STABILIZATION OF SOLUTIONS TO A MODEL OF LANGMUIR-BLODGETT FILMS

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ABSTRACT. We show stabilisation of solutions to one-dimensional advective Cahn–Hilliard equation modeling the Langmuir–Blodgett thin films. This problem has the structure of a gradient flow perturbed by a linear term βu_x . Through application of an abstract result by Carvalho–Langa–Robinson, we show that for small β the equation has the structure of gradient flow in a weak sense. Combining this with the finite number of steady states implies stabilization of solutions.

Key words: stabilization of solutions, gradient-type systems, Cahn–Hilliard type equation, Langmuir–Blodgett transfer.

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1. INTRODUCTION

Langmuir–Blodgett films [2, 14] are obtained by deposition of a thin film of amphiphilic molecules on a solid substrate, through a mechanism known as the Langmuir–Blodgett transfer. In the typical setting, the solid substrate is immersed in a trough, where the amphiphiles float freely in the disordered state at the surface of a liquid. The substrate is lifted vertically with velocity $\mathbf{v}$ and the amphiphiles, whose concentration on the surface of the liquid is kept constant, depose on it. This process has very many industrial applications [20, 22], and also lends itself to theoretical investigation, in particular for some aspects related to pattern formation. Indeed, according to the extraction velocity $\mathbf{v}$, the amphiphiles on the substrate can remain in the disordered (liquid-expanded) phase, align in the ordered (liquid-condensed) phase, or even form striped or checkerboard-like patterns in which the two phases alternate regularly [6, 21, 25].

To model that the liquid expanded and the liquid-condensed phases are preferred ones, one usually resorts to a potential which attains its minimum value precisely at those configurations. From the mathematical viewpoint, the paradigmatic tool to describe this and similar phenomena is the celebrated Cahn–Hilliard equation, originally proposed to study phase separation phenomena [4], see [19] and the references therein, and also [17] for a recent review of this subject. A few years ago, a variation of the classical Cahn–Hilliard equation was proposed in [11, 24] to account for the extraction velocity $\mathbf{v}$; moreover, the typical double-well potential $W_0(s) = \frac{1}{4}(s^2-1)^2$ was also modified to allow modeling a *meniscus*, the phenomenon generated by the asymmetry between adhesion and cohesion forces between a liquid and a solid.

By including these elements in the Cahn–Hilliard equation, the following model for the Langmuir–Blodgett films in a square domain of side length $L > 0$ was presented in [24]

$$
\begin{cases}
c_t = \operatorname{div}\big(\nabla(-\Delta c - c + c^3 + \nu\zeta(\mathbf{x})) - \mathbf{v}c\big) & \text{for } (\mathbf{x},t) \equiv (x,y,t) \in (0,L)^2 \times (0,+\infty), \\
c(0,y,t) = c_0 & \text{for } (y,t) \in [0,L] \times (0,+\infty), \\
c_x(L,y,t) = c_{xx}(0,y,t) = c_{xx}(L,y,t) = 0 & \text{for } (y,t) \in [0,L] \times (0,+\infty), \\
c(x,0,t) = c(x,L,t) & \text{for } (x,t) \in [0,L] \times (0,+\infty),
\end{cases}
\tag{1.1}
$$

where $c\colon (0,L)^2 \times (0,+\infty) \to \mathbb{R}$ describes the time-dependent concentration of the amphiphiles ($c = -1$ corresponding to the disordered phase and $c = 1$ to the ordered one), $\zeta\colon (0,L)^2 \to (-1,0)$

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describes the profile of the meniscus, and $\mathbf{v}$ is the extraction velocity. In [24], the explicit choice

$$\zeta(\mathbf{x}) = -\frac{1}{2}\left[1 + \tanh\left(\frac{x - x_{\mathrm{mns}}}{l_{\mathrm{mns}}}\right)\right]$$

was made, modeling a meniscus at $x = x_{\mathrm{mns}}$ of thickness l_{mns}; despite this specific example, it will always required to be analytic. The vertical axis is denoted by x, the horizontal one by y, and with this convention, the extraction velocity $\mathbf{v} = (\beta, 0)$ $(\beta > 0)$ points upward; the first boundary condition models the constant concentration at the liquid free surface; the second boundary conditions are designed to model the fact that the boundary at $x = L$ has no impact on the outflow of the concentration; finally, the last boundary condition encodes periodicity in the horizontal direction.

By denoting the potential by

$$W(\mathbf{x}, s) = W_0(s) + \nu\zeta(\mathbf{x})s, \qquad \text{where} \qquad W_0(s) := \frac{1}{4}(s^2 - 1)^2, \tag{1.2}$$

the first equation in (1.1) can be written as

$$c_t = \operatorname{div}\big(\nabla(-\Delta c + \partial_s W(\mathbf{x}, c)) - \mathbf{v}c\big) =: \operatorname{div}\big(\nabla\mu - \mathbf{v}c\big),$$

where we have defined the *chemical potential* $\mu := -\Delta c + \partial_s W(\mathbf{x}, c)$.

The one-dimensional version of (1.1) was studied in [3], where the change of variables $u(x,t) = c(x,t) - c_0$ was performed, yielding, see [3, equation (2.2)],

$$\begin{cases} u_t = \big(-u_{xx} + (u + c_0) - (u + c_0)^3 + \zeta(x)\big)_{xx} - \beta u_x & \text{for } (x,t) \in (0,L) \times (0,T), \\ u(0,t) = u_x(L,t) = \mu(0,t) = \mu_x(L,t) = 0 & \text{for } t \in (0,T], \\ u(x,0) = u_0(x) & \text{for } x \in (0,L); \end{cases} \tag{1.3}$$

the coefficient ν is taken to be equal to 1, and the chemical potential μ reads, in this case,

$$\mu = \mu(u) = -u_{xx} + (u + c_0)^3 - (u + c_0) + \zeta. \tag{1.4}$$

In [3, Theorem 2.4], the existence, uniqueness of weak solutions to (1.3), and their regularity was studied. Moreover, the authors established the existence of a global attractor when $\beta < L^{-3}$ (see [3, Theorem 6.5]). However, a number of interesting questions linger:

(i) Do solutions to (1.1) stabilize? In other words, do solutions approximate, for large times, a steady state solution?

(ii) Is it possible to extend the results of [3] to the original two-dimensional model of [24]? Is it possible to prove their stabilization? This latter question, which is possibly the most relevant for the physical application to the LB transfer, requires a thorough analysis of the set of steady states of (1.1), which is not available at present or a new idea.

Question (i) above is particularly interesting when one can leverage the results obtained for $\beta = 0$ (situation in which problem (1.3) is a Cahn–Hilliard equation with potential W given by (1.2)) to the case $\beta > 0$, for which (1.3) features the non-trivial advective term $-\beta u_x$.

In the present paper, we will concentrate on giving an affirmative answer to the first question. Our main result is the following.

