RESTORATION OF COLOR IMAGES BY VECTOR VALUED BV FUNCTIONS AND VARIATIONAL CALCULUS

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Abstract. We analyze a variational problem for the recovery of vector valued functions and we compute its numerical solution. The data of the problem are a small set of complete samples of the vector valued function and a significant incomplete information where the former are missing. The incomplete information is assumed as the result of a distortion, with values in a lower dimensional manifold. For the recovery of the function we minimize a functional which is formed by the discrepancy with respect to the data and total variation regularization constraints. We show existence of minimizers in the space of vector valued BV functions. For the computation of minimizers we provide a stable and efficient method. First we approximate the functional by coercive functionals on $W^{1,2}$ in terms of Γ-convergence. Then we realize approximations of minimizers of the latter functionals by an iterative procedure to solve the PDE system of the corresponding Euler-Lagrange equations. The numerical implementation comes naturally by finite element discretization. We apply the algorithm to the restoration of color images from a limited color information and gray levels where the colors are missing. The numerical experiments show that this scheme is very fast and robust. The reconstruction capabilities of the model are shown, also from very limited (randomly distributed) color data. Several examples are included from the real restoration problem of the A. Mantegna's art frescoes in Italy.

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1. Introduction

This paper concerns with the analysis and the numerical implementation of a variational model for the restoration of vector valued functions. The restoration is obtained from few and sparse *complete* samples of the function and from a significant *incomplete* information. The latter is assumed as the result of a nonlinear distortion and with values in a lower dimensional manifold. The applications we consider are in the field of digital signal and image restoration. Therefore we deal with functional analysis in the space of bounded variation (BV) functions, that are actually considered a reasonable functional model for natural images and signals, usually characterized by discontinuities and piecewise smooth behavior. While in the literature on mathematical image processing real valued BV functions and associated variational problems are mainly discussed, see for example [7, 29], in this contribution we consider vector valued functions.

Since the work of Mumford and Shah [27], and Rudin, Osher and Fatemi [28], variational calculus techniques have been applied in several image processing problems. We refer the reader to the introductory book [6] for a presentation of this field, for more details, and an extended literature.

Let Ω be an open, bounded, and connected subset of $\mathbb{R}^N$, $D \subset \Omega$, and $p \geq 1$. Inspired by the fresco problem described in Figure 1, in [21] one of the authors has proposed the following variational problem

(1)

$$\operatorname{arginf}_{u:\Omega\to\mathbb{R}^M} \left\{ F(u) = \mu \int_{\Omega\setminus D} |u(x) - \bar{u}(x)|^p dx + \lambda \int_D |\mathcal{L}(u(x)) - \bar{v}(x)|^p dx + \int_\Omega \sum_{i=1}^M \phi(|\nabla u_i(x)|) dx \right\}$$

FIGURE 1. Fragmented A. Mantegna's frescoes (1452) by a bombing in the Second World War. Computer based reconstruction by using efficient pattern matching techniques [23]. Unfortunately the surface covered by the original fragments is only 77 m^2, while the original area was of several hundreds. This means that what we can currently reconstruct is just a *fraction* (estimated up to 8%) of what was this inestimable artwork. In particular, for most of the frescoes, the *original color* of the blanks is not known. So, natural questions raise: is it possible to estimate *mathematically* the original colors of the frescoes by using the known fragments information and the gray level of the pictures taken before the damage? And, how *faithful* this estimation is?

to model the reconstruction/restoration of a vector valued function $u : \Omega \to \mathbb{R}^M$ from a given observed couple of functions $(\bar{u}, \bar{v})$. The observed function $\bar{u}$ is assumed to represent correct information on $\Omega \backslash D$, and $\bar{v}$ the result of a *nonlinear distortion* $\mathcal{L} : \mathbb{R}^M \to \mathbb{R}$ on D.

In particular, a digital image can be modeled as a function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3_+$, so that, to each "point" $\mathbf{x}$ of the image, one associates the vector $u(\mathbf{x}) = (r(\mathbf{x}), g(\mathbf{x}), b(\mathbf{x})) \in \mathbb{R}^3_+$ of the color represented by the different channels red, green, and blue. In particular, a digitalization of the image u corresponds to its sampling on a regular lattice $\tau\mathbb{Z}^2$, $\tau > 0$. Let us again write $u : \mathcal{N} \to \mathbb{R}^3_+$, $u(\mathbf{x}) = (r(\mathbf{x}), g(\mathbf{x}), b(\mathbf{x}))$, for $\mathbf{x} \in \mathcal{N} := \Omega \cap \tau\mathbb{Z}^2$.

Usually the gray level of an image can be described as a submanifold $\mathcal{M} \subset \mathbb{R}^3$ by

$$\mathcal{M} := \mathcal{M}_\sigma = \{\sigma(x) : x = \mathcal{L}(r, g, b) := L(\alpha r + \beta g + \gamma b), (r, g, b) \in \mathbb{R}^3_+\},$$

where $\alpha, \beta, \gamma > 0$, $\alpha + \beta + \gamma = 1$, $L : \mathbb{R} \to \mathbb{R}$ is a non-negative increasing function, and $\sigma : \mathbb{R}_+ \to \mathbb{R}^3_+$ is a suitable section such that $\mathcal{L} \circ \sigma = \mathrm{id}_{\mathbb{R}_+}$. The function L is assumed smooth, nonlinear, and normally nonconvex and nonconcave. For example, Figure 1 illustrates a typical situation where this model applies and Figure 2 describes the typical shape of an L function. Here L is estimated by fitting a distribution of data from real color fragments. In fact, in this case, there is an area $\Omega \backslash D$ of the domain $\Omega \subset \mathbb{R}^2$ of the image, where some fragments with colors are placed and complete information is available, and an other area D (which we call the *inpainting region*) where only the gray level information is known, modeled as the image of $\mathcal{L}$.

The solution of the variational problem (1) produces in this case a new color image that extends the colors of the fragments in the gray region. Once the extended color image is transformed by means of $\mathcal{L}$, it is constrained to match the known gray level. We can consider this problem as a generalization of the well known *image inpainting/disocclusion*, see, e.g., [3, 8, 9, 13, 14, 15, 16]. Several heuristic algorithms have been introduced for colorization of gray images and we refer to the recently appeared paper [31] for related literature. Nevertheless, our approach is theoretically

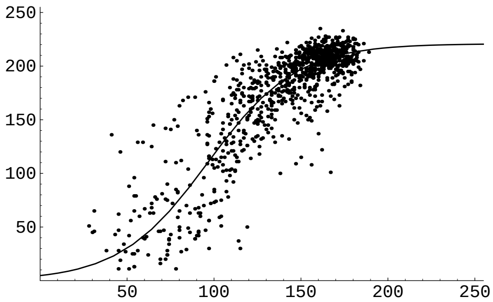

FIGURE 2. Estimate of the nonlinear curve L from a distribution of points with coordinates given by the linear combination $\alpha r + \beta g + \gamma b$ of the (r, g, b) color fragments (abscissa) and by the corresponding underlying gray level of the original photographs dated to 1920 (ordinate). The sensitivity parameters α, β, γ to the different frequencies of red, green, and blue are chosen in order to minimize the total variance of the ordinates.

founded, more general, and fits with many possible applications, for example the recovery of a transmitted multichannel signal affected by a stationary (nonlinear) distortion.

For $N = p = 2$, we can compute the Euler-Lagrange equations associated to the functional F and obtain

$$(2)\quad 0 = -\nabla \cdot \left(\frac{\phi'(|\nabla u_i|)}{|\nabla u_i|} \nabla u_i \right) + 2\mu(u_i - \bar{u}_i) 1_{\Omega \setminus D} + 2\lambda(\mathcal{L}(u) - \bar{v}) \frac{\partial \mathcal{L}}{\partial x_i}(u) 1_D := \mathcal{E}_i(\mathcal{L}, u), \quad i = 1, ..., M,$$

where $u = (u_1, ..., u_M)$ are the components of the function u. This is a system of coupled second order equations and the analysis of the solutions constitutes itself a problem of independent interest. By using (2) and a finite difference approximation, a steepest-descent algorithm can be formulated as in [21]. An application of this numerical scheme for the restoration of a color image from few and sparse fragments and from the constraint given by known gray levels of the missing part is shown in Figure 3.

Encouraged by these numerical evidences, we discuss the existence of minimizers of the functional F in the context of vector valued BV functions. Our second goal is the formulation of efficient and stable algorithms for the computation of minimizers. Although the steepest-descent scheme recalled above gives appreciable results, it lacks of a rigorous analysis and its convergence is usually very slow. For these reasons, we introduce new coercive functionals F_h on $W^{1,2}$ which approximate $\bar{F}$ (the relaxed functional of F with respect to the BV weak-$*$-topology) in terms of Γ-convergence. The computation of minimizers of F_h is performed by an iterative *double-minimization* algorithm, see also [12]. The reconstruction performances are very good, also from very limited (randomly distributed) color data. The virtues of our scheme can be summarized as follows.

- It is derived as the minimization of a functional and its mathematical analysis and foundations are well described.
- It implements a total variation (TV) minimization. It is well known [14, 15] that total variation inpainting is affected by two major drawbacks. The first one is that the TV model is only a linear interpolant, i.e., the broken isophotes are interpolated by straight lines. Thus it can generate corners along the inpainting boundary. The second one is that TV often fails to connect widely separated parts of a whole object, due to the high cost in TV measure of making long-distance connections. Due to the constraint on the gray level in the inpainting region, our scheme does not extend isophotes as straight lines and it does not violate the *connectivity principle.*

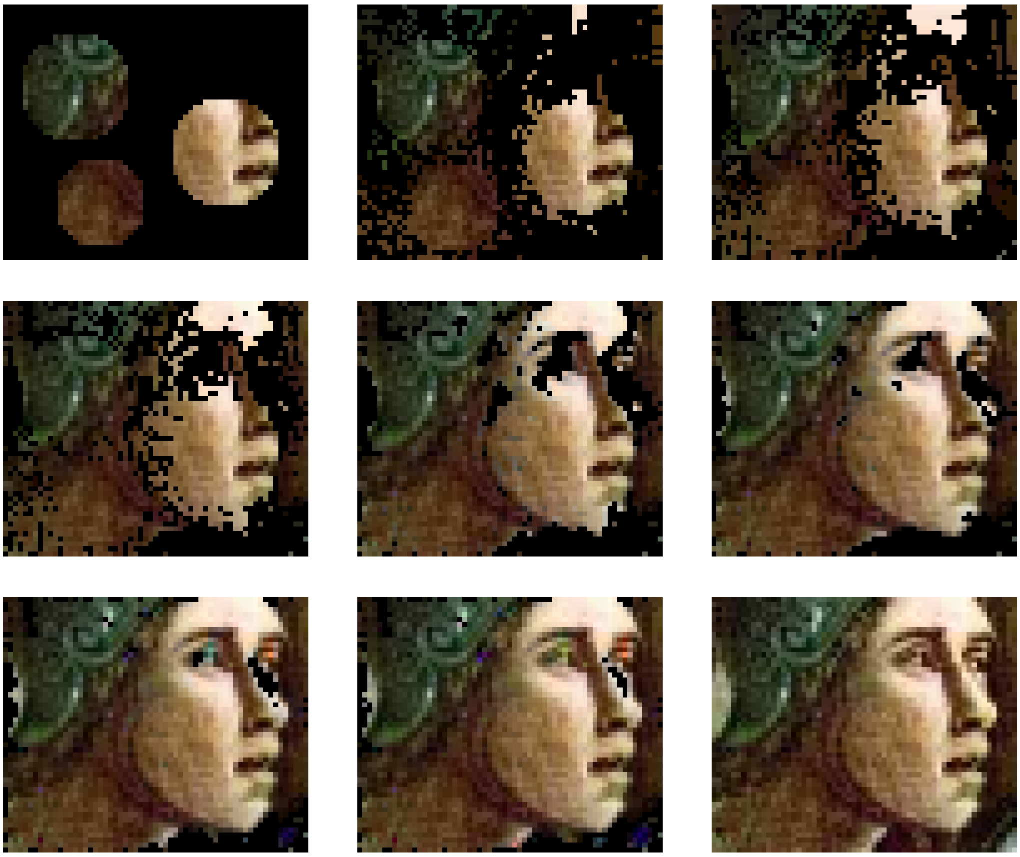

FIGURE 3. Successive iterations of the interpolation-inpainting process. We have chosen here a descent parameter $\Delta t = 0.1$, $\lambda = \mu = 10$, and $\mathcal{L}(r, g, b) = \frac{1}{3}(r+b+g)$. Although the gray level is a fundamental datum, in this figure we have displayed only the color in order to better visualize the diffusion progress.

