ORLICZ MEETS DOUBLE PHASE: STEADY FLUIDS

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Abstract. We revisit and refine the analysis of partial Hölder regularity for nondegenerate nonlinear elliptic systems in divergence form modeling stationary double-phase non-Newtonian fluids developed in [41]. The growth function is of the form $H(x,s)=s^p+\mu(x)s^q$, with $\frac{3n}{n+2}<p\le q$ and $\mu(\cdot)$ a nonnegative $C^{0,\alpha}$-continuous function for some $\alpha\in(0,1]$. Our main result establishes that the gradient $\nabla\mathbf{u}$ of any weak solution is locally Hölder continuous, except on a set of measure zero.

Dedicated to Gioconda Moscariello for her 70th birthday

1. Introduction

The problem and the main assumptions. This article deals with nondegenerate nonlinear elliptic systems in divergence form, modeling double-phase non-Newtonian fluids in the stationary case:

$$\begin{cases}\operatorname{div}\mathbf{a}(x,\boldsymbol{\varepsilon}(\mathbf{u}))+\nabla\pi=\mathbf{u}[\nabla\mathbf{u}]+\mathbf{f}\,,\\ \operatorname{div}\mathbf{u}=0\,,\quad\text{in }\Omega.\end{cases}\tag{1.1}$$

Here $\Omega\subseteq\mathbb{R}^n$, $n\ge 3$, is an open, bounded set representing the reference configuration of the fluid, the vector-valued map $\mathbf{u}:\Omega\to\mathbb{R}^n$ can be interpreted as the stationary velocity field of the fluid, and the scalar function $\pi:\Omega\to\mathbb{R}$ plays the role of the pressure. The stress tensor will have a double phase growth, as a function of the symmetric gradient $\boldsymbol{\varepsilon}(\mathbf{u})$.

For the nonlinear diffusion term $\mathbf{a}:\Omega\times\mathbb{R}^{n\times n}_{\mathrm{sym}}\to\mathbb{R}^{n\times n}_{\mathrm{sym}}$ we are considering a double phase growth condition of the type

$$H(x,s):=s^p+\mu(x)s^q\,,\tag{1.2}$$

for $\frac{3n}{n+2}+2\beta_1<p\le q$, for some $0<\beta_1<\frac{1}{n+2}$, where $\mu\in C^{0,\alpha}(\Omega)$ for some $\alpha\in(0,1]$, $0\le\mu\le L$ and

$$|\mu(x_1)-\mu(x_2)|\le L|x_1-x_2|^\alpha\,,\ \text{for every } x_1,x_2\in\Omega,\quad\text{and}\quad \frac{q}{p}\le 1+\frac{\alpha}{n}\,.\tag{1.3}$$

The lower bound for p is needed in order to take care of the convective term and apply the theory of monotone operators for the existence of weak solutions, in the case of the power law s^p. Notice that in the region $\{\mu=0\}$, we have $H(x,s)=s^p$ so that H has p-phase, while in the region $\{\mu>0\}$ the function H has (p,q)-phase. Moreover, for each $x\in\Omega$ such that $\mu(x)>0$, the function $s\to H(x,s)$ is an N-function (see Section 2.2) complying with (2.4) where $p_1=p$ and $p_2=q$. The precise assumptions are:

$$|\mathbf{a}(x,\boldsymbol{\xi})|+|D_\xi\mathbf{a}(x,\boldsymbol{\xi})|(1+|\boldsymbol{\xi}|)\le LH'(x,1+|\boldsymbol{\xi}|)\,,\tag{A1}$$

$$D_\xi\mathbf{a}(x,\boldsymbol{\xi})\boldsymbol{\lambda}:\boldsymbol{\lambda}\ge\nu H''(x,1+|\boldsymbol{\xi}|)|\boldsymbol{\lambda}|^2\,,\tag{A2}$$

for every $x\in\Omega$ and $\boldsymbol{\xi}\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$, $\boldsymbol{\lambda}\in\mathbb{R}^{n\times n}$ and for some $0<\nu\le L$, where $H'(x,t)$ and $H''(x,t)$ denote the first and second derivatives of $t\to H(x,t)$, respectively, and "$:$" denotes the standard Euclidean inner product on $\mathbb{R}^{n\times n}$.

Note that (A2) implies

$$(\mathbf{a}(x,\boldsymbol{\xi}_1)-\mathbf{a}(x,\boldsymbol{\xi}_2)):(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\ge\tilde\nu H''(x,1+|\boldsymbol{\xi}_1|+|\boldsymbol{\xi}_2|)|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|^2\tag{1.4}$$

for every $x\in\Omega$ and $\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$, where $\tilde\nu=\tilde\nu(p,q,\nu)$. We further assume that

(A3)

$$|\mathbf{a}(x_1,\boldsymbol{\xi})-\mathbf{a}(x_2,\boldsymbol{\xi})|\le L|x_1-x_2|^{\beta_0}\left(H'(x_1,1+|\boldsymbol{\xi}|)+H'(x_2,1+|\boldsymbol{\xi}|)\right)+L|\mu(x_1)-\mu(x_2)|(1+|\boldsymbol{\xi}|)^{q-1}\,,$$

and

$$|D_\xi\mathbf{a}(x,\boldsymbol{\xi}_1)-D_\xi\mathbf{a}(x,\boldsymbol{\xi}_2+\boldsymbol{\xi}_1)|\le L\left(\frac{|\boldsymbol{\xi}_2|}{1+|\boldsymbol{\xi}_1|}\right)^{\beta_0}H''(x,1+|\boldsymbol{\xi}_1|)\,,\quad\text{whenever}\quad|\boldsymbol{\xi}_2|\le\frac{1}{2}(1+|\boldsymbol{\xi}_1|)\,,\tag{A4}$$

for some $\beta_0\in(0,1)$, for every $x,x_1,x_2\in\Omega$ and $\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$.

For the force term $\mathbf{f}$ we require that

$$\mathbf{f}\in L^{n(1+\beta_1)}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)\tag{A5}$$

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for the same $\beta_1 > 0$ as above.

The main result. A pair $(\mathbf{u}, \pi) \in W^{1,1}(\Omega;\mathbb{R}^n) \times W^{1,1}(\Omega,\mathbb{R})$ with $H(\cdot, |\boldsymbol{\varepsilon}(\mathbf{u})|) \in L^1_{\mathrm{loc}}(\Omega)$ is a *weak solution* to (1.1) if and only if $\operatorname{div}\mathbf{u} = 0$ in Ω in the sense of distributions and

$$\int_\Omega [\mathbf{a}(x, \boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) + \pi \operatorname{div}\boldsymbol{\varphi}] \,\mathrm{d}x = \int_\Omega [\mathbf{u}[\nabla \mathbf{u}] + \mathbf{f}] \cdot \boldsymbol{\varphi} \,\mathrm{d}x \tag{1.5}$$

holds for all $\boldsymbol{\varphi} \in C^\infty_0(\Omega, \mathbb{R}^n)$. If we test with divergence free vector fields

$$\boldsymbol{\varphi} \in C^\infty_{0,\mathrm{div}}(\Omega, \mathbb{R}^n) := \{\boldsymbol{\psi} \in C^\infty_0(\Omega, \mathbb{R}^n) : \ \operatorname{div}\boldsymbol{\psi} = 0\},$$

the pressure term in (1.5) vanishes and the system reduces to $\operatorname{div}\mathbf{u} = 0$ in Ω in the sense of distributions and

$$\int_\Omega \mathbf{a}(x, \boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \,\mathrm{d}x = \int_\Omega [\mathbf{u}[\nabla \mathbf{u}] + \mathbf{f}] \cdot \boldsymbol{\varphi} \,\mathrm{d}x \tag{1.6}$$

whenever $\boldsymbol{\varphi} \in C^\infty_{0,\mathrm{div}}(\Omega, \mathbb{R}^n)$. Within this setting, we say that $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^n)$ with $H(\cdot, |\boldsymbol{\varepsilon}(\mathbf{u})|) \in L^1_{\mathrm{loc}}(\Omega)$ is a weak solution to (1.1) if and only if $\operatorname{div}\mathbf{u} = 0$ in Ω in the sense of distributions and (1.6) holds.

The main result of the paper is a partial regularity result:

Theorem 1.1. *Let* $H : \Omega \times [0, +\infty) \to [0, +\infty)$ *be defined as in* (1.2)*, and satisfying* (1.3)*. Assume that* $\mathbf{a} : \Omega \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ *complies with* (A1)–(A4)*, and that* (A5) *holds for* $\mathbf{f}$*. Let* $(\mathbf{u}, \pi) \in W^{1,1}(\Omega;\mathbb{R}^n) \times W^{1,1}(\Omega,\mathbb{R})$ *with* $H(\cdot, |\boldsymbol{\varepsilon}(\mathbf{u})|) \in L^1_{\mathrm{loc}}(\Omega)$ *be a weak solution to* (1.1)*. Then there exists an open subset* $\Omega_0 \subset \Omega$ *such that*

$$\nabla \mathbf{u} \in C^{0,\beta}_{\mathrm{loc}}\left(\Omega_0, \mathbb{R}^{n\times n}\right), \quad \pi \in \mathscr{L}^{p', n(1-\frac{1}{p^*})}_{\mathrm{loc}}(\Omega_0), \qquad |\Omega \setminus \Omega_0| = 0$$

for some $\beta \in (0, 1)$*. Moreover,* $\Omega \setminus \Omega_0 \subset \Sigma_1 \cup \Sigma_2$ *where*

$$\Sigma_1 := \left\{ x_0 \in \Omega : \ \liminf_{r\to 0^+} \fint_{B_r(x_0)} |\mathbf{V}_{H^-_{B_r(x_0)}}(\boldsymbol{\varepsilon}(\mathbf{u})) - (\mathbf{V}_{H^-_{B_r(x_0)}}(\boldsymbol{\varepsilon}(\mathbf{u})))_{x_0,r}|^2 \,\mathrm{d}x > 0 \right\},$$

$$\Sigma_2 := \left\{ x_0 \in \Omega : \ \limsup_{r \searrow 0} \, (|\nabla \mathbf{u}|^p)_{x_0,r} + |(\mathbf{u})_{x_0,r}| \ = +\infty \right\}.$$

Remark 1. Note that $|\Sigma_1| = |\Sigma_2| = 0$. In particular, the set Σ_1 has measure zero since it is contained in the set of non-Lebesgue points of $\mathbf{V}_H(\cdot, \boldsymbol{\varepsilon}(\mathbf{u}))$ (see (2.18) for the definition) in L^2-space. For details, see [37, Remark 1].

Motivation and historical background. The mathematical analysis of non-Newtonian fluids has been an active research area for several decades due to its relevance in continuum mechanics and industrial applications. Unlike classical Newtonian fluids, whose viscosity remains constant, non-Newtonian fluids exhibit rheological properties depending on the deformation rate. As a consequence, the constitutive relation between the stress tensor and the velocity gradient becomes nonlinear, leading to systems of nonlinear partial differential equations.

A typical mathematical model for incompressible non-Newtonian fluids is given by the stationary generalized Navier–Stokes system

$$-\operatorname{div}\mathbf{S}(\boldsymbol{\varepsilon}(\mathbf{u})) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\pi = \mathbf{f}, \qquad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega,$$

where $\mathbf{u} : \Omega \to \mathbb{R}^n$ denotes the velocity field, π the pressure, and $\mathbf{f}$ an external force acting on the fluid. The nonlinear tensor $\mathbf{S}(\boldsymbol{\varepsilon}(\mathbf{u}))$ describes the stress–strain relation and typically depends on the symmetric gradient $\boldsymbol{\varepsilon}(\mathbf{u})$. Models of this type arise naturally in the theory of generalized Newtonian fluids and have been extensively studied in the mathematical literature; see, for instance, the monographs by Fuchs and Seregin [21] and Galdi [22].

A significant class of models corresponds to fluids with power-law type viscosity, where the stress tensor satisfies a growth condition of the form

$$|\mathbf{S}(\mathbf{P})| \sim (\kappa + |\mathbf{P}|)^{\frac{p-2}{2}}\mathbf{P},$$

leading to the so-called p-Stokes systems. The existence theory and regularity properties of weak solutions for such systems have been studied extensively; see, for example, [7] and the references therein.

In parallel, increasing attention has been devoted to models exhibiting nonstandard growth conditions, where the energy density does not follow a single power law. Among these, we mention the analysis of electrorheological fluids, where the viscosity depends on a spatially varying exponent $p(\cdot)$. These models were introduced by Rajagopal and Růžička [39] and further developed in the monograph of Růžička [40]. The first partial regularity results for steady flows of electrorheological fluids were established by Acerbi and Mingione [2], where they proved that $\nabla \mathbf{u} \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_0)$ for some $\alpha \in (0, 1)$ and $\Omega_0 \subset \Omega$ with $|\Omega \setminus \Omega_0| = 0$, assuming $p(\cdot) > \frac{3n}{n+2}$ and Hölder continuous. Later on, additional developments in the case of VMO coefficients and $p(\cdot)$ log-Hölder continuous were obtained by Bögelein, Duzaar, Habermann and Scheven [5].

Still within the framework of nonstandard growth conditions, and beyond the modeling of non-Newtonian fluids, a prominent role is played by problems with (p,q)-growth, originally studied by Marcellini [30]. Since then, this framework has been generalized in several directions; see for instance [20, 32, 8, 33, 12, 31] and the references therein. In these models the ellipticity and growth conditions involve two different exponents p and q, creating a gap between coercivity and growth. This phenomenon introduces substantial analytical difficulties and may lead to loss of regularity or even unbounded solutions if the ratio q/p becomes too large. A possible condition that excludes such pathological behaviour is an upper bound on the ratio between the two exponents: $\frac{q}{p} < 1 + \frac{1}{n}$. Recently, this bound has been improved by Bella and Schäffner to $\frac{q}{p} < 1 + \frac{2}{n-1}$, [4]. Another approach is based on nonlinear potential theory, which provides sharp partial regularity results for relaxed minimizers of degenerate or singular, nonuniformly elliptic quasiconvex functionals with (p,q)-growth; see [11, 13].

The double phase case is genuinely different from the (p,q) one since the ellipticity changes from (p,q)-phase to p-phase exactly where the modulating coefficient annihilates. This structure naturally models heterogeneous materials and fluids whose rheological properties change in space. It is worth mentioning the milestone papers of Colombo and Mingione [10], Baroni, Colombo and Mingione [3], who established $C^{1.\alpha}$ estimates for local (scalar) minimizers under the condition $\frac{q}{p} < 1 + \frac{\alpha}{n}$, where α is the Hölder continuity exponent of the modulating coefficient. In addition, they considered borderline cases (of logarithmic type) and a unified approach to variable exponent. Their method relies on a refined alternative between p-phase and (p,q)-phase at every scale combined with an exit time argument.

Partial regularity for nondegenerate systems with double phase growth has been studied in [35, 36, 41], under the assumption that the smaller exponent p of H is larger that or equal to 2, and more recently for degenerate double-phase systems in [37], where the authors with Ok developed a unified approach independent of the the exponent p.

Description of our results. The literature concerning non-Newtonian double-phase fluids is still rather limited. Recently, the uniqueness of small solutions for steady double phase fluids has been investigated in [1] for $\frac{6}{5} < p < 2 < q$. To the best of our knowledge, the first partial regularity result for these models was obtained in [41], where we established the local partial Hölder continuity of $\mathbf{u}$ for $p > 2$. Nevertheless, the general case of nondegenerate systems with double phase growth that depend only on the symmetric gradient has not yet been explored. In this paper, we consider systems of the form (1.1) with double phase growth and prove partial Hölder regularity of the gradient of their weak solutions and partial Morrey regularity for the pressure (see Theorem 1.1). With this result, we complete the analysis initiated in [41], improving and simplifying it in the case $p > 2$. Since our analysis does not rely on the superquadratic condition, a unified approach that is independent of the exponent p is required.

The proof of the main theorem (Theorem 1.1) is based on the $\mathcal{A}$-Stokes approximation approach introduced in [41]. The double phase growth condition presents different phenomena, as it does not imply uniform ellipticity with respect to the gradient variable. Therefore, we need to develop the approach in our setting. Moreover, the excess functional and methodology employed in [41] are not easily applicable, since we are dealing with both the superquadratic ($p \geq 2$) and the subquadratic ($1 < p < 2$) cases at the same time. To overcome this issue, we build on our recent results [37, 38] and introduce a new excess functional:

$$\Phi(x_0, r, \mathbf{u}) = \fint_{B_r(x_0)} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|}\left(\left|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}\right|\right)\,\mathrm{d}x + r^{\frac{\alpha_3}{2}}\,, \tag{1.7}$$

where α_3 is defined in (5.2) and H^-_σ denotes the shifted N-function of

$$H^-(t) := t^p + \inf_{x\in B_r(x_0)} \mu(x) t^q\,,\ t \geq 0\,.$$

Here, the double phase function H is frozen at the infimum of the modulating coefficient $\mu(x)$ on the ball $B_r(x_0)$. Then a linearization procedure combined with $\mathcal{A}$-Stokes approximation yields local comparison estimates for $\boldsymbol{\varepsilon}(\mathbf{u})$ with the symmetric gradient of a suitably chosen smooth $\mathcal{A}$-Stokes function. We subsequently establish a decay estimate for the excess functional (1.7) in Lemma 5.1. Finally, by iterating this decay at smaller scales, as shown in Lemma 5.2, we obtain the partial regularity result in a standard way, as stated in Section 6: the gradient of a weak solution is locally Hölder continuous, except on a set of measure zero; see Theorem 1.1.

To conclude, we emphasize a key feature of our approach. We provide a unified method to address the problem of partial regularity for Navier–Stokes type systems with double-phase growth depending solely on the symmetric gradient. Unlike most contributions on double-phase problems, where different techniques are employed for the p-phase and the (p,q)-phase, our arguments of approximation, comparison, and iteration do not distinguish between the two regimes, thus leading to significant simplifications.

Outline of the paper. The paper is organized as follows. Section 2 collects basic definitions and auxiliary results used throughout. In particular, Section 2.2 recalls fundamental properties of Orlicz functions, while Sections 2.4–2.6 present standard auxiliary results for systems depending on the symmetrized gradient, including a Bogovskiĭ lemma and Sobolev–Korn-type inequalities. In Section 2.7, we recall the regularity properties of $\mathcal{A}$-Stokes functions, and formulate the $\mathcal{A}$-Stokes approximation lemma, which represents a key tool for the linearization procedure.

The study of partial regularity begins in Section 3, where we establish the necessary Caccioppoli-type estimates and higher integrability results, Lemma 3.3 and Lemma 3.4, respectively. In Section 4 we setup the linearization procedure, and show that any weak solution is almost $\mathcal{A}$-Stokes. Section 5 derives the excess decay estimates, see Lemma 5.1. Finally, Section 6 is devoted to the proof of the main result, Theorem 1.1.