Theorem 1.1. *There exist a positive number* Λ_0 *and an at most countable set* $E \subset [\Lambda_0, \infty)$ *such that, if* $L \in (0,\infty) \setminus E$*, then there is* $\beta^* = \beta^*(L) > 0$ *with the following properties. If* $u_0 \in \{u \in H^1(0,L) \colon u(0) = 0\}$ *and* u *is the corresponding solution to* (1.3) *with* $\beta \in [0, \beta^*]$*, then there exists* $u^\infty \in C^\infty(0,L)$ *such that for all* $m \in \mathbb{N}$ *we have*

$$\lim_{t \to +\infty} \|u(t) - u^\infty\|_{H^m(0,L)} = 0. \tag{1.5}$$

We stress that the notion of solution has to be clarified and we do this later (see Definition 4.1 and formula (4.7)). The set E, which is defined in (2.14) below, of exceptional sample lengths lacks explicit characterization. The origin of its presence comes from the definition of *branch points* (see Definition 2.7), which are those for which the Implicit Function Theorem cannot be applied. Fortunately, Theorem 2.10 ensures that E is discrete and therefore negligible.

In order to prove Theorem 1.1 we will show that, for sufficiently small $\beta > 0$, equation (1.3), which is a perturbation of a gradient flow of the energy functional

$$\mathcal{J}(u) = \int_0^L \left(\frac{1}{2}(u_x)^2 + W(x, u + c_0) \right) \mathrm{d}x,$$

is still a gradient flow, albeit, in a weaker sense, see [5] or [23]. This weaker notion is based on the structure of the global attractor and the lack of homoclinic structures. Once we show this, it suffices to check that there is only a finite number of steady state solutions to (1.3). The strategy of the proof is borrowed by that in [23], where it was applied to study the stabilization of a sixth-order Cahn–Hilliard-type equation featuring a convective term.

We start our work with the analysis of steady states of (1.3) when $\beta = 0$. We first estimate the number of the steady states for any $L > 0$ by means of the shooting method, see Theorem 2.5. We separately show that there exists $L_0 > 0$ such that for all $L \in (0, L_0)$ there is a unique steady state of (1.3), see Proposition 2.6.

Then we prove that if $v_0^1, \dots, v_0^{N(L)}$ is the family of all steady states of (1.3) with $\beta = 0$, then each v_0^i belongs to a family v_β^i for small $\beta \geq 0$. A natural tool for proving this is the Implicit Function Theorem, see [18], which requires triviality of the kernel of the linearization of the right-hand side of (1.3) with $\beta = 0$. This is the main technical challenge in the study of the problem with $\beta = 0$, which we resolve in Propositions 2.8 and 2.9 and in Theorem 2.10. Remarkably, in the course of its proof, we invoke the Weierstrass Preparation Theorem, see [15].

The second step is to invoke [5, Theorem 5.26] to conclude that (1.3) is still a gradient flow for small β. Since this theorem is written in the language of the strongly continuous semigroups, we rewrite the existence result of [3] in this language, which, in particular, proves to be very useful when we have to establish the asymptotic collective compactness of families of semigroups $\{S_\beta\}_{\beta \geq 0}$ generated by (1.3). On this occasion, we prove some estimates on a modified energy functional (see (4.22)) by using some ideas from [7] that were also employed in a similar context in [12]. This yields estimates that are uniform in time and in β on the weak solutions to (1.3).

The last step to prove stabilization is the application of [5, Theorem 5.26], which we state in Theorem 5.3 below for the reader's convenience. This theorem provides a check-list of four conditions to be satisfied to obtain the stability of gradient semigroups. We devote Section 5 to showing that our problem satisfies these conditions.

We notice that the proof strategy that we use is intrinsically one-dimensional and cannot be easily established for the original two-dimensional problem (1.1). In particular, the question of counting the number of steady states of the two-dimensional elliptic problem (1.1) is much more difficult, see, *e.g.*, [1]. Because of these reasons, the analysis of (1.1) is left for future work.

The plan of the paper is the following. In Section 2, we study the steady states in the non advective case, namely problem (1.3) for $\beta = 0$ (see (2.1) below). The main result for this case is Theorem 2.5, where we prove that (2.2) has a finite number of solutions. The brief Section 3 deals with the small-β continuation: by continuity (in fact analyticity) the results of Section 2 can be extended to (1.3) with $\beta \in (0, \beta_0)$, where the existence of $\beta_0 > 0$ is proved in Theorem 3.1. In Section 4, we translate the results in the language of analytic semigroup theory, to prepare the ground for the application of [5, Theorem 5.26]. Finally, in Section 5, we prove stabilization of solutions.

2. The case $\beta = 0$

We begin by studying problem (1.3) in the regime $\beta = 0$, namely

$$\begin{cases} u_t = \big(- u_{xx} + (u + c_0)^3 - (u + c_0) + \zeta \big)_{xx} & \text{for } (x,t) \in (0, L) \times (0, T), \\ u(0, t) = u_x(L, t) = 0 & \text{for } t \in (0, T), \\ \mu(0, t) = \mu_x(L, t) = 0 & \text{for } t \in (0, T), \end{cases} \tag{2.1}$$

where μ is defined by (1.4). We have to address different aspects of solutions to the associated steady-state problem, that is

$$\begin{cases} 0 = \big(- u_{xx} + (u+c_0)^3 - (u+c_0) + \zeta \big)_{xx} & \text{for } x \in (0,L), \\ u(0) = u_x(L) = 0, \\ \mu(0) = \mu_x(L) = 0. \end{cases} \tag{2.2}$$

We start with a preliminary lemma.

Lemma 2.1. *A function u is a solution to* (2.2) *if and only if it is a solution to*

$$\begin{cases} 0 = -u_{xx} + (u+c_0)^3 - (u+c_0) + \zeta \equiv \mu & \textit{for } x \in (0,L), \\ u(0) = u_x(L) = 0. \end{cases} \tag{2.3}$$

Namely, u is steady state if and only if the chemical potential vanishes, $\mu = 0$.

Proof. If u solves (2.3), then it is immediate to see that it solves (2.2). Viceversa, integration of the equation in (2.2) twice over $(0,x)$ yields that the chemical potential is an affine function, that is $\mu(x) = ax + b$, for $a, b \in \mathbb{R}$. Now, the boundary conditions $(2.2)_3$ imply that $a = b = 0$. ☐

We have to address various aspects of solutions to (2.3). In particular, we need estimates on the derivative u_x. We have to bound the number of steady states and we have to expose their dependence on the length L of the domain. We prove estimates on $u_x(0)$ which are uniform in L and will be important for further analysis.

Lemma 2.2. *If $u \in C^2([0,L])$ is a solution of* (2.3)*, then*

$$\sqrt{2} z_\ell := -|c_0^2 - 1| \le \sqrt{2}\, u_x(0) \le \sqrt{5 - 4c_0 + (c_0^2-1)^2} =: \sqrt{2} z_r\,.$$

Proof. Multiplying (2.3) by u_x and integrating from 0 to $x \in [0,L]$ yields the first integral

$$\begin{aligned} 2u_x^2(x) &= 2u_x^2(0) + (u(x)+c_0)^4 - 2(u(x)+c_0)^2 - c_0^4 + 2c_0^2 + 4\int_0^x u_x(s)\zeta(s)\,\mathrm{d}s \\ &= 2u_x^2(0) + \big((u(x)+c_0)^2 - 1\big)^2 - (c_0^2-1)^2 + 4\int_0^x u_x(s)\zeta(s)\,\mathrm{d}s. \end{aligned} \tag{2.4}$$

Let

$$x_* := \inf\{x \in (0,L] : \, u_x(x) = 0\}.$$

This is well-defined since u_x is continuous and $u_x(L) = 0$.

