

A BRUNN-MINKOWSKI INEQUALITY FOR SCHRÖDINGER OPERATORS WITH KATO CLASS POTENTIALS

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ABSTRACT. In this paper we prove a Brunn-Minkowski inequality for the first Dirichlet eigenvalue of a Schrödinger type operator $\mathcal{H}_V := -\operatorname{div}(A\nabla) + V$, where V is convex and Kato decomposable, using the trace class property of the generated semigroup. As a consequence, using the ultracontractivity of the semigroup we obtain the log-concavity of the ground state which is also strong log-concave under additional assumptions on Ω and V .

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1. INTRODUCTION

The Brunn-Minkowski inequality is a classical topic in Geometric Analysis. In its simplest form, it asserts that for any nonempty Borel sets $\Omega_0, \Omega_1 \subset \mathbb{R}^N$, set for any $r \in [0, 1]$ the convex Minkowski sum $\Omega_r := (1-r)\Omega_0 + r\Omega_1$, one has that

$$|\Omega_r|^{1/N} \geq (1-r)|\Omega_0|^{1/N} + r|\Omega_1|^{1/N}, \quad (1.1)$$

where $|\cdot|$ denotes the Lebesgue measure. In other words, for every nonempty Borel set Ω , the function $\Omega \mapsto |\Omega|^{1/N}$ is concave. Inequality (1.1) has far-reaching implications, including the isoperimetric inequality, concentration phenomena and several other functional inequalities. See [3, 19] for several applications.

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Inequality (1.1) is a particular case of the Prékopa-Leindler inequality, established in [23, 27] and generalized in higher dimension in [5, 26], which can be stated as follows.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a measurable and convex set and $f, g, h : \Omega \rightarrow (0, +\infty)$ measurable functions such that*

$$h((1-r)x + ry) \geq f(x)^{1-r} g(y)^r \quad (1.2)$$

for every $x, y \in \Omega$ and every $r \in [0, 1]$. Then

$$\int_{\Omega} h(x) dx \geq \left(\int_{\Omega} f(x) dx \right)^{1-r} \left(\int_{\Omega} g(x) dx \right)^r$$

Theorem 1.1 has been crucial in the study of concavity properties of solutions to elliptic and parabolic equations. In particular, condition (1.2) when $f = g = h$ is the definition of *log-concave* functions. In recent years, numerous extensions and refinements of (1.1) and Theorem 1.1 have been proposed for other geometric functionals arising in Calculus of variations and for other notions of concavity. See [10, 19] and the references therein. Apart from the euclidean setting, in the Gauss space the following inequality

$$\gamma(\Omega_r) \geq \gamma(\Omega_0)^{1-r} \gamma(\Omega_1)^r, \quad (1.3)$$

where γ denotes the standard Gaussian measure $\gamma := \frac{e^{-|\cdot|^2/2}}{(2\pi)^{N/2}} \mathcal{L}^N$, expresses that the Gaussian volume enjoys a *weak* Brunn-Minkowski inequality which emphasizes the log-concavity of the measure γ . With a little abuse of notation, in the sequel we will refer to γ to indicate both the Gaussian measure and its density with respect to the Lebesgue measure.

Inequality (1.3) holds true as an immediate application of Theorem 1.1 and has been proved with other methods by Borell in [4]. An improvement of inequality (1.3) has been given in [17], where in the class of symmetric sets with respect to the origin the authors prove (1.1) with the Lebesgue measure replaced by the Gaussian one. Moving to the spectral framework, let $w \in \{1, \gamma\}$, and let $\lambda_{1,w}(\Omega)$ be the least real number λ such that the problem

$$\begin{cases} -\operatorname{div}(w \nabla v) = \lambda v & \text{in } \Omega \\ v = 0 & \text{in } \partial\Omega \end{cases} \quad (1.4)$$

admits a nontrivial solution u . A Brunn-Minkowski inequality for $\lambda_{1,w}(\Omega)$ goes as follows

$$\lambda_{1,w}(\Omega_r) \leq (1-r)\lambda_{1,w}(\Omega_0) + r\lambda_{1,w}(\Omega_1). \quad (1.5)$$

In the case $w = 1$ the convexity of the first eigenvalue of the Dirichlet Laplacian (1.5) is equivalent to the concavity of the function $\Omega \mapsto \lambda_1^{-1/2}(\Omega)$ proved in [7]. If $w = \gamma$ inequality (1.5) deals with the convexity of the first Dirichlet eigenvalue of the Ornstein-Uhlenbeck operator $-\Delta_{\gamma} := -\Delta + x \cdot \nabla$ recently proved in [12, Theorem 1.2], where the authors also address the case of equality in (1.5). We want to notice that $\lambda_{1,\gamma}(\Omega)$ also enjoys a Faber-Krahn inequality, see [2, 9]. The Brunn-Minkowski inequality for $\lambda_{1,\gamma}(\Omega)$ can also be

restated for the first Dirichlet eigenvalue of the Schrödinger operator $\mathcal{H} := -\Delta + \frac{|\cdot|^2}{4} - \frac{N}{2}$ using the unitary transformation (3.1) defined in the following Section 3, which induces an isospectrality between $-\Delta_\gamma$ and \mathcal{H} . The first Dirichlet eigenvalue of the Schrödinger operator $-\Delta + V$ played a key role in the proof of the fundamental gap conjecture by Andrews and Clutterbuck in [1].