- As pointed out in [11, 22], while it is relatively easy to recover at higher resolution image portions with relatively uniform color, it might be difficult to recover jumps correctly. Not only we should preserve the morphology and enhance the detail of the discontinuities, but these properties must fit through the different color channels. An incorrect or uncoupled recovery in fact produces "rainbow effects" around jumps. In our functional, the constraint on the gray level in the inpainting region is formulated as a coupled combination of the color channels. In practice, this is sufficient to enforce the correct coupling of the channels at edges.
- The numerical implementation of our double-minimization scheme is very simple. Its approximation by finite elements comes in a natural way. The scheme is fast and stable.

The paper is organized as follows. In Section 2 we introduce the mathematical setting. We recall the main properties of BV functions and a definition of the space of BV functions with vector values. Section 3 is dedicated to results on convex functions and relaxed functionals of measures. In Section 4 we collect the assumptions on the nonlinear function $\mathcal{L}$ we will need in our analysis. In Section 5 the representation of the relaxed functional $\bar{F}$ of F with respect to the BV tolopology is given, and existence and uniqueness of minimizers of $\bar{F}$ are discussed. In Section 6 we introduce coercive functionals F_h on $W^{1,2}$ which are shown to Γ-converge to the relaxed functional described above. The double-minimization algorithm to compute minimizers of F_h is illustrated in Section 7. Its numerical implementation is presented in Section 8. We include several numerical experiments and we discuss their results.

Nota on color pictures. This paper introduces methods to recover colors in digital images. Therefore the gray level printout of the manuscript does not allow to appreciate the quality of the illustrated techniques. The authors recommend the interested reader to access the electronic version with color pictures which is available online.

2. Vector Valued BV Functions

In this section we want to introduce notations and preliminary results concerning vector valued BV functions.

We denote by $\mathcal{L}_N$ (and in the integrals dx) the Lebesgue N-dimensional measure in $\mathbb{R}^N$ and by $\mathcal{H}_\alpha$ the α-dimensional Hausdorff measure. Let Ω be an open, bounded, and connected subset of $\mathbb{R}^N$. With $\mathcal{B}(\Omega)$ we denote the family of Borel subsets of $\Omega \subset \mathbb{R}^N$. For a given vector valued measure $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^M$, we denote with $|\mu|$ its total variation, i.e., the finite positive measure

$$|\mu|(A) := \sup \left\{ \sum_{j=1}^{M} \int_\Omega v_j d\mu_j : v = (v_1, ..., v_M) \in C_0(A; \mathbb{R}^M), \|v\|_\infty \leq 1 \right\},$$

where $C_0(A; \mathbb{R}^M) := \overline{C_c(A; \mathbb{R}^M)}^{\|\cdot\|_\infty}$, i.e., the sup-norm closure of the space of continuous function with compact support in A and vector values in $\mathbb{R}^M$. The set of the signed measures on Ω with bounded total variation is denoted by $\mathcal{M}(\Omega)$, coinciding in fact with the topological dual of $(C_0(A; \mathbb{R}^M), \|\cdot\|_\infty)$. Thus, the usual weak-$*$-topology on $\mathcal{M}(\Omega)$ is the weakest topology that makes the maps $\mu \to \int_\Omega f d\mu$ continuous for every continuous function $f \in C_0(A; \mathbb{R}^M)$. In the following we will make use of the notations $x \wedge y := \inf\{x, y\}$ and $x \vee y := \sup\{x, y\}$ for all $x, y \in \mathbb{R}$.

We say that $u \in L^1(\Omega)$ is a real function of bounded variation if its distributional derivative $Du = (D_{x_1} u, ..., D_{x_N} u)$ is in $\mathcal{M}(\Omega)$. Then the space of bounded variation functions is denoted by

$$BV(\Omega) := \{u \in L^1(\Omega) : Du \in \mathcal{M}(\Omega)\},$$

and, endowed with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_1 + |Du|(\Omega),$$

is a Banach space [20]. More general, we are interested in vector valued functions with bounded variation components, whose space is defined by

$$BV(\Omega; \mathbb{R}^M) := \{u = (u_1, ..., u_M) \in L^1(\Omega; \mathbb{R}^M) : u_i \in BV(\Omega)\}.$$

To this space it will turn out to be convenient to attach the norm

$$\|u\|_{BV(\Omega;\mathbb{R}^M)} := \|u\|_{L^1(\Omega;\mathbb{R}^M)} + \sum_{i=1}^{M} |Du_i|(\Omega).$$

With a slight abuse of notation, for $u \in BV(\Omega; \mathbb{R}^M)$ we denote

$$|Du| := \sum_{i=1}^{M} |Du_i|, \tag{3}$$

that is again a finite positive measure for Ω. The space $(BV(\Omega; \mathbb{R}^M), \|\cdot\|_{BV(\Omega;\mathbb{R}^M)})$ is a Banach space, and its norm can be of course equivalently defined by

$$\|u\|_{BV(\Omega;\mathbb{R}^M)} \sim \|u\|_{L^1(\Omega;\mathbb{R}^M)} + \left(\sum_{i=1}^{M} |Du_i(\Omega)|^q \right)^{1/q},$$

for all $q \in [1, \infty)$ and for $q \to \infty$ we have

$$\|u\|_{BV(\Omega;\mathbb{R}^M)} \sim \|u\|_{L^1(\Omega;\mathbb{R}^M)} + \max_{i=1,\dots,M} |Du_i(\Omega)|.$$

Of course $BV(\Omega;\mathbb{R}^M) = BV(\Omega)$ for $M = 1$, and our notations are consistent with this case.
The product topology of the strong topology of $L^1(\Omega;\mathbb{R}^M)$ for u and of the weak-$*$-topology of measures for Du_i (for all $i = 1, ..., M$) will be called the weak-$*$-topology of $BV(\Omega;\mathbb{R}^M)$ or the componentwise BV weak-$*$-topology. In the following, whenever the domain Ω and the dimension M will be clearly understood, we will write L^1 instead of $L^1(\Omega;\mathbb{R}^M)$ and BV instead of $BV(\Omega;\mathbb{R}^M)$.

We further recall the main structure properties of BV functions [1, 2, 20]. If $v \in BV(\Omega)$ then the Lebesgue decomposition of Dv with respect to the Lebesgue measure $\mathcal{L}_N$ is given by

$$Dv = \nabla v \cdot \mathcal{L}_N + D_s v,$$

where $\nabla u = \frac{d(Dv)}{dx} \in L^1(\Omega;\mathbb{R}^N)$ is the Radon-Nikodym derivative of Dv and $D_s v$ is singular with respect to $\mathcal{L}_N$.

For a function $v \in L^1(\Omega)$ one denotes with S_v the complement of the Lebesgue set of v, i.e.,

$$S_v := \{x \in \Omega : v^-(x) < v^+(x)\},$$

where

$$v^+(x) := \inf\left\{t \in \bar{\mathbb{R}} : \lim_{\epsilon\to 0} \frac{\mathcal{L}_N(\{v > t\} \cap B(x,\epsilon))}{\epsilon^N} = 0\right\}$$

and

$$v^-(x) := \sup\left\{t \in \bar{\mathbb{R}} : \lim_{\epsilon\to 0} \frac{\mathcal{L}_N(\{v < t\} \cap B(x,\epsilon))}{\epsilon^N} = 0\right\}.$$

Then S_v is countably rectifiable, and for $\mathcal{H}_{N-1}$-a.e. $x \in \Omega$ we can define the outer normal $\nu(x)$. We denote by $\tilde{v} : \Omega \setminus S_v \to \mathbb{R}$ the approximate limit of v defined as $\tilde{v}(x) = v^+(x) = v^-(x)$.

Following [1, 20] $D_s v$ can be expressed by

$$D_s v = C_v + J_v,$$

where

$$J_v = (v^+ - v^-)\nu \cdot \mathcal{H}_{N-1}|_{S_v},$$

is the *jump part* and C_v is the *Cantor part* of Dv. Therefore, we can express the measure Dv by

$$Dv = \nabla v \cdot \mathcal{L}_N + C_v + (v^+ - v^-)\nu \cdot \mathcal{H}_{N-1}|_{S_v}, \tag{4}$$

and its total variation by

$$|Dv|(E) = \int_E |\nabla v| dx + \int_{E\setminus S_v} |C_v| + \int_{E\cap S_v} (v^+ - v^-) d\mathcal{H}_{N-1}, \tag{5}$$

for every Borel set E in the Borel σ-algebra $\mathcal{B}(\Omega)$ of Ω. For major details we refer the reader to [1]. By these properties of real BV-functions, one obtains the following result for vector valued BV-functions.

Lemma 2.1 (Lebsegue decomposition for vector valued BV-functions)**.** *For $u \in BV(\Omega;\mathbb{R}^N)$, the positive measure $|Du|$ as defined in* (3) *has the following Lebesgue decomposition*

$$|Du| = |D_a u| + |D_s u|, \tag{6}$$

where

$$|D_a u| = \sum_{i=1}^{M} |\nabla u_i| \mathcal{L}_N \tag{7}$$

is the absolutely continuous part and

$$|D_s u| = \sum_{i=1}^{M} |C_{u_i}| + \sum_{i=1}^{M} (u_i^+ - u_i^-) \mathcal{H}_{N-1}|_{S_{u_i}}, \tag{8}$$

is the singular part of $|Du|$, with respect to the Lebesgue measure $\mathcal{L}_N$.

Proof. By definition it is

$$|Du| = \sum_{i=1}^{M} |Du_i|$$

and by the Lebesgue decomposition (5) for each $|Du_i|$ it is

$$|Du| = \sum_{i=1}^{M} \left(|\nabla u_i| \mathcal{L}_N + |C_{u_i}| + (u_i^+ - u_i^-) \mathcal{H}_{N-1}|_{S_{u_i}} \right).$$

Since $\sum_{i=1}^{M} |\nabla u_i| \mathcal{L}_N$ is absolutely continuous and $\sum_{i=1}^{M} |C_{u_i}| + \sum_{i=1}^{M} (u^+ - u^-) \mathcal{H}_{N-1}|_{S_{u_i}}$ is singular with respect to $\mathcal{L}_N$, one concludes the proof by the uniqueness of the Lebesgue decomposition. ■

3. Convex Functions and Functionals of Measures

In the following and throughout the paper we assume that

(A) $\phi : \mathbb{R} \to \mathbb{R}^+$ is an even and convex function, nondecreasing in $\mathbb{R}^+$ such that
   (i) $\phi(0) = 0$;
   (ii) There exists $c > 0$ and $b \geq 0$ such that $cz - b \leq \phi(z) \leq cz + b$, for all $z \in \mathbb{R}$.