2. PRELIMINARIES AND AUXILIARY RESULTS

2.1. **Basic notation.** We denote by Ω an open bounded domain of $\mathbb{R}^n$. For $x_0 \in \mathbb{R}^n$ and $r > 0$, $B_r(x_0)$ is the open ball of radius r centred at x_0. In the case $x_0 = 0$, we will often use the shorthand B_r in place of $B_r(x_0)$. If $f \in L^1(B_r(x_0))$, we denote the average of f by

$$(f)_{x_0,r} := \fint_{B_r(x_0)} f \, \mathrm{d}x \,,$$

and we use the abbreviate notation $(f)_r$ for $(f)_{0,r}$. We denote by $\mathbb{R}^{n\times n}$ the set of all $n \times n$ matrices, and by $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ the set of the symmetric ones. For $x, y \in \mathbb{R}^n$, we denote their tensor product by $x \otimes y := \{x_i y_j\}_{i,j} \in \mathbb{R}^{n\times n}$ ad their tensor symmetric product by $x \odot y := \frac{1}{2}(x \otimes y + y \otimes x) \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. For a function $\mathbf{u} = (u^i) \in L^1(\Omega)$, we denote by $\boldsymbol{\varepsilon}(\mathbf{u})$ and $\boldsymbol{\omega}(\mathbf{u})$ its symmetric and skew-symmetric distributional derivative, respectively:

$$\boldsymbol{\varepsilon}(\mathbf{u}) \equiv (\boldsymbol{\varepsilon}(\mathbf{u}))_{i,j} := \frac{\partial_j u^i + \partial_i u^j}{2} \,, \quad \boldsymbol{\omega}(\mathbf{u}) \equiv (\boldsymbol{\omega}(\mathbf{u}))_{i,j} := \frac{\partial_j u^i - \partial_i u^j}{2} \,.$$

If $p > 1$, then $p' := \frac{p}{p-1}$ denotes the conjugate exponent of p. If $1 < p < n$, the number $p^* := \frac{np}{n-p}$ stands for the Sobolev conjugate exponent of p, whereas p^* is any real number if $p \geq n$.

Let $H : \Omega \times [0, \infty) \to [0, \infty)$ be given in (1.2) with (1.3). For $B_r(x_0) \Subset \Omega$, let $x^-_{x_0,r}, x^+_{x_0,r} \in \overline{B_r(x_0)}$ be such that

$$\mu^-_{x_0,r} := \mu(x^-_{x_0,r}) = \inf_{x\in B_r(x_0)} \mu(x) \quad \text{and} \quad \mu^+_{x_0,r} := \mu(x^+_{x_0,r}) = \sup_{x\in B_r(x_0)} \mu(x) \,. \tag{2.1}$$

Then we write

$$H^-_{B_r(x_0)}(s) := H(x^-_{x_0,r}, s) \quad \text{and} \quad H^+_{B_r(x_0)}(s) := H(x^+_{x_0,r}, s) \,. \tag{2.2}$$

2.2. **Orlicz functions and operators.** We recall basic notation and properties about Orlicz functions. The following definitions and results can be found, e.g., in [27, 29].

A real-valued function $\varphi : [0, \infty) \to [0, \infty)$ is said to be an N-*function* if it is convex and satisfies the following conditions: $\varphi(0) = 0$, φ admits the derivative φ' and this derivative is right continuous, non-decreasing and satisfies $\varphi'(0) = 0$, $\varphi'(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} \varphi'(t) = \infty$.

We say that φ satisfies the Δ_2-*condition* if there exists $c > 0$ such that for all $t \geq 0$ holds $\varphi(2t) \leq c\,\varphi(t)$. We denote the smallest possible such constant by $\Delta_2(\varphi)$. Since $\varphi(t) \leq \varphi(2t)$, the Δ_2-condition is equivalent to $\varphi(2t) \sim \varphi(t)$.

For an N-function φ, we assume that

$$p_1 \leq \inf_{t>0} \frac{t\varphi'(t)}{\varphi(t)} \leq \sup_{t>0} \frac{t\varphi'(t)}{\varphi(t)} \leq p_2 \,, \tag{2.3}$$

for some $1 < p_1 \leq p_2 < \infty$. Furthermore, we can also assume that $\varphi \in C^2((0, \infty))$ satisfies

$$0 < p_1 - 1 \leq \inf_{t>0} \frac{t\varphi''(t)}{\varphi'(t)} \leq \sup_{t>0} \frac{t\varphi''(t)}{\varphi'(t)} \leq p_2 - 1 \,. \tag{2.4}$$

Note that if φ satisfies (2.4), then (2.3) holds hence we have

$$\varphi(t) \sim t\varphi'(t) \quad \text{and} \quad \varphi(t) \sim t^2\varphi''(t) \,, \quad t > 0 \,. \tag{2.5}$$

For instance, $\varphi(t) := t^p$, $1 < p < \infty$, is an N-function satisfying (2.4) with $p_1 = p_2 = p$. Also, for H defined as in (1.2) and each $x \in \Omega$, $\varphi(t) := H(x, t)$ is an N-function satisfying (2.4) with $p_1 = p$ and $p_2 = q$.

We denote the Young-Fenchel-Yosida conjugate function of φ by $\varphi^*(t) := \sup_{s\geq 0}(st - \varphi(s))$. It is again an N-function; it satisfies (2.3) with $\frac{p_2}{p_2-1}$ and $\frac{p_1}{p_1-1}$ in place of p_1 and p_2, respectively. We will denote by $\Delta_2(\varphi, \varphi^*)$ constants depending on $\Delta_2(\varphi)$ and $\Delta_2(\varphi^*)$. Also, it is easy to check that $(\varphi^*)^* = \varphi$.

Moreover,

$$\min\{C^{\frac{p_1}{p_1-1}}, C^{\frac{p_2}{p_2-1}}\}\varphi^*(t) \leq \varphi^*(Ct) \leq \max\{C^{\frac{p_1}{p_1-1}}, C^{\frac{p_2}{p_2-1}}\}\varphi^*(t) \,, \quad C > 0 \,, \tag{2.6}$$

and, from the definition of φ^*, the following Young's inequality

$$t\tilde{t} \le \varphi(\tilde{t}) + \varphi^*(t)\,, \quad t, \tilde{t} \ge 0\,, \tag{2.7}$$

holds true. From now on, we always assume that φ and φ^* satisfy the Δ_2 condition and this is indicated by $\Delta(\varphi, \varphi^*) < \infty$, where $\Delta(\varphi, \varphi^*)$ denotes the constants $\Delta(\varphi)$ and $\Delta(\varphi^*)$. We note that the exact value of φ^* is not always explicitly computable and instead the estimate

$$\varphi^*\left(\frac{\varphi(t)}{t}\right) \sim \varphi^*(\varphi'(t)) \sim \varphi(t) \tag{2.8}$$

will often be useful in computations (see [26, Theorem 2.4.10]).

For an N-function φ and for $a \ge 0$, we define the *shifted N-function* φ_a of φ by

$$\varphi_a(t) := \int_0^t \frac{\varphi'(a+\tilde{t})\tilde{t}}{a+\tilde{t}}\,\mathrm{d}\tilde{t}\,, \quad \text{that is,} \;\; \varphi_a'(t) = \frac{\varphi'(a+t)}{a+t}t\,. \tag{2.9}$$

The condition (2.4) implies that

$$\tilde{p}_1 := \min\{p_1, 2\} \le \frac{t\varphi_a''(t)}{\varphi_a'(t)} + 1 \le \max\{p_2, 2\} =: \tilde{p}_2 \quad \text{for all} \;\; t > 0 \;\; \text{and} \;\; a \ge 0\,.$$

Moreover, we have the following relations (see, e.g., [9, Proposition 2.3] and [14, 19]), which hold uniformly with respect to $a \ge 0$:

$$\varphi_a(t) \sim \varphi_a'(t)\,t\,; \tag{2.10}$$

$$\varphi_a(t) \sim \varphi''(a+t)t^2 \sim \frac{\varphi(a+t)}{(a+t)^2}t^2 \sim \frac{\varphi'(a+t)}{a+t}t^2\,; \tag{2.11}$$

$$\varphi(t) \le \varphi(a+t) \sim \varphi_a(t) + \varphi(a)\,. \tag{2.12}$$

We define vector valued functions $\mathbf{V}_\varphi : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ by

$$\mathbf{V}_\varphi(\mathbf{Q}) := \sqrt{\frac{\varphi_1'(|\mathbf{Q}|)}{|\mathbf{Q}|}}\mathbf{Q} = \sqrt{\frac{\varphi'(1+|\mathbf{Q}|)}{1+|\mathbf{Q}|}}\mathbf{Q}. \tag{2.13}$$

In particular, we write

$$\mathbf{V}_p(\mathbf{Q}) := \mathbf{V}_\varphi(\mathbf{Q}) \quad \text{when} \;\; \varphi(t) = t^p\,.$$

Then we recall equivalent relations in [14, Lemmas 3 and 20] and [18, Lemma 3.1] with φ_1 in place of φ:

$$\frac{\varphi'(1+|\mathbf{P}|+|\mathbf{Q}|)}{1+|\mathbf{P}|+|\mathbf{Q}|}|\mathbf{P}-\mathbf{Q}|^2 \sim |\mathbf{V}_\varphi(\mathbf{P}) - \mathbf{V}_\varphi(\mathbf{Q})|^2 \sim \varphi_{1+|\mathbf{Q}|}(|\mathbf{P}-\mathbf{Q}|)\,, \tag{2.14}$$

$$\frac{\varphi'(1+|\mathbf{P}|+|\mathbf{Q}|)}{1+|\mathbf{P}|+|\mathbf{Q}|} \sim \int_0^1 \frac{\varphi'(1+|\tau\mathbf{P}+(1-\tau)\mathbf{Q}|)}{1+|\tau\mathbf{P}+(1-\tau)\mathbf{Q}|}\,\mathrm{d}\tau\,. \tag{2.15}$$

Moreover, by the same proof of [15, Lemma A.2], we have that for every $\mathbf{g} \in L^\varphi(B_r;\mathbb{R}^{N\times n})$,

$$\fint_{B_r} |\mathbf{V}(\mathbf{g}) - (\mathbf{V}(\mathbf{g}))_{B_r}|^2\,\mathrm{d}x \sim \fint_{B_r} |\mathbf{V}(\mathbf{g}) - \mathbf{V}((\mathbf{g})_{B_r})|^2\,\mathrm{d}x\,. \tag{2.16}$$

Note that all constants concerned with the relation $\sim$ and c in above depend only on p and q.

The following lemma (see [16, Corollary 26]) deals with the *change of shift* for N-functions.

Lemma 2.1 (change of shift)**.** *Let φ be an N-function with $\Delta_2(\varphi), \Delta_2(\varphi^*) < \infty$. Then for any $\eta > 0$ there exists $c_\eta > 0$, depending only on η and $\Delta_2(\varphi)$, such that for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ and $t \ge 0$*

$$\varphi_{|\mathbf{a}|}(t) \le c_\eta \varphi_{|\mathbf{b}|}(t) + \eta\varphi_{|\mathbf{a}|}(|\mathbf{a}-\mathbf{b}|)\,. \tag{2.17}$$

We recall the notion of "*almost convexity*" and a Jensen type inequality (see [26, Lemma 4.3.1]).

Lemma 2.2. *If $\psi : [0,\infty) \to [0,\infty]$ is increasing with $\psi(0) = 0$ and $\frac{\psi(t)}{t}$ is almost increasing; i.e., $\frac{\psi(t)}{t} \le L\frac{\psi(\tilde{t})}{\tilde{t}}$ for every $0 < t \le \tilde{t}$ with constant $L \ge 1$, then the following Jensen's type inequality holds:*

$$\psi\left(\frac{1}{L^2}\fint_U |f|\,dz\right) \le \fint_U \psi(|f|)\,dz.$$

2.3. **Generalized Orlicz function H.** For basic definitions and properties about generalized Orlicz functions and the associated spaces, we refer to the monograph [26].

Let $H : \Omega \to [0, \infty)$ be the function given in (1.2). Note that for each $x \in \Omega$, $\varphi(s) := H(x, s)$ is an N-function satisfying that $\Delta_2(\varphi, \varphi^*) < \infty$. With this H, we define the generalized Orlicz space $L^H(\Omega, \mathbb{R}^N)$ as the set of all measurable functions $f : \Omega \to \mathbb{R}^N$ such that

$$\int_\Omega H(x, |f(x)|)\,\mathrm{d}x < \infty,$$

endowed with the usual Luxembourg type norm

$$\|f\|_{L^H(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega H\left(x, \frac{|f(x)|}{\lambda}\right)\mathrm{d}x \le 1\right\}.$$

We define

$$\mathbf{V}_H(x, \mathbf{Q}) := \sqrt{\frac{H'(x, 1 + |\mathbf{Q}|)}{1 + |\mathbf{Q}|}}\,\mathbf{Q}, \quad x \in \Omega. \tag{2.18}$$

Then it follows from (2.14) that for every $x \in \Omega$,

$$\begin{aligned} |\mathbf{V}_H(x, \mathbf{P}) - \mathbf{V}_H(x, \mathbf{Q})|^2 &\sim H_{1+|\mathbf{Q}|}(x, |\mathbf{P} - \mathbf{Q}|) \\ &\sim \varphi_{p,1+|\mathbf{Q}|}(|\mathbf{P} - \mathbf{Q}|) + \mu(x)\varphi_{q,1+|\mathbf{Q}|}(|\mathbf{P} - \mathbf{Q}|) \\ &\sim |\mathbf{V}_p(\mathbf{P}) - \mathbf{V}_p(\mathbf{Q})|^2 + \mu(x)|\mathbf{V}_q(\mathbf{P}) - \mathbf{V}_q(\mathbf{Q})|^2 \\ &\ge |\mathbf{V}_p(\mathbf{P}) - \mathbf{V}_p(\mathbf{Q})|^2, \end{aligned} \tag{2.19}$$

where $\varphi_{p,\sigma}(s)$ is the shifted N function of the power function s^p with shift $\sigma \ge 0$.

We also notice that for the shifted N-functions $H^\pm_\sigma$ and the corresponding $V_{H^\pm_\sigma}$ function the equivalence (2.16) still holds true. This will be used to show that the set of non Lebesgue points has measure zero.

2.4. **A lemma of Bogowskiĭ.** The following lemma is a key tool to deal with the constraint of divergence free vector fields, as we have to construct testing functions in divergence free form. Below, we recall Bogowskiĭ's lemma both in its p-growth version and in the case of Orlicz modulars.

Lemma 2.3. *Let $B_r(x_0)$ be a ball in $\mathbb{R}^n$ and $f \in L^\gamma(B_r(x_0))$ with $(f)_{x_0,r} = 0$, where $1 < \gamma_1 \le \gamma \le \gamma_2 < +\infty$. Then there exists $\mathbf{w} \in W^{1,\gamma}_0(B_r(x_0), \mathbb{R}^n)$, weak solution to $\operatorname{div}\mathbf{w} = f$ in $B_r(x_0)$, such that*

$$\int_{B_r(x_0)} |\nabla \mathbf{w}|^t\,\mathrm{d}x \le c(n, \gamma_1, \gamma_2)\int_{B_r(x_0)} |f|^t\,\mathrm{d}x \tag{2.20}$$

for every $t \in [\gamma_1, \gamma]$.

Proof. See [6] and [22, Chapter 3, Section 3]. □

Lemma 2.4. *Let $B_r(x_0)$ be a ball in $\mathbb{R}^n$, and φ be an N-function such that $\Delta_2(\varphi) < +\infty$ and $\Delta_2(\varphi^*) < +\infty$. Let $\varphi(|f|) \in L^1(B_r(x_0))$ with $(f)_{x_0,r} = 0$. Then there exists $\mathbf{w} \in W^{1,\varphi}_0(B_r(x_0), \mathbb{R}^n)$, solution to $\operatorname{div}\mathbf{w} = f$ a.e. in $B_r(x_0)$, such that*

$$\int_{B_r(x_0)} \varphi(|\nabla \mathbf{w}|)\,\mathrm{d}x \le c_\varphi \int_{B_r(x_0)} \varphi(|f|)\,\mathrm{d}x. \tag{2.21}$$

Proof. See, e.g., [28, Corollary 4.2]. □

2.5. **Affine functions.** We define the space of *traceless affine functions*

$$\mathcal{T}(\mathbb{R}^n) := \{\boldsymbol{\ell} : \mathbb{R}^n \to \mathbb{R}^n : \quad \boldsymbol{\ell} \text{ is affine and } \operatorname{tr}(\nabla\boldsymbol{\ell}) = 0\},$$

and the space of *rigid motions* or *skew-symmetric affine functions* in $\mathbb{R}^n$ as

$$\mathcal{R}(\mathbb{R}^n) := \{\boldsymbol{\ell} : \mathbb{R}^n \to \mathbb{R}^n : \quad \boldsymbol{\ell} \text{ is affine and } (\nabla\boldsymbol{\ell})^T = -\nabla\boldsymbol{\ell}\}.$$

Note that $\mathcal{R}(\mathbb{R}^n) \subseteq \mathcal{T}(\mathbb{R}^n)$, each $\boldsymbol{\ell} \in \mathcal{T}$ satisfies $\operatorname{div}\boldsymbol{\ell} = 0$, and $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^n)$ belongs to $\mathcal{R}(\mathbb{R}^n)$ if and only if $\boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{0}$.

Let $x_0 \in \Omega$ and $r > 0$. We define the affine function

$$\boldsymbol{\ell}_{x_0,r}(x) := (\mathbf{u})_{x_0,r} + (\nabla\mathbf{u})_{x_0,r}(x - x_0). \tag{2.22}$$

Note that $\boldsymbol{\ell}_{x_0,r} \in \mathcal{T}(\mathbb{R}^n)$ and

$$(\mathbf{u} - \boldsymbol{\ell}_{x_0,r})_{x_0,r} = \mathbf{0}, \quad \boldsymbol{\varepsilon}(\boldsymbol{\ell}_{x_0,r}) = (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}, \quad (\boldsymbol{\omega}(\mathbf{u} - \boldsymbol{\ell}_{x_0,r}))_{x_0,r} = \mathbf{0}.$$

Now let $\mathbf{u} \in L^2(B_r(x_0), \mathbb{R}^n)$ be given. With $\boldsymbol{\ell}^{\mathcal{T}}_{x_0,r} \in \mathcal{T}(\mathbb{R}^n)$ we denote the unique traceless affine function that minimizes

$$\boldsymbol{\ell} \mapsto \int_{B_r(x_0)} |\mathbf{u} - \boldsymbol{\ell}|^2\,\mathrm{d}x \tag{2.23}$$

amongst all $\boldsymbol{\ell} \in \mathcal{T}(\mathbb{R}^n)$. Writing

$$\boldsymbol{\ell}^{\mathcal{T}}_{x_0,r}(x) = \mathbf{b}^{\mathcal{T}}_{x_0,r} + \mathbf{A}^{\mathcal{T}}_{x_0,r}(x - x_0),$$

it is easy to check that

$$\mathbf{b}^{\mathcal{T}}_{x_0,r} \equiv \boldsymbol{\ell}^{\mathcal{T}}_{x_0,r}(x_0) = (\mathbf{u})_{x_0,r}\,, \qquad \text{and} \qquad \mathbf{A}^{\mathcal{T}}_{x_0,r} \equiv \nabla \boldsymbol{\ell}^{\mathcal{T}}_{x_0,r} = \frac{n+2}{r^2} \fint_{B_r(x_0)} \left[\mathbf{u} \otimes (x - x_0) - \frac{1}{n}\, \mathbf{u} \cdot (x - x_0)\, \mathbf{I}_n \right] \mathrm{d}x\,,$$

where $\mathbf{I}_n = (\delta_{ij})_{1 \le i,j \le n}$. Also in this case we have

$$(\mathbf{u} - \boldsymbol{\ell}^{\mathcal{T}}_{x_0,r})_{x_0,r} = \mathbf{0}\,, \;\; (\boldsymbol{\omega}(\mathbf{u} - \boldsymbol{\ell}^{\mathcal{T}}_{x_0,r}))_{x_0,r} = \mathbf{0}\,.$$

Moreover,

$$|\nabla \boldsymbol{\ell}^{\mathcal{T}}_{x_0,r} - \nabla \boldsymbol{\ell}| \le \frac{n+2}{r} \fint_{B_r(x_0)} |\mathbf{u} - \boldsymbol{\ell}|\, \mathrm{d}x\,, \qquad \text{for every } \boldsymbol{\ell} \in \mathcal{T}(\mathbb{R}^n)\,. \tag{2.24}$$

Similarly, we denote by $\boldsymbol{\ell}^{\mathcal{R}}_{x_0,r}$ the affine function that minimizes the functional (2.23) in $\mathcal{R}(\mathbb{R}^n)$. We can prove that $\boldsymbol{\ell}^{\mathcal{R}}_{x_0,r}(x_0) = (\mathbf{u})_{x_0,r}$ and $\nabla \boldsymbol{\ell}^{\mathcal{R}}_{x_0,r} = \boldsymbol{\omega}(\boldsymbol{\ell}^{\mathcal{T}}_{x_0,r})$, so that $(\mathbf{u} - \boldsymbol{\ell}^{\mathcal{R}}_{x_0,r})_{x_0,r} = \mathbf{0}$ and $(\boldsymbol{\omega}(\mathbf{u} - \boldsymbol{\ell}^{\mathcal{R}}_{x_0,r}))_{x_0,r} = \mathbf{0}$

Finally, we recall the following Korn-type inequality [2, eq. (2.21)]:

$$\frac{1}{t} \left(\fint_{B_r} \left| \frac{\mathbf{u} - \boldsymbol{\ell}^{\mathcal{R}}_{x_0,r}}{r} \right|^t \mathrm{d}x \right)^{\frac{1}{t}} \le c \left(\fint_{B_r} |\boldsymbol{\varepsilon}(\mathbf{u})|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \tag{2.25}$$

for every $t \in [1, p^*]$.

