

GENERALIZED BMO-TYPE SEMINORMS AND VECTOR-VALUED SOBOLEV FUNCTIONS

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Dedicated to G. MoscarIELLO on the occasion of her 70th birthday

ABSTRACT. We establish a pointwise limit theorem for a broad class of parameter-dependent BMO-type seminorms as the parameter tends to zero. By introducing novel BMO-type seminorms, we provide a unified framework that extends several existing results and yields non-distributional characterizations of Sobolev-type spaces, both in the scalar and in the vector-valued setting. More precisely, for any open set $\Omega \subset \mathbb{R}^n$ and any $p \in (1, \infty)$, we provide a characterization of the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^m)$. In addition, we characterize the space $E^{1,p}(\Omega; \mathbb{R}^n)$ of L^p maps with p -integrable distributional symmetric gradient.

Finally, for all $p \in [1, \infty)$, we show that these seminorms converge to integral functionals with convex, p -homogeneous integrands associated with the distributional gradient and the symmetric gradient.

CONTENTS

1. Introduction	2
1.1. Plan of the paper	7
2. Notation and preliminary results	7
3. p -core functionals and generalized BMO-type seminorms	12
4. Non-distributional characterization of Sobolev-type spaces	24
5. Examples and applications	26
5.1. Characterization of $W^{1,p}(\Omega; \mathbb{R}^m)$	28
5.2. Characterization of $E^{1,p}(\Omega; \mathbb{R}^n)$	32
5.3. Other examples of p -core functionals	35
References	37

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be an open set. The characterization of Sobolev spaces through integral functionals that do not explicitly involve weak derivatives has been a central theme in analysis and the calculus of variations. Given $u \in L^p(\Omega)$, $p \in [1, +\infty)$, one seeks structural conditions under which the boundedness of suitable non-distributional one-parameter energies characterizes membership in $W^{1,p}(\Omega)$ and yields, in the limit, the $L^p(\Omega)$ norm of the gradient.

A foundational result in this direction is established in [BBM01] where the classical Sobolev seminorm is encoded in the asymptotical behavior of purely nonlocal difference-quotient energies. For $s \in (0, 1)$ define the fractional seminorm

$$[u]_{W^{s,p}(\Omega)}^p = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy.$$

If $1 < p < \infty$ and Ω is bounded with Lipschitz boundary, then for every $u \in W^{1,p}(\Omega)$

$$\lim_{s \uparrow 1} (1 - s) [u]_{W^{s,p}(\Omega)}^p = K_{n,p} \int_{\Omega} |\nabla u|^p dx,$$

where $K_{n,p} > 0$ depends only on n and p . Conversely, if

$$\liminf_{s \uparrow 1} (1 - s) [u]_{W^{s,p}(\Omega)}^p < +\infty,$$

then $u \in W^{1,p}(\Omega)$.

An alternative formulation, introduced in [BBM01, Theorem 2], replaces fractional powers by radial kernels ρ_ε concentrating at the origin. In this case one obtains

$$(1.1) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \int_{\Omega} \frac{|u(y) - u(x)|^p}{|y - x|^p} \rho_\varepsilon(|y - x|) dy dx = K_{p,n} \int_{\Omega} |\nabla u|^p dx,$$

with the convention that the RHS in (1.1) is equal to $+\infty$ if $u \notin W^{1,p}(\Omega)$. These results provide a derivative-free characterization of Sobolev spaces based purely on nonlocal quantities.

A conceptually different approach was introduced in [FMS18], where p -mean oscillation-type functionals inspired by BMO seminorms were considered. Given a function $u \in L_{\text{loc}}^p(\Omega)$ (here Ω is an arbitrary open set) and any $\varepsilon > 0$, they define

$$\kappa_\varepsilon(u; p; \Omega) := \varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \int_{Q'} \left| u - \int_{Q'} u \right|^p dx,$$

where the supremum is taken over all families \mathcal{G}_ε of disjoint ε -cubes Q' of side length ε contained in Ω and arbitrary orientation. They proved that for every $u \in L_{\text{loc}}^p(\Omega)$,

$$\nabla u \in L^p(\Omega; \mathbb{R}^n) \iff \liminf_{\varepsilon \rightarrow 0} \kappa_\varepsilon(u; p; \Omega) < +\infty.$$

Moreover, they identified the limit

$$\lim_{\varepsilon \rightarrow 0^+} \kappa_\varepsilon(u; p; \Omega) = \gamma(n, p) \|\nabla u\|_{L^p(\Omega)}^p,$$

where $\gamma(n, p)$ is a positive constant that depends only on n and p .

Variants of this result were subsequently developed in [FFGLBS20], [FGLBS20, DFF20]. In these works two main generalizations are introduced. First, the cubes Q' are replaced by ε -dilation of a bounded connected open set D (called the reference cell). Second, the families of sets \mathcal{G}_ε can either be made of sets with arbitrary

orientation (isotropic case) or with fixed orientation (anisotropic case). When rotations are not allowed, it is proved in [FGLBS20] that by considering

$$H_\varepsilon^D(u, p, \Omega) := \varepsilon^{n-p} \sup_{\mathcal{H}_\varepsilon} \sum_{D' \in \mathcal{H}_\varepsilon} \int_{D'} \left| u - \int_{D'} u \right|^p dx,$$

where \mathcal{H}_ε is any pairwise disjoint family of translations D' of εD contained in Ω , there exists a locally Lipschitz continuous p -homogeneous function $\psi_p^D : \mathbb{R}^n \rightarrow [0, +\infty)$ such that

$$\lim_{\varepsilon \rightarrow 0^+} H_\varepsilon^D(u, p, \Omega) = \int_{\mathbb{R}^n} \psi_p^D(\nabla u(x)) dx.$$

In the vector-valued setting, the situation becomes richer. Given $\Omega \subset \mathbb{R}^n$ open bounded with sufficiently smooth boundary, for vector fields $u : \Omega \rightarrow \mathbb{R}^n$ a non-local limiting integral formula, avoiding the direct manipulation of distributional derivatives, was obtained in the spirit of a representation of the BBM difference quotient for Sobolev functions. In particular, in the seminal work [Men12], the author introduced a nonlocal functional of the form

$$\Phi_\varepsilon^p[u] = \int_{\Omega} \int_{\Omega} \frac{|(u(y) - u(x)) \cdot (y - x)|^p}{|y - x|^{2p}} \rho_\varepsilon(y - x) dy dx,$$

where ρ_ε are standard radial mollifiers.

A variant of this operator appears in continuum mechanics within the framework of peridynamics, where equations are formulated using integral operators rather than differential ones (see [MS15, DFS25, MD15] and the references therein). It was proved in [Men12, Theorem 2.2], that for $p \in (1, \infty)$ and $u \in L^p(\Omega, \mathbb{R}^n)$,

$$u \in W^{1,p}(\Omega, \mathbb{R}^n) \iff \liminf_{\varepsilon \rightarrow 0^+} \Phi_\varepsilon^p[u] < +\infty.$$

More precisely,

$$\lim_{\varepsilon \rightarrow 0^+} \Phi_\varepsilon^p[u] = \int_{\Omega} \mathbf{Q}_p(\mathcal{E}u(x)) dx,$$

where

$$\mathcal{E}u := \frac{\nabla u + \nabla u^T}{2}$$

is the symmetric gradient and $\mathbf{Q}_p(A)$ is a norm in the space of symmetric matrices defined by

$$\mathbf{Q}_p(A) = \left((2p + n) \int_{B_1(0)} |\langle Aw, w \rangle|^p dw \right)^{1/p},$$

where $B_1(0) \subset \mathbb{R}^n$ is the Euclidean ball centered at zero of radius 1. The nonlocal energy therefore controls only the symmetric part $\mathcal{E}u$ of the gradient and the finiteness of the limiting functional implies $\mathcal{E}u \in L^p(\Omega)$. For bounded Lipschitz domains, Korn's inequality yields

$$W^{1,p}(\Omega; \mathbb{R}^n) = \{u \in L^p(\Omega; \mathbb{R}^n) : \mathcal{E}u \in L^p(\Omega; \mathbb{R}^{n \times n})\},$$

together with the estimate

$$(1.2) \quad \inf_{B \in \mathbb{R}_{\text{skew}}^{n \times n}, b \in \mathbb{R}^n} \|u - (Bx + b)\|_{W^{1,p}(\Omega; \mathbb{R}^n)} \leq C(\Omega) \|\mathcal{E}u\|_{L^p(\Omega; \mathbb{R}^{n \times n})},$$

where $\mathbb{R}_{\text{skew}}^{n \times n}$ is the space of real valued skew-symmetric $n \times n$ matrices (see [DM04, Men12]).

This identifies a structural principle: a nonlocal energy may encode only a prescribed linear component of the gradient, while the passage to full Sobolev regularity

is enforced by a Korn-type inequality. This strategy is possible due to the regularity assumptions on Ω .

In our work, we do not require any regularity assumptions on Ω considering an arbitrary open set (in the spirit of [FMS18, FGLBS20, DFF20]) and we introduce a broad class of one-parameter families of BMO-type seminorms originated by suitable functionals, which we call *p-core functionals* (see Def. 3.1). Through our definition, we can recover the classical scalar *p*-mean oscillation functionals available in the literature and introduce novel ones, defined also for vector-valued functions.

The goal of this paper is to present a general and unifying theorem ensuring the existence of the pointwise limit of these one-parameter families as the parameter goes to zero. We will also prove that, when this limit functional is evaluated at a generic function u , it admits an integral representation determined by the value that the limit functional itself assumes on linear maps.

More precisely, given a bounded open set $D \subset \mathbb{R}^n$ and a group Γ of affine transformations on \mathbb{R}^n , a *p*-core functional α_p is a positive functional defined on pairs of the form (u, D') , where $u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ and $D' \subset \mathbb{R}^n$ is a transformation of D according to Γ , satisfying certain axioms. These include *p*-homogeneity and convexity in the first variable and the validity of the following crucial inequality:

$$(1.3) \quad \exists c_1 > 0 \text{ s.t. } \alpha_p(u, D) \leq c_1 \int_D |\nabla u|_2^p dx \quad \forall u \in W^{1,p}(D; \mathbb{R}^m).$$

We observe that in the *p*-mean oscillation case, assuming also D Lipschitz and connected, (1.3) corresponds to the Poincaré inequality.

In defining BMO-type seminorms originated by *p*-core functionals, we introduce a further level of generality compared to [FFGLBS20, FGLBS20, DFF20] by allowing the presence of an “intermediate rotation” group \mathcal{G} between $\{\text{id}_{\mathbb{R}^n}\}$ and $\text{SO}(n)$. More precisely, given an open set $\Omega \subseteq \mathbb{R}^n$ (which we call the *ambient space*) and a subgroup $\mathcal{G} \subset \text{SO}(n)$, we denote by $\mathcal{G}_\varepsilon^D(\Omega)$ the class of all the subsets of Ω obtained by translating, ε -dilating and rotating, according to the rotations in \mathcal{G} , the reference set D , and by $\mathcal{G}^D(\Omega) = \bigcup_{\varepsilon > 0} \mathcal{G}_\varepsilon^D(\Omega)$ (see Def. 3.2). For every $u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ we define the following general *BMO-type seminorm*

$$G_\varepsilon^{\alpha_p}(u, \Omega) := \varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(u, D'),$$

where each \mathcal{G}_ε is a family made of pairwise disjoint elements of $\mathcal{G}_\varepsilon^D(\Omega)$ and the supremum is taken with respect to all of these families \mathcal{G}_ε . Our main result is the following.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $m \geq 1$ a natural number, $p \in [1, \infty)$, \mathcal{G} a subgroup of $\text{SO}(n)$ and $D \subseteq \mathbb{R}^n$ be a bounded open set. Let α_p be a *p*-core functional with domain $L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m) \times \mathcal{G}^D(\mathbb{R}^n)$ and $u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$.*

Then, if $\nabla u|_\Omega \in L^p(\Omega; \mathbb{R}^{m \times n})$,

$$(1.4) \quad \lim_{\varepsilon \rightarrow 0^+} G_\varepsilon^{\alpha_p}(u, \Omega) = \int_\Omega \psi_{\alpha_p}(\nabla u) dx,$$

where ψ_{α_p} is a convex and *p*-homogeneous function defined by

$$\psi_{\alpha_p}(A) := \lim_{\varepsilon \rightarrow 0^+} G_\varepsilon^{\alpha_p} \left(l^A, \left(-\frac{1}{2}, \frac{1}{2} \right)^n \right)$$

for every $A \in \mathbb{R}^{m \times n}$ and $l^A(x) := Ax$ for every $x \in \mathbb{R}^n$.

Moreover, if $u|_{\Omega} \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^m)$ and $\psi_{\alpha_p} \neq 0$, (1.4) still holds.

In order to prove Theorem 1.1 we follow the strategy introduced in [FFGLBS20, Theorem 4.1] and [FGLBS20, Theorem 1.2]. We highlight that in the cases considered in the literature, the condition $\psi_{\alpha_p} \neq 0$ is always satisfied (actually, $\psi_{\alpha_p} > 0$ on $\mathbb{R}^{m \times n} \setminus \{0\}$). However, we remark that in general when $u|_{\Omega} \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^m)$ the condition $\psi_{\alpha_p} \neq 0$ is necessary. Indeed, if $u|_{\Omega} \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^m)$ with $\nabla u|_{\Omega} \notin L^p(\Omega; \mathbb{R}^{m \times n})$ and $\psi_{\alpha_p} \equiv 0$ then (1.4) may or may not hold (see Ex. 5.18 and Ex. 5.19).

In Thm. 1.1, under suitable assumptions on the p -core functional, ψ_{α_p} can be written more explicitly as in Prop. 5.1. In Section 5 we provide several examples for which ψ_{α_p} can be computed explicitly.

A further structural property of the energy arises by considering the decomposition of the matrix space $\mathbb{R}^{m \times n} = \mathcal{P} \oplus \mathcal{N}_{\psi_{\alpha_p}}$, where $\mathcal{N}_{\psi_{\alpha_p}} = \{A \in \mathbb{R}^{m \times n} : \psi_{\alpha_p}(A) = 0\}$ and \mathcal{P} is a linear subspace of $\mathbb{R}^{m \times n}$ (see Rem. 3.11). Let $\pi_{\mathcal{P}} : \mathbb{R}^{m \times n} \rightarrow \mathcal{P}$ denote the projection onto \mathcal{P} associated to the direct sum decomposition. Then, for every $A \in \mathbb{R}^{m \times n}$, it holds

$$\psi_{\alpha_p}(\pi_{\mathcal{P}}(A)) = \psi_{\alpha_p}(A).$$

Moreover, the norm induced by ψ_{α_p} is equivalent to the L^p -norm of $\pi_{\mathcal{P}}(\nabla u)$. More precisely, there exist constants $c, C > 0$ depending on α_p such that

$$c \int_{\Omega} |\pi_{\mathcal{P}}(\nabla u)|_2^p dx \leq \int_{\Omega} \psi_{\alpha_p}(\pi_{\mathcal{P}}(\nabla u)) dx \leq C \int_{\Omega} |\pi_{\mathcal{P}}(\nabla u)|_2^p dx.$$

If $\psi_{\alpha_p} > 0$ on $\mathbb{R}^{m \times n} \setminus \{0\}$, then $\mathcal{N}_{\psi_{\alpha_p}} = \{0\}$ and the above inequality yields the equivalence

$$\int_{\Omega} \psi_{\alpha_p}(\pi_{\mathcal{P}}(\nabla u)) dx \approx \int_{\Omega} |\nabla u|_2^p dx.$$

In this case, assuming also a lower semicontinuity property on α_p we obtain the characterization

$$(1.5) \quad \liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega) < +\infty \iff \nabla u|_{\Omega} \in L^p(\Omega; \mathbb{R}^{m \times n})$$

(see Cor. 4.2). Examples of this situation are discussed in Ex. 5.4 and Ex. 5.8 (see also Cor. 5.11), where the functional α_p corresponds respectively to the p -mean oscillation functional

$$\alpha_p(u, D') = \int_{D'} |u(x) - \int_{D'} u|_2^p dx,$$

and to the functional

$$\alpha_p(u, D') = \int_{D'} \int_{D'} |u(x) - u(y)|_2^p dx dy.$$

However, in general, the equivalence in (1.5) does not hold. Nevertheless, thanks to Cor. 4.1, assuming a lower semicontinuity property on α_p and denoting by $u_{\sigma} = \rho_{\sigma} * u$ the standard mollification of u , it is still possible to obtain the estimate

$$(1.6) \quad \begin{aligned} c \limsup_{\sigma \rightarrow 0^+} \int_{\tilde{\Omega}} |\pi_{\mathcal{P}}(\nabla u_{\sigma})|_2^p dx &\leq \lim_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u_{\sigma}, \tilde{\Omega}) \\ &\leq \liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega) \leq C \int_{\Omega} |\nabla u|_2^p dx, \end{aligned}$$

where $\tilde{\Omega} \subset \mathbb{R}^n$ is an open set such that $\tilde{\Omega} \Subset \Omega$. Assuming $\liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega) < +\infty$, since $u_{\sigma} \rightarrow u$ in $L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ as $\sigma \rightarrow 0^+$, one can only conclude that $\pi_{\mathcal{P}}(\nabla u_{\sigma}|_{\Omega}) \rightarrow D_{\pi_{\mathcal{P}}}(u|_{\Omega})$ as $\sigma \rightarrow 0^+$ in the sense of distributions (here $D_{\pi_{\mathcal{P}}}(u|_{\Omega})$ is the distribution

associated to the projection of the gradient according to $\pi_{\mathcal{P}}$, see Section 2 for the precise definition). This leads to the implication

$$(1.7) \quad \liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega) < +\infty \implies D_{\pi_{\mathcal{P}}}(u|_{\Omega}) \in L^p(\Omega; \mathbb{R}^{m \times n}).$$

The converse implication does not hold in general, as shown in Ex. 5.19,

$$\alpha_p(u, D') = \inf_{A \in \mathbb{R}^{m \times n}} \int_{D'} |u(x) - A(x - \text{bar}(D')) - \int_{D'} u|_2^p dx,$$

where $\text{bar}(D')$ is the barycenter of D' . Indeed, in this case $\lim_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega) = +\infty$ but $\psi_{\alpha_p} \equiv 0$ implies automatically $D_{\pi_{\mathcal{P}}}(u) = 0 \in L^p(\Omega, \mathbb{R}^{m \times n})$.

To recover the reverse implication in (1.7), we introduce the concept of *strong p -core functional*, i.e. a p -core functional satisfying a stronger version of (1.3). More precisely, we require that

$$\exists c_2 > 0 \text{ s.t. } \alpha_p(u, D) \leq c_2 \int_D |D_{\pi_{\mathcal{P}}} u|_2^p dx \quad \forall u \in W^{\pi_{\mathcal{P}}, p}(D; \mathbb{R}^m),$$

where $W^{\pi_{\mathcal{P}}, p}(D; \mathbb{R}^m) = \{u \in L^p(D; \mathbb{R}^m) : D_{\pi_{\mathcal{P}}} u \in L^p(D; \mathbb{R}^{m \times n})\}$. Under this assumption, the upper bound in the previous estimate (1.6) improves to

$$\liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega) \leq c_2 \int_{\Omega} |D_{\pi_{\mathcal{P}}} u|_2^p dx,$$

which implies that

$$(1.8) \quad D_{\pi_{\mathcal{P}}}(u|_{\Omega}) \in L^p(\Omega; \mathbb{R}^{m \times n}) \implies \liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega) < +\infty$$

(see Cor. 4.4).

In this framework we obtain the following analogue of Thm. 1.1.