Under such conditions the asymptotic recession function ϕ^∞ defined by

$$\phi^\infty(z) := \lim_{y \to \infty} \frac{\phi(yz)}{y}$$

is well defined and bounded. It is $c = \lim_{y \to \infty} \frac{\phi(y)}{y} = \phi^\infty(1)$ and $\phi^\infty(z) = cz \cdot \text{sign}(z)$.

Following [17, 24] we can define convex functions of measures. In particular if $\mu \in \mathcal{M}(\Omega)$ then we can define

$$\phi(|\mu|) = \phi(|\mu_a|) \mathcal{L}_N + \phi^\infty(1) |\mu_s|,$$

where μ_a and μ_s are the absolutely continuous and singular parts of μ respectively, with respect to $\mathcal{L}_N$. Therefore, according to Lemma 2.1, if $u \in BV(\Omega; \mathbb{R}^M)$ then

$$\sum_{i=1}^{M} \phi(|Du_i|) = \sum_{i=1}^{M} \phi(|\nabla u_i|) \mathcal{L}_N + \phi^\infty(1) \left(\sum_{i=1}^{M} |C_{u_i}| + \sum_{i=1}^{M} (u^+ - u^-) \mathcal{H}_{N-1}|_{S_{u_i}} \right). \tag{9}$$

Definition 1. Let (X, τ) be a topological space satisfying the first axiom of countability and $F : X \to \bar{\mathbb{R}}$. The relaxed functional of F with respect to the topology τ is defined for every $x \in X$ as $\bar{F}(x) := \sup\{G(x) : G$ is τ-lower semicontinuous and $G \leq F\}$. In other words $\bar{F}$ is the maximal τ-lower semicontinuous functional that is smaller than F. We may also write

$$\bar{F}(u) = \inf_{u^{(n)} \in X, u^{(n)} \xrightarrow{\tau} u} \{\liminf_n F(u^{(n)})\}$$

we have the following result

Lemma 3.1. *If $u \in BV(\Omega; \mathbb{R}^M)$ and ϕ is as in assumption (A), then*

$$E(u) := \int_\Omega \sum_{i=1}^{M} \phi(|Du_i|) := \sum_{i=1}^{M} \phi(|Du_i|)(\Omega) = \int_\Omega \sum_{i=1}^{M} \phi(|\nabla u_i|) dx + c \left(\sum_{i=1}^{M} \int_{\Omega \setminus S_{u_i}} |C_{u_i}| + \int_{S_{u_i}} (u_i^+ - u_i^-) d\mathcal{H}_{N-1} \right)$$

is lower semicontinuous with respect to the componentwise BV weak-$*$-topology.

Proof. It is known that

$$u_i \to E_i(u_i) := \int_\Omega \phi(|\nabla u_i|)dx + c\left(\int_{\Omega\setminus S_{u_i}} |C_{u_i}| + \int_{S_{u_i}} (u_i^+ - u_i^-)d\mathcal{H}_{N-1}\right)$$

is lower semicontinuous for the BV weak-$*$-topology on $BV(\Omega)$ [24]. One concludes simply by observing that $E(u) = \sum_{i=1}^M E_i(u_i)$. ∎

4. Assumptions on the Evaluation Map $\mathcal{L}$.

In the following we assume that

(L1) $\mathcal{L} : \mathbb{R}^M \to \mathbb{R}_+$ is a non-decreasing continuous function in the sense that $\mathcal{L}(x) \leq \mathcal{L}(y)$ for any $x, y \in \mathbb{R}^M$ such that $|x_i| \leq |y_i|$ for any $i \in \{1, \dots, M\}$;

(L2) $\mathcal{L}(x) \leq a + b|x|^s$, for all $x \in \mathbb{R}^M$ and for fixed $s \geq p^{-1}$, $b > 0$, and $a \geq 0$.

Moreover, one of the two following conditions holds

(L3-a) $\lim_{x\to\infty} \mathcal{L}(x) = +\infty$;

(L3-b) $\mathcal{L}(x) = \mathcal{L}(x_1, ..., x_M) = \mathcal{L}((\ell_1 \wedge x_1 \vee -\ell_1), ..., (\ell_M \wedge x_M \vee -\ell_M))$, for a suitable fixed vector $\ell = (\ell_1, ..., \ell_M) \in \mathbb{R}_+^M$.

Observe that condition (L3-a) is equivalent to say that for every $C > 0$ the set $\{\mathcal{L} \leq C\}$ is bounded. Therefore there exists $A \in \mathbb{R}^M$, with $A_i \geq 0$ for any $i \in \{1, \dots, M\}$, such that $\{\mathcal{L} \leq C\} \subseteq \prod_{i=1}^M [-A_i, A_i]$.

In the following and throughout the paper D denotes a measurable subset of Ω, and we are given the couple $(\bar{u}, \bar{v})$ of bounded functions such that $\bar{u} : \Omega \setminus D \to \mathbb{R}^M$ and $\bar{v} : D \to \mathbb{R}$.

If the condition (L3-a) holds, for any measurable function $u : \Omega \to \mathbb{R}^M$, we define the *truncation or clipping operator* as follows:

$$\mathrm{tr}(u, \bar{u}, \Omega, D)(x) := ((\|\bar{u}_i|\Omega \setminus D\|_\infty \vee A_i) \wedge u_i(x) \vee (-\|\bar{u}_i|\Omega \setminus D\|_\infty \wedge -A_i))_{i=1}^M, \tag{10}$$

where $A \in \mathbb{R}^M$ is determined so that $\{\mathcal{L} \leq \|\bar{v}|D\|_\infty\} \subseteq \prod_{i=1}^M [-A_i, A_i]$. Analogously we define the truncation operator in the case of the assumption (L3-b):

$$\mathrm{tr}(u, \bar{u}, \bar{v}, \Omega, D)(x) := ((\|\bar{u}_i|\Omega \setminus D\|_\infty \vee \ell_i) \wedge u_i(x) \vee (-\|\bar{u}_i|\Omega \setminus D\|_\infty \wedge -\ell_i))_{i=1}^M. \tag{11}$$

In the case it is clear which of the assumptions (L3-a,b) holds, and the set D and the functions $\bar{u}, \bar{v}$ are given, then it will be convenient the shorter notation $\hat{u} := \mathrm{tr}(u, \bar{u}, \bar{v}, \Omega, D)$.

For any measurable function $u : \Omega \to \mathbb{R}^M$ we define:

$$G_1(u) = \int_{\Omega\setminus D} |u(x) - \bar{u}(x)|^p dx, \tag{12}$$

$$G_2(u) = \int_D |\mathcal{L}(u(x)) - \bar{v}(x)|^p dx. \tag{13}$$

Lemma 4.1. *For any $u \in BV(\Omega; \mathbb{R}^M)$ the truncation operator has the property that $\hat{u} \in BV(\Omega; \mathbb{R}^M)$, and*

$$G_i(\hat{u}) \leq G_i(u), \quad i = 1, 2, \text{ and } E(\hat{u}) \leq E(u). \tag{14}$$

Proof. Let us assume that the condition (L3-a) holds. If $x \in \Omega \setminus D$ the definition of the truncation operator implies that $|\hat{u}(x) - \bar{u}(x)| \leq |u(x) - \bar{u}(x)|$, from which it follows

$$G_1(\hat{u}) \leq G_1(u).$$

If $x \in D$ is such that $u(x) \in \prod_{i=1}^M [-\|\bar{u}_i|\Omega \setminus D\|_\infty \wedge -A_i, \|\bar{u}_i|\Omega \setminus D\|_\infty \vee A_i]$, then $\hat{u}(x) = u(x)$. Otherwise, $x \notin \prod_{i=1}^M [-A_i, A_i]$ and $|u_i(x)| \geq |\hat{u}_i(x)| \geq |\xi_i|$, for any ξ such that $\mathcal{L}(\xi) \leq \|\bar{v}|D\|_\infty$ and any $i \in \{1, \dots, M\}$. Therefore, by the monotonicity assumption (L1) $\mathcal{L}(u(x)) \geq \mathcal{L}(\hat{u}(x)) \geq \|\bar{v}|D\|_\infty$, which implies that $|\mathcal{L}(\hat{u}(x)) - \bar{v}(x)| \leq |\mathcal{L}(u(x)) - \bar{v}(x)|$ for any $x \in D$, and

$$G_2(\hat{u}) \leq G_2(u).$$

The proof is analogous if the condition (L3-b) holds.

We now prove the corresponding statement for the functional E. Fix $i \in \{1, \dots, M\}$. By definition of the truncation operator, we have $\hat{u}_i = g_i \circ u_i$, where $g_i : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function such that

$$g_i(t) = \begin{cases} t, & -c_i \leq t \leq d_i \\ d_i, & t > d_i \\ -c_i, & t < -c_i, \end{cases}$$

where $c_i, d_i > 0$ are determined by (10,11). Using the chain rule for real-valued BV functions (Theorem 3.99 of [2]), we have that $\hat{u} \in BV(\Omega; \mathbb{R}^M)$ and

$$D\hat{u}_i = g_i'(u_i)\nabla u_i \cdot \mathcal{L}_N + g_i'(\tilde{u}_i)C_{u_i} + \left(g_i(u_i^+) - g_i(u_i^-)\right) \nu_i \cdot \mathcal{H}_{N-1}|_{S_{u_i}},$$

where $\tilde{u}_i$ is the approximate limit of u_i. Then $\nabla\hat{u}_i(x) = \nabla u_i(x)$ if $-c_i < u_i(x) < d_i$, and $\nabla\hat{u}_i(x) = 0$ if either $u_i(x) > d_i$ or $u_i(x) < -c_i$. Moreover, by Proposition 3.73 (c) of [2] it follows that $\nabla u_i(x) = 0$ for a.e. $x \in \{u_i(x) = d_i\}$ and a.e. $x \in \{u_i(x) = -c_i\}$. Hence $|\nabla\hat{u}_i(x)| \leq |\nabla u_i(x)|$ a.e., so that from the assumption (A) of the function ϕ we get

$$\int_\Omega \phi(|\nabla\hat{u}_i|)dx \leq \int_\Omega \phi(|\nabla u_i|)dx. \tag{15}$$

Since $u_i^+(x) \geq u_i^-(x)$ for any $x \in S_{u_i}$, by the definition of the function g_i we have

$$S_{\hat{u}_i} \subseteq S_{u_i}, \qquad g_i(u_i^+(x)) - g_i(u_i^-(x)) \leq u_i^+(x) - u_i^-(x) \quad \text{for any } x \in S_{u_i}.$$

Then it follows

$$\int_{S_{\hat{u}_i}} (\hat{u}_i^+ - \hat{u}_i^-)d\mathcal{H}_{N-1} \leq \int_{S_{u_i}} (u_i^+ - u_i^-)d\mathcal{H}_{N-1}. \tag{16}$$

By the definition of g_i we then have $0 \leq g_i'(\tilde{u}_i(x)) \leq 1$ for any $x \in \{x : \tilde{u}_i(x) \neq d_i\} \cap \{x : \tilde{u}_i(x) \neq -c_i\}$. Moreover, by Proposition 3.92 (c) of [2], the Cantor part C_{u_i} vanishes on sets of the form $\tilde{u}_i^{-1}(Q)$ with $Q \subset \mathbb{R}$, $\mathcal{H}_1(Q) = 0$. It follows that C_{u_i} vanishes on the set $\{x : \tilde{u}_i(x) = d_i\} \cup \{x : \tilde{u}_i(x) = -c_i\}$, so that we get $|C_{\hat{u}_i}|(\Omega) \leq |C_{u_i}|(\Omega)$, i.e.,

$$\int_{\Omega \setminus S_{\hat{u}_i}} |C_{\hat{u}_i}| \leq \int_{\Omega \setminus S_{u_i}} |C_{u_i}|. \tag{17}$$

Collecting the inequalities (15-17) and summing over $i = 1, \dots, M$, we obtain

$$E(\hat{u}) \leq E(u),$$

which concludes the proof. ■

Remark 4.2. *The truncation operator maps C_0^1 functions into $W^{1,q}$, i.e., for any $u \in C_0^1(\Omega; \mathbb{R}^M)$ we have $\mathrm{tr}(u, \bar{u}, \bar{v}, \Omega, D) \in W^{1,q}(\Omega; \mathbb{R}^M)$ for any $1 \leq q \leq \infty$.*

5. Relaxation and Existence of Minimizers

The functional F is well defined in $L^\infty(\Omega; \mathbb{R}^M) \cap W^{1,1}(\Omega; \mathbb{R}^M)$. Since this space is not reflexive, and sequences that are bounded in $W^{1,1}$ are also bounded in BV, we extend F to the space $BV(\Omega; \mathbb{R}^M)$ in such a way that the extended functional is lower semicontinuous. By using the relaxation method of the Calculus of Variations, the natural candidate for the extended functional is the relaxed functional $\bar{F}$ of F with respect to the componentwise BV weak-$*$-topology [6].

