MEASURE PRESERVING MAPS WITH BOUNDED TOTAL VARIATION

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Abstract. Consider a piecewise affine Lipschitz map $\phi : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^d$ is an open set, and assume that $x \mapsto x + t\nabla\phi(x)$ is injective for almost every $t > 0$. In (J.-G. Liu, R. L. Pego, *Rigidly breaking potential flows and a countable Alexandrov theorem for polytopes*, Pure Appl. Anal., **7**(4), 2025) the authors conjecture that every such ϕ must be locally convex. We prove the result assuming additionally $\nabla\phi \in BV_{loc}(\Omega)$, for a more general class of measure preserving maps.

1. Introduction

In [LP25] the authors consider the following problem: let $\Omega \subset \mathbb{R}^d$ be an open set, and consider masses $\{m_i\}_{i\in\mathbb{N}}$ and velocities $\{v_i\}_{i\in\mathbb{N}}$ such that

$$\sum_i m_i = \mathscr{L}^d(\Omega)$$

where $\mathscr{L}^d$ is the d-dimensional Lebesgue measure. It is a classical result in optimal transport [Bre91] that there is a unique convex map $\nabla\phi$ such that

$$\mathscr{L}^d\Big(\big\{x \in \Omega \mid \nabla\phi(x) = v_i\big\}\Big) = m_i.$$

Since ϕ is convex, the map $x \mapsto x + t\nabla\phi(x)$ is injective for all $t > 0$ on the set $F \subset \Omega$, where

$$F = \Big\{x \in \Omega \mid \phi \text{ is differentiable at } x\Big\}.$$

Clearly F is of full measure in Ω, i.e. $|\Omega \setminus F| = 0$.

Conversely, let a partition of Ω be given, up to a negligible set,

$$\Omega = \bigcup_i A_i, \qquad A_i \text{ open sets}$$

together with a Lipschitz map ϕ, affine on each A_i, such that $x \mapsto x + t\nabla\phi(x)$ is injective on F for a.e. $t > 0$. The following conjecture was put forward in [LP25]:

Conjecture 1.1. *Every such map ϕ must necessarily be locally convex in Ω.*

Our aim here is to prove the result for maps ϕ with $\nabla\phi \in BV_{loc}$:

Theorem 1.2. *In the above setting, if $\nabla\phi \in BV_{loc}(\Omega)$, ϕ must be locally convex.*

Proving the result for $\nabla\phi \notin BV$ is presently an almost completely open problem. The only non-BV case is proved in [LP25] assuming only continuity of the gradient, i.e. $\nabla\phi \in C(\Omega; \mathbb{R}^d)$.

The paper is structured as follows. In Section 3 we show that every such ϕ must be subharmonic, this holds without the BV assumption. In Section 2 we conclude the proof by showing, using the theory of BV functions, that ϕ must be locally convex. In Section 4 we give a different proof of the subharmonicity of ϕ, based on optimal transport tools, which yields an additional property of ϕ in the case $\nabla\phi \notin BV$.

2. MEASURE PRESERVING MAPS AND MAIN RESULT

Our Theorem 1.2 will be a consequence of a more general result, valid for a class of measure preserving maps.

In all of the following, $\Omega \subset \mathbb{R}^d$ is an open set.

Definition 2.1. Let T be a measurable map $T : \Omega \to \mathbb{R}^d$ and $F \subset \Omega$ measurable be given. We say that T is *measure preserving* on F if for all $A \subset \Omega$ measurable, $T(F \cap A)$ is measurable and

$$T_\sharp \mathscr{L}^d \llcorner (F \cap A) = \mathscr{L}^d \llcorner T(F \cap A). \tag{2.1}$$

Definition 2.2. Let $\phi : \Omega \to \mathbb{R}$ be a $\mathbf{W}^{1,1}(\Omega)$ function. We say that ϕ satisfies the *measure preserving condition* (in short (**MPC**)) if there is a full measure set $F \subset \Omega$ for which

$$T_t : \Omega \to \mathbb{R}^d, \qquad T_t(x) := x + t\nabla\phi(x) \tag{2.2}$$

is measure preserving on F for a.e. $t \geq 0$.

We prove the following theorem.

Theorem 2.3. *Let $\phi \in \mathbf{W}^{1,1}(\Omega)$ be a map satisfying* (**MPC**)*, and assume that $\nabla\phi \in BV_{loc}(\Omega, \mathbb{R}^d)$. Then ϕ is locally convex in Ω.*

Clearly, Theorem 1.2 follows from (2.3). We would expect every function $\phi \in \mathbf{W}^{1,1}(\Omega)$ belonging to the more general class of maps satisfying the measure preserving condition of Definition 2.2 to be locally convex, but we have been unable to prove it.

The proof of Theorem 2.3 consists of two steps. As a first point, we prove the following proposition, which is valid without the BV assumption.

