EXISTENCE AND REGULARITY IN THE SMALL-MASS REGIME FOR A HARTREE–OHTA-KAWASAKI SHAPE OPTIMIZATION PROBLEM

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Abstract. We consider a shape optimization problem for a hybrid energy combining local confinement and nonlocal Coulomb repulsion. Specifically, for any open set $\Omega \subseteq \mathbb{R}^3$ of prescribed volume, we consider the ground state energy of an L^2-normalized function supported in Ω, defined as a linear combination of its homogeneous $\dot{H}^1$ and $\dot{H}^{-1}$ seminorms. We show that in the small mass regime, volume-constrained minimizers of this geometric functional exist and are $C^{2,\alpha}$ perturbations of a ball. The proof relies on a combination of surgery techniques, Γ-convergence, elliptic PDE theory, and one-phase free boundary regularity.

A key novelty of this paper lies in the treatment of the Coulombic repulsive term: unlike standard competitive models, the lack of (a priori) sign constraints on the optimal functions forces the nonlocal term to exhibit two natures: it acts both as a *scattering* and an *homogenizing* force.

Contents

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1. Introduction

1.1. **Foreword.** Consider the energy functional

$$\widetilde{E}(\Omega) = \min\left\{[u]^2_{\dot{H}^1} + [u]^2_{\dot{H}^{-1}} \,:\, \|u\|_{L^2(\Omega)} = 1,\ u \in H^1_0(\Omega)\right\}.$$

We aim to study the minimizers of $\widetilde{E}$ among open subsets of $\mathbb{R}^3$ with mass constraint $|\Omega| = m$. Since we are interested in an optimal design problem where the set Ω is free to move on $\mathbb{R}^3$, it is convenient to consider the seminorm as defined on the whole space, that is, we extend $u \in H^1_0(\Omega)$ to zero outside Ω and let, up to a multiplicative constant,

$$[u]^2_{\dot{H}^1} = \int_\Omega |\nabla u|^2\,dx, \qquad \text{and} \qquad [u]^2_{\dot{H}^{-1}} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u(x)u(y)}{|x-y|}\,dxdy = \int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|}\,dxdy.$$

Then, we define

$$E_q(u,\Omega) := \int_\Omega |\nabla u(x)|^2\,dx + \frac{q}{2}\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|}\,dxdy \tag{1}$$

and we consider the energy functional

$$E_q(\Omega) := \min\left\{E_q(u,\Omega) : u \in H^1_0(\Omega),\ \int_\Omega u^2(x)\,dx = 1\right\}. \tag{2}$$

Up to take $q = 2\left(\frac{m}{|B_1|}\right)^{4/3} > 0$, thanks to a scaling argument, the original problem is equivalent to

$$\min\left\{E_q(\Omega) : \Omega \subseteq \mathbb{R}^3,\ \text{open},\ |\Omega| = |B_1|\right\}. \tag{3}$$

We are interested in this paper in the small mass regime $m \ll 1$, that is, $q \ll 1$.

When $q = 0$, $E_q(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian. The well-known Faber-Krahn inequality assures that it is uniquely minimized by balls, up to null capacity, among sets of fixed volume. The case $q > 0$ is less clear, as the second addend in the energy may act as a scattering term. Note in particular that among *positive* functions, the second term in the energy is maximized by balls, by the Riesz rearrangement inequality, see [32]; instead, the minimization of the second term alone without sign constraints leads to an ill posed problem, as it is shown in Appendix A. In this case we expect the positive and negative parts of the optimal functions to homogeneize in order to favor cancellations, see also Section 1.2 below. These formal observations reveal a sharp contrast between the two addends in the energy.

Nevertheless, also in view of the recent results on the stability of isoperimetric inequalities [20, 18, 13], and their spectral counterparts [6, 17], and since the ball is a critical point for the nonlocal term, it can be expected that for q sufficiently small the ball is a stable minimizer.

In this paper we show that in the small mass regime well posedness is restored. Namely we show that minimizers exist and that they do converge in $C^{2,\alpha}$−norm to a ball. Specifically, this is the main result of the paper.

Theorem 1.1. *There exist universal*[1] *constants $q^* > 0$ and $\alpha \in (0,1)$ such that, for all $0 < q \le q^*$, there exists an optimal set for problem (3). Furthermore, every optimal set Ω_q is $C^{2,\alpha}$-nearly spherical, namely there is a function $\varphi_q\colon \partial B_1 \to \mathbb{R}$ of class $C^{2,\alpha}$ such that $\|\varphi_q\|_{C^{2,\alpha}}$ vanishes as $q \to 0$ and*

$$\partial\Omega_q = \left\{(1+\varphi_q(x))x : x \in \partial B_1\right\}.$$

1.2. **Motivation and background.** The functional we study can be viewed as an analytical hybrid model combining a spectral confinement energy with a long-range Coulombic self-interaction. From this perspective, the optimization over the shape Ω describes the search for geometries minimizing the energy of a confined state under weak nonlocal repulsion.

The competition between a local regularizing term and a nonlocal long-range interaction is a standard feature in pattern formation and microphase separation, as it arises in actual models such as optimal design of charged quantum devices [35] or Ohta-Kawasaki models, see [12, 2, 16] and references therein. Let us make a comparison between our case and the well-known Ohta-Kawasaki energy for diblock copolymers. Analytically, our problem acts as a diffuse spectral counterpart to the sharp-interface model studied in the seminal work of Alberti, Choksi, and Otto [2]. In the classical Ohta-Kawasaki setting, the competition between a local perimeter term and a nonlocal Coulombic repulsion under an L^1 mass constraint leads to pattern formation for large masses. In our functional, the local regularizing role of the perimeter is carried out by the Dirichlet energy.

The lack of an a priori sign constraint on the admissible functions $u \in H^1_0(\Omega)$ implies a deep difference in how the system reacts to the nonlocal repulsion. In the regime $q \gg 1$ (corresponding to the large mass regime in [2]), the Coulombic term dominates so that, to minimize this repulsive term, the energy does not just separate the domain into periodic patterns as in [2], but the optimal function u tends to oscillate between positive and negative values to maximize cancellations in the double integral. In this regime, the nonlocal term acts as a strongly homogenizing force. We show, at least formally, this phenomenon in Appendix A.

Conversely, for $q \ll 1$, the spectral confinement dominates the behavior of the energy. The optimal function is a posteriori forced to be strictly positive, as the energy is close, for $q \ll 1$, to the first Dirichlet eigenvalue. At this point, as the positivity is established, the Coulombic term changes completely its nature, and acts as a (more standard) scattering force.

[1] In this paper we call *universal constant* a constant possibly depending only on the dimension of the ambient space, which, in our case, is $d = 3$.

One expects that in this weak-coupling regime ($q \ll 1$), the spectral term dominates, making the ball optimal. A common strategy to prove such a stability for local/nonlocal interactions was first exploited in [30, 31], see also [14], following the technique -referred to as *selection principle*- developed in [13] in order to prove the quantitative isoperimetric inequality. Such a strategy roughly consists of dividing the proof into two steps: show rigidity (i.e. that the ball is the unique minimizer) in a suitably chosen class of regular competitors, the nearly spherical sets, and prove independently that a minimizer (or any minimizing sequence) lies in such a class.

In this paper we tackle the latter part of this plan: Theorem 1.1 shows that all minimizers are a $C^{2,\alpha}$-small perturbation of a ball. In fact, a fundamental byproduct of our proof is that in the small$-q$ regime, the optimal functions in (1) are positive so that the ball is in fact a critical point of the energy, thanks to Pólya-Szegö inequality and Riesz rearrangement inequality. This, together with preliminary computations, leads us to formulate the following conjecture.

Conjecture: *There exists a universal constant $q_0 > 0$ such that for $q < q_0$ balls are rigid minimizers for Problem (3).*

The above conjecture deserves some comments. In order to show stability of the ball among nearly spherical sets, there are two techniques available that we are aware of. Either to perform a second order shape derivative of the full energy and to show that this is positive in a suitable neighborhood of a ball, or to show that the Euler-Lagrange equation for a minimizer produces an overdetermined problem which can be satisfied only by balls. The first idea has been successfully exploited in several recent results, but only in the case in which the two addends in the functional are independent shape functionals, i.e. when the full energy is the sum of an aggregating shape functional term $\mathcal{A}$ and a repulsive one $\mathcal{B}$, i.e. of the form

$$\Omega \mapsto \mathcal{A}(\Omega) + \mathcal{B}(\Omega).$$

See, for instance, [31, 27, 26, 6, 37]. However, the derivation of a second order shape derivative for nonlocal functionals is a challenging open problem. The second strategy appears hard, too. Indeed, despite the huge literature developed from the pioneering works by Serrin and Weinberger [39, 41], there is not a clear understanding of how to approach overdetermined problems with competing non-local terms. Even the new free boundary approaches recently developed for example in [19] do not seem to work in our setting.

1.3. Detailed strategy.

In order to make the reading more clear, we offer here below the detailed outline of the strategy of the proof of Theorem 1.1. As mentioned just above, this is inspired by that developed in [35, 37] with a major technical difference due to the fact that an optimal function u is not a priori positive. Neglecting the spectral confinement, i.e. formally picking $q = +\infty$, a minimizing sequence driven only by the nonlocal term would seek to maximize cancellations, leading to a homogenization problem. This sign-changing phenomenon completely prevents the direct application of standard shape optimization techniques or classical overdetermined problem strategies. To bridge this gap, we develop a strategy mixing Γ-convergence and quasi-minimizer regularity à-la Giaquinta-Giusti.

Hence, this marks a substantial difference between our stability result and *all* optimal design problems with competitive terms that we are aware of in the literature. In order to deal with this issue, we need to strongly use the fact that Ω solves a minimization problem, and often merge standard variational techniques with PDE ones.

The key points of our strategy are the following:

1. First, we prove some basic properties of optimal functions for $E_q(\Omega)$, in particular that they are uniformly bounded when $q \in (0, 1]$, and we show that we can get rid of the L^2 integral constraint by adding a suitable Lagrange multiplier.
2. Then, we prove a surgery result that allows to work in an equibounded setting, that is, where the competing shapes have equibounded diameter. This result is somehow independent from all of the other parts of the proofs. This part is quite long and technical but essentially follows the surgery technique developed in [36] with some nontrivial technical modifications.
3. After this, we follow a nowadays classical approach which was first developed by Aguilera, Alt and Caffarelli in [1] to get rid of the measure constraint.
4. At this point we work with a complete unconstrained functional. Now, with a simple yet delicate argument mixing up Γ-convergence tools and a notion of quasi-minimizers for the Dirichlet energy à-la Giaquinta-Giusti, we are able to prove that the optimal functions u_q for optimal shapes Ω_q can be chosen to be non-negative, if q is small enough.

The last point is a necessary bridge. The remaining part of the scheme of the proof is as follows. We prove regularity of the boundary by means of regularity tools developed in the one-phase free boundary theory. Also the understanding of this second part may be eased by a detailed strategy.

5. We begin with some regularity result obtained, as mentioned above, via tools of the regularity for one-phase free boundary problems: existence of minimizers, finiteness of their perimeter, their non-degeneracy, and Lipschitz regularity of the optimal functions.
6. Now, via a scaling argument we are ready to show the equivalence between the original problem with the measure constraint and this unconstrained problem (for q small).
7. At this point we perform a first shape variation argument around points of the reduced boundary (which covers almost all of the boundary, as our sets are of finite perimeter after point 5.) to show an optimality condition at the free boundary. This may be read as a weak Euler-Lagrange equation for the energy.
8. Finally we apply the improvement of flatness machinery by Alt and Caffarelli to prove the main result of the paper [3]. The regularity, together with a simple localization argument which is a consequence of the quantitative stability of the Faber-Krahn inequality [6], will be enough to prove Theorem 1.1.

Figure 1 represents the tree of the steps of the proof detailed here above.

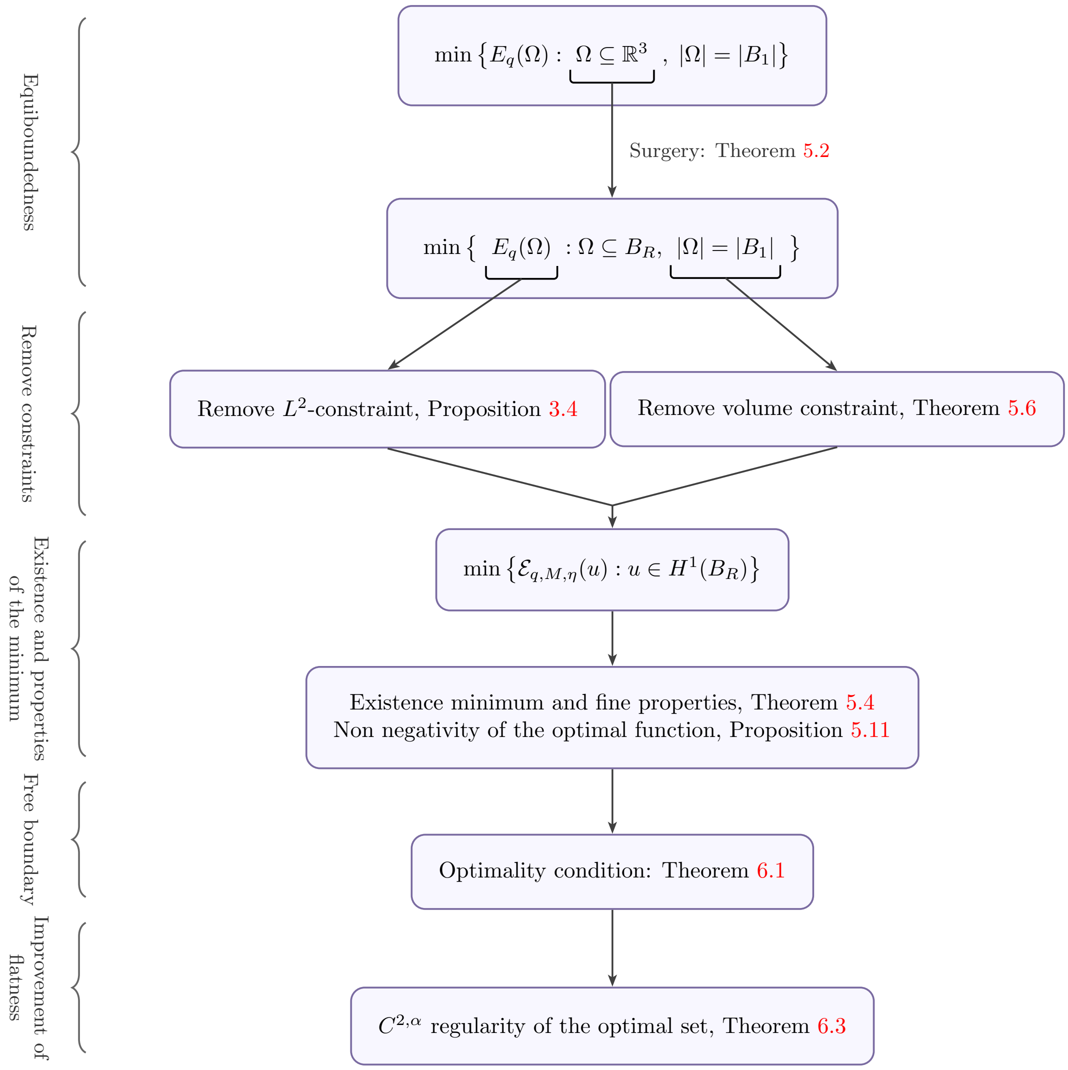

FIGURE 1. Scheme of the proof

We conclude this introduction with some observations.

Remark 1.2. Another way of looking at our problem, which was used, in a different setting, in [4], is to see the energy $E_q(\Omega)$ as an eigenvalue of a competitive mixture of $-\Delta$ and $(-\Delta)^{-1}$. The study of mixtures of operators, one local and the other nonlocal, has been the subject of several works in the last few years, mostly in homogeneous settings. See for instance [5] and references therein. We just remark that our proof does not apply to nonintegrable kernels in the nonlocal term, as the s-Laplacian would be.

Remark 1.3. This paper is settled in the physical three dimensional case. From the purely mathematical point of view, one could of course try to generalize the problem to other dimensions, or one could consider more general Riesz kernels, a general power p for the gradient term and general powers q and r in the non-local term and the constraint, respectively. Such generalizations often appear in the literature and lead to a great variety of bifurcations as the dimension and other parameters are varied. While we believe that our results may be generalized to higher dimensions and more general Riesz kernels (with suitably chosen exponents), such generalization would, highly complicate all the notations of our paper obscuring its main points. Therefore, instead of carrying out such an extension, we prefer to limit ourselves to the only physically relevant case of $N = 3$ and the Newtonian potential.

Structure of the paper. After this Introduction, Section 2 is devoted to setting notations and preliminary results needed in the rest of the paper. Section 3 deals with the study of the minimization of $E_q(u, \Omega)$ with respect to u only and to the reformulation without the L^2 constraint. Then in Section 4 we prove the surgery result mentioned in point 2. above. Section 5 is devoted to introducing the unconstrained functional (point 3.), to prove that optimal functions are positive for q small (point 4.) and to obtain first mild regularity properties (point 5.). Finally, in Section 6 an optimality condition at the free boundary is proved and this allows to conclude by the classical improvement of flatness for one-phase problems (points 6. − 8.).