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $m \geq 1$ a natural number, $p \in [1, \infty)$, $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$ and $D \subseteq \mathbb{R}^n$ be a bounded open set. Let α_p be a strong p -core functional with domain $L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m) \times \mathcal{G}^D(\mathbb{R}^n)$ and $u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$.*

Then, if $D_{\pi_{\mathcal{P}}}(u|_{\Omega}) \in L^p(\Omega; \mathbb{R}^{m \times n})$,

$$(1.9) \quad \lim_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega) = \int_{\Omega} \psi_{\alpha_p}(D_{\pi_{\mathcal{P}}}(u|_{\Omega})) dx.$$

Moreover, if $u|_{\Omega} \in W_{\text{loc}}^{\pi_{\mathcal{P}}, p}(\Omega; \mathbb{R}^m)$ and $\psi_{\alpha_p} \neq 0$, (1.9) still holds.

We remark that in Thm. 1.2 we assume that $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$ because, a priori, the differential operator $D_{\pi_{\mathcal{P}}}$ imposes more rigidity conditions than the canonical distributional gradient ∇ .

In Section 5 we present several examples of p -core and strong p -core functionals. We observe that when $\mathcal{N}_{\psi_{\alpha_p}} = \{0\}$ the two notions coincide, while if $\mathcal{N}_{\psi_{\alpha_p}} = \mathbb{R}^{m \times n}$ no strong p -core functional exists (except the trivial case of $\alpha_p \equiv 0$ in Ex. 5.18). Intermediate situations are illustrated in Ex. 5.12, where

$$\alpha_p(u, D') := \inf_{A \in \mathbb{R}_{\text{skew}}^{n \times n}} \int_{D'} |u(x) - A(x - \text{bar}(D')) - \int_{D'} u|_2^p dx,$$

and one finds that

$$\mathcal{N}_{\psi_{\alpha_p}} = \mathbb{R}_{\text{skew}}^{n \times n}, \quad D_{\pi_{\mathcal{P}}}(u|_{\Omega}) = \mathcal{E}u.$$

As a consequence in this case thanks to (1.7) and (1.8) we are able to obtain the following (cfr. Cor. 5.17): if $p \in (1, \infty)$, for every $u \in L^p(\Omega; \mathbb{R}^n)$,

$$\liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega) < +\infty \iff u \in E^{1,p}(\Omega; \mathbb{R}^n),$$

where the space $E^{1,p}(\Omega; \mathbb{R}^n) := \{u \in L^p(\Omega; \mathbb{R}^n) : \mathcal{E}u \in L^p(\Omega; \mathbb{R}^{n \times n})\}$.

Finally, as a consequence of Thm. 1.1 and Thm. 1.2, the detour in the characterization of Sobolev spaces leads us to obtain a version of the so-called "Constancy Theorem" in the spirit of [Bre02]. This class of results is presented as open problem in [Bre02] and it is instrumental in related topics such as the degree theory for classes of discontinuous maps.

In our abstract setting for a connected open set Ω , we obtain for $p \in (1, \infty)$ (see Cor. 4.5) that assuming a lower semicontinuity property on a strong p -core functional α_p then

$$\liminf_{\varepsilon \rightarrow 0^+} G_\varepsilon^{\alpha_p}(u, \Omega) = 0 \implies D_{\pi_{\mathcal{P}}}(u|_\Omega) = 0.$$

Thus, if $\psi_{\alpha_p} > 0$ in $\mathbb{R}^{m \times n}$, then u is constant a.e. on Ω . On the other hand if we consider the p -core functional in Ex. 5.12 for $p \in (1, \infty)$ we obtain that if $u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ satisfies $\liminf_{\varepsilon \rightarrow 0^+} G_\varepsilon^{\alpha_p}(u, \Omega) = 0$, then $u|_\Omega \in E^{1,p}(\Omega; \mathbb{R}^n)$ and $\mathcal{E}(u|_\Omega) = 0$. Then, by the Korn-Poincaré inequality (1.2) we find that $u(x)$ identify a rigid motion, i.e. $u(x) = Ax + h$ for every $x \in \mathbb{R}^n$, where $A \in \mathbb{R}^{n \times n}_{\text{skew}}$ and $h \in \mathbb{R}^n$.

1.1. Plan of the paper. The plan of the paper is the following.

In Section 2 we set the notation and we collect preliminary results.

In Section 3 we introduce the notion of p -core functional (cfr. Def. 3.1) and define the associated one-parameter family of generalized BMO-type seminorms (see Def. 3.4). After establishing the main properties of these generalized BMO-type seminorms, for any p -core functional α_p we define the p -homogeneous and convex integrand ψ_{α_p} (see Def. 3.9) and we study its main properties. Then, we prove Thm. 1.1 and, after introducing the concept of strong p -core functional (cfr. Def. 3.12), Thm. 1.2.

In Section 4, we discuss some consequences of Thm. 1.1 and Thm. 1.2. We first prove Cor. 4.2, involving p -core functionals, highlighting that the implication in its thesis cannot be reversed in general (cfr. Rem. 4.3). Therefore, after reinforcing its hypotheses, we then prove Cor. 4.4, the characterization result involving strong p -core functionals. We finally deduce the constancy theorem Cor. 4.5.

In Section 5, we present several examples of (strong) p -core functionals to which we apply the results proved in Section 3 and Section 4. In particular, we show how these examples can be used to characterize through a non-distributional approach the spaces $W^{1,p}(\Omega; \mathbb{R}^m)$ and $E^{1,p}(\Omega; \mathbb{R}^n)$ (see Cor. 5.11 and Cor. 5.17).

2. NOTATION AND PRELIMINARY RESULTS

We fix some notation. We denote by $\#S$ the cardinality of a set S . We denote by $|\cdot|_2$ the Euclidean norm of \mathbb{R}^n and if $x, y \in \mathbb{R}^n$ their standard scalar product is denoted by $x \cdot y$. Given $x \in \mathbb{R}^n$ and $r > 0$, we let $B_r(x)$ be the Euclidean ball of radius r centered at x , $Q := (-\frac{1}{2}, \frac{1}{2})^n$ and $Q(x; r) := rQ + x$. If $\Omega \subseteq \mathbb{R}^n$ is an open set, we denote by \mathcal{A}_Ω the family of all the open subsets of Ω .

Let $\text{Aff}(\mathbb{R}^n)$ be the affine group of the Euclidean space \mathbb{R}^n , i.e. the set of all invertible affine transformations from \mathbb{R}^n into itself.

We denote by $\mathbb{R}^{m \times n}$ the space of real valued $m \times n$ matrices, by $\mathbb{R}^{n \times n}_{\text{skew}}$, the space of real valued skew-symmetric $n \times n$ matrices, by $O(n) \subset \mathbb{R}^{n \times n}$ the group of all orthogonal matrices and by $SO(n) \subset O(n)$ the group of all orthogonal matrices with determinant equal to 1.

If $\mathcal{V}_1, \mathcal{V}_2$ are linear subspaces of $\mathbb{R}^{m \times n}$, we write $\mathbb{R}^{m \times n} = \mathcal{V}_1 \oplus \mathcal{V}_2$ if $\mathbb{R}^{m \times n} = \mathcal{V}_1 + \mathcal{V}_2$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$.

For $A \in \mathbb{R}^{m \times n}$ we put,

$$|A|_2 := \sup\{|Ax|_2 : x \in \mathbb{R}^n, |x|_2 \leq 1\}.$$

Given $A \in \mathbb{R}^{m \times n}$, we denote the linear function associated to A by $l^A(x) := Ax$ for every $x \in \mathbb{R}^n$.

It is easy to verify that for $A, B \in \mathbb{R}^{m \times n}$ and $p \in [1, \infty)$ it holds,

$$(2.1) \quad \left| |A|_2^p - |B|_2^p \right| \leq p(|A|_2 + |B|_2)^{p-1} |A - B|_2,$$

and for $A, B \in \mathbb{R}^{m \times n} \setminus \{0\}$ it holds,

$$(2.2) \quad \left| \frac{A}{|A|_2} - \frac{B}{|B|_2} \right| \leq 2 \frac{|A - B|_2}{|A|_2}.$$

We denote by \mathcal{L}^n the Lebesgue measure on \mathbb{R}^n and for a (Lebesgue) measurable set $S \subseteq \mathbb{R}^n$ we put $|S| := \mathcal{L}^n(S)$. For a measurable set $S \subseteq \mathbb{R}^n$ of finite and strictly positive measure and an integrable function $u : S \rightarrow \mathbb{R}^d$ we put,

$$u_S = \int_S u \, dx := \frac{1}{|S|} \int_S u \, dx.$$

If S is a measurable set of strictly positive and finite measure, we define its *barycenter* by,

$$\text{bar}(S) := \int_S x \, dx.$$

Throughout the paper we will always denote by $\{\rho_\sigma\}_{\sigma>0} \subset C_c^\infty(\mathbb{R}^n)$ a family of standard radial mollifiers, i.e.,

$$\rho_\sigma(x) := \sigma^{-n} \rho(x/\sigma) \quad \forall x \in \mathbb{R}^n,$$

where $\rho \in C_c^\infty(\mathbb{R}^n)$ is a standard radial mollifier with $\text{supp}(\rho) \subset B_1(0)$.

The following lemma is a useful Jensen-type inequality.

Lemma 2.1. *Let $p \in [1, \infty)$, $\sigma > 0$, $u \in L^p(\mathbb{R}^n; \mathbb{R}^m)$ and $f : L^p(\mathbb{R}^n; \mathbb{R}^m) \rightarrow [0, +\infty)$ be a proper convex and lower semicontinuous function. Then,*

$$(2.3) \quad f(u_\sigma) \leq \int_{B_1(0)} f(u(\cdot - \sigma y)) \rho(y) \, dy,$$

where $u_\sigma := \rho_\sigma * u$.

Proof. Let $\sigma > 0$ and $u_\sigma := \rho_\sigma * u \in L^p(\mathbb{R}^n; \mathbb{R}^m)$. We consider the probability space $(\mathbb{R}^n, \mathcal{A}, \mu)$, where $\mu := \rho \mathcal{L}^n = \rho \mathcal{L}^n \llcorner B_1(0)$ and \mathcal{A} is the σ -algebra of Lebesgue measurable sets of \mathbb{R}^n .

We also define $X := L^p(\mathbb{R}^n; \mathbb{R}^m)$ and consider the measurable space (X, \mathcal{B}) , where \mathcal{B} is the σ -algebra of Borel sets of X . We recall that X , being a Banach space, is *evenly convex* (see the definition for instance in [Ves17, Sec. 2]). We define,

$$G : \mathbb{R}^n \longrightarrow X \\ y \mapsto u(\cdot - \sigma y).$$

Moreover, thanks to the L^p -continuity of translations, G is continuous. This then implies that $G : (\mathbb{R}^n, \mathcal{A}) \rightarrow (X, \mathcal{B})$ is measurable. We consider the following *Pettis integral* (see the definition for instance in [Ves17, Sec. 2]),

$$(2.4) \quad \int_{\mathbb{R}^n} G \, d\mu \in X.$$

In particular (2.4) is well-defined, since for every $\phi \in X' \simeq L^{p'}(\mathbb{R}^n; \mathbb{R}^m)$ the map $\phi \circ G : (\mathbb{R}^n, \mathcal{A}) \rightarrow \mathbb{R}$ is Lebesgue μ -integrable and there exists $v \in X$ such that

$$\int_{\mathbb{R}^n} \phi \circ G \, d\mu = \phi(v) \quad \forall \phi \in X'.$$

Indeed, fixed $\phi \in X'$ (and letting $\tilde{\phi} \in L^{p'}(\mathbb{R}^n; \mathbb{R}^m)$ be the function representing ϕ) for every $y \in \mathbb{R}^n$,

$$(\phi \circ G)(y) = \int_{\mathbb{R}^n} \tilde{\phi} \cdot G(y) \, d\mathcal{L}^n = \int_{\mathbb{R}^n} \tilde{\phi}(x) \cdot u(x - \sigma y) \, dx.$$

Moreover, applying Tonelli's theorem

$$\int_{\mathbb{R}^n} (\phi \circ G)(y) \, d\mu(y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{\phi}(x) \cdot u(x - \sigma y) \rho(y) \, dx dy = \int_{\mathbb{R}^n} \tilde{\phi}(x) \cdot u_\sigma(x) \, dx.$$

This implies that

$$(2.5) \quad \int_{\mathbb{R}^n} G \, d\mu = u_\sigma \in X.$$

By Jensen's inequality ([Ves17, Thm. 3]), we get that

$$(2.6) \quad f\left(\int_{\mathbb{R}^n} G \, d\mu\right) \leq \int_{\mathbb{R}^n} f \circ G \, d\mu.$$

Combining (2.5) and (2.6) we obtain (2.3). \square

Lemma 2.2. *Let X be a real vector space, $p \geq 1$ and $f : X \rightarrow [0, +\infty)$ be a convex and p -homogeneous function:*

$$(2.7) \quad f(tu) = |t|^p f(u) \quad \forall u \in X, \forall t \in \mathbb{R}.$$

Then, the set

$$\mathcal{N}_f := \{u \in X : f(u) = 0\},$$

is a linear subspace of X . Moreover,

$$(2.8) \quad f(u) = f(u + v),$$

for every $u \in X$ and $v \in \mathcal{N}_f$.

Finally, $f^{1/p}$ is a seminorm on X . Thus, for every W linear subspace of X , the restriction of $f^{1/p}$ to W is a norm whenever $W \cap \mathcal{N}_f = \{0\}$.

Proof. We observe that (2.7) implies that f is an even function and that $f(0) = 0$. We prove that \mathcal{N}_f is a linear subspace of X . Let $t, s \in \mathbb{R}$ and $u, v \in \mathcal{N}_f$. Then, by the p -homogeneity and the convexity of f we deduce,

$$\begin{aligned} f(tu + sv) &= (|t| + |s|)^p f\left(\frac{|t|}{|t| + |s|} \operatorname{sign}(t)u + \frac{|s|}{|t| + |s|} \operatorname{sign}(s)v\right) \\ &\leq (|t| + |s|)^p \left(\frac{|t|}{|t| + |s|} f(u) + \frac{|s|}{|t| + |s|} f(v)\right) = 0, \end{aligned}$$

if $s \neq 0$. If $s = 0$, $f(tu + sv) = |t|^p f(u) = 0$.

Finally, we prove (2.8). Let $u \in X$ and $v \in \mathcal{N}_f$ and $t \in (0, 1)$.

$$\begin{aligned} f(u) &= f\left(t \frac{u+v}{t} + (1-t) \frac{-v}{(1-t)}\right) \\ &\leq t f\left(\frac{u+v}{t}\right) + (1-t) f\left(\frac{-v}{(1-t)}\right) = t^{1-p} f(u+v). \end{aligned}$$

Letting $t \rightarrow 1$ we deduce that

$$(2.9) \quad f(u) \leq f(u+v) \quad \forall u \in X \forall v \in \mathcal{N}_f.$$

Then, we let $u \in X$ and $v \in \mathcal{N}_f$ and applying (2.9), we observe that $f(u+v) \leq f(u+v+(-v)) = f(u)$, since $-v \in \mathcal{N}_f$. This together with (2.9) proves (2.8).

Finally, we define $C := \{x \in X : f(x) < 1\}$ and observe that $tC = \{x \in X : f(x) < t^p\}$ for every $t > 0$, by the p -homogeneity of f . Taking into account this information, we compute the *Minkowski functional* of C (cfr. [Bre11]) and obtain that it coincides with $f^{1/p}$. Therefore, by [Bre11, Lem 1.2] we deduce that $f^{1/p}$ is a seminorm. \square

The following lemma is a crucial tool in the proof of our main theorem (cfr. [FGLBS20, Lemma 3.1]).

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $u \in C^1(\overline{\Omega}; \mathbb{R}^m)$ and for every $t > 0$ let $U_t := \{x \in \Omega : |\nabla u(x)|_2 > t\}$.*

Let $t > 0$ and $\sigma > 0$. Then, there exist $r > 0$ and a finite family of pairwise disjoint open cubes $\{Q(x_i; r)\}_{1 \leq i \leq l}$ contained in U_t , such that,

- a) $|U_t \setminus \bigcup_{i=1}^l Q(x_i; r)| < \sigma$,
- b) $|\nabla u(x) - \nabla u(y)|_2 < \sigma \quad \forall x, y \in Q(x_i; r) \forall 1 \leq i \leq l$.

Moreover, if $|\partial U_t| = 0$ there exists $W_{t,\sigma} \subset \Omega$ open s.t. $\Omega \cap (\overline{U_t} \setminus \bigcup_{i=1}^l Q(x_i; r)) \subset W_{t,\sigma}$ and $|W_{t,\sigma}| < \sigma$.

Proof. Let $t > 0$, $\sigma > 0$. Since ∇u is uniformly continuous in $\overline{\Omega}$, there exists $\delta > 0$ s.t. $|\nabla u(x) - \nabla u(y)|_2 < \sigma$ for every cube $Q^\delta \subset \overline{\Omega}$ of side length δ and for every $x, y \in Q^\delta$. Since U_t is open, there exists a sequence of pairwise disjoint open dyadic cubes $\{Q_k\}_{k \in \mathbb{N}}$ with side length less than δ , such that $U_t = \bigcup_{k \in \mathbb{N}} Q_k$ up to a set of zero Lebesgue measure. Moreover, by the inner regularity of the Lebesgue measure, we can find a compact set $K_{t,\sigma} \subset U_t$ s.t. $|U_t \setminus K_{t,\sigma}| < \sigma/2$. Thanks to the compactness of $K_{t,\sigma}$ from $\{Q_k\}_{k \in \mathbb{N}}$ we can extract a finite subfamily $\{Q_k\}_{1 \leq k \leq l_0}$ such that $K_{t,\sigma} \subset \bigcup_{k=1}^{l_0} Q_k$. We define $r > 0$ as the minimum side length among all the cubes in $\{Q_k\}_{1 \leq k \leq l_0}$. For every $1 \leq k \leq l_0$ we subdivide in a dyadic way Q_k with respect to the length r : we observe that $Q_k = \bigcup_{i=1}^{j_k} Q_k^i$ (up to a set of zero Lebesgue measure), where $\{Q_k^i\}_{1 \leq i \leq j_k}$ is a finite family of pairwise disjoint open dyadic cubes of side length r . $\{Q_k^i\}_{1 \leq k \leq l_0, 1 \leq i \leq j_k}$ is (up to relabeling the indexes) the desired family $\{Q(x_i; r)\}_{1 \leq i \leq l}$, which guarantees that a) and b) hold.

Finally, if $|\partial U_t| = 0$, then $|\overline{U_t} \setminus \bigcup_{1 \leq i \leq l} Q(x_i; r)| = |U_t \setminus \bigcup_{1 \leq i \leq l} Q(x_i; r)| < \sigma/2$. Therefore, by the outer regularity of the Lebesgue measure, we can find an open set $O_{t,\sigma}$ s.t. $\overline{U_t} \setminus \bigcup_{1 \leq i \leq l} Q(x_i; r) \subset O_{t,\sigma}$ and $|O_{t,\sigma}| < \sigma$.

We can then define $W_{t,\sigma} := \Omega \cap O_{t,\sigma}$ to conclude the proof. \square

For $\Omega \subseteq \mathbb{R}^n$ open, $p \in [1, \infty]$ and $u \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^n)$ we denote by ∇u the weak Jacobian matrix of u and by $\mathcal{E}u$ the symmetric part of ∇u , also referred to as the *symmetric gradient* of u :

$$\mathcal{E}u := \frac{\nabla u + \nabla u^T}{2}.$$

Throughout this work we will need the following generalization of the notion of weak gradient.

Definition 2.4. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $u \in L^1_{\text{loc}}(\Omega; \mathbb{R}^m)$ and $\mathfrak{L} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ be a linear map:

$$\mathfrak{L}(A)_{ij} := \sum_{1 \leq k \leq m, 1 \leq l \leq n} \mathfrak{L}_{ijkl} A_{kl},$$

for every $A \in \mathbb{R}^{m \times n}$ and for every $1 \leq i \leq m$ and $1 \leq j \leq n$. We define the $m \times n$ matrix of distributions $D_{\mathfrak{L}}u$ as,

$$\langle (D_{\mathfrak{L}}u)_{ij}, \phi \rangle := - \int_{\Omega} \sum_{1 \leq k \leq m, 1 \leq l \leq n} \mathfrak{L}_{ijkl} u_k \frac{\partial \phi}{\partial x_l} dx = - \int_{\Omega} \mathfrak{L}(u \otimes \nabla \phi)_{ij} dx,$$

for every $1 \leq i \leq m$, $1 \leq j \leq n$ and for every $\phi \in C_c^\infty(\Omega)$.