We conclude this section with a technical tool useful in the sequel (see [38, Lemma 5.1]).

Lemma 2.5. *Let φ be an N-function satisfying $\Delta_2(\varphi, \varphi^*) < \infty$, and let $\mathbf{u} \in L^{\varphi}(B_r(x_0), \mathbb{R}^n)$. There exists a constant $c = c(n, \Delta_2(\varphi, \varphi^*)) > 0$ such that for every affine traceless function $\boldsymbol{\ell} : \mathbb{R}^n \to \mathbb{R}^n$,*

$$\fint_{B_r(x_0)} \varphi\left(\frac{|\mathbf{u} - \boldsymbol{\ell}^{\mathcal{T}}_{x_0,r}|}{r} \right) \mathrm{d}x \le c \fint_{B_r(x_0)} \varphi\left(\frac{|\mathbf{u} - \boldsymbol{\ell}|}{r} \right) \mathrm{d}x\,. \tag{2.26}$$

Moreover, if $\mathbf{u} \in W^{1,\varphi}(B_r(x_0), \mathbb{R}^n)$, for every affine function ℓ

$$\fint_{B_r(x_0)} \varphi_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|}\left(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|\right) \mathrm{d}x \le c \fint_{B_r(x_0)} \varphi_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}\left(|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|\right) \mathrm{d}x\,. \tag{2.27}$$

2.6. **Sobolev-Korn inequalities and related results.**

A standard tool for obtaining local bounds of $\nabla \mathbf{u}$ in terms of $\boldsymbol{\varepsilon}(\mathbf{u})$ are the Sobolev-Korn inequalities. We first recall the classical Sobolev-Korn inequality in L^p-spaces (see, e.g., [34]).

Lemma 2.6. *Let $1 < p \le s \le q$ and $\mathbf{u} \in L^p(B_r(x_0), \mathbb{R}^n)$ be such that $\boldsymbol{\varepsilon}(\mathbf{u}) \in L^s(B_r(x_0), \mathbb{R}^{n\times n}_{\mathrm{sym}})$. Then $\nabla \mathbf{u} \in L^s(B_r(x_0), \mathbb{R}^{n\times n})$, and for some constant $c = c(n, p, q)$ it holds that*

$$\fint_{B_r(x_0)} |\nabla \mathbf{u}|^s \, \mathrm{d}x \le c \fint_{B_r(x_0)} |\boldsymbol{\varepsilon}(\mathbf{u})|^s \, \mathrm{d}x + c \left(\fint_{B_r(x_0)} \left| \frac{\mathbf{u} - (\mathbf{u})_{x_0,r}}{\rho} \right| \mathrm{d}x \right)^s. \tag{2.28}$$

If, in addition, $\mathbf{u} = \mathbf{0}$ on $\partial B_r(x_0)$, then

$$\fint_{B_r(x_0)} |\nabla \mathbf{u}|^s \, \mathrm{d}x \le c \fint_{B_r(x_0)} |\boldsymbol{\varepsilon}(\mathbf{u})|^s \, \mathrm{d}x\,. \tag{2.29}$$

We also need the following version of the Korn's inequality in Orlicz spaces (see [15, Lemma 3.3] and [17, Theorem 6.13]).

Lemma 2.7. *Let φ be an N-function such that both φ and φ^* satisfy the Δ_2-condition. Then for all $\mathbf{u} \in W^{1,\varphi}(B_r(x_0), \mathbb{R}^n)$, the inequality*

$$\int_{B_r(x_0)} \varphi(|\nabla \mathbf{u} - (\nabla \mathbf{u})_{x_0,r}|) \, \mathrm{d}x \le c_{\mathrm{Korn}} \int_{B_r(x_0)} \varphi(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|) \, \mathrm{d}x \tag{2.30}$$

holds, and $c_{\mathrm{Korn}} = c_{\mathrm{Korn}}(n, \Delta_2(\varphi, \varphi^*))$. If, in addition, either $(\boldsymbol{\omega}(\mathbf{u}))_{x_0,r} = \mathbf{0}$ or $\mathbf{u} = \mathbf{0}$ on $\partial B_r(x_0)$ in the sense of traces, then the inequality

$$\int_{B_r(x_0)} \varphi(|\nabla \mathbf{u}|) \, \mathrm{d}x \le \tilde{c}_{\mathrm{Korn}} \int_{B_r(x_0)} \varphi(|\boldsymbol{\varepsilon}(\mathbf{u})|) \, \mathrm{d}x \tag{2.31}$$

holds, and $\tilde{c}_{\mathrm{Korn}} = \tilde{c}_{\mathrm{Korn}}(n, \Delta_2(\varphi, \varphi^*))$.

We end the section with the following Sobolev-Poincaré type inequality for shifted N-functions of H, which can be proved as [38, Lemma 2.2].

Lemma 2.8 (Sobolev-Poincaré inequality)**.** *Let H be defined as in* (1.2) *with* (1.3)*, and let $a \geq 0$ and $0 < r \leq 1$. Then there exists $\theta = \theta(n,p,q) \in (0,1)$ such that for $B_r(x_0) \subset \Omega$, if $\mathbf{w} \in W^{1,1}(B_r(x_0);\mathbb{R}^n)$ with $(\boldsymbol{\omega}(\mathbf{w}))_{x_0,r} = \mathbf{0}$ and $\|H_a^-(|\boldsymbol{\varepsilon}(\mathbf{w})|)\|_{L^1(B_r(x_0))} < \infty$, then we have*

$$\fint_{B_r(x_0)} H_a^+\left(\frac{|\mathbf{w}-(\mathbf{w})_{x_0,r}|}{r}\right) \mathrm{d}x \leq c_P \left(\fint_{B_r(x_0)} H_a^-(|\boldsymbol{\varepsilon}(\mathbf{w})|)^\theta \,\mathrm{d}x\right)^{\frac{1}{\theta}} + c_P\left(r^{\alpha-\frac{n(q-p)}{p}} + r^\alpha a^{q-p}\right) a^p\,,$$

for some $c_P = c_P(n,p,q,L) \geq 1$, where $H_a^\pm$ denote the shifted N-functions of $H^\pm_{B_r(x_0)}$ (see (2.2)*) with shift $a \geq 0$. Moreover, if $\mathbf{w} \in W_0^{1,1}(B_r(x_0);\mathbb{R}^n)$, the same estimate holds with $\mathbf{w}$ in place of $\mathbf{w}-(\mathbf{w})_{x_0,r}$, namely*

$$\fint_{B_r(x_0)} H_a^+\left(\frac{|\mathbf{w}|}{r}\right) \mathrm{d}x \leq c_P \left(\fint_{B_r(x_0)} H_a^-(|\boldsymbol{\varepsilon}(\mathbf{w})|)^\theta \,\mathrm{d}x\right)^{\frac{1}{\theta}} + c_P\left(r^{\alpha-\frac{n(q-p)}{p}} + r^\alpha a^{q-p}\right) a^p$$

holds, where c_P and θ have the same dependencies as above.

Proof. Let $x^\pm_{0,r}$ be defined as in (2.1) and, for ease of notation, set $x^\pm := x^\pm_{0,r}$ and $B_r := B_r(x_0)$. First, by (2.11) and (1.3) we get

$$\begin{aligned} H_a^+(s) &\sim (a+s)^{p-2}s^2 + \mu(x^+)(a+s)^{q-2}s^2 \\ &\lesssim (a+s)^{p-2}s^2 + \mu(x^-)(a+s)^{q-2}s^2 + r^\alpha(a+s)^{q-2}s^2 \\ &\lesssim H_a^-(s) + r^\alpha(a^q+s^q)\,. \end{aligned}$$

By using Sobolev-Poincaré inequalities for the shifted N-function $H_a^-(s)$ (see, e.g., [14, Theorem 7]) and for the function $s \mapsto s^q$, and Korn's inequality (2.31) for $(H_a^-(t))^{\tilde\theta}$ and t^{q_*} we get

$$\begin{aligned} \fint_{B_r} H_a\left(x, \frac{|\mathbf{w}-(\mathbf{w})_{x_0,r}|}{r}\right) \mathrm{d}x &\lesssim \fint_{B_r} H_a^-\left(\frac{|\mathbf{w}-(\mathbf{w})_{x_0,r}|}{r}\right) \mathrm{d}x + r^\alpha \fint_{B_r} \left(\frac{|\mathbf{w}-(\mathbf{w})_{x_0,r}|}{r}\right)^q \mathrm{d}x + r^\alpha a^q \\ &\lesssim \left(\fint_{B_r} H_a^-(|\nabla \mathbf{w}|)^{\tilde\theta}\,\mathrm{d}x\right)^{\frac{1}{\tilde\theta}} + r^\alpha \left(\fint_{B_r} |\nabla\mathbf{w}|^{q_*}\,\mathrm{d}x\right)^{\frac{q}{q_*}} + r^\alpha a^q \\ &\lesssim \left(\fint_{B_r} H_a^-(|\boldsymbol{\varepsilon}(\mathbf{w})|)^{\tilde\theta}\,\mathrm{d}x\right)^{\frac{1}{\tilde\theta}} + r^\alpha \left(\fint_{B_r} |\boldsymbol{\varepsilon}(\mathbf{w})|^{q_*}\,\mathrm{d}x\right)^{\frac{q}{q_*}} + r^\alpha a^q\,, \end{aligned}$$

where $\tilde\theta \in (0,1)$ depends on n,p,q and $q_* := \max\{1, \frac{nq}{n+q}\} \leq \max\{1, q\frac{n}{n+1}\} < p$. Set $\theta := \max\{\tilde\theta, q_*/p\} \in (0,1)$. Note that, by (2.12) applied to the N-function $s \mapsto s^p$, we have

$$s^p \lesssim (a+s)^{p-2}s^2 + a^p \lesssim H_a^-(s) + a^p\,. \tag{2.32}$$

Then using Hölder's inequality, (2.32), (1.3) and the assumption that $\|\boldsymbol{\varepsilon}(\mathbf{w})\|_{L^p(B_r;\mathbb{R}^{n\times n}_{\mathrm{sym}})} \leq 1$, we obtain

$$\begin{aligned} r^\alpha \left(\fint_{B_r} |\boldsymbol{\varepsilon}(\mathbf{w})|^{q_*}\,\mathrm{d}x\right)^{\frac{q}{q_*}} &\leq r^\alpha \left(\fint_{B_r} |\boldsymbol{\varepsilon}(\mathbf{w})|^{p\theta}\,\mathrm{d}z\right)^{\frac{1}{\theta}} \left(\fint_{B_r} |\boldsymbol{\varepsilon}(\mathbf{w})|^{p}\,\mathrm{d}x\right)^{\frac{q-p}{p}} \\ &\leq r^{\alpha-\frac{n(q-p)}{p}} \left(\fint_{B_r} |\boldsymbol{\varepsilon}(\mathbf{w})|^{p\theta}\,\mathrm{d}x\right)^{\frac{1}{\theta}} \\ &\leq c\left(\fint_{B_r} H_a^-(|\boldsymbol{\varepsilon}(\mathbf{w})|)^{\theta}\,\mathrm{d}x\right)^{\frac{1}{\theta}} + c r^{\alpha-\frac{n(q-p)}{p}} a^p\,. \end{aligned}$$

This completes the proof. ☐

2.7. $\mathcal{A}$-Stokes functions and $\mathcal{A}$-Stokes approximation. Let $\mathcal{A}$ be a bilinear form on $\mathbb{R}^{n\times n}_{\mathrm{sym}}$. We assume that $\mathcal{A}$ is coercive and bounded; i.e., there holds

$$\mathcal{A}\boldsymbol{\zeta} : \boldsymbol{\zeta} \ge \kappa_A |\boldsymbol{\zeta}|^2 \,, \qquad \text{and} \qquad |\mathcal{A}\boldsymbol{\zeta} : \boldsymbol{\eta}| \le L_{\mathcal{A}} |\boldsymbol{\zeta}||\boldsymbol{\eta}| \,, \tag{2.33}$$

for some $L_{\mathcal{A}} \ge \kappa_{\mathcal{A}} > 0$ and for all $\boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. We recall that a Sobolev function $\mathbf{w}$ on a ball B is *$\mathcal{A}$-Stokes* on B if

$$\int_B \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\varepsilon}(\boldsymbol{\psi}) \, \mathrm{d}x = 0 \,, \qquad \text{for all } \boldsymbol{\psi} \in C^\infty_{0,\mathrm{div}}(B, \mathbb{R}^n) \,.$$

It is well known from the linear theory (see, e.g. [21, Lemma 3.0.5], [24]) that $\mathbf{w}$ is smooth in the interior of $B_r(x_0)$, and it satisfies the estimate

$$\sup_{B_{r/2}(x_0)} |\nabla \mathbf{w}| + r \sup_{B_{r/2}(x_0)} |\nabla^2 \mathbf{w}| \le c(n, \kappa_{\mathcal{A}}, L_{\mathcal{A}}) \fint_{B_r(x_0)} |\boldsymbol{\varepsilon}(\mathbf{w})| \, \mathrm{d}x \,. \tag{2.34}$$

We formulate the $\mathcal{A}$-Stokes approximation result for Stokes systems with general growth, see [41, Theorem 2.10 and Remark 1].

Theorem 2.9. *Let $\mathcal{A}$ be a bilinear form on $\mathbb{R}^{n\times n}_{\mathrm{sym}}$, complying with* (2.33)*. Let $B \subset \mathbb{R}^n$ be a ball with radius r_B and let $\widetilde{B} \subset \mathbb{R}^n$ denote either B or $2B$. Let φ be an N-function with $\Delta_2(\varphi, \varphi^*) < \infty$ and let $s > 0$. Then for every $\kappa > 0$, there exists $\delta > 0$ only depending on n, κ_A, $\|\mathcal{A}\|$, $\Delta_2(\varphi, \varphi^*)$ and s such that the following holds. Let $\mathbf{v} \in W^{1,\varphi}_{\mathrm{div}}(\widetilde{B}; \mathbb{R}^n)$ satisfy $(\boldsymbol{\omega}(\mathbf{v}))_{\widetilde{B}} = \mathbf{0}$ and*

$$\fint_{\widetilde{B}} \varphi(|\boldsymbol{\varepsilon}(\mathbf{v})|) \, \mathrm{d}x \le \left(\fint_{\widetilde{B}} [\varphi(|\boldsymbol{\varepsilon}(\mathbf{v})|)]^{1+s} \, \mathrm{d}x \right)^{\frac{1}{1+s}} \le \varphi(\eta)$$

for some exponent $s > 0$ and for a constant $\eta > 0$, and let it be almost $\mathcal{A}$-Stokes *on B in the sense that*

$$\left| \fint_B \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta}) \, \mathrm{d}x \right| \le \delta \eta \|\nabla \boldsymbol{\zeta}\|_{L^\infty(B; \mathbb{R}^{n\times n})} \,,$$

for all $\boldsymbol{\zeta} \in C^\infty_{0,\mathrm{div}}(B; \mathbb{R}^n)$. Then the unique solution $\mathbf{w} \in W^{1,\varphi^{1+s}}_{0,\mathrm{div}}(B; \mathbb{R}^n)$ of

$$\int_B \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\varepsilon}(\boldsymbol{\psi}) \, \mathrm{d}x = \int_B \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\boldsymbol{\psi}) \, \mathrm{d}x \,, \qquad \text{for all } \boldsymbol{\psi} \in C^\infty_{0,\mathrm{div}}(B, \mathbb{R}^n) \,,$$

satisfies

$$\fint_B \varphi\left(\frac{|\mathbf{w}|}{r_B} \right) \mathrm{d}x + \fint_B \varphi(|\nabla \mathbf{w}|) \, \mathrm{d}x \le \kappa \varphi(\eta) \,.$$

It holds $\kappa = \kappa(\varphi, s, \delta)$ and $\lim_{\delta \to 0} \kappa(\varphi, s, \delta) = 0$. The function $\mathbf{h} = \mathbf{v} - \mathbf{w}$ is called the $\mathcal{A}$-Stokes approximation of $\mathbf{v}$.

3. CACCIOPPOLI-TYPE INEQUALITIES AND HIGHER INTEGRABILITY

We prove a self improving property for double phase Stokes systems (1.1) with lower order terms. Namely, a higher integrability result for any solution $\mathbf{u}$, which will play an important role in the sequel. First, note that since $|\boldsymbol{\varepsilon}(\mathbf{u})|^p \in L^1(\Omega)$ and $p < n$, by Korn's inequality and the Sobolev embedding theorem it holds that

$$|\nabla \mathbf{u}|^p + |\mathbf{u}|^{p^*} \in L^1(\Omega) \,. \tag{3.1}$$

Lemma 3.1 (Higher integrability)**.** *Let $H : \Omega \times [0, \infty) \to [0, \infty)$ be defined as in* (1.2)*, and satisfying* (1.3)*. Let $\mathbf{a} : \Omega \times \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ be such that* (A1) *and* (A2) *hold for all $x \in \Omega$ and $\boldsymbol{\xi} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, and for some $0 < \nu \le L < \infty$. Let $\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^n)$ with $H(\cdot, |\boldsymbol{\varepsilon}(\mathbf{u})|) \in L^1(\Omega)$ be a weak solution to* (1.1)*. Then there exists $\sigma_0 > 0$, depending on n, p, q, ν, L, β_1, such that $H(\cdot, |\boldsymbol{\varepsilon}(\mathbf{u})|) \in L^{1+\sigma_0}_{\mathrm{loc}}(\Omega)$ and, for every $\sigma \in (0, \sigma_0]$ and every $B_{2r}(x_0) \subset\subset \Omega$, $r < 1$,*

$$\left(\fint_{B_r(x_0)} [H(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)]^{1+\sigma} \, \mathrm{d}x \right)^{\frac{1}{1+\sigma}} \le c \fint_{B_{2r}(x_0)} H(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|) \, \mathrm{d}x + c \fint_{B_{2r}(x_0)} (r^{\frac{p}{2}} |\nabla \mathbf{u}|^p + |\mathbf{u}|^{p^*} + 1) \, \mathrm{d}x \tag{3.2}$$

for some $c = c(n, p, q, L, \nu,) > 0$.