2. Preliminaries

Throughout the paper, we adopt the following notations: $\Omega \subset \mathbb{R}^3$ is a quasi-open set[2] of finite measure and $u \in H^1_0(\Omega)$ denoting with $*$ the usual convolution, we define

$$v_u(x) := \left(u * \frac{1}{|\cdot|}\right)(x) = \int_\Omega \frac{u(y)}{|x-y|}\,dy,$$

with $v_u \in W^{2,6}_{\mathrm{loc}}(\mathbb{R}^3)$ since $u \in H^1_0(\mathbb{R}^3) \hookrightarrow W^{1,6}(\mathbb{R}^3)$ and by [25, Theorem 9.9]. We will use several times the following Hardy-Sobolev inequality

$$\int_{\mathbb{R}^3} \frac{|\varphi(x)|}{|x-y|}dx \le C \int_{\mathbb{R}^3} |\nabla\varphi(x)|dx \quad \forall \varphi \in C^\infty_c(\mathbb{R}^3), \tag{4}$$

for a universal constant $C > 0$, see for instance [34, Corollary 2 of Section 2.1.7]. The inequality holds also for $u \in H^1_0(\Omega)$ by approximation and Fatou lemma. We introduce, for $\varphi, \psi : \Omega \to \mathbb{R}$, the Coulomb energy

$$D(\varphi, \psi) := \int_\Omega \int_\Omega \frac{\varphi(x)\psi(y)}{|x-y|}dxdy.$$

Note that it is a well defined bilinear form as soon as $D(|\varphi|, |\psi|) < +\infty$. Let us state some basic properties of the Coulomb interaction.

Lemma 2.1. *It holds that $D(u, u) \ge 0$ for all $u \in H^1_0(\Omega)$. Moreover it holds that $D(u, u) > 0$ if $u \in H^1_0(\Omega) \setminus \{0\}$ and the map $u \to D(u, u)$ is strictly convex in H^1_0.*

For the proof of the above Lemma, we refer to [32, Theorem 9.8].

Proposition 2.2. *Let Ω be a quasi-open set of finite measure. Then the functional $D : L^2(\Omega) \to \mathbb{R}$ defined by $D(u) = D(u, u)$ is continuous in $L^2(\Omega)$.*

Proof. Let $u \in L^2(\Omega)$ and $(u_n)_n$ be a sequence such that $u_n \to u$ in $L^2(\Omega)$. Then, up to subsequence, there exists $g \in L^2(\Omega)$ such that $|u_n(x)| \le g(x)$ and $u_n(x) \to u(x)$ almost everywhere in $x \in \Omega$. Since

$$\frac{|u_n(x)u_n(y)|}{|x-y|} \le \frac{|g(x)g(y)|}{|x-y|} \qquad \text{for a.e. } (x, y) \in \Omega \times \Omega,$$

then if $\frac{|g(x)g(y)|}{|x-y|} \in L^1(\Omega \times \Omega)$, the claim holds by Dominated Convergence Theorem. Let $v_g(x) := \int_\Omega \frac{g(y)}{|x-y|}dy$ for $x \in \Omega$, then $v_g \in W^{2,6}(\Omega)$ by [25, Theorem 9.9] and so

$$\int_\Omega \int_\Omega \frac{|g(x)g(y)|}{|x-y|}dxdy = \int_\Omega v_g(x)g(x)dx \le ||v_g||_{L^2(\Omega)}||g||_{L^2(\Omega)} < \infty. \qquad \square$$

[2] See the end of this section for the precise notion of quasi-open sets

2.1. **Quasi-open sets.** In this section, we briefly recall some standard definitions and properties concerning quasi-open sets, which define the natural setting on which our shape minimization problem is well posed. We refer to [29] for further details on the subject.

Definition 2.3. A measurable set $\Omega \subset \mathbb{R}^3$ is called *quasi-open* if, for every $\varepsilon > 0$, there exists a compact set K_ε whose Newtonian capacity satisfies $\text{cap}(K_\varepsilon) < \varepsilon$ such that $\Omega \setminus K_\varepsilon$ is an open set. Here $\text{cap}(\cdot)$ stands for the standard Sobolev capacity.

Along the same lines, a function $u : \Omega \to \mathbb{R}$ is defined as *quasi-continuous* if for every $\varepsilon > 0$ one can find a compact set K_ε with $\text{cap}(K_\varepsilon) < \varepsilon$ such that the restriction of u to $\Omega \setminus K_\varepsilon$ is continuous. A property is said to hold *quasi-everywhere* (q.e.) if the set where it fails to hold has null capacity.

It is a classical result that any function $u \in H^1(\Omega)$ admits a quasi-continuous representative $\tilde{u} : \Omega \to \mathbb{R}$. Furthermore, any two quasi-continuous representatives of u coincide quasi-everywhere. Consequently, it is customary to identify every Sobolev function with its quasi-continuous representative. In this framework, a quasi-open set can simply be understood as a superlevel set of a function $u \in H^1(\mathbb{R}^3)$. Notice that if a function $u \in H^1(\mathbb{R}^3)$ is sign-changing, clearly $\{u \neq 0\}$ is a quasi-open set.

Finally, for a quasi-open set $\Omega \subset \mathbb{R}^3$, the associated Sobolev space $H^1_0(\Omega)$ is naturally defined as:

$$H^1_0(\Omega) = \{u \in H^1(\mathbb{R}^3) : u = 0 \text{ quasi-everywhere in } \mathbb{R}^3 \setminus \Omega\}.$$

We stress that this definition coincides with the classical one whenever Ω is a standard open set.

2.2. **Fraenkel asymmetry and quantitative Faber-Krahn inequality.** We conclude this preliminary section by recalling the sharp quantitative version of the Faber-Krahn inequality. In order to do that we first remind the notion of *Fraenkel asymmetry*: for a measurable set $E \subset \mathbb{R}^3$ with finite measure we define

$$\mathcal{A}(E) = \inf_{x \in \mathbb{R}^3} \frac{|E \Delta (B + x)|}{|E|}, \tag{5}$$

where B denotes the ball of measure $|E|$ centered at the origin. We also recall that we denote by $\lambda_0(\Omega)$ the first eigenvalue of the Dirichlet Laplacian of a quasi-open set $\Omega \subset \mathbb{R}^3$.

Theorem 2.4 ([6]). *There exists a universal positive constant $\widehat{\sigma} > 0$ such that for all quasi-open sets $\Omega \subset \mathbb{R}^3$ with finite measure we have*

$$|\Omega|^{2/3}\lambda_0(\Omega) - |B_1|^{2/3}\lambda_0(B_1) \geq \widehat{\sigma}\mathcal{A}(\Omega)^2.$$

3. Minimization of $E_q(u, \Omega)$ in u

We are now in a position to address the study of the energy functional for a fixed domain Ω. Relying on the nonnegativity and continuity properties established in the previous section, the first natural result guarantees the existence of a minimizing function.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a quasi-open set of finite measure and let $q \geq 0$. Then*

$$E_q(\Omega) = \inf\left\{E_q(v, \Omega) : v \in H^1_0(\Omega), \int_\Omega v^2\,dx = 1\right\}.$$

admits a minimizer.

Proof. Let $(u_n)_n$ be a minimizing sequence for the energy. By Lemma 2.1 all the terms in the definition of $E_q(u, \Omega)$ are nonnegative, thus we infer that $(u_n)_n$ is bounded in $H^1_0(\Omega)$. Then up to passing to a subsequence, $(u_n)_n$ converges weakly in $H^1_0(\Omega)$, strongly in $L^2(\Omega)$ and pointwise a.e. to some function $u \in H^1_0(\Omega)$. Observe that in the case of finite measure unbounded set, the Sobolev's embedding holds by [9]. In particular the L^2-convergence implies that $\|u\|_{L^2(\Omega)} = 1$. By lower semicontinuity with respect to the weak convergence, we have that

$$\int_\Omega |\nabla u|^2\,dx \leq \liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^2\,dx$$

and, by Proposition 2.2

$$D(u, u) = \iint_{\Omega\times\Omega} \frac{u(x)u(y)}{|x-y|}\,dxdy = \lim_{n\to+\infty} D(u_n, u_n).$$

Hence, the functional is lower semicontinuous

$$E_q(u, \Omega) \leq \liminf_{n\to+\infty} E_q(u_n, \Omega),$$

and u is a minimizer. $\square$

Having ensured the existence of an optimal function, the next step is to derive its optimality conditions. So now we study the associated Euler-Lagrange equation, analyzing some of its consequences.

Theorem 3.2. *Let $q \in [0,1)$ and Ω be a quasi-open set of finite measure such that $E_q(\Omega) \le E_1(B_1)$. Then every optimal function $u_q \in H^1_0(\Omega)$ of $E_q(\Omega)$ satisfies in a weak sense*

$$-\Delta u_q - \lambda_{q,\Omega} u_q = -\frac{q}{2}\Big(u_q * \frac{1}{|\cdot|}\Big) \quad \text{in } H^1_0(\Omega),$$

where

$$\lambda_{q,\Omega} = \int_\Omega |\nabla u_q|^2\,dx + \frac{q}{2} D(u_q, u_q).$$

Furthermore $u_q \in C^\infty(\Omega)$ and

$$||u_q||_{L^\infty(\Omega)} \le C, \qquad \|v_{u_q}\|_{L^\infty(\mathbb{R}^3)} \le C$$

for a constant $C = C(|\Omega|)$.

Proof. By Lemma 3.1, let u_q be an optimal function arising from the minimisation of $E_q(\Omega)$. By the direct computation of its first variation we have that it satisfies in weak sense

$$-\Delta u_q - \lambda_{q,\Omega} u_q = -\frac{q}{2}\Big(u_q * \frac{1}{|\cdot|}\Big).$$

Let us now call $f(x) := -\frac{q}{2}\left(u_q * \frac{1}{|\cdot|}\right)(x)$, then by (4)

$$|f(x)| \le C||\nabla u_q||_{L^1(\Omega)} \le C|\Omega|^{1/2}||\nabla u_q||_{L^2(\Omega)} \le C|\Omega|^{1/2}\max\{\lambda_{q,\Omega}; 1\},$$

so $f(x) \in L^\infty(\Omega)$. Moreover by the Faber-Krahn inequality

$$||u_q||_{L^2(\Omega)} \le \frac{1}{\lambda_0(\Omega)}||\nabla u_q||_{L^2(\Omega)} \le \frac{|\Omega|^{2/3}}{\lambda_0(B)}\lambda_{q,\Omega},$$

where B is a ball of measure one. By the definition of $\lambda_{q,\Omega}$, it holds

$$\lambda_{q,\Omega} = E_q(\Omega) \le E_1(\Omega) \le E_1(B_1),$$

so that, by classical elliptic regularity theory (see [25, Theorem 8.8 and Theorem 8.15]), $u_q \in W^{2,2}_{\mathrm{loc}}(\Omega)$ and

$$||u_q||_{L^\infty(\Omega)} \le C(|\Omega|).$$

Eventually, recalling also that $-\Delta v_{u_q} = u_q$ in Ω, we conclude that $\|v_{u_q}\|_{L^\infty(\mathbb{R}^3)} \le C$ and by a bootstrap argument that $u_q \in C^\infty(\Omega)$. ☐

Remark 3.3. It is not difficult to check that the scale invariant functional

$$\widetilde{E}_q(v,\Omega) := \frac{\int_\Omega |\nabla v|^2\,dx}{\|v\|^2_{L^2}} + \frac{q}{2}\frac{\int_\Omega\int_\Omega \frac{v(x)v(y)}{|x-y|}\,dx\,dy}{\|v\|^2_{L^2}}, \qquad v \in H^1_0(\Omega),$$

leads to an equivalent, yet unconstrained, minimization problem

$$\min\Big\{E_q(v,\Omega) : v \in H^1_0(\Omega),\ \int_\Omega v^2\,dx = 1\Big\} = \min\Big\{\widetilde{E}_q(v,\Omega) : v \in H^1_0(\Omega), v \not\equiv 0\Big\}.$$

In particular we can always choose an optimal function $u \in H^1_0(\Omega)$ for the unconstrained problem which satisfies, a posteriori, the constraint $\|u\|_{L^2} = 1$.

Now we introduce an equivalent formulation of problem (2) without any L^2 constraint, in a different spirit with respect to Remark 3.3, and which is more suited for the regularity arguments which will follow in next sections. We recall that $\Omega \subset \mathbb{R}^3$ is quasi-open set with finite measure, then we define

$$E_{q,M}(\Omega) := \min\Big\{E_{q,M}(v,\Omega) : v \in H^1_0(\Omega)\Big\}, \tag{6}$$

where

$$E_{q,M}(v,\Omega) := E_q(v,\Omega) + M\Big|\int_\Omega v^2\,dx - 1\Big|.$$

Reasoning exactly as in Lemma 3.1 we can prove that there exists a minimizer for $E_{q,M}(\Omega)$. We show now that for some value of M, problems (6) and (2) are equivalent.

Proposition 3.4. *Let $\Omega \subseteq \mathbb{R}^3$ be a quasi-open set with finite measure. Assume that $M \ge 5E_{q,M}(\Omega) + 5E_1(B_1)$. Then the minimizers of the problem (2) are the same as those of (6).*

Proof. First of all, it is easy to check that $E_{q,M}(\Omega) \le E_q(\Omega)$, since optimal functions for the latter are admissible in the minimization of the former and are L^2-normalized. Now, let $\hat{u}$ be an optimal function for $E_{q,M}(\Omega)$. Observe that $||\hat{u}||_{L^2(\Omega)} \ge 1/2$. Indeed if by contradiction $||\hat{u}||_{L^2(\Omega)} < 1/2$, then (using also the assumption on M)

$$E_{q,M}(\Omega) = E_{q,M}(\Omega, \hat{u}) = E_q(\hat{u}, \Omega) + M\Big(1 - \int_\Omega \hat{u}^2 dx\Big) > \frac{3}{4}M \ge E_1(B_1) + E_{q,M}(\Omega) \ge E_{q,M}(\Omega).$$

Let us now call $\sigma := ||\hat{u}||_{L^2(\Omega)} - 1 \ge -1/2$ and $u := \frac{\hat{u}}{1+\sigma}$. We can then compute

$$\begin{aligned} E_q(\Omega) \ge E_{q,M}(\Omega) = E_{q,M}(\hat{u}, \Omega) &= (1+\sigma)^2 \int_\Omega |\nabla u|^2 dx + \frac{q}{2}(1+\sigma)^2 D(u,u) + M\left|(1+\sigma)^2 - 1\right| \\ &\ge E_q(u, \Omega) + 2\sigma\Big(\int_\Omega |\nabla u|^2 dx + \frac{q}{2} D(u,u)\Big) + M|2\sigma + \sigma^2| \end{aligned}$$

At this point, if $\sigma > 0$ we immediately find a contradiction for all $M > 0$. On the other hand, if $\sigma \in (-1/2, 0)$, we notice that $|2\sigma + \sigma^2| \ge |\sigma|$ and then

$$E_q(\Omega) \ge E_q(\Omega) + 2\sigma E_q(u, \Omega) + M|\sigma|,$$

so we obtain a contradiction since $E_q(u, \Omega) \le 4E_{q,M}(\Omega)$ (recall that $(1+\sigma)^2 \ge 1/4$) and $M \ge 5E_{q,M}(\Omega)$. Thus $\sigma = 0$. $\square$

Remark 3.5. From now on, we fix once and for all a constant

$$M > 10(E_1(B_1) + |B_1|).$$

Clearly, if Ω is a quasi-open set of finite measure such that $E_{q,M}(\Omega) \le E_1(B_1) + |B_1|$, then $M > 10(E_1(B_1) + |B_1|) \ge 5E_{q,M}(\Omega) + 5E_1(B_1)$ and so by Proposition 3.4, we obtain that problems $E_q(\Omega)$ and $E_{q,M}(\Omega)$ are equivalent. Now let us focus on the case when $q \in (0,1)$, when the monotonicity of the functionals with respect to q entails that

$$\inf\{E_{q,M}(\Omega) : \Omega \subseteq \mathbb{R}^3 \text{ quasi-open}, \ |\Omega| = |B_1|\} \le \inf\{E_q(\Omega) : \Omega \subseteq \mathbb{R}^3 \text{ quasi-open}, \ |\Omega| = |B_1|\} \le E_1(B_1),$$

thus considering in the above minimizations only quasi-open sets Ω with $E_{q,M}(\Omega) \le E_1(B_1) + |B_1|$ is not restrictive. In conclusion, for $q \in (0,1)$ and M chosen as above,

$$\inf\{E_q(\Omega) : \Omega \subseteq \mathbb{R}^3 \text{ quasi-open}, \ |\Omega| = |B_1|\} = \inf\{E_{q,M}(\Omega) : \Omega \subseteq \mathbb{R}^3 \text{ quasi-open}, \ |\Omega| = |B_1|\}.$$

4. A surgery result

In this section, we prove a surgery result that allows to remove the equiboundedness assumption. The surgery strategy that we employ is similar to the one proposed in [36] (see also [10]) and used for problems more similar to the current one in in [37, 35]. Nevertheless some non-trivial technical changes are needed to adapt it to our case, whence we report the full proof here below. In particular some additional technicalities are due to the PDE solved by optimal functions and to the fact that they can change sign.