We will also make use of the following *Sobolev-type spaces*.

Definition 2.5. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $p \in [1, \infty)$ and $\mathfrak{L} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ be a linear map. Given $u \in L^p_{\text{loc}}(\Omega; \mathbb{R}^m)$, whenever the matrix of distributions $D_{\mathfrak{L}}u$ is represented by a map in $L^p_{\text{loc}}(\Omega; \mathbb{R}^{m \times n})$, by a slight abuse of notation, we will denote that function using the same symbol $D_{\mathfrak{L}}u$. We define,

$$\begin{aligned} W^{\mathfrak{L},p}_{\text{loc}}(\Omega; \mathbb{R}^m) &:= \{u \in L^p_{\text{loc}}(\Omega; \mathbb{R}^m) : D_{\mathfrak{L}}u \in L^p_{\text{loc}}(\Omega; \mathbb{R}^{m \times n})\}, \\ W^{\mathfrak{L},p}(\Omega; \mathbb{R}^m) &:= \{u \in L^p(\Omega; \mathbb{R}^m) : D_{\mathfrak{L}}u \in L^p(\Omega; \mathbb{R}^{m \times n})\}. \end{aligned}$$

For a more general version of the spaces in Def. 2.5 and their properties we refer for instance to [DG25, GR19].

We observe that $W^{\mathfrak{L},p}_{\text{loc}}(\Omega; \mathbb{R}^m) = L^p_{\text{loc}}(\Omega; \mathbb{R}^m)$ and $W^{\mathfrak{L},p}(\Omega; \mathbb{R}^m) = L^p(\Omega; \mathbb{R}^m)$ if $\mathfrak{L} = 0$, while $W^{\mathfrak{L},p}_{\text{loc}}(\Omega; \mathbb{R}^m) = W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^m)$ and $W^{\mathfrak{L},p}(\Omega; \mathbb{R}^m) = W^{1,p}(\Omega; \mathbb{R}^m)$ if $\mathfrak{L} = \text{id}_{\mathbb{R}^{m \times n}}$.

Moreover, if $u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^m)$, then for every $\mathfrak{L} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ linear map $D_{\mathfrak{L}}u = \mathfrak{L}(\nabla u) \in L^p_{\text{loc}}(\Omega; \mathbb{R}^{m \times n})$, being

$$\langle (D_{\mathfrak{L}}u)_{ij}, \phi \rangle = \int_{\Omega} \mathfrak{L}(\nabla u)_{ij} \phi dx,$$

for every $\phi \in C_c^\infty(\Omega)$.

Therefore,

$$(2.10) \quad W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^m) \subset W^{\mathfrak{L},p}_{\text{loc}}(\Omega; \mathbb{R}^m), \quad W^{1,p}(\Omega; \mathbb{R}^m) \subset W^{\mathfrak{L},p}(\Omega; \mathbb{R}^m),$$

and the set inclusions in (2.10) become equalities when \mathfrak{L} coincides with the identity operator.

We also observe that if

$$(2.11) \quad \mathfrak{L}(A) := \frac{A + A^T}{2},$$

for every $A \in \mathbb{R}^{n \times n}$, then $D_{\mathfrak{L}}u = \mathfrak{L}u$ for every $u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^n)$.

Definition 2.6. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $p \in [1, \infty)$ and \mathfrak{L} as in (2.11). We define,

$$E^{1,p}(\Omega; \mathbb{R}^n) := W^{\mathfrak{L},p}(\Omega; \mathbb{R}^n), \quad E^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^n) := W^{\mathfrak{L},p}_{\text{loc}}(\Omega; \mathbb{R}^n).$$

We remark that for $p \in (1, \infty)$ the space $E^{1,p}(\Omega; \mathbb{R}^n)$ coincides with $W^{1,p}(\Omega; \mathbb{R}^n)$ if Ω is sufficiently regular (cfr. [DFS25, Thm. 2]) while in general the two spaces are different (cfr. [ADLG13]). Therefore, the set inclusions in (2.10) can be equalities even for operators \mathfrak{L} different from the identity. Instead, for $p = 1$ the spaces $W^{1,1}(\Omega; \mathbb{R}^n)$ and $E^{1,1}(\Omega; \mathbb{R}^n)$ are different (cfr. [DFS25]). For a discussion related to this subject we refer to [VS13] and the references therein.

Lemma 2.7. *Let $p \in [1, \infty)$ and $\mathfrak{L} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ be a linear operator. Let $u \in W^{\mathfrak{L}, p}(\mathbb{R}^n; \mathbb{R}^m)$. Then,*

$$\mathfrak{L}(\nabla(\rho_\sigma * u)) = D_{\mathfrak{L}}(\rho_\sigma * u) = \rho_\sigma * D_{\mathfrak{L}}u \in L^p(\mathbb{R}^n; \mathbb{R}^{m \times n})$$

Proof. The strategy is the same one used in the proof of [Bre11, Lem 8.4] \square

Lemma 2.8. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $p \in [1, \infty)$ and $\mathfrak{L} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ be a linear operator. Let $u \in W_{\text{loc}}^{\mathfrak{L}, p}(\Omega; \mathbb{R}^m)$ and $v \in C_c^1(\Omega)$. Then, $\overline{uv} \in W^{\mathfrak{L}, p}(\mathbb{R}^n; \mathbb{R}^m)$ and,*

$$D_{\mathfrak{L}}\overline{uv} = \overline{vD_{\mathfrak{L}}u + uD_{\mathfrak{L}}v} \in L^p(\mathbb{R}^n; \mathbb{R}^{m \times n}),$$

where \overline{f} denotes the null extensions of a function f defined on Ω to the whole space \mathbb{R}^n .

Proof. The strategy is the same one used in the proof of [Bre11, Ch. 9, Rem. 4, part (ii)]. \square

Lemma 2.9. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $p \in [1, \infty)$ and $\mathfrak{L} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ be a linear operator. Let $u \in W_{\text{loc}}^{\mathfrak{L}, p}(\Omega; \mathbb{R}^m)$. Then, there exists $\{u_\sigma\}_{\sigma > 0} \subset C_c^\infty(\mathbb{R}^n; \mathbb{R}^m)$ s.t. $u_\sigma \rightarrow u$ in $L_{\text{loc}}^p(\Omega; \mathbb{R}^m)$ as $\sigma \rightarrow 0^+$ and,*

$$\lim_{\sigma \rightarrow 0^+} \|D_{\mathfrak{L}}u - D_{\mathfrak{L}}u_\sigma\|_{L^p(\tilde{\Omega}; \mathbb{R}^{m \times n})} = 0,$$

for every $\tilde{\Omega} \in \mathcal{A}_\Omega$ s.t. $\tilde{\Omega} \Subset \Omega$.

Moreover, if $u \in W^{\mathfrak{L}, p}(\Omega; \mathbb{R}^m)$, then $u_\sigma \rightarrow u$ in $L^p(\Omega; \mathbb{R}^m)$ and $u_\sigma \rightarrow u$ in $W^{\mathfrak{L}, p}(\tilde{\Omega}; \mathbb{R}^m)$ for every $\tilde{\Omega} \in \mathcal{A}_\Omega$ s.t. $\tilde{\Omega} \Subset \Omega$ as $\sigma \rightarrow 0^+$.

Proof. The strategy is the same one used in the proof of [Bre11, Thm. 9.2].

For every $\sigma > 0$ we define $u_\sigma := \zeta_\sigma(\rho_\sigma * \bar{u})$ (where \bar{u} is the null extension of u to \mathbb{R}^n) and $\zeta \in C_c^\infty(\mathbb{R}^n; [0, 1])$ is a cut-off function null in $\mathbb{R}^n \setminus B_2(0)$ and identically equal to 1 in $\overline{B_1(0)}$ and $\zeta_\sigma(x) := \zeta(\sigma x)$ for every $\sigma > 0$ and $x \in \mathbb{R}^n$.

We fix $\tilde{\Omega}, \Omega' \in \mathcal{A}_\Omega$ s.t. $\tilde{\Omega} \Subset \Omega' \Subset \Omega$ and a function $v \in C_c^\infty(\mathbb{R}^n; [0, 1])$ with $\text{supp}(v) \subset \Omega'$ s.t. $v \equiv 1$ on a neighborhood of $\tilde{\Omega}$. Then let \overline{uv} be the null extension of uv to \mathbb{R}^n . We then observe that $(\rho_\sigma * \overline{uv})|_{\tilde{\Omega}} = (\rho_\sigma * \bar{u})|_{\tilde{\Omega}}$ for every $\sigma > 0$.

By Lem. 2.7 and Lem. 2.8 we obtain that $\{u_\sigma\}_{\sigma > 0}$ is the desired sequence. \square

3. p -CORE FUNCTIONALS AND GENERALIZED BMO-TYPE SEMINORMS

Given two open sets $\Omega, D \subseteq \mathbb{R}^n$ and $\Gamma \subseteq \text{Aff}(\mathbb{R}^n)$, we denote by,

$$\mathcal{E}_\Gamma^D(\Omega) := \{F(D) : F \in \Gamma \text{ and } F(D) \subseteq \Omega\},$$

the class of all subsets of Ω obtained by transforming D according to the elements of Γ .

We start stating the main definition of our work.

Definition 3.1 (p -core functional). Let $D \subseteq \mathbb{R}^n$ be a bounded open set, $m \geq 1$ a natural number, $p \in [1, \infty)$ and Γ a subgroup of $\text{Aff}(\mathbb{R}^n)$.

We define $\text{dom}(\alpha_{p, D, \Gamma, m}) := L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m) \times \mathcal{E}_\Gamma^D(\mathbb{R}^n)$ and we let,

$$\alpha_{p, D, \Gamma, m} : \text{dom}(\alpha_{p, D, \Gamma, m}) \rightarrow [0, +\infty)$$

be a positive function.

To avoid heavy notation, when the set D , the subgroup Γ and m are clear from the context we shall drop some subscripts in $\alpha_{p, D, \Gamma, m}$. For instance, below we will write α_p instead of $\alpha_{p, D, \Gamma, m}$.

We say that α_p is a *p-core functional* (subordinated to the *reference set* D , the subgroup Γ and m) if it satisfies the following properties.

Given $D' \in \mathcal{E}_\Gamma^D(\mathbb{R}^n)$ and a certain function in $L^p(D'; \mathbb{R}^m)$, α_p is *extension independent* in the following sense:

$$(EI) \quad \alpha_p(u, D') = \alpha_p(v, D') \quad \forall D' \in \mathcal{E}_\Gamma^D(\mathbb{R}^n) \forall u, v \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m) \text{ s.t. } u|_{D'} = v|_{D'}.$$

α_p is *convex* in its first entry:

$$(CX) \quad \alpha_p(\cdot, D') \text{ is convex } \forall D' \in \mathcal{E}_\Gamma^D(\mathbb{R}^n).$$

α_p is *p-homogeneous* in its first entry:

$$(pH) \quad \alpha_p(tu, D') = |t|^p \alpha_p(u, D') \quad \forall (u, D') \in \text{dom}(\alpha_p) \forall t \in \mathbb{R}.$$

α_p is *translation invariant* in its first entry:

$$(T1) \quad \alpha_p(u + h, D') = \alpha_p(u, D') \quad \forall (u, D') \in \text{dom}(\alpha_p) \forall h \in \mathbb{R}^m.$$

α_p satisfies the following *change of variable* property:

$$(CV) \quad \alpha_p(u \circ F, D') = \alpha_p(u, F(D')) \quad \forall (u, D') \in \text{dom}(\alpha_p) \forall F \in \Gamma.$$

When $\alpha_p(\cdot, D)$ is tested on a Sobolev function it is *bounded* from above by the $L^p(D; \mathbb{R}^{m \times n})$ norm of its *gradient* (up to a multiplicative constant):

$$(GB) \quad \begin{aligned} & \exists c_1 > 0 \text{ s.t. } \alpha_p(u, D) \leq c_1 \int_D |\nabla u|_2^p dx \\ & \forall u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m) \text{ s.t. } u|_D \in W^{1,p}(D; \mathbb{R}^m). \end{aligned}$$

We observe that if α_p is a *p-core functional* subordinated to a certain Γ , subgroup of $\text{Aff}(\mathbb{R}^n)$, then it can be naturally seen also as a *p-core functional* subordinated to every subgroup of Γ .

Let us collect some properties of a *p-core functional* α_p as in Def. 3.1. Let $D' \in \mathcal{E}_\Gamma^D(\mathbb{R}^n)$. Then, for every $u, v \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$,

$$(3.1) \quad \alpha_p(u + v, D') \leq 2^{p-1} (\alpha_p(u, D') + \alpha_p(v, D')),$$

thanks to (CX) and (pH). In particular, if $p = 1$, (3.1) amounts to a triangular inequality for $u \mapsto \alpha_1(u, D')$. This then allows to conclude (using also the symmetry inherited from (pH)), that

$$|\alpha_1(u, D') - \alpha_1(v, D')| \leq \alpha_1(u - v, D'),$$

for every $u, v \in L_{\text{loc}}^1(\mathbb{R}^n; \mathbb{R}^m)$.

The following inequalities hold for any $p \in [1, \infty)$ and are another consequence of (CX) and (pH).

$$(3.2) \quad \begin{aligned} \alpha_p(u + v, D') & \leq (1 + \delta)^p \alpha_p(u, D') + \left(\frac{1 + \delta}{\delta}\right)^p \alpha_p(v, D') \\ \alpha_p(u - v, D') & \geq \frac{1}{(1 + \delta)^p} \alpha_p(u, D') - \frac{1}{\delta^p} \alpha_p(v, D') \end{aligned}$$

for every $u, v \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ and $\delta \in (0, 1)$.

Furthermore, α_p is zero when evaluated on constant functions,

$$(ZC) \quad \alpha_p(h, D') = 0 \quad \forall h \in \mathbb{R}^m,$$

thanks to (T1) and (pH).

In addition to that, (CV), (EI) and (GB) imply

$$(3.3) \quad \begin{aligned} \alpha_p(u, F(D)) &\leq c_1 \frac{|\nabla F|_2^p}{|\det(\nabla F)|} \int_{F(D)} |\nabla(u|_{F(D)})|_2^p dx \\ \forall F \in \Gamma \forall u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m) \text{ s.t. } \nabla(u|_{F(D)}) &\in L^p(F(D); \mathbb{R}^{m \times n}). \end{aligned}$$

In the next definition we will introduce the subgroups of $\text{Aff}(\mathbb{R}^n)$ which will be fundamental in defining our class of generalized BMO-type seminorms.

Definition 3.2. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $D \subseteq \mathbb{R}^n$ a bounded open set and \mathcal{G} a subgroup of $\text{SO}(n)$. For every $\varepsilon > 0$ we define,

$$\begin{aligned} \Gamma_\varepsilon^{\mathcal{G}} &:= \{\mathbb{R}^n \ni x \mapsto \varepsilon \mathcal{R}(x) + h : \mathcal{R} \in \mathcal{G}, h \in \mathbb{R}^n\} \subset \text{Aff}(\mathbb{R}^n), \quad \Gamma^{\mathcal{G}} := \bigcup_{\varepsilon > 0} \Gamma_\varepsilon^{\mathcal{G}} \subset \text{Aff}(\mathbb{R}^n), \\ \mathcal{G}_\varepsilon^D(\Omega) &:= \mathcal{E}_{\Gamma_\varepsilon^{\mathcal{G}}}^D(\Omega), \quad \mathcal{G}^D(\Omega) := \mathcal{E}_{\Gamma^{\mathcal{G}}}^D(\Omega) = \bigcup_{\varepsilon > 0} \mathcal{G}_\varepsilon^D(\Omega). \end{aligned}$$

We observe that $\mathcal{G}_\varepsilon^D(\Omega) \neq \emptyset$ if $\varepsilon > 0$ is small enough.

Let \mathcal{G} be a subgroup of $\text{SO}(n)$. Let us collect some properties of a p -core functional α_p be as in Def. 3.1 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2.

(CV) implies that α_p satisfies the following property connected to *translations*,

$$(T2) \quad \alpha_p(u, D') = \alpha_p(u(\cdot - h), D' + h) \quad \forall (u, D') \in \text{dom}(\alpha_p) \quad \forall h \in \mathbb{R}^n,$$

and that α_p satisfies the following *scaling* property:

$$(SC) \quad \alpha_p\left(u\left(\frac{\cdot}{t}\right), D'\right) = \alpha_p\left(u, \frac{D'}{t}\right) \quad \forall (u, D') \in \text{dom}(\alpha_p) \quad \forall t > 0.$$

Finally, from (3.3) we can infer that,

$$(GB+) \quad \begin{aligned} \alpha_p(u, D') &\leq c_2 \text{diam}(D')^{p-n} \int_{D'} |\nabla(u|_{D'})|_2^p dx \\ \forall D' \in \mathcal{G}^D(\mathbb{R}^n) \forall u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m) \text{ s.t. } \nabla(u|_{D'}) &\in L^p(D'; \mathbb{R}^{m \times n}), \end{aligned}$$

where $c_2 = c_1 \text{diam}(D)^{n-p}$.

Proposition 3.3. Let α_p be a p -core functional as in Def. 3.1. Then,

$$(3.4) \quad \mathbb{R}^{m \times n} \ni A \mapsto \alpha_p(l^A, D') \text{ is convex } \forall D' \in \mathcal{E}_\Gamma^D(\mathbb{R}^n).$$

Moreover,

$$(3.5) \quad \mathbb{R}^{m \times n} \ni A \mapsto \alpha_p(l^A, D') \text{ is locally Lipschitz continuous } \forall D' \in \mathcal{E}_\Gamma^D(\mathbb{R}^n).$$

Assume now that α_p is subordinated to $\Gamma = \Gamma^{\mathcal{G}}$ where \mathcal{G} is a subgroup of $\text{SO}(n)$. Then,

$$(3.6) \quad |\alpha_p(l^A, D') - \alpha_p(l^B, D')| \leq \varepsilon^p C_\Sigma |A - B|_2 \quad \forall A, B \in \Sigma \quad \forall \varepsilon > 0 \quad \forall D' \in \mathcal{G}_\varepsilon^D(\mathbb{R}^n),$$

where

$$(3.7) \quad \Sigma := \{A \in \mathbb{R}^{m \times n} : |A|_2 \leq 1\},$$

and $C_\Sigma > 0$ is a positive constant (depending on Σ and α_p).

Proof. Thanks to (CX) and the linearity of the map $\mathbb{R}^{m \times n} \ni A \mapsto l^A \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$, we observe that (3.4) holds. Since $\mathbb{R}^{m \times n}$ is a finite-dimensional normed vector space, (3.4) implies (3.5).

Assume now that $\Gamma = \Gamma^{\mathcal{G}}$ where \mathcal{G} is a subgroup of $\text{SO}(n)$. Thanks to (3.5) we can find a constant $C_{\Sigma} > 0$ (depending on Σ and α_p) such that

$$(3.8) \quad |\alpha_p(l^A, D) - \alpha_p(l^B, D)| \leq C_{\Sigma} |A - B|_2 \quad \forall A, B \in \Sigma.$$

Let $\varepsilon > 0$ and $D' \in \mathcal{G}_{\varepsilon}^D(\mathbb{R}^n)$. Then, $D' = \varepsilon \mathcal{R}D + h$ for some $\mathcal{R} \in \mathcal{G}$ and $h \in \mathbb{R}^n$. By (CV), (T1) and (pH) we find,

$$(3.9) \quad \alpha_p(l^A, D') = \alpha_p(l^A(\varepsilon \mathcal{R}(\cdot) + h), D) = \varepsilon^p \alpha_p(l^{A\mathcal{R}}, D) \quad \forall A \in \mathbb{R}^{m \times n}.$$

Combining (3.8) and (3.9) we obtain (3.6), noting that $A\mathcal{R} \in \Sigma$ for every $\mathcal{R} \in \text{SO}(n)$ and $A \in \Sigma$. \square

We are now ready to define the generalized BMO-type seminorms which will be the focus of our work.