Proof. Our argument partly follows that of [41, Lemma 2.11]. Let $\eta \in C^\infty_0(B_{2r}(x_0))$ be a cut-off function such that $\eta \equiv 1$ on $B_r(x_0)$, $0 \le \eta \le 1$ and $|\nabla \eta| \le \frac{c}{2r}$, and let $\boldsymbol{\ell}^{\mathcal{R}} := \boldsymbol{\ell}^{\mathcal{R}}_{x_0, 2r}$ be the rigid displacement minimizing (2.23). We consider

$$\boldsymbol{\psi} := \eta^q (\mathbf{u} - \boldsymbol{\ell}^{\mathcal{R}}) + \mathbf{w} \,,$$

where the function $\mathbf{w}$ is defined through Lemma 2.3 as a solution to

$$\operatorname{div}\mathbf{w} = -\operatorname{div}\left(\eta^q(\mathbf{u}-\boldsymbol{\ell}^{\mathcal{R}})\right) = -q\eta^{q-1}\nabla\eta\cdot(\mathbf{u}-\boldsymbol{\ell}^{\mathcal{R}})\,.$$

Such a $\mathbf{w}$ exists since $\operatorname{div}\mathbf{u}=0$ and

$$\int_{B_{2r}(x_0)}(\mathbf{u}-\boldsymbol{\ell}^{\mathcal{R}})\cdot\nabla(\eta^q)\,\mathrm{d}x = 0\,,$$

and by (1.3) and the summability properties of $\mathbf{u}$ it holds that $\mathbf{w}\in W^{1,p}_0(B_{2r}(x_0);\mathbb{R}^n)$. We recall also that, by (2.20), we have the estimate

$$\fint_{B_{2r}(x_0)}|\nabla\mathbf{w}|^t\,\mathrm{d}x \le c\fint_{B_{2r}(x_0)}\left|\frac{\mathbf{u}-\boldsymbol{\ell}^{\mathcal{R}}}{2r}\right|^t\,\mathrm{d}x \tag{3.3}$$

for every exponent t for which the right hand side is finite. Taking $\boldsymbol{\psi}$ as a test function in (1.1) we get

$$\begin{aligned}
J_1 := \fint_{B_{2r}}\eta^q\mathbf{a}(x,\boldsymbol{\varepsilon}(\mathbf{u})):\boldsymbol{\varepsilon}(\mathbf{u})\,\mathrm{d}x &= -q\fint_{B_{2r}}\eta^{q-1}\mathbf{a}(x,\boldsymbol{\varepsilon}(\mathbf{u})):(\nabla\eta\odot(\mathbf{u}-\boldsymbol{\ell}^{\mathcal{R}}))\,\mathrm{d}x\\
&\quad-\fint_{B_{2r}}\mathbf{a}(x,\boldsymbol{\varepsilon}(\mathbf{u})):\boldsymbol{\varepsilon}(\mathbf{w})\,\mathrm{d}x+\fint_{B_{2r}}\eta^q\mathbf{u}[\nabla\mathbf{u}]\cdot(\mathbf{u}-\boldsymbol{\ell}^{\mathcal{R}})\,\mathrm{d}x\\
&\quad+\fint_{B_{2r}}\mathbf{u}[\nabla\mathbf{u}]\cdot\mathbf{w}\,\mathrm{d}x+\fint_{B_{2r}}\eta^q\mathbf{f}\cdot(\mathbf{u}-\boldsymbol{\ell}^{\mathcal{R}})\,\mathrm{d}x+\fint_{B_{2r}}\mathbf{f}\cdot\mathbf{w}\,\mathrm{d}x\\
&=: J_2+J_3+J_4+J_5+J_6+J_7\,.
\end{aligned}$$

It can be shown that (1.4) implies

$$\mathbf{a}(x,\boldsymbol{\xi}):\boldsymbol{\xi}\ge\nu H(x,|\boldsymbol{\xi}|)-\nu_0H(x,1)\,,$$

where $\nu_0=\nu_0(p,q,\nu,L)$. By this condition we have

$$J_1\ge\nu\fint_{B_{2r}}\eta^qH(x,|\boldsymbol{\varepsilon}(\mathbf{u})|)\,\mathrm{d}x-\nu_0\fint_{B_{2r}}\eta^qH(x,1)\,\mathrm{d}x\,,$$

while from (A1) we get

$$|J_2|\le c\fint_{B_{2r}}\eta^qH'(x,1+|\boldsymbol{\varepsilon}(\mathbf{u})|)\frac{|\mathbf{u}-\boldsymbol{\ell}^{\mathcal{R}}|}{2r}\,\mathrm{d}x\,.$$

Now, by using Young's inequality for $\varphi(t)=H(x,t)$ and (2.8), for any $\kappa\in(0,1)$ we obtain

$$|J_2|\le\kappa\fint_{B_{2r}}\eta^qH(x,1+|\boldsymbol{\varepsilon}(\mathbf{u})|)\,\mathrm{d}x+c_\kappa\fint_{B_{2r}}H\left(x,\frac{|\mathbf{u}-\boldsymbol{\ell}^{\mathcal{R}}|}{2r}\right)\,\mathrm{d}x\,.$$

As for J_3, using again (A1), Young's inequality as above and Corollary 2.4, we have

$$|J_3|\le c\fint_{B_{2r}}H'(x,1+|\boldsymbol{\varepsilon}(\mathbf{u})|)|\boldsymbol{\varepsilon}(\mathbf{w})|\,\mathrm{d}x\le\kappa\fint_{B_{2r}}H(x,1+|\boldsymbol{\varepsilon}(\mathbf{u})|)\,\mathrm{d}x+c_\kappa\fint_{B_{2r}}H\left(x,\frac{|\mathbf{u}-\boldsymbol{\ell}^{\mathcal{R}}|}{2r}\right)\,\mathrm{d}x\,.$$

We now proceed to the estimation of the lower-order terms, which follows exactly [41, Lemma 2.11]. However, for clarity, we prefer to outline the main estimates here.

We set

$$\gamma:=1+\frac{\beta_1}{2}\left(\frac{n+2}{n}\right)\,,\quad\sigma:=\left[\frac{1}{2}\left(\frac{p}{\gamma}\right)^*\right]'\,,$$

and note that

$$1\le\frac{p}{\gamma}<n\,,\quad 1<\sigma<\sigma\gamma\le p\,,\quad\left(\frac{p}{\gamma}\right)^*\le\frac{p^*}{\gamma}\,,\tag{3.4}$$

since

$$2\beta_1+\frac{3n}{n+2}\le p<n<n\gamma\,.$$

By using Hölder's inequality, the Poincaré inequality for $\mathbf{u}-\boldsymbol{\ell}^{\mathcal{R}}$ and (3.4) we have

$$\begin{aligned}
|J_4|&\le 2r\left(\fint_{B_{2r}}|\mathbf{u}|^{(\frac{p}{\gamma})^*}\,\mathrm{d}x\right)^{\frac{1}{(\frac{p}{\gamma})^*}}\left(\fint_{B_{2r}}|\nabla\mathbf{u}|^\sigma\,\mathrm{d}x\right)^{\frac{1}{\sigma}}\left(\fint_{B_{2r}}\left|\frac{\mathbf{u}-\boldsymbol{\ell}^{\mathcal{R}}}{2r}\right|^{(\frac{p}{\gamma})^*}\,\mathrm{d}x\right)^{\frac{1}{(\frac{p}{\gamma})^*}}\\
&\le cr\left(\fint_{B_{2r}}|\mathbf{u}|^{\frac{p^*}{\gamma}}\,\mathrm{d}x\right)^{\frac{\gamma}{p^*}}\left(\fint_{B_{2r}}|\nabla\mathbf{u}|^{\frac{p}{\gamma}}\,\mathrm{d}x\right)^{\frac{2\gamma}{p}}
\end{aligned}$$

whence, with Young's inequality applied twice, we get

$$|J_4| \le c \left(\fint_{B_{2r}} |\mathbf{u}|^{\frac{p^*}{\gamma}} \,\mathrm{d}x + r^{\frac{p}{2\gamma}} \fint_{B_{2r}} |\nabla \mathbf{u}|^{\frac{p}{\gamma}} \,\mathrm{d}x + 1 \right) .$$

Again using Hölder's inequality with the fact that (3.4) implies $1^* \le (\frac{p}{\gamma})^*$, (2.25) for $t = p^*$ and Young's inequality, we obtain

$$\begin{aligned} |J_6| &\le cr \left(\fint_{B_{2r}} |\mathbf{f}|^n \,\mathrm{d}x \right)^{\frac{1}{n}} \left(\fint_{B_{2r}} \left| \frac{\mathbf{u} - \boldsymbol{\ell}^{\mathcal{R}}}{2r} \right|^{p^*} \mathrm{d}x \right)^{\frac{1}{p^*}} \le c \|\mathbf{f}\|_{L^n(\Omega)} \left(\fint_{B_{2r}} |\boldsymbol{\varepsilon}(\mathbf{u})|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \\ &\le \kappa \fint_{B_{2r}} H^-_{2r}(1 + |\boldsymbol{\varepsilon}(\mathbf{u})|) \,\mathrm{d}x + c_\kappa \end{aligned}$$

for every $\kappa \in (0,1)$. Using Hölder's inequality, recalling that $\mathbf{w} \in W^{1,p}_0(B_{2r};\mathbb{R}^n)$, using Poincaré inequality, (3.3) and then Young's inequality we obtain the estimate

$$\begin{aligned} |J_5| &\le \left(\fint_{B_{2r}} |\mathbf{u}|^{(\frac{p}{\gamma})^*} \,\mathrm{d}x \right)^{\frac{1}{(\frac{p}{\gamma})^*}} \left(\fint_{B_{2r}} |\nabla \mathbf{u}|^{\sigma} \,\mathrm{d}x \right)^{\frac{1}{\sigma}} \left(\fint_{B_{2r}} |\mathbf{w}|^{(\frac{p}{\gamma})^*} \,\mathrm{d}x \right)^{\frac{1}{(\frac{p}{\gamma})^*}} \\ &\le c \left(\fint_{B_{2r}} |\mathbf{u}|^{\frac{p^*}{\gamma}} \,\mathrm{d}x + r^{\frac{p}{2\gamma}} \fint_{B_{2r}} |\nabla \mathbf{u}|^{\frac{p}{\gamma}} \,\mathrm{d}x + 1 \right) . \end{aligned}$$

Arguing as above and, with (2.20) and Korn's inequality (2.25) for $t = p$, we get

$$\begin{aligned} |J_7| &\le c \left(\fint_{B_{2r}} |\mathbf{f}|^n \,\mathrm{d}x \right)^{\frac{1}{n}} \left(\fint_{B_{2r}} |\mathbf{w}|^{p^*} \,\mathrm{d}x \right)^{\frac{1}{p^*}} \le cr \|\mathbf{f}\|_{L^n(\Omega)} \left(\fint_{B_{2r}} |\boldsymbol{\varepsilon}(\mathbf{u})|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \\ &\le \kappa \fint_{B_{2r}} H^-_{2r}(1 + |\boldsymbol{\varepsilon}(\mathbf{u})|) \,\mathrm{d}x + c_\kappa \, , \end{aligned}$$

where in the last inequality we applied Young's inequality for some $\kappa \in (0,1)$.

Collecting the previous estimates and taking into account (3.4), we finally have

$$\begin{aligned} \fint_{B_r} H(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|) \,\mathrm{d}x &\le \kappa \fint_{B_{2r}} H(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|) \,\mathrm{d}x + c_\kappa \fint_{B_{2r}} H\left(x, \frac{|\mathbf{u} - \boldsymbol{\ell}^{\mathcal{R}}|}{2r} \right) \mathrm{d}x \\ &\quad + c_\kappa \fint_{B_{2r}(x_0)} (r^{\frac{p}{2\gamma}} |\nabla \mathbf{u}|^{\frac{p}{\gamma}} + |\mathbf{u}|^{\frac{p^*}{\gamma}} + 1) \,\mathrm{d}x \, , \end{aligned} \tag{3.5}$$

for every $\kappa \in (0,1)$. Note that setting

$$g := r^{\frac{p}{2\gamma}} |\nabla \mathbf{u}|^{\frac{p}{\gamma}} + |\mathbf{u}|^{\frac{p^*}{\gamma}} + 1$$

we have $g \in L^{\gamma}(B_{2r})$ by (3.1).

Now, combining (3.5) and Lemma 2.8 applied to $\mathbf{u} - \boldsymbol{\ell}^{\mathcal{R}}$ with $a = 0$ we can write

$$\fint_{B_r} H(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|) \,\mathrm{d}x \le \kappa \fint_{B_{2r}} H(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|) \,\mathrm{d}x + c_\kappa \left(\fint_{B_{2r}} [H^-_{2r}(1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)]^{\theta} \,\mathrm{d}x \right)^{\frac{1}{\theta}} + c_\kappa \fint_{B_{2r}(x_0)} g \,\mathrm{d}x \, .$$

This estimate holds for every $\kappa \in (0,1)$ and every ball $B_{2r} \subset\subset \Omega$, and the constants c_κ only depend on the data. Thus, by Gehring's lemma (see, e.g., [23]) we get (3.2). This concludes the proof. □

Remark 2. Lemma 3.1 implies $H(\cdot, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|) \in L^{1+\sigma}_{\mathrm{loc}}(\Omega)$. Then for each $\Omega' \Subset \Omega$, there exists $R_0 \in (0,1]$ such that for any $B_{2r}(x_0) \subset \Omega'$ with $r \in (0, R_0]$,

$$|B_{2r}(x_0)| \le 1 \quad \text{and} \quad \int_{B_{2r}(x_0)} H(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)^{1+\sigma} \,\mathrm{d}x \le 1 \, . \tag{3.6}$$

Moreover, by Young's inequality,

$$\int_{B_{2r}(x_0)} H(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|) \,\mathrm{d}x \le \frac{1}{1+\sigma} \int_{B_{2r}(x_0)} H(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)^{1+\sigma} \,\mathrm{d}x + \frac{\sigma}{1+\sigma} |B_{2r}(x_0)| \le 1 \, . \tag{3.7}$$

Remark 3. Let $B_{2r}(x_0) \Subset \Omega$, with $r \leq 1$, and assume that

$$(|\nabla \mathbf{u}|^p)_{x_0,2r} + |(\mathbf{u})_{x_0,2r}| \leq M \tag{3.8}$$

holds for some constant $M \geq 1$. We have, by Korn's inequality (2.30) with $\varphi(t) = t^p$,

$$\begin{aligned}\fint_{B_{2r}(x_0)} |\nabla \mathbf{u}|^p \,\mathrm{d}x &\lesssim \fint_{B_{2r}(x_0)} |\nabla \mathbf{u} - (\nabla \mathbf{u})_{x_0,2r}|^p \,\mathrm{d}x + |(\nabla \mathbf{u})_{x_0,2r}|^p \\ &\lesssim \fint_{B_{2r}(x_0)} |\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,2r}|^p \,\mathrm{d}x + M \\ &\lesssim 1 + \fint_{B_{2r}(x_0)} |\boldsymbol{\varepsilon}(\mathbf{u})|^p \,\mathrm{d}x \,.\end{aligned}$$

Then, by the Poincaré inequality,

$$\begin{aligned}\fint_{B_{2r}(x_0)} |\mathbf{u}|^{p^*} \,\mathrm{d}x &\leq c(n,p) \left(\fint_{B_{2r}(x_0)} |\mathbf{u} - (\mathbf{u})_{x_0,2r}|^{p^*} \,\mathrm{d}x + |(\mathbf{u})_{x_0,2r}|^{p^*} \right) \\ &\leq \tilde{c}(n,p) \left((2r)^{p^*} (|\nabla \mathbf{u}|^p)_{x_0,2r}^{\frac{p^*}{p}} + |(\mathbf{u})_{x_0,2r}|^{p^*} \right) \\ &\leq \tilde{c}(n,p,M) \left((|\nabla \mathbf{u}|^p)_{x_0,2r} + 1 \right) ,\end{aligned}$$

so that

$$\fint_{B_{2r}(x_0)} (r^{\frac{p}{2}} |\nabla \mathbf{u}|^p + |\mathbf{u}|^{p^*} + 1) \,\mathrm{d}x \leq C_M \fint_{B_{2r}(x_0)} (1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)^p \,\mathrm{d}x$$

for some $C_M = C_M(n,p,M) \geq 1$.

Therefore, for each ball $B_{2r}(x) \subset \Omega$ complying with (3.8), the estimate (3.2) reads as

$$\left(\fint_{B_r(x_0)} [H(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)]^{1+\sigma} \,\mathrm{d}x \right)^{\frac{1}{1+\sigma}} \leq c \fint_{B_{2r}(x_0)} H(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|) \,\mathrm{d}x \tag{3.9}$$

for some $c = c(n,p,q,L,\nu,\beta_1,M) > 0$.

A consequence of the previous higher integrability result, in the form (3.9), are the following estimates, which can be proved arguing as for [37, Lemma 3.3 and Lemma 3.6].

Lemma 3.2. *Let $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^n)$ with $H(\cdot, |\boldsymbol{\varepsilon}(\mathbf{u})|) \in L^1(\Omega)$ be a weak solution to* (1.1), *and let $\sigma > 0$ be the exponent of Lemma 3.1. Let $M \geq 1$ and $B_{2r}(x_0) \Subset \Omega$ satisfying* (3.6) *and* (3.8) *with $r \leq \frac{1}{2}$. Then*

(i) *there exists a constant $c = c(n,p,q,\nu,L,\alpha,\beta_1,M) > 0$ such that*

$$\fint_{B_r(x_0)} (H^-_{B_{2r}(x_0)})'(1 + |\boldsymbol{\varepsilon}(\mathbf{u})|) \,\mathrm{d}x \leq c (H^-_{B_{2r}(x_0)})' \Bigg(\fint_{B_{2r}(x_0)} 1 + |\boldsymbol{\varepsilon}(\mathbf{u})| \,\mathrm{d}x \Bigg) ; \tag{3.10}$$

(ii) *there exists $\alpha_2 = \alpha_2(n,p,q,\nu,L,\alpha) > 0$ such that*

$$\fint_{B_r(x_0)} (\mu(x) - \mu^-_{x_0,2r})(1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)^{q-1} \,\mathrm{d}x \leq c r^{\alpha_2} (H^-_{B_{2r}(x_0)})' \Bigg(\fint_{B_{2r}(x_0)} 1 + |\boldsymbol{\varepsilon}(\mathbf{u})| \,\mathrm{d}x \Bigg) , \tag{3.11}$$

for some $c = c(n,p,q,\nu,L,\alpha,\beta_1,M) > 0$.

The first key tool for the linearization is the following Caccioppoli type estimate for $\mathbf{u} - \boldsymbol{\ell}$, where $\boldsymbol{\ell}$ is any affine traceless function. It will provide, in a standard way, a higher integrability result for $\boldsymbol{\varepsilon}(\mathbf{u} - \boldsymbol{\ell})$.