In this article we establish a Brunn-Minkowski-type inequality for the first Dirichlet eigenvalue of a Schrödinger type operator: Namely, we consider the following boundary value problem

$$\begin{cases} \mathcal{H}_V u = \lambda_{1,V}(\Omega)u & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases} \quad (1.6)$$

being

$$\mathcal{H}_V := -\operatorname{div}(A\nabla) + V$$

where A is a constant symmetric matrix such that $A\xi \cdot \xi \geq a_1|\xi|^2$ for every $\xi \in \mathbb{R}^N$ and for some $a_1 > 0$. Denoting for $k \in \mathbb{R}$ $k^+ := \max\{k, 0\}$ and $k^- := \max\{-k, 0\}$ we also assume that the potential $V : \Omega \rightarrow \mathbb{R}$ satisfies the following assumptions:

A.1 V is convex in Ω .

A.2 V is *Kato decomposable*, namely $V^+ \in L^1_{\text{loc}}(\Omega)$ and $V^- \in \mathcal{K}(\Omega)$, where $\mathcal{K}(\Omega)$ denotes the Kato class. See Definition 2.1 below.

A.3 $e^{-tV} \in L^1(\Omega)$ for every $t > 0$.

Main Theorem. *Let $r \in [0, 1]$ and Ω_0, Ω_1 be non-empty convex sets in \mathbb{R}^N and*

$$\Omega_r := (1 - r)\Omega_0 + r\Omega_1.$$

Then

$$\lambda_{1,V}(\Omega_r) \leq (1 - r)\lambda_{1,V}(\Omega_0) + r\lambda_{1,V}(\Omega_1). \quad (1.7)$$

Some comments are in order. Our approach relies on the pioneering papers by Brascamp and Lieb [6, 7]. The assumption of taking constant second order coefficients is sharp in order to ensure the log-concavity of the heat kernel. See [20, Theorem 1.2, Proposition 1.3] and Section 3. Concerning the potential term V assumptions **A.1**, **A.2** and **A.3** are classical. More specifically, Assumption **A.1** ensures that the heat kernel of \mathcal{H}_V is log-concave in the spatial variables; Assumption **A.2** guarantees that the associated quadratic form is coercive and that the heat kernel enjoys upper Gaussian estimates; finally, Assumption **A.3** implies both that \mathcal{H}_V has discrete spectrum and that the semigroup is trace class. We observe that one can also address the problem of optimizing the first eigenvalue $\lambda_{1,V}(\Omega)$ with respect to the potential V under the constraint given by the $L^1(\Omega)$ norm of e^{-tV} for some $t > 0$, as recently done in [18]. For a more complete treatment on Schrödinger semigroups we refer the interested reader to the fundamental work [32].

The paper is organized as follows. In Section 2 we introduce the notation, the geometric setting, and the functional framework required for our analysis. In Section 3 we prove Theorem 1 and we apply it in Theorem 3.5 exploiting some properties of the semigroup generated by the Schrödinger operator with homogeneous Dirichlet boundary conditions in order to prove the log-concavity of the ground state $\psi_{1,V}$. We conclude the paper by proving that if both Ω and V satisfy stronger regularity and convexity assumptions the ground state $\psi_{1,V}$ enjoys a strong log-concavity.

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2. PRELIMINARY RESULTS

In the sequel we assume that $D \subseteq \mathbb{R}^N$ is a nonempty open set and that $\Omega \subset \mathbb{R}^N$ is a nonempty connected convex set in \mathbb{R}^N . We will also denote with L^Ω the realization of a second order operator endowed with homogeneous Dirichlet boundary conditions in Ω .

Definition 2.1. Let $N \geq 2$. We say that the function W belongs to the *Kato class* $\mathcal{K}(D)$ if

$$\limsup_{r \rightarrow 0^+} \sup_{x \in D} \int_{B_r(x) \cap D} |W(y)| \mathbb{G}_N(x-y) dy = 0,$$

where \mathbb{G}_N denotes the Green function for the Laplacian in \mathbb{R}^N .

Otherwise, if $N = 1$ we say $W \in \mathcal{K}(D)$ if

$$\sup_{x \in D} \int_{x-1}^{x+1} |W(y)| dy < \infty.$$

Functions in the Kato class enjoy a Hardy-type inequality with remainder term as stated in the following Lemma. We refer to [30, Theorem 9.3] for a proof.

Lemma 2.2. *Let $W \in \mathcal{K}(D)$. For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\int_D |W| \varphi^2 dx \leq \varepsilon \int_D |\nabla \varphi|^2 dx + C_\varepsilon \int_D \varphi^2 dx$$

for every $\varphi \in H_0^1(D)$.

2.1. Trace class semigroups. Let H be an infinite dimension separable Hilbert space and let $\mathcal{B}(H)$ denote the algebra of bounded linear operators on H . For any operator $T \in \mathcal{B}(H)$, we define its absolute value as the positive operator $|T| = \sqrt{T^*T}$.

Definition 2.3 (Trace Class Operator). An operator $T \in \mathcal{B}(H)$ is said to be *trace class* (or strictly nuclear) if, for some orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of H , the following sum converges:

$$\sum_{k=1}^{\infty} (|T|e_k, e_k)_H < \infty$$

If T is trace class, the value of the sum does not depend on the choice of the orthonormal basis. We denote the space of all trace class operators by $\mathcal{S}_1(H)$. For any $T \in \mathcal{S}_1(H)$, the *trace* of T is defined as:

$$\mathrm{Tr}(T) = \sum_{k=1}^{\infty} (Te_k, e_k)_H.$$

This series converges absolutely and is independent of the chosen basis. Trace class operators are necessarily compact.