Proposition 2.4. *Let $\phi \in \mathbf{W}^{1,1}(\Omega)$ be a map satisfying* (**MPC**)*. Then ϕ is subharmonic in Ω, that is,*

$$-\int_\Omega \nabla\phi(x) \cdot \nabla\psi(x)\,\mathrm{d}x \geq 0 \qquad \text{for all } \psi \in C^1_c(\Omega), \qquad \psi \geq 0. \tag{2.3}$$

Unfortunately, it is well known that the gradient of subharmonic functions does not need to lie in BV_{loc}, as it can be readily seen by looking at the fundamental solution of the Poisson equation

$$\Delta\Phi = \delta_0, \qquad \Phi = \frac{1}{(d-2)\omega_d}\frac{1}{|x|^{d-2}}.$$

Therefore presently the BV_{loc} assumption in Theorem 2.3 cannot be removed.

We will prove Proposition 2.4 in Section 3, for a more general class of maps which we call *expanding*.

Using Proposition 2.4 we can prove Theorem 2.3.

Proof of Theorem 2.3. By the standard theory of BV functions, see [AFP00], since by assumption $\nabla\phi \in BV_{loc}(\Omega; \mathbb{R}^d)$, we can decompose the derivative $D(\nabla\phi)$ as

$$D(\nabla\phi) = D^a(\nabla\phi) + D^s(\nabla\phi) \in \mathscr{M}(\Omega, \mathbb{R}^{d\times d})$$

where $D^a(\nabla\phi)$ is the absolutely continuous part of $D(\nabla\phi)$ with respect to the Lebesgue measure $\mathscr{L}^d$, and $D^s(\nabla\phi)$ is the singular part. Here $\mathscr{M}(\Omega, \mathbb{R}^{d\times d})$ is the space of locally finite matrix valued measures. In the following we analyze them separately.

1. *(Singular part).* We have that

$$D^s(\nabla\phi) = \frac{\mathrm{d}D^s(\nabla\phi)}{\mathrm{d}|D^s(\nabla\phi)|}|D^s(\nabla\phi)|$$

where $\frac{\mathrm{d}D^s(\nabla\phi)}{\mathrm{d}|D^s(\nabla\phi)|}$ is the polar matrix, defined for $|D^s(\nabla\phi)|$-a.e. x. By the Alberti's rank 1 theorem [AFP00, Theorem 3.94] for $|D^s(\nabla\phi)|$ a.e. $x\in\Omega$ there exist $a(x), b(x)\in\mathbb{R}^n\setminus\{0\}$ such that

$$\frac{\mathrm{d}D^s(\nabla\phi)}{\mathrm{d}|D^s(\nabla\phi)|} = a(x)\otimes b(x)$$

Since the polar matrix $\frac{\mathrm{d}D^s(\nabla\phi)}{\mathrm{d}|D^s(\nabla\phi)|}$ must be symmetric, we deduce the existence of $\lambda(x)\in\mathbb{R}\setminus\{0\}$ and $\vec{n}(x)\in\mathbb{R}^d\setminus\{0\}$ such that $\lambda(x)\vec{n}(x)\otimes\vec{n}(x) = a(x)\otimes b(x)$ for $|D^s(\nabla\phi)|$ a.e. $x\in\Omega$, therefore

$$D^s(\nabla\phi) = \lambda\vec{n}\otimes\vec{n}\;|D^s(\nabla\phi)| \tag{2.4}$$

and the Laplacian of ϕ is

$$\Delta\phi = \mathrm{Tr}(D^a(\nabla\phi)) + \lambda|D^s(\nabla\phi)|.$$

By Proposition 2.4 we deduce in particular that

$$\lambda(x)\geq 0 \qquad \text{for } |D^s(\nabla\phi)|\text{-almost every } x\in\Omega$$

therefore the polar $\frac{\mathrm{d}D^s(\nabla\phi)}{\mathrm{d}|D^s(\nabla\phi)|}$ is a positive definite matrix and by (2.4) $D^s(\nabla\phi)$ is a positive definite matrix valued measure, since for every continuous vector field $\xi\in C_c(\Omega;\mathbb{R}^d)$, there holds

$$\int_\Omega \xi^\perp(x)\cdot\frac{\mathrm{d}D^s(\nabla\phi)}{\mathrm{d}|D^s(\nabla\phi)|}(x)\cdot\xi(x)\,\mathrm{d}\,|D^s(\nabla\phi)|(x)\geq 0.$$

2. *(Absolutely continuous part).* We will show that in fact the absolutely continuous part is zero. We will use the following well-known Lusin-Lipschitz properties for BV functions (see [AFP00, Theorem 5.34]). In our setting, since $\nabla\phi\in BV(\Omega,\mathbb{R}^n)$, there exists a constant κ (depending only on the dimension d) such that for every $\Lambda>0$ there exists a Lipschitz function f^Λ with Lipschitz constant less then $\kappa\Lambda$, such that

$$\mathscr{L}^d\Big(\big\{x\in\Omega \mid \nabla\phi(x)\neq f^\Lambda(x)\big\}\Big)\leq\frac{\kappa}{\lambda}|D\nabla\phi|(\Omega).$$