Lemma 3.3. *Let $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^n)$ with $H(\cdot, |\boldsymbol{\varepsilon}(\mathbf{u})|) \in L^1(\Omega)$ be a weak solution to* (1.1). *Let $B_{2r}(x_0) \Subset \Omega$ with $r \leq 1$, and $\boldsymbol{\ell} \in \mathcal{T}(\mathbb{R}^n)$ be any traceless affine function such that $(2r)^\alpha (1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^{q-p} \leq 1$. Then we have*

$$\begin{aligned}&\fint_{B_r(x_0)} H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(x, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \,\mathrm{d}x \leq c \fint_{B_{2r}(x_0)} H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|} \left(x, \frac{|\mathbf{u} - \boldsymbol{\ell}|}{2r} \right) \mathrm{d}x \\ &\qquad + c(r^{\beta_0} + r^\alpha (1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^{q-p})^{\frac{q}{q-1}} + (r^{\frac{\beta_1 p'}{2(\beta_1+1)}} + r^{\frac{\beta_1 p}{2(\beta_1+1)}}) H^+(1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \\ &\qquad + c r^{\frac{p'}{2}} \left(\fint_{B_{2r}} |\mathbf{u}|^{p^*} + |\nabla \mathbf{u}|^p \,\mathrm{d}x \right)^{\frac{p'}{(p^*)'}}\end{aligned} \tag{3.12}$$

for some constant $c = c\left(n, p, q, \nu, L, \|\mathbf{f}\|_{L^{n(1+\beta_1)}(\Omega)}\right) > 0$.

Proof. We adapt the arguments of [41, Lemma 3.1] and [37, Lemma 3.1]. We consider a cut-off function $\eta \in C_0^\infty(B_{2r};[0,1])$ such that $\eta \equiv 1$ on B_r and $|\nabla\eta| \le \frac{c(n)}{r}$. Correspondingly, we define the function $\boldsymbol{\psi} := \eta^q(\mathbf{u}-\boldsymbol{\ell}) \in W^{1,p}(B_{2r};\mathbb{R}^n)$. Since

$$\operatorname{div}\boldsymbol{\psi} = \nabla(\eta^q)\cdot(\mathbf{u}-\boldsymbol{\ell}) + \eta^q \operatorname{div}(\mathbf{u}-\boldsymbol{\ell}) = q\eta^{q-1}\nabla\eta\cdot(\mathbf{u}-\boldsymbol{\ell}), \tag{3.13}$$

where we used that $\operatorname{div}\mathbf{u} = 0$ and $\boldsymbol{\ell} \in \mathcal{T}(\mathbb{R}^n)$, we conclude that $\boldsymbol{\psi}$ is not a divergence-free vector field. By virtue of Lemma 2.3 we can find $\mathbf{w} \in W_0^{1,p}(B_{2r};\mathbb{R}^n)$ such that, setting

$$\boldsymbol{\zeta} := \boldsymbol{\psi} - \mathbf{w}, \tag{3.14}$$

we have $\operatorname{div}\boldsymbol{\zeta} = 0$. Taking $\boldsymbol{\zeta}$ as a test function in (1.6) and using the identity

$$\fint_{B_{2r}} \mathbf{a}(x_{2r}^-, \boldsymbol{\varepsilon}(\boldsymbol{\ell})) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta})\,\mathrm{d}x = 0 \tag{3.15}$$

we get

$$\begin{aligned}
0 &= \fint_{B_{2r}} (\mathbf{a}(x,\boldsymbol{\varepsilon}(\mathbf{u})) - \mathbf{a}(x_{2r}^-,\boldsymbol{\varepsilon}(\boldsymbol{\ell})) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta})\,\mathrm{d}x + \fint_{B_{2r}} \mathbf{u}[D\mathbf{u}]\cdot\boldsymbol{\zeta}\,\mathrm{d}x + \fint_{B_{2r}} \mathbf{f}\cdot\boldsymbol{\zeta}\,\mathrm{d}x \\
&= \fint_{B_{2r}} (\mathbf{a}(x,\boldsymbol{\varepsilon}(\mathbf{u})) - \mathbf{a}(x,\boldsymbol{\varepsilon}(\boldsymbol{\ell})) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta})\,\mathrm{d}x + \fint_{B_{2r}} (\mathbf{a}(x,\boldsymbol{\varepsilon}(\boldsymbol{\ell})) - \mathbf{a}(x_{2r}^-,\boldsymbol{\varepsilon}(\boldsymbol{\ell}))) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta})\,\mathrm{d}x \\
&\quad + \fint_{B_{2r}} \mathbf{u}[D\mathbf{u}]\cdot\boldsymbol{\zeta}\,\mathrm{d}x + \fint_{B_{2r}} \mathbf{f}\cdot\boldsymbol{\zeta}\,\mathrm{d}x,
\end{aligned}$$

whence, recalling the definition of $\boldsymbol{\zeta}$ and noting that $\boldsymbol{\varepsilon}(\boldsymbol{\zeta}) = \eta^q\,\boldsymbol{\varepsilon}(\mathbf{u}-\boldsymbol{\ell}) + q\,\eta^{q-1}\,\nabla\eta\odot(\mathbf{u}-\boldsymbol{\ell}) - \boldsymbol{\varepsilon}(\mathbf{w})$, we obtain

$$\begin{aligned}
J_1 :&= \fint_{B_{2r}} \eta^q\,(\mathbf{a}(x,\boldsymbol{\varepsilon}(\mathbf{u})) - \mathbf{a}(x,\boldsymbol{\varepsilon}(\boldsymbol{\ell}))) : (\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell}))\,\mathrm{d}x \\
&= \fint_{B_{2r}} \big(\mathbf{a}(x_{2r}^-,\boldsymbol{\varepsilon}(\boldsymbol{\ell})) - \mathbf{a}(x,\boldsymbol{\varepsilon}(\boldsymbol{\ell}))\big) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta})\,\mathrm{d}x \\
&\quad - q\fint_{B_{2r}} \eta^{q-1}\,(\mathbf{a}(x,\boldsymbol{\varepsilon}(\mathbf{u})) - \mathbf{a}(x,\boldsymbol{\varepsilon}(\boldsymbol{\ell}))) : (\nabla\eta\odot(\mathbf{u}-\boldsymbol{\ell}) - \boldsymbol{\varepsilon}(\mathbf{w}))\,\mathrm{d}x \\
&\quad + \fint_{B_{2r}} \mathbf{u}[D\mathbf{u}]\cdot(\eta^q(\mathbf{u}-\boldsymbol{\ell}) - \mathbf{w})\,\mathrm{d}x + \fint_{B_{2r}} \mathbf{f}\cdot(\eta^q(\mathbf{u}-\boldsymbol{\ell}) - \mathbf{w})\,\mathrm{d}x \\
&=: J_2 + J_3 + J_4 + J_5\,.
\end{aligned} \tag{3.16}$$

We now estimate each term on the right-hand side separately. Using (1.4) and (2.11) we get

$$\begin{aligned}
J_1 &\ge \tilde{\nu}\fint_{B_{2r}} \eta^q H''(x, 1+|\boldsymbol{\varepsilon}(\mathbf{u})| + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|^2\,\mathrm{d}x \\
&\ge \frac{1}{\tilde{c}}\fint_{B_{2r}} \eta^q H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(x, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)\,\mathrm{d}x,
\end{aligned} \tag{3.17}$$

for some $\tilde{c} \ge 1$.

To estimate J_3 we use (A1), (2.21) with $\varphi(t) = H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(x,t)$, Young's inequality for φ, φ^*, (2.8), (2.10), (2.15) and we get

$$\begin{aligned}
|J_3| &\le c\fint_{B_{2r}} \left(\int_0^1 |D_\xi\mathbf{a}(x, s(\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})) + \boldsymbol{\varepsilon}(\boldsymbol{\ell}))|\,\mathrm{d}s\right)|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{2r}|\left(\frac{|\mathbf{u}-\boldsymbol{\ell}|}{2r} + |\boldsymbol{\varepsilon}(\mathbf{w})|\right)\mathrm{d}x \\
&\le c\fint_{B_{2r}} \frac{H'(x, 1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})| + |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})| + |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|}|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|\left(\frac{|\mathbf{u}-\boldsymbol{\ell}|}{2r} + |\boldsymbol{\varepsilon}(\mathbf{w})|\right)\mathrm{d}x \\
&\le c\fint_{B_{2r}} \frac{H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(x, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|}\left(\frac{|\mathbf{u}-\boldsymbol{\ell}|}{2r} + |\boldsymbol{\varepsilon}(\mathbf{w})|\right)\mathrm{d}x \\
&\le \frac{1}{4\tilde{c}}\fint_{B_{2r}} H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(x, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)\,\mathrm{d}x + c\fint_{B_{2r}} H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}\left(x, \frac{|\mathbf{u}-\boldsymbol{\ell}|}{2r}\right)\mathrm{d}x\,.
\end{aligned}$$

As for J_2, from (A3) we use Young's inequality (2.7) for $\varphi(t) = H_{|\mathbf{Q}|}(x,t)$ for each $x \in \Omega$ to $|\boldsymbol{\varepsilon}(\boldsymbol{\zeta})|$ and its conjugate to the other term, taking into account (3.13) and (3.14). Then, we use the growth condition for the conjugate and (2.8) to obtain:

$$
\begin{aligned}
|J_2| &\lesssim \fint_{B_{2r}} \left(r^{\beta_0} H'(x, 1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) + r^{\alpha}(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^{q-1}\right) |\boldsymbol{\varepsilon}(\boldsymbol{\zeta})| \, \mathrm{d}x \\
&\lesssim \fint_{B_{2r}} \left\{r^{\beta_0} + r^{\alpha}(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^{q-p}\right\} (H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|})'(x, 1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) |\boldsymbol{\varepsilon}(\boldsymbol{\zeta})| \, \mathrm{d}x \\
&\le \frac{1}{4\tilde{c}} \fint_{B_{2r}} \eta^q H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(x, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \, \mathrm{d}x + c \fint_{B_{2r}} H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}\left(x, \frac{|\mathbf{u}-\boldsymbol{\ell}|}{2r}\right) \mathrm{d}x \\
&\quad + c \fint_{B_{2r}} (H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|})^*\Big(x, \left\{r^{\beta_0} + r^{\alpha}(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^{q-p}\right\} (H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|})'(x, 1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)\Big) \, \mathrm{d}x \\
&\le \frac{1}{4\tilde{c}} \fint_{B_{2r}} \eta^q H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(x, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \, \mathrm{d}x + c \fint_{B_r} H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}\left(x, \frac{|\mathbf{u}-\boldsymbol{\ell}|}{2r}\right) \mathrm{d}x \\
&\quad + c(r^{\beta_0} + r^{\alpha}(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^{q-p})^{\frac{q}{q-1}} H(x_{2r}^+, 1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)\,,
\end{aligned} \tag{3.18}
$$

where in the latter we used the fact that $r^{\alpha}(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^{q-p} \le 1$ and Hölder's inequality.

Now, we proceed with the estimate of the lower order terms. Using Young's inequality with exponents p^*-1 and $p(1-\frac{1}{p^*})$, and Hölder's inequality with exponents p^* and $(p^*)'$, and Young's inequality with exponents p and p', we have

$$
\begin{aligned}
|J_4| &\lesssim \fint_{B_{2r}} \left(|\mathbf{u}|^{p^*-1} + |\nabla \mathbf{u}|^{p(1-\frac{1}{p^*})}\right) (|\eta^q(\mathbf{u}-\boldsymbol{\ell})| + |\mathbf{w}|) \, \mathrm{d}x \\
&\lesssim \sqrt{r} \left(\fint_{B_{2r}} \left(|\mathbf{u}|^{p^*-1} + |\nabla \mathbf{u}|^{p(1-\frac{1}{p^*})}\right)^{(p^*)'} \mathrm{d}x\right)^{\frac{1}{(p^*)'}} \sqrt{r} \left(\fint_{B_{2r}} \left|\frac{\eta^q(\mathbf{u}-\boldsymbol{\ell})}{2r}\right|^{p^*} + \left|\frac{\mathbf{w}}{2r}\right|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \\
&\lesssim \sqrt{r} \left(\fint_{B_{2r}} |\mathbf{u}|^{p^*} + |\nabla \mathbf{u}|^{p} \, \mathrm{d}x\right)^{\frac{1}{(p^*)'}} \sqrt{r} \left(\fint_{B_{2r}} \left|\frac{\eta^q(\mathbf{u}-\boldsymbol{\ell})}{2r}\right|^{p^*} + \left|\frac{\mathbf{w}}{2r}\right|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \\
&\lesssim c_{\kappa} r^{\frac{p'}{2}} \left(\fint_{B_{2r}} |\mathbf{u}|^{p^*} + |\nabla \mathbf{u}|^{p} \, \mathrm{d}x\right)^{\frac{p'}{(p^*)'}} + \kappa r^{\frac{p}{2}} \left(\fint_{B_{2r}} \left|\frac{\eta^q(\mathbf{u}-\boldsymbol{\ell})}{2r}\right|^{p^*} + \left|\frac{\mathbf{w}}{2r}\right|^{p^*} \mathrm{d}x\right)^{\frac{p}{p^*}}
\end{aligned}
$$

for every $\kappa \in (0,1)$.

In order to estimate the second integral on the right hand side, we use Poincaré's inequality, Lemma 2.3 and (2.32) to get

$$
\begin{aligned}
\fint_{B_{2r}} \left|\frac{\mathbf{w}}{2r}\right|^{p^*} \mathrm{d}x &\le c \left(\fint_{B_{2r}} |\nabla \mathbf{w}|^p \, \mathrm{d}x\right)^{\frac{p^*}{p}} \le c \left(\fint_{B_{2r}} \left|\frac{\eta^{q-1}(\mathbf{u}-\boldsymbol{\ell})}{2r}\right|^p \mathrm{d}x\right)^{\frac{p^*}{p}} \\
&\qquad c \left(\fint_{B_{2r}} H^-_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}\left(\frac{|\mathbf{u}-\boldsymbol{\ell}|}{2r}\right) \mathrm{d}x + (1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^p\right)^{\frac{p^*}{p}}.
\end{aligned} \tag{3.19}
$$

As for the first integral on the right hand side, we use Korn's inequality and (2.32) to obtain

$$
\begin{aligned}
\fint_{B_{2r}} \left|\frac{\eta^q(\mathbf{u}-\boldsymbol{\ell})}{2r}\right|^{p^*} \mathrm{d}x &\le c \left(\fint_{B_{2r}} |\boldsymbol{\varepsilon}(\eta^q(\mathbf{u}-\boldsymbol{\ell}))|^p \, \mathrm{d}x\right)^{\frac{p^*}{p}} \\
&\le c \left(\fint_{B_{2r}} \eta^q H^-_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \, \mathrm{d}x + \fint_{B_{2r}} H^-_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}\left(\frac{|\mathbf{u}-\boldsymbol{\ell}|}{2r}\right) \mathrm{d}x + (1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^p\right)^{\frac{p^*}{p}}.
\end{aligned} \tag{3.20}
$$

Collecting these estimates, we find

$$
\begin{aligned}
|J_4| &\lesssim c_\kappa r^{\frac{p'}{2}} \left(\fint_{B_{2r}} |\mathbf{u}|^{p^*} + |\nabla \mathbf{u}|^p \,\mathrm{d}x \right)^{\frac{p'}{(p^*)'}} + \kappa \fint_{B_{2r}} \eta^q H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}\left(x, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|\right) \mathrm{d}x \\
&\quad + \fint_{B_{2r}} H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}\left(x, \frac{|\mathbf{u}-\boldsymbol{\ell}|}{2r}\right) \mathrm{d}x + r^{\frac{p}{2}} H^+(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)\,.
\end{aligned}
\tag{3.21}
$$

We are left to estimate J_5. Using Hölder's inequality, $p^* > \frac{n}{n-1}$ and Young's inequality with exponents p and p' we have

$$
\begin{aligned}
|J_5| &\le c \left(\fint_{B_{2r}} |\mathbf{f}|^n \,\mathrm{d}x \right)^{\frac{1}{n}} r \left(\fint_{B_{2r}} \left| \frac{\eta^q(\mathbf{u}-\boldsymbol{\ell})}{2r} \right|^{\frac{n}{n-1}} + \left| \frac{\mathbf{w}}{2r} \right|^{\frac{n}{n-1}} \mathrm{d}x \right)^{\frac{n-1}{n}} \\
&\le c r^{1-\frac{1}{\beta_1+1}} \|\mathbf{f}\|_{L^{n(1+\beta_1)}(\Omega)} \left(\fint_{B_{2r}} \left| \frac{\eta^q(\mathbf{u}-\boldsymbol{\ell})}{2r} \right|^{p^*} + \left| \frac{\mathbf{w}}{2r} \right|^{p^*} \mathrm{d}x \right)^{\frac{1}{p^*}} \\
&\lesssim c_\kappa r^{\frac{\beta_1 p'}{2(\beta_1+1)}} + \kappa r^{\frac{\beta_1 p}{2(\beta_1+1)}} \left(\fint_{B_{2r}} \left| \frac{\eta^q(\mathbf{u}-\boldsymbol{\ell})}{2r} \right|^{p^*} + \left| \frac{\mathbf{w}}{2r} \right|^{p^*} \mathrm{d}x \right)^{\frac{p}{p^*}} .
\end{aligned}
$$

for every $\kappa \in (0,1)$.

Now, using (3.19) and (3.20), we finally obtain

$$
\begin{aligned}
|J_5| &\lesssim \kappa \fint_{B_{2r}} \eta^q H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}\left(x, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|\right) \mathrm{d}x \\
&\quad + \fint_{B_{2r}} H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}\left(x, \frac{|\mathbf{u}-\boldsymbol{\ell}|}{2r}\right) \mathrm{d}x + \left(r^{\frac{\beta_1 p'}{2(\beta_1+1)}} + r^{\frac{\beta_1 p}{2(\beta_1+1)}} \right) H^+(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)\,.
\end{aligned}
\tag{3.22}
$$

Plugging the estimates (3.17)–(3.18) and (3.21)–(3.22) into (3.16) and reabsorbing some terms we obtain (3.12). The proof of (3.12) is then concluded. □

Now, we are in position to establish a higher integrability result for $H^-_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)$. The result follows from Lemma 2.8 with $\mathbf{w} := \mathbf{u} - \boldsymbol{\ell}$ and Lemma 3.3 as a consequence of Gehring's lemma.