Lemma 4.1. *There exist universal constants D, $\overline{\delta} < 1$ and $q_1 \in (0,1)$ such that if $q \le q_1$ then for any open and connected set $\Omega \subset \mathbb{R}^3$ of measure $|B_1|$ satisfying $E_q(\Omega) - \lambda_0(B) \le \overline{\delta}$ there exists an open, connected set $\widehat{\Omega}$ of measure $|B_1|$ with diameter bounded by D and such that*

$$E_q(\widehat{\Omega}) \le E_q(\Omega).$$

We show this result in several steps. Let us introduce some notation. Let Ω be a connected set of measure $|B_1|$ such that $\lambda_0(\Omega) - \lambda_0(B_1) \le E_q(\Omega) - \lambda_0(B_1) \le \overline{\delta}$, with $\overline{\delta} \in (0,1)$ to be chosen, and fix B_1 the ball attaining the minimum in the Fraenkel asymmetry for Ω (see (5)). We assume, up to a translation of Ω, that B_1 is centered at the origin. Then, by the quantitative Faber–Krahn inequality (see Theorem 2.4), we have

$$|\Omega \Delta B_1| = \mathcal{A}(\Omega) \le |B_1|^{1/3} \left(\frac{\overline{\delta}}{\widehat{\sigma}}\right)^{1/2},$$

where $\widehat{\sigma}$ is the constant provided by Theorem 2.4. By defining

$$K := \lambda_0(B_1) + 1 \ge \lambda_0(B_1) + \overline{\delta}$$

we obtain immediately

$$E_q(\Omega) \le K, \qquad \text{and in particular,} \qquad \int_\Omega |\nabla u|^2 \, dx \le K,$$

where $u = u_{q,\Omega}$ from now on is the function attaining $E_q(\Omega)$. We then note that (since B_1 has unit radius)

$$|\Omega \setminus [-t,t]^3| \le |\Omega \Delta B_1| = \mathcal{A}(\Omega), \qquad \text{for all } t \ge 1.$$

Let $\widehat{m} \in (0, 1/4)$ be such that

$$\frac{(4\widehat{m})^{\frac{2}{3}}}{\lambda_0(B_1)|B_1|^{\frac{2}{3}}} K \le \frac{1}{2}\,. \tag{7}$$

Moreover, we choose $\overline{\delta}$ small enough so that

$$|\Omega \setminus [-1,1]^3| \le \mathcal{A}(\Omega) \le |B_1| \left(\frac{\overline{\delta}}{\widehat{\sigma}}\right) \le \frac{\widehat{m}}{2^6}. \tag{8}$$

We first focus on the direction e_1 and detail the construction in this case. We shall denote $z = (x, y) \in \mathbb{R} \times \mathbb{R}^2$ and by z_i the i-th component of $z \in \mathbb{R}^3$. For any $t \in \mathbb{R}$, we define

$$\Omega_t := \Big\{ y \in \mathbb{R}^2 : (t, y) \in \Omega \Big\},$$

and given any set $\Omega \subseteq \mathbb{R}^3$, we define its 1-dimensional projections for $p \in \{1, 2, 3\}$ as

$$\pi_p(\Omega) := \Big\{ t \in \mathbb{R} : \exists\, (z_1,\, z_2, z_3) \in \Omega,\ z_p = t \Big\}.$$

For every $t \le -1$ we call

$$\Omega^+(t) := \Big\{ (x, y) \in \Omega : x > t \Big\}, \qquad \Omega^-(t) := \Big\{ (x, y) \in \Omega : x < t \Big\}, \qquad \varepsilon(t) := \mathcal{H}^2(\Omega_t)\,.$$

Observe that by (8)

$$m(t) := \big|\Omega^-(t)\big| = \int_{-\infty}^{t} \varepsilon(s)\, ds \le 2\widehat{m}\,.$$

We call u a optimizer for $E_q(\Omega)$. We define then also, for every $t \le -1$,

$$\delta(t) := \int_{\Omega_t} |\nabla u(t, y)|^2\, d\mathcal{H}^2(y)\,, \qquad \mu(t) := \int_{\Omega_t} u(t, y)^2\, d\mathcal{H}^2(y)\,,$$

which makes sense since u is smooth inside Ω, see Theorem 3.2. Applying the Faber–Krahn inequality in $\mathbb{R}^2$ to the set Ω_t, and using the rescaling property of eigenvalues on $\mathbb{R}^2$, we know that

$$\varepsilon(t)\lambda_0(\Omega_t) = \mathcal{H}^2(\Omega_t)\lambda_0(\Omega_t) \ge \lambda_0(B_{\mathbb{R}^2})\,,$$

calling $B_{\mathbb{R}^2}$ the ball of unit measure in $\mathbb{R}^2$. As a trivial consequence, we can estimate μ in terms of ε and δ: in fact, noting that $u(t, \cdot) \in H^1_0(\Omega_t)$ and writing $\nabla u = (\nabla_1 u, \nabla_y u)$, we have

$$\mu(t) = \int_{\Omega_t} u(t, \cdot)^2\, d\mathcal{H}^2 \le \frac{1}{\lambda_0(\Omega_t)} \int_{\Omega_t} |\nabla_y u(t, \cdot)|^2\, d\mathcal{H}^2 \le C\varepsilon(t)\delta(t). \tag{9}$$

We can now present two estimates which assure that u and ∇u cannot be too big in $\Omega^-(t)$.

Lemma 4.2. *Let $\Omega \subseteq \mathbb{R}^3$ and u be as in Lemma 4.1. For every $t \le -1$ the following inequalities hold:*

$$\int_{\Omega^-(t)} u^2\, dx \le C_1 \varepsilon(t)^{\frac{1}{2}} \delta(t) + qC_m m(t)\,, \qquad \int_{\Omega^-(t)} |\nabla u|^2\, dx \le C_1 \varepsilon(t)^{\frac{1}{2}} \delta(t) + qC_m m(t)\,, \tag{10}$$

for some universal constants $C_1, C_m > 0$.

The proof of the above Lemma follows, up to a few minor changes, as in [36, Lemma 2.3], by working on u. The main difference is that u solves the PDE

$$\begin{cases} -\Delta u = \lambda_q u - \frac{q}{2} v_u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{11}$$

instead of being first eigenfunction of the Dirichlet Laplacian in Ω. In particular, if $u \ge 0$, we note that u solves the differential inequality $-\Delta u \le \lambda_q u$, in Ω and things work as in [35] without the need of the additional term $C_m m(t)$. We reproduce fully the proof here for the sake of completeness.

Proof. Let us fix $t \leq -1$. Consider the set Ω_S^- obtained by the union of $\Omega^-(t)$ and its reflection with respect to the plane $\{x = t\}$, and call $u_S \in H_0^1(\Omega_S)$ the function obtained by reflecting u. Using the Faber-Krahn inequality, we find then

$$\frac{\lambda_0(B_1)|B_1|^{\frac{2}{3}}}{\big(2m(t)\big)^{\frac{2}{3}}} = \frac{\lambda_0(B_1)|B_1|^{\frac{2}{3}}}{|\Omega_S^-|^{\frac{2}{3}}} \leq \lambda_0(\Omega_S^-) \leq \frac{\displaystyle\int_{\Omega_S^-} |\nabla u_S|^2\,dx}{\displaystyle\int_{\Omega_S^-} u_S^2\,dx} = \frac{\displaystyle\int_{\Omega^-(t)} |\nabla u|^2\,dx}{\displaystyle\int_{\Omega^-(t)} u^2\,dx}$$

by the symmetry of Ω_S^-, and using the scaling. This estimate gives

$$\int_{\Omega^-(t)} u^2\,dx \leq \frac{\big(2m(t)\big)^{\frac{2}{3}}}{\lambda_0(B_1)|B_1|^{\frac{2}{3}}} \int_{\Omega^-(t)} |\nabla u|^2\,dx \tag{12}$$

which in particular, being $m(t) \leq 2\widehat{m}$ and recalling (7), implies

$$\int_{\Omega^-(t)} u^2\,dx \leq \frac{1}{2}\,.$$

On the other hand, recalling (11), by Schwarz inequality, the uniform L^∞ bound on v_u and u, recalling $m(t) = |\Omega^-(t)|$ and using (9) we have

$$\begin{aligned}
\int_{\Omega^-(t)} |\nabla u|^2\,dx &= \int_{\Omega^-(t)} \lambda_q u^2 - \frac{q}{2} v_u u\,dx + \int_{\Omega_t} u\,\frac{\partial u}{\partial \nu}\,d\mathcal{H}^2 \\
&\leq K \int_{\Omega^-(t)} u^2\,dx + qC_m m(t) + \sqrt{\int_{\Omega_t} u^2\,d\mathcal{H}^2 \int_{\Omega_t} |\nabla u|^2\,d\mathcal{H}^2} \\
&\leq K \int_{\Omega^-(t)} u^2\,dx + C\varepsilon(t)^{\frac{1}{2}}\delta(t) + qC_m m(t)\,.
\end{aligned} \tag{13}$$

It is now easy to obtain (10) combining (12) and (13). In fact, by inserting the latter into the first, we find

$$\int_{\Omega^-(t)} u^2\,dx \leq \frac{\big(2m(t)\big)^{\frac{2}{3}}}{\lambda_0(B_1)|B_1|^{\frac{2}{3}}} \left(K \int_{\Omega^-(t)} u^2\,dx + C\varepsilon(t)^{\frac{1}{2}}\delta(t) + qC_m m(t)\right),$$

which by (7) again yields

$$\frac{1}{2}\int_{\Omega^-(t)} u^2\,dx \leq \frac{\big(2m(t)\big)^{\frac{2}{3}}}{\lambda_0(B_1)|B_1|^{\frac{2}{3}}} \Big[C\varepsilon(t)^{\frac{1}{2}}\delta(t) + qC_m m(t)\Big] \leq C\varepsilon(t)^{\frac{1}{2}}\delta(t) + qC_m m(t)\,. \tag{14}$$

The left estimate in (10) is then obtained. To obtain the right one, one has then just to insert (14) into (13). □

Let us go further into the construction, giving some additional definitions. For any $t \leq -1$ and $\sigma(t) > 0$, we define the cylinder $Q(t)$ as

$$Q(t) := \Big\{(x, y) \in \mathbb{R}^3 : t - \sigma(t) < x < t,\ (t, y) \in \Omega\Big\} = \big(t - \sigma(t), t\big) \times \Omega_t\,,$$

where for any $t \leq -1$ we set

$$\sigma(t) = \varepsilon(t)^{\frac{1}{2}}\,.$$

We let also $\widetilde{\Omega}(t) = \Omega^+(t) \cup Q(t)$, and we introduce $\widetilde{u} \in H_0^1\big(\widetilde{\Omega}(t)\big)$ as

$$\widetilde{u}(x, y) := \begin{cases} u(x, y) & \text{if } (x, y) \in \Omega^+(t)\,, \\ \dfrac{x - t + \sigma(t)}{\sigma(t)}\,u(t, y) & \text{if } (x, y) \in Q(t)\,. \end{cases}$$

The fact that $\widetilde{u}$ vanishes on $\partial\widetilde{\Omega}(t)$ is obvious; moreover, $\nabla u = \nabla \widetilde{u}$ on $\Omega^+(t)$, while on $Q(t)$ one has

$$\nabla \widetilde{u}(x, y) = \left(\frac{u(t, y)}{\sigma(t)}\,,\ \frac{x - t + \sigma(t)}{\sigma(t)}\,\nabla_y u(t, y)\right).$$

A simple calculation allows us to estimate the integrals of $\widetilde{u}$ and $\nabla \widetilde{u}$ on $Q(t)$.

Lemma 4.3. *For every $t \leq -1$, one has*

$$\int_{Q(t)} |\nabla \widetilde{u}|^2 \, dx \leq C_2 \varepsilon(t)^{\frac{1}{2}} \delta(t) \,, \qquad \int_{Q(t)} \widetilde{u}^2 \, dx \leq C_2 \varepsilon(t)^{\frac{3}{2}} \delta(t) \,,$$

for a universal constant $C_2 > 0$.

The proof of the above Lemma follows as [36, Lemma 2.4], as it does not depend on the PDE solved by u.

Another simple but useful estimate concerns the Rayleigh quotients of the functions $\widetilde{u}$ on the sets $\widetilde{\Omega}(t)$: notice that, while u has unit L^2 norm, the modifed function $\widetilde{u}$ in general is not normalized so we need to take care also of its norm.

Lemma 4.4. *There exists a universal constant $C_3 > 0$ such that for every $t \leq -1$, one has*

$$\int_{\widetilde{\Omega}(t)} |\nabla \widetilde{u}|^2 \, dx \leq \int_{\Omega} |\nabla u|^2 \, dx + C_3 \varepsilon(t)^{\frac{1}{2}} \delta(t) \,, \qquad \int_{\widetilde{\Omega}(t)} \widetilde{u}^2 \, dx \geq \int_{\Omega} u^2 \, dx - C_3 \varepsilon(t)^{\frac{1}{2}} \delta(t) - q C_m m(t) \,.$$

Proof. It is enough to note that, by definition of $\widetilde{\Omega}(t)$ and using Lemma 4.2 and 4.3, we obtain for the gradient term

$$\begin{aligned} \int_{\widetilde{\Omega}(t)} |\nabla \widetilde{u}|^2 \, dx &= \int_{\Omega^+(t)} |\nabla u|^2 \, dx + \int_{Q(t)} |\nabla \widetilde{u}|^2 \, dx \\ &= \int_{\Omega} |\nabla u|^2 \, dx + \int_{Q(t)} |\nabla \widetilde{u}|^2 \, dx - \int_{\Omega^-(t)} |\nabla u|^2 \, dx \leq \int_{\Omega} |\nabla u|^2 \, dx + C_2 \varepsilon(t)^{\frac{1}{2}} \delta(t) \,, \end{aligned}$$

while for the function, we have

$$\int_{\widetilde{\Omega}(t)} \widetilde{u}^2 \, dx = \int_{\Omega^+(t)} u^2 \, dx + \int_{Q(t)} \widetilde{u}^2 \, dx = \int_{\Omega} u^2 \, dx + \int_{Q(t)} \widetilde{u}^2 \, dx - \int_{\Omega^-(t)} u^2 \, dx \geq \int_{\Omega} u^2 \, dx - C_1 \varepsilon(t)^{\frac{1}{2}} \delta(t) - q C_m m(t) \,.$$

□

We can now enter in the central part of our construction. Basically, we aim to show that either Ω already has bounded left "tail" in direction e_1, or some rescaling of $\widetilde{\Omega}(t)$ has energy lower than that of Ω.

Lemma 4.5. *Let Ω be as in the assumptions of Lemma 4.2, and let $t \leq -1$. There exist universal $q_1 \in (0,1)$ and $C_4 > 2$ such that, for all $q \leq q_1$ exactly one of the three following conditions hold:*

(1) $\max\big\{\varepsilon(t), \, \delta(t)\big\} > 1$*;*

(2) *(1) does not hold and* $m(t) \leq C_4 \big(\varepsilon(t) + \delta(t)\big) \varepsilon(t)^{\frac{1}{2}}$*;*

(3) *(1) and (2) do not hold and there holds*

$$\frac{\int_{\widehat{\Omega}(t)} |\nabla \widehat{u}|^2 \, dx}{\int_{\widehat{\Omega}(t)} \widehat{u}^2 \, dx} \leq \int_{\Omega} |\nabla u|^2 \, dx, \qquad \textit{and} \qquad E_q\big(\widehat{\Omega}(t)\big) < E_q(\Omega),$$

where for $t \leq -1$ we set

$$\widehat{\Omega}(t) := \big|B_1\big|^{\frac{1}{3}} \big|\widetilde{\Omega}(t)\big|^{-\frac{1}{3}} \widetilde{\Omega}(t), \qquad \textit{and} \qquad \widehat{u}(x) = \widetilde{u}\big(|B_1|^{-\frac{1}{3}} |\widetilde{\Omega}(t)|^{\frac{1}{3}} x\big), \quad \textit{for } x \in \widehat{\Omega}(t).$$

Proof. Assume (1) is false. Then it is possible to apply Lemma 4.4, to obtain

$$\begin{aligned} \int_{\widetilde{\Omega}(t)} |\nabla \widetilde{u}|^2 \, dx &\leq \int_{\Omega} |\nabla u|^2 \, dx + C_3 \varepsilon(t)^{\frac{1}{2}} \delta(t) \,, \\ \int_{\widetilde{\Omega}(t)} \widetilde{u}^2 \, dx &\geq \int_{\Omega} u^2 \, dx - C_3 \varepsilon(t)^{\frac{1}{2}} \delta(t) - q C_m m(t) = 1 - C_3 \varepsilon(t)^{\frac{1}{2}} \delta(t) - q C_m m(t) \,. \end{aligned} \tag{15}$$

By the scaling properties of the eigenvalue and the fact that $\big|\widehat{\Omega}(t)\big| = |B_1|$, we know that

$$\frac{\int_{\widehat{\Omega}(t)} |\nabla \widehat{u}|^2 \, dx}{\int_{\widehat{\Omega}(t)} \widehat{u}^2 \, dx} = \frac{\big|\widetilde{\Omega}(t)\big|^{\frac{2}{3}}}{|B_1|^{\frac{2}{3}}} \frac{\int_{\widetilde{\Omega}(t)} |\nabla \widetilde{u}|^2 \, dx}{\int_{\widetilde{\Omega}(t)} \widetilde{u}^2 \, dx} \,.$$

By construction,

$$\big|\widetilde{\Omega}(t)\big| = \big|\Omega^+(t)\big| + \big|Q(t)\big| = |B_1| - m(t) + \varepsilon(t)^{\frac{3}{2}} \,,$$

hence the above estimates, the scaling of the integrals due to the definition of $\widehat{u}$ and (15) lead to

$$\begin{aligned}
\frac{\int_{\widehat{\Omega}(t)}|\nabla\widehat{u}|^2\,dx}{\int_{\widehat{\Omega}(t)}\widehat{u}^2\,dx} &= \Big(1-\frac{m(t)}{|B_1|}+\frac{\varepsilon(t)^{\frac32}}{|B_1|}\Big)^{\frac23}\,\frac{\int_{\widetilde{\Omega}(t)}|\nabla\widetilde{u}|^2\,dx}{\int_{\widetilde{\Omega}(t)}\widetilde{u}^2\,dx},\\
&\le \Big(1-\frac{2}{3|B_1|}\,m(t)+\frac{2}{3|B_1|}\,\varepsilon(t)^{\frac32}\Big)\Big(1+C_3\varepsilon^{\frac12}(t)\delta(t)+qC_m m(t)\Big)\Big(\int_\Omega|\nabla u|^2\,dx+C_3\varepsilon(t)^{\frac12}\delta(t)\Big)\\
&\le \int_\Omega|\nabla u|^2\,dx-\frac{2\lambda_0(B_1)}{3|B_1|}\,m(t)+qKC_m m(t)+\frac{2K}{3|B_1|}\,\varepsilon(t)^{\frac32}+\Big(2C_3+KC_3+\frac{2}{3|B_1|}\Big)\varepsilon(t)^{\frac12}\delta(t)\\
&\le \int_\Omega|\nabla u|^2\,dx-\frac{49}{100}\pi\,m(t)+\frac{2K}{3|B_1|}\,\varepsilon(t)^{\frac32}+\Big(2C_3+KC_3+\frac{2}{3|B_1|}\Big)\varepsilon(t)^{\frac12}\delta(t)\,,
\end{aligned}$$

noting that (in three dimension) $\frac{\pi}{2}=\frac{2\lambda_0(B_1)}{3|B_1|}$ and up to take q_1 such that

$$q_1KC_m<\frac{\pi}{100}=\frac{2\lambda_0(B_1)}{150|B_1|}.$$

At this point, defining $C_4:=\max\{\frac{2(K+1)}{3|B_1|}+2C_3+KC_3,2\}$, if

$$m(t)\le C_4\big(\varepsilon(t)+\delta(t)\big)\varepsilon(t)^{\frac12},$$

then condition (2) holds true. Otherwise, we immediately have that

$$\frac{\int_{\widehat{\Omega}(t)}|\nabla\widehat{u}|^2\,dx}{\int_{\widehat{\Omega}(t)}\widehat{u}^2\,dx}\le\int_\Omega|\nabla u|^2\,dx-\Big(\frac{49}{100}\pi-1\Big)\,m(t)\le\int_\Omega|\nabla u|^2\,dx-C_5m(t), \tag{16}$$

for a universal constant $C_5>0$, therefore the first part of the third claim is verified.