Letting $G^\Lambda := \{x\in\Omega \mid f(x) = \nabla\phi(x)\}$, we also have the property

$$G_1^\Lambda\subset G_2^\Lambda \qquad \text{if } \Lambda_1\leq\Lambda_2.$$

We have that $f^\Lambda(x)-\nabla\phi(x)=0$ for a.e. $x\in G^\Lambda$, therefore by the locality of the gradient of BV functions we obtain

$$Df^\Lambda \llcorner G^\Lambda = D(\nabla\phi)\llcorner G^\Lambda$$

so that in particular $D^s(\nabla\phi)\llcorner G^\Lambda = 0$, and moreover for the absolutely continuous part there holds

$$Df^\Lambda(x) = D^a(\nabla\phi)(x) \qquad \text{for } \mathscr{L}^d \text{ almost every } x\in G^\Lambda. \tag{2.5}$$

Define now the Lipschitz function

$$T_t^\Lambda(x) := x + tf^\Lambda(x).$$

We have now that, using the standard area formula for the Lipschitz function f^Λ, for every measurable $A \subset \Omega$ there holds

$$\begin{aligned}\int_{A\cap G^\Lambda} |\det D^a T_t(x)|\mathrm{d}x &= \int_{A\cap G^\Lambda} |\det DT_t^\Lambda(x)|\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \sharp\{x \in G^\Lambda \cap A \mid T_t^\Lambda(x) = y\}\mathrm{d}y \qquad (2.6)\\ &= \int_{\mathbb{R}^d} \sharp\{x \in G^\Lambda \cap A \mid T_t(x) = y\}\mathrm{d}y.\end{aligned}$$

Notice that for every disjoint measurable $A, B \subset F$, the measure preserving condition (2.2) implies that $|T_t(A) \cup T_t(B)| = |T_t(A)| + |T_t(B)|$ and therefore $T_t(A)$ and $T_t(B)$ are essentially disjoint; in particular this yields

$$\sharp\{x \in G^\Lambda \cap A \mid T_t(x) = y\} = 1 \qquad \text{for a.e. } y \in \mathbb{R}^d.$$

In fact, assume by contradiction that for a positive measure set $B \subset \mathbb{R}^d$ there are at least two elements $x_1(y) \neq x_2(y)$ such that $T_t(x_1(y)) = T_t(x_2(y)) = y$. Then, by the measurable selection theorem, there are two measurable sets $A_1, A_2 \subset \Omega$ such that $T_t(A_1) = B = T_t(A_2)$ up to negligible sets, which is a contradiction. Therefore we conclude, by letting $\Lambda \to +\infty$, that

$$\int_A |\det D^a T_t(x)|\mathrm{d}x = \int_{T_t(A)} 1\mathrm{d}y = |T_t(A)| = |A|$$

which implies that

$$|\det D^a T_t(x)| = 1 \qquad \text{for a.e. } x \in \Omega.$$

Therefore we deduce that for almost every $x \in \Omega$

$$1 = |\det(\mathrm{Id}_{\mathbb{R}^d} + tD^a(\nabla\phi)(x))| = \prod_i |(1 + t\lambda_i(x))| \qquad (2.7)$$

where $\lambda_i(x)$, $i = 1, \dots, d$ are the eigenvalues of $D^a(\nabla\phi)(x)$. Repeating the argument for a dense and countable set of times $t_j \geq 0$, and using the continuity in t of the expressions in (2.7), we deduce that for $\mathscr{L}^d$ almost every $x \in \Omega$

$$1 = \prod_i |(1 + t\lambda_i(x))| \qquad \text{for all } t \geq 0. \qquad (2.8)$$

This clearly implies $\lambda_i(x) = 0$ for all $i = 1, \dots, d$. Since this holds for $\mathscr{L}^d$-almost every $x \in \Omega$, we deduce that $D^a(\nabla\phi) = 0$.

Finally, using the first step, we conclude

$$D(\nabla\phi) = D^s(\nabla\phi) \geq 0.$$

is a positive definite matrix valued measure. It is well known that positive definite distributional Hessian of ϕ is equivalent to ϕ being locally convex, therefore this concludes the proof. □

3. SUBHARMONICITY AND EXPANDING MAPS

The aim of this Section is to provide a proof of Proposition 2.4. We will prove it in the more general setting of *expanding maps*, that we now introduce.

Definition 3.1. We say that a measurable map $T : \Omega \to \mathbb{R}^d$ is *expanding* if

$$\rho := T_\sharp \mathscr{L}^d \llcorner \Omega \leq \mathscr{L}^d \qquad (3.1)$$

Remark 3.2. Clearly, if T is measure preserving on $F \subset \Omega$ (Definition 2.2) and F is of full measure in Ω, there holds

$$T_\sharp \mathscr{L}^d \llcorner F = T_\sharp \mathscr{L}^d \llcorner \Omega = \mathscr{L}^d \llcorner T(F) \leq \mathscr{L}^d$$

therefore measure preserving maps are also expanding according to Definition 3.1.

In the next proposition, we show that if a gradient $\nabla\phi$ is such that $x + t\nabla\phi(x)$ is expanding for a.e. $t \geq 0$, then ϕ is subharmonic:

Proposition 3.3. *Let $\phi : \Omega \to \mathbb{R}$ be in $\mathbf{W}^{1,1}(\Omega)$ and assume that $T_t(x) = x + t\nabla\phi(x)$ is expanding for a.e. $t \geq 0$. Then it holds*

$$\Delta\phi \geq 0 \qquad \text{in } \mathscr{D}'(\Omega). \tag{3.2}$$

By Remark 3.2, this also proves Proposition 2.4.