Lower bound. If $u_x(0) \ge 0$, then $\sqrt{2}\, u_x(0) \ge 0 \ge -|c_0^2 - 1|$ and we are done. Assume $u_x(0) < 0$. Then by definition of x_* we have $u_x < 0$ on $[0, x_*)$, hence $u_x(s)\zeta(s) \ge 0$ for every $s \in (0, x_*)$ because $\zeta \le 0$. Evaluating (2.4) at $x = x_*$ and using $u_x(x_*) = 0$ gives

$$0 = 2u_x^2(x_*) \ge 2u_x^2(0) - (c_0^2-1)^2,$$

so $2u_x^2(0) \le (c_0^2-1)^2$, and since $u_x(0) < 0$,

$$\sqrt{2}\, u_x(0) \ge -|c_0^2 - 1|.$$

Upper bound. If $u_x(0) \le 0$, then $\sqrt{2}\, u_x(0) \le 0 \le \sqrt{5 - 4c_0 + (c_0^2-1)^2}$ and we are done. Assume $u_x(0) > 0$. Then $u_x > 0$ on $[0, x_*)$. Using $-1 \le \zeta \le 0$ we infer, for any $x \in (0, x_*]$,

$$4\int_0^x u_x(s)\zeta(s)\,\mathrm{d}s \;\ge\; -4\int_0^x u_x(s)\,\mathrm{d}s \;=\; -4u(x),$$

where we used $u(0) = 0$ in the last identity. Insert this estimate into (2.4) and evaluate at $x = x_*$ to obtain

$$0 = 2u_x^2(x_*) \ge 2u_x^2(0) + \big((u(x_*)+c_0)^2 - 1\big)^2 - (c_0^2-1)^2 - 4u(x_*). \tag{2.5}$$

Set $t := u(x_*) + c_0$ (so $u(x_*) = t - c_0$). Then the sum of the second and last term in the right-hand side of (2.5) become

$$(t^2-1)^2 - 4(t - c_0) = \big((t^2-1)^2 - 4t + 5\big) + 4c_0 - 5.$$

The key point is the global algebraic inequality (valid for all $t \in \mathbb{R}$)

$$(t^2-1)^2 - 4t + 5 = t^4 - 2t^2 - 4t + 6 = (t^2-2)^2 + 2(t-1)^2 \ \geq\ 0. \tag{2.6}$$

Using (2.6) in (2.5) therefore yields $0 \geq 2u_x^2(0) + (4c_0 - 5) - (c_0^2-1)^2$, *i.e.*,

$$2u_x^2(0) \leq (c_0^2-1)^2 - 4c_0 + 5.$$

Since $u_x(0) > 0$, taking square roots gives

$$\sqrt{2}\, u_x(0) \leq \sqrt{5 - 4c_0 + (c_0^2-1)^2},$$

which is the desired upper bound. □

In order to expose the dependence of solutions to (2.3) on L we will rewrite this problem in a fixed domain. For this purpose we set $x = Ly$ with $y \in [0,1]$ and $\tilde{v}(y) := u(Ly) + c_0$. The steady problem (2.3) becomes

$$\begin{cases} -\tilde{v}_{yy} + L^2\big(\tilde{v}^3 - \tilde{v} + \tilde{\zeta}(y,L)\big) = 0, & \text{for } y \in (0,1), \\ \tilde{v}(0) = c_0, \qquad \tilde{v}_y(1) = 0, \end{cases} \tag{2.7}$$

where $\tilde{\zeta}(y,L) := \zeta(Ly)$. Of course, problems (2.3) and (2.7) are equivalent.

We will use the shooting method. For this purpose we set up the initial value problem for the first equation in (2.7),

$$\begin{cases} -\tilde{v}_{yy}(y) + L^2\big(\tilde{v}(y)^3 - \tilde{v}(y) + \tilde{\zeta}(y,L)\big) = 0, & \text{for } y \in (0,1), \\ \tilde{v}(0) = c_0, \qquad \tilde{v}_y(0) = \tilde{z}. \end{cases} \tag{2.8}$$

In order to proceed, we have to make sure that solutions to (2.8) exist on $[0,1]$. This is the content of the lemma below.

Lemma 2.3. *For any value of c_0 and $\tilde{z} \in [Lz_\ell\,, Lz_r]$ there exists a unique solution $\tilde{v}(\cdot; L, \tilde{z})$ to* (2.8) *on $[0,1]$. Moreover, if $\tilde{v}$ has a maximum in $[0,1)$ or $\tilde{v}_y(1) = 0$ when $\tilde{v}_y(0) > 0$, (respectively $\tilde{v}$ has a minimum in $[0,1)$ or $\tilde{v}_y(1) = 0$ when $\tilde{v}_y(0) < 0$), then we have the following bounds,*

$$\sup_{y\in[0,1]} |\tilde{v}(y; L, \tilde{z})| \leq \max\{1, v_M\}, \qquad \sup_{y\in[0,1]} |\partial_y \tilde{v}(y; L, \tilde{z})| \leq L \max\{|z_\ell|, z_r\} + L^2 \max\{1, v_M\}.$$

where $v_M > 0$ is the (only real) solution to $v_M^3 - v_M = 1$.

Proof. We will consider only the case $\tilde{v}_y(0) > 0$, since the case $\tilde{v}_y(0) < 0$ can be handled in a similar way and $\tilde{v}_y(0) = 0$ is a limit of both cases and does not bring any additional difficulty.

Set $y_0 = 0$ and suppose $\tilde{v}_y(0) > 0$; then the function $\tilde{v}$ is increasing on an interval $[0, y_1]$ and it attains its maximum at $y_1 \in (0,1]$. We claim that $\tilde{v}_{yy}(y_1) \leq 0$, to show which we distinguish two cases: $y_1 = 1$ and $y_1 < 1$. In the former case, we have that $\tilde{v}_y \geq 0$ on $[0,1]$, hence the condition $\tilde{v}_y(1) = 0$ implies that $\tilde{v}_y$ attains at $y = 1$ its minimum and our claim follows. In the latter case, we deduce that $\tilde{v}_{yy}(y_1) \leq 0$ and equation (2.8) implies

$$\tilde{v}(y_1)^3 - \tilde{v}(y_1) + \tilde{\zeta}(y_1, L) \leq 0.$$

Since $-1 < c_0 < \tilde{v}(y_1)$ we deduce that $\tilde{v}(y_1) \leq v_M$, where v_M is a solution to

$$v_M^3 - v_M = 1.$$

If $y_1 = 1$ our argument is finished, whereas, if $y_1 < 1$, we may suppose that $\tilde{v}$ is decreasing on $[y_1, y_2]$ and it achieves a local minimum at $y_2 \in (y_1, 1]$. If $y_2 < 1$, then obviously $\tilde{v}_{yy}(y_2) \geq 0$. If $y_2 = 1$, we notice that $\tilde{v}_y \leq 0$ on $[y_1, y_2]$, hence $\tilde{v}_y$ attains a maximum at $y = 1$, hence $\tilde{v}_{yy}(x_2) \geq 0$.

Then, equation (2.8) yields

$$\tilde{v}(y_2)^3 - \tilde{v}(y_2) + \tilde{\zeta}(y_2, L) \geq 0.$$

Since $\tilde{\zeta} < 0$, we deduce that $\tilde{v}$ must be greater than or equal to $v_m < 0$, the solution to $v_m^3 - v_m = 0$. Hence $v_m = -1$.