Now, let $(T(t))_{t \geq 0}$ be a positive, selfadjoint strongly continuous C_0 -semigroup of bounded linear operators on H , and let $-L$ with domain $D(L)$ be its infinitesimal generator. When the generator is defined we denote by $(e^{-tL})_{t > 0}$ the associated semigroup.

If $(e^{-tL})_{t > 0}$ is also compact, the spectrum of $-L$, denoted by $\sigma(-L)$, consists entirely of isolated eigenvalues of finite algebraic multiplicity:

$$\sigma(-L) = \{\lambda_k\}_{k \in \mathbb{N}} \subset [0, \infty)$$

with $\lambda_k \rightarrow +\infty$ as $k \rightarrow \infty$.

To link the trace of the semigroup $T(t)$ directly to the eigenvalues of its generator $-L$, we employ the Spectral Mapping Theorem.

Theorem 2.4 (Spectral Mapping Theorem for Point Spectrum). *For a strongly continuous semigroup $(e^{-tL})_{t > 0}$ generated by $-L$, the point spectrum obeys the following relation:*

$$e^{t\sigma_p(-L)} \subseteq \sigma_p(e^{-tL}) \setminus \{0\}. \quad (2.1)$$

If $(e^{-tL})_{t > 0}$ is also compact, the equality holds in (2.1).

A useful criterion to satisfy the trace class property is given by the following Proposition.

Proposition 2.5 (Trace of a Semigroup). *Let $(e^{-tL})_{t > 0}$ be a strongly continuous, self-adjoint and compact semigroup generated by $-L$, and let $\{\lambda_k\}_{k \in \mathbb{N}}$ be the sequence of eigenvalues of $-L$ (repeated according to algebraic multiplicity). The semigroup $(e^{-tL})_{t > 0}$ is trace class if and only if:*

$$\sum_{k=1}^{\infty} e^{-t\lambda_k} < \infty$$

If this condition holds, the trace is given by the formula

$$\mathrm{Tr}(e^{-tL}) = \sum_{k=1}^{\infty} e^{-t\lambda_k}.$$

Remark 2.6. Let $H = L^2(D)$ and let $\mathbf{a} : D(\mathbf{a}) \times D(\mathbf{a}) \rightarrow (0, \infty)$ be a symmetric, closed, densely defined and coercive sesquilinear form with $D(\mathbf{a}) \subset H$, and L the operator associated to \mathbf{a} by the formula

$$(-Lu, v)_H := \mathbf{a}(u, v),$$

for every $u, v \in D(\mathbf{a})$ where $(\cdot, \cdot)_H$ denotes the scalar product in H . By the standard Theory the operator $-L$ generates a selfadjoint strongly continuous semigroup of contractions in H , and the embedding

$$D(\mathbf{a}) \hookrightarrow H$$

is compact. Therefore, there exists a heat kernel $p_L : (0, \infty) \times D \times D \rightarrow (0, \infty)$ such that, for any $\{\psi_k\}_{k \in \mathbb{N}}$ orthonormal basis of eigenfunctions

$$p_L(t, x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \psi_k(x) \psi_k(y).$$

Therefore, we can equivalently say that the semigroup is trace class if and only if the *partition function*

$$Z_D(t) := \int_D p_L(t, x, x) dx, \quad (2.2)$$

is finite for every $t > 0$.

Another class of semigroups enjoying good spectral properties is the class of *irreducible semigroups*. We state this notion in the setting of L^2 -spaces.

Definition 2.7. Let μ a σ -finite measure. A C_0 -semigroup $(T(t))_{t \geq 0}$ on $L^2(D) := L^2(D, \mu)$ is called *irreducible* if, for each measurable set $\omega \subseteq D$, the inclusion

$$T(t)L^2(\omega) \subseteq L^2(\omega) \quad (t > 0)$$

implies that either $\mu(\omega) = 0$ or $\mu(D \setminus \omega) = 0$.

A criterion to ensure irreducibility is given by the following result. See [25, Corollary 2.11].

Theorem 2.8. Let \mathbf{a} be a local, densely defined, coercive, bounded and closed form in $L^2(D, \mu)$ and assume that the associated semigroup $(e^{-tL})_{t > 0}$ is positive. The following assertions are equivalent

- (1) The semigroup $(e^{-tL})_{t > 0}$ is irreducible
- (2) If $\omega \subset D$ is such that $\chi_\omega(D(\mathbf{a})) \subseteq D(\mathbf{a})$ then either $\mu(\omega) = 0$ or $\mu(D \setminus \omega) = 0$
- (3) If $\omega \subset D$ is such that $\chi_\omega(C) \subseteq D(\mathbf{a})$ for some core C of \mathbf{a} then either $\mu(\omega) = 0$ or $\mu(D \setminus \omega) = 0$.

The following corollary ensures that positive and irreducible semigroups are strictly positive.

Corollary 2.9. *Let μ be a σ -finite measure and $(T(t))_{t \geq 0}$ be a positive C_0 semigroup in $L^2(D, \mu)$. Then, $T(t)$ is irreducible if and only if for every nonzero $f \in L^2(D)$, $f \geq 0$, it holds that $T(t)f(x) > 0$ for μ -almost every $x \in D$.*

2.2. The Schrödinger semigroup in $L^2(\Omega)$ with Kato class potential. In this Subsection we recall some fundamental results of the semigroup generated by $-\mathcal{H}_V$ in $L^2(\mathbb{R}^N)$ and in $L^2(\Omega)$ with Dirichlet conditions.