In the following, without loosing generality, we set $\mu = \lambda = 1$.

5.1. **Relaxation.** We set

$$X = \{u \in BV(\Omega;\mathbb{R}^M) : \|u_i\|_\infty \le K_i,\ i = 1,\dots,M\},$$

where, for any $i \in \{1,\dots,M\}$, the constant $K_i > 0$ is defined by

$$K_i = \max\{A_i, \|\bar{u}_i|\Omega \setminus D\|_\infty\},$$

if the condition (L3-a) holds, and by

$$K_i = \max\{\ell_i, \|\bar{u}_i|\Omega \setminus D\|_\infty\},$$

if the condition (L3-b) holds.

The following theorem extends to our case the relaxation result proved in Theorem 3.2.1 of [6].

Theorem 5.1. *The relaxed functional of F in X with respect to the componentwise BV weak-$*$-topology is given by*

$$\begin{aligned}\bar{F}(u) &= \int_{\Omega\setminus D} |u(x) - \bar{u}(x)|^p dx + \int_D |\mathcal{L}(u(x)) - \bar{v}(x)|^p dx \\ &+ \int_\Omega \sum_{i=1}^M \phi(|\nabla u_i|)dx + c\left(\sum_{i=1}^M \int_{\Omega\setminus S_{u_i}} |C_{u_i}| + \int_{S_{u_i}} (u_i^+ - u_i^-) d\mathcal{H}_{N-1}\right).\end{aligned}$$

Proof. Let us define

$$f(u) := \begin{cases} F(u), & u \in X \setminus W^{1,1}(\Omega;\mathbb{R}^M) \\ +\infty, & u \in X \setminus W^{1,1}(\Omega;\mathbb{R}^M).\end{cases}$$

Observe that $f(u) = \bar{F}(u)$ for $u \in W^{1,1}(\Omega;\mathbb{R}^M)$.

By the property (L2) we have that $G_1(u), G_2(u) < +\infty$ for all $u \in X$. By using Fatou's lemma the functionals G_1 and G_2 are lower semicontinuous with respect to the strong L^1 topology, and hence with respect to the componentwise BV weak-$*$-topology. Therefore, by Lemma 3.1, $\bar{F}$ is lower semicontinuous in X with respect to such topology.

Let $\bar{f}$ denote the relaxed functional of f in X with respect to the same topology. Since $\bar{F}(u) \le f(u)$ for any $u \in X$, and $\bar{f}$ is the greatest lower semicontinuous functional less than or equal to f, we have $\bar{f}(u) \ge \bar{F}(u)$ for any $u \in X$. Then we have to show that $\bar{f}(u) \le \bar{F}(u)$.

By [17, Theorem 2.2 and Theorem 2.3] for any $u \in X$ there exists a sequence $\{u^{(n)}\}_n \subset C_0^\infty(\Omega;\mathbb{R}^M) \cap W^{1,1}(\Omega;\mathbb{R}^M)$ such that $u^{(n)}$ converges to u in the componentwise BV weak-$*$-topology and $E(u) = \lim_n E(u^{(n)})$.

Let us now consider the sequence $\{\hat{u}^{(n)}\}_n$ of the truncated functions. By Lemma 4.1 we have

$$E(u) = \lim_n E(u^{(n)}) \ge \limsup_n E(\hat{u}^{(n)}). \tag{18}$$

With similar computations as those in the proof of Lemma (4.1)

$$\int_\Omega |\hat{u}^{(n)}(x) - u(x)| dx \le \int_\Omega |u^{(n)}(x) - u(x)| dx \to 0, \quad n \to \infty.$$

Moreover, since the truncated functions $\hat{u}^{(n)}$ are uniformly bounded in $L^\infty(\Omega;\mathbb{R}^M)$, then $\hat{u}^{(n)}$ converges to u in $L^q(\Omega;\mathbb{R}^M)$ for any $1 \le q < \infty$.

Now the functional G_1 is continuous with respect to the strong $L^p(\Omega\setminus D;\mathbb{R}^M)$ topology. Moreover, since $\mathcal{L}$ is continuous, the functional G_2 is continuous with respect to the strong $L^q(D;\mathbb{R}^M)$ topology, with $q = sp \ge 1$ (see [19, Lemma 3.2, Chapter 9]).

Then, using (18), the continuity properties of G_1 and G_2, and Remark 4.2, we have $\hat{u}^{(n)} \in W^{1,1}(\Omega;\mathbb{R}^M)$, $\bar{F}(\hat{u}^{(n)}) = f(\hat{u}^{(n)})$, and

$$\begin{aligned}\bar{F}(u) = G_1(u) + G_2(u) + E(u) &\ge \lim_n (G_1(\hat{u}^{(n)}) + G_2(\hat{u}^{(n)})) + \limsup_n E(\hat{u}^{(n)}) \ge \limsup_n f(\hat{u}^{(n)}) \\ &\ge \liminf_n f(\hat{u}^{(n)}) \ge \inf_{u^{(n)} \in BV, u^{(n)} \xrightarrow{BV-w^*} u} \{\liminf_n f(u^{(n)})\} = \bar{f}(u).\end{aligned}$$

Then we have $\bar{F}(u) = \bar{f}(u)$ and the statement is proved. ■

5.2. **Existence and uniqueness of minimizers.**

Theorem 5.2. *There exists a solution $u \in X$ of the following variational problem:*

$$\begin{aligned} \min\{\bar{F}(v) &= \int_{\Omega\setminus D} |v(x) - \bar{u}(x)|^p dx + \int_D |\mathcal{L}(v(x)) - \bar{v}(x)|^p dx \\ &+ \int_\Omega \sum_{i=1}^M \phi(|\nabla v_i|)dx + c\left(\sum_{i=1}^M \int_{\Omega\setminus S_{v_i}} |C_{v_i}| + \int_{S_{v_i}} (v_i^+ - v_i^-)d\mathcal{H}_{N-1}\right)\}. \end{aligned}$$

In particular we have

$$\min_{v\in X} \bar{F}(v) = \inf_{v\in X} F(v).$$

Moreover, if $D \subsetneq \Omega$ and G_2 is a strictly convex functional then the solution is unique.

Proof. Let $\{u^{(n)}\}_n$ be a minimizing sequence in BV. By assumption (A) (ii) there exists a constant $C > 0$ such that

$$|Du^{(n)}|(\Omega) \leq C,$$

uniformly with respect to n. By Lemma 4.1 we can modify the minimizing sequence by truncation, obtaining a new minimizing sequence $\{\hat{u}^{(n)}\}_n \subset X$. By Lemma 4.1 this sequence is uniformly bounded in $BV(\Omega;\mathbb{R}^M)$, i.e.,

$$\|\hat{u}^{(n)}\|_\infty \leq \max_{i=1,\ldots,M} K_i, \qquad |D\hat{u}^{(n)}|(\Omega) \leq C,$$

for any n. Therefore there exists a subsequence $\{\hat{u}^{(n_k)}\}_k$ converging with respect to the componentwise BV weakly-$*$-topology to a function $u \in X$. Since the relaxed functional $\bar{F}$ is lower semicontinuous in X with respect to such a topology, we have

$$\bar{F}(u) \leq \liminf_k \bar{F}(u^{(n_k)}).$$

From the compactness and lower semicontinuity properties of $\bar{F}$ it follows that $u \in X$ is a minimizer of $\bar{F}$. Moreover, if $D \subsetneq \Omega$ and G_2 is a strictly convex functional, then $\bar{F}$ is strictly convex and the solution u is unique. Since F is coercive in X one concludes by an application of Theorem A.3. ■

6. Approximation by Γ-Convergence

In this section we endow the space X with the L^1 strong topology, and we show that minimizers of $\bar{F}$ can be approximated in X by minimum points of functionals that are defined in $W^{1,2}(\Omega;\mathbb{R}^M)$.

For a positive decreasing sequence $\{\varepsilon_h\}_{h\in\mathbb{N}}$ such that $\lim_{h\to\infty} \varepsilon_h = 0$, and for $\phi \in C^1(\mathbb{R})$, we define

$$F_h(u) = \begin{cases} G_1(u) + G_2(u) + \int_\Omega \sum_{i=1}^M \phi_h(|\nabla u_i(x)|)dx & u \in W^{1,2}(\Omega;\mathbb{R}^M) \\ +\infty, & u \in X \setminus W^{1,2}(\Omega;\mathbb{R}^M), \end{cases} \tag{19}$$

where

$$\phi_h(z) = \begin{cases} \dfrac{\phi'(\varepsilon_h)}{2\varepsilon_h} z^2 + \phi(\varepsilon_h) - \dfrac{\varepsilon_h \phi'(\varepsilon_h)}{2} & 0 \leq z \leq \varepsilon_h \\ \phi(z) & \varepsilon_h \leq z \leq 1/\varepsilon_h \\ \dfrac{\varepsilon_h \phi'(1/\varepsilon_h)}{2} z^2 + \phi(1/\varepsilon_h) - \dfrac{\phi'(1/\varepsilon_h)}{2\varepsilon_h} & z \geq 1/\varepsilon_h. \end{cases}$$

If $z \mapsto \frac{\phi'(z)}{z}$ is continuously decreasing, then $\phi_h(z) \geq \phi(z) \geq 0$ for any h and any z, and $\lim_h \phi_h(z) = \phi(z)$ for any z.

By means of standard arguments we have that for any h the functional F_h has a minimizer in $X \cap W^{1,2}(\Omega;\mathbb{R}^M)$, see, e.g., [29, Proposition 6.1]. Moreover, if $D \subsetneq \Omega$ and G_2 is a strictly convex functional then the minimizer is unique. The following theorem extends to our case the Γ-convergence result proved in [29, Proposition 6.1], see also Theorem 3.2.3 of [6].

Theorem 6.1. *Let $\{u^{(h)}\}_h$ be a sequence of minimizers of F_h. Then $\{u^{(h)}\}_h$ is relatively compact in $L^1(\Omega;\mathbb{R}^M)$, each of its limit points minimizes the functional $\bar{F}$, and*

$$\min_{u\in X} \bar{F}(u) = \lim_{h\to\infty} \min_{u\in X\cap W^{1,2}} F_h(u).$$

Moreover, if $D \subsetneq \Omega$ and G_2 is a strictly convex functional, we have

$$\lim_{h\to\infty} u^{(h)} = u^{(\infty)} \text{ in } X, \quad \lim_{h\to\infty} F_h(u^{(h)}) = \bar{F}(u^{(\infty)}), \tag{20}$$

where $u^{(\infty)}$ is the unique minimizer of $\bar{F}$ in X.