On the other hand, we note that, using the L^∞ bound of u, see Theorem 3.2, the fact that $\|\widetilde{u}\|_{L^\infty}\le\|u\|_{L^\infty}$ by construction and also by [21, Lemma 2.4],

$$\begin{aligned}
D(\widetilde{u},\widetilde{u}) &= D(u,u)-2\int_{\Omega^-(t)}\int_{\Omega^+(t)}\frac{u(x)u(y)}{|x-y|}\,dxdy-\int_{\Omega^-(t)}\int_{\Omega^-(t)}\frac{u(x)u(y)}{|x-y|}\,dxdy\\
&\qquad+2\int_{\Omega^+(t)}\int_{Q(t)}\frac{\widetilde{u}(x)\widetilde{u}(y)}{|x-y|}\,dxdy+\int_{Q(t)}\int_{Q(t)}\frac{\widetilde{u}(x)\widetilde{u}(y)}{|x-y|}\,dxdy\\
&\le D(u,u)+\tilde{C}_m m(t)+C_{fp}\varepsilon^{\frac32}(t),
\end{aligned}$$

where $\tilde{C}_m$ and C_{fp} are positive universal constants. Then we can estimate, using the appropriate scalings,

$$\begin{aligned}
\frac{D(\widehat{u},\widehat{u})}{\int_{\widehat{\Omega}(t)}\widehat{u}^2\,dx} &\le \frac{D(\widetilde{u},\widetilde{u})}{\int_{\widetilde{\Omega}(t)}\widetilde{u}^2\,dx}\left(1-\frac{m(t)}{|B_1|}+\frac{\varepsilon(t)^{\frac32}}{|B_1|}\right)^{-\frac23}\le\Big(1+\frac{2}{3|B_1|}m(t)\Big)\frac{D(\widetilde{u},\widetilde{u})}{\int_{\widetilde{\Omega}(t)}\widetilde{u}^2\,dx}\\
&\le\Big(1+\frac{2}{3|B_1|}m(t)\Big)\Big(1+C_3\varepsilon^{\frac12}(t)\delta(t)\Big)\Big(D(u,u)+\tilde{C}_m m(t)+C_{fp}\varepsilon^{\frac32}(t)\Big)\\
&\le D(u,u)+C\|u\|^2_{L^\infty}m(t)+\tilde{C}_m m(t)+C_{fp}\varepsilon^{\frac32}(t)+C_3\|u\|^2_{L^\infty}\varepsilon^{\frac12}(t)\delta(t)\\
&\le D(u,u)+C\|u\|^2_{L^\infty}m(t)+\tilde{C}_m m(t)+(C_{fp}+C_3\|u\|^2_{L^\infty})m(t)\\
&= D(u,u)+C_6m(t).
\end{aligned} \tag{17}$$

Then, putting together (16) and (17), recalling also Remark 3.3 for the equivalence of the scale invariant energy,

$$\begin{aligned}
E_q(\widehat{\Omega}(t)) &\le \frac{\int_{\widehat{\Omega}(t)}|\nabla\widehat{u}|^2\,dx}{\int_{\widehat{\Omega}(t)}\widehat{u}^2\,dx}+\frac{q}{2}\frac{D(\widehat{u},\widehat{u})}{\int_{\widehat{\Omega}(t)}\widehat{u}^2\,dx}\\
&\le\int_\Omega|\nabla u|^2\,dx+\frac{q}{2}D(u,u)-(C_5-\frac{q}{2}C_6)m(t)\\
&\le\int_\Omega|\nabla u|^2\,dx+\frac{q}{2}D(u,u)-\frac{C_5}{2}m(t),
\end{aligned}$$

up to taking $q\le q_1<\min\Big\{\sqrt{\frac{C_5}{2C_6}},\frac{\pi}{100KC_m}\Big\}$, so that in this case condition (3) holds and the proof is concluded. □

Once we have Lemma 4.5, the rest of the proof follows as in [36] or [37] as we detail here below.

Proof of Lemma 4.1. It is enough to repeat the analogs of [37, Lemma 8.7, Lemma 8.8, Proposition 8.1 and Section 9.2], noting that it is only a geometric argument and having $\int_\Omega |\nabla u|^2\,dx$ instead of $\lambda_0(\Omega)$ does not change anything. ☐

5. The auxiliary problem

To avoid the restriction imposed by the measure constraint, we employ the strategy first suggested by Aguilera, Alt, and Caffarelli [1] adding the following penalization term. Let $\eta \in (0,1)$ and consider the piecewise linear function

$$f_\eta\colon \mathbb{R}^+ \to \mathbb{R}, \qquad f_\eta(s) = \begin{cases} \eta(s-|B_1|), & \text{if } s \le |B_1|,\\ \frac{1}{\eta}(s-|B_1|), & \text{if } s \ge |B_1|. \end{cases}$$

It is easy to check that, for all $0 \le s_2 \le s_1$, there holds

$$\eta(s_1-s_2) \le f_\eta(s_1) - f_\eta(s_2) \le \frac{1}{\eta}(s_1-s_2). \tag{18}$$

Let us now consider the problem

$$\inf_{\substack{\Omega\subseteq\mathbb{R}^3\\ \text{quasi-open}}} \ \inf_{u\in H^1_0(\Omega)} E_{q,M,\eta}(\Omega,u). \tag{19}$$

where

$$E_{q,M,\eta}(\Omega,u) = E_{q,M}(u,\Omega) + f_\eta(|\Omega|).$$

Even without knowing at this point whether existence of an optimal set for problem (19) holds true, we point out that we can at least select one *good* minimizing sequence.

Lemma 5.1. *Let $q \in [0,1)$. Then there exists a universal constant $\eta_1 > 0$ such that for all $\eta \in (0,\eta_1)$ there exists a minimizing sequence $(\Omega_n)_n$ of problem (19) such that Ω_n are connected and*

$$C \le |\Omega_n| \le |B_2|$$

for a universal constant C.

Proof. Let us suppose for the sake of contradiction that there exists a minimizing sequence $(\Omega_n)_n$ of problem (19) such that $|\Omega_n| > |B_2|$ for all $n \in \mathbb{N}$. We are then going to reach a contradiction as long as

$$1/\eta \ge E_1(B_1).$$

Indeed it holds

$$E_{q,M,\eta}(\Omega_n) \le \inf\{E_{q,M,\eta}(\Omega) : \Omega \subseteq \mathbb{R}^3, \text{quasi-open}\} + 1 \le E_1(B_1) + 1,$$

On the other hand, by Lemma 2.1, since $|\Omega_n| > |B_2|$ we have

$$E_{q,M,\eta}(\Omega_n) \ge \frac{1}{\eta}(|\Omega_n| - |B_1|) \ge \frac{1}{\eta}(|B_2|-|B_1|).$$

By choosing η_1 such that $\eta_1 < 1$ and

$$\frac{(|B_2|-|B_1|)}{\eta_1} > E_1(B_1) + 1,$$

we reach the desired contradiction.

Let us now fix

$$\widetilde{R} = \left(\frac{\lambda_0(B_1)}{2(1+E_1(B_1)+|B_1|)}\right)^{1/2}$$

and suppose by contradiction that there exists a minimizing sequence $(\Omega_n)_n$ of problem (19) such that $|\Omega_n| < |B_{\widetilde{R}}|$. Then, by similar computations as before, by the Faber-Krahn inequality and the monotonocity of the first eigenvalue it holds

$$\lambda_0(B_{\widetilde{R}}) + f_\eta(|\Omega_n|) \le \lambda_0(\Omega_n) + f_\eta(|\Omega_n|) \le E_{q,M,\eta}(\Omega_n) \le E_1(B_1) + 1.$$

Since $f_\eta(|\Omega_n|) \ge -\eta|B_1|$, $\lambda_0(B_{\widetilde{R}}) = \frac{1}{\widetilde{R}^2}\lambda_0(B_1)$ and $\eta \in (0,1)$, we deduce

$$\widetilde{R}^2 \ge \frac{\lambda_0(B_1)}{1+E_1(B_1)+|B_1|}$$

which is a contradiction with the definition of $\widetilde{R}$. Then we can take $C = |B_{\widetilde{R}}|$ in the statement.

We focus now on the connectedness. Let $\Omega\subseteq\mathbb{R}^3$ be a term of a minimizing sequence such that $C\le|\Omega|\le|B_2|$, $E_{q,M,\eta}(\Omega)\le E_1(B_1)+|B_1|$ and made up of at most countably many connected components

$$\Omega=\bigcup_{k\in\mathbb{N}}\Omega^k.$$

For all $\vartheta\in(0,1)$, we consider a segment S_k connecting the components Ω^k and Ω^{k+1} and we consider $T_{k,\vartheta}=\cup_{x\in S_k}B_\zeta(x)$, choosing ζ so that $|T_{k,\vartheta}|\le\frac{\vartheta}{2^k}$. We call now

$$\Omega_\vartheta:=\bigcup_{k\in\mathbb{N}}T_{k,\vartheta}\cup\Omega\supset\Omega,\qquad\widehat{\Omega}_\vartheta:=\left(\frac{|\Omega|}{|\Omega_\vartheta|}\right)^{1/3}\Omega_\vartheta,$$

and note that

$$|\Omega_\vartheta|\le|\Omega|+\vartheta,\qquad|\widehat{\Omega}_\vartheta|=|\Omega|.$$

To simplify the notation, let us set $t=\left(\frac{|\Omega|}{|\Omega_\vartheta|}\right)^{1/3}\le1$. Let $u\in H_0^1(\Omega_\vartheta)$ be an optimal function for $E_q(\Omega_\vartheta)$, then $v(x)=t^{-3/2}u(x/t)\in H_0^1(\widehat{\Omega}_\vartheta)$. It holds, by a rescaling argument, that

$$\begin{aligned}E_q(\widehat{\Omega}_\vartheta)\le E_q(v,\widehat{\Omega}_\vartheta)&=t^{-2}\int_{\Omega_\vartheta}|\nabla u|^2dx+t^2\frac{q}{2}\int_{\Omega_\vartheta}\int_{\Omega_\vartheta}\frac{u(x)u(y)}{|x-y|}dxdy\\&\le(1+C\vartheta)\int_{\Omega_\vartheta}|\nabla u|^2dx+\frac{q}{2}\int_{\Omega_\vartheta}\int_{\Omega_\vartheta}\frac{u(x)u(y)}{|x-y|}dxdy=E_q(\Omega_\vartheta)+C\vartheta\int_{\Omega_\vartheta}|\nabla u|^2dx\end{aligned}$$

for a universal constant C. Observe that by set inclusion

$$E_q(\Omega_\vartheta)\le\inf\{E_q(u,\Omega_\vartheta):u\in H_0^1(\Omega),\ \int_\Omega u^2\,dx=1\}=\inf\{E_q(u,\Omega):u\in H_0^1(\Omega),\ \int_\Omega u^2\,dx=1\}=E_q(\Omega)$$

and so

$$\int_{\Omega_\vartheta}|\nabla u|^2dx\le E_q(\Omega_\vartheta)\le E_q(\Omega)\le E_1(B_1)+1,$$

thus $E_q(\widehat{\Omega}_\vartheta)\le E_q(\Omega)+C\vartheta$. Since

$$E_{q,M}(\Omega)-|B_1|\le E_{q,M,\eta}(\Omega)\le E_1(B_1)+|B_1|,$$

by Proposition 3.4 (see also Remark 3.5), it holds $E_q(\Omega)=E_{q,M}(\Omega)$. Recalling $|\widehat{\Omega}_\vartheta|=|\Omega|$, it holds

$$E_{q,M,\eta}(\widehat{\Omega}_\vartheta)\le E_{q,M,\eta}(\Omega)+C\vartheta.$$

By arbitrariety of ϑ, by applying this procedure to all of the elements of the minimizing sequence, it is not restrictive to assume that they are connected. □

We show now how the surgery argument of Section 4 can be extended also to this unconstrained functional.

Theorem 5.2. *Let $q\in[0,q_1)$ and $\eta\in(0,\eta_1)$. Then there exists a universal constant R such that*

$$\inf\{E_{q,M,\eta}(\Omega):\Omega\subseteq\mathbb{R}^3,\ \textit{quasi-open}\}=\inf\{E_{q,M,\eta}(\Omega):\Omega\subseteq B_R,\ \textit{quasi-open}\}.$$

Proof of Theorem 5.2. It is trivial that

$$\inf\{E_{q,M,\eta}(\Omega):\Omega\subseteq\mathbb{R}^3,\ \text{quasi-open}\}\le\inf\{E_{q,M,\eta}(\Omega):\Omega\subseteq B_R,\ \text{quasi-open}\}.$$

By Lemma 5.1 there exists a minimizing sequence for problem (19) made of connected sets with measure uniformly bounded from above and below, thus all the constants in the surgery argument (Lemma 4.1) will depend only on these universal bounds and not on the measure of the optimal unconstrained set. Thus by Lemma 4.1, we can construct another minimizing sequence made of sets with uniformly bounded diameter D and with the same measure. Observe that the term f_η does not interfere since the new sets have the same volume of the corresponding ones. Taking $R=\max\{2D,10\}$, this is a minimizing sequence also for the same problem but in the box B_R. □

Remark 5.3. Observe that the radius R in the proof of Theorem 5.2 has been taken as $R=\max\{2D,10\}$, where D comes from 4.1. Hence, hereafter we are allowed to consider R to be fixed.

5.1. **Existence and some properties.** Due to the surgery result, we can now restrict us to work in a fixed box, a ball of radius R (see Remark 5.3). Let us now consider the problem

$$\inf\left\{\mathcal{E}_{q,M,\eta}(u) : u \in H^1_0(B_R)\right\}, \tag{20}$$

where

$$\mathcal{E}_{q,M,\eta}(u) = \int_{B_R} |\nabla u(x)|^2\, dx + \frac{q}{2}\int_{B_R}\int_{B_R} \frac{u(x)u(y)}{|x-y|}\, dxdy + M\left|\int_{B_R} u^2 dx - 1\right| + f_\eta(|\{u \neq 0\}|).$$

Observe that problem (20) is equivalent to

$$\inf\{E_{q,M,\eta}(\Omega) : \Omega \subseteq B_R, \text{ quasi-open}\} \tag{21}$$

Indeed let $\{u_n\}_n$ be a minimizing sequence for problem (20). Then $\{u_n \neq 0\}$ is a valid candidate in (21). Thus

$$\inf_{\substack{\Omega\subseteq B_R \\ \text{quasi-open}}} \inf_{u\in H^1_0(\Omega)} E_{q,M,\eta}(\Omega, u) \leq \mathcal{E}_{q,M,\eta}(u_n),$$

and by passing to the limit we obtain

$$\inf_{\substack{\Omega\subseteq B_R \\ \text{quasi-open}}} \inf_{u\in H^1_0(\Omega)} E_{q,M,\eta}(\Omega, u) \leq \inf_{u\in H^1_0(B_R)} \mathcal{E}_{q,M,\eta}(u),$$

Suppose by contradiction that there exists a quasi-open set $\widetilde{\Omega} \subseteq B_R$ for which the inequality is strict, namely $E_{q,M,\eta}(\widetilde{\Omega}) < \inf_{u\in H^1_0(B_R)} \mathcal{E}_{q,M,\eta}(u)$. Then there exists an optimal $\widetilde{u} \in H^1_0(\widetilde{\Omega})$ for $E_{q,M}(\widetilde{\Omega})$. Since $\{\widetilde{u} \neq 0\} \subseteq \widetilde{\Omega}$ and f_η is increasing, then we reach a contradiction by

$$\mathcal{E}_{q,M,\eta}(\tilde{u}) \leq E_{q,M}(\tilde{u}, \widetilde{\Omega}) + f_\eta(|\widetilde{\Omega}|) < \inf_{u\in H^1_0(B_R)} \mathcal{E}_{q,M,\eta}(u).$$

Since we are now in an equibounded setting, we can address the existence of an optimizer for problem (20). We also show that (its support) has finite perimeter, and this proof, although inspired by the one proposed first in [8], needs to take care of the sing-changing nature of the functions involved in this problem.

