi_j \quad \text{in } \Omega, \qquad \phi_j = 0 \quad \text{on } \partial\Omega.$$

If

$$a_j := \int_\Omega \phi_j,$$

then

$$u(x, t) = \sum_{j=1}^\infty a_j e^{-\lambda_j t} \phi_j(x), \tag{6.2}$$

and, for every $\tau > 0$,

$$\partial_\nu u(t) = \sum_{j=1}^\infty a_j e^{-\lambda_j t} \partial_\nu \phi_j \qquad \text{in } C^\infty(\partial\Omega) \text{ for } t \geq \tau, \tag{6.3}$$

see [8, Chapters VI–VII].

Let

$$\mathcal{C}(\partial\Omega) := \{\text{constant functions on } \partial\Omega\}.$$

The boundary function $\partial_\nu u(t)$ belongs to $\mathcal{C}(\partial\Omega)$ if and only if

$$\int_{\partial\Omega} \psi\,\partial_\nu u(t) = 0 \qquad \text{for every } \psi \in C^\infty(\partial\Omega) \text{ such that } \int_{\partial\Omega} \psi = 0.$$

Proposition 6.1. *Assume that there exists a sequence* $(t_m)_m \subset (0,\infty)$ *such that*

$$\partial_\nu u(t_m) \in \mathcal{C}(\partial\Omega) \qquad \textit{for all } m \in \mathbb{N},$$

and either

$$t_m \to t_* \in (0,\infty), \qquad \textit{or} \qquad t_m \to \infty.$$

Then Ω *has the* constant flow property, *namely*

$$\partial_\nu u(t) \in \mathcal{C}(\partial\Omega) \qquad \textit{for every fixed } t > 0.$$

Proof. Fix $\psi \in C^\infty(\partial\Omega)$ with zero average on $\partial\Omega$ and set

$$F_\psi(t) := \int_{\partial\Omega} \psi\,\partial_\nu u(t).$$

By (6.3),

$$F_\psi(t) = \sum_{j=1}^{\infty} b_j e^{-\lambda_j t}, \qquad b_j := a_j \int_{\partial\Omega} \psi\,\partial_\nu \phi_j, \tag{6.4}$$

with absolute and uniform convergence on $[\tau,\infty)$ for every $\tau > 0$. Hence F_ψ is real-analytic on $(0,\infty)$.

If $t_m \to t_* \in (0,\infty)$, then $F_\psi(t_m) = 0$ for every $m \in \mathbb{N}$ by the assumption $\partial_\nu u(t_m) \in \mathcal{C}(\partial\Omega)$. Since the zeros accumulate at an interior point of $(0,\infty)$ and F_ψ is real-analytic, we conclude that $F_\psi \equiv 0$ on $(0,\infty)$.

Assume now that $t_m \to \infty$. Let

$$0 < \Lambda_1 < \Lambda_2 < \Lambda_3 < \cdots$$

be the distinct Dirichlet eigenvalues of Δ_g on Ω, and rewrite (6.4) as

$$F_\psi(t) = \sum_{\ell=1}^{\infty} \beta_\ell e^{-\Lambda_\ell t}, \qquad \beta_\ell := \sum_{\lambda_j = \Lambda_\ell} b_j.$$

We claim that $\beta_\ell = 0$ for every ℓ. Since the series converges absolutely on $[T, \infty)$ for every $T > 0$, we can choose $T > 0$ so that $t_m \geq 2T$ for all m large enough. Then

$$0 = e^{\Lambda_1 t_m} F_\psi(t_m) = \beta_1 + \sum_{\ell \geq 2} \beta_\ell e^{-(\Lambda_\ell - \Lambda_1) t_m}.$$

Moreover,

$$\begin{aligned} \sum_{\ell \geq 2} |\beta_\ell| e^{-(\Lambda_\ell - \Lambda_1) t_m} &= \sum_{\ell \geq 2} |\beta_\ell| e^{-\Lambda_\ell T} e^{-\Lambda_\ell (t_m - T)} e^{\Lambda_1 t_m} \\ &\leq e^{-(\Lambda_2 - \Lambda_1) t_m + \Lambda_2 T} \sum_{\ell \geq 2} |\beta_\ell| e^{-\Lambda_\ell T} \xrightarrow[m \to \infty]{} 0. \end{aligned}$$

Hence $\beta_1 = 0$. Subtracting the first term and iterating the same argument, we obtain $\beta_\ell = 0$ for every ℓ, hence $F_\psi \equiv 0$ also in this case.