The following Gaussian upper bound is stated and proved in [32, Prop. B. 6.7].

Theorem 2.10. *Let \mathcal{H}_V be the Schrödinger operator on $L^2(\mathbb{R}^N)$ with A constant, symmetric and positive definite and $V : \mathbb{R}^N \rightarrow \mathbb{R}$ with V Kato decomposable. Then, there exist $C_1, C_2 > 0$ such that for every $t > 0$ and $x, y \in \mathbb{R}^N$*

$$0 \leq p_V(t, x, y) \leq C_1 \frac{e^{-C_2 \frac{|x-y|^2}{t} + \omega_0 t}}{\min\{1, t^{N/2}\}} \quad (2.3)$$

where ω_0 is the semigroup growth bound defined by

$$\omega_0 := \inf \left\{ \omega \in \mathbb{R} : \exists M \geq 1 \text{ such that } \|e^{-t\mathcal{H}_V}\| \leq M e^{\omega t} \text{ for all } t \geq 0 \right\}. \quad (2.4)$$

Corollary 2.11. *The semigroup $e^{-t\mathcal{H}_V} : L^1(\mathbb{R}^N) \rightarrow L^\infty(\mathbb{R}^N)$ is bounded and the following ultracontractivity estimate holds true*

$$\|e^{-t\mathcal{H}_V}\|_{\mathcal{B}(L^1(\mathbb{R}^N), L^\infty(\mathbb{R}^N))} = \sup_{x, y \in \mathbb{R}^N} p_V(t, x, y) \leq C \frac{e^{\omega_0 t}}{\min\{1, t^{N/2}\}} \quad t > 0. \quad (2.5)$$

Now, let $V : \Omega \rightarrow \mathbb{R}$ satisfy assumptions **A.1**, **A.2** and **A.3**, and set $V_\Omega := V\chi_\Omega$. By Theorem 2.10 the operator \mathcal{H}_{V_Ω} has a heat kernel $p_{V_\Omega}(t, x, y) : \mathbb{R}^N \times \mathbb{R}^N \times (0, \infty) \rightarrow (0, \infty)$ which enjoys the Gaussian estimate (2.3). Moreover, the semigroup $(e^{-t\mathcal{H}_{V_\Omega}^\Omega})_{t > 0}$ generated by $-\mathcal{H}_{V_\Omega}^\Omega$ in $L^2(\Omega)$ is a positive and selfadjoint C_0 semigroup such that, for every $f \in L^2(\Omega)$ and $t > 0$

$$|e^{-t\mathcal{H}_{V_\Omega}^\Omega} f| \leq e^{-t\mathcal{H}_{V_\Omega}} |f|.$$

Therefore there exists $p_{V_\Omega}^\Omega : (0, \infty) \times \Omega \times \Omega \rightarrow (0, \infty)$ such that

$$(e^{-t\mathcal{H}_{V_\Omega}^\Omega} f)(x) = \int_\Omega p_{V_\Omega}^\Omega(t, x, y) f(y) dy, \quad (2.6)$$

see [29, Chapter IV]. Moreover, since $\mathcal{Q}_{V_\Omega} = \mathcal{Q}_V$ we have that $p_{V_\Omega}^\Omega(t, x, y) = p_V^\Omega(t, x, y) \leq p_{V_\Omega}(t, x, y)$ for every $t > 0$ and $x, y \in \Omega$. This implies that $e^{-t\mathcal{H}_V^\Omega} f = e^{-t\mathcal{H}_{V_\Omega}^\Omega} f$ for every $f \in L^2(\Omega)$ and for every $t > 0$. The semigroup $(e^{-t\mathcal{H}_V^\Omega})_{t > 0}$ is also trace class by the Golden-Thompson-Symanzik estimate, (see e.g. [31])

$$\mathrm{Tr}(e^{-t\mathcal{H}_V^\Omega}) \leq \mathrm{Tr}(e^{t \operatorname{div}(A\nabla)^\Omega} e^{-tV}) \leq \frac{C}{\min\{1, t^{N/2}\}} \int_\Omega e^{-tV(x)} dx, \quad t > 0,$$

where the right-hand side is finite thanks to Assumption **A.3**.

Now, we consider the bilinear form

$$\mathcal{E}_V(u, v) := \frac{1}{2} \int_{\Omega} A \nabla u \cdot \nabla v \, dx + \int_{\Omega} V u v \, dx,$$

and we set $\mathcal{Q}_V(u) := \mathcal{E}_V(u, u)$. Since we do not require V to be nonnegative in Ω we need sufficient conditions to ensure the coercivity of \mathcal{Q}_V . Using the assumptions on A and Lemma 2.2 with $W = V^-$ we have that for every $0 < \varepsilon < \min\{1, a_1\}$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} \mathcal{Q}_V(u) &\geq a_1 \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} V^+ u^2 \, dx - \varepsilon \int_{\Omega} |\nabla u|^2 \, dx - C_\varepsilon \int_{\Omega} u^2 \, dx \\ &= (a_1 - \varepsilon) \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} (V^+ - C_\varepsilon) u^2 \, dx \\ &\geq (a_1 - \varepsilon) \lambda_1(\Omega) \int_{\Omega} u^2 \, dx + \int_{\Omega} (V^+ - C_\varepsilon) u^2 \, dx, \end{aligned} \tag{2.7}$$