Theorem 5.4. *Let $\eta \in (0, \eta_1)$ and $q \in [0, 1)$. Then there exists a minimizer for problem (20). Moreover for all minimizers $u \in H^1_0(B_R)$, the quasi-open set $\{u \neq 0\}$ has perimeter[3] uniformly bounded by a constant depending on η.*

Proof. Let $(u_n)_n$ be a sequence such that

$$\mathcal{E}_{q,M,\eta}(u_n) \leq \inf\left\{\mathcal{E}_{q,M,\eta}(u) : u \in H^1_0(B_R)\right\} + \frac{1}{n}.$$

By the nonnegativity of the terms in $\mathcal{E}_{q,M,\eta}(u_n)$, $(u_n)_n$ is bounded in $H^1_0(B_R)$ and up to a subsequence there exists $u \in H^1_0(B_R)$ such that

$$u_n \rightharpoonup u \text{ in } H^1_0(B_R), \quad u_n \to u \text{ in } L^2(B_R), \quad u_n \to u \text{ pointwise a.e. in } B_R.$$

This implies that

$$\chi_{\{u\neq 0\}}(x) \leq \liminf_{n\to+\infty} \chi_{\{u_n\neq 0\}}(x) \quad \text{for a.e. } x \in B_R,$$

so that by Fatou's Lemma there holds

$$|\{u \neq 0\}| \leq \liminf_{n\to+\infty} |\{u_n \neq 0\}|. \tag{22}$$

If $|\{u \neq 0\}| < |B_1|$, then

$$f_\eta(|\{u \neq 0\}|) = \eta(|\{u \neq 0\}| - |B_1|) \leq \liminf_{n\to+\infty} \eta(|\{u_n \neq 0\}| - |B_1|) \leq \liminf_{n\to+\infty} f_\eta(|\{u_n \neq 0\}|).$$

On the other hand, if $|\{u \neq 0\}| > |B_1|$, then by (22) $|\{u_n \neq 0\}| \geq |B_1|$ for n large enough and so

$$f_\eta(|\{u \neq 0\}|) = \frac{1}{\eta}(|\{u \neq 0\}| - |B_1|) \leq \lim_{n\to+\infty} f_\eta(|\{u_n \neq 0\}|).$$

Finally if $|\{u \neq 0\}| = |B_1|$,

$$f_\eta(|\{u \neq 0\}|) = 0 \leq \liminf_{n\to+\infty}(|\{u_n \neq 0\}| - |B_1|) \leq \liminf_{n\to+\infty} f_\eta(|\{u_n \neq 0\}|).$$

So

$$f_\eta(|\{u \neq 0\}|) \leq \liminf f_\eta(|\{u_n \neq 0\}|). \tag{23}$$

[3]Here we mean that $\{u \neq 0\}$ is a set of finite perimeter in the distributional sense. See [33] for details.

Thus by lower semicontinuity with respect to the weak convergence, Proposition 2.2 and (23), u is a minimum of $\mathcal{E}_{q,M,\eta}$. Let us show that $\{u \neq 0\}$ has finite perimeter. Define $u_\varepsilon := (u-\varepsilon)_+ - (u+\varepsilon)_-$ for $\varepsilon > 0$ small. Since u is a minimum, it holds $\mathcal{E}_{q,M,\eta}(u) \le \mathcal{E}_{q,M,\eta}(u_\varepsilon)$, which implies (using also (18))

$$\int_{\{-\varepsilon<u<\varepsilon\}} |\nabla u|^2\,dx + \eta|\{-\varepsilon<u<\varepsilon\}| \le -\frac{q}{2}(D(u,u)-D(u_\varepsilon,u_\varepsilon)) + M\Big(\Big|\int_{B_R} u_\varepsilon^2\,dx - 1\Big| - \Big|\int_{B_R} u^2\,dx - 1\Big|\Big). \tag{24}$$

Let us analyze each term. Let us start with the right hand side:

$$\begin{aligned}\Big|\int_{B_R} u_\varepsilon^2 dx - 1\Big| - \Big|\int_{B_R} u^2 dx - 1\Big| &\le \Big|\int_{\{u>\varepsilon\}} (u-\varepsilon)^2 dx + \int_{\{u<-\varepsilon\}} (u+\varepsilon)^2 dx - \int_{B_R} u^2 dx\Big| \\ &\le \Big|\int_{\{-\varepsilon<u<\varepsilon\}} u^2 dx\Big| + \varepsilon^2|B_R| + 4\varepsilon|B_R|^{1/2}||u||_{L^2(B_R)} \\ &\le 2|B_R|(\varepsilon^2 + 2\varepsilon||u||_{L^2(B_R)}).\end{aligned}$$

Then, using also the Hardy-Sobolev inequality (4), it holds

$$\begin{aligned} D(u,u) - D(u_\varepsilon,u_\varepsilon) &= \int_{\{u\neq0\}}\int_{\{u\neq0\}} \frac{u(x)u(y)}{|x-y|}\,dxdy - \int_{\{u_\varepsilon\neq0\}}\int_{\{u_\varepsilon\neq0\}} \frac{u_\varepsilon(x)u_\varepsilon(y)}{|x-y|}\,dxdy \\ &\ge \int_{\{-\varepsilon<u<\varepsilon\}}\int_{\{-\varepsilon<u<\varepsilon\}} \frac{u(x)u(y)}{|x-y|}\,dxdy + 2\int_{\{u<-\varepsilon\}}\int_{\{-\varepsilon<u<\varepsilon\}} \frac{u(x)u(y)}{|x-y|}\,dxdy \\ &\quad + 2\int_{\{u>\varepsilon\}}\int_{\{-\varepsilon<u<\varepsilon\}} \frac{u(x)u(y)}{|x-y|}\,dxdy + 2\int_{\{u<-\varepsilon\}}\int_{\{u>\varepsilon\}} \frac{u(x)u(y)}{|x-y|}\,dxdy \\ &\quad - 2\int_{\{u<-\varepsilon\}}\int_{\{u>\varepsilon\}} \frac{(u(x)-\varepsilon)(u(y)-\varepsilon)}{|x-y|}\,dxdy \\ &\ge -C(R)\varepsilon^2 - C(R)||\nabla u||_{L^1(B_R)}\varepsilon. \end{aligned}$$

Thus (24) implies that

$$\int_{\{-\varepsilon<u<\varepsilon\}} |\nabla u|^2 dx + \eta|\{-\varepsilon<u<\varepsilon\}| \le C(R)(1+||\nabla u||_{L^1(B_R)})\varepsilon.$$

Furthermore using the Cauchy-Schwarz inequality we have

$$\Big(\int_{\{0<u<\varepsilon\}} |\nabla u|\,dx\Big)^2 \le |\{\{0<u<\varepsilon\}\}|\Big(\int_{\{0<u<\varepsilon\}} |\nabla u|^2\,dx\Big) \le \frac{1}{\eta}(C(R)(1+||\nabla u||_{L^1(B_R)})\varepsilon)^2.$$

By Coarea formula

$$\int_0^\varepsilon P(\{u>s\})ds = \int_{\{0<u<\varepsilon\}} |\nabla u|\,dx \le \frac{1}{\sqrt{\eta}}C(R)(1+||\nabla u||_{L^1(B_R)})\varepsilon.$$

Finally we find $\delta_n > 0$ such that $\delta_n \to 0$ and

$$P(\{u>\delta_n\}) \le \frac{1}{\sqrt{\eta}}C(R)(1+||\nabla u||_{L^1(B_R)}).$$

Passing to the limit

$$P(\{u>0\}) \le \frac{1}{\sqrt{\eta}}C(R)(1+||\nabla u||_{L^1(B_R)}).$$

Similar computations hold for $P(\{u<0\})$ while $P(\{u\neq0\})$ is bounded as consequence of

$$P(\{u\neq0\}) \le P(\{u<0\}) + P(\{u>0\}).$$ □

Remark 5.5. In light of Theorem 5.4, Lemma 4.1, Theorem 5.2 and Remark 5.3, we can suppose there exists an optimal set Ω of problem (21) and it is well separated from ∂B_R, in the sense that $\Omega \subseteq B_{R/2} \subseteq B_R$. Moreover any optimal set is connected, otherwise by moving the connected components apart, the functional decreases.

We can now show the equivalence between the constrained and the unconstrained problems. We recall that the constant M has been already fixed, see Remark 3.5.

Theorem 5.6. *There exist universal constant $q_2 \in (0, q_1]$ and $\eta_2 \in (0, \eta_1]$ such that, for all $\eta \in (0, \eta_2]$ and $q \in [0, q_2)$, we have that*

$$\min\left\{E_{q,M,\eta}(\Omega) : \Omega \subset B_R\right\} = \inf\left\{E_q(\Omega) : \Omega \subseteq \mathbb{R}^3,\ |\Omega| = |B_1|\right\}.$$

As a consequence, problems (21) and (3) are equivalent for these values of q and η.

Proof. It is easy to check that

$$\min\left\{E_{q,M,\eta}(\Omega) : \Omega \subseteq \mathbb{R}^3\right\} \leq \inf\left\{E_q(\Omega) : \Omega \subseteq \mathbb{R}^3,\ |\Omega| = |B_1|\right\} =: \mu(q),$$

as the two functionals coincide on sets of measure $|B_1|$, thanks to the definition of f_η. So by Theorem 5.2,

$$\min\left\{E_{q,M,\eta}(\Omega) : \Omega \subseteq B_R\right\} \leq \mu(q),$$

Then, if the reverse inequality holds, it follows that on the set of minimizers (of the first or of the second problem) the two functionals coincide. We prove the claim of the theorem arguing by contradiction. Let

$$\Omega_{q,M,\eta} \subset B_R, \quad \sigma_{q,M,\eta} \in \mathbb{R}, \quad |\Omega_{q,M,\eta}| = |B_1| + \sigma_{q,M,\eta}, \quad E_{q,M,\eta}(\Omega_{q,M,\eta}) < \mu(q),$$

and we also note that, $\mu(q) \leq E_q(B_1)$, by definition of infimum. We moreover assume, without loss of generality, that $\Omega_{q,M,\eta}$ are minimizers for problem (21). We treat separately the case $\sigma_{q,M,\eta} > 0$ and $\sigma_{q,M,\eta} < 0$. Observe that repeating the computation of Lemma 5.1 and up to taking η_2 small enough, we can suppose that $|\Omega_{q,M,\eta}|$ is uniformly bounded from above and below, thus there exist universal constants $\tilde{C}, \tilde{c} \in (0, 1)$ such that $-\tilde{c}|B_1| \leq \sigma_{q,M,\eta} \leq \tilde{C}|B_1|$.

Case $\sigma_{q,M,\eta} > 0$. Let now $\rho_{q,M,\eta} < 1$ be such that $|\rho_{q,M,\eta}\Omega_{q,M,\eta}| = |B_1|$, so that

$$\rho_{q,M,\eta} = 1 - \frac{\sigma_{q,M,\eta}}{3|B_1|} + C\sigma_{q,M,\eta}^2,$$

for some $C = C(\sigma_{q,M,\eta}) \in \mathbb{R}$ such that $|C| \leq C_0$ for all $|\sigma_{q,M,\eta}| < \tilde{C}|B_1|$ some $C_0 > 0$ universal.

We call $u = u_{q,M,\eta}$ an optimal normalized function attaining $E_q(\Omega_{q,M,\eta})$, thus the function

$$\widetilde{u}(y) = {\rho_{q,M,\eta}}^{-\frac{3}{2}} u\Big(\frac{y}{\rho_{q,M,\eta}}\Big), \qquad y \in \rho_{q,M,\eta}\Omega_{q,M,\eta},$$

is an admissible competitor with unitary L^2-norm for $E_q(\rho_{q,M,\eta}\Omega_{q,M,\eta})$. We have the following scalings

$$\int_{\rho_{q,M,\eta}\Omega_{q,M,\eta}} |\nabla\widetilde{u}(y)|^2\,dy = {\rho_{q,M,\eta}}^{-2} \int_{\Omega_{q,M,\eta}} |\nabla u(x)|^2\,dx,$$
$$D(\widetilde{u}, \widetilde{u}) = {\rho_{q,M,\eta}}^2 D(u, u).$$

Since the new set $\rho_{q,M,\eta}\Omega_{q,M,\eta}$ is now admissible in the constrained minimization problem (3), using the above scaling we obtain

$$\begin{aligned}
E_{q,M,\eta}(\Omega_{q,M,\eta}) &= E_q(\Omega_{q,M,\eta}) + \frac{\sigma_{q,M,\eta}}{\eta} < \mu \leq E_q(\rho_{q,M,\eta}\Omega_{q,M,\eta})\\
&\leq \int_{\rho_{q,M,\eta}\Omega_{q,M,\eta}} |\nabla\widetilde{u}(y)|^2\,dy + \frac{q}{2}D(\widetilde{u}, \widetilde{u})\\
&= {\rho_{q,M,\eta}}^{-2} \int_{\Omega_{q,M,\eta}} |\nabla u(x)|^2\,dx + {\rho_{q,M,\eta}}^2 \frac{q}{2} D(u, u)\\
&= \int_{\Omega_{q,M,\eta}} |\nabla u(x)|^2\,dx \left(1 + \frac{2\sigma_{q,M,\eta}}{3|B_1|} + C\sigma_{q,M,\eta}^2\right) + \frac{q}{2}D(u, u)\left(1 - \frac{2\sigma_{q,M,\eta}}{3|B_1|} + C\sigma_{q,M,\eta}^2\right),
\end{aligned}$$

we deduce that (up to increasing C, recalling also that $E_q(\Omega_{q,M,\eta}) \leq E_q(B_1)$)

$$\begin{aligned}
\frac{\sigma_{q,M,\eta}}{\eta} &< \int_{\Omega_{q,M,\eta}} |\nabla u(x)|^2\,dx \left(\frac{2\sigma_{q,M,\eta}}{3|B_1|}\right) - \frac{q}{2}D(u, u)\left(\frac{2\sigma_{q,M,\eta}}{3|B_1|}\right) + 2E_q(B_1)C\sigma_{q,M,\eta}^2\\
&\leq \frac{\sigma_{q,M,\eta}}{3|B_1|} 2E_q(\Omega_{q,M,\eta}) + C\sigma_{q,M,\eta}^2.
\end{aligned}$$

Thus, for some universal $C > 0$,

$$\frac{1}{\eta} \leq CE_q(\Omega_{q,M,\eta}) + C\sigma \leq CE_q(B_1) \leq CE_1(B_1),$$

which leads to a contradiction as soon as $\eta_2 < \frac{1}{C\,E_1(B_1)}$.

Case $\sigma_{q,M,\eta} < 0$. Let now $\rho_{q,M,\eta} > 1$ be such that $|\rho_{q,M,\eta}\Omega_{q,M,\eta}| = |B_1|$ and consider the function $g\colon [1, \rho_{q,M,\eta}] \to \mathbb{R}$ defined by

$$g(r) = \int_{r\Omega_{q,M,\eta}} |\nabla u_r|^2 \, dx + \frac{q}{2} \int_{r\Omega_{q,M,\eta}} \int_{r\Omega_{q,M,\eta}} \frac{u_r(x)u_r(y)}{|x-y|} \, dxdy + \eta(r^3|\Omega_{q,M,\eta}| - |B_1|),$$

where

$$u_r(y) = r^{-\frac{3}{2}} u\Big(\frac{y}{r}\Big), \qquad y \in r\Omega_{q,M,\eta},$$

We show that the minimum of the function g is attained at $r = \rho := \rho_{q,M,\eta}$. Then the proof is concluded by Proposition 3.4 because this implies that $E_q(\rho_{q,M,\eta}\Omega_{q,M,\eta}) = E_{q,M,\eta}(\rho_{q,M,\eta}\Omega_{q,M,\eta}) \le E_{q,M,\eta}(\Omega_{q,M,\eta}) < \mu(q)$ which leads to a contradiction. Proving that g has a minimum at $r = \rho$ is equivalent to show that for some η the inequality

$$g(r) \ge \int_{\rho\Omega_{q,M,\eta}} |\nabla \widetilde{u}|^2 \, dx + \frac{q}{2} \int_{\rho\Omega_{q,M,\eta}} \int_{\rho\Omega_{q,M,\eta}} \frac{\widetilde{u}(x)\widetilde{u}(y)}{|x-y|} \, dxdy, \qquad \text{for all } r \in [1, \rho],$$

holds true, where $\widetilde{u}$ is defined as before. Up to rearranging the terms, and by the rescaling of the involved integrals, such an inequality reads as

$$\eta\left(1 - \left(\frac{r}{\rho}\right)^3\right) \le \int_{\rho\Omega_{q,M,\eta}} |\nabla \widetilde{u}|^2 \left(\left(\frac{r}{\rho}\right)^{-2} - 1\right) + \frac{q}{2} D(\widetilde{u}, \widetilde{u}) \left(\left(\frac{r}{\rho}\right)^2 - 1\right).$$

Setting $t := \frac{r}{\rho} < 1$, and observing that $r^3|\Omega_{q,M,\eta}| = t^3$, the last inequality is equivalent to

$$\eta \le \frac{\int_{\rho\Omega_{q,M,\eta}} |\nabla \widetilde{u}|^2 \, dx (t^{-2} - 1) - \frac{q}{2} D(\widetilde{u}, \widetilde{u})(1 - t^2)}{1 - t^3}.$$

Moreover by scaling and $|\sigma_{q,M,\eta}| \ge -\tilde{c}|B_1|$, it holds

$$D(\widetilde{u}, \widetilde{u}) = {\rho_{q,M,\eta}}^2 D(u, u) \le (1 - \tilde{c})^{-2/3} E_1(B_1) = \overline{C}$$

where $\overline{C}$ is a universal constant. It is easy to check that the right hand side is bounded from below by the function

$$t \mapsto \frac{\lambda_0(B_2)(t^{-2} - 1) - \frac{q}{2}\overline{C}(1 - t^2)}{1 - t^3}, \qquad t \in (0, 1),$$

which is a function strictly decreasing in $(0, 1)$ for $q \le 2/3\overline{C}$ and with infimum given by

$$\lim_{t \to 1^-} \frac{\lambda_0(B_2)(t^{-2} - 1) - q\overline{C}(1 - t^2)}{1 - t^3} = \frac{2\lambda_0(B_2) - q\overline{C}}{3} > 0,$$

as $q \le q_2$ where $q_2 = \frac{2}{\overline{C}} \min\{\lambda_0(B_2), 1/3\}$. Thus it is enough to take $\eta \le \eta_2 \le \frac{2\lambda_0(B_2) - q_2\overline{C}}{3}$ and we immediately deduce that g has minimum for $r = \rho$. This concludes the proof. □

Remark 5.7. Hereafter we fix $\eta = \frac{\eta_2}{2}$. In light of Theorem 5.6, we know that all optimizers Ω of problem (21) satisfy $|\Omega| = |B_1|$.