Proof. We need to show that for every $0 \leq \psi \in C_c^1(\Omega)$, there holds

$$\int_\Omega \nabla\phi(x)\nabla\psi(x)dx \leq 0. \tag{3.3}$$

Let $\rho_t = (T_t)_\sharp \mathscr{L}^d \llcorner \Omega$. Since T_t is expanding, we have that for all $\psi : \mathbb{R}^d \to \mathbb{R}^+$ such that $\psi = 0$ in $\overline{\Omega^c}$, there holds

$$\int_{\mathbb{R}^d} \psi(x)d\rho_t \leq \int_{\mathbb{R}^d} \psi(x)d\mathscr{L}^d = \int_{\mathbb{R}^d} \psi(x)d\rho_0. \tag{3.4}$$

We show that for every $\psi \in C_c^1(\Omega)$ compactly supported in Ω, the function $g^\psi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$g^\psi(t) := \int_{\mathbb{R}^d} \psi(x)d\rho_t$$

is Lipschitz and we compute its derivative. We have for all $\delta > 0$

$$\frac{1}{\delta}|\psi(x + (t+\delta)\nabla\phi(x)) - \psi(x + t\nabla\phi(x))| \leq L|\nabla\phi(x)| \tag{3.5}$$

where L is the Lipschitz constant of ψ. Then we have, using the definition of ρ_t,

$$\begin{aligned}
\lim_{\delta\to 0^+} \frac{g^\psi(t+\delta) - g^\psi(t)}{\delta} &= \lim_{\delta\to 0^+} \int_{\mathbb{R}^d} \psi(x)d\rho_{t+\delta} - \int_{\mathbb{R}^d} \psi(x)d\rho_t \\
&= \lim_{\delta\to 0^+} \frac{1}{\delta} \int_\Omega \psi(x + (t+\delta)\nabla\phi(x)) - \psi(x + t\nabla\phi(x))dx \\
&= \int_\Omega \lim_{\delta\to 0^+} \frac{1}{\delta} \psi(x + (t+\delta)\nabla\phi(x)) - \psi(x + t\nabla\phi(x))dx \\
&= \int_\Omega \nabla\psi(x + t\nabla\phi(x)) \cdot \nabla\phi(x)dx
\end{aligned}$$

where the penultimate equality follows from the dominated convergence theorem, which we could apply because of (3.5) since $|\nabla\phi| \in \mathbf{L}^1(\Omega)$.

Now using (3.4) we compute

$$\begin{aligned}
0 &\geq g^\psi(t) - g^\psi(0) \\
&= \int_0^t (g^\psi)'(s)ds \\
&= \int_0^t \int_\Omega \nabla\psi(x + s\nabla\phi(x)) \cdot \nabla\phi(x)dxds.
\end{aligned}$$

In particular we have

$$\begin{aligned} 0 &\geq \lim_{t\to 0^+} \frac{1}{t}\int_0^t \int_\Omega \nabla\psi(x+s\nabla\phi(x))\cdot\nabla\phi(x)dxds \\ &= \lim_{t\to 0^+} \int_\Omega \nabla\phi(x)\cdot\frac{1}{t}\int_0^t \nabla\psi(x+s\nabla\phi(x))dsdx \\ &= \int_\Omega \nabla\phi(x)\nabla\psi(x)dx \end{aligned}$$

where the last equality follows from the fact that

$$\frac{1}{t}\int_0^t \nabla\psi(x+s\nabla\phi(x))ds \longrightarrow \nabla\psi(x) \qquad \text{strongly in } \mathbf{L}^1(\Omega) \text{ as } t\to 0^+$$

because again by (3.5) we have

$$\begin{aligned} \int_\Omega \left|\frac{1}{t}\int_0^t \nabla\psi(x+s\nabla\phi(x)) - \nabla\psi(x)ds\right| dx &\leq \int_\Omega \frac{1}{t}\int_0^t Ls|\nabla\phi(x)|dsdx \\ &\leq Lt\int_\Omega |\nabla\phi(x)|dx. \end{aligned}$$

This concludes the proof.

□

4. Optimal transport approach

Here we provide a different proof of Proposition 2.4 based on optimal transport techniques.

Let us define the measures, for all $t>0$,

$$\mu := \mathscr{L}^d \llcorner \Omega, \qquad \nu_t := (T_t)_\sharp \mathscr{L}^d \llcorner \Omega.$$

In this section we need to assume that the first moment is finite, that is

$$\int |x|^2 \mathrm{d}\mu, \qquad \int |y|^2 \mathrm{d}\nu_t < \infty$$

which amounts to assume that $T_t \in \mathbf{L}^2(\Omega,\mathbb{R}^d)$, or equivalently $\nabla\phi \in \mathbf{L}^2(\Omega,\mathbb{R}^d)$, and that

$$\int_\Omega |x|^2 \mathrm{d}x < \infty.$$

From [Bre91] we know that there exists a unique convex function f_t such that $(\nabla f_t)_\sharp \mu = \nu_t$. As above, this means that there exists a set of full measure (which we can take to be equal to F, recall Definition 2.2) such that ∇f_t exists in F and

$$(\nabla f_t)_\sharp \mathscr{L}^d = \mathscr{L}^d \llcorner_{F_t}$$

where $F_t = \nabla f_t(F)$.