Lemma 3.4. *Let $\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^n)$ with $H(\cdot, |\boldsymbol{\varepsilon}(\mathbf{u})|) \in L^1(\Omega)$ be a weak solution to (1.1). Let $B_{2r}(x_0) \Subset \Omega$, and $\boldsymbol{\ell} \in \mathcal{T}(\mathbb{R}^n)$ be any traceless affine function such that $(2r)^\alpha (1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^{q-p} \le 1$, $(\mathbf{u}-\boldsymbol{\ell})_{x_0,2r} = \mathbf{0}$ and $(\boldsymbol{\omega}(\mathbf{u}-\boldsymbol{\ell}))_{x_0,2r} = \mathbf{0}$. Then there exist a constant $c_{\mathrm{high}} = c_{\mathrm{high}}(n,p,q,\nu,L,\alpha,\|\mathbf{f}\|_{L^{n(1+\beta_1)}(\Omega)}) > 0$ and $\sigma = \sigma(n,p,q,\nu,L,\alpha,\|\mathbf{f}\|_{L^{n(1+\beta_1)}(\Omega)}) > 0$ such that*

$$
\begin{aligned}
&\left(\fint_{B_r(x_0)} \left[H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(x, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \right]^{1+\sigma} \mathrm{d}x \right)^{\frac{1}{1+\sigma}} \le c_{\mathrm{high}} \fint_{B_{2r}(x_0)} (H^-_{B_{2r}(x_0)})_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \,\mathrm{d}x \\
&\quad + c_{\mathrm{high}} \left[(r^{\beta_0} + r^\alpha (1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^{q-p})^{\frac{q}{q-1}} + r^{\alpha - \frac{n(q-p)}{p}} + r^\alpha (1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^{q-p} + r^{\frac{\beta_1 p'}{2(\beta_1+1)}} + r^{\frac{\beta_1 p}{2(\beta_1+1)}} \right] H^+_{B_{2r}(x_0)}(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \\
&\quad + c_{\mathrm{high}} r^{\frac{p'}{2}} \left(\fint_{B_{2r}} |\mathbf{u}|^{p^*} + |\nabla \mathbf{u}|^p \,\mathrm{d}x \right)^{\frac{p'}{(p^*)'}} .
\end{aligned}
\tag{3.23}
$$

Remark 4. Let $B_{2r}(x_0) \Subset \Omega$ be such that (3.6) holds and $r \le \frac{1}{2}$. Then, by (3.7) it follows that $\|1+|\boldsymbol{\varepsilon}(\mathbf{u})|\|_{L^p(B_{2r}(x_0))} \le 1$ whence, by Hölder's inequality and (1.3), we get

$$
(2r)^\alpha (1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,2r}|)^{q-p} \le (2r)^\alpha (1+|\boldsymbol{\varepsilon}(\mathbf{u})|^p)_{x_0,2r}^{\frac{q-p}{p}} \le (2r)^{\alpha - n\frac{q-p}{p}} |B_1|^{-\frac{q-p}{p}} \le 1\,.
$$

Now, assume that also (3.8) holds for some $M \geq 1$. Choosing in Lemma 3.4 $\boldsymbol{\ell} = \boldsymbol{\ell}_{x_0,2r}$ (see (2.22)) and noting that $1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})| = 1 + |(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,2r}| \sim 1$, we may rewrite (3.23) as

$$\left(\fint_{B_r(x_0)} \left[H_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,2r}|}(x, |\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,2r}|) \right]^{1+\sigma} \mathrm{d}x \right)^{\frac{1}{1+\sigma}} \leq \tilde{c}_{\text{high}} \fint_{B_{2r}(x_0)} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,2r}|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,2r}|) \, \mathrm{d}x + \tilde{c}_{\text{high}} r^{\bar{\gamma}} \,, \tag{3.24}$$

where

$$\bar{\gamma} := \min \left\{ \frac{\beta_0 q}{q-1}, \frac{\alpha q}{q-1}, \alpha - \frac{n(q-p)}{p}, \frac{\beta_1 p'}{2(\beta_1 + 1)}, \frac{\beta_1 p}{2(\beta_1+1)} \right\} , \tag{3.25}$$

and the constant $\tilde{c}_{\text{high}}$ now depends also on M.

4. LINEARIZATION

We can start with the linearization procedure for system (1.1). Given $B_{2r}(x_0) \Subset \Omega$ and $\boldsymbol{\ell} \in \mathcal{T}(\mathbb{R}^n)$, we introduce the bilinear form

$$\mathcal{A} := D_\xi \mathbf{a}(x^-_{2r}, \boldsymbol{\varepsilon}(\boldsymbol{\ell})) \,. \tag{4.1}$$

We aim to show that $\mathbf{w} := \mathbf{u} - \boldsymbol{\ell}$ is approximately $\mathcal{A}$-Stokes. This fact, together with the higher integrability estimate (3.23) will enable us to apply the $\mathcal{A}$-Stokes approximation theorem.

We set

$$E_{\boldsymbol{\ell}}(x_0, r) := \fint_{B_r(x_0)} \frac{H^-_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{H^-(1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)} \, \mathrm{d}x \,. \tag{4.2}$$

Lemma 4.1. *Let $B_{2r}(x_0) \Subset \Omega$, $r \in (0, R_0]$, where R_0 is given in Remark 2. Let $\alpha_2, \beta_0, \beta_1$ be the exponents of Lemma 3.2,* (A4), (A5)*, respectively. Assume that*

$$(|\nabla \mathbf{u}|^p)_{x_0,2r} + |(\mathbf{u})_{x_0,2r}| \leq M \tag{4.3}$$

holds for some $M \geq 1$. Then there exists a constant $c_{\mathrm{S}} = c_{\mathrm{S}}(n, p, q, \nu, L, \alpha, \|\mathbf{f}\|_{L^{n(1+\beta_1)}(\Omega)}, M) > 0$ such that

$$\left| \fint_{B_r(x_0)} \mathcal{A} \boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta}) \, \mathrm{d}x \right| \leq c_{\mathrm{S}} (H^-_{2r})'(1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \left\{ E_{\boldsymbol{\ell}}(x_0, 2r) + [E_{\boldsymbol{\ell}}(x_0, 2r)]^{\frac{1+\beta_0}{2}} + (r^{\alpha_2} + r^{\beta_0})[1 + E_{\boldsymbol{\ell}}(x_0, 2r)]^{\frac{q-1}{p}} + r^{\frac{\beta_1}{\beta_1+1}} \right\} \|\boldsymbol{\varepsilon}(\boldsymbol{\zeta})\|_\infty \tag{4.4}$$

for every $\boldsymbol{\zeta} \in C^\infty_{0,\mathrm{div}}(B_r(x_0))$.

Proof. We follow the argument of [37, Lemma 4.1]. Up to a standard normalization argument, it will suffice to prove (4.4) for $\boldsymbol{\zeta} \in C^\infty_0(B_r(x_0))$ with $\operatorname{div} \boldsymbol{\zeta} = 0$ such that $\|\boldsymbol{\varepsilon}(\boldsymbol{\zeta})\|_{L^\infty(B_r(x_0))} \leq 1$. To enlighten notation, we omit the explicit dependence on x_0.

From the definitions of $\mathcal{A}$ and $\mathbf{w}$ we have

$$\begin{aligned} \fint_{B_r} \mathcal{A} \boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta}) \, \mathrm{d}x &= \fint_{B_r} D_\xi \mathbf{a}(x^-_{2r}, \boldsymbol{\varepsilon}(\boldsymbol{\ell})) \boldsymbol{\varepsilon}(\mathbf{u} - \boldsymbol{\ell}) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta}) \, \mathrm{d}x \\ &= \fint_{B_r} \int_0^1 \left[D_\xi \mathbf{a}(x^-_{2r}, \boldsymbol{\varepsilon}(\boldsymbol{\ell}))) - D_\xi \mathbf{a}(x^-_{2r}, \boldsymbol{\varepsilon}(\boldsymbol{\ell}) + t \boldsymbol{\varepsilon}(\mathbf{u} - \boldsymbol{\ell})) \right] \boldsymbol{\varepsilon}(\mathbf{u} - \boldsymbol{\ell}) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta}) \, \mathrm{d}t \, \mathrm{d}x \\ &\quad + \fint_{B_r} \int_0^1 \left[D_\xi \mathbf{a}(x^-_{2r}, \boldsymbol{\varepsilon}(\boldsymbol{\ell}) + t \boldsymbol{\varepsilon}(\mathbf{u} - \boldsymbol{\ell})) \right] \boldsymbol{\varepsilon}(\mathbf{u} - \boldsymbol{\ell}) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta}) \, \mathrm{d}t \, \mathrm{d}x \\ &=: J_1 + J_2 \,. \end{aligned} \tag{4.5}$$

We start with an estimate for J_1, which can be written as

$$
\begin{aligned}
J_1 &= \fint_{B_r} \mathbb{1}_E(x) \int_0^1 \big[D_\xi \mathbf{a}(x^-_{2r}, \boldsymbol{\varepsilon}(\boldsymbol{\ell}))) - D_\xi \mathbf{a}(x^-_{2r}, \boldsymbol{\varepsilon}(\boldsymbol{\ell}) + t\boldsymbol{\varepsilon}(\mathbf{u}-\boldsymbol{\ell}))\big]\, \boldsymbol{\varepsilon}(\mathbf{u}-\boldsymbol{\ell}) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta})\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \fint_{B_r} \mathbb{1}_F(x) \int_0^1 \big[D_\xi \mathbf{a}(x^-_{2r}, \boldsymbol{\varepsilon}(\boldsymbol{\ell})) - D_\xi \mathbf{a}(x^-_{2r}, \boldsymbol{\varepsilon}(\boldsymbol{\ell}) + t\boldsymbol{\varepsilon}(\mathbf{u}-\boldsymbol{\ell}))\big]\, \boldsymbol{\varepsilon}(\mathbf{u}-\boldsymbol{\ell}) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta})\,\mathrm{d}t\,\mathrm{d}x\,, \\
&=: J_{1,E} + J_{1,F}\,,
\end{aligned}
$$

where $E := \{x \in B_r :\ |\boldsymbol{\varepsilon}(\mathbf{u})(x) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})| > \frac{1}{2}\,(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)\}$, and $F := B_r \backslash E$.

For what concerns $J_{1,E}$, from (A1) and (2.5),

$$
\begin{aligned}
|J_{1,E}| &\le c \fint_{B_r} \mathbb{1}_E(x) \bigg(\int_0^1 \big[(H^-_{2r})''(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) + (H^-_{2r})''(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell}) + t(\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell}))|)\big]\ \mathrm{d}t \bigg) |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|\,\mathrm{d}x \\
&\lesssim \fint_{B_r} \mathbb{1}_E(x) \big[(H^-_{2r})''(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) + (H^-_{2r})''(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})| + |\boldsymbol{\varepsilon}(\mathbf{u})|)\big]\, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|\,\mathrm{d}x \\
&\lesssim \fint_{B_r} \mathbb{1}_E(x) (H^-_{2r})'(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})| + |\boldsymbol{\varepsilon}(\mathbf{u})|) \frac{|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|}{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}\,\mathrm{d}x\,.
\end{aligned}
$$

For a.e. $x \in E$, it holds

$$
1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})| + |\boldsymbol{\varepsilon}(\mathbf{u})| \le |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})| + 2(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \le 5|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|\,,
$$

whence

$$
(H^-_{2r})'(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})| + |\boldsymbol{\varepsilon}(\mathbf{u})|) \lesssim \frac{(H^-_{2r})'(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})| + |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})| + |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|} |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|\,.
$$

Now, using (2.9), (2.10) and (2.5) for $\varphi(t) := H^-_{2r}(t)$, we finally get

$$
|J_{1,E}| \lesssim \frac{1}{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|} \fint_{B_r} H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(x^-_{2r}, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)\,\mathrm{d}x\,.
$$

As for $J_{1,F}$, by assumption (A4) we note that, for every $t \in [0,1]$,

$$
\big|D_\xi \mathbf{a}(x^-_{2r}, 1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) - D_\xi \mathbf{a}(x^-_{2r}, 1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell}) + t(\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell}))|)\big| \lesssim (H^-_{2r})''(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \left(\frac{|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|}{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|} \right)^{\beta_0}\,,
$$

so that

$$
|J_{1,F}| \lesssim (1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)(H^-_{2r})''(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \fint_{B_{2r}} \mathbb{1}_F(x) \left(\frac{|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|}{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|} \right)^{1+\beta_0} \mathrm{d}x\,.
$$

For a.e. $x \in F$, we have

$$
1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})| + |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})| < \frac{3}{2}(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)\,,
$$

whence, using again (2.9), (2.10) for $\varphi(t) = H^-_{2r}(t)$, we obtain

$$
\begin{aligned}
\frac{|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|^2}{(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^2} = \frac{(H^-_{2r})'(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{(H^-_{2r})'(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)} \cdot \frac{|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|^2}{(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^2} &\lesssim \frac{(H^-_{2r})'(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})| + |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|^2}{H^-_{2r}(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})| + |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)} \\
&\sim \frac{H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(x^-_{2r}, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{H^-_{2r}(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}\,.
\end{aligned}
$$

Combining the previous estimates and using Jensen's inequality with $\frac{1+\beta_0}{2} < 1$, we get

$$
|J_{1,F}| \lesssim (H^-_{2r})'(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \fint_{B_{2r}} \left(\frac{H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(x^-_{2r}, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{H^-_{2r}(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)} \right)^{\frac{1+\beta_0}{2}} \mathrm{d}x\,.
$$

Collecting the estimates for $J_{1,E}$ and $J_{1,F}$, we then infer

$$
|J_1| \lesssim (H^-_{2r})'(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \left[\fint_{B_{2r}} \frac{H^-_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{H^-_{2r}(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}\,\mathrm{d}x + \left(\fint_{B_{2r}} \frac{H^-_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{H^-_{2r}(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}\,\mathrm{d}x \right)^{\frac{1+\beta_0}{2}} \right]. \tag{4.6}
$$

In order to estimate J_2, we use (3.15), the definition of weak solution, (A3) and we preliminarily obtain

$$
\begin{aligned}
|J_2| &= \left| \fint_{B_r} \langle \mathbf{a}(x_{2r}^-, \boldsymbol{\varepsilon}(\mathbf{u})) - \mathbf{a}(x_{2r}^-, \boldsymbol{\varepsilon}(\boldsymbol{\ell})) | \boldsymbol{\varepsilon}(\boldsymbol{\zeta}) \rangle \, \mathrm{d}x \right| \\
&= \left| \fint_{B_r} \langle \mathbf{a}(x_{2r}^-, \boldsymbol{\varepsilon}(\mathbf{u})) - \mathbf{a}(x, \boldsymbol{\varepsilon}(\mathbf{u})) | \boldsymbol{\varepsilon}(\boldsymbol{\zeta}) \rangle \, \mathrm{d}x + \fint_{B_r} [\mathbf{u}[\nabla \mathbf{u}] + \mathbf{f}] \cdot \boldsymbol{\zeta} \, \mathrm{d}x \right| \\
&\leq c \fint_{B_r} |\mu(x) - \mu(x_{2r}^-)| (1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)^{q-1} \, \mathrm{d}x + c r^{\beta_0} \fint_{B_r} H'(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|) \, \mathrm{d}x + \fint_{B_r} |\mathbf{u}||\nabla \mathbf{u}||\boldsymbol{\zeta}| \, \mathrm{d}x + \fint_{B_r} |\mathbf{f}||\boldsymbol{\zeta}| \, \mathrm{d}x \\
&=: J_{2,1} + J_{2,2} + J_{2,3} + J_{2,4}.
\end{aligned}
$$

By a similar argument as for [37, eq. (4.9)], we can prove that

$$
|J_{2,1}| + |J_{2,2}| \lesssim (r^{\beta_0} + r^{\alpha_2}) [E_{\boldsymbol{\ell}}(x_0, 2r)^{\frac{q-1}{p}} + 1] (H_{2r}^-)'(1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \, .
$$

We briefly sketch the proof. Arguing as in [37, eq. (4.8)], we first can show that

$$
(1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)^p \lesssim (1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)^p \left(\mathbb{1}_E(x) 4 \frac{H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|}(x_{2r}^-, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{H_{2r}^-(1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)} + 1 \right) . \tag{4.7}
$$

Using (4.7) and (3.10) with Hölder's inequality, and (2.5) for $\varphi = H_{2r}^-$, we have

$$
\begin{aligned}
|J_{2,1}| &\lesssim r^{\beta_0} (H_{2r}^-)' \bigg(\bigg[\fint_{B_{2r}} (1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)^p \, \mathrm{d}x \bigg]^{1/p} \bigg) \\
&\lesssim r^{\beta_0} (H_{2r}^-)' \bigg((1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \bigg[\fint_{B_{2\rho}} \frac{H_{2\rho}^-(|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{H_{2\rho}^-(1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)} \, \mathrm{d}x + 1 \bigg]^{1/p} \bigg) \\
&\lesssim \rho^{\beta_0} \bigg[\bigg(\fint_{B_{2\rho}} \frac{H_{2\rho}^-(|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{H_{2\rho}^-(1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)} \, \mathrm{d}x \bigg)^{\frac{q-1}{p}} + 1 \bigg] (H_{2\rho}^-)'(1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \, .
\end{aligned}
$$

As for $J_{2,2}$, by (3.11) and the previous estimation, we have

$$
\begin{aligned}
|J_{2,2}| &\lesssim r^{\alpha_2} (H_{2r}^-)' \bigg(\fint_{B_{2r}} 1 + |\boldsymbol{\varepsilon}(\mathbf{u})| \, \mathrm{d}x \bigg) \\
&\lesssim r^{\alpha_2} \bigg[\bigg(\fint_{B_{2r}} \frac{H_{2r}^-(|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{H_{2r}^-(1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)} \, \mathrm{d}x \bigg)^{\frac{q-1}{p}} + 1 \bigg] (H_{2r}^-)'(1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell})|) \, .
\end{aligned}
$$

In order to estimate $J_{2,3}$ and $J_{2,4}$, we may argue as in the proof of Lemma 3.3, exploiting the smallness assumptions in (4.3), so we briefly sketch the proof.

Setting $\gamma := \left(\frac{p^*}{2}\right)'$ again, and using Hölder's inequality we have

$$
|J_{2,3}| \leq \bigg(\fint_{B_r} |\mathbf{u}|^{p^*} \, \mathrm{d}x \bigg)^{\frac{1}{p^*}} \bigg(\fint_{B_r} |\nabla \mathbf{u}|^{\gamma} \, \mathrm{d}x \bigg)^{\frac{1}{\gamma}} \bigg(\fint_{B_r} |\boldsymbol{\zeta}|^{p^*} \, \mathrm{d}x \bigg)^{\frac{1}{p^*}} ,
$$

then the proof is similar. The only difference is the use of the Sobolev-Korn inequality (2.29) for $\boldsymbol{\zeta}$ with $\|\boldsymbol{\varepsilon}(\boldsymbol{\zeta})\|_\infty \leq 1$. We then obtain

$$
|J_{2,3}| \leq cr \, .
$$

For what concerns $J_{2,4}$, using Hölder's inequality, the fact that $p^* > \frac{n}{n-1}$ and the Sobolev-Korn inequality for $\boldsymbol{\zeta}$ with $\|\boldsymbol{\varepsilon}(\boldsymbol{\zeta})\|_\infty \leq 1$, we have

$$
\begin{aligned}
|J_{2,4}| &\leq \bigg(\fint_{B_r} |\mathbf{f}|^n \, \mathrm{d}x \bigg)^{\frac{1}{n}} \bigg(\fint_{B_r} |\boldsymbol{\zeta}|^{\frac{n}{n-1}} \, \mathrm{d}x \bigg)^{\frac{n-1}{n}} \\
&\leq r^{-\frac{1}{\beta_1 + 1}} \|\mathbf{f}\|_{L^{n(1+\beta_1)}(\Omega)} \bigg(\fint_{B_r} |\boldsymbol{\zeta}|^{p^*} \, \mathrm{d}x \bigg)^{\frac{1}{p^*}} \\
&\leq c r^{\frac{\beta_1}{\beta_1 + 1}} \, .
\end{aligned}
$$

Collecting the previous estimates, we then obtain

$$|J_2| \le c\left((r^{\alpha_2} + r^{\beta_0}) \left[1 + \fint_{B_{2r}} \frac{H^-_{2r}(|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell})|)}{H^-_{2r}(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)} \,\mathrm{d}x \right]^{\frac{q-1}{p}} + c r^{\frac{\beta_1}{\beta_1+1}} \right) (H^-_{2r})'(1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell})|)\,. \tag{4.8}$$

The final estimate (4.4) then follows inserting (4.6) and (4.8) into (4.5). □

5. Excess decay estimate

In this section, we prove an excess decay estimate for weak solutions to (1.1). This will be the content of Lemma 5.1.