Definition 3.4 (BMO-type seminorm functional). Let $\Omega \subseteq \mathbb{R}^n$ be an open set, which we will refer to as the *ambient space*, and \mathcal{G} be a subgroup of $\text{SO}(n)$. Recalling Def. 3.2, let $\alpha_p = \alpha_{p,D,\Gamma^{\mathcal{G}},m}$ be a p -core functional as in Def. 3.1 with $\text{dom}(\alpha_p) = L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m) \times \mathcal{G}^D(\mathbb{R}^n)$. For every $u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ we define,

$$(3.10) \quad G_{\varepsilon}^{\alpha_p}(u, \Omega) := \varepsilon^{n-p} \sup_{\mathcal{G}_{\varepsilon}} \sum_{D' \in \mathcal{G}_{\varepsilon}} \alpha_p(u, D'),$$

where each $\mathcal{G}_{\varepsilon}$ is a family made of pairwise disjoint elements of $\mathcal{G}_{\varepsilon}^D(\Omega)$ and the supremum is taken with respect to all of these families $\mathcal{G}_{\varepsilon}$. We observe that the cardinality of every family $\mathcal{G}_{\varepsilon}$ is at most countable and it is bounded from above by $|\Omega|/(\varepsilon^n |D|)$. For convenience, if $\mathcal{G}_{\varepsilon} = \emptyset$, we let the supremum in (3.10) be equal to 0. We define,

$$G_{+}^{\alpha_p}(u, \Omega) := \limsup_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega),$$

$$G_{-}^{\alpha_p}(u, \Omega) := \liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega),$$

for every $u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$. We also define,

$$G^{\alpha_p}(u, \Omega) := \lim_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega),$$

for every $u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ such that $G_{-}^{\alpha_p}(u, \Omega) = G_{+}^{\alpha_p}(u, \Omega)$.

We finally observe that for every $\varepsilon > 0$, thanks to (EI),

$$(3.11) \quad G_{\varepsilon}^{\alpha_p}(u, \Omega) = G_{\varepsilon}^{\alpha_p}(v, \Omega), \quad G_{\pm}^{\alpha_p}(u, \Omega) = G_{\pm}^{\alpha_p}(v, \Omega),$$

for all $u, v \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ such that $u|_{\Omega} = v|_{\Omega}$.

Remark 3.5. Under the assumptions of Def. 3.4, if $u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ and $\nabla u|_{\Omega} \in L^p(\Omega; \mathbb{R}^{m \times n})$ thanks to (GB+) we obtain that

$$\begin{aligned} G_{\varepsilon}^{\alpha_p}(u, \Omega) &\leq \varepsilon^{n-p} \sup_{\mathcal{G}_{\varepsilon}} \sum_{D' \in \mathcal{G}_{\varepsilon}} c_2 \text{diam}(D')^{p-n} \int_{D'} |\nabla u|_2^p dx \\ &\leq c_1 \int_{\Omega} |\nabla u|_2^p dx, \end{aligned}$$

for every $\varepsilon > 0$. Therefore,

$$G_{\pm}^{\alpha_p}(u, \Omega) \leq c_1 \int_{\Omega} |\nabla u|_2^p dx.$$

We list some properties of $G_\varepsilon^{\alpha_p}$ and $G_\pm^{\alpha_p}$, omitting the elementary proofs.

Proposition 3.6. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $m \geq 1$ a natural number, $p \in [1, \infty)$, \mathcal{G} a subgroup of $\text{SO}(n)$ and $D \subseteq \mathbb{R}^n$ be a bounded open set. Let $\alpha_p = \alpha_{p,D,\Gamma^\mathcal{G},m}$ be a p -core functional as in Def. 3.1 subordinated to $\Gamma^\mathcal{G}$ as in Def. 3.2. Let $\varepsilon > 0$ and $u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ and $A \in \mathbb{R}^{m \times n}$. Recalling Def. 3.4, for every $\lambda > 0$ it holds,*

$$\lambda^{n-p} G_\varepsilon^{\alpha_p, \lambda D, \Gamma^\mathcal{G}}(u, \Omega) = G_{\varepsilon\lambda}^{\alpha_p}(u, \Omega), \quad \lambda^{n-p} G_\pm^{\alpha_p, \lambda D, \Gamma^\mathcal{G}}(u, \Omega) = G_\pm^{\alpha_p}(u, \Omega),$$

where $\alpha_{p,\lambda D, \Gamma^\mathcal{G}}$ is the p -core functional defined by,

$$\alpha_{p,\lambda D, \Gamma^\mathcal{G}}(u, D') := \alpha_p\left(u, \frac{1}{\lambda} D'\right),$$

for every $D' \in \mathcal{G}^{\lambda D}(\mathbb{R}^n) = \mathcal{G}^D(\mathbb{R}^n)$.

For every $t > 0$,

$$(3.12) \quad \begin{aligned} t^{n-p} G_\varepsilon^{\alpha_p}(u, \Omega) &= G_{t\varepsilon}^{\alpha_p}(u(\cdot/t), t\Omega), & t^{n-p} G_\pm^{\alpha_p}(u, \Omega) &= G_\pm^{\alpha_p}(u(\cdot/t), t\Omega), \\ t^n G_\varepsilon^{\alpha_p}(l^A, \Omega) &= G_{t\varepsilon}^{\alpha_p}(l^A, t\Omega), & t^n G_\pm^{\alpha_p}(l^A, \Omega) &= G_\pm^{\alpha_p}(l^A, t\Omega). \end{aligned}$$

For every $h \in \mathbb{R}^n$,

$$(3.13) \quad \begin{aligned} G_\varepsilon^{\alpha_p}(u, \Omega) &= G_\varepsilon^{\alpha_p}(u(\cdot - h), \Omega + h), & G_\pm^{\alpha_p}(u, \Omega) &= G_\pm^{\alpha_p}(u(\cdot - h), \Omega + h), \\ G_\varepsilon^{\alpha_p}(l^A, \Omega) &= G_\varepsilon^{\alpha_p}(l^A, \Omega + h), & G_\pm^{\alpha_p}(l^A, \Omega) &= G_\pm^{\alpha_p}(l^A, \Omega + h). \end{aligned}$$

Proposition 3.7. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $m \geq 1$ a natural number, $p \in [1, \infty)$, \mathcal{G} a subgroup of $\text{SO}(n)$ and $D \subseteq \mathbb{R}^n$ be a bounded open set. Let $\alpha_p = \alpha_{p,D,\Gamma^\mathcal{G},m}$ be a p -core functional as in Def. 3.1 subordinated to $\Gamma^\mathcal{G}$ as in Def. 3.2. Let $\varepsilon > 0$ and $u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$.*

Recalling Def. 3.4, we define the functionals acting on open subsets of Ω ,

$$\mathcal{A}_\Omega \ni \tilde{\Omega} \mapsto g_\varepsilon(\tilde{\Omega}) := G_\varepsilon^{\alpha_p}(u, \tilde{\Omega}), \quad \mathcal{A}_\Omega \ni \tilde{\Omega} \mapsto g_\pm(\tilde{\Omega}) := G_\pm^{\alpha_p}(u, \tilde{\Omega}).$$

Then,

- (1) g_ε, g_\pm are monotone non-decreasing with respect to set inclusion,
- (2) g_ε, g_- are superadditive:

$$g_\varepsilon(\Omega_1 \cup \Omega_2) \geq g_\varepsilon(\Omega_1) + g_\varepsilon(\Omega_2), \quad g_-(\Omega_1 \cup \Omega_2) \geq g_-(\Omega_1) + g_-(\Omega_2),$$

for every $\Omega_1, \Omega_2 \in \mathcal{A}_\Omega$ s.t. $\Omega_1 \cap \Omega_2 = \emptyset$.

- (3) if $\nabla u|_\Omega \in L^p(\Omega; \mathbb{R}^{m \times n})$, then,

(i) g_+ is inner-regular in the following sense:

$$(3.14) \quad g_+(\tilde{\Omega}) = \sup\{g_+(\Omega') : \Omega' \Subset \tilde{\Omega}, \Omega' \in \mathcal{A}_\Omega\} \text{ for all } \tilde{\Omega} \in \mathcal{A}_\Omega.$$

(ii) g_+ is σ -subadditive:

$$g_+\left(\bigcup_{k \in \mathbb{N}} \Omega_k\right) \leq \sum_{k \in \mathbb{N}} g_+(\Omega_k),$$

for all sequences $\{\Omega_k\}_{k \in \mathbb{N}} \subset \mathcal{A}_\Omega$.

Proof. It is elementary to obtain (1) and (2). For the proof of (i) and (ii) we refer to the proof of [FFGLBS20, Prop. 3.1] taking into account Rem. 3.5. \square

We observe that, for a given linear map $l^A(x) = Ax$, the functionals $G_\pm^{\alpha_p}(l^A, \cdot)$ coincide if they act on any unitary cube centered in the origin. The following proposition is obtained by retracing [FFGLBS20, Prop. 3.2].

Proposition 3.8. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $m \geq 1$ a natural number, $p \in [1, \infty)$, \mathcal{G} a subgroup of $\text{SO}(n)$ and $D \subseteq \mathbb{R}^n$ be a bounded open set. Let $\alpha_p = \alpha_{p,D,\Gamma^{\mathcal{G}},m}$ be a p -core functional as in Def. 3.1 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2. For any unitary cube \tilde{Q} centered in the origin,*

$$\sup_{0 < \varepsilon \leq 1} G_{\varepsilon}^{\alpha_p}(l^A, \tilde{Q}) = G^{\alpha_p}(l^A, \tilde{Q}) = G^{\alpha_p}(l^A, Q),$$

for all $A \in \mathbb{R}^{m \times n}$.

We can now define the function $\psi_{\alpha_p} : \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$.

Definition 3.9. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $m \geq 1$ a natural number, $p \in [1, \infty)$, \mathcal{G} a subgroup of $\text{SO}(n)$ and $D \subseteq \mathbb{R}^n$ be a bounded open set. Let $\alpha_p = \alpha_{p,D,\Gamma^{\mathcal{G}},m}$ be a p -core functional as in Def. 3.1 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2. We define,

$$\psi_{\alpha_p}(A) := G^{\alpha_p}(l^A, Q),$$

for every $A \in \mathbb{R}^{m \times n}$.

The next result shows that ψ_{α_p} is well defined, i.e. the values of the function do not depend on the choice of the cube Q at the right hand side of (3.15), and lists some useful properties.

Proposition 3.10. *Under the assumptions of Def. 3.9, ψ_{α_p} is a well-defined function with values in $[0, +\infty)$ and for any unitary cube \tilde{Q} centered in the origin*

$$(3.15) \quad \psi_{\alpha_p}(A) = G^{\alpha_p}(l^A, \tilde{Q}),$$

for all $A \in \mathbb{R}^{m \times n}$.

Moreover, ψ_{α_p} is p -homogeneous, convex and locally Lipschitz. Precisely,

$$(3.16) \quad |\psi_{\alpha_p}(A) - \psi_{\alpha_p}(B)| \leq \frac{C_{\Sigma}}{|D|} |A - B|_2 \quad \forall A, B \in \Sigma,$$

where Σ and $C_{\Sigma} > 0$ are as in Prop. 3.3.

Furthermore, the set

$$\mathcal{N}_{\psi_{\alpha_p}} := \{A \in \mathbb{R}^{m \times n} : \psi_{\alpha_p}(A) = 0\},$$

is a linear subspace of $\mathbb{R}^{m \times n}$.

Finally, $\psi_{\alpha_p}^{1/p}$ is a seminorm and

$$(3.17) \quad \psi_{\alpha_p}(A) = \psi_{\alpha_p}(A + B),$$

for every $A \in \mathbb{R}^{m \times n}$ and $B \in \mathcal{N}_{\psi_{\alpha_p}}$.

Proof. ψ_{α_p} is well-defined and (3.15) holds as a direct consequence of Prop. 3.8, while $\psi_{\alpha_p}(A) \in [0, +\infty)$ for every $A \in \mathbb{R}^{m \times n}$ by Rem. 3.5.

We now prove that ψ_{α_p} is p -homogeneous. We fix $t \in \mathbb{R}$ and we notice that

$$\psi_{\alpha_p}(tA) = \lim_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(l^{tA}, Q) = |t|^p \lim_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(l^A, Q) = |t|^p \psi_{\alpha_p}(A),$$

thanks to (pH).

We prove that ψ_{α_p} is convex. Let $t \in [0, 1]$ and $A, B \in \mathbb{R}^{m \times n}$. Let $\varepsilon > 0$ and \mathcal{G}_ε be a family made of pairwise disjoint elements of $\mathcal{G}_\varepsilon^D(Q)$. Then,

$$\begin{aligned} & \varepsilon^{n-p} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(tl^A + (1-t)l^B, D') \\ & \leq t\varepsilon^{n-p} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(l^A, D') + (1-t)\varepsilon^{n-p} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(l^B, D') \\ & \leq tG_\varepsilon^{\alpha_p}(l^A, Q) + (1-t)G_\varepsilon^{\alpha_p}(l^B, Q), \end{aligned}$$

where we used (CX). Taking the supremum with respect to \mathcal{G}_ε we conclude that $\mathbb{R}^{m \times n} \ni A \mapsto G_\varepsilon^{\alpha_p}(l^A, Q) \in [0, +\infty)$ is convex. Letting $\varepsilon \rightarrow 0^+$ this implies that ψ_{α_p} is convex.

From the convexity of ψ_{α_p} , being $\mathbb{R}^{m \times n}$ a finite dimension space, we immediately deduce that ψ_{α_p} is locally Lipschitz.

Nevertheless, we now prove (3.16) to show how the locally Lipschitz constant of ψ_{α_p} on the set $\Sigma := \{A \in \mathbb{R}^{m \times n} : |A|_2 \leq 1\}$ is related to the one of $\mathbb{R}^{m \times n} \ni A \mapsto \alpha_p(l^A, D)$ on Σ .

Let $A, B \in \Sigma$ and $\delta > 0$. By definition of ψ_{α_p} there exists $\varepsilon_0 > 0$ such that,

$$(3.18) \quad \psi_{\alpha_p}(A) - \psi_{\alpha_p}(B) \leq G_\varepsilon^{\alpha_p}(l^A, Q) - G_\varepsilon^{\alpha_p}(l^B, Q) + \delta \quad \forall 0 < \varepsilon < \varepsilon_0.$$

Moreover, by definition of $G_\varepsilon^{\alpha_p}(l^A, Q)$ and $G_\varepsilon^{\alpha_p}(l^B, Q)$, for every $\varepsilon > 0$ there exists a family \mathcal{G}_ε , made of pairwise disjoint elements of $\mathcal{G}_\varepsilon^D(Q)$, whose cardinality does not exceed $(\varepsilon^n |D|)^{-1}$ and

$$(3.19) \quad G_\varepsilon^{\alpha_p}(l^A, Q) - G_\varepsilon^{\alpha_p}(l^B, Q) \leq \varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \sum_{D' \in \mathcal{G}_\varepsilon} (\alpha_p(l^A, D') - \alpha_p(l^B, D')) + \delta.$$

Combining (3.18) and (3.19) with (3.6) we find,

$$\psi_{\alpha_p}(A) - \psi_{\alpha_p}(B) \leq \frac{C_\Sigma}{|D|} |A - B|_2 + 2\delta,$$

where C_Σ is as in Prop. 3.3. By letting $\delta \rightarrow 0^+$ and then interchanging the role of A and B we deduce that (3.16) holds.

The conclusion of the proof is an immediate consequence of Lem. 2.2. \square

Remark 3.11. Under the assumptions of Def. 3.9, as a consequence of Prop. 3.10, for every \mathcal{P} linear subspace of $\mathbb{R}^{m \times n}$ such that,

$$(3.20) \quad \mathbb{R}^{m \times n} = \mathcal{P} \oplus \mathcal{N}_{\psi_{\alpha_p}}$$

it follows that $\psi_{\alpha_p}^{1/p}$ restricted to \mathcal{P} is a norm and for every $A \in \mathbb{R}^{m \times n}$,

$$\psi_{\alpha_p}(\pi_{\mathcal{P}}(A)) = \psi_{\alpha_p}(A),$$

where $\pi_{\mathcal{P}} : \mathbb{R}^{m \times n} \rightarrow \mathcal{P}$ is the projection on \mathcal{P} (associated to the direct sum decomposition in (3.20)). Then, under the assumptions of Thm. 1.1, (1.4) is equivalent to

$$(3.21) \quad \lim_{\varepsilon \rightarrow 0^+} G_\varepsilon^{\alpha_p}(u, \Omega) = \int_{\Omega} \psi_{\alpha_p}(\pi_{\mathcal{P}}(\nabla u)) \, dx.$$

Moreover, making use of Prop. 3.10 again, for every $u \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^m)$,

$$c \int_{\Omega} |\pi_{\mathcal{P}}(\nabla u)|_2^p \, dx \leq \int_{\Omega} \psi_{\alpha_p}(\pi_{\mathcal{P}}(\nabla u)) \, dx \leq C \int_{\Omega} |\pi_{\mathcal{P}}(\nabla u)|_2^p \, dx,$$

for some $c, C > 0$ depending on α_p .

Now we are ready to prove our main result.

Proof of Thm. 1.1. We divide the proof in several steps.

Step 1. We first assume that Ω is a bounded open set and $u|_{\overline{\Omega}} \in C^1(\overline{\Omega}; \mathbb{R}^m)$. For every $t > 0$, we define $U_t := \{x \in \Omega : |\nabla u(x)|_2 > t\}$. Let $\sigma \in (0, 1)$ and $t > 0$ s.t $|\partial U_t| = 0$ (this happens for every $t > 0$ except for countably many). Let $r > 0$, $\{Q(x_i; r)\}_{1 \leq i \leq l}$ and $W_{t, \sigma}$ as in Lem. 2.3.

In order to prove (1.4) for u , we consider two substeps.

Step 1.1. First we prove that

$$(3.22) \quad G_{\varepsilon}^{\alpha_p}(u, \Omega) \geq \int_{\Omega} \psi_{\alpha_p}(\nabla u(x)) dx.$$

Fix $i \in \{1, \dots, l\}$ and $\varepsilon > 0$ and consider a family $\mathcal{G}_{\varepsilon}$ as in Def. 3.4 with $Q(x_i; r)$ as ambient space. This means that $\mathcal{G}_{\varepsilon}$ is a family of pairwise disjoint sets $\{D_j\}_{1 \leq j \leq k}$, where for every $j \in \{1, \dots, k\}$ we have $D_j = z_j + \varepsilon \mathcal{R}_j D \subset Q(x_i; r)$, with $\mathcal{R}_j \in \mathcal{G}$, $z_j \in \mathbb{R}^n$ and

$$(3.23) \quad k \leq |Q(x_i; r)| / (\varepsilon^n |D|) = r^n / (\varepsilon^n |D|).$$

For every $j \in \{1, \dots, k\}$ and $x \in D_j$ we may write

$$u(x) = u(z_j) + \nabla u(z_j)(x - z_j) + R_j(x),$$

with $R_j(x) := (\nabla u(\bar{x}) - \nabla u(z_j))(x - z_j)$ for all $x \in \mathbb{R}^n$, where $\bar{x} \in Q(x_i; r)$. Thus,

$$(3.24) \quad \begin{aligned} G_{\varepsilon}^{\alpha_p}(u, Q(x_i; r)) &\geq \varepsilon^{n-p} \sum_{j=1}^k \alpha_p(u, D_j) \\ &= \varepsilon^{n-p} \sum_{j=1}^k \alpha_p(\nabla u(z_j)(\cdot - z_j) + R_j, D_j) \\ &\geq \frac{\varepsilon^{n-p}}{(1+\delta)^p} \sum_{j=1}^k \alpha_p(\nabla u(x_i)(\cdot - z_j), D_j) - \frac{\varepsilon^{n-p}}{\delta^p} \sum_{j=1}^k \alpha_p(\bar{R}_j, D_j) \\ &= \frac{\varepsilon^{n-p}}{(1+\delta)^p} \sum_{j=1}^k \alpha_p(t^{\nabla u(x_i)}, D_j) - \frac{\varepsilon^{n-p}}{\delta^p} \sum_{j=1}^k \alpha_p(\bar{R}_j, D_j) \end{aligned}$$

where in the first equality we use (T1) and (EI), in the last equality we use (T1), while in the last inequality we use (3.2) having set,

$$\begin{aligned} \bar{R}_j(x) &:= (\nabla u(x_i) - \nabla u(z_j))(x - z_j) - R_j(x) \\ &= (\nabla u(x_i) - \nabla u(\bar{x}))(x - z_j), \end{aligned}$$

for every $x \in \mathbb{R}^n$.