Proof. We define

$$g(u) = \begin{cases} F(u) & u \in X \cap W^{1,2}(\Omega;\mathbb{R}^M) \\ +\infty, & u \in X \setminus W^{1,2}(\Omega;\mathbb{R}^M). \end{cases}$$

Observe that g is the restriction of F to functions $u \in W^{1,2}(\Omega;\mathbb{R}^M)$.

By construction we have that $\{F_h\}_h$ is a decreasing sequence of functionals that converges pointwise to g in $X \cap W^{1,2}(\Omega;\mathbb{R}^M)$. Therefore, by Proposition A.1, F_h Γ-converges to the relaxed functional $\bar{g}$ of g in X with respect to the $L^1(\Omega;\mathbb{R}^M)$ topology. Then we have to show that $\bar{F} = \bar{g}$.

Let $\{u^{(n)}\}_n \subset X$ be a sequence such that $u^{(n)} \to u$ in $L^1(\Omega;\mathbb{R}^M)$ and $\liminf_n \bar{F}(u^{(n)}) < +\infty$. Up to the extraction of a subsequence we may assume that $\liminf_n \bar{F}(u^{(n)}) = \lim_n \bar{F}(u^{(n)})$. Then $\bar{F}(u^{(n)})$ is uniformly bounded with respect to n, so that $\{u^{(n)}\}_n$ is uniformly bounded in BV. Then, up to a subsequence, $u^{(n)}$ converges to u in the componentwise BV weak-$*$-topology and, by Theorem 5.1, we have $\liminf_n \bar{F}(u^{(n)}) \geq \bar{F}(u)$. Hence $\bar{F}$ is lower-semicontinuous in X with respect to the $L^1(\Omega;\mathbb{R}^M)$ topology.

Then, arguing as in the proof of Theorem 5.1, for any function $u \in X$ there exists a sequence of truncated functions $\hat{u}^{(n)} \in W^{1,2}(\Omega;\mathbb{R}^M) \cap X$ such that

$$\hat{u}^{(n)} \to u \text{ in } L^1(\Omega;\mathbb{R}^M), \quad \text{and } \bar{F}(u) \geq \liminf_{n\to\infty} g(\hat{u}^{(n)}). \tag{21}$$

Since $g \geq \bar{F}$, property (21) implies that $\bar{F} \geq \bar{g}$. Then, by the lower-semicontinuity of $\bar{F}$ with respect to the $L^1(\Omega;\mathbb{R}^M)$ topology, we have $\bar{F} = \bar{g}$. Therefore F_h Γ-converges to $\bar{F}$.

By construction $\phi_h(z) \geq \phi(z)$ for any $z \geq 0$, so that $F_h(u) \geq \bar{F}(u)$ for any h and any $u \in X$. Since $\bar{F}$ is coercive and lower semicontinuous in $L^1(\Omega;\mathbb{R}^M)$, it follows that the sequence $\{F_h\}_h$ is equi-coercive in $L^1(\Omega;\mathbb{R}^M)$. In particular, any family $\{u^{(h)}\}_h$ of minimizers of F_h is relatively compact in $L^1(\Omega;\mathbb{R}^M)$. Then, using Theorem A.2, the limit points of sequences of minimizers of F_h minimize $\bar{F}$ and $\min_{u\in X} \bar{F}(u) = \lim_h \min_{u\in W^{1,2}} F_h(u)$.

Finally, if $D \subsetneq \Omega$ and G_2 is a strictly convex functional, by Theorem 5.2 there exists a unique minimizer of $\bar{F}$ in X. Therefore the limits (20) follow from Corollary 7.24 of [25]. ∎

Remark 6.2. *So far we have considered evaluation maps $\mathcal{L} : \mathbb{R}^M \to \mathbb{R}$. However the whole analysis can be generalized to the case $\mathcal{L} : \mathbb{R}^M \to \mathcal{M}$, $\mathcal{L}(x) = (\mathcal{L}_1(x), \ldots, \mathcal{L}_D(x))$, where $\mathcal{M} \subset \mathbb{R}^M$ is a $D \leq M$ dimensional submanifold.*

In certain problems, as that of the reconstruction of the colors of an image from the colors of some fragments and the gray levels of the blanks, we can model the evaluation map by

$$\mathcal{L}(r,g,b) := L(\alpha(0 \vee r \wedge 255) + \beta(0 \vee g \wedge 255) + \gamma(0 \vee b \wedge 255)),$$

where $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma = 1$ and $L : [0, 255] \to [0, 255]$ is an increasing continuous function such that $\mathrm{ran}(\bar{v}|D) \subset L([0, 255])$. The evaluation map $\mathcal{L}$ satisfies the condition (L3-b) of Section 4. Hence, in this case, instead of minimizing directly (1) we may minimize

$$\begin{aligned} F^*(u) &= \mu \int_{\Omega \setminus D} |u(x) - \bar{u}(x)|^2 dx \\ &+ \lambda \int_D |\left(\alpha(0 \vee u_1(x) \wedge 255) + \beta(0 \vee u_2(x) \wedge 255) + \gamma(0 \vee u_3(x) \wedge 255)\right) - L^{-1}(\bar{v}(x))|^2 dx \\ &+ \sum_{i=1}^M \int_\Omega \phi(|\nabla u_i(x)|) dx, \end{aligned}$$

since L is invertible on $L([0, 255])$. If L is also differentiable (or at least a Lipschitz function), then certainly we have

$$\begin{aligned} |\mathcal{L}(u(x)) - \bar{v}(x)| &= |L(\alpha(0 \vee u_1(x) \wedge 255) + \beta(0 \vee u_2(x) \wedge 255) + \gamma(0 \vee u_3(x) \wedge 255)) - \bar{v}(x)| \\ &= |L(\alpha(0 \vee u_1(x) \wedge 255) + \beta(0 \vee u_2(x) \wedge 255) + \gamma(0 \vee u_3(x) \wedge 255)) - L(L^{-1}(\bar{v}(x)))| \\ &\leq C|\alpha(0 \vee u_1(x) \wedge 255) + \beta(0 \vee u_2(x) \wedge 255) + \gamma(0 \vee u_3(x) \wedge 255) - (L^{-1}(\bar{v}(x))|, \end{aligned}$$

for a suitable constant $C > 0$. Therefore we have

$$F(u) \leq \max\{1, C^2\} F^*(u),$$

where F^* is convex on X.

7. Euler-Lagrange Equations and a Relaxation Algorithm

In this section we want to provide an algorithm to compute efficiently minimizers of the approximating functionals F_h. First, we want to derive the Euler-Lagrange equations associated to F_h. In the following we assume that both ϕ_h and $\mathcal{L}$ are continuously differentiable, and that Ω is an open, bounded, and connected subset of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. Moreover $p = 2$ if $N = 1$ and $p = \frac{N}{N-1}$ for $N > 1$, $1/p + 1/p' = 1$. By standard arguments we have the following result.

Proposition 7.1. *If u is a minimizer in $W^{1,2}(\Omega; \mathbb{R}^M)$ of F_h, then u solves the following system of Euler-Lagrange equations*

$$\tag{22} \begin{cases} 0 = -\mathrm{div}\left(\frac{\phi_h'(|\nabla u_i|)}{|\nabla u_i|} \nabla u_i\right) + p|u - \bar{u}|^{p-2}(u_i - \bar{u}_i) 1_{\Omega \setminus D} + p|\mathcal{L}(u) - \bar{v}|^{p-2}(\mathcal{L}(u) - \bar{v}) \frac{\partial \mathcal{L}}{\partial x_i}(u) 1_D, \\ \frac{\phi_h'(|\nabla u_i|)}{|\nabla u_i|} \frac{\partial u_i}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad i = 1, ..., M. \end{cases}$$

The former equalities hold in the sense of distributions and in $L^{p'}(\Omega; \mathbb{R}^M)$.

Equations (22) yield a necessary condition for the computation of minimizers of F_h. Again we are not ensured of the uniqueness in general, unless G_2 is strictly convex. The system (22) is composed by M second order *nonlinear* equations which are coupled on terms of order 0. Both the nonlinear term $\mathrm{div}\left(\frac{\phi_h'(|\nabla u_i|)}{|\nabla u_i|} \nabla u_i\right)$ and the coupled terms of order 0 constitute a complication for the numerical solution of these equations.

Based on the work [12, 18, 30], we propose in the following a method to compute efficiently solutions of (22), which simplifies the problem of the nonlinearity. Since we want to illustrate concrete applications for color image recovery, for simplicity, we limit our analysis to the case $N = p = 2$ and $\phi(t) = |t|$, for all $t \in \mathbb{R}$. Let us introduce a new functional given by

$$\tag{23} \mathcal{E}_h(u, v) := 2\left(G_1(u) + G_2(u)\right) + \int_\Omega \sum_{i=1}^M \left(v_i |\nabla u_i(x)|^2 + \frac{1}{v_i}\right) dx,$$

where $u \in W^{1,2}(\Omega;\mathbb{R}^M)$, and $v \in L^2(\Omega;\mathbb{R}^M)$ is such that $\varepsilon_h \le v_i \le \frac{1}{\varepsilon_h}$, $i = 1,\dots,M$. While the variable u is again the function to be reconstructed, we call the variable v the *gradient weight*. In the following, since we assume h fixed, we drop the index h from the functional $\mathcal{E}_h$.

For any given $u^{(0)} \in X \cap W^{1,2}(\Omega;\mathbb{R}^M)$ and $v^{(0)} \in L^2(\Omega;\mathbb{R}^M)$ (for example $v^{(0)} := 1$), we define the following iterative double-minimization algorithm:

$$
\left\{ \begin{array}{l} u^{(n+1)} = \arg \min\limits_{u \in W^{1,2}(\Omega;\mathbb{R}^M)} \mathcal{E}(u, v^{(n)}) \\ v^{(n+1)} = \arg\min_{\varepsilon_h \le v \le \frac{1}{\varepsilon_h}} \mathcal{E}(u^{(n+1)}, v). \end{array} \right. \tag{24}
$$

We have the following convergence result.