Thus, we see that $\tilde{v}$ varies on intervals $\{[y_{k-1}, y_k]\}_{k=1}^N$ between -1 and v_M, where y_k and y_{k+1} are consecutive extrema. If $y_N < 1$, then we can continue $\tilde{v}$ past y_N to reach the next extremum. Thus, we conclude that $\tilde{v}(y) \in [-1, v_M]$ for all $y \in [0, 1]$.

The bound on $\tilde{v}_y$ follows from the integration of equation (2.8). $\square$

We now define the function $f\colon (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ by

$$f(L, \tilde{z}) := \tilde{v}_y(1; L, \tilde{z}). \tag{2.9}$$

Remark 2.4. The function f defined in (2.9) is analytic in $L > 0$ and $\tilde{z}$. This is a consequence of the analyticity of ζ and of the fact that the nonlinearity in (2.8) is polynomial. Indeed, we can assume that $L, \tilde{z} \in \mathbb{C}$ and therefore standard differentiability with respect to parameters yields analyticity of f with respect to L and $\tilde{z}$. Restricting L and $\tilde{z}$ to the real line completes the argument.

Here is the main observation of this section.

Theorem 2.5. *For any given $L > 0$ and c_0, there is a finite number of solutions to* (2.3)*. They correspond to the zeros of the analytic function $\tilde{z} \mapsto f(L, \tilde{z})$, where f is defined in* (2.9)*.*

Proof. Solutions to (2.8) depend analytically on three parameters c_0, L, and $\tilde{z}$ (see Remark 2.4). Solution to (2.3) correspond to zeros of the function $\tilde{z} \mapsto f(L, \tilde{z})$. From Lemma 2.2, we know that these zeros may belong only to the interval $[Lz_\ell, Lz_r]$, which is compact. Therefore, analyticity with respect to $\tilde{z}$ implies that there is only a finite number of them. $\square$

We may be more precise when the sample length, L, small. In this case we may prove uniqueness of steady states.

Proposition 2.6. *There is $\Lambda_0 > 0$ such that if $L \in (0, \Lambda_0)$, then there exists a unique solution to* (2.3)*.*

Proof. Since problems (2.3) and (2.7) are equivalent, we deal with (2.7) which is more manageable.

Let us notice that when $L = 0$, then (2.7) has a unique solution given by $\tilde{v}(y) = c_0$. We shall consider (2.7) as a problem

$$G(L, \tilde{v}(L)) = 0, \tag{2.10}$$

where

$$G\colon \mathbb{R} \times C^2_{\mathrm{bc}}([0, 1]) \to C([0, 1])$$

and $C^2_{\mathrm{bc}}([0, 1]) = \{\tilde{v} \in C^2([0, 1]) : \tilde{v}(0) = c_0,\ \tilde{v}_y(1) = 0\}$. In order to apply the Implicit Function Theorem, we have to check that $\partial_{\tilde{v}} G(0, c_0)$ is an isomorphism. Indeed,

$$\frac{\partial G}{\partial \tilde{v}}(0, c_0)h = h_{yy}\,.$$

This operator with the boundary conditions $h(0) = h_y(1) = 0$ has a trivial kernel and it is an isomorphism. Hence, there is $\Lambda_0 > 0$ such that for all $L \in (-\Lambda_0, \Lambda_0)$ there is a function $(-\Lambda_0, \Lambda_0) \ni L \mapsto \tilde{v}(L)$ is a unique solution to (2.10). $\square$

In view of Theorem 2.5, for each $L > 0$, we denote by $\tilde{z}^1(L), \dots, \tilde{z}^{N(L)}(L)$ the zeros of $\tilde{z} \mapsto f(L, \tilde{z})$ and by $\tilde{v}^1(L), \dots, \tilde{v}^{N(L)}(L)$ the corresponding solutions to (2.3).

We now analyze the set $\{\tilde{z}^j(L)\}_{j=1}^{N(L)}$ in more detail. In particular, we partition the pairs $(L, \tilde{z}^j(L))$ in two classes.

Definition 2.7 (Nondegenerate solution and branch point)**.** Let $L_0 > 0$ and consider a zero $\tilde{z}_0 \in \big\{\tilde{z}^j(L_0)\big\}_{j=1}^{N(L_0)}$.

- We call $(L_0, \tilde{z}_0)$ *nondegenerate* if $\partial_{\tilde{z}} f(L_0, \tilde{z}_0) \neq 0$.
- We call $(L_0, \tilde{z}_0)$ a *branch point* if $\partial_{\tilde{z}} f(L_0, \tilde{z}_0) = 0$.

Our aim is to relate nondegeneracy of pairs $(L_0, \tilde{z}_0)$ to the triviality of the kernel of the linearized operator corresponding to problem (2.2). This will be developed in the following results.

We start by considering the linearized problem associated with (2.7). For a given $L_0 > 0$, let $\tilde{v}_*$ be a solution to (2.7) for $L = L_0$. We set $D(\mathcal{M}_{L_0}) = \{u \in H^2(0,1) \colon u(0) = 0 = u_x(1)\}$ and define the linearized operator $\mathcal{M}_{L_0} \colon D(\mathcal{M}_{L_0}) \subset L^2(0,1) \to L^2(0,1)$ as

$$\mathcal{M}_{L_0} h := -h_{yy} + L_0^2(3\tilde{v}_*^2(y) - 1)h. \tag{2.11}$$

Proposition 2.8 (Nondegenerate points yield unique analytic branches)**.** *Let* $L_0 > 0$ *and let* $\tilde{z}_0 \in \big\{\tilde{z}^j(L_0)\big\}_{j=1}^{N(L_0)}$. *If* $(L_0, \tilde{z}_0)$ *is non degenerate, then there exist* $\varepsilon > 0$ *and a unique real-analytic function*

$$\tilde{z} \colon (L_0 - \varepsilon, L_0 + \varepsilon) \to \mathbb{R}$$

such that $f(L, \tilde{z}(L)) = 0$ *for* $|L - L_0| < \varepsilon$ *and* $\tilde{z}(L_0) = \tilde{z}_0$. *Consequently* $L \mapsto \tilde{v}(\cdot; L, \tilde{z}(L)) \in C^1([0,1])$ *is a unique real-analytic branch of steady states near* L_0.

Proof. The real-analytic Implicit Function Theorem (see, *e.g.*, [13, Section 6.1]) applies to the analytic map $f \colon (0,\infty) \times \mathbb{R} \to \mathbb{R}$ defined in (2.9), at $(L_0, \tilde{z}_0)$, since $\partial_{\tilde{z}} f \neq 0$. Analytic dependence of $L \mapsto \tilde{v}(\cdot; L, \tilde{z}(L))$ follows from Remark 2.4. ☐

In the next proposition, we prove that the loss of nondegeneracy of pairs $(L_0, \tilde{z}_0)$ corresponds to the nontriviality of the kernel of the operator $\mathcal{M}_{L_0}$.