Using (GB+) and part b) in Lem. 2.3, we estimate

$$(3.25) \quad \begin{aligned} \frac{1}{\delta^p} \sum_{j=1}^k \alpha_p(\bar{R}_j, D_j) &\leq \frac{c_2}{\delta^p} \sum_{j=1}^k \text{diam}(D_j)^{p-n} \int_{D_j} |\nabla u(x_i) - \nabla u(\bar{x})|_2^p dx \\ &\leq \frac{c_2}{\delta^p} \varepsilon^{p-n} \sum_{j=1}^k \text{diam}(D)^{p-n} \sigma^p \varepsilon^n |D| \\ &\leq \frac{c_2}{\delta^p} \text{diam}(D)^{p-n} \varepsilon^{p-n} \sigma^p r^n, \end{aligned}$$

where in the last inequality we used (3.23) and $c_2 > 0$ is as in (GB+). Thus combining (3.24) and (3.25), taking the supremum with respect to all families \mathcal{G}_ε and the lim inf with respect to ε , we have

$$(3.26) \quad \begin{aligned} G_-^{\alpha_p}(u, Q(x_i; r)) &\geq \frac{1}{(1+\delta)^p} G_-^{\alpha_p}(l^{\nabla u(x_i)}, Q(x_i; r)) - \frac{c_2}{\delta^p} \text{diam}(D)^{p-n} \sigma^p r^n \\ &= \frac{1}{(1+\delta)^p} r^n \psi_{\alpha_p}(\nabla u(x_i)) - \frac{c_2}{\delta^p} \text{diam}(D)^{p-n} \sigma^p r^n, \end{aligned}$$

where in the last inequality we used (3.12) and (3.13).

Now we observe that,

$$(3.27) \quad \begin{aligned} r^n \psi_{\alpha_p}(\nabla u(x_i)) &= \int_{Q(x_i; r)} \psi_{\alpha_p}(\nabla u(x)) \, dx \\ &\quad + \int_{Q(x_i; r)} \psi_{\alpha_p}(\nabla u(x_i)) - \psi_{\alpha_p}(\nabla u(x)) \, dx \end{aligned}$$

Recalling Prop. 3.10, (2.1) and (2.2) we observe that,

$$\begin{aligned} &\int_{Q(x_i; r)} |\psi_{\alpha_p}(\nabla u(x_i)) - \psi_{\alpha_p}(\nabla u(x))| \, dx \\ &= \int_{Q(x_i; r)} \left| |\nabla u(x_i)|_2^p \psi_{\alpha_p} \left(\frac{\nabla u(x_i)}{|\nabla u(x_i)|_2} \right) - |\nabla u(x)|_2^p \psi_{\alpha_p} \left(\frac{\nabla u(x)}{|\nabla u(x)|_2} \right) \right| \, dx \\ &\leq \int_{Q(x_i; r)} |\nabla u(x_i)|_2^p \left| \psi_{\alpha_p} \left(\frac{\nabla u(x_i)}{|\nabla u(x_i)|_2} \right) - \psi_{\alpha_p} \left(\frac{\nabla u(x)}{|\nabla u(x)|_2} \right) \right| \, dx \\ &\quad + \int_{Q(x_i; r)} \psi_{\alpha_p} \left(\frac{\nabla u(x)}{|\nabla u(x)|_2} \right) \left| |\nabla u(x_i)|_2^p - |\nabla u(x)|_2^p \right| \, dx \\ &\leq \frac{C_\Sigma}{|D|} \int_{Q(x_i; r)} |\nabla u(x_i)|_2^p \left| \frac{\nabla u(x_i)}{|\nabla u(x_i)|_2} - \frac{\nabla u(x)}{|\nabla u(x)|_2} \right| \, dx \\ &\quad + \|\psi_{\alpha_p}\|_{L^\infty(\Sigma)} \int_{Q(x_i; r)} \left| |\nabla u(x_i)|_2^p - |\nabla u(x)|_2^p \right| \, dx, \\ &\leq 2 \frac{C_\Sigma}{|D|} \|\nabla u\|_{L^\infty(\Omega)}^{p-1} \int_{Q(x_i; r)} |\nabla u(x_i) - \nabla u(x)|_2 \, dx \\ &\quad + p \left(2 \|\nabla u\|_{L^\infty(\Omega)} \right)^{p-1} \|\psi_{\alpha_p}\|_{L^\infty(\Sigma)} \int_{Q(x_i; r)} |\nabla u(x_i) - \nabla u(x)|_2 \, dx, \\ &\leq C \sigma r^n \end{aligned}$$

where $\Sigma := \{A \in \mathbb{R}^{m \times n} : |A|_2 \leq 1\}$, $C_\Sigma > 0$ is as in the proof of Prop. 3.10 and $C > 0$ is a constant depending on α_p and ∇u .

Recalling (3.27) and (3.26), we then get

$$G_-^{\alpha_p}(u, Q(x_i; r)) \geq \frac{1}{(1+\delta)^p} \int_{Q(x_i; r)} \psi_{\alpha_p}(\nabla u(x)) \, dx - C \sigma r^n,$$

where $C > 0$ is a constant depending on α_p , ∇u and δ .

Thus, summing up with respect to i , by the monotonicity and the superadditivity of $G_-^{\alpha_p}(u, \cdot)$ (recall Prop. 3.7), we get

$$\begin{aligned} G_-^{\alpha_p}(u, \Omega) &\geq \sum_{i=1}^l G_-^{\alpha_p}(u, Q(x_i; r)) \geq \frac{1}{(1+\delta)^p} \sum_{i=1}^l \int_{Q(x_i; r)} \psi_{\alpha_p}(\nabla u(x)) \, dx - C \sigma |\Omega| \\ &\geq \frac{1}{(1+\delta)^p} \int_{U_t} \psi_{\alpha_p}(\nabla u(x)) \, dx - C \sigma, \end{aligned}$$

where the last inequality follows from part a) in Lem. 2.3 and the constant C depends on α_p , ∇u , $|\Omega|$ and δ . Then (3.22) follows at once letting first $\sigma \rightarrow 0$, $t \rightarrow 0$ and then $\delta \rightarrow 0$ in the previous inequality.

Step 1.2. We prove now,

$$(3.28) \quad G_+^{\alpha_p}(u, \Omega) \leq \int_{\Omega} \psi_{\alpha_p}(\nabla u(x)) \, dx.$$

Using the subadditivity of $G_+^{\alpha_p}(u, \cdot)$ (recall Prop. 3.7) we have

$$(3.29) \quad G_+^{\alpha_p}(u, \Omega) \leq \sum_{i=1}^l G_+^{\alpha_p}(u, Q(x_i; r)) + G_+^{\alpha_p}(u, \Omega \setminus \overline{U}_t) + G_+^{\alpha_p}(u, W_{t, \sigma}).$$

Let us estimate the three terms in the last inequality. Arguing as in the previous step, for every $i = 1, \dots, l$ we get that,

$$(3.30) \quad G_+^{\alpha_p}(u, Q(x_i; r)) \leq (1 + \delta)^p \int_{Q(x_i; r)} \psi_{\alpha_p}(\nabla u(x)) \, dx + C\sigma r^n,$$

for some positive constant C depending only on α_p , ∇u and δ .

By using Rem. 3.5 we get

$$(3.31) \quad \begin{aligned} & G_+^{\alpha_p}(u, \Omega \setminus \overline{U}_t) + G_+^{\alpha_p}(u, W_{t, \sigma}) \\ & \leq c_1 \int_{\{|\nabla u|_2 \leq t\}} |\nabla u|_2^p \, dx + c_1 \int_{W_{t, \sigma}} |\nabla u|_2^p \, dx \leq c_1 \left(|\Omega| t^p + \sigma \|\nabla u\|_{L^\infty(\Omega)}^p \right) \end{aligned}$$

Then (3.28) follows combining (3.29), (3.30), (3.31) and letting first $\sigma \rightarrow 0$, $t \rightarrow 0$ and then $\delta \rightarrow 0$.

Finally, (3.22) and (3.28) imply that (1.4) holds under the assumptions on u made in this first step.

Step 2. We assume that Ω is a bounded open set and that $u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ with $\nabla u|_{\Omega} \in L^p(\Omega; \mathbb{R}^{m \times n})$.

Thanks to the previous step we know that,

$$(3.32) \quad \lim_{\varepsilon \rightarrow 0^+} G_\varepsilon^{\alpha_p}(v, \Omega') = \int_{\Omega'} \psi_{\alpha_p}(\nabla v) \, dx,$$

holds for every $\Omega' \in \mathcal{A}_\Omega$ and for every $v \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ s.t. $v|_{\overline{\Omega'}} \in C^1(\overline{\Omega'}; \mathbb{R}^m)$.

Let $\Omega' \in \mathcal{A}_\Omega$ s.t. $\Omega' \Subset \Omega$ and $\sigma > 0$. Then, there exists $v_\sigma^{\Omega'} \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^m)$ s.t. $\|u - v_\sigma^{\Omega'}\|_{W^{1,p}(\Omega'; \mathbb{R}^m)} < \sigma$.

Fix $\varepsilon > 0$ and consider a family \mathcal{G}_ε of pairwise disjoint elements of $\mathcal{G}_\varepsilon^D(\Omega')$. Then,

$$(3.33) \quad \begin{aligned} G_\varepsilon^{\alpha_p}(u, \Omega') & \geq \varepsilon^{n-p} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(u, D') \\ & \geq \frac{1}{(1 + \delta)^p} \varepsilon^{n-p} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(v_\sigma^{\Omega'}, D') - \frac{1}{\delta^p} \varepsilon^{n-p} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(u - v_\sigma^{\Omega'}, D'), \end{aligned}$$

where we used (3.2).

Using (GB+), we estimate,

$$(3.34) \quad \begin{aligned} \frac{1}{\delta^p} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(u - v_\sigma^{\Omega'}, D') & \leq \frac{c_2}{\delta^p} \sum_{D' \in \mathcal{G}_\varepsilon} \text{diam}(D')^{p-n} \int_{D'} |\nabla u - \nabla v_\sigma^{\Omega'}|_2^p \, dx \\ & \leq \frac{c_2}{\delta^p} \text{diam}(D)^{p-n} \varepsilon^{p-n} \int_{\Omega'} |\nabla u - \nabla v_\sigma^{\Omega'}|_2^p \, dx \\ & \leq \frac{c_1}{\delta^p} \varepsilon^{p-n} \sigma^p, \end{aligned}$$

where in the last inequalities $c_2 > 0$ is as in (GB+) and c_1 is as in (GB). Combining (3.33) with (3.34), taking the supremum with respect to \mathcal{G}_ε and letting $\varepsilon \rightarrow 0^+$, we obtain,

$$(3.35) \quad \begin{aligned} G_-^{\alpha_p}(u, \Omega) &\geq G_-^{\alpha_p}(u, \Omega') \geq \frac{1}{(1+\delta)^p} G_-^{\alpha_p}(v_\sigma^{\Omega'}, \Omega') - \frac{C\sigma^p}{\delta^p} \\ &= \frac{1}{(1+\delta)^p} \int_{\Omega'} \psi_{\alpha_p}(\nabla v_\sigma^{\Omega'}(x)) dx - \frac{C\sigma^p}{\delta^p}, \end{aligned}$$

where in the first inequality we used Prop. 3.7, $C > 0$ is a constant depending on α_p and in the last equality we made use of (3.32).

Similarly it holds,

$$(3.36) \quad G_+^{\alpha_p}(u, \Omega') \leq (1+\delta)^p \int_{\Omega'} \psi_{\alpha_p}(\nabla v_\sigma^{\Omega'}(x)) dx + \frac{C(1+\delta)^p \sigma^p}{\delta^p}.$$

Thanks to the dominated convergence theorem,

$$(3.37) \quad \lim_{\sigma \rightarrow 0^+} \int_{\Omega'} \psi_{\alpha_p}(\nabla v_\sigma^{\Omega'}(x)) dx = \int_{\Omega'} \psi_{\alpha_p}(\nabla u(x)) dx,$$

since $\lim_{\sigma \rightarrow 0^+} \|u - v_\sigma^{\Omega'}\|_{W^{1,p}(\Omega'; \mathbb{R}^m)} = 0$, ψ_{α_p} is continuous and $\psi_{\alpha_p}(A) \leq C|A|_2^p$ (cfr. Prop. 3.10) where $C > 0$ is a constant depending on α_p . Combining (3.35), (3.36), (3.37) and letting first $\sigma \rightarrow 0^+$, $\delta \rightarrow 0^+$ and $\Omega' \uparrow \Omega$ (recalling also (3.14) in Prop. 3.7) we deduce that (1.4) holds for u .

Step 3. We assume that $u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ with $\nabla u|_\Omega \in L^p(\Omega; \mathbb{R}^{m \times n})$ and $\Omega \subseteq \mathbb{R}^n$ is an arbitrary nonempty open set. Let $\Omega' \in \mathcal{A}_\Omega$ s.t. $\Omega' \Subset \Omega$. By the previous step and Prop. 3.7 we find,

$$(3.38) \quad G_+^{\alpha_p}(u, \Omega') = \int_{\Omega'} \psi_{\alpha_p}(\nabla u(x)) dx = G_-^{\alpha_p}(u, \Omega') \leq G_-^{\alpha_p}(u, \Omega).$$

(1.4) follows by letting $\Omega' \uparrow \Omega$ in (3.38) and using (3.14) in Prop. 3.7.

Step 4. We assume that $\psi_{\alpha_p} \not\equiv 0$ and $u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ with $u|_\Omega \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^m)$ and $\Omega \subseteq \mathbb{R}^n$ is an arbitrary nonempty open set.

We may also assume that $\nabla u|_\Omega \notin L^p(\Omega; \mathbb{R}^{m \times n})$, otherwise we can apply the previous step. Let \mathcal{P} be a linear subspace of $\mathbb{R}^{m \times n}$ as in (3.20) and $\pi_{\mathcal{P}}: \mathbb{R}^{m \times n} \rightarrow \mathcal{P}$ as in Rem. 3.11. Since $\psi_{\alpha_p} \not\equiv 0$, \mathcal{P} is a non-trivial linear subspace of $\mathbb{R}^{m \times n}$. Then, by Prop. 3.10 and Rem. 3.11,

$$(3.39) \quad \begin{aligned} &\int_{\Omega} \psi_{\alpha_p}(\nabla u) dx \\ &= \int_{\Omega \setminus \{|\nabla u|_2=0\}} |\nabla u|_2^p \psi_{\alpha_p} \left(\pi_{\mathcal{P}} \left(\frac{\nabla u}{|\nabla u|_2} \right) \right) dx \geq C \int_{\Omega} |\nabla u|_2^p dx = +\infty, \end{aligned}$$

where,

$$C := \min_{A \in \mathcal{P}, |A|_2=1} \psi_{\alpha_p}(\pi_{\mathcal{P}}(A)) > 0,$$

is a positive constant.

Let $\Omega' \in \mathcal{A}_\Omega$ s.t. $\Omega' \Subset \Omega$. By the previous step and the monotonicity of $G_\pm^{\alpha_p}(u, \cdot)$ (cfr. Prop. 3.7) we find that (3.38) holds.

(1.4) follows by letting $\Omega' \uparrow \Omega$ in (3.38) and recalling (3.39). This concludes the proof. \square

We now introduce a stronger version of the concept of p -core functional seen in Def. 3.1. This stronger definition will allow us to extend Thm. 1.1 to functions less regular than Sobolev ones.

Definition 3.12 (strong p -core functional). Let α_p be a p -core functional as in Def. 3.1. Recalling Def. 3.9 and Prop. 3.10, we say that α_p is a strong p -core functional if there exists \mathcal{P} , a linear subspace of $\mathbb{R}^{m \times n}$, such that (3.20) holds and,

$$(GB_{\mathcal{P}}) \quad \begin{aligned} &\exists c_3 > 0 \text{ s.t. } \alpha_p(u, D) \leq c_3 \int_D |D_{\pi_{\mathcal{P}}} u|_2^p dx \\ &\forall u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m) \text{ s.t. } u|_D \in W^{\pi_{\mathcal{P}}, p}(D; \mathbb{R}^m). \end{aligned}$$

We observe that $(GB_{\mathcal{P}})$ implies (GB). Indeed, if $v = u|_D \in W^{1,p}(D; \mathbb{R}^m)$, then,

$$|D_{\pi_{\mathcal{P}}} v|_2 = |\pi_{\mathcal{P}}(\nabla v)|_2 \leq |\pi_{\mathcal{P}}| |\nabla v|_2,$$

where $|\pi_{\mathcal{P}}| := \sup\{|\pi_{\mathcal{P}}(A)|_2 : A \in \mathbb{R}^{m \times n}, |A|_2 \leq 1\}$. Therefore, $(GB_{\mathcal{P}})$ implies that (GB) holds with $c_1 = |\pi_{\mathcal{P}}|^p c_3$ if $|\pi_{\mathcal{P}}| > 0$. If $|\pi_{\mathcal{P}}| = 0$ (GB) holds for any $c_1 > 0$ (cfr. Rem. 3.13).

Remark 3.13. Every p -core functional such that $\mathcal{N}_{\psi_{\alpha_p}} = \{0\}$ (cfr. Prop. 3.10) is a strong p -core functional, since this condition forces \mathcal{P} to be equal to $\mathbb{R}^{m \times n}$ and then $D_{\pi_{\mathcal{P}}} = \nabla$.

We also highlight that if a strong p -core functional is such that $\mathcal{N}_{\psi_{\alpha_p}} = \mathbb{R}^{m \times n}$, then $\mathcal{P} = \{0\}$. Therefore, $D_{\pi_{\mathcal{P}}} = 0$ and from $(GB_{\mathcal{P}})$ and (CV) we deduce that α_p is the trivial p -core functional (cfr. Ex. 5.18).

Some consequences of (GB) are reinforced with $(GB_{\mathcal{P}})$. If $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$ and $\alpha_p = \alpha_{p,D,\Gamma^{\mathcal{G}},m}$ is a strong p -core functional as in Def. 3.12 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2, then, (CV), (EI) and $(GB_{\mathcal{P}})$ imply,

$$(GB_{\mathcal{P}+}) \quad \begin{aligned} &\alpha_p(u, D') \leq c_4 \text{diam}(D')^{p-n} \int_{D'} |D_{\pi_{\mathcal{P}}}(u|_{D'})|_2^p dx \\ &\forall D' \in \mathcal{G}^D(\mathbb{R}^n) \forall u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m) \text{ s.t. } D_{\pi_{\mathcal{P}}}(u|_{D'}) \in L^p(D'; \mathbb{R}^{m \times n}), \end{aligned}$$

where $c_4 = c_3 \text{diam}(D)^{n-p}$.