Theorem 7.2. *The sequence $\{u^{(n)}\}_{n\in\mathbb{N}}$ has subsequences that converge strongly in $L^2(\Omega;\mathbb{R}^M)$ and weakly in $W^{1,2}(\Omega;\mathbb{R}^M)$ to a stationary point $u^{(\infty)}$ of F_h, i.e., $u^{(\infty)}$ solves* (22). *Moreover, if F_h has a unique minimizer u^*, then $u^{(\infty)} = u^*$ and the full sequence $\{u^{(n)}\}_{n\in\mathbb{N}}$ converges to u^*.*

Proof. Observe that

$$
\begin{aligned}
\mathcal{E}(u^{(n)}, v^{(n)}) - \mathcal{E}(u^{(n+1)}, v^{(n+1)}) \quad &= \quad \underbrace{\Big(\mathcal{E}(u^{(n)}, v^{(n)}) - \mathcal{E}(u^{(n+1)}, v^{(n)})\Big)}_{A_n} \\
&+ \quad \underbrace{\Big(\mathcal{E}(u^{(n+1)}, v^{(n)}) - \mathcal{E}(u^{(n+1)}, v^{(n+1)})\Big)}_{B_n} \ge 0.
\end{aligned}
$$

Therefore $\mathcal{E}(u^{(n)}, v^{(n)})$ is a nonincreasing sequence and moreover it is bounded from below, since

$$
\inf_{\varepsilon_h \le v \le 1/\varepsilon_h} \int_\Omega \sum_{i=1}^M \left(v_i |\nabla u_i(x)|^2 + \frac{1}{v_i} \right) dx \ge 0.
$$

This implies that $\mathcal{E}(u^{(n)}, v^{(n)})$ converges. Moreover, we can write

$$
B_n = \int_\Omega \sum_{i=1}^M c(v_i^{(n)}(x), |\nabla u_i^{(n+1)}(x)|) - c(v_i^{(n+1)}(x), |\nabla u_i^{(n+1)}(x)|) dx,
$$

where

$$
c(z,w) := zw^2 + \frac{1}{z}.
$$

By Taylor's formula, we have

$$
c(v_i^{(n)}, w) = c(v_i^{(n+1)}, w) + \frac{\partial c}{\partial z}(v_i^{(n+1)}, w)(v_i^{(n)} - v_i^{(n+1)}) + \frac{1}{2}\frac{\partial^2 c}{\partial z^2}(\xi, w)|v_i^{(n)} - v_i^{(n+1)}|^2,
$$

for $\xi \in \mathrm{conv}(v_i^{(n)}, v_i^{(n+1)})$. By definition of $v_i^{(n+1)}$, and taking into account that $\varepsilon_h \le v_i^{(n+1)} \le \frac{1}{\varepsilon_h}$, we have

$$
\frac{\partial c}{\partial z}(v_i^{(n+1)}, |\nabla u_i^{(n+1)}(x)|)(v_i^{(n)} - v_i^{(n+1)}) \ge 0,
$$

and $\frac{\partial^2 c}{\partial z^2}(z,w) = \frac{2}{z^3} \ge 2\varepsilon_h^3$, for any $z \le 1/\varepsilon_h$. This implies that

$$
\mathcal{E}(u^{(n)}, v^{(n)}) - \mathcal{E}(u^{(n+1)}, v^{(n+1)}) \ge B_n \ge \varepsilon_h^3 \int_\Omega \sum_{i=1}^M |v_i^{(n)}(x) - v_i^{(n+1)}(x)|^2 dx,
$$

and since $\mathcal{E}(u^{(n)}, v^{(n)})$ is convergent, we have $\sum_{i=1}^M \int_\Omega |v_i^{(n)}(x) - v_i^{(n+1)}(x)|^2 dx \to 0$ for $n \to \infty$. In fact it holds

$$
\|v_i^{(n)} - v_i^{(n+1)}\|_{L^q} \to 0, \quad i = 1, ..., M, \tag{25}
$$

for $n \to \infty$, for any $1 \le q < \infty$. Since $u^{(n+1)}$ is a minimizer of $\mathcal{E}(u, v^{(n)})$ it solves the following system of variational equations

$$\begin{aligned} &\int_\Omega \Big(v_i^{(n)} \nabla u_i^{(n+1)}(x) \cdot \nabla \varphi_i(x) + 2(u_i^{(n+1)}(x) - \bar{u}_i(x)) 1_{\Omega \setminus D}(x) \\ &+ \quad 2(\mathcal{L}(u^{(n+1)}(x)) - \bar{v}(x)) \frac{\partial \mathcal{L}}{\partial x_i}(u^{(n+1)}(x)) 1_D(x) \Big) \varphi_i(x) dx = 0 \end{aligned}$$

for $i = 1, ..., M$, for all $\varphi \in W^{1,2}(\Omega; \mathbb{R}^M)$. Therefore we can write

$$\begin{aligned} & \int_\Omega \Big(v_i^{(n+1)} \nabla u_i^{(n+1)}(x) \cdot \nabla \varphi_i(x) \\ + \quad & 2(u_i^{(n+1)}(x) - \bar{u}_i(x)) 1_{\Omega \setminus D}(x) + 2(\mathcal{L}(u^{(n+1)}(x)) - \bar{v}(x)) \frac{\partial \mathcal{L}}{\partial x_i}(u^{(n+1)}(x)) 1_D(x) \Big) \varphi_i(x) dx \\ = \quad & \int_\Omega (v_i^{(n+1)} - v_i^{(n)}) \nabla u_i^{(n+1)}(x) \cdot \nabla \varphi_i(x) dx. \end{aligned}$$

For $\frac{1}{q} + \frac{1}{q'} + \frac{1}{2} = 1$, we have

$$\begin{aligned} & \Big| \int_\Omega \Big(v_i^{(n+1)} \nabla u_i^{(n+1)}(x) \cdot \nabla \varphi_i(x) \\ + \quad & 2(u_i^{(n+1)}(x) - \bar{u}_i(x)) 1_{\Omega \setminus D}(x) + 2(\mathcal{L}(u^{(n+1)}(x)) - \bar{v}(x)) \frac{\partial \mathcal{L}}{\partial x_i}(u^{(n+1)}(x)) 1_D(x) \Big) \varphi_i(x) dx \Big| \\ \le \quad & \|v_i^{(n+1)} - v_i^{(n)}\|_{L^q} \|\nabla u_i^{(n+1)}\|_{L^{q'}} \|\nabla \varphi_i\|_{L^2}. \end{aligned}$$

Since $u^{(n+1)}$ is a minimizers of $\mathcal{E}(u, v^{(n)})$, we may assume without loss of generality that $\hat{u}_i^{(n+1)} = u_i^{(n+1)}$, for all $i = 1, ..., M$ where $\hat{\cdot}$ is the truncation operator. Consequently $\|u_i^{(n+1)}\|_\infty \le C < +\infty$ uniformly with respect to n. We can use the results in [26] to show that there exists $q' > 2$ such that

$$\|\nabla u_i^{(n+1)}\|_{L^{q'}} \le C < +\infty$$

uniformly with respect to n (see also [4, 5, 12] for similar arguments). Therefore, using (25), we can conclude that

$$-\mathrm{div}(v_i^{(n+1)} \nabla u_i^{(n+1)}) + 2\left((u_i^{(n+1)} - \bar{u}_i) 1_{\Omega \setminus D} + (\mathcal{L}(u^{(n+1)}) - \bar{v}) \frac{\partial \mathcal{L}}{\partial x_i}(u^{(n+1)}) 1_D \right) \to 0,$$

for $n \to \infty$, in $(W^{1,2}(\Omega; \mathbb{R}^M))'$. This also shows that $\{u^{(n)}\}_n$ is uniformly bounded in $W^{1,2}(\Omega; \mathbb{R}^M)$. Therefore there exists a subsequence $\{u^{(n_k)}\}_k$ that converges strongly in L^2 and weakly in $W^{1,2}(\Omega; \mathbb{R}^M)$ to a function $u^{(\infty)} \in W^{1,2}(\Omega; \mathbb{R}^M)$. Since $v_i^{(n+1)} = \frac{\phi_h'(|\nabla u_i^{(n+1)}|)}{|\nabla u_i^{(n+1)}|}$, with standard arguments for monotone operators (see the proof of [12, Proposition 3.1] and [10]), we show that in fact

$$-\mathrm{div}\left(\frac{\phi_h'(|\nabla u_i^{(\infty)}|)}{|\nabla u_i^{(\infty)}|} \nabla u_i^{(\infty)} \right) + 2\left((u_i^{(\infty)} - \bar{u}_i) 1_{\Omega \setminus D} + (\mathcal{L}(u^{(\infty)}) - \bar{v}) \frac{\partial \mathcal{L}}{\partial x_i}(u^{(\infty)}) 1_D \right) = 0, \tag{26}$$

for $i = 1, ..., M$, in $(W^{1,2}(\Omega; \mathbb{R}^M))'$. The latter are the Euler-Lagrange equations associated to the functional F_h and therefore $u^{(\infty)}$ is a stationary point for F_h.

Assume now that F_h has a unique minimizer u^*. Then necessarily $u^{(\infty)} = u^*$. Since every subsequence of $\{u^{(n)}\}_n$ has a subsequence converging to u^*, the full sequence $\{u^{(n)}\}_n$ converges to u^*. ■

Since both F_h and $\mathcal{E}_h(\cdot, v)$ admit minimizers, their uniqueness is equivalent to the uniqueness of the solutions of the corresponding Euler-Lagrange equations. If uniqueness of the solution is satisfied, then the algorithm (24) can be equivalently reformulated as the following two-step iterative procedure:

- Find $u^{(n+1)}$ which solves

$$\begin{aligned}&\int_\Omega \Big(v_i^{(n)}(x) \nabla u_i^{(n+1)}(x) \cdot \nabla \varphi_i(x) + 2(u_i^{(n+1)}(x) - \bar{u}_i(x)) 1_{\Omega \setminus D}(x) \\ + \quad & 2(\mathcal{L}(u^{(n+1)}(x)) - \bar{v}(x)) \frac{\partial \mathcal{L}}{\partial x_i}(u^{(n+1)}(x)) 1_D(x) \Big) \varphi_i(x) dx = 0\end{aligned}$$

  for $i = 1, ..., M$, for all $\varphi \in W^{1,2}(\Omega; \mathbb{R}^M)$.
- Compute directly $v^{(n+1)}$ by

$$v_i^{(n+1)} = \varepsilon_h \vee \frac{1}{|\nabla u_i^{(n+1)}|} \wedge \frac{1}{\varepsilon_h}, \quad i = 1, \ldots, M.$$

There are cases for which one can ensure uniqueness of solutions:

1. If G_2 is strictly convex then the minimizers are unique as well as the solutions of the equations.
2. Modify the equations by inserting again the parameters $\lambda, \mu > 0$

$$\begin{aligned}&\int_\Omega \Big(v_i^{(n)} \nabla u_i^{(n+1)}(x) \cdot \nabla \varphi_i(x) + 2\mu(u_i^{(n+1)}(x) - \bar{u}_i(x)) 1_{\Omega \setminus D}(x) \\ + \quad & 2\lambda(\mathcal{L}(u^{(n+1)}(x)) - \bar{v}(x)) \frac{\partial \mathcal{L}}{\partial x_i}(u^{(n+1)}(x)) 1_D(x) \Big) \varphi_i(x) dx = 0\end{aligned}$$

for $i = 1, ..., M$, for all $\varphi \in W^{1,2}(\Omega; \mathbb{R}^M)$. By a standard fixed point argument, it is not difficult to show that for $\mu \sim \lambda \sim \varepsilon_h$ the solution of the previous equations is unique. Unfortunately the condition $\mu \sim \lambda \sim \varepsilon_h$ is acceptable only for those applications where the constraints on the data are weak. For example, when the data are affected by a strong noise.

3. In the following section we illustrate the finite element approximation of the Euler-Lagrange equations. Since we are interested in color image applications, we restrict the numerical experiments to the case $\mathcal{L}(u_1, u_2, u_3) = \frac{1}{3}(u_1 + u_2 + u_3)$. By this choice, the numerical results confirm that the linear systems arising from the finite element discretization are uniquely solvable for a rather large set of possible parameters λ, μ.

8. Numerical Implementation and Results

In this section we want to present the numerical implementation of the iterative double-minimization algorithm (24) for color image restoration. As the second step of the scheme (which amounts in the up-date of the gradient weight) can be explicitly done once $u^{(n+1)}$ is computed, we are left essentially to provide a numerical implementation of the first step, i.e., the solution of the Euler-Lagrange equations.