We recall the Polar factorization theorem by Brenier [Bre91].

Theorem 4.1. *Let $\xi : \Omega \to \mathbb{R}^d$ be a measurable map in $\mathbf{L}^2(\Omega;\mathbb{R}^d)$, such that $\xi_\sharp \mathscr{L}^d \llcorner \Omega$ is absolutely continuous with respect to the Lebesgue measure. There exists a unique decomposition*

$$\xi = \nabla u \circ \Phi$$

were $u : \Omega \to \mathbb{R}$ is a convex function and $\Phi : \Omega \to \Omega$ is measure preserving (i.e. $\Phi_\sharp \mathscr{L}^d \llcorner \Omega = \mathscr{L}^d \llcorner \Omega$). Moreover, ∇u is the optimal map for the quadratic transport cost pushing forward $\mathscr{L}^d \llcorner \Omega$ to $\xi_\sharp \mathscr{L}^d \Omega$.

We have the following proposition.

Proposition 4.2. *Let $\{X_t\}_{t>0}$ be a family of maps $X_t \in \mathbf{L}^2(\Omega, \mathbb{R}^d)$. Let $\mathbf{b} \in \mathbf{L}^2(\Omega, \mathbb{R}^d)$ be a vector field such that*

$$\lim_{t\to 0^+} \int_\Omega \left(\frac{X_t(x) - x}{t} - \mathbf{b}(x) \right)^2 \mathrm{d}x = 0. \tag{4.1}$$

Letting

$$X_t = \nabla f_t \circ \Phi_t$$

be the Polar Decomposition of X_t given by Theorem 4.1, it holds

$$\nabla h_t := \frac{\nabla f_t - \mathrm{id}}{t} \to \mathtt{P}_\nabla \mathbf{b}, \qquad \frac{\Phi_t - \mathrm{id}}{t} \to (\mathrm{id} - \mathtt{P}_\nabla)\mathbf{b} \quad \textit{in } \mathbf{L}^2 \textit{ as } t \to 0^+$$

where $\mathtt{P}_\nabla : \mathbf{L}^2(\Omega, \mathbb{R}^d) \to \mathbf{L}^2(\Omega, \mathbb{R}^d)$ is the projection on the space of gradients.

Proof. **1.** Let h_t be defined, up to constants, by

$$\nabla h_t := \frac{\nabla f_t - \mathrm{id}}{t}.$$

As a first step, we compute the weak limit of h_t as $t \to 0^+$. For any function $\psi \in C^2(\Omega)$, we compute

$$\begin{aligned} &\left| \int_\Omega \psi(x) \mathrm{d}[(\nabla f_t)_\sharp \mathscr{L}^d \llcorner \Omega](x) - \int_\Omega \Big(\psi(x) + t \nabla\psi(x) \cdot \nabla h_t(x) \Big) \mathrm{d}x \right| \\ &\qquad = \left| \int_\Omega \psi(x + t\nabla h_t(x)) \mathrm{d}x - \int_\Omega \Big(\psi(x) + t\nabla\psi(x) \cdot \nabla h_t(x) \Big) \mathrm{d}x \right| \\ &\qquad \leq \frac{t^2}{2} \|\nabla^2 \psi\|_\infty \|\nabla h_t\|^2_{\mathbf{L}^2} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\left| \int_\Omega \psi(x) \mathrm{d}[(X_t)_\sharp \mathscr{L}^d \llcorner \Omega](x) - \int_\Omega \Big(\psi(x) + \nabla\psi(x) \cdot (X_t(x) - x) \Big) \mathrm{d}x \right| \\ &\qquad \left| \int_\Omega \psi(X_t(x)) \mathrm{d}x - \int_\Omega \Big(\psi(x) + \nabla\psi(x) \cdot (X_t(x) - x) \Big) \mathrm{d}x \right| \\ &\qquad \leq \quad \frac{t^2}{2} \|\nabla^2 \psi\|_\infty \|X_t - \mathrm{id}\|^2_{\mathbf{L}^2}. \end{aligned} \tag{4.3}$$