If $u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ and $D_{\pi_{\mathcal{P}}}(u|_{\Omega}) \in L^p(\Omega; \mathbb{R}^{m \times n})$, thanks to $(GB_{\mathcal{P}+})$, we obtain that

$$\begin{aligned} G_{\varepsilon}^{\alpha_p}(u, \Omega) &\leq \varepsilon^{n-p} \sup_{\mathcal{G}_{\varepsilon}} \sum_{D' \in \mathcal{G}_{\varepsilon}} c_4 \text{diam}(D')^{p-n} \int_{D'} |D_{\pi_{\mathcal{P}}}(u|_{D'})|_2^p dx \\ &\leq c_3 \int_{\Omega} |D_{\pi_{\mathcal{P}}}(u|_{\Omega})|_2^p dx, \end{aligned}$$

for every $\varepsilon > 0$. Therefore,

$$(3.40) \quad G_{\pm}^{\alpha_p}(u, \Omega) \leq c_3 \int_{\Omega} |D_{\pi_{\mathcal{P}}}(u|_{\Omega})|_2^p dx.$$

Remark 3.14. Under the above assumptions, g_+ is inner-regular and σ -subadditive: that is, (i) and (ii) in Prop. 3.7 hold (requiring only the milder assumption that $D_{\pi_{\mathcal{P}}}(u|_{\Omega})$ belongs to $L^p(\Omega; \mathbb{R}^{m \times n})$ instead of the p -integrability of the whole gradient). Indeed, to prove that g_+ is inner-regular and σ -subadditive we refer again to the proof of [FFGLBS20, Prop. 3.1] taking into account (3.40).

We now prove the extension of Thm. 1.1.

Proof of Thm. 1.2. Let $u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ such that $D_{\pi_{\mathcal{P}}}(u|_{\Omega}) \in L^p(\Omega; \mathbb{R}^{m \times n})$. Thanks to Thm. 1.1 and Rem. 3.11 we know that,

$$(3.41) \quad \lim_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(v, \Omega') = \int_{\Omega'} \psi_{\alpha_p}(D_{\pi_{\mathcal{P}}}(v|_{\Omega'})) \, dx,$$

holds for every $\Omega' \in \mathcal{A}_{\Omega}$ and for every $v \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ s.t. $\nabla(v|_{\Omega'}) \in L^p(\Omega'; \mathbb{R}^{m \times n})$.

Let $\Omega' \in \mathcal{A}_{\Omega}$ s.t. $\Omega' \Subset \Omega$ and $\sigma > 0$. Then, thanks to Lem. 2.9, there exists $v_{\sigma}^{\Omega'} \in C_c^{\infty}(\mathbb{R}^n; \mathbb{R}^m)$ s.t. $\|u - v_{\sigma}^{\Omega'}\|_{W^{\pi_{\mathcal{P}}, p}(\Omega'; \mathbb{R}^m)} < \sigma$.

Now we argue as in **Step 2** in the proof of Thm. 1.1, making use of (3.41), (GB $_{\mathcal{P}}$ +) (instead of (GB+)) and Prop. 3.7 (reinforced with Rem. 3.14) to obtain (1.9).

Finally, we assume that $u|_{\Omega} \in W^{\pi_{\mathcal{P}}, p}_{\text{loc}}(\Omega; \mathbb{R}^m)$ and $\psi_{\alpha_p} \not\equiv 0$. Arguing as in **Step 4** in the proof of Thm. 1.1 we deduce that,

$$\begin{aligned} & \int_{\Omega} \psi_{\alpha_p}(D_{\pi_{\mathcal{P}}}(u|_{\Omega})) \, dx \\ &= \int_{\Omega \setminus \{|D_{\pi_{\mathcal{P}}}(u|_{\Omega})|_2=0\}} |D_{\pi_{\mathcal{P}}}(u|_{\Omega})|_2^p \psi_{\alpha_p} \left(\pi_{\mathcal{P}} \left(\frac{D_{\pi_{\mathcal{P}}}(u|_{\Omega})}{|D_{\pi_{\mathcal{P}}}(u|_{\Omega})|_2} \right) \right) \, dx \\ &\geq C \int_{\Omega} |D_{\pi_{\mathcal{P}}}(u|_{\Omega})|_2^p \, dx, \end{aligned}$$

where,

$$C := \min_{A \in \mathcal{P}, |A|_2=1} \psi_{\alpha_p}(\pi_{\mathcal{P}}(A)) > 0,$$

is a positive constant. At this point, the same argument in **Step 4** in the proof of Thm. 1.1 allows us to conclude that (1.9) holds. \square

4. NON-DISTRIBUTIONAL CHARACTERIZATION OF SOBOLEV-TYPE SPACES

As a consequence of Thm. 1.1 we obtain the following.

Corollary 4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $m \geq 1$ a natural number, $p \in [1, \infty)$, \mathcal{G} a subgroup of $\text{SO}(n)$ and $D \subseteq \mathbb{R}^n$ be a bounded open set. Let $\alpha_p = \alpha_{p, D, \Gamma^{\mathcal{G}}, m}$ be a p -core functional as in Def. 3.1 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2. Assume also that for every $D' \in \mathcal{G}^D(\mathbb{R}^n)$ the functional $\alpha_p(\cdot, D')|_{L^p(\mathbb{R}^n; \mathbb{R}^m)} : L^p(\mathbb{R}^n; \mathbb{R}^m) \rightarrow [0, +\infty)$ is lower semicontinuous.*

Let \mathcal{P} be a linear subspace of $\mathbb{R}^{m \times n}$ as in (3.20) and $\pi_{\mathcal{P}} : \mathbb{R}^{m \times n} \rightarrow \mathcal{P}$ as in Rem. 3.11. Let $\{\rho_{\sigma}\}_{\sigma>0} \subset C_c^{\infty}(\mathbb{R}^n)$ be a family of standard mollifiers and $\tilde{\Omega} \in \mathcal{A}_{\Omega}$ s.t. $\tilde{\Omega} \Subset \Omega$. For every $u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ and for every $0 < \sigma < \text{dist}(\tilde{\Omega}, \partial\Omega)$,

$$(4.1) \quad C \int_{\tilde{\Omega}} |\pi_{\mathcal{P}}(\nabla u_{\sigma})|_2^p \, dx \leq \int_{\tilde{\Omega}} \psi_{\alpha_p}(\nabla u_{\sigma}) \, dx = G^{\alpha_p}(u_{\sigma}, \tilde{\Omega}) \leq G^{\alpha_p}(u, \Omega),$$

where $u_{\sigma} := \rho_{\sigma} * u$ for every $\sigma > 0$ and $C > 0$ is a constant depending on ψ_{α_p} .

Proof. Let $0 < \sigma < \text{dist}(\tilde{\Omega}, \partial\Omega)$. Without loss of generality, having (3.11) in mind, we may assume that $u|_{\mathbb{R}^n \setminus \tilde{\Omega}} = 0$ so that $u \in L^p(\mathbb{R}^n; \mathbb{R}^m)$. Let $\varepsilon > 0$ and $\mathcal{G}_{\varepsilon}$ be a family made of pairwise disjoint elements of $\mathcal{G}_{\varepsilon}^D(\tilde{\Omega})$ and we observe that $\sigma > 0$ is so small that $D' - \sigma y \subset \Omega$ for every $y \in B_1(0)$. Applying Lem. 2.1 (with $f(v) := \alpha_p(v, D')$ for

all $v \in L^p(\mathbb{R}^n; \mathbb{R}^m)$) and (T2), we get,

$$\begin{aligned} \varepsilon^{n-p} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(u_\sigma, D') &\leq \varepsilon^{n-p} \sum_{D' \in \mathcal{G}_\varepsilon} \int_{B_1(0)} \alpha_p(u(\cdot - \sigma y), D') \rho(y) dy \\ &= \varepsilon^{n-p} \sum_{D' \in \mathcal{G}_\varepsilon} \int_{B_1(0)} \alpha_p(u, D' - \sigma y) \rho(y) dy \leq G_\varepsilon^{\alpha_p}(u, \Omega). \end{aligned}$$

Taking the supremum with respect to \mathcal{G}_ε and letting $\varepsilon \rightarrow 0^+$, we obtain

$$G_-^{\alpha_p}(u_\sigma, \tilde{\Omega}) \leq G_-^{\alpha_p}(u, \Omega).$$

(4.1) follows by applying Thm. 1.1 to u_σ and invoking Prop. 3.10 and Rem. 3.11. \square

The following result is a first step in characterizing Sobolev-type spaces.

Corollary 4.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $m \geq 1$ a natural number, $p \in (1, \infty)$, \mathcal{G} a subgroup of $\text{SO}(n)$ and $D \subseteq \mathbb{R}^n$ be a bounded open set. Let $\alpha_p = \alpha_{p, D, \Gamma^\mathcal{G}, m}$ be a p -core functional as in Def. 3.1 subordinated to $\Gamma^\mathcal{G}$ as in Def. 3.2. Assume also that for every $D' \in \mathcal{G}^D(\mathbb{R}^n)$ the functional $\alpha_p(\cdot, D')|_{L^p(\mathbb{R}^n; \mathbb{R}^m)} : L^p(\mathbb{R}^n; \mathbb{R}^m) \rightarrow [0, +\infty)$ is lower semicontinuous.*

Let \mathcal{P} be a linear subspace of $\mathbb{R}^{m \times n}$ as in (3.20) and $\pi_{\mathcal{P}} : \mathbb{R}^{m \times n} \rightarrow \mathcal{P}$ as in Rem. 3.11. Let $u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$. Then,

$$(4.2) \quad G_-^{\alpha_p}(u, \Omega) < +\infty \implies D_{\pi_{\mathcal{P}}}(u|_\Omega) \in L^p(\Omega; \mathbb{R}^{m \times n}).$$

Moreover, if $\psi_{\alpha_p} > 0$ in $\mathbb{R}^{m \times n} \setminus \{0\}$, then $\nabla u|_\Omega \in L^p(\Omega; \mathbb{R}^{m \times n})$ if and only if $G_-^{\alpha_p}(u, \Omega) < +\infty$.

Remark 4.3. Under the assumptions of Cor. 4.2, if $u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ is such that $D_{\pi_{\mathcal{P}}}(u|_\Omega) \in L^p(\Omega; \mathbb{R}^{m \times n})$, in general we cannot conclude that $G_-^{\alpha_p}(u, \Omega) < +\infty$ (cfr. Ex. 5.19).

Proof of Cor. 4.2. Let $p \in (1, \infty)$, $u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ and $u_\sigma := \rho_\sigma * u$ for every $\sigma > 0$. If $G_-^{\alpha_p}(u, \Omega) < +\infty$, using (4.1) we obtain that

$$(4.3) \quad \limsup_{\sigma \rightarrow 0^+} \int_{\tilde{\Omega}} |\pi_{\mathcal{P}}(\nabla u_\sigma)|_2^p dx \leq \frac{1}{C} G_-^{\alpha_p}(u, \Omega),$$

for every $\tilde{\Omega} \in \mathcal{A}_\Omega$ s.t. $\tilde{\Omega} \Subset \Omega$. Since $u_\sigma \rightarrow u$ in $L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ as $\sigma \rightarrow 0^+$, we deduce that $\pi_{\mathcal{P}}(\nabla u_\sigma|_\Omega) \rightarrow D_{\pi_{\mathcal{P}}}(u|_\Omega)$ as $\sigma \rightarrow 0^+$ in the sense of distributions. Then, from (4.3) (since it provides a uniform bound with respect to $\tilde{\Omega}$) we get that the distribution $D_{\pi_{\mathcal{P}}}(u|_\Omega)$ is a linear and continuous functional on $L^{p'}(\Omega; \mathbb{R}^m)$. This proves that $D_{\pi_{\mathcal{P}}}(u|_\Omega) \in L^p(\Omega; \mathbb{R}^{m \times n})$.

Finally, if $\psi_{\alpha_p} > 0$ in $\mathbb{R}^{m \times n} \setminus \{0\}$, then $\mathcal{P} = \mathbb{R}^{m \times n}$ and $\pi_{\mathcal{P}} = \text{id}_{\mathbb{R}^{m \times n}}$. Invoking what proved above and Thm. 1.1 we deduce that $\nabla u|_\Omega = \pi_{\mathcal{P}}(\nabla u|_\Omega) \in L^p(\Omega; \mathbb{R}^{m \times n})$ if and only if $G_-^{\alpha_p}(u, \Omega) < +\infty$. \square

Now for strong p -core functionals as a consequence of Thm. 1.2 we obtain the following characterization.

Corollary 4.4. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $m \geq 1$ a natural number, $p \in (1, \infty)$, $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$ and $D \subseteq \mathbb{R}^n$ be a bounded open set. Let $\alpha_p = \alpha_{p, D, \Gamma^\mathcal{G}, m}$ be a strong p -core functional as in Def. 3.12 subordinated to $\Gamma^\mathcal{G}$ as in Def. 3.2. Assume also that for every $D' \in \mathcal{G}^D(\mathbb{R}^n)$ the functional $\alpha_p(\cdot, D')|_{L^p(\mathbb{R}^n; \mathbb{R}^m)} : L^p(\mathbb{R}^n; \mathbb{R}^m) \rightarrow [0, +\infty)$ is lower semicontinuous.*

Let \mathcal{P} be a linear subspace of $\mathbb{R}^{m \times n}$ as in (3.20) and $\pi_{\mathcal{P}} : \mathbb{R}^{m \times n} \rightarrow \mathcal{P}$ as in Rem. 3.11. Let $u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$. Then,

$$(4.4) \quad G^{-\alpha_p}(u, \Omega) < +\infty \iff D_{\pi_{\mathcal{P}}}(u|_{\Omega}) \in L^p(\Omega; \mathbb{R}^{m \times n}).$$

Moreover, if one of the equivalent conditions in (4.4) is in force, then (1.9) holds.

Proof. We first assume that $D_{\pi_{\mathcal{P}}}(u|_{\Omega}) \in L^p(\Omega; \mathbb{R}^{m \times n})$. Then, by (3.40) in Rem. 3.14 we obtain that $G^{-\alpha_p}(u, \Omega) < +\infty$.

The reversed implication follows directly from Cor. 4.2.

Finally, if one of the equivalent conditions in (4.4) is in force, (1.9) follows by Thm. 1.2. \square

The following result is a constancy theorem.

Corollary 4.5. *Let $\Omega \subseteq \mathbb{R}^n$ be an open connected set, $m \geq 1$ a natural number, $p \in (1, \infty)$, \mathcal{G} a subgroup of $\text{SO}(n)$ and $D \subseteq \mathbb{R}^n$ be a bounded open set. Let $\alpha_p = \alpha_{p, D, \Gamma^{\mathcal{G}}, m}$ be a p -core functional as in Def. 3.1 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2. Let $u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$. Assume also that for every $D' \in \mathcal{G}^D(\mathbb{R}^n)$ the functional $\alpha_p(\cdot, D')|_{L^p(\mathbb{R}^n; \mathbb{R}^m)} : L^p(\mathbb{R}^n; \mathbb{R}^m) \rightarrow [0, +\infty)$ is lower semicontinuous and that $G^{-\alpha_p}(u, \Omega) = 0$.*

Then, $D_{\pi_{\mathcal{P}}}(u|_{\Omega}) = 0$. As a consequence, if $\psi_{\alpha_p} > 0$ in $\mathbb{R}^{m \times n}$, then $\nabla u|_{\Omega} = 0$ and u is constant a. e. on Ω .

Moreover, if we also assume that $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$ and α_p is a strong p -core functional, then $\alpha_p(u, D') = 0$ for every $D' \in \mathcal{G}^D(\Omega)$.

Proof. We let $u_{\sigma} := \rho_{\sigma} * u$ for every $\sigma > 0$. Since $u_{\sigma} \rightarrow u$ in $L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ as $\sigma \rightarrow 0^+$, as in the proof of Cor. 4.2, we deduce that $\pi_{\mathcal{P}}(\nabla u_{\sigma}|_{\Omega}) \rightarrow D_{\pi_{\mathcal{P}}}(u|_{\Omega}) \in L^p(\Omega; \mathbb{R}^{m \times n})$. Since $G^{-\alpha_p}(u, \Omega) = 0$, by Cor. 4.1 we obtain that $\pi_{\mathcal{P}}(\nabla u_{\sigma}|_{\Omega})$ vanishes on every $\tilde{\Omega} \in \mathcal{A}_{\Omega}$ s.t. $\tilde{\Omega} \Subset \Omega$ for every $0 < \sigma < \text{dist}(\tilde{\Omega}, \partial\Omega)$. Therefore, $D_{\pi_{\mathcal{P}}}(u|_{\Omega}) = 0$. Moreover, if $\psi_{\alpha_p} > 0$ in $\mathbb{R}^{m \times n}$, then $\nabla u|_{\Omega} = D_{\pi_{\mathcal{P}}}(u|_{\Omega}) = 0$.

Finally, if we also assume that $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$ and α_p is a strong p -core functional, then, by (GB $_{\mathcal{P}}$ +) we deduce that $\alpha_p(u, D') = 0$ for every $D' \in \mathcal{G}^D(\Omega)$. \square

5. EXAMPLES AND APPLICATIONS

In this section we discuss some examples of p -core functionals which can be used to apply the results in Section 3 and Section 4. We start with a proposition which will be used to provide more explicit representation of the function ψ_{α_p} for some p -core functionals α_p .

Proposition 5.1. *Let $m \geq 1$ be a natural number, $p \in [1, \infty)$, \mathcal{G} a subgroup of $\text{SO}(n)$ and $D \subseteq \mathbb{R}^n$ be a bounded open set. Let $\alpha_p = \alpha_{p, D, \Gamma^{\mathcal{G}}, m}$ be a p -core functional as in Def. 3.1 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2.*

Then, for every $A \in \mathbb{R}^{m \times n}$, ψ_{α_p} is constant on $A\mathcal{G} := \{AO : O \in \mathcal{G}\}$ and

$$(5.1) \quad \psi_{\alpha_p}(A) \leq \frac{1}{|D|} \max_{B \in A\mathcal{G}} \alpha_p(l^B, D),$$

where ψ_{α_p} is defined in Def. 3.9.

Moreover, assume that for every $\mathcal{R} \in \mathcal{G}$ there exists a family $\{\mathcal{G}_{\varepsilon}\}_{0 < \varepsilon < 1}$ such that:

(1) for every $\varepsilon \in (0, 1)$, \mathcal{G}_ε is a family made of pairwise disjoint elements of $\mathcal{G}_\varepsilon^D(Q)$ of the form:

$$D' = \varepsilon \mathcal{R}D + h_{D'},$$

where $h_{D'} \in \mathbb{R}^n$ depends on D' ;

(2) the following no-gaps condition holds,

$$(5.2) \quad \lim_{\varepsilon \rightarrow 0^+} |\mathcal{G}_\varepsilon^\cup| = |Q| = 1,$$

where, for every $\varepsilon \in (0, 1)$, $\mathcal{G}_\varepsilon^\cup \subset Q$ is the union of all the sets belonging to \mathcal{G}_ε .

Then, the equality holds in (5.1).

Remark 5.2. We first observe that the condition (5.2) is equivalent to,

$$(5.3) \quad \lim_{\varepsilon \rightarrow 0^+} |Q \setminus \mathcal{G}_\varepsilon^\cup| = 0,$$

since $\mathcal{G}_\varepsilon^\cup \subset Q$ for every $\varepsilon \in (0, 1)$. (5.2) is also equivalent to,

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^n \#\mathcal{G}_\varepsilon = \frac{1}{|D|},$$

being $|\mathcal{G}_\varepsilon^\cup| = |D| \varepsilon^n \#\mathcal{G}_\varepsilon$ for every $\varepsilon \in (0, 1)$.

We also observe that (5.2) and (5.3) are related to [DFF20, Def. 2]. Indeed, by Lebesgue's dominated convergence theorem, the condition

$$(5.4) \quad \lim_{\varepsilon \rightarrow 0^+} \chi_{\mathcal{G}_\varepsilon^\cup} = \chi_Q \text{ a. e.},$$

implies (5.3). Moreover, we also observe that the weaker condition

$$\liminf_{\varepsilon \rightarrow 0^+} \chi_{\mathcal{G}_\varepsilon^\cup} \geq \chi_Q \text{ a. e.},$$

implies (5.4), since $\mathcal{G}_\varepsilon^\cup \subset Q$ for every $\varepsilon \in (0, 1)$.