8.1. **Finite element approximation of the Euler-Lagrange equations.** For the solution of the Euler-Lagrange equations we use a finite element approximation. We illustrate the implementation with the concrete aim of the reconstruction of a digital color image supported in $\Omega = [0, 1]^2$ from few color fragments supported in $\Omega \setminus D$ and the gray level information where colors are missing. By the nature of this problem, we can choose a regular triangulation $\mathcal{T}$ of the domain Ω with nodes distributed on a regular grid $\mathcal{N} := \tau\mathbb{Z}^2 \cap \Omega$, corresponding to the pixels of the image. Associated to $\mathcal{T}$ we fix the following finite element spaces:

$$\begin{aligned}\mathcal{U} &= \{u \in C^0(\Omega) : u|T \in \mathbb{P}^1, T \in \mathcal{T}\}, \\ \mathcal{V} &= \{v \in L^2(\Omega) : v|T \in \mathbb{P}^0, T \in \mathcal{T}\}.\end{aligned}$$

The space $\mathcal{U}$ induces the finite element space of color images given by

$$U := \{u \in W^{1,2}(\Omega, \mathbb{R}^3) : u_i \in \mathcal{U}, i = 1, 2, 3\}.$$

The space $\mathcal{V}$ induces the finite element space of *gradient weights* given by

$$V := \{v \in L^2(\Omega, \mathbb{R}^3) : v_i \in \mathcal{V}, i = 1, 2, 3\}.$$

In order to avoid the nonlinearity in the coupled terms of order 0, we restrict our functional to the case $\mathcal{L}(u_1,u_2,u_3)=\frac{1}{3}(u_1+u_2+u_3)$. For further simplicity we have not considered truncations which in fact are not necessary in practice.

For a given $v^{(n)}\in V$, the first step of our approximation of the double-minimization scheme amounts in the computation of $u^{(n+1)}\in U$ which solves

$$\begin{aligned}(27)\qquad &\int_\Omega \Big(v_i^{(n)}(x)\nabla u_i^{(n+1)}(x)\cdot\nabla\varphi_i(x)+2\mu(u_i^{(n+1)}(x)-\bar u_i(x))1_{\Omega\setminus D}(x)\\ &+\frac{2}{3}\lambda(\frac{1}{3}(u_1^{(n+1)}(x)+u_2^{(n+1)}(x)+u_3^{(n+1)}(x))-\bar v(x))1_D(x)\Big)\varphi_i(x)dx=0\end{aligned}$$

for $i=1,2,3$, for all $\varphi\in U$. To the spaces $\mathcal{U}$ and $\mathcal{V}$ are attached the corresponding nodal bases $\{\varphi_k\}_{k\in\mathcal{N}}$ and $\{\chi_k\}_{k\in\mathcal{N}}$ respectively. Therefore, we have also that

$$U=\left\{u:u=\left(\sum_{k\in\mathcal{N}}u_{i,k}\varphi_k\right)_{i=1,2,3}\right\},\quad V=\left\{v:v=\left(\sum_{k\in\mathcal{N}}v_{i,k}\chi_k\right)_{i=1,2,3}\right\}.$$

With these bases we can construct the following matrices:

$$(28)\qquad \mathbf{K}_i^{(n+1)} := \left(\int_\Omega v_i^{(n)}(x)\nabla\varphi_k(x)\cdot\nabla\varphi_h(x)dx\right)_{k,h\in\mathcal{N}},$$

$$(29)\qquad \mathbf{M}_{\Omega\setminus D} := \left(2\mu\int_\Omega 1_{\Omega\setminus D}(x)\varphi_k(x)\varphi_h(x)dx\right)_{k,h\in\mathcal{N}},$$

$$(30)\qquad \mathbf{M}_D := \left(\frac{2\lambda}{9}\int_\Omega 1_D(x)\varphi_k(x)\varphi_h(x)dx\right)_{k,h\in\mathcal{N}}.$$

By these building blocks, we can assemble

$$(31)\quad \mathbf{K}^{(n+1)}:=\begin{pmatrix}\mathbf{K}_1^{(n+1)}+\mathbf{M}_{\Omega\setminus D}+\mathbf{M}_D & \mathbf{M}_D & \mathbf{M}_D\\ \mathbf{M}_D & \mathbf{K}_2^{(n+1)}+\mathbf{M}_{\Omega\setminus D}+\mathbf{M}_D & \mathbf{M}_D\\ \mathbf{M}_D & \mathbf{M}_D & \mathbf{K}_3^{(n+1)}+\mathbf{M}_{\Omega\setminus D}+\mathbf{M}_D\end{pmatrix},$$

and

$$(32)\qquad \mathbf{M}:=\begin{pmatrix}\mathbf{M}_{\Omega\setminus D}+\mathbf{M}_D & \mathbf{M}_D & \mathbf{M}_D\\ \mathbf{M}_D & \mathbf{M}_{\Omega\setminus D}+\mathbf{M}_D & \mathbf{M}_D\\ \mathbf{M}_D & \mathbf{M}_D & \mathbf{M}_{\Omega\setminus D}+\mathbf{M}_D\end{pmatrix}.$$

Furthermore, let us denote the vector of the nodal values of the solution by

$$(33)\qquad \mathbf{u}^{(n+1)}=(u_{1,k_1}^{(n+1)},...,u_{1,k_{\#\mathcal{N}}}^{(n+1)},u_{2,k_1}^{(n+1)},...,u_{2,k_{\#\mathcal{N}}}^{(n+1)},u_{3,k_1}^{(n+1)},...,u_{3,k_{\#\mathcal{N}}}^{(n+1)})^T$$

assembled as a column vector containing the nodal values of each channel in order, where $k_i\in\mathcal{N}$ are nodes which are suitably ordered. In a similar way the nodal values of the data $\bar u,\bar v$ are assembled in the vector

(34)

$$\bar{\mathbf{u}}=(\bar u_{1,k_1},...,\bar u_{1,k_j},\bar v_{1,k_{j+1}},...,\bar v_{1,k_{\#\mathcal{N}}},\bar u_{2,k_1},...,\bar u_{2,k_j},\bar v_{2,k_{j+1}},...,\bar v_{2,k_{\#\mathcal{N}}},\bar u_{3,k_1},...,\bar u_{3,k_j},\bar v_{3,k_{j+1}},...,\bar v_{3,k_{\#\mathcal{N}}})^T.$$

For the right-hand side we have the additional requirement that $\bar v_{i,k}=\bar v_{\ell,k}$ for $i\neq\ell$, representing the gray level values. Moreover, the order of the nodes $\{k_l : l=1,\dots,\#\mathcal{N}\}$ is such that

$$(\mathbf{M}_{\Omega\setminus D}+\mathbf{M}_D)(\bar u_{i,k_1},...,\bar u_{i,k_j},\bar v_{i,k_{j+1}},...,\bar v_{i,k_{\#\mathcal{N}}})^T=\mathbf{M}_{\Omega\setminus D}\begin{pmatrix}\bar{\mathbf{u}}_{\mathbf{i}}\\0\end{pmatrix}+\mathbf{M}_D\begin{pmatrix}0\\ \bar{\mathbf{v}}_{\mathbf{i}}\end{pmatrix}.$$

With this notations and conventions, the solution of the system of equations (27) is equivalent to the solution of the following algebraic linear system

$$(35)\qquad \mathbf{K}^{(n+1)}\mathbf{u}^{(n+1)}=\mathbf{M}\bar{\mathbf{u}}.$$

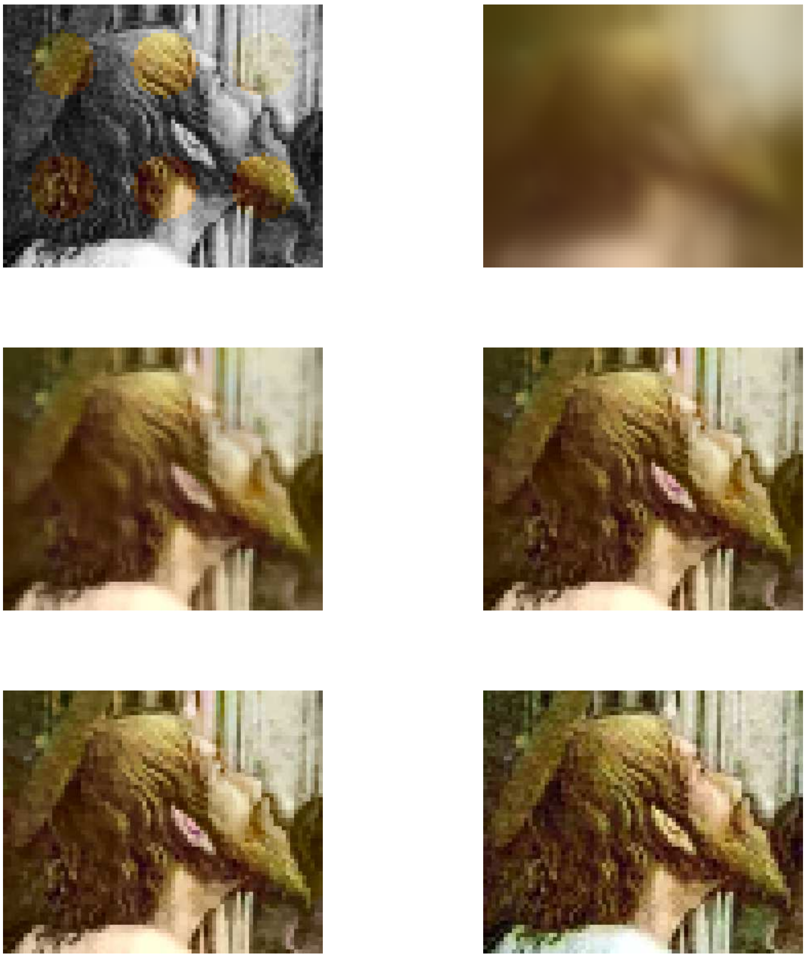

FIGURE 4. The datum $(\bar{u}, \bar{v})$ is illustrated on the top-left position. The first five iterations of the algorithms are listed from left to right, starting from the first row. The original color image (A. Mantegna's frescoes) to be reconstructed is illustrated on the right-bottom position. The parameters we have used are $\varepsilon_h = 10^{-4}$, $\lambda = \mu = 150$.

8.2. **Numerical implementation of the double-minimization algorithm.** We have now all the ingredients to assemble our numerical scheme into the following algorithm.

Algorithm 1. DOUBLE_MINIMIZATION

Input:	*Data vector* $\bar{\mathbf{u}}$, $\varepsilon_h > 0$, *initial gradient weight* $v^{(0)}$ *with* $\varepsilon_h \leq v^{(0)}_{i,k} \leq 1/\varepsilon_h$, *number* $n_{\max}$ *of outer iterations.*
Parameters:	*positive weights* $\lambda, \mu \geq 0$.
Output:	*Approximation* u^* *of the minimizer of* F_h

$\mathbf{u}^{(0)} := 0$;
$\mathbf{f} := \mathbf{M}\bar{\mathbf{u}}$;
`for` $n := 0$ `to` $n_{\max}$ `do`

Assemble the matrix $\mathbf{K}^{(n+1)}$ *as in* (28)*;*
Compute $\mathbf{u}^{(n+1)}$ *such that* $\mathbf{K}^{(n+1)}\mathbf{u}^{(n+1)} := \mathbf{f}$*;*
Assemble the solution $u^{(n+1)} = (\sum_{k\in\mathcal{N}} u_{i,k}^{(n+1)}\varphi_k)_{i=1,2,3}$*;*
Compute the gradient $\nabla u^{(n+1)} = (\sum_{k\in\mathcal{N}} u_{i,k}^{(n+1)}\nabla\varphi_k)_{i=1,2,3}$*;*
$v_i^{(n+1)} := \varepsilon_h \vee \frac{1}{|\nabla u_i^{(n+1)}|} \wedge \frac{1}{\varepsilon_h}, \quad i = 1, \ldots, M$*;*
`endfor`
$u^* := u^{(n+1)}$.