Since ∇f_t and X_t have the same image measure

$$(\nabla f_t)_\sharp \mathscr{L}^d \llcorner \Omega = X_{t\sharp} \mathscr{L}^d \llcorner \Omega,$$

and using that, by optimality of ∇f_t,

$$\|\nabla h_t\|_{\mathbf{L}^2(\Omega)} = \left\| \frac{\nabla f_t - \mathrm{id}}{t} \right\|_{\mathbf{L}^2(\Omega)} \leq \frac{1}{t} \|X_t - \mathrm{id}\|_{\mathbf{L}^2(\Omega)}$$

we obtain from (4.3), (4.2) that

$$\begin{aligned} \left| \int_\Omega \nabla\psi(x) \cdot \left(\nabla h_t(x) - \frac{X_t(x) - x}{t} \right) \mathrm{d}x \right| &= \left| \int_\Omega \nabla\psi(x) \cdot \left(\nabla h_t - \mathtt{P}_\nabla \left(\frac{X_t - \mathrm{id}}{t} \right)(x) \right) \mathrm{d}x \right| \\ &\leq \; t \, \|\nabla^2 \psi\|_\infty \left\| \frac{X_t - \mathrm{id}}{t} \right\|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Finally, letting $t \to 0$ we conclude

$$\nabla h_t \longrightarrow \mathtt{P}_\nabla \mathbf{b} \quad \text{weakly in } \mathbf{L}^2(\Omega, \mathbb{R}^d) \text{ as } t \to 0^+ \tag{4.4}$$

2. The aim of this step is to show that ∇h_t converges strongly to $\mathtt{P}_\nabla \mathbf{b}$ by showing that

$$\limsup_{t\to 0^+} \|\nabla h_t\|_{\mathbf{L}^2} \le \|\mathtt{P}_\nabla \mathbf{b}\|_{\mathbf{L}^2}. \tag{4.5}$$

In order to prove (4.15), we first consider the Helmhotz decomposition of $\mathbf{b}$ as

$$\mathbf{b} = \nabla g + \mathbf{b}_1$$

where ∇g, for some $g \in \mathbf{W}^{1,2}(\Omega)$, is the $\mathbf{L}^2$-projection of $\mathbf{b}$ on the subspace of gradients, and $\mathbf{b}_1$ is a divergence free vector field.

Let $\mathbf{b}_{1,n} \in \mathbf{L}^2(\Omega, \mathbb{R}^d)$ be a sequence of smooth divergence free vector fields such that

$$\|\mathbf{b}_1 - \mathbf{b}_{1,n}\|_{\mathbf{L}^2(\Omega)} \le 2^{-n}, \quad \mathbf{b}_{1,n} = 0 \quad \text{on } \partial\Omega \tag{4.6}$$

and consider the corresponding flow map $\tilde{\Phi}^n_t : \Omega \to \Omega$, defined by

$$\frac{d}{dt}\tilde{\Phi}^n_t(x) = \mathbf{b}_{1,n}(\tilde{\Phi}^n_t(x)), \quad \tilde{\Phi}_n(0,x) = x.$$

Clearly, $\tilde{\Phi}^n_t$ is measure preserving,

$$(\tilde{\Phi}^n_t)_\sharp \mathscr{L}^2 \llcorner_\Omega = \mathscr{L}^2 \llcorner_\Omega$$

since $\mathbf{b}_{1,n}$ are divergence free. Moreover, define maps $\tilde{X}^n_t \in \mathbf{L}^2(\Omega, \mathbb{R}^d)$ by

$$\tilde{X}_n(t,x) = \tilde{\Phi}^n_t(x) + t\nabla g(\tilde{\Phi}^n_t(x)).$$

By definition we have

$$\tilde{X}^n_t = \nabla \tilde{f}_t \circ \tilde{\Phi}^n_t, \quad \tilde{f}(t,x) := \frac{x^2}{2} + tg(x).$$

Note that there holds by triangular inequality

$$\left(\int_\Omega \left|\tilde{\Phi}^n_t(x) + t\nabla g(\tilde{\Phi}^n_t(x)) - \Big(x + t\mathbf{b}_{1,n}(x) + t\nabla g(x)\Big)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}}$$
$$\le t\left(\int_\Omega \left|\frac{\tilde{\Phi}^n_t(x) - x}{t} - \mathbf{b}_{1,n}(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} + t\left(\int_\Omega |\nabla g(\tilde{\Phi}^n_t(x)) - \nabla g(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

Since $\mathbf{b}_{1,n}$ is smooth, one clearly has

$$\lim_{t\to 0^+}\left(\int_\Omega \left|\frac{\tilde{\Phi}^n_t(x) - x}{t} - \mathbf{b}_{1,n}(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} = 0.$$

We claim that also

$$\lim_{t\to 0^+}\left(\int_\Omega |\nabla g(\tilde{\Phi}^n_t(x)) - \nabla g(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} = 0. \tag{4.7}$$

This can be easily seen since $\tilde{\Phi}^n_t$ is measure preserving, via a standard argument as follows. Let $\varepsilon > 0$, then by density there is a Lipschitz function $F : \Omega \to \mathbb{R}^d$ such that

$$\|F - \nabla g\|_{\mathbf{L}^2} \le \varepsilon.$$

Therefore we estimate

$$\begin{aligned}\lim_{t\to 0^+}\left(\int_\Omega |\nabla g(\tilde{\Phi}^n_t(x)) - \nabla g(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} &\leq 2\left(\int_\Omega |\nabla g(x) - F(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\quad + \lim_{t\to 0^+}\left(\int_\Omega |F(\tilde{\Phi}^n_t(x)) - F(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq 2\varepsilon + \mathrm{Lip}\,(F)\lim_{t\to 0^+}\left(\int_\Omega |\tilde{\Phi}^n_t(x) - x|^2 \mathrm{d}x\right)^{\frac{1}{2}} = 2\varepsilon\end{aligned}$$