5.2. **Non-negativity of the optimal function.** Before delving into the regularity of the optimal sets, it is crucial to determine whether the optimal function provided by Theorem 5.4 is nonnegative or sign-changing. This distinction is fundamental: if the function is nonnegative, the problem can be treated using one-phase free boundary techniques; otherwise, a two-phase free boundary argument is required. The idea is to obtain the nonnegativity of our optimal function by testing its minimality against its positive part. To do so, it is useful to know some other information about the optimal function, such as some convergences as q goes to zero.

Proposition 5.8. *Let $q \in (0, q_2)$. Then $\mathcal{E}_{q,M,\eta}$ Γ-converges to $\mathcal{E}_{0,M,\eta}$ as $q \to 0$ with respect to the weak topology of $H^1_0(B_R)$ and*

$$\inf\left\{\mathcal{E}_{q,M,\eta}(u) : u \in H^1_0(B_R)\right\} \to \lambda_0(B_1).$$

Moreover for all $(u_q)_q \subseteq H^1_0(B_R)$ such that each u_q minimizes $\mathcal{E}_{q,M,\eta}$, then every weak limit of a subsequence in $H^1_0(B_R)$ is either u_{B_1} or $-u_{B_1}$, denoting u_{B_1} is the first positive normalized eigenfunction of the Dirichlet Laplacian on B_1.

Proof. Let $(v_q)_q \subseteq H_0^1(B_R)$ be a sequence weakly converging to $v \in H_0^1(B_R)$ in $H_0^1(B_R)$ as $q \to 0$. Then $(v_q)_q$ is bounded in $H_0^1(B_R)$ and so, up to a subsequence, $(v_q)_q$ converges pointwise and in $L^2(B_R)$ to v as $q \to 0$. Since $|v \neq 0| \leq \liminf_{q\to 0} |\{v_q \neq 0\}|$ and by Proposition 2.2, it holds

$$\mathcal{E}_{0,M,\eta}(v) \leq \liminf_{q\to 0} \mathcal{E}_{q,M,\eta}(v_q),$$

so the *liminf inequality* holds. Instead the *limsup inequality* holds by taking the sequence $v_q = v$ for all $q \in (0,1)$. Thus $\mathcal{E}_{q,M,\eta}$ Γ-converges to $\mathcal{E}_{0,M,\eta}$ as $q \to 0$ with respect to the weak $H_0^1(B_R)$ topology. By Theorem 5.4, for all $q \in (0,1)$, there exists $u_q \in H_0^1(B_R)$ such that

$$\inf\left\{\mathcal{E}_{q,M,\eta}(u) : u \in H_0^1(B_R)\right\} = \mathcal{E}_{q,M,\eta}(u_q).$$

Since for all $q \in (0,1)$ it holds (recalling also Lemma 2.1 and the fact that $|\{u_q \neq 0\}| = |B_1|$)

$$\int_{B_R} |\nabla u_q|^2 dx \leq \mathcal{E}_{q,M,\eta}(u_q) \leq E_1(B_1) + |B_1|,$$

so $(u_q)_q$ is bounded in $H_0^1(B_R)$ and by reflexivity of $H_0^1(B_R)$, there exists a subsequence which weakly converging in $H_0^1(B_R)$. Then by a well-known property of the Γ-convergence (see for example [15, Chapter 7]), the minimum and the minimizers of $\mathcal{E}_{q,M,\eta}$ converge respectively to the minimum and the minimizers of $\mathcal{E}_{0,M,\eta}$, respectively. By Theorem 5.6 and the Faber-Krahn inequality,

$$\inf_{u\in H_0^1(B_R)} \mathcal{E}_{0,M,\eta}(u) = \lambda_0(B_1).$$

Since u_{B_1} and $-u_{B_1}$ are the only minimizers of $\mathcal{E}_{0,M,\eta}$ on $H_0^1(B_R)$, where u_{B_1} is the first positive normalized eigenfunction of the Dirichlet Laplacian, the proof is concluded. □

Despite Proposition 5.8 gives a lot of information about some convergences of the optimal function, we need a stronger (uniform) one. In the following proposition we prove uniform convergence of minimizers as a consequence of the following regularity result, inspired by the regularity theory introduced by Giaquinta and Giusti in [23].

Proposition 5.9. *Let $q \in (0, q_2)$. Then for all $(u_q)_q \subseteq H_0^1(B_R)$ such that u_q minimizes $\mathcal{E}_{q,M,\eta}$, up to possibly taking $-u_q$ for some q, $(u_q)_q$ uniformly converges to u_{B_1} in B_R as $q \to 0$.*

Proof. Let $u \in H_0^1(B_R)$ be an optimal function of problem (20). Let $\widehat{x} \in B_{R-1}$ and $0 < r < 1$. Let us consider a function $\varphi \in H_0^1(B_r(\widehat{x}))$ and define $v = u + \varphi$. Then by minimality it holds

$$\begin{aligned}(25)\qquad \int_{B_R} |\nabla u|^2 + \frac{q}{2}D(u,u) + M\Big|\int_{B_R} u^2 dx - 1\Big| + f_\eta(|\{u \neq 0\}|)\\ \leq \int_{B_R} |\nabla v|^2 + \frac{q}{2}D(v,v) + M\Big|\int_{B_R} v^2 dx - 1\Big| + f_\eta(|\{v \neq 0\}|).\end{aligned}$$

By (18), $\Big|f_\eta(|\{v \neq 0\}|) - f_\eta(|\{u \neq 0\}|)\Big| \leq C(\eta) r^3$ where $C(\eta) = C\min\{\eta, 1/\eta\}$ and C is a universal constant possibly increasing from line to line. Considering the L^2 term, we obtain, for a universal constant $C > 0$,

$$\left|\Big|\int_{B_R} u^2 dx - 1\Big| - \Big|\int_{B_R} v^2 dx - 1\Big|\right| \leq \left|\int_{B_r(\widehat{x})} u^2 - v^2 dx\right| \leq C(||u||^2_{L^\infty(B_R)} + ||v||^2_{L^\infty(B_R)}) r^3.$$

Finally the non local term gives

$$\begin{aligned} D(v,v) - D(u,u) &= D(u+\varphi, u+\varphi) - D(u,u) = 2D(u,\varphi) + D(\varphi,\varphi)\\ &= 2\int_{B_r(\widehat{x})} v_u \varphi dx + \int_{B_r(\widehat{x})}\int_{B_r(\widehat{x})} \frac{\varphi(x)\varphi(y)}{|x-y|}\, dxdy.\end{aligned}$$

Since

$$\int_{B_r(\widehat{x})} v_u \varphi dx \leq C\|v_u\|_{L^\infty(B_R)} \|\varphi\|_{L^\infty(B_R)} r^3$$

and by (4) and [21, Lemma 2.4]

$$D(\varphi,\varphi) \leq C\|\varphi\|^2_{L^\infty(B_R)} r^3.$$

Thus (25) becomes

$$\int_{B_r(\widehat{x})} |\nabla u|^2 \le \int_{B_r(\widehat{x})} |\nabla v|^2 + C\Big((\|u\|^2_{L^\infty(B_R)} + \|v\|^2_{L^\infty(B_R)}) + C(\eta)\Big) r^3. \tag{26}$$

for a universal constant C. Let us now consider a suitable explicit test function v, namely the solution to

$$\begin{cases} \Delta v = 0 & \text{in } B_r(\widehat{x}) \\ v = u & \text{on } B_r^c(\widehat{x})\,. \end{cases}$$

Observe that $\int_{B_r(\widehat{x})} |\nabla v|^2 dx \le \int_{B_r(\widehat{x})} |\nabla u|^2 dx$ and $||v||_{L^\infty(B_R)} \le ||u||_{L^\infty(B_R)}$ by weak maximum principle (see [25, Theorem 8.1]). Moreover $u - v \in H^1_0(B_r(\widehat{x}))$, so v can be taken as test function in (25). Then (26) becomes, recalling that u is uniformly bounded in L^∞, see Theorem 3.2,

$$\int_{B_r(\widehat{x})} |\nabla u|^2 \le \int_{B_r(\widehat{x})} |\nabla v|^2 + C(\eta) r^3. \tag{27}$$

By the equation solved by v, it holds $\int_{B_r(\widehat{x})} \nabla v \cdot \nabla(u - v) dx = 0$, so $\int_{B_r(\widehat{x})} |\nabla v|^2 dx = \int_{B_r(\widehat{x})} \nabla v \cdot \nabla u\, dx$. Finally we deduce

$$\int_{B_r(\widehat{x})} |\nabla(u-v)|^2 dx = \int_{B_r(\widehat{x})} |\nabla v|^2 dx + \int_{B_r(\widehat{x})} |\nabla u|^2 dx - 2\int_{B_r(\widehat{x})} \nabla u \cdot \nabla v dx = \int_{B_r(\widehat{x})} |\nabla u|^2 dx - \int_{B_r(\widehat{x})} |\nabla v|^2 dx.$$

Thus by (27) we obtain

$$\int_{B_r(\widehat{x})} |\nabla(u-v)|^2 dx \le C(\eta) r^3. \tag{28}$$

Consider $0 < \rho < r$. Since v is harmonic, then also $\partial_i v$ for $i = 1, 2, 3$ is harmonic. So by the mean value theorem for harmonic functions and Jensen's inequality, there holds

$$|\nabla v(x)|^2 = \sum_{i=1}^{3} |\partial_i v|^2 = \sum_{i=1}^{3} \Big(\fint_{B_r(\widehat{x})} \partial_i v\, dy \Big)^2 \le \sum_{i=1}^{3} \Big(\fint_{B_r(\widehat{x})} |\partial_i v|^2\, dy \Big) = \fint_{B_r(\widehat{x})} |\nabla v|^2\, dx.$$

Then, integrating over $B_\rho(\widehat{x})$ the above chain of inequalities, we obtain

$$\int_{B_\rho(\widehat{x})} |\nabla v|^2\, dx \le \frac{|B_\rho|}{|B_r|} \int_{B_r(\widehat{x})} |\nabla v|^2\, dx. \tag{29}$$

Thus by (28) and (29), there holds

$$\begin{aligned} \int_{B_\rho(\widehat{x})} |\nabla u|^2 dx = \int_{B_\rho(\widehat{x})} |\nabla((u-v)+v)|^2 dx &\le \int_{B_r(\widehat{x})} |\nabla(u-v)|^2 dx + \int_{B_\rho(\widehat{x})} |\nabla v|^2 dx \\ &\le C(\eta) r^3 + \Big(\frac{\rho}{r}\Big)^3 \int_{B_r(\widehat{x})} |\nabla v|^2 dx \le C r^3 + \Big(\frac{\rho}{r}\Big)^3 \int_{B_r(\widehat{x})} |\nabla u|^2 dx. \end{aligned} \tag{30}$$

Set $\psi(s) := \int_{B_s(\widehat{x})} |\nabla u|^2 dx$, then (30) can be rewritten as

$$\psi(\rho) \le C(\eta) r^3 + \Big(\frac{\rho}{r}\Big)^3 \psi(r).$$

In light of [22, Lemma 2.1, Chapter 3] or [24, Lemma 5.13], then for all $\beta \in (0, 3)$, it holds

$$\int_{B_\rho(\widehat{x})} |\nabla u|^2 = \psi(\rho) \le C(\eta, \beta) \rho^\beta. \tag{31}$$

where $C(\eta, \beta)$ is a universal constant depending only on η, β. Let us choose $\beta = 2$. Finally by the classical Campanato criterion (for further details see [22, Chapter 3] or [24, Section 5]),

$$\|u\|_{C^{0,1/2}(B_r(\widehat{x}))} \le C(\eta), \tag{32}$$

where $C = C(\eta)$ is a constant independent of q, r and $\widehat{x}$. Then we can consider a finite covering of $B_{R/2}$ with balls $B_r(x_i)$ for some $x_i \in B_{R-1}$. Since by Remark 5.5, $\{u \neq 0\} \subseteq B_{R/2} \subseteq B_R$, then we can cover $\{u \neq 0\}$ with a finite number of balls of radius r, to obtain

$$\|u\|_{C^{0,1/2}(B_{R/2})} \le C(\eta), \tag{33}$$

where $C = C(\eta)$. Finally by Proposition 5.8 there exists a sequence of functions (u_q) minimizing $\mathcal{E}_{q,M,\eta}$ that weakly converge as $q \to 0$ to u_{B_1} (or $-u_{B_1}$), the first positive normalized eigenfunction of the Dirichlet Laplacian on B_1. For each element of this sequence, (33) holds. Then by Ascoli-Arzelà, up to a subsequence, there exists $v \in H^1_0(B_R)$ such that u_q converge uniformly to v. By uniqueness of the limit $v = u_{B_1}$ (or $-u_{B_1}$). Since $-u_q$ is still a minimizer for $\mathcal{E}_{q,M,\eta}$, then up to changing some elements in $(u_q)_q$ with its opposite, without loss of generality we can suppose that the uniform limit is u_{B_1}. □

Remark 5.10. Observe that a local Hölder estimate follows also by the $C^\infty(\Omega_q)$ regularity proven in Theorem 3.2. The difference is that (32) holds locally in B_R and not only in Ω_q, allowing to obtain a global estimate. Moreover dealing with the d-dimensional version of this problem, Proposition 5.9 still holds taking $\beta = d-1$ in (31).

Thanks to the aforementioned convergences, now we can prove the nonnegativity of an optimal function.

Proposition 5.11. *There exists $q_3 = q_3(\eta)$ such that for all $q \le q_3$ all optimal functions $u_q \in H^1_0(B_R)$ attaining the infimum of $\mathcal{E}_{q,M,\eta}$ on $H^1_0(B_R)$ have constant sign.*

Proof. By Theorem 5.4, we know that there exists a minimizer $u \in H^1_0(B_R)$ for $\mathcal{E}_{q,M,\eta}$ on $H^1_0(B_R)$. Let us show that testing the minimality of u against u^+ leads to the claim. By minimality it holds

$$\begin{aligned}\int_{B_R} |\nabla u(x)|^2\,dx &+ \frac{q}{2}\int_{B_R}\int_{B_R}\frac{u(x)u(y)}{|x-y|}\,dxdy + M\left|\int_{B_R} u^2dx - 1\right| + f_\eta(|\{u\neq 0\}|)\\ &\le \int_{B_R}|\nabla u^+(x)|^2\,dx + \frac{q}{2}\int_{B_R}\int_{B_R}\frac{u^+(x)u^+(y)}{|x-y|}\,dxdy + M\left|\int_{B_R}(u^+)^2dx - 1\right| + f_\eta(|\{u^+\neq 0\}|),\end{aligned}$$

Moreover

$$\left|\int_{B_R} u^2dx - 1\right| - \left|\int_{B_R}(u^+)^2dx - 1\right| \le \left|\int_{B_R} u^2dx - \int_{B_R}(u^+)^2dx\right| = \int_{\{u<0\}} u^2dx,$$

and by (18)

$$f_\eta(|\{u\neq 0\}|) - f_\eta(|\{u^+\neq 0\}|) \ge \eta|\{u<0\}|.$$

So

$$\begin{aligned}\int_{\{u<0\}}|\nabla u(x)|^2\,dx &+ \frac{q}{2}\int_{\{u<0\}}\int_{\{u<0\}}\frac{u(x)u(y)}{|x-y|}\,dxdy\\ &+ q\int_{\{u>0\}}\int_{\{u<0\}}\frac{u(x)u(y)}{|x-y|}\,dxdy - M\int_{\{u<0\}}u^2dx + \eta|\{u<0\}| \le 0.\end{aligned}$$

By Hardy-Sobolev and Hölder inequalities there holds

$$\begin{aligned}\left|\int_{\{u>0\}}\int_{\{u<0\}}\frac{u(x)u(y)}{|x-y|}dxdy\right| &\le \int_{\{u>0\}}\int_{\{u<0\}}\frac{|u(x)||u(y)|}{|x-y|}dxdy\\ &\le C||\nabla u||_{L^1(B_R)}\int_{\{u<0\}}|u(x)|dx\\ &\le C||\nabla u||_{L^1(B_R)}||u^-||_{L^\infty(B_R)}|\{u<0\}|\\ &\le C(R)||u^-||_{L^\infty(B_R)}||\nabla u||_{L^2(B_R)}|\{u<0\}|.\end{aligned}$$

Since u minimizes $\mathcal{E}_{q,M,\eta}$, then $||\nabla u||_{L^2(B_R)} \le \mathcal{E}_{q,M,\eta}(u) \le E_1(B_1) + |B_1|$, implying

$$-q\int_{\{u>0\}}\int_{\{u<0\}}\frac{u(x)u(y)}{|x-y|}dxdy \ge -C(R)||u^-||_{L^\infty(B_R)}|\{u<0\}|. \tag{34}$$

So it holds

$$\int_{\{u<0\}}|\nabla u(x)|^2\,dx + \frac{q}{2}\int_{\{u<0\}}\int_{\{u<0\}}\frac{u(x)u(y)}{|x-y|}\,dxdy + \Big(\eta - M||u^-||^2_{L^\infty(B_R)} - C(R)||u^-||_{L^\infty(B_R)}\Big)|u<0| \le 0.$$

Finally, by Proposition 5.9 and up to taking $-u$, there exists $q_3(\eta)$ such that for all $q \in (0, q_3)$ it holds

$$\eta - M||u^-||^2_{L^\infty(B_R)} - C(R)||u^-||_{L^\infty(B_R)} > 0.$$

Thus by (34), $|\{u<0\}| = 0$. □

Remark 5.12. Since u_q and $-u_q$ are minimizers of $\mathcal{E}_{q,M,\eta}$, by Proposition 5.11 and without loss of generality, we can consider the optimal functions to be nonnegative.