Let $x_0 \in \Omega$, $B_r(x_0) \subset \Omega$ and $\boldsymbol{\ell}_{x_0,r}$ be the traceless affine function defined in (2.22). We introduce the following *Campanato-type* excess functionals, measuring the oscillations of $\boldsymbol{\varepsilon}(\mathbf{u})$:

$$\Phi(x_0, r) := \fint_{B_r(x_0)} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|)\,\mathrm{d}x + r^{\frac{\alpha_3}{2}} \tag{5.1}$$

where $H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|} := (H^-_{B_r(x_0)})_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|}$ denotes the shifted N function $(H^-_{B_r(x_0)})_a$ with shift $a := 1 + |(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|$, and

$$\alpha_3 := \min\left\{ \alpha_2, \beta_0, \bar{\gamma}, \frac{\beta_1}{\beta_1+1} \right\}, \tag{5.2}$$

where α_2, β_0, $\bar{\gamma}$ and β_1 are from Lemma 3.2, (A4) and (3.25) and (A5), respectively.

We are now in position to prove the decay estimate.

Lemma 5.1. *Let* $\mathbf{u}$ *be a weak solution to* (1.1)*,* $M \ge 1$*, and* $\beta \in (0, \frac{\alpha_3}{4})$ *where* α_3 *is given in* (5.2)*. There exist* $\widetilde{R}_0 \in (0, R_0]$ *and* $\delta_0, \tau \in (0,1)$ *depending on* $n, p, q, \alpha, L, \nu, \beta_0, \beta_1, M$ *and* β *such that if* $B_R(x_0) \Subset \Omega$ *with* $R \in (0, \widetilde{R}_0]$*,*

$$(|\nabla \mathbf{u}|^p)_{x_0,R} + |(\mathbf{u})_{x_0,R}| \le M \tag{5.3}$$

and

$$\Phi(x_0, R) \le \delta_0\,, \tag{5.4}$$

then

$$\begin{aligned} (|\nabla \mathbf{u}|^p)_{x_0,\tau R} &\le c_1 \tau^{-n} \Phi(x_0,R)^{\frac{p}{2}} + (|\nabla \mathbf{u}|^p)_{x_0,R}, \\ |(\mathbf{u})_{x_0,\tau R}| &\le c_1 R \tau^{-n} \Phi(x_0,R)^{\frac{p}{2}} + |(\mathbf{u})_{x_0,R}|\,. \end{aligned} \tag{5.5}$$

for some $c_1 = c_1(n,p,q) > 0$*, and*

$$\Phi(x_0, \tau R) \le \tau^{2\beta} \Phi(x_0, R)\,. \tag{5.6}$$

Proof. In order to enlighten notation, we will omit the explicit dependence on x_0. Let $\boldsymbol{\ell}_R$ be the affine function defined as in (2.22) with $r = R$, and define $E_{\boldsymbol{\ell}_R}(R)$ as in (4.2) with $\boldsymbol{\ell} = \boldsymbol{\ell}_R$. We also set $\widetilde{R}_0 := \min\{R_0, \frac{1}{2}\}$ and

$$\psi^{\pm}(s) := (H^{\pm}_R)_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}(s)\,, \quad s \ge 0\,. \tag{5.7}$$

Note that (5.3) and Hölder's inequality imply

$$1 + |(\boldsymbol{\varepsilon}(\mathbf{u}))_R| \sim 1\,, \tag{5.8}$$

whence $E_{\boldsymbol{\ell}_R}(R) \sim \Phi(R)$ and, using (2.10)–(2.11),

$$\frac{s^2}{\tilde{c}_2} \le \psi^{\pm}(s) \le c s^2\,, \quad s \in (0,1]\,, \tag{5.9}$$

for some $\tilde{c}_2 \ge 1$.

Now, we define the bilinear form $\mathcal{A}$ as in (4.1) with $\boldsymbol{\ell} = \boldsymbol{\ell}_R$. Then, by (4.4) and (5.4), we get

$$\begin{aligned} \left| \fint_{B_{R/2}} \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_R) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta})\,\mathrm{d}x \right| &\le c_{\mathrm{S}} \left\{ \Phi(R)^{\frac{1}{2}} + \Phi(R)^{\frac{\beta_0}{2}} + R^{\frac{\alpha_3}{2}} \right\} \sqrt{\Phi(R)} \|\boldsymbol{\varepsilon}(\boldsymbol{\zeta})\|_{L^\infty} \\ &\le \tilde{c}_1 \left\{ \delta_0^{\frac{1}{2}} + \delta_0^{\frac{\beta_0}{2}} + R^{\frac{\alpha_3}{2}} \right\} \sqrt{\Phi(R)} \|\boldsymbol{\varepsilon}(\boldsymbol{\zeta})\|_{L^\infty} \end{aligned} \tag{5.10}$$

for every $\boldsymbol{\zeta} \in C^\infty_0(B_{R/2}; \mathbb{R}^n)$.

Moreover, we observe from Lemma 3.4 and Remark 4 that

$$\left(\fint_{B_{R/2}} [\psi^-(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_R|)]^{1+\sigma}\,\mathrm{d}x\right)^{\frac{1}{1+\sigma}} \le \tilde{c}_3 \varPhi(R) \tag{5.11}$$

holds for some constant $\tilde{c}_3 \ge 1$.

With the constants $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 \ge 1$ determined above and $\tilde{c}_5 \ge 1$ determined below, we define

$$\bar{\eta} := \max\left\{\tilde{c}_1, \sqrt{\tilde{c}_2\tilde{c}_3(2\tilde{c}_5)^{1/p}}\right\}\sqrt{\varPhi(R)}\,.$$

Then, choosing δ_0 sufficiently small, we see that

$$\bar{\eta} \le \max\left\{\tilde{c}_1, \sqrt{\tilde{c}_2\tilde{c}_3(2\tilde{c}_5)^{1/p}}\right\}\delta_0^{\frac{1}{2}} < 1\,. \tag{5.12}$$

Combining the previous estimates, we obtain

$$\left(\fint_{B_{R/2}} [\psi^-(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_R|)]^{1+\sigma}\,\mathrm{d}x\right)^{\frac{1}{1+\sigma}} \le \frac{1}{\tilde{c}_2(2\tilde{c}_5)^{1/p}}\bar{\eta}^2 \le \frac{1}{(2\tilde{c}_5)^{1/p}}\psi^-(\bar{\eta})\,. \tag{5.13}$$

For given η and ψ^- defined as above, we determine the constant δ as the one in Theorem 2.9. Then choosing δ_0 sufficiently small such that

$$\delta_0^{\frac{1}{2}} + \delta_0^{\frac{\beta_0}{2}} + \delta_0 \le \delta \tag{5.14}$$

and inserting (5.12) and (5.14) into (5.10), we obtain

$$\fint_{B_{R/2}} \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_R) : \boldsymbol{\varepsilon}(\boldsymbol{\zeta})\,\mathrm{d}x \le \frac{\tilde{c}_1(\delta_0^{\frac{1}{2}} + \delta_0^{\frac{\beta_0}{2}} + \delta_0)}{\max\left\{\tilde{c}_1, \sqrt{\tilde{c}_2\tilde{c}_3(2\tilde{c}_5)^{1/p}}\right\}}\bar{\eta}\|\boldsymbol{\varepsilon}(\boldsymbol{\zeta})\|_\infty \le \delta\bar{\eta}\|\boldsymbol{\varepsilon}(\boldsymbol{\zeta})\|_\infty.$$

Note that, taking into account (5.8), $\mathcal{A}$ complies with (2.33) by virtue of (A1) and (A2). Therefore, we can apply Theorem 2.9 to the function $\mathbf{u} - \boldsymbol{\ell}_R$ in place of $\mathbf{v}$ and $\varphi = \psi^-$, so that recalling the definition of ψ^- in (5.7) we have

$$\fint_{B_{R/2}} \psi^-(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_R - \boldsymbol{\varepsilon}(\mathbf{h})|)\,\mathrm{d}x \le \eta\psi^-(\bar{\eta})\,,$$

where $\mathbf{h}$ is the $\mathcal{A}$-Stokes function in $B_{R/2}$ with $\mathbf{h} = \mathbf{u} - \boldsymbol{\ell}_R$ on $\partial B_{R/2}$. Moreover, again by (5.9) and (5.12), we obtain

$$\fint_{B_{R/2}} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_R - \boldsymbol{\varepsilon}(\mathbf{h})|)\,\mathrm{d}x \le \tilde{c}_4\eta\varPhi(R) \tag{5.15}$$

for a suitable constant $\tilde{c}_4 > 0$.

Now, since $\mathbf{u} - \boldsymbol{\ell}_R - \mathbf{h} = \mathbf{0}$ on $\partial B_{R/2}$, by Lemma 2.8, (5.15), (5.2), and (5.8),

$$\begin{aligned}
\fint_{B_{R/2}} H^+_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}\left(\frac{|\mathbf{u} - \boldsymbol{\ell}_R - \mathbf{h}|}{R}\right)\mathrm{d}x &\le c\left(\fint_{B_{R/2}} \left[H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_R - \boldsymbol{\varepsilon}(\mathbf{h})|)\right]^\theta \mathrm{d}x\right)^{1/\theta} \\
&\quad + c\left(R^{\alpha - \frac{n(q-p)}{p}} + R^\alpha(1 + |(\boldsymbol{\varepsilon}(\mathbf{u}))_R|)^{q-p}\right)(1 + |(\boldsymbol{\varepsilon}(\mathbf{u}))_R|)^p \\
&\le c\eta\varPhi(R) + R^{\alpha - \frac{n(q-p)}{p}} \\
&\le c\big(\eta + R^{\frac{1}{2}[\alpha - \frac{n(q-p)}{p}]}\big)\varPhi(R) \\
&\le c\tilde{\eta}\varPhi(R)\,,
\end{aligned} \tag{5.16}$$

where

$$\tilde{\eta} := \eta + R^{\frac{1}{2}[\alpha - \frac{n(q-p)}{p}]}\,.$$

Let $\tau \in (0, \frac{1}{4})$ be a sufficiently small constant to be determined later. We will prove that

$$\fint_{B_{2\tau R}} H^+_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell}_{2\tau R})|}\left(\frac{|\mathbf{u} - \boldsymbol{\ell}_{2\tau R}|}{2\tau R}\right)\mathrm{d}x \le c_2\left(\tau^{-(n+\tilde{q})}\tilde{\eta} + \tau^2\right)\varPhi(R)\,, \tag{5.17}$$

for some constant $c_2 \ge 1$, where $\boldsymbol{\ell}_{2\tau R} := \boldsymbol{\ell}^{\mathcal{T}}_{x_0,2\tau R}$ is the affine function minimizing (2.23) with $r = 2\tau R$.

To this end, let $\mathbf{h}$ be the $\mathcal{A}$-Stokes map above and define, correspondingly,

$$\tilde{\boldsymbol{\ell}}_{r,\mathbf{h}} := \tilde{\boldsymbol{\ell}}_{x_0,r,\mathbf{h}}(x) := (\nabla\mathbf{h})_{B_r}(x - x_0)\,.$$

Then, for $x \in B_{R/2}$, from (2.34) and (5.15) we get

$$
\begin{aligned}
\mathbf{h}(x) - \mathbf{h}(x_0) &\le R\|\nabla \mathbf{h}\|_{L^\infty(B_{R/2})} \\
&\le cR \fint_{B_R} |\boldsymbol{\varepsilon}(\mathbf{h})|\,\mathrm{d}x \le cR\left(\fint_{B_R} |\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_R - \boldsymbol{\varepsilon}(\mathbf{h})|\,\mathrm{d}x + \fint_{B_R} |\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_R|\,\mathrm{d}x \right) \\
&\le cR\left((H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|})^{-1}(\eta \Phi(R)) + (H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|})^{-1}(\Phi(R)) \right) \\
&\le cR\sqrt{\Phi(R)}\,.
\end{aligned} \tag{5.18}
$$

Similarly, we also obtain that

$$
\begin{aligned}
\sup_{B_{2\tau R}} |\mathbf{h} - \tilde{\boldsymbol{\ell}}_{2\tau R,\mathbf{h}}| &\le 2\tau R \sup_{B_{2\tau R}} |\nabla \mathbf{h} - (\nabla \mathbf{h})_{B_{2\tau R}}| \\
&\le (2\tau R)^2 \sup_{B_{2\tau R}} |\nabla^2 \mathbf{h}| \\
&\le (2\tau R)^2 \sup_{B_{R/2}} |\nabla^2 \mathbf{h}| \\
&\le c\tau^2 R \left(\fint_{B_R} |\boldsymbol{\varepsilon}(\mathbf{h})|\,\mathrm{d}x \right) \le c\tau^2 R\sqrt{\Phi(R)}\,.
\end{aligned} \tag{5.19}
$$

Observe now that by (2.24) with $r = 2\tau R$ and $\boldsymbol{\ell} = \boldsymbol{\ell}_R + \mathbf{h}(x_0)$, (5.9), (5.16) and (5.18),

$$
\begin{aligned}
|\boldsymbol{\varepsilon}(\boldsymbol{\ell}_{2\tau R})| &\le |(\nabla \mathbf{u})_R| + (n+2) \fint_{B_{2\tau R}} \frac{|\mathbf{u} - \boldsymbol{\ell}_R - \mathbf{h}(x_0)|}{2\tau R}\,\mathrm{d}x \\
&\le |(\nabla \mathbf{u})_R| + \frac{c}{\tau^{n+1}} \left(\fint_{B_{R/2}} \left| \frac{\mathbf{u} - \boldsymbol{\ell}_R - \mathbf{h}}{R} \right| \mathrm{d}x + \fint_{B_{R/2}} \left| \frac{\mathbf{h} - \mathbf{h}(x_0)}{R} \right| \mathrm{d}x \right) \\
&\le |(\nabla \mathbf{u})_R| + c\tau^{-n-1}\sqrt{\Phi(R)} \\
&\le |(\nabla \mathbf{u})_R| + c\tau^{-n-1}\delta_0^{\frac{1}{2}} \le M + 1\,.
\end{aligned}
$$

In the last inequality, we have chosen $\delta_0 > 0$ sufficiently small such that $c(\frac{\tau}{2})^{-n-1}\delta_0^{1/2} \le 1$ with $\tau > 0$ that will be determined in the end of the proof. Then it follows that $1 + |\boldsymbol{\varepsilon}(\boldsymbol{\ell}_{2\tau R})| \sim 1 \sim 1 + |(\boldsymbol{\varepsilon}(\mathbf{u}))_R|$, and hence $H^+_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell}_{2\tau R})|}(s) \sim H^+_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}(s)$. Moreover, by (2.26) with $\varphi = H^+_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}$ and $\boldsymbol{\ell} = \boldsymbol{\ell}_R + \tilde{\boldsymbol{\ell}}_{2\tau R,\mathbf{h}}$,

$$
\begin{aligned}
&\fint_{B_{2\tau R}} H^+_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell}_{2\tau R})|}\left(\frac{|\mathbf{u} - \boldsymbol{\ell}_{2\tau R}|}{2\tau R} \right) \mathrm{d}x \le c \fint_{B_{2\tau R}} H^+_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}\left(\frac{|\mathbf{u} - \boldsymbol{\ell}_{2\tau R}|}{2\tau R} \right) \mathrm{d}x \\
&\le c \fint_{B_{2\tau R}} H^+_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}\left(\frac{|\mathbf{u} - \boldsymbol{\ell}_R - \tilde{\boldsymbol{\ell}}_{2\tau R,\mathbf{h}}|}{2\tau R} \right) \mathrm{d}x \\
&\le c \fint_{B_{2\tau R}} H^+_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}\left(\frac{|\mathbf{u} - \boldsymbol{\ell}_R - \mathbf{h}|}{2\tau R} \right) \mathrm{d}x + c \fint_{B_{2\tau R}} H^+_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}\left(\frac{|\mathbf{h} - \tilde{\boldsymbol{\ell}}_{2\tau R,\mathbf{h}}|}{2\tau R} \right) \mathrm{d}x \\
&\le c\tau^{-(n+\tilde{q})} \fint_{B_{R/2}} H^+_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}\left(\frac{|\mathbf{u} - \boldsymbol{\ell}_R - \mathbf{h}|}{R} \right) \mathrm{d}x + c \fint_{B_{2\tau R}} H^+_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}\left(\frac{|\mathbf{h} - \tilde{\boldsymbol{\ell}}_{2\tau R,\mathbf{h}}|}{2\tau R} \right) \mathrm{d}x\,.
\end{aligned} \tag{5.20}
$$

Combining the estimates (5.20), (5.16), (5.19) and (5.9) yields

$$
\begin{aligned}
\fint_{B_{2\tau R}} H^+_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell}_{2\tau R})|}\left(\frac{|\mathbf{u} - \boldsymbol{\ell}_{2\tau R}|}{2\tau R} \right) \mathrm{d}x &\le c\tau^{-(n+\tilde{q})}\tilde{\eta}\Phi(R) + cH^+_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}(\tau\sqrt{\Phi(R)}) \\
&\le c\tau^{-(n+\tilde{q})}\tilde{\eta}\Phi(R) + c\tau^2\Phi(R)\,,
\end{aligned}
$$

and the proof of (5.17) is concluded.

Now, we prove the inequality (5.6). By (5.1), (2.27) for $\mathbf{Q} = \boldsymbol{\varepsilon}(\boldsymbol{\ell}_{2\tau R})$, the Caccioppoli inequality (3.12) for $\boldsymbol{\ell} = \boldsymbol{\ell}_{2\tau R}$, and (5.17), we obtain

$$\begin{aligned}\Phi(\tau R) &= \fint_{B_{\tau R}} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{\tau R}|}\left(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{\tau R}|\right) \mathrm{d}x + (\tau R)^{\frac{\alpha_3}{2}} \\ &\le c \fint_{B_{\tau R}} H_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell}_{2\tau r})|}(x, |\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\boldsymbol{\ell}_{2\tau R})|) \, \mathrm{d}x + (\tau R)^{\frac{\alpha_3}{2}} \\ &\le c \fint_{B_{2\tau R}} H^+_{1+|\boldsymbol{\varepsilon}(\boldsymbol{\ell}_{2\tau R})|}\left(\frac{|\mathbf{u} - \boldsymbol{\ell}_{2\tau R}|}{2\tau R}\right) \mathrm{d}x + (\tau r)^{\alpha_3} + (\tau R)^{\frac{\alpha_3}{2}} \\ &\le c_3 \left(\tau^{-(n+\tilde{q})}\tilde{\eta} + \tau^2\right) \Phi(R) + c_3 \tau^{\frac{\alpha_3}{2}} R^{\frac{\alpha_3}{2}} \\ &\le \tau^{2\beta} \Phi(R) \,,\end{aligned}$$

where we chose $\tau \in (0, \frac{1}{4})$ and $\tilde{\eta} \in (0,1)$ such that

$$c_3 \tau^{\frac{\alpha_3}{2} - 2\beta} \le \frac{1}{2} \,, \quad 4c_3 \tau^{2-2\beta} \le \frac{1}{4} \,, \quad \text{and} \quad c_3 \tau^{-(n+\tilde{q})} \tilde{\eta} \le \frac{1}{4} \tau^{2\beta}.$$

With this choice, δ_0 is determined as well. This estimate corresponds to (5.6).