Proposition 2.9. *Let* $(L_0, \tilde{z}_0)$ *satisfy* $f(L_0, \tilde{z}_0) = 0$ *and let* $\tilde{v}_*(y) := \tilde{v}(y; L_0, \tilde{z}_0)$ *denote the corresponding steady state,* i.e., $\tilde{v}_*$ *is a solution to* (2.7). *Consider the boundary-value problem*

$$\begin{cases} \mathcal{M}_{L_0} h = 0 & \text{for } y \in (0,1), \\ h(0) = 0, \quad h_y(1) = 0, \end{cases} \tag{2.12}$$

for the linearized operator introduced in (2.11). *Then* $\partial_{\tilde{z}} f(L_0, \tilde{z}_0) = 0$ *if and only if* $\ker \mathcal{M}_{L_0} \neq \{0\}$.

Proof. Differentiate the initial-value problem (2.8) with respect to $\tilde{z}$ at $(L_0, \tilde{z}_0)$. The variation $h(y) = \partial_{\tilde{z}} \tilde{v}(y; L_0, \tilde{z}_0)$ solves

$$\begin{cases} \mathcal{M}_{L_0} h = 0 & \text{for } y \in (0,1), \\ h(0) = 0, \quad h_y(0) = 1. \end{cases} \tag{2.13}$$

Hence $\partial_{\tilde{z}} f(L_0, \tilde{z}_0) = h_y(1)$. Thus, $\partial_{\tilde{z}} f(L_0, \tilde{z}_0) = 0$ if and only if the solution to (2.13) is also a solution to (2.12), *i.e.*, if and only if (2.12) admits a nontrivial solution. Conversely, if k is a nontrivial solution to (2.12), then $k_y(0) \neq 0$ by uniqueness; defining $h := k/k_y(0)$ gives $h(0) = 0$, $h'(0) = 1$, $h'(1) = 0$, hence $\partial_{\tilde{z}} f(L_0, \tilde{z}_0) = 0$. ☐

Therefore, branch points correspond to nontriviality of the kernel of the linearized operator $\mathcal{L}_{L_0}$. In the next theorem, we prove that there are at most countably many branch points.

Theorem 2.10 (Rarity of branch points in L)**.** *The set*

$$E := \big\{L > 0 : \exists\, j \in \{1, \dots, N(L)\} : (L, \tilde{z}^j(L)) \text{ is a branch point}\big\} \tag{2.14}$$

is discrete (in particular, of measure zero).

Proof. Define the zero level sets, for $i = 0, 1$,

$$\mathcal{S}_i := \big\{(L, \tilde{z}) \in (0,\infty) \times [Lz_\ell, Lz_r] \;:\; \partial_{\tilde{z}}^i f(L, \tilde{z}) = 0\big\}.$$

We immediately notice that the set $\mathcal{S}_0 \setminus \mathcal{S}_1$ is a smooth manifold of dimension 1 as a consequence of the Implicit Function Theorem. Consider any compact set $K \subset (0, +\infty)$ and the projection $\pi_L \colon (0, +\infty) \times \mathbb{R} \to (0, +\infty)$ on the first component.

We claim that $\pi_L\big(\mathcal{S}_0 \cap \mathcal{S}_1 \cap (K \times [Lz_\ell, Lz_r])\big) \subset K$ is finite. If the set $\mathcal{S}_0 \cap \mathcal{S}_1 \cap (K \times [Lz_\ell, Lz_r])$ is finite, then the claim follows. Otherwise, it has an accumulation point and two cases may occur.

(a) There exists $L_0 \in K$ such that $\mathcal{S}_0 \cap \mathcal{S}_1 \cap (\{L_0\} \times [Lz_\ell\,, Lz_r])$ has an accumulation point, hence the same is true about $\mathcal{S}_0 \cap (\{L_0\} \times [Lz_\ell\,, Lz_r])$. As a result, the analytic function of one variable $\tilde{z} \mapsto f(L_0, \tilde{z})$ vanishes on a set having an accumulation point. This means that $f(L_0, \tilde{z}) \equiv 0$, which implies that there are infinitely many steady states for $L = L_0$, contradicting Theorem 2.5.

(b) The set $\pi_L\big(\mathcal{S}_0 \cap \mathcal{S}_1 \cap (K \times [Lz_\ell\,, Lz_r])\big)$ has an accumulation point. By invoking the Weierstrass Preparation Theorem (see, *e.g.*, [15, Chapter C, §2]), and possibly up to a local analytic change of coordinates, we can factor

$$f(L, \tilde{z}) = U(L, \tilde{z}) P_L(\tilde{z}),$$

where U is an analytic nonvanishing function, and P_L is a monic polynomial of degree n, whose coefficients are analytic in L. We recall that the discriminant of P_L is a function of L and is defined by

$$d(L) = (-1)^{\binom{n}{2}} \prod_{j=1}^{n} \frac{\partial P_L}{\partial \tilde{z}}(\tilde{z}_j),$$

where $\tilde{z}_j\,, j = 1, \dots, n$ is the complete set of zeros of the polynomial $\tilde{z} \mapsto P_L(\tilde{z})$.

In this case, $L \mapsto d(L)$ is an analytic function of one variable vanishing on a set with an accumulation point in a compact set (in the intersection $\mathcal{S}_0 \cap \mathcal{S}_1$, zeros are multiple). Thus, d must be identical to zero on $(0, \infty)$. However, we know that for small values of L there is a unique and simple zero of $f(L, \cdot)$, which contradicts Proposition 2.6.

The theorem is proved. □

We noted that problems (2.3) and (2.7) are equivalent, *i.e.*, u is a solution to (2.3) if and only if $\tilde{v}$, the scaling of u, is a solution to (2.7). Moreover, problem (2.12) has a nontrivial solution if and only if there is a non-zero solution of

$$\begin{cases} -h_{xx} + (3u^2(x) - 1)h = 0, & \text{for } x \in (0, L), \\ h(0) = 0, \quad h_x(L) = 0. \end{cases} \tag{2.15}$$

We can summarize the results of this section as follows.

Theorem 2.11. *Let us suppose $L \in (0, \infty) \backslash E$, where E is defined in (2.14) and $c_0 \in (-1, 0)$. Then there exists a finite family $\{u^1, \dots, u^{N(L)}\}$ of steady stated of (2.2) and the linearized operator at each u^i, i.e., (2.15) has a trivial kernel.*

Proof. We noticed in Lemma 2.1 that equations (2.2) and (2.3) are equivalent. Existence of a finite family of solutions to (2.3) for any positive L and c_0 follows from Theorem 2.5 after taking into account that (2.3) and the rescaled equation (2.7) are equivalent. Proposition 2.9 and Theorem 2.10 combined show that if $L \in (0, \infty) \setminus E$, then all solutions to (2.3) lead to linearized operators (2.15) with a trivial kernel. □

3. Continuation to small β

We first address existence of steady state solution of (2.2) for a range of β. Later, in the course of proof of Lemma 5.6, we will study the spectrum of the linearized operator. We return to the fourth-order formulation with β and we define the operator $\mathcal{F}\colon X \times (-1, 1) \to L^2(0, L)$ by formula

$$\mathcal{F}(u, \beta) := \big(- u_{xx} + (u + c_0)^3 - (u + c_0) + \zeta \big)_{xx} - \beta u_x\,,$$

where X is the following complete metric space

$$X := \{u \in H^4(0, L) \colon u(0) = 0,\ u_x(L) = 0,\ \mu(0) = \mu_x(L) = 0\}. \tag{3.1}$$

At a steady state u_* for $\beta = 0$, the linearisation $D_u \mathcal{F}(u_*, 0) = \Delta \mathcal{M}_L$ has kernel $\{0\}$ provided the second-order linearised problem (2.13) is nondegenerate, see Proposition 2.9. Then, we can prove the following result.