Remark 5.3. Let $D \subset \mathbb{R}^n$ be a bounded open set, $\varepsilon_0 > 0$ and assume that D is a reference cell for an ε_0 -tessellation of \mathbb{R}^n (cfr. [DFF20, Def. 1]). Then, $\mathcal{R}(D)$ is also a reference cell for an ε_0 -tessellation of \mathbb{R}^n for every $\mathcal{R} \in \text{SO}(n)$. Therefore, there exists a family $\{\mathcal{G}_\varepsilon\}_{0 < \varepsilon < 1}$ satisfying (1) and (2) in Prop. 5.1 (cfr. [DFF20, Def. 2 and Prop. 1]).

Proof of Prop. 5.1. Let $A \in \mathbb{R}^{m \times n}$. Recalling Prop. 3.10,

$$(5.5) \quad \psi_{\alpha_p}(A) = G^{\alpha_p}(l^A, Q) = \lim_{\varepsilon \rightarrow 0^+} G_\varepsilon^{\alpha_p}(l^A, Q) = \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(l^A, D'),$$

where for every $\varepsilon > 0$, each \mathcal{G}_ε is a family made of pairwise disjoint elements of $\mathcal{G}_\varepsilon^D(Q)$ and the supremum is taken with respect to all of these families \mathcal{G}_ε . We recall that the cardinality of every family \mathcal{G}_ε is bounded from above by $(\varepsilon^n |D|)^{-1}$. Let $A \in \mathbb{R}^{m \times n}$ and $B, C \in \mathcal{AG}$. Then, $B = CO$ with $O \in \mathcal{G}$. By (CV) we obtain,

$$\alpha_p(l^B, D') = \alpha_p(l^C, OD'),$$

for every $D' \in \mathcal{G}_\varepsilon^D(\mathbb{R}^n)$. In particular,

$$\psi_{\alpha_p}(B) = \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(l^C, OD') \leq G^{\alpha_p}(l^C, OQ) = \psi_{\alpha_p}(C),$$

where we used (3.15) in Prop. 3.10, with $\tilde{Q} = OQ$, and the fact that for every $\varepsilon > 0$, if \mathcal{G}_ε is a family made of pairwise disjoint elements of $\mathcal{G}_\varepsilon^D(Q)$, then the family

$\{OD' : D' \in \mathcal{G}\}$ is made of pairwise disjoint elements of $\mathcal{G}_\varepsilon^D(OQ)$. Interchanging the role of B and C we obtain that ψ_{α_p} is constant on AG .

We now prove (5.1). Let $\varepsilon > 0$ and $D' \in \mathcal{G}_\varepsilon^D(\mathbb{R}^n)$. Then, $D' = \varepsilon \mathcal{R}_{D'} D + h_{D'}$ for some $\mathcal{R}_{D'} \in \mathcal{G}$ and $h_{D'} \in \mathbb{R}^n$. By (3.9), we get that

$$(5.6) \quad \alpha_p(l^A, D') = \varepsilon^p \alpha_p(l^{A\mathcal{R}_{D'}}, D).$$

Then,

$$\varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(l^A, D') = \varepsilon^n \sup_{\mathcal{G}_\varepsilon} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(l^{A\mathcal{R}_{D'}}, D) \leq \varepsilon^n \sup_{\mathcal{G}_\varepsilon} \#\mathcal{G}_\varepsilon \max_{B \in AG} \alpha_p(l^B, D).$$

Therefore, from (5.5) we get,

$$(5.7) \quad \psi_{\alpha_p}(A) = \lim_{\varepsilon \rightarrow 0^+} \varepsilon^n \sup_{\mathcal{G}_\varepsilon} \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(l^{A\mathcal{R}_{D'}}, D) \leq \frac{1}{|D|} \max_{B \in AG} \alpha_p(l^B, D).$$

Then, (5.1) holds.

Finally, let $\mathcal{R} \in \mathcal{G}$ and consider a family $\{\mathcal{G}_\varepsilon\}_{0 < \varepsilon < 1}$ satisfying (1) and (2). Then, $\lim_{\varepsilon \rightarrow 0^+} \varepsilon^n \#\mathcal{G}_\varepsilon = \frac{1}{|D|}$, and consequently from (5.5) and (5.6) we obtain,

$$\psi_{\alpha_p}(A) \geq \lim_{\varepsilon \rightarrow 0^+} \varepsilon^n \sum_{D' \in \mathcal{G}_\varepsilon} \alpha_p(l^{A\mathcal{R}}, D) = \lim_{\varepsilon \rightarrow 0^+} \varepsilon^n \#\mathcal{G}_\varepsilon \alpha_p(l^{A\mathcal{R}}, D) = \frac{1}{|D|} \alpha_p(l^{A\mathcal{R}}, D).$$

Taking the maximum with respect to $\mathcal{R} \in \mathcal{G}$, we obtain,

$$(5.8) \quad \psi_{\alpha_p}(A) \geq \frac{1}{|D|} \max_{B \in AG} \alpha_p(l^B, D).$$

Combining (5.8) and (5.7) we conclude the proof. \square

5.1. Characterization of $W^{1,p}(\Omega; \mathbb{R}^m)$.

In this section we will consider examples of p -core functionals used to characterize the space $W^{1,p}(\Omega; \mathbb{R}^m)$ of (vector-valued) Sobolev functions when $p \in (1, \infty)$ and to approximate the following L^p norm of the gradient of functions in $W^{1,p}(\Omega; \mathbb{R}^m)$ for $p \in [1, \infty)$:

$$\int_{\Omega} |\nabla u|_{\alpha_p}^p dx,$$

where $|\cdot|_{\alpha_p} := \psi_{\alpha_p}^{1/p}(\cdot)$ is a (possibly anisotropic) norm in $\mathbb{R}^{m \times n}$ (cfr. Cor. 5.11).

In this section $m \geq 1$ will always denote a natural number and $D \subseteq \mathbb{R}^n$ a bounded connected open set with Lipschitz boundary since we will need the classical *Poincaré* inequality.

This first example is a generalization of the BMO-type seminorms considered in the literature for the scalar case (see [FMS16, FMS18, FFGLBS20, FGLBS20, DFF20]).

Example 5.4. Let $p \in [1, \infty)$. We define,

$$\alpha_p(u, D') := \int_{D'} |u(x) - u_{D'}|_2^p dx.$$

Then, $\alpha_p : L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m) \times \mathcal{E}_\Gamma^D(\mathbb{R}^n) \rightarrow [0, +\infty)$ is a p -core functional for any Γ subgroup of $\text{Aff}(\mathbb{R}^n)$. Moreover, by Fatou's lemma, $\alpha_p(\cdot, D')|_{L^p(\mathbb{R}^n; \mathbb{R}^m)} : L^p(\mathbb{R}^n; \mathbb{R}^m) \rightarrow [0, +\infty)$ is lower semicontinuous for every $D' \in \mathcal{E}_\Gamma^D(\mathbb{R}^n)$.

Proposition 5.5. *Let $p \in [1, \infty)$ and \mathcal{G} be a subgroup of $\text{SO}(n)$. Let $\alpha_p = \alpha_{p,D,\Gamma^{\mathcal{G}},m}$ be the p -core functional in Ex. 5.4 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2.*

Then, α_p is a strong p -core functional as in Def. 3.12 and,

$$(5.9) \quad \mathcal{N}_{\psi_{\alpha_p}} = \{0\},$$

$$(5.10) \quad \alpha_p(l^{\mathcal{R}A}, D') = \alpha_p(l^A, D'), \quad \psi_{\alpha_p}(\mathcal{R}A) = \psi_{\alpha_p}(A),$$

for every $A \in \mathbb{R}^{m \times n}$ and $\mathcal{R} \in \text{O}(m)$.

Moreover, recalling Def. 3.9, for every $A \in \mathbb{R}^{m \times n}$,

(1) if $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$ and D is as in Rem. 5.3,

$$\psi_{\alpha_p}(A) = \frac{1}{|D|^2} \| |l^A|_2 \|_{L^p(D - \text{bar}(D))}^p \quad \forall A \in \mathbb{R}^{m \times n}.$$

(2) if $\mathcal{G} = \text{SO}(n)$ and $m = n$,

$$\psi_{\alpha_{p,D,\Gamma^{\mathcal{G}},n}}(A) = \psi_{\alpha_{p,D,\Gamma^{\mathcal{G}},1}}(\lambda),$$

where $\lambda \in \mathbb{R}^n$ is a vector obtained collecting (in an arbitrary order) the n eigenvalues (counted with multiplicity) of the symmetric and positive definite matrix $\sqrt{AA^T}$.

Remark 5.6. Thanks to Prop. 5.1, the function $\mathbb{R}^n \ni \nu \rightarrow \psi_{\alpha_{p,D,\Gamma^{\mathcal{G}},1}}(\nu)$ in Prop. 5.5 is constant on the unit sphere \mathbb{S}^{n-1} (therefore it does not depend on permutations of the components of ν). Moreover, $\psi_{\alpha_{p,D,\Gamma^{\mathcal{G}},1}}$ coincides with the function $\tilde{\psi}_p^D$ in [FGLBS20, Sec. 4]. In particular (recalling also Prop. 5.1) it coincides with the function $\mathbb{R}^n \ni \nu \rightarrow |\nu|_2^p$ up to a multiplicative constant, which can be explicitly computed for some $D \subset \mathbb{R}^n$ and some values of n and p [FMS18, cfr. Rem. 2.1]. For instance, if D is equal to the cube Q , then for every $\nu \in \mathbb{R}^n$ $\tilde{\psi}_p^Q(\nu) = \gamma(n,p)|\nu|_2^p$, where

$$\gamma(n,p) = \max_{\nu \in \mathbb{S}^{n-1}} \int_Q |x \cdot \nu|^p dx.$$

We also recall that $\gamma(n,1) = \frac{1}{4}$ and $\gamma(n,2) = \frac{1}{12}$ for every $n \geq 1$.

More generally, if D is as in Rem. 5.3, thanks to Prop. 5.1 we get that,

$$\tilde{\psi}_p^D(\mu) = |\mu|_2^p \frac{1}{|D|^2} \max_{\nu \in \mathbb{S}^{n-1}} \int_D |(x - \text{bar}(D)) \cdot \nu|^p dx,$$

for every $\mu \in \mathbb{R}^n$.

Proof of Prop. 5.5. (5.10) follows by the fact that $|\mathcal{R}z|_2 = |z|_2$ for every $z \in \mathbb{R}^m$.

We now observe that for every $D' \in \mathcal{G}^D(\mathbb{R}^n)$,

$$(5.11) \quad \begin{aligned} \alpha_p(l^A, D') &= \int_{D'} \left| l^A(x) - \int_{D'} l^A(y) dy \right|_2^p dx \\ &= \int_{D'} \left| A \left(x - \int_{D'} y dy \right) \right|_2^p dx = \int_{D'} |A(x - \text{bar}(D'))|_2^p dx. \end{aligned}$$

We prove (5.9). Let $A \in \mathcal{N}_{\psi_{\alpha_p}}$. Thanks to Prop. 3.8 and (5.11), we get

$$0 = \psi_{\alpha_p}(A) \geq G_{1/2}^{\alpha_p}(l^A, Q) \geq 2^{p-n} \int_{D'} |A(x - \text{bar}(D'))|_2^p dx,$$

where D' is an arbitrary set in $\mathcal{G}_{1/2}^D(Q)$. This implies that the affine function $A(\cdot - \text{bar}(D'))$ is identically equal to 0 in D' (which has positive measure). Then

$A = 0$ and (5.9) follows. Then, recalling Rem. 3.13, we deduce that α_p is a strong p -core functional.

We prove (1). By Prop. 5.1 and (5.11) we obtain,

$$\psi_{\alpha_p}(A) = \frac{1}{|D|} \int_D |A(x - \text{bar}(D))|_2^p dx = \frac{1}{|D|^2} \| |A|_2 \|_{L^p(D - \text{bar}(D))}^p.$$

We now prove (2).

Let $\alpha_p = \alpha_{p,D,\Gamma^\sigma,n}$ and $A \in \mathbb{R}^{n \times n}$. We now consider a (left) polar decomposition of A :

$$A = \sqrt{AA^T}O,$$

where $O \in O(n)$. Now, let $U \in O(n)$ such that $UO \in SO(n)$ and

$$\sqrt{AA^T} = U^T \Lambda U,$$

where Λ is a diagonal matrix. Then, by (5.10) and Prop. 5.1,

$$(5.12) \quad \psi_{\alpha_p}(A) = \psi_{\alpha_p}(U^T \Lambda U O) = \psi_{\alpha_p}(\Lambda).$$

Let $\lambda \in \mathbb{R}^n$ be the vector obtained from the diagonal of Λ , so that,

$$(5.13) \quad |\Lambda x|_2 = |\lambda \cdot x|,$$

for every $x \in \mathbb{R}^n$. Then, recalling Prop. 3.10, by (5.13) and (5.11),

$$(5.14) \quad \begin{aligned} \psi_{\alpha_p}(\Lambda) &= \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \int_{D'} \left| \Lambda \left(x - \int_{D'} y dy \right) \right|_2^p dx \\ &= \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \int_{D'} \left| \left(\lambda \cdot x - \int_{D'} \lambda \cdot y dy \right) \right|^p dx = \psi_{\alpha_{p,D,\Gamma^\sigma,1}}(\lambda), \end{aligned}$$

where for every $\varepsilon > 0$, each \mathcal{G}_ε is a family made of pairwise disjoint elements of $\mathcal{G}_\varepsilon^D(Q)$ and the supremum is taken with respect to all of these families \mathcal{G}_ε . Combining (5.14) and (5.12) we conclude the proof. \square

The following example is similar to Ex. 5.4 (actually when $p = 2$ it coincides with Ex. 5.4).

Example 5.7. Let $p \in [1, \infty)$. We define,

$$\alpha_p(u, D') := \inf_{c \in \mathbb{R}^m} \int_{D'} |u(x) - c|_2^p dx.$$

Then, $\alpha_p : L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m) \times \mathcal{E}_\Gamma^D(\mathbb{R}^n) \rightarrow [0, +\infty)$ is a p -core functional for any Γ subgroup of $\text{Aff}(\mathbb{R}^n)$. Moreover, by Fatou's lemma, $\alpha_p(\cdot, D')|_{L^p(\mathbb{R}^n; \mathbb{R}^m)} : L^p(\mathbb{R}^n; \mathbb{R}^m) \rightarrow [0, +\infty)$ is lower semicontinuous for every $D' \in \mathcal{E}_\Gamma^D(\mathbb{R}^n)$.

The next example is inspired by [PS17].

Example 5.8. Let $p \in [1, \infty)$. We define,

$$\alpha_p(u, D') := \int_{D'} \int_{D'} |u(x) - u(y)|_2^p dx dy.$$

Then, $\alpha_p : L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m) \times \mathcal{E}_\Gamma^D(\mathbb{R}^n) \rightarrow [0, +\infty)$ is a p -core functional for any Γ subgroup of $\text{Aff}(\mathbb{R}^n)$. Moreover, by Fatou's lemma, $\alpha_p(\cdot, D')|_{L^p(\mathbb{R}^n; \mathbb{R}^m)} : L^p(\mathbb{R}^n; \mathbb{R}^m) \rightarrow [0, +\infty)$ is lower semicontinuous for every $D' \in \mathcal{E}_\Gamma^D(\mathbb{R}^n)$.

Proposition 5.9. *Let $p \in [1, \infty)$ and \mathcal{G} be a subgroup of $\text{SO}(n)$. Let $\alpha_p = \alpha_{p,D,\Gamma^{\mathcal{G}},m}$ be the p -core functional in Ex. 5.8 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2.*

Then, α_p is a strong p -core functional,

$$(5.15) \quad \mathcal{N}_{\psi_{\alpha_p}} = \{0\},$$

$$(5.16) \quad \alpha_p(l^{\mathcal{R}A}, D') = \alpha_p(l^A, D'), \quad \psi_{\alpha_p}(\mathcal{R}A) = \psi_{\alpha_p}(A),$$

for every $A \in \mathbb{R}^{m \times n}$ and $\mathcal{R} \in \text{O}(m)$.

Moreover, recalling Def. 3.9, for every $A \in \mathbb{R}^{m \times n}$,

(1) if $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$ and D is as in Rem. 5.3,

$$\psi_{\alpha_p}(A) = \frac{1}{|D|} \int_D \int_D |A(x-y)|_2^p dx dy.$$

(2) if $\mathcal{G} = \text{SO}(n)$ and $m = n$,

$$\psi_{\alpha_{p,D,\Gamma^{\mathcal{G}},n}}(A) = \psi_{\alpha_{p,D,\Gamma^{\mathcal{G}},1}}(\lambda),$$

where $\lambda \in \mathbb{R}^n$ is a vector obtained collecting (in an arbitrary order) the n eigenvalues (counted with multiplicity) of the symmetric and positive definite matrix $\sqrt{AA^T}$.

Remark 5.10. Thanks to Prop. 5.1, the function $\mathbb{R}^n \ni \nu \rightarrow \psi_{\alpha_{p,D,\Gamma^{\mathcal{G}},1}}(\nu)$ in Prop. 5.9 is constant on the unit sphere \mathbb{S}^{n-1} (therefore it does not depend on permutations of the components of ν).

More generally, if D is as in Rem. 5.3, thanks to Prop. 5.1 we get that,

$$\psi_{\alpha_{p,D,\Gamma^{\mathcal{G}},1}}(\mu) = |\mu|_2^p \frac{1}{|D|^3} \max_{\nu \in \mathbb{S}^{n-1}} \int_D \int_D |(x-y) \cdot \nu|^p dx dy \quad \forall A \in \mathbb{R}^{m \times n},$$

for every $\mu \in \mathbb{R}^n$.

Proof of Prop. 5.9. (5.16) follows by the fact that $|\mathcal{R}z|_2 = |z|_2$ for every $z \in \mathbb{R}^m$.

We now observe that for every $D' \in \mathcal{G}^D(\mathbb{R}^n)$,

$$(5.17) \quad \alpha_p(l^A, D') = \int_{D'} \int_{D'} |A(x-y)|_2^p dx dy \quad \forall A \in \mathbb{R}^{m \times n}.$$

(1) follows from Prop. 5.1 and (5.17). The proof of (5.15), (2) and the fact that α_p is a strong p -core functional follows the same argument used in the proof of Prop. 5.5. \square

The following result shows how Ex. 5.4, Ex. 5.7 and Ex. 5.8 can be used in the study of $W^{1,p}(\Omega; \mathbb{R}^m)$.

Corollary 5.11. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $m \geq 1$ a natural number, $p \in [1, \infty)$, \mathcal{G} a subgroup of $\text{SO}(n)$ and $D \subseteq \mathbb{R}^n$ be a bounded connected open set with Lipschitz boundary. Let $\alpha_p = \alpha_{p,D,\Gamma^{\mathcal{G}},m}$ one of the p -core functional in Ex. 5.4, Ex. 5.7 or Ex. 5.8 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2.*

Then,

(1) for every $u \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^m)$,

$$\lim_{\varepsilon \rightarrow 0^+} G_\varepsilon^{\alpha_p}(u, \Omega) = \int_\Omega \psi_{\alpha_p}(\nabla u),$$

where ψ_{α_p} is as in Def. 3.9 and $G_\varepsilon^{\alpha_p}$ is as in Def. 3.4.

Furthermore, if $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$ or $\mathcal{G} = \text{SO}(n)$ and D is as in Rem. 5.3 the function ψ_{α_p} can be written more explicitly according to Prop. 5.5 or Prop. 5.9.

(2) if $p \in (1, \infty)$, for every $u \in L^p(\Omega; \mathbb{R}^n)$,

$$\liminf_{\varepsilon \rightarrow 0^+} G_\varepsilon^{\alpha_p}(u, \Omega) < +\infty \iff u \in W^{1,p}(\Omega; \mathbb{R}^m).$$

(3) if $\Omega \subseteq \mathbb{R}^n$ is a connected open set, $p \in (1, \infty)$ and $u \in L^p_{\text{loc}}(\Omega; \mathbb{R}^n)$ satisfies $\liminf_{\varepsilon \rightarrow 0^+} G_\varepsilon^{\alpha_p}(u, \Omega) = 0$, then

$$u(x) = h \quad \forall x \in \Omega,$$

where $h \in \mathbb{R}^m$.