Since $F_\psi(t) = 0$ for every zero-average ψ and every $t > 0$, it follows that $\partial_\nu u(t)$ is orthogonal to all zero-average smooth functions on $\partial\Omega$, and therefore $\partial_\nu u(t) \in \mathcal{C}(\partial\Omega)$ for every $t > 0$. ☐

Corollary 6.2. *Assume in addition that (M, g) is real-analytic. Under the assumptions of Proposition 6.1, the domain Ω is an isoparametric tube. In particular, if $M = \mathbb{R}^N$ or $M = \mathbb{H}^N$, then Ω is a geodesic ball.*

Proof. By Proposition 6.1, Ω has the constant flow property. Savo proved that, on a compact real-analytic Riemannian manifold with smooth boundary, the constant flow property is equivalent to being an isoparametric tube [33]. This gives the first claim.

If $M = \mathbb{R}^N$ or $M = \mathbb{H}^N$, then compact isoparametric hypersurfaces are geodesic spheres; equivalently, bounded isoparametric tubes are geodesic balls. This follows from the classification of isoparametric hypersurfaces in real space forms; see, for instance, the survey [7, Section 1]. ☐

The conclusion of Corollary 6.2 is sharp: on general real-analytic manifolds, one cannot replace “isoparametric tube” with “geodesic ball,” not even on the round sphere.

Proposition 6.3. *Let $N \geq 3$, $2 \leq k \leq N - 1$, and $0 < r < 1$. Then, the set*

$$\Omega := \left\{ (x, y) \in \mathbb{R}^k \times \mathbb{R}^{N-k} \;\middle|\; |x|^2 + |y|^2 = 1, \quad |y| < r \right\} \subset \mathbb{S}^{N-1}$$

is a domain of $\mathbb{S}^{N-1}$ with the constant flow property that is not a geodesic ball.

Proof. By construction, Ω is an isoparametric tube around a totally geodesic sphere $\mathbb{S}^{k-1}$ on $\mathbb{S}^{N-1}$. Its boundary is given by

$$\partial\Omega = \left\{(x,y) \in \mathbb{R}^k \times \mathbb{R}^{N-k} \;\middle|\; |x| = \sqrt{1-r^2}, \quad |y| = r\right\} \simeq \mathbb{S}^{k-1} \times \mathbb{S}^{N-k-1}.$$

As $\mathbb{S}^{k-1} \times \mathbb{S}^{N-k-1}$ is not diffeomorphic to a sphere, this readily implies that Ω cannot be a geodesic ball. In what follows, we will show that, despite that, Ω exhibits enough symmetry to ensure the constant flow property. To this end, consider the group $\Gamma := O(k)\times O(N-k)$, acting (in the natural way) on the set $\Omega \subset \mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$ as a subgroup of the orthogonal group in dimension N. First, notice that both Ω and $\partial\Omega$ are Γ-invariant, and that the differential operators Δ_g and ∂_ν are Γ-equivariant. This, combined with the fact that the initial and boundary conditions in (6.1) are Γ-invariant, yields that for any solution $u = u(x,t)$ of (6.1) and $\gamma \in \Gamma$, then $u_\gamma(x,t) := u(\gamma(x),t)$ also solves (6.1). In particular, by unique solvability, it follows that any solution to (6.1) must be Γ-invariant in space for all $t > 0$. Finally, since Γ acts transitively on $\partial\Omega$, then $\partial_\nu u(t)$ must be a constant function on $\partial\Omega$ for all $t > 0$. □

Proposition 6.3 shows that the conclusion “Ω is a ball” cannot hold in a naïve Riemannian extension of the finite-time or large-time rigidity result, even when the overdetermined condition holds for *every* $t > 0$.