Theorem 3.1 (small-β continuation)**.** *Suppose* $0 < L \notin E$, *where* E *is defined in* (2.14), *and let* $u^j(L)$, *for some* $j \in \{1, \dots, N(L)\}$, *be a steady state of* (2.1). *Then the linearized operator at* $u^j(L)$ *has trivial kernel and there exist* $\beta_0 > 0$ *and a* C^1 *function* $(-\beta_0, \beta_0) \ni \beta \mapsto u^j_\beta \in H^4(0, L)$ *with* $u^j_0 = u^j(L)$ *and* $\mathcal{F}(u^j_\beta, \beta) = 0$. *Moreover, there exists* $C > 0$ *depending only on* $\mathcal{F}$ *and* u^i_β *such that*

$$\|u^j_\beta(L) - u^j(L)\|_{H^4(0,L)} \le C|\beta|. \tag{3.2}$$

Proof. The existence of the function $\beta \mapsto u^j_\beta$ is a direct application of the Implicit Function Theorem in Banach spaces (see, *e.g.*, [18, Theorem 2.7.2]). Indeed, the mapping $\mathcal{F}$ is of class C^1 in a neighborhood of $(u^j(L), 0)$ in X defined in (3.1). Moreover, due to Proposition 2.9, $D_u\mathcal{F}(u^j(L), 0)$ has trivial kernel by the nondegeneracy $(L, \tilde{z}^j(L))$. Since $D_u\mathcal{F}(u^j(L), 0)$ is the inverse of a compact operator, its spectrum consists of eigenvalues, hence $D_u\mathcal{F}(u^j(L), 0)$ is an isomorphism. ☐

4. Semigroup background

In this section we will recall the existence result of [3]. We want to recast it in the language of the analytic semigroup theory as exposed in [8] or [10] (to which we redirect the reader for basic facts). For this purpose we introduce in a concise way the basics of this theory. We have to do this, because our main tool, *i.e.*, [5, Theorem 5.26], is written in this language. In fact, this theorem provides a check-list of properties of the semigroup generated by an equation to be established.

4.1. Existence of weak solutions to (1.3).

Here, we recall the statement of the main existence result in [3]. We first introduce a number of function spaces. We set

$$V := \{u \in H^1(0, L) : u(0) = 0\}$$

and we endow it with the scalar product $(u, v)_V := \int_0^L u'(x)v'(x)\,\mathrm{d}x$. We introduce the notion of weak solution to (1.5) as follows.

Definition 4.1. Let $u_0 \in V$ be a given initial datum. A function u is a weak solution to (1.3) in $[0, T]$ corresponding to u_0 if $u(x, 0) = u_0(x)$ for almost every $x \in (0, L)$, and the following conditions hold:

1. $u \in H^1(0, T; V') \cap L^2(0, T; H^3(0, L))$;
2. the chemical potential $\mu := -u_{xx} + \partial_s W(x, u + c_0) \in L^2(0, L; V)$;
3. for every $\psi \in V$ and almost every $t \in (0, T)$, the function u satisfies
$$\langle \partial_t u, \psi\rangle_{V,V'} = -(\mu, \psi)_V - \beta \int_0^L u_x(x)\psi(x)\,\mathrm{d}x;$$
4. for almost every $t \in (0, T)$, the function u satisfies the boundary conditions $u(0, t) = u_x(L, t) = 0$.

The following result was established in [3].

Proposition 4.2. *(*[3, Theorem 2.4]*) Let* W *be given by* (1.2) *and let* $u_0 \in V$. *For any* $T > 0$, *there exists a unique weak solution* u *to the problem* (1.3) *in* $[0, T]$ *corresponding to* u_0 *in the sense of Definition 4.1. Furthermore, setting*

$$\mathcal{E}(v) := \frac{1}{2}\int_0^L |v_x(x)|^2\,\mathrm{d}x + \int_0^L W(x, v(x))\,\mathrm{d}x \qquad \text{for } v \in V, \tag{4.1}$$

the following energy equality holds true for almost every $t \in (0, T)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u(t)) + \|\mu(t)\|_V^2 = -\beta \int_0^L u_x(x, t)\mu(x, t)\,\mathrm{d}x, \tag{4.2}$$

where μ *is defined in* (1.4). *Finally, the solution* u *depends continuously on the initial data, in the following sense: given any* $M > 0$, *there exists a constant* $C_M > 0$ *(depending on* β, L, M, *and* T*) such that for every initial data* $u_0, \bar{u}_0 \in V$ *with* $\|u_0\|_V, \|\bar{u}_0\|_V \le M$ *the corresponding weak solutions* u, $\bar{u}$ *satisfy, for all* $t \in [0, T]$,

$$\|u(t) - \bar{u}(t)\|_V \le C_M\|u_0 - \bar{u}_0\|_V.$$

4.2. **Basics of the analytic semigroup theory.** By considering a Hilbert space $\mathscr{H}$, we rewrite the problem in (1.3) as

$$w_t + Aw = F_\beta(w), \qquad w(0) = w_0 \tag{4.3}$$

where $A\colon D(A) \subset \mathscr{H} \to \mathscr{H}$ is the (one-dimensional) bilaplacian operator $A = \Delta^2 = (-\Delta)^2$, and where the meaning of F_β will be explained momentarily. Here,

$$D(A) \coloneqq \{w \in H^4(0,L) : w(0) = w_{xx}(0) = 0,\ w_x(L) = w_{xxx}(L) = 0\}.$$

It is not difficult to check that A is a positive definite self-adjoint operator. Hence, it is sectorial on $\mathscr{H} = L^2(0,L)$ and $-A$ is a generator of an analytic semigroup. In particular, the fractional powers A^α are well-defined. We write $\mathscr{H}^\alpha \coloneqq D(A^\alpha)$; in particular, $A^{1/2} = -\Delta$ with

$$\mathscr{H}^{1/2} = D(A^{1/2}) = \{w \in H^2(0,L) :\ w(0) = 0 = w_x(L)\}$$

and

$$\mathscr{H}^{1/4} = V, \qquad \|(-\Delta)^{1/2}w\| = \|w_x\|,$$

where, here and hencefort, we use $\|\cdot\|$ to denote the $L^2(0,L)$-norm $\|\cdot\|_{L^2(0,L)}$. Indeed,

$$\|(-\Delta)^{1/2}w\|^2 = (w, -\Delta w) = (w_x, w_x). \tag{4.4}$$

We also recall existence of constants C_α for all $\alpha \in (0,1]$ such that, see [10, Theorem 1.4.3],

$$\|A^\alpha e^{-At}\| \le C_\alpha t^{-\alpha} e^{-\lambda t}, \qquad t > 0, \tag{4.5}$$

where $\lambda \in (0, \inf \sigma(A))$, $\sigma(A)$ being the spectrum of A.