Since this holds for every $\varepsilon > 0$, (4.7) is proved. Therefore we have

$$\lim_{t\to 0^+}\frac{1}{t}\left(\int_\Omega \left|\tilde{\Phi}^n_t(x) + t\nabla g(\tilde{\Phi}^n_t(x)) - \Big(x + t\mathbf{b}_{1,n}(x) + t\nabla g(x)\Big)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} = 0 \tag{4.8}$$

We thus compute

$$\begin{aligned}\frac{1}{t}\|X_t - \tilde{X}^n_t\|_{\mathbf{L}^2} &\leq \frac{1}{t}\|X_t - (\mathrm{id} + t\nabla\mathbf{b})\|_{\mathbf{L}^2} \\ &\quad + \frac{1}{t}\|(\mathrm{id} + t\nabla\mathbf{b}) - (\mathrm{id} + t(\nabla g + \mathbf{b}_{1,n}))\|_{\mathbf{L}^2} \\ &\quad + \frac{1}{t}\|(\mathrm{id} + t(\nabla g + \mathbf{b}_{1,n})) - \tilde{X}^n_t\|_{\mathbf{L}^2} \\ &\leq o(1) + 2^{-n} + o(1)\end{aligned} \tag{4.9}$$

where $o(1)$ is a quantity approaching zero with $t \to 0$, and we used respectively (4.1) to estimate the first term, (4.6) to estimate the second term, and (4.8) to estimate the third term. Then we can estimate the 2-Wasserstein distance between $X_{t\sharp}\mathscr{L}^d\llcorner\Omega$ and $(\tilde{X}^n_t)_\sharp\mathscr{L}^d\llcorner\Omega$ as

$$\mathrm{dist}_{W_2}\left(X_{t\sharp}\mathscr{L}^d\llcorner\Omega, (\tilde{X}^n_t)_\sharp\mathscr{L}^d\llcorner\Omega\right) = \|X_t - \tilde{X}^n_t\|^2_{\mathbf{L}^2} \leq t^2 2^{-n+1} \tag{4.10}$$

for all $t > 0$ and sufficiently small. By the triangular inequality for the 2-Wasserstain distance, we obtain that

$$\begin{aligned}\mathrm{dist}_{W_2}\left(X_{t\sharp}\mathscr{L}^d\llcorner\Omega,\ \mathscr{L}^d\llcorner\Omega\right) &\leq \mathrm{dist}_{W_2}\left(X_{t\sharp}\mathscr{L}^d\llcorner\Omega, (\tilde{X}^n_t)_\sharp\mathscr{L}^d\llcorner\Omega\right) \\ &\quad + \mathrm{dist}_{W_2}\left(\mathscr{L}^d\llcorner\Omega,\ (\tilde{X}^n_t)_\sharp\mathscr{L}^d\llcorner\Omega\right) \\ &\leq t^2 2^{-n+1} + t^2\|\nabla g\|^2_{\mathbf{L}^2}\end{aligned} \tag{4.11}$$

where we have used that

$$\mathrm{dist}_{W_2}\left(\mathscr{L}^d\llcorner\Omega,\ (\tilde{X}^n_t)_\sharp\mathscr{L}^d\llcorner\Omega\right) \leq t^2\|\nabla g\|^2_{\mathbf{L}^2} \tag{4.12}$$

To see that (4.12) holds, first notice that since $(\tilde{\Phi}^n_t)$ is measure preserving, we deduce

$$(\nabla\tilde{f}_t)_\sharp\mathscr{L}^d\llcorner\Omega = (\nabla\tilde{f}_t)_\sharp \circ (\tilde{\Phi}^n_t)_\sharp\mathscr{L}^d\llcorner\Omega = (\tilde{X}^n_t)_\sharp\mathscr{L}^d\llcorner\Omega. \tag{4.13}$$

In particular $\nabla\tilde{f}_t$ is an admissible map between $\mathscr{L}^d\llcorner\Omega$ and $(\tilde{X}^n_t)_\sharp\mathscr{L}^d\llcorner\Omega$, with cost

$$\|\nabla\tilde{f}_t - \mathrm{id}\|^2_{\mathbf{L}^2} = t^2\|\nabla g\|^2_{\mathbf{L}^2}$$

and this proves (4.12). Since ∇f_t is the optimal map between $\mathscr{L}^d\llcorner\Omega$ and $(X_t)_\sharp\mathscr{L}^d\llcorner\Omega$, we deduce that for all $t > 0$ small

$$\|\nabla f_t - \mathrm{id}\|^2_{\mathbf{L}^2} = \mathrm{dist}_{W_2}\left(X_{t\sharp}\mathscr{L}^d\llcorner\Omega,\ \mathscr{L}^d\llcorner\Omega\right) \leq t^2\|\nabla g\|^2_{\mathbf{L}^2} + t^2 2^{-n+1} \tag{4.14}$$

We thus conclude that

$$\limsup_{t\to 0}\int |\nabla h_t|^2_2 \leq \int_\Omega |\nabla g|^2 \mathrm{d}x + 2^{-n}.$$

This holds for all n, so that

$$\limsup_{t\to 0} \|\nabla h_t\|_{\mathbf{L}^2(\Omega)} \leq \|\nabla g\|_{\mathbf{L}^2(\Omega)}. \tag{4.15}$$

Using that $\nabla h_t \rightharpoonup \nabla g$ weakly, we thus conclude from (4.15) that the convergence is strong by the uniform convexity of $\mathbf{L}^2$.