5.3. **Free boundary formulation.** In this section we want to improve the regularity for our optimal set.

Lemma 5.13. *Let $q \in (0, q_3]$, let Ω be an optimal set for problem (21), and let $u \in H_0^1(\Omega)$ be any (nonnegative) function attaining $E_q(\Omega) = E_{q,M}(\Omega)$. Then for every $\kappa \in (0,1)$ there are positive constants K_0, ρ_0 depending only on κ, η such that the following assertion holds: if $\rho \le \rho_0$ and $x_0 \in \overline{B_R}$, then*

$$\fint_{\partial B_\rho(x_0)\cap B_R} u\, d\mathcal{H}^2 \le K_0\rho \implies u \equiv 0 \text{ in } B_{\kappa\rho}(x_0) \cap B_R.$$

Proof. The proof is the same of [35, Lemma 3.9]. Let us just point out that

$$\gamma_1 := 2 \sup_x (\lambda_q u(x) - q v_u(x)) > 0.$$

Indeed suppose by contradiction that $\gamma_1 \le 0$. Then $u(x) \le q\frac{v_u(x)}{\lambda_q}$ for all $x \in \mathbb{R}^3$. Recalling that $\|v_u(x)\|_{L^\infty(\mathbb{R}^3)} \le CE_1(B_1)$ by (4) where C is a universal constant, and $\lambda_q \ge \lambda_0(\Omega) \ge \lambda_0(B_1)$ since $|\Omega| \le |B_1|$. Then by nonnegativity of u,

$$||u||_{L^\infty(\mathbb{R}^3)} \le Cq.$$

Finally u has unitary L^2 norm and $|\Omega| = |B_1|$, so we obtain a contradiction as q is small enough. □

Lemma 5.14. *Let η, q, Ω and u be as in Lemma 5.13. The function u can be extended to a Lipschitz continuous function defined in the whole B_R, with Lipschitz constant $L = L(\eta)$. In particular, $\Omega = \{u > 0\} \subset B_R$ is an open set.*

Proof. We follow the approach of [40, Section 3.2], first proposed in [7].
Step 1. We prove an estimate on the nonnegative Radon measure $|\Delta u|$, namely

$$|\Delta u|(B_r(x_0)) \le Cr^2, \qquad \text{for all } x_0 \in B_R \text{ and } 0 < r < 1 \text{ such that } B_{2r}(x_0) \subset B_R$$

for a universal constant $C > 0$. Let $\psi \in C_c^\infty(B_{2r}(x_0))$ for some $B_{2r}(x_0) \subset B_R$, with $\|\psi\|_{L^\infty} \le c$, and we test the optimality of u against $u + \psi$, obtaining:

$$\int_{B_R} |\nabla u|^2\, dx + \frac{q}{2} D(u,u) + f_\eta(|\{u > 0\}|) \le \int_{B_R} |\nabla(u+\psi)|^2\, dx + \frac{q}{2} D\Big(u+\psi, u+\psi\Big) + f_\eta(|\{u+\psi \ne 0\}|),$$

which implies

$$-2\int_{B_{2r}(x_0)} \nabla u \cdot \nabla\psi\, dx \le \int_{B_{2r}(x_0)} |\nabla\psi|^2\, dx + C_\eta|\{u = 0\} \cap B_{2r}(x_0)| + \frac{q}{2}\int_{B_{2r}(x_0)}\int_{B_R(0)} P(x,y)\, dxdy$$

where

$$P(x,y) = \frac{\psi(x)\psi(y) + 2u(x)\psi(y)}{|x-y|}$$

Recalling that $\|\psi\|_{L^\infty} \le c$ and using Theorem 3.2, we can control the nonlocal term as

$$\int_{B_{2r}(x_0)}\int_{B_R(0)} P(x,y)\, dxdy \le C_1 \int_{B_{2r}(x_0)}\int_{B_R(0)} \frac{1}{|x-y|}\, dxdy \le C_2 R^2 |B_{2r}(x_0)| \le C_3 r^3.$$

Thus we obtain

$$-2\int_{B_{2r}(x_0)} \nabla u \cdot \nabla\psi\, dx \le \int_{B_{2r}(x_0)} |\nabla\psi|^2\, dx + C_\eta|\{u = 0\} \cap B_{2r}(x_0)| + Cqr^3. \tag{35}$$

We now set, for all $\varphi \in C_c^\infty(B_{2r}(x_0))$, $\psi = \pm r^{3/2}\|\nabla\varphi\|_{L^2}^{-1}\varphi$ and from (35) we deduce, for some $\widetilde{C} > 0$

$$\Big|\int_{B_{2r}(x_0)} \nabla u \cdot \nabla\varphi\, dx\Big| \le \widetilde{C} r^{3/2} \|\nabla\varphi\|_{L^2(B_{2r}(x_0))}$$

It is then enough to choose $\varphi \in C_c^\infty(B_{2r}(x_0))$ with $\varphi \ge 0$ and $\varphi = 1$ in $B_r(x_0)$ and with $\|\nabla\varphi\|_{L^\infty(B_{2r})} \le \frac{2}{r}$ (notice that this is compatible with the requirement $\|\psi\|_{L^\infty} \le c$ independently of r) to obtain, for some constant $C > 0$:

$$|\Delta u|(B_r(x_0)) \le |\Delta u|(\varphi) = \left|\int_{B_{2r}(x_0)} \nabla u \cdot \nabla\varphi\, dx\right| \le Cr^2. \tag{36}$$

Step 2. We prove that the Laplacian estimate (36) of Step 1 entails (recall that $\mathcal{H}^2(\partial B_r) = 4\pi r^2$)

$$\frac{1}{4\pi r^2}\int_{\partial B_r(x_0)} u\, d\mathcal{H}^2 \le u(x_0) + Cr \qquad \text{for all } x_0 \in B_R, \tag{37}$$

for some constant $C > 0$. This follows from [7, Lemma 3.6], which assures that, for all $x_0 \in B_R$, it holds

$$\frac{1}{4\pi r^2}\int_{\partial B_r(x_0)} u\, d\mathcal{H}^2 - u(x_0) = \int_0^r \frac{1}{4\pi s^2}\Delta u(B_s(x_0))\, ds. \tag{38}$$

It is then enough to put together (38) and (36) to obtain (37). Now, let us take $x_0 \in \partial\{u > 0\} \cap B_R$ and a sequence of $x_n \to x_0$ such that $u(x_n) = 0$ for all n and with $x_n \in B_{r_1}(x_0) \subset B_R$ for some $r_1 > 0$. For those points (37) reads as

$$\frac{1}{4\pi r^2}\int_{\partial B_r(x_n)} u\, d\mathcal{H}^2 \le u(x_n) + Cr = Cr, \qquad \text{for all } r < r_1, \tag{39}$$

and the constant C does not depend on n. Since $u \in H^1(B_R)$, the map $x \mapsto \frac{1}{4\pi r^2}\int_{\partial B_r(x)} u\, d\mathcal{H}^2$ is continuous, see [40, Remark 3.6]. We can then pass to the limit as $n \to \infty$ in (39) to deduce

$$\frac{1}{4\pi r^2}\int_{\partial B_r(x_0)} u\, d\mathcal{H}^2 \le Cr, \qquad \text{for all } r < r_1.$$

Finally, passing to the limit as $r \to 0$, we obtain that $u(x_0) = 0$ (recalling that we are considering the quasi continuous representative of the Sobolev function u), thus $\Omega \cap \partial\Omega = \{u > 0\} \cap \partial\{u > 0\} = \emptyset$, hence $\Omega = \{u > 0\}$ is an open set.

Step 3. By previous steps and [28, Lemma 2.4], it holds

$$|\nabla u(x_0)| \le C\left[\frac{1}{r}\fint_{\partial B_r(x_0)} u(x)dx + r^3\right] \le C$$

for a universal constant C. □

Lemma 5.15. *There exists a universal constant $q_4 \in (0, q_3]$ such that for all $q \in (0, q_4]$, calling Ω an optimal set for problem (20) and u a positive normalized function attaining $E_{q,M,\eta}(\Omega)$, there exist positive constants $\theta = \theta(\eta)$ and $\rho_0 = \rho_0(\eta) < 1$ such that for every $x_0 \in \partial\Omega$ and every $\rho \le \rho_0$, we have*

$$\theta \le \frac{|\Omega \cap B_\rho(x_0)|}{|B_\rho|} \le (1 - \theta).$$

Proof. The proof follows from [35, Lemma 3.12]. □

Lemma 5.16. *For all $\delta > 0$ there exists $q_\delta = q_\delta(\eta) \in (0, q_4]$ such that for all $q \le q_\delta$, we have*

$$\mathrm{dist}_H(\partial\Omega_{q,M,\eta}, \partial B_{q,M,\eta}) \le \delta.$$

where $\Omega_{q,M,\eta}$ is an optimal domain for problem (21).

Proof. It is a trivial adaptation of [35, Lemmas 3.14,3.15,3.16] hence we just sketch the proof. By the quantitative version of the Faber-Krahn inequality we first deduce that minimizers are close in L^1-topology to a ball for $q-$small. Such an L^1-closeness can then be improved into a proximity in Hausdorff distance by means of the uniform density estimates just obtained. □

6. Optimality condition and improvements of flatness

We show now the following result.

Theorem 6.1. *Let $q \in (0, q_4]$, let Ω be a minimizer of (3), and let u be an optimal nonnegative function attaining $E_q(\Omega)$. Then we have that:*

(i) There is a Borel function $\mu_u \colon \partial\Omega \to \mathbb{R}$ such that, in the sense of the distributions, one has

$$-\Delta u = \lambda_q u - qv_u - \mu_u \mathcal{H}^2 \llcorner \partial\Omega, \qquad \text{in } B_R. \tag{40}$$

(ii) There exist constants $0 < c < C < +\infty$, depending on R, such that $c \le \mu_u \le C$.

(iii) For all points $\overline{x} \in \partial^*\Omega = \partial^*\{u > 0\}$, the measure theoretic inner unit normal $\nu_u(\overline{x})$ is well defined and, as $\rho \to 0$,

$$\frac{\Omega - \overline{x}}{\rho} \to \{x : x \cdot \nu_u(\overline{x}) \ge 0\}, \qquad \text{in } L^1(B_R).$$

(iv) For $\mathcal{H}^2$ almost all $\overline{x} \in \partial^*\{u > 0\}$ we have

$$\frac{u(\overline{x} + \rho x)}{\rho} \longrightarrow \mu_u(\overline{x})(x \cdot \nu_u(\overline{x}))_+, \qquad \text{in } W^{1,p}(B_R) \text{ for every } p \in [1, +\infty).$$

(v) $\mathcal{H}^2(\partial\Omega \setminus \partial^*\Omega) = 0$.

Moreover $\mu_u : \partial\Omega \to \mathbb{R}$ *is constant on* $\partial^*\Omega$.

Proof. The proof is essentially identical to that in [3, Section 4]. We only have to check that our hypotheses match with those in [3]. First by Theorem 3.2, u satisfies

$$-\Delta u - Q(x) = 0 \qquad \text{in } \mathcal{D}'(\Omega),$$

where $Q = \lambda_q u - q v_u \in L^\infty(\Omega)$ and $u \in H^1_0(\Omega)$. Hence, by repeating the proof of [3, Theorem 4.5] or by directly applying [11, Proposition 2.3] one obtains that there exists a positive Radon measure concentrated on $\partial\Omega$ that we denote $\mu_u \mathcal{H}^2 \llcorner \partial\Omega$. Moreover, thanks to the non-degeneracy, see [38, Remark 2.8], and the Lipschtiz continuity of u we have that there exist constant $C > c > 0$ depending on q and R such that

$$c \le \frac{1}{r} \int_{\partial B_r} u \, d\mathcal{H}^2 \le C.$$

Hence we can work under the hypotheses of [3, Theorem 4.5] so that μ_u is a density of a Radon measure on $\partial\Omega$ and, denoting still with μ_u the function defininig it, μ_u satisfies $(i) - (v)$.

Let us now prove that μ_u is constant on $\partial^*\Omega$. The proof follows the path of [37, Theorem 6.5], in turn inspired by [1]. Due to the nonlocal term, we will have to perform some new and non-straightforward computations.

We reason by contradiction and we assume that there exists $x_0, x_1 \in \partial^*\Omega$ such that

$$\mu_u(x_0) < \mu_u(x_1).$$

Then we construct a family of volume preserving diffeomorphisms as follows: let $\kappa < 1$ and $\rho < 1$ and let $\varphi \in C^1_0(B_1(0))$ be a non-null, radially symmetric function supported in $B_1(0)$. We define

$$\tau_{\rho,\kappa}(x) = \tau(x) = x + \sum_{i \in \{0,1\}} (-1)^i \kappa\rho\varphi\left(\frac{|x - x_i|}{\rho}\right) \nu_{x_i} \chi_{B_\rho(x_i)},$$

where ν_{x_i} are the measure theoretic inner normals to $\partial^*\Omega$ at x_i, $i = 1, 2$.

It is easy to notice that τ is indeed a diffeomorphism for ρ and κ small enough and that $\tau(x) - x$ vanishes outside $B_\rho(x_0) \cup B_\rho(x_1)$. Moreover we have:

$$\nabla\tau(x) = Id + \sum_{i \in \{0,1\}} (-1)^i \kappa\varphi'\left(\frac{|x - x_i|}{\rho}\right) \frac{x - x_i}{|x - x_i|} \otimes \nu_{x_i} \chi_{B_\rho(x_i)},$$

so that[4]

$$\det(\nabla\tau(x)) = 1 + \sum_{i \in \{0,1\}} (-1)^i \kappa\varphi'\left(\frac{|x - x_i|}{\rho}\right) \frac{x - x_i}{|x - x_i|} \cdot \nu_{x_i} \chi_{B_\rho(x_i)} + o(\kappa). \tag{41}$$

We call $\Omega_\rho = \tau(\Omega)$. We aim to show that for κ, ρ small enough we obtain a contradiction with the minimality of Ω. To do that, we deal with the first variation of each term of the sum defining $E_{q,M,\eta}$, see Remark 5.7 for the choice of η. We stress that the computations regarding the volume and the Dirichlet energy contributions are identical to those performed originally in [1] (see also [6] and [17], where the same idea is applied). Moreover, exactly as in the proof of [37, Theorem 6.5] one obtains that

$$f_\eta(\Omega_\rho) - f_\eta(\Omega) = o(\rho^3), \qquad \text{as } \rho \to 0,$$

and that

$$\frac{1}{\rho^3}\Big(\int_{\Omega_\rho} |\nabla\widetilde{u}_\rho|^2 \, dx - \int_\Omega |\nabla u|^2 \, dx\Big) \le \kappa(\mu_u^2(x_0) - \mu_u^2(x_1))C(\varphi) + o_\rho(1) + o(\kappa), \tag{42}$$

where u is the function attaining $E_q(\Omega) = E_{q,M,\eta}(\Omega)$, $\widetilde{u}_\rho = u \circ \tau^{-1}$ and

$$C(\varphi) = \int_{B_1(0)\cap\{y\cdot\nu=0\}} \varphi(|y|) \, d\mathcal{H}^2(y) = -\int_{B_1(0)\cap\{y\cdot\nu>0\}} \varphi'(|y|)\frac{y\cdot\nu}{|y|} \, dy,$$

with the last equality that follows from the Divergence Theorem, recalling that ν is a inner normal and

$$\operatorname{div}(\varphi(|y|)\nu) = \varphi'(|y|)\frac{y\cdot\nu}{|y|}.$$

[4] We are using the formula $\det(Id + \xi A) = 1 + trace(A)\xi + o(\xi)$ for a matrix $A \in \mathbb{R}^{3\times3}$.