where in the last inequality we used the Poincaré inequality and $\lambda_1(\Omega)$ denotes the first Dirichlet eigenvalue of $-\Delta$ in Ω . This implies that the form \mathcal{Q}_V is semibounded from below. If Ω is unbounded, the condition $e^{-tV} \in L^1(\Omega)$ also implies that V is confining, i.e. $V(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$, which means that there exists $R_\varepsilon > 0$ such that for every $x \in \Omega \setminus B_{R_\varepsilon}(0)$ it holds that $V^+(x) - C_\varepsilon \geq 0$. Therefore,

$$\mathcal{Q}_V(u) \geq (a_1 - \varepsilon) \lambda_1(\Omega) \|u\|_{L^2(\Omega)}^2.$$

Conversely, if Ω is bounded we equivalently consider the form $\mathcal{Q}_{V_\varepsilon}$ associated with the shifted operator $\mathcal{H}_{V_\varepsilon} = -\operatorname{div}(A \nabla) + V_\varepsilon$, with $V_\varepsilon := V + C_\varepsilon$. In this case the form $\mathcal{Q}_{V_\varepsilon}$ is trivially coercive. In any case we have that the embedding

$$D(\mathcal{Q}_V) := H_0^1(\Omega) \cap L^2(\Omega, V \mathcal{L}^N) \hookrightarrow L^2(\Omega) \tag{2.8}$$

is compact and the spectrum is discrete. Therefore, the eigenvalues of \mathcal{H}_V define an increasing sequence

$$0 < \lambda_{1,V}(\Omega) \leq \lambda_{2,V}(\Omega) \leq \dots \leq \lambda_{k,V}(\Omega) \leq \dots,$$

and we can write $\lambda_{1,V}(\Omega)$ in terms of the Rayleigh quotient

$$\lambda_{1,V}(\Omega) = \inf_{u \in D(\mathcal{Q}_V) \setminus \{0\}} \frac{\mathcal{Q}_V(u)}{\|u\|_{L^2(\Omega)}^2}.$$

We refer to [24] for a complete characterization of the spectrum of \mathcal{H}_V .

We conclude this Section with the following results that allow to obtain further information on the spectral properties of the Schrödinger semigroup

Proposition 2.12. *Let $D \subset \mathbb{R}^N$ an open connected set and $(e^{-t\mathcal{H}_V^D})_{t>0}$ the semigroup generated by $-\mathcal{H}_V$ with Dirichlet condition in $L^2(D)$. Then, $(e^{-t\mathcal{H}_V^D})_{t>0}$ is irreducible.*

Proof. The statement follows using the fact that the bilinear form \mathcal{E}_V is a perturbation of \mathcal{E}_0 and from Theorem 2.8. \square

Remark 2.13. Since $(e^{-t\mathcal{H}_V^D})_{t>0}$ is a positive, selfadjoint and irreducible semigroup in $L^2(D)$ it follows that

$$\omega_0 = s(-\mathcal{H}_V^D) := \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(-\mathcal{H}_V^D)\} = -\lambda_{1,V}(D)$$

where $\lambda_{1,V}(D)$ is strictly positive and simple by Krein-Rutman Theorem. See, for instance, [16, Proposition 3.4, Chapter VI].

3. LOG-CONCAVITY OF THE HEAT KERNEL

In this Section we give the proof of Theorem 1. As already said in Section 1, the constancy assumption on the second order coefficients is sharp in order to ensure the log concavity of the heat kernel as given by the following Theorem (See [20, Theorem 1.2]).

Theorem 3.1. *Let $Q(x) = (q_{ij}(x))_{i,j=1,\dots,N}$ and $\beta(x) = (\beta_h(x))_{h=1,\dots,N}$ be such that $q_{ij}, \beta_h \in C^{2+\delta}(B_R)$ for some $\delta > 0$ and for every $R > 0$, $i, j, h = 1, \dots, N$. Let $L = \operatorname{Tr}(Q(\cdot)D^2) + \beta(\cdot) \cdot \nabla$ and let $(e^{tL})_{t>0}$ be the associated semigroup. Assume that $(e^{tL})_{t>0}$ sends log-concave functions to log-concave functions for every $t > 0$. Then Q is constant and β is affine.*

Now we are ready to prove our main result

Proof of Theorem 1. We follow the computations done in [6]. Let $p_{V,r}(t, x, y)$ be the fundamental solution of

$$\begin{cases} U_t + \mathcal{H}_V U = 0 & \text{in } (0, +\infty) \times \Omega_r \times \Omega_r \\ U(0, x, y) = \delta(x - y) & \text{in } \Omega_r \times \Omega_r \\ U(t, x, y) = 0 & \text{in } (0, +\infty) \times \partial\Omega_r \times \Omega_r \\ U(t, x, y) = 0 & \text{in } (0, +\infty) \times \Omega_r^c \times \Omega_r \cup (0, +\infty) \times \Omega_r \times \Omega_r^c, \end{cases}$$

where Ω_r is the convex sum of Ω_0 and Ω_1 and $\mathcal{H}_{V,r} := \mathcal{H}_V^{\Omega_r}$ denotes the realization of \mathcal{H}_V with Dirichlet boundary conditions on Ω_r . We notice that since A is constant, symmetric and positive definite, the heat kernel of $\operatorname{div}(A\nabla)$ in \mathbb{R}^N is given for every $t > 0$ and $x, y \in \mathbb{R}^N$ by

$$p_A(t, x, y) = \frac{1}{\sqrt{\det A}} \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|A^{-1/2}(x-y)|^2}{4t}}.$$

With this notation in force the heat kernel of $-\mathcal{H}_{V,r}$ is given by the Trotter perturbation formula, (see e.g. [15, Corollary 5.8])

$$p_{V,r}(t, x, y) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^{N(n-1)}} \left(\prod_{k=1}^n p_A \left(\frac{t}{n}, x_k, x_{k-1} \right) e^{-\frac{t}{n} V(x_k)} \chi_{\Omega_r}(x_k) \right) dx_1 \dots dx_{n-1}$$

where $x_0 := x$ and $x_n := y$.