3. Notice that we have

$$\left\|\frac{\Phi_t - \mathrm{id}}{t}\right\|_{\mathbf{L}^2(\Omega)} \leq \|h_t\|_{\mathbf{L}^2(\Omega)} + \left\|\frac{X_t - \mathrm{id}}{t}\right\|_{\mathbf{L}^2(\Omega)} \leq 2\left\|\frac{X_t - \mathrm{id}}{t}\right\|_{\mathbf{L}^2(\Omega)}. \tag{4.16}$$

Then we deduce

$$\frac{\nabla X_t - \mathrm{id}}{t} = \frac{\nabla f_t \circ \Phi_t - \Phi_t}{t} + \frac{\Phi_t - \mathrm{id}}{t} = h_t(\Phi_t) + \frac{\Phi_t - \mathrm{id}}{t},$$

and we conclude that

$$\lim_{t\to 0} \frac{\Phi_t - \mathrm{id}}{t} = \lim_{t\to 0} \frac{X_t - \mathrm{id}}{t} - \lim_{t\to 0} h_t \circ \Phi_t = \mathbf{b} - \mathtt{P}_{\nabla}\mathbf{b}. \qquad \square$$

where we used that since (4.16) holds, then $h_t \circ \Phi_t \to \mathtt{P}_{\nabla}\mathbf{b}$ strongly in $\mathbf{L}^2$ (this can done as in the proof of (4.7)).

Applying the result to our setting, we obtain:

Corollary 4.3. *Let $\phi \in \mathbf{W}^{1,2}(\Omega)$ such that* (**MPC**) *holds. Let $\nabla f_t, \Phi_t$ be the maps in the polar decomposition of T_t, i.e.*

$$T_t = \nabla f_t \circ \Phi_t.$$

Then

$$\lim_{t\to 0} \left\|\frac{\nabla f_t - \mathrm{id}}{t} - \nabla\phi\right\|_{\mathbf{L}^2} = 0, \quad \lim_{t\to 0} \left\|\frac{\Phi_t - \mathrm{id}}{t}\right\|_{\mathbf{L}^2} = 0,$$

We also obtain a different proof of the following Proposition.

Proposition 4.4. *Let $\phi \in \mathbf{W}^{1,2}(\Omega)$ such that* (**MPC**) *holds. Then*

$$\Delta\phi \geq 0.$$

Proof. Let $\nabla f_t, \Phi_t$ be as above. Being ∇f_t measure preserving and monotone, hence BV, it holds

$$\det\big(\mathrm{id} + tD^a\nabla h_t(x)\big) = 1,$$

for $\mathscr{L}^d$-a.e. $x \in \Omega$, where we recall

$$\nabla h_t = \frac{\nabla f_t - \mathrm{id}}{t}.$$

If $\lambda_i(t,x)$ are the eigenvalues of $D^a\nabla h_t(x)$ one gets, using the arithmetic-geometric inequality

$$1 = \left(\prod_i (1 + t\lambda_i(t,x))\right)^{\frac{1}{d}} \leq \frac{1}{d}\sum_i (1 + t\lambda_i(t,x)) = 1 + \frac{t}{d}\sum_i \lambda_i(t,x). \tag{4.17}$$

By (4.17), there holds

$$\mathrm{Tr}(D^a\nabla h_t) \geq 0$$

while for the singular part of the Laplacian, one has

$$t(\Delta h_t)^s = (\Delta f_t)^s \geq 0,$$

being f_t convex. Hence $\Delta h_t \geq 0$ in the sense of distributions,

$$\int_\Omega \nabla\psi(x) \cdot \nabla h_t(x)\mathrm{d}x \leq 0 \qquad \text{for all } \psi \in C_c^1(\Omega),\ \psi \geq 0.$$

Since by Corollary 4.3 we have the convergence $\nabla h_t \to \nabla \phi$ in $\mathbf{L}^2$, we deduce for every $\psi \in C^1_c(\Omega)$ that

$$0 \geq \lim_{t \to 0^+} \int_\Omega \nabla \psi(x) \cdot \nabla h_t(x) \mathrm{d}x = \lim_{t \to 0^+} \int_\Omega \nabla \psi(x) \cdot \nabla \phi(x) \mathrm{d}x$$

and this proves the result. □

REFERENCES

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Science Publications. Clarendon Press, 2000.

[Bre91] Y. Brenier, *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math. **44**, 1991.

[LP25] J.-G. Liu and R. L. Pego, Rigidly breaking potential flows and a countable Alexandrov theorem for polytopes, *Pure and Applied Analysis*, vol. 7, no. 4, pp. 927–956, 2025,

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