Notice also that by the radial symmetry of φ the value of $C(\varphi)$ is not affected by the choice of ν. Moreover

$$
\begin{aligned}
(43)\qquad \frac{1}{\rho^3}\left|\int_{\Omega_\rho}\widetilde{u}_\rho^2-\int_\Omega u^2\right| =&\frac{1}{\rho^3}\left|\int_\Omega (u^2\,\det\nabla\tau-u^2)dx\right|\\
=&\left|\sum_{i\in\{0,1\}}\int_{B_1(0)\cap(\frac{\Omega-x_i}{\rho})}u^2(x_i+\rho y)\,\det\nabla\tau(x_i+\rho y)-u^2(x_i+\rho y)dy\right|\\
=&\left|\sum_{i\in\{0,1\}}\int_{B_1(0)\cap(\frac{\Omega-x_i}{\rho})}(-1)^i\kappa\varphi'(|y|)\frac{y}{|y|}\frac{u^2(x_i+\rho y)}{\rho^2}\rho^2dy\right|+o(\kappa)\\
=&o(\rho)+o(\kappa)
\end{aligned}
$$

where we performed the change of variable $x=x_i+\rho y$, we used (41) and Theorem 6.1 points (iii) and (iv).

We are left to compute the variation of the nonlocal term $D(\cdot,\cdot)$. We claim that

$$
(44)\qquad \frac{1}{\rho^3}(D(u_\rho,u_\rho)-D(u,u)=o(\kappa)+o_\rho(1).
$$

Once (44) is proved, the conclusion follows: by minimality of Ω, (42), (43) and (44) (recalling also that $E_{q,M,\eta}(\Omega_\rho)\le E_{q,M,\eta}(\Omega_\rho,\widetilde{u}_\rho)$) it holds

$$
\begin{aligned}
0\le&E_{q,M,\eta}(\Omega_\rho)-E_{q,M,\eta}(\Omega)\\
\le&\kappa\rho^3C(\varphi)\Bigg((\mu_u(x_0))^2-(\mu_u(x_1))^2\Bigg)+o(\rho^3)+\rho^3o(\kappa).
\end{aligned}
$$

Since we suppose $(\mu_u(x_0))^2-(\mu_u(x_1))^2<0$, then we obtain a contradiction as soon as ρ and κ are small enough.

It remains to show (44). Setting $w(x):=v_u(x)u(x)$ and $\widetilde{w}(x)=v_{\widetilde{u}_\rho}(x)\widetilde{u}_\rho(x)$ and recalling (41), it holds

$$
\begin{aligned}
(45)\quad \frac{1}{\rho^3}\Bigg(D(u_\rho,u_\rho)-D(u,u)\Bigg)=&\frac{1}{\rho^3}\Bigg(\int_{\Omega_\rho}\widetilde{w}-\int_\Omega w\Bigg)=\frac{1}{\rho^3}\Bigg(\int_\Omega\widetilde{w}(\tau(x))\,\det\nabla\tau(x)-w(x)dx\Bigg)\\
\le&\frac{1}{\rho^3}\int_\Omega\widetilde{(w\circ\tau-w)}dx+\frac{1}{\rho^3}\int_\Omega\Bigg(\sum_{i\in\{0,1\}}(-1)^i\kappa\varphi'\Big(\frac{|x-x_i|}{\rho}\Big)\frac{x-x_i}{|x-x_i|}\cdot\nu_{x_i}\chi_{B_\rho(x_i)}+o(\kappa)\Bigg)\widetilde{w}\circ\tau dx.
\end{aligned}
$$

Recalling that $v_{\widetilde{u}_\rho}$ is uniformly bounded (Theorem 3.2) and $\widetilde{u}_\rho$ is Lipschitz continuous (Lemma 5.14), then $|\widetilde{w}(\tau(x))|\le C\rho$ in $\Omega\cap B_\rho(x_i)$ since $\widetilde{u}_\rho(\tau(x_i))=u(x_i)=0$ as $x_i\in\partial^*\Omega$, for some universal constant $C>0$. This implies that

$$
(46)\qquad \left|\frac{1}{\rho^3}\int_\Omega\Bigg(\sum_{i\in\{0,1\}}(-1)^i\kappa\varphi'\Big(\frac{|x-x_i|}{\rho}\Big)\frac{x-x_i}{|x-x_i|}\cdot\nu_{x_i}\chi_{B_\rho(x_i)}\Bigg)\widetilde{w}\circ\tau dx\right|\le\frac{C}{\rho^2}|B_\rho|=o_\rho(1).
$$

Moreover

$$
\begin{aligned}
\frac{1}{\rho^3}\int_\Omega\widetilde{w}\circ\tau-w\,dx=&\frac{1}{\rho^3}\int_\Omega u(x)\Bigg(\int_{\Omega_\rho}\frac{\widetilde{u}_\rho(y)}{|\tau(x)-y|}dy-\int_\Omega\frac{u(y)}{|x-y|}dy\Bigg)dx\\
=&\frac{1}{\rho^3}\int_\Omega u(x)\Bigg(\int_\Omega\frac{u(y)}{|\tau(x)-\tau(y)|}\,\det\nabla\tau dy-\int_\Omega\frac{u(y)}{|x-y|}dy\Bigg)dx\\
=&\frac{1}{\rho^3}\int_\Omega u(x)\int_{\Omega\cap(B_\rho(x_0)\cup B_\rho(x_1))}u(y)\Bigg(\frac{1}{|\tau(x)-\tau(y)|}-\frac{1}{|x-y|}\Bigg)dydx\\
&+\frac{1}{\rho^3}\int_\Omega u(x)\int_\Omega\frac{u(y)}{|\tau(x)-\tau(y)|}\Bigg(\sum_{i\in\{0,1\}}(-1)^i\kappa\varphi'\Big(\frac{|y-x_i|}{\rho}\Big)\frac{y-x_i}{|x-x_i|}\cdot\nu_{x_i}\chi_{B_\rho(x_i)}+o(\kappa)\Bigg)dydx
\end{aligned}
$$

By computations analogous to the ones in (46), it holds

$$\begin{aligned}
&\left|\frac{1}{\rho^3}\int_\Omega u(x)\int_\Omega \frac{u(y)}{|\tau(x)-\tau(y)|}\left(\sum_{i\in\{0,1\}}(-1)^i\kappa\varphi'\Big(\frac{|y-x_i|}{\rho}\Big)\frac{y-x_i}{|x-x_i|}\cdot\nu_{x_i}\chi_{B_\rho(x_i)}\right)dydx\right|\\
&\quad\le\frac{1}{\rho^3}\int_\Omega u(x)\int_{\Omega\cap(B_\rho(x_0)\cup B_\rho(x_1))}\frac{u(y)}{|\tau(x)-\tau(y)|}\le\frac{C}{\rho^2}\int_\Omega\int_{B_\rho(x_0)\cup B_\rho(x_1)}\frac{1}{|\tau(x)-\tau(y)|}dydx\\
&\quad\le\frac{C}{\rho^2}\int_{\Omega_\rho}\int_{B_\rho(x_0)\cup B_\rho(x_1)}\frac{1}{|x-y|}\le\frac{C}{\rho^2}|B_\rho|=o_\rho(1)
\end{aligned}$$

where we used Theorem 3.2 and [21, Lemma 2.4]. Eventually we obtain

$$\begin{aligned}
(47)\quad \left|\frac{1}{\rho^3}\int_\Omega \widetilde{w}\circ\tau-wdx\right| &\le\frac{1}{\rho^3}\left|\int_\Omega u(x)\int_{\Omega\cap(B_\rho(x_0)\cup B_\rho(x_1))}u(y)\left(\frac{1}{|\tau(x)-\tau(y)|}-\frac{1}{|x-y|}\right)dydx\right|+o_\rho(1)\\
&\le\frac{C}{\rho^2}\int_\Omega\int_{B_\rho(x_0)\cup B_\rho(x_1)}\left(\frac{1}{|\tau(x)-\tau(y)|}+\frac{1}{|x-y|}\right)dydx+o_\rho(1)=o_\rho(1)
\end{aligned}$$

where the last equality holds again by [21, Lemma 2.4]. By (45), (46) and (47) we deduce (44) and this concludes the proof. □

We are now in position to show $C^{2,\alpha}-$regularity of the boundary of a minimizer Ω. This can be done in two steps: first one shows that such a boundary is locally the graph of a $C^{2,\alpha}$ function defined on the boundary of a ball. To do that one exploits the improvement of flatness technique from [3, Section 7 and 8], readapted with minimal changes to our setting with a right hand side as in [28, Appendix]. Then, as we already know by the previous section that the boundary of Ω is close in Hausdorff distance to that of a ball, we obtain that the local parametrization is a global parametrization of class $C^{2,\alpha}$ on the boundary of the ball. We first need a definition (see [3, Definition 7.1]).

Definition 6.2. Let $\gamma_\pm\in(0,1]$ and $k>0$. A weak solution u of (40) is of class $F(\gamma_-,\gamma_+,k)$ in $B_\rho(x_0)$ with respect to direction $\nu\in\mathbb{S}^{N-1}$ if

(a) $x_0\in\partial\{u>0\}$ and

$$\begin{aligned}
u&=0, &&\text{for } (x-x_0)\cdot\nu\le-\gamma_-\rho, \quad x\in B_\rho(x_0),\\
u(x)&\ge\mu_u(x_0)[(x-x_0)\cdot\nu-\gamma_+\rho], &&\text{for } (x-x_0)\cdot\nu\ge\gamma_+\rho, \quad x\in B_\rho(x_0).
\end{aligned}$$

(b) $|\nabla u(x_0)|\le\mu_u(x_0)(1+k)$ in $B_\rho(x_0)$ and $\operatorname{osc}_{B_\rho(x_0)}\mu_u\le k\mu_u(x_0)$.

We note that when $k=+\infty$, then condition (b) is automatically satisfied. We can show the following result.

Theorem 6.3. *Let $q\in(0,q_4]$, Ω be an optimal set for (3), and u a function attaining $E_q(\Omega)$ and a weak solution to (40) in B_R. Then there are constants $\overline{\gamma}$ and $\overline{k}$, depending only on R, μ_u, such that if u is of class $F(\gamma,1,+\infty)$ in $B_{4\rho}(x_0)$ with respect to some direction $\nu\in\mathbb{S}^{N-1}$ with $\gamma\le\overline{\gamma}$ and $\rho\le\overline{k}\gamma^2$, then there exists a $C^{2,\alpha}$ function $f\colon\mathbb{R}^2\to\mathbb{R}$ with $\|f\|_{C^{2,\alpha}}\le C(R,\mu_u)$ such that, calling*

$$\operatorname{graph}_\nu f:=\{x\in\mathbb{R}^3: x\cdot\nu=f(x-(x\cdot\nu)\nu)\},$$

then

$$\partial\{u>0\}\cap B_\rho(x_0)=(x_0+\operatorname{graph}_\nu(f))\cap B_\rho(x_0).$$

Moreover for all $\varepsilon_0>0$ there exists $q_\varepsilon\in(0,q_4]$ such that if $q<q_\varepsilon$ then

$$\partial\{u>0\}=\left\{\left(r+\varphi\left(\tfrac{x}{|x|}\right)\right)\frac{x}{|x|}\ :\ x\in\partial B_r\right\}$$

where $\varphi\colon\partial B_1\to\mathbb{R}$ is a function with the same regularity of f and $\|\varphi\|_{C^{2,\alpha}}\le\varepsilon_0$.

We omit the proof which is identical to that in [37, Theorems 1.2 and 6.8] (which is in turn inspired by [3, Theorem 8.1] and [28, Theorem 2.17 and Appendix]). We note that in our setting μ_u is constant, thus the requirement to be $C^{1,\alpha}$ regular is trivially satisfied.

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. The existence of a minimizer follows from Theorem 5.4 and Theorem 5.6. On the other hand, the fact that any optimal set is $C^{2,\alpha}$ nearly spherical follows from Theorem 6.3. □

Appendix A. Remarks on $D(\cdot,\cdot)$

In this Appendix we show how the minimization of the nonlocal term $D(\cdot,\cdot)$ alone is ill posed.

Lemma A.1. *We have that*

$$\inf\{D(u,u):\ u\in L^2(\mathbb{R}^3),\ ||u||_{L^2(\mathbb{R}^3)}=1\}=0$$

and it is not attained.

Proof. Let us show the existence of a sequence $(\varphi_\varepsilon)\subset L^2(\mathbb{R}^3)$ such that $||\varphi_\varepsilon||_{L^2(\mathbb{R}^3)}=1$ and $D(\varphi_\varepsilon,\varphi_\varepsilon)\to 0$ as $\varepsilon\to 0$. Let $\varphi\in C_c^\infty(\mathbb{R}^3)$, $\varphi\geq 0$, $\Omega:=\operatorname{supp}(\varphi)\subset B(0,1)$ and $||\varphi||_{L^2(\Omega)}=1$. Let $\varepsilon>0$, then we define

$$\varphi_\varepsilon(x):=\varepsilon^{-3/2}\varphi\Big(\frac{x-x_0}{\varepsilon}\Big),$$

so $||\varphi_\varepsilon||_{L^2(\mathbb{R}^3)}=1$ and $\varphi_\varepsilon\in L^2(\mathbb{R}^3)$. Evaluating the energy we obtain

$$D(\varphi_\varepsilon,\varphi_\varepsilon)=\varepsilon^2\int_\Omega\int_\Omega\frac{\varphi(x)\varphi(y)}{|x-y|}\,dxdy.$$

Thus as $\varepsilon\to 0$, $D(\varphi_\varepsilon,\varphi_\varepsilon)\to 0$, so we proved that the infimum under study is equal to zero.

To show that the infimum is not attained, it is enough to notice that any admissible $\tilde u\in L^2(\mathbb{R}^3)$ must be $\tilde u\neq 0$ by the L^2 constraint. Hence $D(\tilde u,\tilde u)>0$ by Lemma 2.1. □

Remark A.2. Let Ω be a (quasi-)open set in $\mathbb{R}^3$ such that there exists a point x_0 and a radius $r>0$ for which $B_r(x)\subseteq\Omega$, then by the same proof of Lemma A.1 we deduce that

$$\inf\{D(u,u):\ u\in H_0^1(\Omega),||u||_{L^2(\Omega)}=1\}=\inf\{D(u,u):\ u\in H_0^1(\Omega),||u||_{L^2(\Omega)}=1,\ u\geq 0\}=0.$$

Lemma A.1 does not really show the homogenization phenomenon that we expect, essentially because of scaling properties of the functional. We can impose an additional L^∞ bound to see the importance of considering sign-changing functions.

Lemma A.3. *For all $L\geq 1$, then*

$$\inf_{\substack{\Omega\subseteq\mathbb{R}^3,\,\textit{quasi-open}\\ |\Omega|=|B_1|}}\ \inf\left\{D(u,u):u\in H_0^1(\Omega),\int_\Omega u^2dx=1,||u||_{L^\infty(\Omega)}\leq L\right\}=0.$$

Proof. Take $\varphi\in C_c^\infty(B_1)$ such that $||\varphi||_{L^2(B_1)}=1$ and $||\varphi||_{L^\infty(B_1)}\leq 1$. Consider $n\in\mathbb{N}$, $\varepsilon=(2n)^{-1/3}$ and define

$$\varphi_{\varepsilon,n}(x):=\frac{1}{\sqrt{2n}}\left(\sum_{i=1}^{n}\varepsilon^{-3/2}\varphi\Big(\frac{x-x_i}{\varepsilon}\Big)-\sum_{i=n+1}^{2n}\varepsilon^{-3/2}\varphi\Big(\frac{x-x_i}{\varepsilon}\Big)\right),$$

where $x_i\in\mathbb{R}^3$ such that $\operatorname{supp}\left\{\varphi\Big(\frac{\cdot-x_i}{\varepsilon}\Big)\right\}\cap\operatorname{supp}\left\{\varphi\Big(\frac{\cdot-x_j}{\varepsilon}\Big)\right\}=\emptyset$ whenever $i\neq j$ and $\min\{|x_i-x_j|:i\neq j\}$ is diverging as n diverges. Then it is easy to see that $||\varphi_{\varepsilon,n}||_{L^2(\mathbb{R}^3)}=1$ and

$$|\operatorname{supp}\{\varphi_{\varepsilon,n}\}|=2n|B_\varepsilon|=2n\varepsilon^3|B_1|=|B_1|.$$

Moreover

$$||\varphi_{\varepsilon,n}||_{L^\infty(\mathbb{R}^3)}\leq\frac{\varepsilon^{-3/2}}{\sqrt{2n}}=1,$$

so $\varphi_{\varepsilon,n}$ satisfies the L^∞ constraint since true since $L\geq 1$. Then arguing as in Lemma A.1,

$$D(\varphi_{\varepsilon,n},\varphi_{\varepsilon,n})=n^{1/2}(2n)^{-2/3}D(\varphi,\varphi)\to 0,$$

as n diverges. Since

$$\inf_{\substack{\Omega\subseteq\mathbb{R}^3,\,\text{quasi-open}\\ |\Omega|=1}}\ \inf\left\{D(u,u):u\in H_0^1(\Omega),\int_\Omega u^2dx=1,||u||_{L^\infty(\Omega)}\leq L\right\}\leq D(\varphi_{\varepsilon,n},\varphi_{\varepsilon,n})\to 0,$$

then by Lemma 2.1 we conclude. □

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