We recall that the trace of the semigroup $(e^{-t\mathcal{H}_{V,r}})_{t>0}$ is given by

$$\mathrm{Tr}(e^{-t\mathcal{H}_{V,r}}) = \int_{\Omega_r} p_{V,r}(t, x, x) dx =: Z(r, t),$$

and the function $Z(r, t)$ is log-concave in r for every $t > 0$, which means that

$$Z(r, t) \geq Z(0, t)^{1-r} Z(1, t)^r,$$

for every $r \in [0, 1]$ and $t > 0$. Since the semigroup $(e^{-t\mathcal{H}_{V,r}})_{t>0}$ is trace class, we have that

$$Z(r, t) = \sum_{k=1}^{\infty} e^{-t\lambda_{k,V}(\Omega_r)} < \infty,$$

for every $t > 0$. In particular

$$\lambda_{1,V}(\Omega_r) = - \lim_{t \rightarrow \infty} \frac{\log Z(r, t)}{t}$$

and $\lambda_1(r)$ is convex in r since it is the pointwise limit of convex functions. \square

Example 3.2 (Kolmogorov Operators). Let $A, B \in \mathbb{R}^{N,N}$ be constant and symmetric matrices with A positive definite and $b_0 \in \mathbb{R}^N$. Consider the Kolmogorov operator

$$\mathcal{L} = \mathrm{div}(A\nabla) - b \cdot \nabla,$$

being $b(x) := Bx + b_0$, endowed with Dirichlet boundary condition on $\partial\Omega$ and we refer to it by \mathcal{L}^Ω . We notice that the semigroup $(e^{t\mathcal{L}^\Omega})_{t>0}$ satisfies the hypotheses of Theorem 3.1. Furthermore, the transformation

$$\begin{aligned} U_\varphi : L^2(\Omega, e^{-2\varphi} \mathcal{L}^N) &\rightarrow L^2(\Omega) \\ f &\mapsto e^{-\varphi} f, \end{aligned} \tag{3.1}$$

where

$$\varphi(x) := \frac{A^{-1}Bx \cdot x}{4} + \frac{A^{-1}b_0 \cdot x}{2}, \tag{3.2}$$

defines an isometry between $L^2(\Omega, e^{-2\varphi} \mathcal{L}^N)$ and $L^2(\Omega)$.

Set $V_\varphi(x) := |\nabla\varphi(x)|^2 - \Delta\varphi(x) = \frac{|A^{-1}b(x)|^2}{4} - \frac{\mathrm{Tr}(A^{-1}B)}{2}$. It is easy to check that $\mathcal{H}_{V_\varphi}(e^{-\varphi}) = 0$ in \mathbb{R}^N , and so the isometry U_φ yields

$$-\mathcal{L}^\Omega = U_\varphi^{-1} \mathcal{H}_{V_\varphi}^\Omega U_\varphi.$$

Since A and V_φ fulfill the hypotheses given in Section 2 and \mathcal{L}^Ω has the same spectrum as of $\mathcal{H}_{V_\varphi}^\Omega$, Theorem 1 applies to the first Dirichlet eigenvalue of $-\mathcal{L}$. In particular, when $A = B = Id$ and $b_0 = 0$ we have that $V_\varphi(x) = \frac{|x|^2}{4} - \frac{N}{2}$, which is the potential of the shifted harmonic oscillator. In this case \mathcal{L}^Ω reduces to the Dirichlet Ornstein-Uhlenbeck operator Δ_γ^Ω and we recover the Brunn-Minkowski inequality for $\lambda_{1,\gamma}(\Omega)$ proved in [12, 13].

Example 3.3 (Schrödinger operator with singular potential). Let $N \geq 3$, $x_0 \in \mathbb{R}^N$ and $V_{x_0}(x) := \frac{C}{|x-x_0|^2}$ for some $C > 0$ and assume Ω to be bounded and that $x_0 \in \Omega$. Consider the associated quadratic form $\mathcal{Q}_{V_{x_0}}$ which is coercive thanks to the Poincaré inequality. Moreover, from the Hardy inequality

$$\int_{\Omega} V_{x_0} u^2 dx \leq \left(\frac{N-2}{2} \right)^2 \int_{\Omega} |\nabla u|^2 dx,$$

it follows that $D(\mathcal{Q}_V) = H_0^1(\Omega)$, and the Rellich-Kondrachov Theorem ensures the compactness of the resolvent and hence the discreteness of the spectrum. Finally, it is an easy task to check that V satisfies Assumptions [A.1](#), [A.2](#), [A.3](#). Therefore, Theorem [1](#) holds true.