8.3. **Numerical experiments in color image restoration and results.** In this section we show numerical results dealing with applications of the algorithm to color image restoration. We assume as in Figure 1 to have at disposal few color fragments of the image and the gray levels of the missing parts.

The algorithm converges to a stationary situation in a very limited number of iterations. In our numerical tests 3-4 iterations are sufficient, see Figure 4 and Figure 7. The quality of the reconstruction increases for increasing amount of correct color information of the datum. Nevertheless we observe that the geometrical distribution of the color datum is more crucial for a better reconstruction. A remarkable result is illustrated in Figure 5 and Figure 6. In the bottom-left positions we illustrate data with only the 3% of original color information, randomly distributed. From this very limited complete information the algorithm anyway produces a rather good reconstruction of the original color images. Let us emphasize this once more:

It is sufficient to have a very limited guess of possible colors which are nicely distributed in the image to re-color all the image.

This result has a significant impact for several possible applications. Besides the problem of the restoration of the fresco colors (where we dispose of the 8% of the total color surface), we can use this algorithm in old black and white video and image restoration, and for extreme compression of color images.

To conclude, in Figure 7 we show the history of the residual error with respect to the original color image for increasing choices of the parameters λ, μ. These numerical results confirm the regularization effect due to the total variation constraint.

Appendix A

This appendix collects several useful concepts and results on Γ-convergence from [25]. In the following X is a topological space and $(F_h)_{h\in\mathbb{N}}$ is a sequence of functions from X to $\bar{\mathbb{R}}$. We denote $\mathcal{N}(x)$ the set of all open neighborhoods of x in X.

Definition 2. The Γ-lower-limit and the Γ-lower-limit of a sequence $(F_h)_{h\in\mathbb{N}}$ are the functions from X to $\bar{\mathbb{R}}$ defined by

$$\begin{aligned}\left(\Gamma - \liminf_{h\to\infty} F_h\right)(x) &= \sup_{U\in\mathcal{N}(x)} \liminf_{h\to\infty} \inf_{y\in U} F_h(y)\\ \left(\Gamma - \limsup_{h\to\infty} F_h\right)(x) &= \sup_{U\in\mathcal{N}(x)} \limsup_{h\to\infty} \inf_{y\in U} F_h(y)\end{aligned}$$

If there exists a function $F : X \to \bar{\mathbb{R}}$ such that

$$\left(\Gamma - \liminf_{h\to\infty} F_h\right)(x) = F(x) = \left(\Gamma - \limsup_{h\to\infty} F_h\right)(x),$$

for all $x \in X$, then we write $F = \Gamma - \lim_{h\to\infty} F_h$ and we say that the sequence $(F_h)_{h\in\mathbb{N}}$ Γ-converges to F.

Proposition A.1. *(*[25, Proposition 5.7] *If* $(F_h)_{h\in\mathbb{N}}$ *is a decreasing sequence converging to* $F : X \to \bar{\mathbb{R}}$ *pointwise, then* $(F_h)_{h\in\mathbb{N}}$ Γ*-converges to the relaxed function* $\bar{F}$*.*

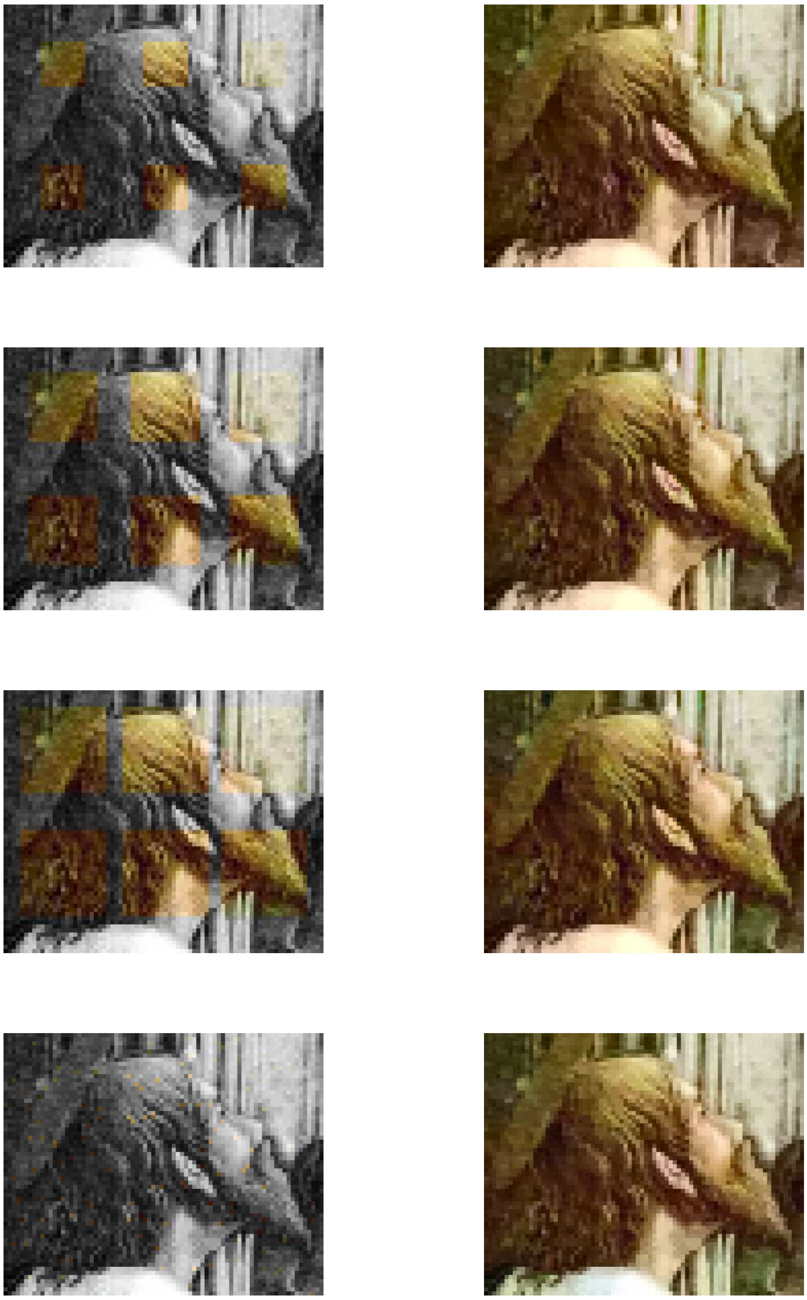

FIGURE 5. The first column illustrates a sequence of different data. The second column illustrates the corresponding 10^{th} iteration of the algorithm. The parameters we have used are $\varepsilon_h = 10^{-4}$, $\lambda = \mu = 150$. In the bottom-left position we illustrate a datum with only the 3% of original color information, randomly distributed.

FIGURE 6. The first column illustrates a sequence of different data. The second column illustrates the corresponding 10^{th} iteration of the algorithm. The parameters we have used are $\varepsilon_h = 10^{-4}$, $\lambda = \mu = 150$. In the bottom-left position we illustrate a datum with only the 3% of original color information, randomly distributed.

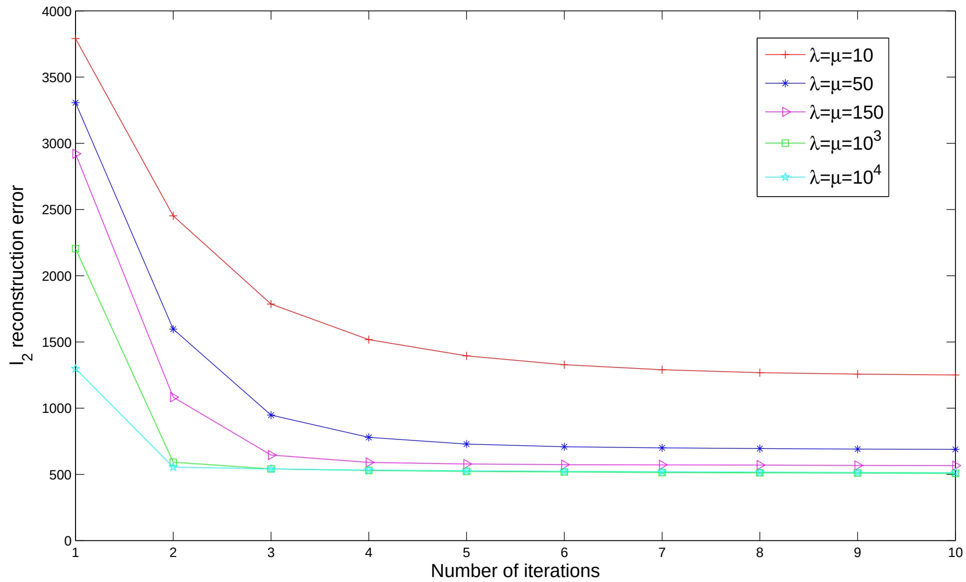

FIGURE 7. History of the $\ell^2(\mathcal{N})$-error with respect to the original color image, and for different values of the parameters λ, μ

Definition 3. We say that a function $F : X \to \bar{\mathbb{R}}$ is coercive on X, if the closure of the set $\{F \leq t\}$ is countably compact (for example if it is sequentially compact) in X for every $t \in \mathbb{R}$. A sequence $(F_h)_{h\in\mathbb{N}}$ is called equi-coercive if there exists a lower-semicontinuous coercive function $F : X \to \bar{\mathbb{R}}$ such that $F_h \geq F$ on X for every $h \in \mathbb{N}$.

Theorem A.2. *([25, Theorem 7.8]) Suppose that $(F_h)_{h\in\mathbb{N}}$ is equi-coercive in X and it Γ-converges to a function $F : X \to \bar{\mathbb{R}}$ in X, then F is coercive and*

$$\min_{x\in X} F(x) = \lim_{h\to\mathbb{N}} \inf_{x\in X} F_h(x).$$

Moreover, the limit points of sequences of minimizers of F_h minimize F.

Theorem A.3. *([25, Theorem 3.8]) Assume that $F : X \to \bar{\mathbb{R}}$ is coercive in X and that $\bar{F}$ is its relaxed functional. Then the following properties hold:*

(i) *$\bar{F}$ is coercive and lower-semicontinuous;*
(ii) *$\bar{F}$ has a minimum point;*
(iii) $\min_{x\in X} \bar{F}(x) = \inf_{x\in X} F(x)$.

Putting together the previous results we obtain the following:

Theorem A.4. *If $(F_h)_{h\in\mathbb{N}}$ is a decreasing sequence converging to a coercive function $F : X \to \bar{\mathbb{R}}$ pointwise, then*

$$\inf_{x\in X} F(x) = \min_{x\in X} \bar{F}(x) = \lim_{h\to\infty} \inf_{x\in X} F_h(x).$$

Proof. Since $F \geq \bar{F}$ it is also $F_h \geq F \geq \bar{F}$. By Theorem A.3 (i) $\bar{F}$ is coercive and lower-semicontinuous, therefore $(F_h)_{h\in\mathbb{N}}$ is equi-coercive. By Theorem A.1 $(F_h)_{h\in\mathbb{N}}$ Γ-converges to $\bar{F}$ and by Theorem A.2 it is

$$\min_{x\in X} \bar{F}(x) = \lim_{h\to\infty} \inf_{x\in X} F_h(x).$$

By Theorem A.3 (iii) it is also

$$\inf_{x \in X} F(x) = \min_{x \in X} \bar{F}(x).$$

$\blacksquare$

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