In order to accommodate the boundary conditions coming from $\mu(0) = 0 = \mu_x(L)$, we introduce $w = u - \eta$, where η is a smooth function with the following properties,

$$\eta(0) = 0 = \eta_x(L), \quad \eta_{xx}(0) = c_0^3 - c_0 + \zeta(0), \quad \eta_{xxx}(L) = \zeta_x(L).$$

Then $F_\beta \colon \mathscr{H}^{1/2} \to \mathscr{H}$ given by

$$F_\beta(w) = \Delta \partial_s W(x, w + \eta + c_0) - \beta w_x - \Delta^2 \eta - \beta \eta_x \tag{4.6}$$

is well-defined, and one can easily check that F_β is locally Lipschitz continuous. As a result, the basic existence result [10, Theorem 3.3.3] applies, yielding local-in-time strong solutions for data $w_0 \in \mathscr{H}^{1/2}$, meaning that

$$w \in C((0,T); D(A)) \cap C([0,T); \mathscr{H}^{1/2}) \cap C^1((0,T); \mathscr{H}). \tag{4.7}$$

Let us note that the semigroup theory provides locally more regular solutions than the weak ones constructed in [3].

Since it suffices that the initial condition u_0 is in $V = \mathscr{H}^{1/4}$ to construct a weak solution, we cannot immediately claim that weak and strong solutions coincide. However, Proposition 4.2 assures us that for a.e. $t_0 > 0$ the weak solution $u(t_0)$ is in $H^3(0,L)$ and satisfies the appropriate boundary conditions, hence it is in $\mathscr{H}^{1/2}$. Thus, the strong solution of (4.3) with the nonlinearity (4.6) and initial condition $w_0 = u(t_0) - \eta$ exists. By the uniqueness part of Proposition 4.2, we see that the strong solution $w(t) + \eta$ and the weak solutions $u(t)$ must coincide on $[t_0, T)$. Since $t_0 > 0$ and $t_0 < T$ are arbitrary, we see that both types of solutions agree on $(0, \infty)$.

We need to establish a number of properties of the solution. The following integral representation of strong solutions will be very useful for us, see [8, 10],

$$\begin{aligned} w(t) &= e^{-\Delta^2(t-t_0)} w_0 + \int_{t_0}^t e^{-\Delta^2(t-\sigma)} \big(\Delta \partial_s W(x, w + \eta + c_0) - \beta w_x - \Delta^2 \eta - \beta \eta_x\big)\, \mathrm{d}\sigma \\ &= e^{-\Delta^2(t-t_0)} w_0 + (Id - e^{-\Delta^2(t-t_0)})\eta \\ &\quad + \int_{t_0}^t e^{-\Delta^2(t-\sigma)} \big(\Delta \partial_s W(x, w + \eta + c_0) - \beta w_x - \beta \eta_x\big)\, \mathrm{d}\sigma. \end{aligned} \tag{4.8}$$

Since we want to use it to deduce estimates of weak solutions, t_0 must be greater than zero.

Let us note the following smoothing effect.

Lemma 4.3 (Instantaneous smoothing)**.** *Let* $u_0 \in V$ *and let* u *be the corresponding weak solution to* (1.3)*. Then, for every* $0 < \tau < T < \infty$ *and every* $m \in \mathbb{N}$*, there exists a positive constant* $C_{m,\tau,T} = C_{m,\tau,T}(u_0, L, \beta, \eta, \zeta)$ *such that*

$$\sup_{t\in[\tau,T]} \|u(t)\|_{H^m(0,L)} \le C_{m,\tau,T}.$$

In particular,

$$u(t) \in C^\infty(0,L) \qquad \text{for every } t > 0.$$

Proof. Fix $0 < \tau < T$. Since $u \in L^2(0,T;H^3(0,L))$ as per Definition 4.1(1), there exists $t_0 \in (0,\tau/2)$ such that $u(t_0) \in H^3(0,L)$ and satisfies the boundary conditions.

Set $w = u - \eta$. By the discussion preceding (4.8), the weak solution coincides on $[t_0,T]$ with the strong solution of (4.3) issued from $w(t_0)$. In particular, $w \in C\big((t_0,T];D(A)\big) \cap C^1\big((t_0,T];\mathscr{H}\big)$, and, since $D(A) \hookrightarrow H^4(0,L)$, we infer that there exists a positive constant $C_{\tau,T}(u_0,L,\beta,\eta,\zeta)$ depending on u_0 , L, β, η, and ζ such that

$$\sup_{t\in[\tau,T]} \|w(t)\|_{H^4(0,L)} \le C_{\tau,T}(u_0,L,\beta,\eta,\zeta).$$

We now assume that for some $m \ge 4$ there exists a constant $C_{m,\tau,T}(u_0,L,\beta,\eta,\zeta) > 0$ such that

$$\sup_{t\in[\tau,T]} \|w(t)\|_{H^m(0,L)} \le C_{m,\tau,T}(u_0,L,\beta,\eta,\zeta)$$

and bootstrap. Since $H^m(0,L)$ is a Banach algebra in one space dimension and F_β is a differential polynomial of order at most two, see (4.6), we have that there exists a positive constant $\widetilde{C}_{m,\tau,T}(u_0,L,\beta,\eta,\zeta)$ such that

$$\sup_{t\in[\tau,T]} \|F_\beta(w(t))\|_{H^{m-2}(0,L)} \le \widetilde{C}_{m,\tau,T}(u_0,L,\beta,\eta,\zeta). \tag{4.9}$$

Writing (4.8) with initial time t_0 and applying $A^{(m+1)/4}$, we obtain, thanks to (4.5), that

$$\begin{aligned}\|w(t)\|_{H^{m+1}(0,L)} &\le C_{m,\tau,T}(u_0,L,\beta,\eta,\zeta)\|w(t_0)\|_{\mathscr{H}} + C_{3/4}\int_{t_0}^{t}(t-s)^{-3/4}\|F_\beta(w(s))\|_{H^{m-2}(0,L)}\,\mathrm{d}s\\ &\le C_{m,\tau,T}(u_0,L,\beta,\eta,\zeta)\|w(t_0)\|_{\mathscr{H}} + 4T^{1/4}C_{3/4}\widetilde{C}_{m,\tau,T}(u_0,L,\beta,\eta,\zeta)\\ &=: C_{m+1,\tau,T}(u_0,L,\beta,\eta,\zeta),\end{aligned}$$

for every $t \in [\tau,T]$.

Hence w is bounded in $H^m(0,L)$ on $[\tau,T]$ for every $m \in \mathbb{N}$. Since η is smooth, the same is true for u, possibly redefining the constant $C_{m,\tau,T}$. □

4.3. **Estimates uniform in β.**

Our convergence analysis depends on uniform-in-time estimates on solutions in various Sobolev norms. On the way we will revisit the proof of existence a global attractor for (1.3) given in [3, Theorem 6.5]. We will improve it by making it independent of L and $\beta \in (0,1]$. The first step in this direction in a new proof of existence of an absorbing ball. We will use, for this purpose, an argument based on the ideas of [7], which were developed in [12]. Then, we show uniform-in-time estimates in the Sobolev norms H^m for all $m \ge 2$. This is more than enough to establish the existence of a global compact attractor $\mathcal{A}_\beta$ in the H^1-topology. However, the uniform bounds in higher Sobolev norms will be useful when we want to infer stabilization not only in $H^1(0,L)$, but also in $H^m(0,L)$, for $m \ge 2$, see Theorem 5.7 below.