Proof. The proof of (1) and (2) follows combining Prop. 5.5, Prop. 5.9, Thm. 1.1 and Cor. 4.2. To prove (3) we apply Cor. 4.5. \square

5.2. Characterization of $E^{1,p}(\Omega; \mathbb{R}^n)$.

In this section we will consider examples of p -core functionals used to characterize the space $E^{1,p}(\Omega; \mathbb{R}^n)$, of $L^p(\Omega; \mathbb{R}^n)$ functions whose distributional symmetric gradient is p -integrable, when $p \in (1, \infty)$ and to approximate the following L^p norm of the distributional symmetric gradient of functions in $E^{1,p}(\Omega; \mathbb{R}^n)$ for $p \in [1, \infty)$:

$$\int_{\Omega} |\mathcal{E}u|_{\alpha_p}^p dx,$$

where $|\cdot|_{\alpha_p} := \psi_{\alpha_p}^{1/p}(\cdot)$ is a (possibly anisotropic) norm in $\mathbb{R}^{m \times n}$ (Cor. 5.17).

In this section $D \subseteq \mathbb{R}^n$ will always denote a bounded connected open set with Lipschitz boundary.

Example 5.12. Let $p \in [1, \infty)$ and $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$. We define,

$$(5.18) \quad \alpha_p(u, D') := \inf_{A \in \mathbb{R}_{\text{skew}}^{n \times n}} \int_{D'} |u(x) - A(x - z) - u_{D'}|_2^p dx,$$

where $z = \text{bar}(D')$ is the barycenter of D' . Then, $\alpha_p = \alpha_{p,D,\Gamma^{\mathcal{G}},m}$ is a p -core functional as in Def. 3.1 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2. Moreover, if $p \in (1, \infty)$ α_p is a strong p -core functional as in Def. 3.12, thanks to the *Korn inequality* (cfr. [DM04, Eq. (2.2)]).

Remark 5.13. Under the assumptions of Ex. 5.12, we observe that the infimum in (5.18) is actually a minimum. Indeed, let $u \in L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ and $D' \in \mathcal{G}^D(\mathbb{R}^n)$. We define,

$$(5.19) \quad f(A) := \int_{D'} |u(x) - A(x - \text{bar}(D')) - u_{D'}|_2^p dx \text{ for every } A \in \mathbb{R}^{n \times n}.$$

We observe that f is coercive, since for every $\eta \in (0, 1)$,

$$f(A) \geq \frac{1}{(1+\eta)^p} \int_{D'} |A(x - \text{bar}(D'))|_2^p dx - \frac{1}{\eta^p} \int_{D'} |u(x) - u_{D'}|_2^p dx,$$

$$\lim_{|A|_2 \rightarrow +\infty} \int_{D'} |A(x - \text{bar}(D'))|_2^p dx = +\infty.$$

Moreover f is convex and $\mathbb{R}_{\text{skew}}^{n \times n}$ is a closed subset of \mathbb{R}^n . This, implies that the minimum of $f|_{\mathbb{R}_{\text{skew}}^{n \times n}}$ is attained.

Proposition 5.14. *Let $p \in [1, \infty)$ and $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$. Let $\alpha_p = \alpha_{p,D,\Gamma^{\mathcal{G}},m}$ be the p -core functional in Ex. 5.12 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2. Then, $\alpha_p(\cdot, D')|_{L^p(\mathbb{R}^n; \mathbb{R}^n)} : L^p(\mathbb{R}^n; \mathbb{R}^n) \rightarrow [0, +\infty)$ is lower semicontinuous for every $D' \in \mathcal{G}^D(\mathbb{R}^n)$.*

Proof. Let $D' \in \mathcal{G}^D(\mathbb{R}^n)$, $z = \text{bar}(D')$, $u \in L^p(\mathbb{R}^n; \mathbb{R}^n)$ and $\{u_k\}_{k \in \mathbb{N}} \subset L^p(\mathbb{R}^n; \mathbb{R}^n)$ such that $u_k \rightarrow u$ in $L^p(\mathbb{R}^n; \mathbb{R}^n)$. Without loss of generality we assume that $u_k \rightarrow u$ a. e. in \mathbb{R}^n and,

$$(5.20) \quad \lim_{k \rightarrow +\infty} \alpha_p(u_k, D') = \liminf_{k \rightarrow +\infty} \alpha_p(u_k, D') < +\infty.$$

For every $k \in \mathbb{N}$ there exist $A_k \in \mathbb{R}_{\text{skew}}^{n \times n}$ such that,

$$(5.21) \quad \int_{D'} \left| u_k(x) - A_k(x - z) - \int_{D'} u_k \right|_2^p dx \leq \alpha_p(u_k, D') + 1/k.$$

Since $\{u_k\}_{k \in \mathbb{N}}$ is bounded in $L^p(\mathbb{R}^n; \mathbb{R}^n)$, combining (5.20) and (5.21) we obtain that

$$\sup_{k \in \mathbb{N}} |A_k|_2 < +\infty.$$

Then, up to subsequences, $A_k \rightarrow A_\infty$ for some $A_\infty \in \mathbb{R}_{\text{skew}}^{n \times n}$. Then, by Fatou's lemma and (5.21) we get that

$$\begin{aligned} \alpha_p(u, D') &\leq \int_{D'} \left| u(x) - A_\infty(x - z) - \int_{D'} u \right|_2^p dx \\ &\leq \liminf_{k \rightarrow +\infty} \int_{D'} \left| u_k(x) - A_k(x - z) - \int_{D'} u_k \right|_2^p dx \leq \liminf_{k \rightarrow +\infty} \alpha_p(u_k, D'). \end{aligned}$$

□

Proposition 5.15. *Let $p \in [1, \infty)$ and $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$. Let $\alpha_p = \alpha_{p,D,\Gamma^{\mathcal{G}},m}$ be the p -core functional in Ex. 5.12 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2. Recalling Def. 3.9, if D is as in Rem. 5.3, then for every $B \in \mathbb{R}^{n \times n}$,*

$$(5.22) \quad \psi_{\alpha_p}(B) = \frac{1}{|D|} \inf_{A \in \mathbb{R}_{\text{skew}}^{n \times n}} \int_D |(B - A)(x - \text{bar}(D))|_2^p dx,$$

and

$$(5.23) \quad \mathcal{N}_{\psi_{\alpha_p}} = \mathbb{R}_{\text{skew}}^{n \times n},$$

so that

$$\psi_{\alpha_p}(B) = \psi_{\alpha_p} \left(\frac{B + B^T}{2} \right),$$

for every $B \in \mathbb{R}^{n \times n}$.

Moreover, if $p = 2$ and $D = Q(x; r)$ for some $x \in \mathbb{R}^n$ and $r > 0$, then,

$$(5.24) \quad \psi_{\alpha_2}(B) = \frac{1}{|D|^2} \left\| \int_D l^A \right\|_{L^2(D - \text{bar}(D))}^2,$$

where

$$A = \frac{B + B^T}{2}.$$

Proof. We observe that for every $D' \in \mathcal{G}^D(\mathbb{R}^n)$ and $B \in \mathbb{R}^{n \times n}$,

$$(5.25) \quad \begin{aligned} \alpha_p(l^B, D') &= \inf_{A \in \mathbb{R}_{\text{skew}}^{n \times n}} \int_{D'} \left| l^B(x) - A(x - \text{bar}(D')) - \int_{D'} l^B(y) dy \right|_2^p dx \\ &= \inf_{A \in \mathbb{R}_{\text{skew}}^{n \times n}} \int_{D'} |(B - A)(x - \text{bar}(D'))|_2^p dx. \end{aligned}$$

Then, (5.22) follows from Prop. 5.1. Moreover, (5.23) follows from Rem. 5.13 and (5.22).

Let $B \in \mathbb{R}^{m \times n}$ and define,

$$f_B(A) := \int_{D - \text{bar}(D)} |(B - A)x|_2^p dx \quad \forall A \in \mathbb{R}^{m \times n}.$$

Then, by (5.22) and Rem. 5.13,

$$\psi_{\alpha_p}(B) = \frac{1}{|D|} \min_{A \in \mathbb{R}_{\text{skew}}^{n \times n}} f_B(A) = \frac{1}{|D|} \min_{A \in \mathbb{R}^{n \times n}} f_B(A - A^T).$$

If $p > 1$ we can then find the minimum point of the strictly convex function $f_B|_{\mathbb{R}_{\text{skew}}^{n \times n}}$ by differentiating $\mathbb{R}^{n \times n} \ni A \mapsto f_B(A - A^T)$ and finding its stationary point.

If particular, if $p = 2$ we obtain the following conditions,

$$(5.26) \quad \begin{aligned} &\sum_{j=1}^n (A_{kj} - A_{jk} - B_{kj}) \int_{D - \text{bar}(D)} x_j x_l dx \\ &= \sum_{j=1}^n (A_{lj} - A_{jl} - B_{lj}) \int_{D - \text{bar}(D)} x_j x_k dx, \end{aligned}$$

for every $k, l \in \{1, \dots, n\}$.

Without loss of generality we may assume $\text{bar}(D) = 0$. Then, D is equal to the cube $(-\frac{\tau}{2}, \frac{\tau}{2})^n$, so that,

$$(5.27) \quad \int_D x_i x_j dx = 0 \text{ for every } i, j \in \{1, \dots, n\},$$

$$(5.28) \quad \int_D x_i^2 dx = \int_D x_j^2 dx \text{ for every } i, j \in \{1, \dots, n\}.$$

Thanks to (5.27) then (5.26) becomes,

$$(5.29) \quad (A_{kl} - A_{lk} - B_{kl}) \int_D x_l^2 dx = (A_{lk} - A_{kl} - B_{lk}) \int_D x_k^2 dx.$$

Finally, thanks to (5.28), (5.29) becomes,

$$(5.30) \quad A_{kl} - A_{lk} = \frac{B_{kl} - B_{lk}}{2} \text{ for every } k, l \in \{1, \dots, n\}.$$

(5.30) implies that the minimum of $f_B|_{\mathbb{R}_{\text{skew}}^{n \times n}}$ is attained when $A = (B - B^T)/2$. Therefore,

$$\psi_{\alpha_p}(B) = \frac{1}{|D|} \int_{D - \text{bar}(D)} \left| \frac{B + B^T}{2}(x) \right|_2^2 dx,$$

so (5.24) is proved. \square

The following example is similar to Ex. 5.12

Example 5.16. Let $p \in [1, \infty)$ and $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$. We define,

$$\alpha_p(u, D') := \inf_{A \in \mathbb{R}_{\text{skew}}^{n \times n}, c \in \mathbb{R}^m} \int_{D'} |u(x) - A(x - z) - c|_2^p dx,$$

where $z = \text{bar}(D')$ is the barycenter of D' . Then, $\alpha_p = \alpha_{p, D, \Gamma^{\mathcal{G}}, m}$ is a p -core functional as in Def. 3.1 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2. Moreover, if $p \in (1, \infty)$ α_p is a strong p -core functional as in Def. 3.12, thanks to the *Korn inequality* (cfr. [DM04, Eq. (2.2)]).

The following result shows how Ex. 5.12 can be used in the study of $E^{1,p}(\Omega; \mathbb{R}^n)$.

Corollary 5.17. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $p \in [1, \infty)$, $\mathcal{G} = \{\text{id}_{\mathbb{R}^n}\}$ and $D \subseteq \mathbb{R}^n$ be a bounded connected open set with Lipschitz boundary. Let $\alpha_p = \alpha_{p, D, \Gamma^{\mathcal{G}}, m}$ be one of the p -core functionals in Ex. 5.12 or Ex. 5.16 subordinated to $\Gamma^{\mathcal{G}}$ as in Def. 3.2.

Then,

(1) for every $u \in E_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^n)$,

$$\lim_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega) = \int_{\Omega} \psi_{\alpha_p}(\mathcal{E}u),$$

where ψ_{α_p} is as in Def. 3.9 and $G_{\varepsilon}^{\alpha_p}$ is as in Def. 3.4. Furthermore, if D is as in Rem. 5.3 the function ψ_{α_p} can be written more explicitly according to Prop. 5.15.

(2) if $p \in (1, \infty)$, for every $u \in L^p(\Omega; \mathbb{R}^n)$,

$$\liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega) < +\infty \iff u \in E^{1,p}(\Omega; \mathbb{R}^n).$$

(3) if $\Omega \subseteq \mathbb{R}^n$ is a connected open set, $p \in (1, \infty)$ and $u \in L_{\text{loc}}^p(\Omega; \mathbb{R}^n)$ satisfies $\liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon}^{\alpha_p}(u, \Omega) = 0$, then

$$(5.31) \quad u(x) = Ax + h \quad \forall x \in \Omega,$$

where $A \in \mathbb{R}_{\text{skew}}^{n \times n}$ and $h \in \mathbb{R}^n$.

Proof. The proof of (1) and (2) follows combining Prop. 5.15, Prop. 5.14, Thm. 1.2 and Cor. 4.4.

To prove (3) we apply Cor. 4.5 and Prop. 5.15 to deduce that for every $D' \in \mathcal{G}^D(\Omega)$ there exist $A \in \mathbb{R}_{\text{skew}}^{n \times n}$ and $h \in \mathbb{R}^n$ (possibly depending on D') s.t.

$$u(x) = Ax + h \quad \forall x \in D'.$$

At this point exploiting the connectedness of Ω we can employ the simple argument in [AC17, Lem. 8] to deduce that (5.31) holds for $A \in \mathbb{R}_{\text{skew}}^{n \times n}$ and $h \in \mathbb{R}^n$. \square

5.3. Other examples of p -core functionals.

Example 5.18. Let $m \geq 1$ be a natural number and $p \in [1, \infty)$ and Γ a subgroup of $\text{Aff}(\mathbb{R}^n)$. Let $D \subseteq \mathbb{R}^n$ be a bounded open set. We define,

$$\alpha_p(u, D') := 0 \quad \forall u \in L^p(\mathbb{R}^n; \mathbb{R}^m) \forall D' \in \mathcal{E}_{\Gamma}^D(\mathbb{R}^n).$$

$\alpha_p \equiv 0$ is a (trivial) strong p -core functional (cfr. Rem. 3.13).

If a p -core functional α_p is identically equal to zero (Ex. 5.18), then $\psi_{\alpha_p} \equiv 0$, but the vice-versa does not hold (see Ex. 5.19).

Example 5.19. The following example is somehow pathological: linear maps are not able to entirely describe the behavior of the p -core functional α_p when we are interested in computing the pointwise limit of $G_\varepsilon^{\alpha_p}(\cdot, \Omega)$ as $\varepsilon \rightarrow 0^+$.

We consider a p -core functional subordinated to $p \in [1, \infty)$, the reference set $D := Q = (-1/2, 1/2)^n$, $m \geq 1$ and the subgroup $\Gamma^{\mathcal{G}}$ with $\mathcal{G} := \{\text{id}_{\mathbb{R}^n}\}$. The p -core functional is defined by,

$$\alpha_p(u, D') := \inf_{A \in \mathbb{R}^{m \times n}} \int_{D'} |u(x) - A(x - \text{bar}(D')) - u_{D'}|_2^p dx.$$

Then, $\psi_{\alpha_p} \equiv 0$, since

$$\alpha_p(l^B, D') = \inf_{A \in \mathbb{R}^{m \times n}} \int_{D'} |(B - A)(x - \text{bar}(D'))|_2^p dx = 0,$$

for all $B \in \mathbb{R}^{m \times n}$ and $D' \in \mathcal{G}^D(\mathbb{R}^n)$.

Being $\psi_{\alpha_p} \equiv 0$ and $\alpha_p \not\equiv 0$, having in mind Rem. 3.13, we deduce that α_p is not a strong p -core functional as in Def. 3.12.

Let $\Omega = D$ be the ambient space, $\delta \in (n-p, n) \setminus \{0\}$ and $u(x) := |x|_2^{-\delta/p} e_1$ for every $x \in \Omega \setminus \{0\}$ and $u(x) := 0$ for every $x \in \mathbb{R}^n \setminus \Omega$. We observe that $u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ but $\nabla u|_{\Omega} \notin L^p(\Omega; \mathbb{R}^{m \times n})$. Let,

$$f(A) := \int_D |u(x) - A(x - \text{bar}(D)) - u_D|_2^p dx \quad \forall A \in \mathbb{R}^{m \times n}.$$

We observe that f is convex and strictly positive. Indeed, if f vanished at some point $A \in \mathbb{R}^{m \times n}$, then u would be equal a.e. in D to the map $l^A(\cdot - \text{bar}(D)) - u_D$. We also observe, using the argument in Rem. 5.13, that f is coercive. Therefore, $\min_{A \in \mathbb{R}^{m \times n}} f(A) > 0$ and for every $\varepsilon > 0$,

$$\begin{aligned} G_\varepsilon^{\alpha_p}(u, \Omega) &\geq \varepsilon^{n-p} \alpha_p(u, \varepsilon D) = \varepsilon^{n-p} \alpha_p(u(\varepsilon \cdot), D) \\ &= \varepsilon^{n-p-\delta} \min_{A \in \mathbb{R}^{m \times n}} f(\varepsilon^{\delta/p} A) = \varepsilon^{n-p-\delta} \min_{A \in \mathbb{R}^{m \times n}} f(A), \end{aligned}$$

thanks to (SC) and the fact that $u(\varepsilon x) = \varepsilon^{-\delta/p} u(x)$ for every $x \in D$. Therefore,

$$G_+^{\alpha_p}(u, \Omega) \geq G_-^{\alpha_p}(u, \Omega) \geq \left(\min_{A \in \mathbb{R}^{m \times n}} f(A) \right) \left(\lim_{\varepsilon \rightarrow 0^+} \varepsilon^{n-p-\delta} \right) = +\infty,$$

which implies that $G^{\alpha_p}(u, \Omega) = +\infty$. Then, being $\psi_{\alpha_p} \equiv 0$, (1.4) and (3.21) cannot hold.

Finally, we highlight that $\mathcal{N}_{\psi_{\alpha_p}} = \mathbb{R}^{m \times n}$, so that $\mathcal{P} = \{0\}$ trivially satisfies (3.20) and $\pi_{\mathcal{P}} = 0$. Therefore, $D_{\pi_{\mathcal{P}}}(u|_{\Omega}) = 0 \in L^p(\Omega; \mathbb{R}^{m \times n})$, but $G_-^{\alpha_p}(u, \Omega) = G^{\alpha_p}(u, \Omega) = +\infty$. This then shows that the implication (4.2) in Cor. 4.2 cannot be reversed in general.

This also proves in a different way that α_p is not a strong p -core functional, since this would be in contradiction with Thm. 1.2.

Example 5.20. Let $p \in [1, \infty)$, $m \geq 1$, $D \subset \mathbb{R}^n$ be a bounded open connected set with Lipschitz boundary and $S \subset \mathbb{R}^m$ be a nonempty set. We define,

$$\alpha_p(u, D') := \sup_{\nu \in S} \int_{D'} |(u - u_{D'}) \cdot \nu|^p dx,$$

Then, $\alpha_p : L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m) \times \mathcal{E}_\Gamma^D(\mathbb{R}^n) \rightarrow [0, +\infty)$ is a p -core functional for any Γ subgroup of $\text{Aff}(\mathbb{R}^n)$. Moreover, $\alpha_p(\cdot, D')|_{L^p(\mathbb{R}^n; \mathbb{R}^m)} : L^p(\mathbb{R}^n; \mathbb{R}^m) \rightarrow [0, +\infty)$ is lower

semicontinuous for every $D' \in \mathcal{E}_\Gamma^D(\mathbb{R}^n)$ being supremum of lower semicontinuous functionals.

In this case, it is easy to observe that $\mathcal{N}_{\psi_{\alpha_p}} = \{A \in \mathbb{R}^{m \times n} : A(\mathbb{R}^n)^\perp \supset S\}$. Thus, if S contains m linearly independent vectors, then $\mathcal{N}_{\psi_{\alpha_p}} = \{0\}$ and can be used to characterize $W^{1,p}$. Otherwise, the null set is non trivial and can be used to approximate larger Sobolev-type spaces.

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