3.1. Log-concavity of the ground state. In this subsection we prove, as a consequence of Theorem [1](#), the log-concavity of the first eigenfunction $\psi_{1,V}$ of \mathcal{H}_V^Ω . As a consequence of example [3.2](#) we have that $U_\varphi^{-1}\psi_{1,V} = e^\varphi\psi_{1,V}$, where U_φ denotes the isometry [\(3.1\)](#), is the first Dirichlet eigenfunction for the Kolmogorov operator $-\operatorname{div}(A\nabla) + b \cdot \nabla$. In the particular case $A = Id$ and $b(x) = x$ the log-concavity of $-\Delta_\gamma^\Omega$ has been proved with different methods in [\[8, 11–13\]](#). Moreover, in the very recent paper [\[28\]](#) the author proves that $e^\varphi\psi_{1,V}$ enjoys a strong log-concavity property if Ω is bounded and convex. In all these papers, the set Ω is assumed to be bounded because of the unboundedness of the weight e^φ . Here, we prove the log-concavity of $\psi_{1,V}$ skipping the boundedness assumption on Ω and exploiting the ultracontractivity of the semigroup $(e^{-t\mathcal{H}_V^\Omega})_{t>0}$ which is not preserved by the isometry [\(3.1\)](#) and does not hold for Ornstein-Uhlenbeck type semigroups without the further assumption of *intrinsic* ultracontractivity. We refer e.g. to the work [\[14\]](#).

Before to state and prove our Theorem, we recall the following result due to Prékopa, [\[27, Theorem 6\]](#).

Theorem 3.4. *Let $N_1, N_2 \in \mathbb{N}$, $A \subseteq \mathbb{R}^{N_1}$, $B \subseteq \mathbb{R}^{N_2}$ two convex sets and $f : A \times B \rightarrow (0, \infty)$ a log-concave function. Then, the function*

$$F(x) := \int_B f(x, y) dy,$$

is log-concave in A .

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^N$ be an open and connected convex set and let $\psi_{1,V}$ be the first eigenfunction of \mathcal{H}_V^Ω in Ω . Then $\psi_{1,V}$ is strictly positive in Ω and log-concave in $\bar{\Omega}$.*

Proof. Consider the following Cauchy-Dirichlet problem

$$\begin{cases} u_t + \mathcal{H}_V u = 0 & \text{in } \Omega \times (0, \infty) \\ u(x, t) = 0 & \text{in } \partial\Omega \times (0, \infty) \\ u(x, 0) = f(x) & \text{in } \Omega \end{cases} \quad (3.3)$$

for some $f \in L^2(\Omega)$, $f \geq 0$, f log-concave in Ω . The solution to (3.3) is then given by

$$u(x, t) = e^{-t\mathcal{H}_V^\Omega} f(x) = \int_{\Omega} p_V^\Omega(t, x, y) f(y) dy.$$

By Theorem 3.4, since the function $\Omega \times \Omega \ni (x, y) \mapsto p_V^\Omega(t, x, y) f(y)$ is log-concave for every $t > 0$, so does $u(x, t)$ for every $x \in \Omega$, $t > 0$. Using the spectral representation of p_V^Ω it follows that

$$e^{\lambda_{1,V}(\Omega)t} u(x, t) = (f, \psi_{1,V}) \psi_{1,V}(x) + \sum_{k=2}^{\infty} e^{-(\lambda_{k,V}(\Omega) - \lambda_{1,V}(\Omega))t} (f, \psi_{k,V})_{L^2(\Omega)} \psi_{k,V}(x).$$

Therefore, by Cauchy-Schwarz inequality we have that

$$|e^{\lambda_{1,V}(\Omega)t} u(x, t) - (f, \psi_{1,V})_{L^2(\Omega)} \psi_{1,V}(x)| \leq M(t) \|f\|_{L^2(\Omega)} \sqrt{Z_\Omega(1)} \left(\sum_{k=2}^{\infty} e^{-\lambda_{k,V}(\Omega)t} |\psi_{k,V}(x)|^2 \right)^{1/2},$$

where $M(t) := \sup_{k \geq 2} e^{-(\lambda_{k,V}(\Omega) - \lambda_{1,V}(\Omega))t + \lambda_{k,V}(\Omega)t}$, for every $t > 0$ and every $x \in \Omega$. In particular, set $I(x, t) := e^{\lambda_{1,V}(\Omega)t} u(x, t) - (f, \psi_{1,V})_{L^2(\Omega)} \psi_{1,V}(x)$, since the sequence $(\lambda_{k,V}(\Omega))_{k \in \mathbb{N}}$ is increasing and the ultracontractivity estimate (2.5) holds true, for every $t \geq 1$ we have that

$$\begin{aligned} |I(x, t)| &\leq e^{-(\lambda_{2,V}(\Omega) - \lambda_{1,V}(\Omega))t + \lambda_{2,V}(\Omega)t} \|f\|_{L^2(\Omega)} \sqrt{Z_\Omega(1)} \sqrt{p_V^\Omega(1, x, x)} \\ &\leq C(\|f\|_{L^2(\Omega)}, \|e^{-V}\|_{L^1(\Omega)}, \omega_0, a_1) e^{-(\lambda_{2,V}(\Omega) - \lambda_{1,V}(\Omega))t + \lambda_{2,V}(\Omega)t}. \end{aligned} \quad (3.4)$$