We now turn to the proof of (5.5). Using the Jensen type inequality in Lemma 2.2 with $\psi(t) = H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}(t^{1/p})$, Korn's inequality and (5.9), we obtain

$$\begin{aligned}(|\nabla \mathbf{u}|^p)_{\tau R} &\le \tau^{-n} \fint_{B_R} |\nabla \mathbf{u} - (\nabla \mathbf{u})_R|^p \, \mathrm{d}x + |(\nabla \mathbf{u})_R|^p \\ &\le c\tau^{-n} \left[(H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|})^{-1} \left(\fint_{B_R} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}(|\nabla \mathbf{u} - (\nabla \mathbf{u})_R|) \, \mathrm{d}x \right) \right]^p + (|\nabla \mathbf{u}|^p)_R \\ &\le c\tau^{-n} \left[(H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|})^{-1} \left(\fint_{B_R} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_R|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_R|) \, \mathrm{d}x \right) \right]^p + (|\nabla \mathbf{u}|^p)_R \\ &\le c\tau^{-n} \Phi(R)^{\frac{p}{2}} + (|\nabla \mathbf{u}|^p)_R \,.\end{aligned} \tag{5.21}$$

Let $\mathbf{v} := \mathbf{u} - (\nabla \mathbf{u})_R (x - x_0)$. Then, using Poincaré inequality, we have

$$\begin{aligned}|(\mathbf{v})_{\tau R}| \le \tau^{-n} \fint_{B_R} |\mathbf{v} - (\mathbf{v})_R| \, \mathrm{d}x + |(\mathbf{v})_R| &\le \tau^{-n} \left(\fint_{B_R} |\mathbf{v} - (\mathbf{v})_R|^{p^*} \, \mathrm{d}x \right)^{\frac{1}{p^*}} + |(\mathbf{v})_R| \\ &\lesssim \tau^{-n} R \left(\fint_{B_R} |\nabla \mathbf{v}|^p \, \mathrm{d}x \right)^{\frac{1}{p}} + |(\mathbf{v})_R| \,.\end{aligned}$$

Now, since $(\mathbf{v})_{\tau R} = (\mathbf{u})_{\tau r}$, $(\mathbf{v})_R = (\mathbf{u})_R$ and $\nabla \mathbf{v} = \nabla \mathbf{u} - (\nabla \mathbf{u})_R$, we deduce that

$$|(\mathbf{u})_{\tau R}| \lesssim \tau^{-n} R \left(\fint_{B_R} |\nabla \mathbf{u} - (\nabla \mathbf{u})_R|^p \, \mathrm{d}x \right)^{\frac{1}{p}} + |(\mathbf{u})_R| \,, \tag{5.22}$$

whence the second estimate in (5.5) follows arguing as in (5.21). □

Now, we prove that the previous excess improvement estimate can be iterated at each scale.

Lemma 5.2 (Iteration)**.** *Let* $\mathbf{u}$ *be a weak solution to* (1.1)*,* $M \ge 1$*, and* $\beta \in (0, \frac{\alpha_3}{4})$ *where* α_3 *is given in* (5.2)*, and* $\widetilde{R}_0$ *as in Lemma 5.1. There exist* $\delta_0, \tau \in (0,1)$ *depending on* $n, p, q, \alpha, L, \nu, \beta_0, M$ *and* β *such that if* $B_R(x_0) \Subset \Omega$ *with* $R \in (0, \widetilde{R}_0]$*,*

$$1 + (|\nabla \mathbf{u}|^p)_{x_0,R} + |(\mathbf{u})_{x_0,R}| \le M \qquad \text{and} \qquad \Phi(x_0, R) \le \delta_1 \,,$$

then for each $j \in \mathbb{N} \cup \{0\}$ *we have*

(i) $\Phi(x_0, \tau^j R) \le \tau^{2\beta j} \Phi(x_0, R) \,,$

(ii) $(|\nabla \mathbf{u}|^p)_{x_0,\tau^j R} + |(\mathbf{u})_{x_0,\tau^j R}| \le M - \tau^{\beta j} \,,$

and moreover, for every $0 < r \le R$

$$\fint_{B_r(x_0)} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|) \, \mathrm{d}x \le \bar{c} \left(\frac{r}{R}\right)^{2\beta} \Phi(x_0, R) \tag{5.23}$$

for some $\bar{c} = \bar{c}(n, p, q, \alpha, L, \nu, \beta_0, \beta_1, M, \beta)$.

Proof. Recall the constants δ_0, τ and c_1 determined in Lemma 5.1, and set

$$\delta_1 := \left[\min\left\{\delta_0,\ \left((2c_1)^{-1}\tau^n\left(1-\tau^\beta\right)\right)^{\frac{1}{p}}\right\}\right]^2 . \tag{5.24}$$

Note that the inequalities in (i) and (ii) are trivial when $j = 0$. Then, given $j_0 \geq 0$, we assume that the inequalities in (i) and (ii) hold for every $j = 0, 1, \dots, j_0$. By the definition of δ_1 in (5.24), (i) and (ii) for $j = j_0$, we deduce that

$$(|\nabla \mathbf{u}|^p)_{x_0,\tau^{j_0}R} + |(\mathbf{u})_{x_0,\tau^{j_0}R}| \leq M \qquad \text{and} \qquad \Phi(x_0, \tau^{j_0}R) \leq \Phi(x_0, R) \leq \delta_1 \leq \delta_0 ,$$

so we can apply Lemma 5.1 with $\tau^{j_0}R$ in place of r to obtain

$$\Phi(x_0, \tau^{j_0+1}R) \leq \tau^{2\beta}\Phi(x_0, \tau^{j_0}R) \leq \tau^{2\beta(j_0+1)}\Phi(x_0, R) ,$$

which proves (i) for $j = j_0 + 1$. For what concerns (ii), we have from (5.5) with $\tau^{j_0}R$ in place of r and (i) for $j = j_0$ that

$$\begin{aligned}
(|\nabla \mathbf{u}|^p)_{x_0,\tau^{j_0+1}R} + |(\mathbf{u})_{x_0,\tau^{j_0+1}R}| &\leq (|\nabla \mathbf{u}|^p)_{x_0,\tau^{j_0}R} + |(\mathbf{u})_{x_0,\tau^{j_0}R}| + 2^{1-n}c_1\tau^{-n}\Phi(x_0, \tau^{j_0}R)^{\frac{p}{2}} \\
&\leq M - \tau^{\beta j_0} + 2c_1\tau^{-n}\tau^{\beta j_0}\delta_1^{\frac{p}{2}} \\
&\leq M - \tau^{\beta j_0} + \tau^{\beta j_0}\left(1-\tau^\beta\right) = M - \tau^{\beta(j_0+1)} ,
\end{aligned}$$

which corresponds to (ii) for $j = j_0 + 1$. Therefore, by induction, assertions (i) and (ii) hold for every integer $j \geq 1$. We now prove (5.23). Let $r \in (0, R]$. There exists $j \geq 0$ such that $\tau^{j+1}R < r \leq \tau^j R$. Using (2.27) and (i) above, we can write

$$\begin{aligned}
\fint_{B_r} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|)\,\mathrm{d}x &\leq c\fint_{B_r} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,\tau^j R}|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,\tau^j R}|)\,\mathrm{d}x \\
&\leq \frac{c}{\tau^n}\fint_{B_{\tau^j R}} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,\tau^j R}|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,\tau^j R}|)\,\mathrm{d}x \\
&\leq \frac{c}{\tau^n}\Phi(x_0, \tau^j R) \\
&\leq \frac{c}{\tau^n}\tau^{2\beta j}\Phi(x_0, R) \\
&\leq \bar{c}\left(\frac{r}{R}\right)^{2\beta}\Phi(x_0, R) ,
\end{aligned}$$

then the proof is concluded. □

6. Proof of Theorem 1.1

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $x_1 \in \Omega$ be such that

$$\liminf_{r\to 0^+} \fint_{B_r(x_1)} |\mathbf{V}_{H^-_{B_r(x_1)}}(\boldsymbol{\varepsilon}(\mathbf{u})) - (\mathbf{V}_{H^-_{B_r(x_1)}}(\boldsymbol{\varepsilon}(\mathbf{u})))_{x_1,r}|^2\,\mathrm{d}x = 0$$

and

$$K_{x_1} := \limsup_{r\to 0^+}\left((|\nabla \mathbf{u}|^p)_{x_1,r} + |(\mathbf{u})_{x_1,r}|\right) < +\infty .$$

In particular, taking into account (2.16) and (2.19), the first limit implies

$$\liminf_{r\to 0}\fint_{B_r} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_1,r}|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_1,r}|)\,\mathrm{d}x = 0 ,$$

where $H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_1,r}|}$ is the shifted N-function of $H^-_{B_r}$ with shift $1 + |(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_1,r}|$. Let $M := 2(2 + K_{x_1})$ and, with δ_1 and $\widetilde{R}_0$ as in Lemma 5.2,

$$\varepsilon := \frac{\delta_1}{4} , \qquad \bar{R} := \min\left\{\left(\frac{\delta_1}{4}\right)^{\frac{2}{\alpha_3}}, \widetilde{R}_0\right\} .$$

Then we can find $R \leq \bar{R}$ small enough such that $B_R(x_1) \Subset \Omega$ and

$$\fint_{B_R(x_1)} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_1,R}|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_1,R}|)\,\mathrm{d}x < \varepsilon \qquad \text{and} \qquad (|\nabla \mathbf{u}|^p)_{x_1,R} + |(\mathbf{u})_{x_1,R}| < K_{x_1} + 1 ,$$

which implies that

$$\Phi(x_1, R) \leq \frac{\delta_1}{2} \qquad \text{and} \qquad 1 + (|\nabla \mathbf{u}|^p)_{x_1,R} + |(\mathbf{u})_{x_1,R}| \leq \frac{M}{2} .$$

From the absolute continuity of the integrals, there exists $R_1 \in (0, R]$ such that for every $x_0 \in B_{R_1}(x_1) \Subset \Omega$,

$$\Phi(x_0, R) \leq \delta_1 \quad \text{and} \quad 1 + (|\nabla \mathbf{u}|^p)_{x_0,R} + |(\mathbf{u})_{x_0,R}| \leq M\,.$$

Therefore, by applying Lemma 5.2 to each $B_R(x_0) \Subset \Omega$, with $x_0 \in B_{R_1}(x_1)$, and recalling (2.19), we deduce that for each $\beta \in (0, \frac{\alpha_3}{4})$,

$$\begin{aligned}
\fint_{B_r(x_0)} \frac{|\mathbf{V}_p(\boldsymbol{\varepsilon}(\mathbf{u})) - (\mathbf{V}_p(\boldsymbol{\varepsilon}(\mathbf{u})))_{x_0,r}|^2}{r^{2\beta}}\,\mathrm{d}x &\leq \frac{c}{r^{2\beta}} \fint_{B_r(x_0)} |\mathbf{V}_{H^-_{B_r(x_0)}}(\boldsymbol{\varepsilon}(\mathbf{u})) - (\mathbf{V}_{H^-_{B_r(x_0)}}(\boldsymbol{\varepsilon}(\mathbf{u})))_{x_0,r}|^2\,\mathrm{d}x \\
&\leq \frac{c}{r^{2\beta}} \fint_{B_r(x_0)} H^-_{1+|(\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|}(|\boldsymbol{\varepsilon}(\mathbf{u}) - (\boldsymbol{\varepsilon}(\mathbf{u}))_{x_0,r}|)\,\mathrm{d}x \\
&\leq \frac{c\delta_1}{R^{2\beta}}\,,
\end{aligned}$$

for every $r \in (0, R]$ and $x_0 \in B_{R_1}(x_1)$.

This implies $\mathbf{V}_p(\boldsymbol{\varepsilon}(\mathbf{u})) \in C^{0,\beta}(B_{R_1}(x_1), \mathbb{R}^{n\times n})$. Since $\mathbf{V}_p^{-1} \in C^{0,\sigma}(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n})$, for some $\sigma \in (0, 1]$ (see, e.g., [19, Lemma 2.10]), we obtain $\boldsymbol{\varepsilon}(\mathbf{u}) \in C^{0,\beta\sigma}(B_{R_1}(x_1), \mathbb{R}^{n\times n}_{\mathrm{sym}})$. Finally, by Korn's inequality, $\nabla \mathbf{u} \in C^{0,\beta\sigma}(B_{R_1}(x_1), \mathbb{R}^{n\times n})$.

To conclude, we prove the Morrey regularity for the pressure π on the regular set. For this, we fix a ball $B_r = B_r(x_0) \subset \Omega$, with $r \in (0, R]$ and $x_0 \in B_{R_1}(x_1)$ as above, and consider the linear functional $F : W^{1,p}_0(B_r, \mathbb{R}^n) \to \mathbb{R}$ defined as

$$F(\boldsymbol{\varphi}) := -\int_{B_r} \mathbf{a}(x, \boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi})\,\mathrm{d}x + \int_{B_r} [\mathbf{u}[\nabla \mathbf{u}] + \mathbf{f}] \cdot \boldsymbol{\varphi}\,\mathrm{d}x\,.$$

Then, by (1.6), we have $F(\boldsymbol{\varphi}) = 0$ for all $\boldsymbol{\varphi} \in C^\infty_{0,\mathrm{div}}(B_r, \mathbb{R}^n)$. Using (A1), for every $\boldsymbol{\varphi} \in W^{1,p}_0(B_r, \mathbb{R}^n)$ we can write

$$\begin{aligned}
|F(\boldsymbol{\varphi})| &\leq L \int_{B_r} H'(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)|\boldsymbol{\varepsilon}(\boldsymbol{\varphi})|\,\mathrm{d}x + \int_{B_r} |\mathbf{u}||\nabla \mathbf{u}||\boldsymbol{\varepsilon}(\boldsymbol{\varphi})|\,\mathrm{d}x + \int_{B_r} |\mathbf{f}||\boldsymbol{\varphi}|\,\mathrm{d}x \\
&=: J_1 + J_2 + J_3\,.
\end{aligned}$$

We start by estimating J_1. Since $1 + |\boldsymbol{\varepsilon}(\mathbf{u})| \in C^{0,\beta\sigma}(B_{R_1}(x_1))$, it follows that $|\boldsymbol{\varepsilon}(\mathbf{u})| \in L^\infty(B_{R_1}(x_1))$, which implies $1 + |\boldsymbol{\varepsilon}(\mathbf{u})| \in \mathscr{L}^{p,\lambda}(B_{R_1}(x_1))$, for every $0 \leq \lambda < n + p\beta\sigma$. Recall also that $\mathscr{L}^{p,\lambda}(B_{R_1}(x_1)) \simeq L^{p,\lambda}(B_{R_1}(x_1))$ for every $0 \leq \lambda < n$ (see, e.g., [25, Chapter II]).

Hence, by Hölder's inequality, we get

$$\begin{aligned}
J_1 &\leq L \left(\int_{B_r} [H'(x, 1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)]^{p'}\,\mathrm{d}x \right)^{\frac{1}{p'}} \|\boldsymbol{\varepsilon}(\boldsymbol{\varphi})\|_{L^p(B_r)} \\
&\lesssim \left(\int_{B_r} (1 + |\boldsymbol{\varepsilon}(\mathbf{u})|)^p\,\mathrm{d}x \right)^{\frac{1}{p'}} \|\boldsymbol{\varepsilon}(\boldsymbol{\varphi})\|_{L^p(B_r)} \\
&\lesssim r^{\frac{\lambda}{p'}} \|\boldsymbol{\varepsilon}(\boldsymbol{\varphi})\|_{L^p(B_r)}
\end{aligned} \tag{6.1}$$

for every $0 \leq \lambda < n$.

As for J_2, using $\|\mathbf{u}\|_{L^\infty(B_r)}$, $\|\nabla \mathbf{u}\|_{L^\infty(B_r)} < +\infty$, Hölder's inequality, Poincarè and Korn's inequalities, we get

$$\begin{aligned}
J_2 &\lesssim \|\mathbf{u}\|_{L^\infty(B_r)} \|\nabla \mathbf{u}\|_{L^\infty(B_r)} \int_{B_r} |\boldsymbol{\varphi}|\,\mathrm{d}x \\
&\lesssim r^{n(1-\frac{1}{p^*})} \left(\int_{B_r} |\boldsymbol{\varphi}|^{p^*} \right)^{\frac{1}{p^*}} \\
&\lesssim r^{n(1-\frac{1}{p^*})+1} \|\boldsymbol{\varepsilon}(\boldsymbol{\varphi})\|_{L^p(B_r)}\,.
\end{aligned} \tag{6.2}$$

Similarly,

$$\begin{aligned}
J_3 &\lesssim r^{n(1-\frac{1}{p^*}-\frac{1}{n})} \left(\int_{B_r} |\mathbf{f}|^n\,\mathrm{d}x \right)^{\frac{1}{n}} \left(\int_{B_r} |\boldsymbol{\varphi}|^{p^*} \right)^{\frac{1}{p^*}} \\
&\lesssim r^{n(1-\frac{1}{p^*})} \|\mathbf{f}\|_{L^n(\Omega)} \|\boldsymbol{\varepsilon}(\boldsymbol{\varphi})\|_{L^p(B_r)} \\
&\lesssim r^{n(1-\frac{1}{p^*})} \|\boldsymbol{\varepsilon}(\boldsymbol{\varphi})\|_{L^p(B_r)}\,.
\end{aligned} \tag{6.3}$$

Therefore, F can be identified with $\mathbf{F} \in (W^{1,p}_0(B_r, \mathbb{R}^n))^* = W^{-1,p'}(B_r, \mathbb{R}^n)$. Since $\langle \mathbf{F}, \boldsymbol{\varphi} \rangle = 0$ for all $\boldsymbol{\varphi} \in C^\infty_{0,\mathrm{div}}(B_r, \mathbb{R}^n)$, by [42, Chapter II, Lemma 2.2.2] we infer the existence of $\widetilde{\pi} \in L^{p'}(B_r)$ such that $\nabla \widetilde{\pi} = \mathbf{F}$ in the sense of distributions, $\int_{B_r} \widetilde{\pi}\,\mathrm{d}x = 0$ and

$$\|\widetilde{\pi}\|_{L^{p'}(B_r)} \leq \|\mathbf{F}\|_{W^{-1,p'}(B_r,\mathbb{R}^n)}\,.$$

Now, since $(\mathbf{u},\pi)$ is a weak solution to (1.1), we have also $\nabla\pi = \mathbf{F}$ in the sense of distributions. Thus, the pressure is given by $\pi = \widetilde{\pi} + (\pi)_{x_0,r}$, and it complies with

$$\|\pi - (\pi)_{x_0,r}\|_{L^{p'}(B_r)} \le \|\mathbf{F}\|_{W^{-1,p'}(B_r,\mathbb{R}^n)}\,.$$

Now, collecting the previous estimates (6.1)–(6.3), and choosing $\lambda = np'(1-\frac{1}{p^*})$, we have

$$\|\pi - (\pi)_{x_0,r}\|_{L^{p'}(B_r)} \lesssim r^{\frac{\lambda}{p'}} + r^{n(1-\frac{1}{p^*})} \lesssim r^{n(1-\frac{1}{p^*})}\,,$$

whence $\pi \in \mathscr{L}^{p',n(1-\frac{1}{p^*})}(B_{R_1}(x_1))$. This concludes the proof. □

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