Corollary 6.4. *Assume that the hypotheses of Proposition 6.1 hold with $t_m \to \infty$. Then the function*

$$v(x) := \int_0^\infty u(x,t)$$

is well defined, belongs to $C^\infty(\overline{\Omega})$, and solves the overdetermined torsion problem

$$\begin{cases} \Delta_g v = 1 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \partial_\nu v = const & \text{on } \partial\Omega. \end{cases}$$

In particular, Ω is a Serrin domain.

Proof. By Proposition 6.1, Ω has the constant flow property, i.e.

$$\partial_\nu u(t) = c(t) \qquad \text{on } \partial\Omega$$

for some function $c : (0,\infty) \to \mathbb{R}$. By (6.2), the solution u decays exponentially in $C^\infty(\overline{\Omega})$ as $t \to \infty$, so the integral defining v converges in $C^\infty(\overline{\Omega})$. Since $u = 0$ on $\partial\Omega \times (0,\infty)$, we have $v = 0$ on $\partial\Omega$. Moreover

$$\partial_\nu v = \int_0^\infty \partial_\nu u(t) = \int_0^\infty c(t),$$

which is constant on $\partial\Omega$.

Finally, integrating the heat equation in time gives

$$\Delta_g v = -\int_0^\infty \partial_t u(t) = u(0) - \lim_{t\to\infty} u(t) = 1 \qquad \text{in } \Omega.$$

$\square$

7 Conjectures and open problems

In this section, we state some open problems and conjectures related to the overdetermined problem presented in this paper.

Problem 1. *For what unbounded domains* $\Omega \subset \mathbb{R}^N$ *(*$N \geq 2$*) does the solution to the following problem satisfy* (1.4)*?*

$$\begin{cases} \partial_t u = \Delta u \quad \text{in } \Omega \times (0, \infty), \\ u = 1 \quad \text{on } \Omega \times \{0\}, \\ u = 0 \quad \text{on } \partial\Omega \times (0, \infty), \\ u(x,t) \to 0 \quad \text{uniformly in } t \text{ as } |x| \to \infty. \end{cases}$$

Conjecture 7.1. *Complements of closed balls are the only solutions to Problem 1 among* exterior domains.

Remark 7.2. *Conjecture 7.1 is in line with the results of [17]. There, the authors make use of the method of moving planes in a cylindrical region of* $\mathbb{R}^N \times \mathbb{R}$*, thus fully exploiting a continuous-time overdetermination. As a result, it is not clear how to adapt their methods to the case of a discrete-time overdetermination, as in* (1.4)*. We also remark that the eigenfunction expansion technique employed in the proof of Theorem I cannot be applied in the unbounded domain case due to the lack of compactness.*

Problem 2. *Generalize Theorem II to milder regularity conditions.*

Remark 7.3. *Our current methods rely on the following two ingredients: precise heat content asymptotics and Aleksandrov's soap bubble theorem. While the latter is known to hold (in a geometric-measure-theoretical sense) for sets of finite perimeter, the heat content* *asymptotics provided in [19] (as well as the analogous point-wise asymptotics given in [32]) used in the proof of Theorem II rely on an infinite family of recurrence relations, which can only be derived when the boundary $\partial\Omega$ is smooth.*

Problem 3. *For what bounded domains $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ does the solution u to* (1.1) *admit* a finite number *of times $t_1, \dots, t_m > 0$ satisfying the following?*

$$\partial_\nu u(t_n) \equiv b_n \quad \text{on } \partial\Omega \quad \text{for } n = 1, \dots, m.$$

Conjecture 7.4. *For any $m \in \mathbb{N}$, Problem 3 admits nontrivial solutions (solutions that are not Euclidean balls).*

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