Passing to the supremum with respect to $x \in \Omega$ in (3.4) and letting $t \rightarrow \infty$ we have that $\psi_{1,V}$ is log-concave in $\bar{\Omega}$ since is the uniform limit of a sequence of log-concave functions. To conclude, $\psi_{1,V}$ is also strictly positive since for every $t > 0$ and every $x \in \Omega$ we have that

$$\psi_{1,V}(x) = e^{\lambda_{1,V}(\Omega)t} (e^{-t\mathcal{H}_V^\Omega} \psi_{1,V})(x).$$

Then, the result plainly follows by Proposition 2.12. \square

Corollary 3.6. *Assume all the hypotheses of Theorem 3.5 are satisfied and also that Ω is bounded. Then, the first Dirichlet eigenfunction of the Kolmogorov operator $-\operatorname{div}(A\nabla) + b \cdot \nabla$ is log-concave in Ω .*

Proof. The proof follows by Theorem 3.1, Example 3.2 and Theorem 3.5. \square

We conclude the paper with the following Proposition 3.7 which is an enhancement of Theorem 3.5. We refer the reader to [12, Section 4] to remark how the geometrical assumptions therein can be partially rephrased in our framework in terms of the potential V .

Proposition 3.7. *Assume in Theorem 3.5 that Ω is a C^2 bounded set with strictly positive Gauss curvature and that the potential V is strongly convex, with $V \in L^\infty(\Omega) \cap C_{\text{loc}}^{2,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. Then $\psi_{1,V}$ is strongly log-concave in $\bar{\Omega}$.*

Proof. Since $V \in L^\infty(\Omega) \cap C_{\text{loc}}^{2,\alpha}(\Omega)$ by classical Schauder Theory and bootstrap regularity we have that $\psi_{1,V} \in C_{\text{loc}}^{4,\alpha}(\Omega) \cap C^{1,\beta}(\bar{\Omega})$, for some $\beta \in (0, 1)$ then, $w := -\log \psi_{1,V} \in C_{\text{loc}}^{4,\alpha}(\Omega) \cap C^{1,\beta}(\Omega)$. Moreover, w solves the following problem

$$\begin{cases} \operatorname{div}(A\nabla v) = A\nabla v \cdot \nabla v + \lambda_{1,V}(\Omega) - V & \text{in } \Omega \\ \lim_{x \rightarrow y \in \partial\Omega} v(x) = +\infty. \end{cases} \quad (3.5)$$

Setting $W := D^2w$, we already know that $W(x)\xi \cdot \xi \geq 0$ for every $x \in \Omega$ and every $\xi \in \mathbb{R}^N$. By differentiating twice in x equation (3.5) we obtain that the entries of W solve

$$\operatorname{div}(A\nabla W_{ij}) - 2A\nabla w \cdot \nabla W_{ij} - 2(AW^2)_{ij} + V_{ij} = 0 \quad \text{in } \Omega \quad (3.6)$$

for every $i, j = 1, \dots, N$.

We want to prove that $W(x)\xi \cdot \xi > 0$ for every $x \in \Omega$ and every $\xi \in \mathbb{R}^N \setminus \{0\}$. Let $\mu(x)$ be the smallest eigenvalue of $W(x)$. Suppose by contradiction that there exists an interior point $x_0 \in \Omega$ such that $\mu(x_0) = 0$.

Possibly by a linear change of variables we may assume without loss of generality that at the point x_0 , the Hessian $W(x_0)$ is diagonal, and the x_1 -direction corresponds to the minimum eigenvalue $\mu(x_0)$. Therefore:

$$W_{11}(x_0) = \mu(x_0) = 0. \quad (3.7)$$

Since x_0 is a local minimum for the function $W_{11}(x)$, necessary conditions of minimality imply:

$$\nabla W_{11}(x_0) = 0, \quad (3.8)$$

$$\operatorname{div}(A\nabla W_{11})(x_0) \geq 0. \quad (3.9)$$

We now evaluate equation (3.6) for $(i, j) = (1, 1)$ at the point x_0 :

$$\operatorname{div}(A\nabla W_{11})(x_0) - 2A\nabla w(x_0) \cdot \nabla W_{11}(x_0) - 2(AW^2)_{11}(x_0) + V_{11}(x_0) = 0. \quad (3.10)$$

Using that $(W^2)_{11}(x_0) = W_{11}^2(x_0) = 0$ and putting (3.7), (3.8) into (3.10) the equation reduces to:

$$\operatorname{div}(A\nabla W_{11})(x_0) = -V_{11}(x_0) < 0, \quad (3.11)$$

where $V_{11}(x_0) > 0$ by the strong convexity of V . But (3.11) is in contradiction with (3.9). Therefore, by the Constant Rank Theorem, (see e.g. [22, Theorem 1]), there exists $\rho \in \{0, \dots, N-1\}$ such that $\operatorname{Rank}(W) = \rho$ in Ω , and so the function w is constant along $N - \rho$ directions or affine in at least one direction. In particular, if $\rho = 0$ the function w is constant along N coordinate directions, otherwise if $\rho > 0$, for every $x \in \Omega$ there exists at least a line r_x on which the function w is affine. Since $w \in C^2(\Omega)$ and $w(x) \rightarrow +\infty$, as $x \rightarrow y \in \partial\Omega$ it follows that $\operatorname{Rank}(W) = N$ in Ω which implies that $w = -\log \psi_{1,V}$ is strongly convex in Ω .

To conclude, $\mu(x) \rightarrow +\infty$ as $x \rightarrow y \in \partial\Omega$ and so it cannot degenerate on $\partial\Omega$ as proved in [21, Section 2]. \square

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