Let us call the semigroup of weak solutions generated by (1.3) by S_β . Here is our basic result on S_β .

Proposition 4.4 (Absorbing ball in $H^1(0,L)$)**.** *Let us suppose that* $\beta \in (0,1]$*. The semigroup* $S_\beta(t)\colon V \to V$*,* $u_0 \mapsto S_\beta(t)u_0 = u(t)$*, generated by the unique weak solutions to equation to (1.3) for* $u_0 \in V$ *has an* H^1 *absorbing ball* $\mathcal{B} = \{u \in V : \|u\|_{H^1(0,L)} \le M_1(L,\zeta)\}$, i.e., *for a set* $B \subset V$ *bounded in the* H^1*-topology, there is* $t_B > 0$ *such that* $S(t)u_0 = u(t) \in \mathcal{B}$ *for* $u_0 \in B$ *and* $t \ge t_B$ *. The value of the constant* $M_1(L,\zeta) > 0$ *is provided at the end of the proof.*

Proof. We will proceed in a number of steps developing a priori estimates.
Step 1. We claim that weak solutions, in the sense of Definition 4.1, to equation (1.3) with $u_0 \in V$ fulfill, for every $\beta \leq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L W_0(u+c_0)\,\mathrm{d}x + \frac{1}{2}\|u_x\|^2\right] + \frac{1}{2}\|(-\Delta)^{-1/2}u_t\|^2 \leq \beta\|u\|^2 + \|\zeta_x\|^2, \tag{4.10}$$

where we recall that W_0 is the classical double-well potential defined in (1.2). Indeed, application of the integral operator $(-\Delta)^{-1}\colon L^2 \to D(-\Delta) \subset H^2$ to both sides of equation (1.3) yields

$$(-\Delta)^{-1}u_t - u_{xx} + (u+c_0)^3 - (u+c_0) + \zeta - \beta(-\Delta)^{-1}u_x = 0\,, \tag{4.11}$$

Since $(-\Delta)^{-1}u_t$ and u_{xx} belong to H^1, we may pair the above equation with u_t . Next, integration by parts and rearranging yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{4}\|u+c_0\|_{L^4}^4 - \frac{1}{2}\|u+c_0\|^2 + \frac{1}{2}\|u_x\|^2\right] + \|(-\Delta)^{-1/2}u_t\|^2 \leq \beta|\langle(-\Delta)^{-1}u_x, u_t\rangle| + |\langle\zeta, u_t\rangle|\,, \tag{4.12}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V and V'. We notice that the right-hand side can be handled by using the Cauchy inequality with $\epsilon = 1/2$; indeed,

$$\begin{aligned} |\langle(-\Delta)^{-1}u_x, u_t\rangle| = |\langle(-\Delta)^{-1/2}u_x, (-\Delta)^{-1/2}u_t\rangle| &\leq \frac{1}{4}\|(-\Delta)^{-1/2}u_t\|^2 + \|(-\Delta)^{-1/2}u_x\|^2 \\ &\leq \frac{1}{4}\|(-\Delta)^{-1/2}u_t\|^2 + \|u\|^2, \end{aligned} \tag{4.13a}$$

the last inequality being justified by having $\|(-\Delta)^{1/2}\| = 1$, which follows from (4.4), and the fact that the norm $u \mapsto \|u_x\|$ on V is equivalent to the standard one in H^1. In an analogous fashion, we estimate

$$|\langle\zeta, u_t\rangle| = |\langle(-\Delta)^{1/2}\zeta, (-\Delta)^{-1/2}u_t\rangle| \leq \frac{1}{4}\|(-\Delta)^{-1/2}u_t\|^2 + \|\zeta_x\|^2. \tag{4.13b}$$

By using inequalities (4.13) in (4.12) and taking into account that $\beta \leq 1$, we obtain (4.10).
Step 2. Estimate (4.10) was a preliminary result. The next one will give the first uniform-in-time estimate. We claim that the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(-\Delta)^{-1/2}u\|^2 + \frac{1}{8}\|u+c_0\|_{L^4}^4 + \|u_x\|^2 \leq 2K_1(L) \tag{4.14}$$

holds true for all weak solutions u of (1.3) with $u_0 \in V$, where $K_1(L)$ is a positive constant whose explicit expression is given by formula (4.21) below.

Indeed, by using u itself as a test function in (4.11), we obtain

$$\begin{aligned} 0 &= \langle(-\Delta)^{-1}u_t, u\rangle - \langle\Delta u, u\rangle + \langle(u+c_0)^3, u\rangle - \langle u+c_0, u\rangle + \langle\zeta, u\rangle - \beta\langle(-\Delta)^{-1}u_x, u\rangle \\ &= \langle(-\Delta)^{-1/2}u_t, (-\Delta)^{-1/2}u\rangle + \|u_x\|^2 + \|u+c_0\|_{L^4}^4 - \|u+c_0\|^2 + \langle\zeta, u+c_0\rangle \\ &\quad - \langle(u+c_0)^3, c_0\rangle + \langle u+c_0, c_0\rangle - \langle\zeta, c_0\rangle - \beta\langle(-\Delta)^{-1}u_x, u\rangle. \end{aligned} \tag{4.15}$$

This is legitimate, because all the ingredients are in L^2. Subsequently, we use a series of estimates based on Young's inequality and on the fact that $|c_0| \leq 1$, namely,

$$\begin{aligned} &c_0(u+c_0)^3 \leq \frac{1}{8}(u+c_0)^4 + 54, \qquad (u+c_0)^2 \leq \frac{1}{2}(u+c_0)^4 + \frac{1}{2}, \\ &c_0(u+c_0) \leq \frac{1}{2}(c_0^2 + (u+c_0)^2) \leq \frac{1}{2}c_0^2 + \frac{1}{4}(u+c_0)^4 + \frac{1}{4} \leq \frac{1}{4}(u+c_0)^4 + \frac{3}{4}. \end{aligned} \tag{4.16}$$

If we combine these inequalities with (4.15) and take into account that $|c_0| < 1$, then we reach

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(-\Delta)^{-1/2}u\|^2 + \|u_x\|^2 + \|u+c_0\|_{L^4}^4 - \frac{1}{2}\|u+c_0\|_{L^4}^4 - \frac{L}{2} \\ &\leq \frac{1}{8}\|u+c_0\|_{L^4}^4 + 54L + \frac{1}{4}\|u+c_0\|_{L^4}^4 + \frac{3L}{4} + |\langle\zeta, c_0\rangle| + |\langle\zeta, u+c_0\rangle| + \beta|\langle(-\Delta)^{-1}u_x, u\rangle|. \end{aligned} \tag{4.17}$$

We can now use Young's inequality with $\epsilon = 1/8$ and the fact that $|\zeta| \le 1$ to estimate

$$\zeta(u+c_0) \le \frac{1}{2}|\zeta|^2 + \frac{1}{2}(u+c_0)^2 \le \frac{1}{2} + \frac{1}{2}\left(\frac{\epsilon}{2}(u+c_0)^4 + \frac{1}{2\epsilon}\right) = \frac{1}{32}(u+c_0)^4 + \frac{5}{2},$$

whence

$$|\langle \zeta, u+c_0\rangle| \le \frac{1}{32}\|u+c_0\|_{L^4}^4 + \frac{5L}{2}. \tag{4.18}$$

Moreover, by observing that

$$\|(-\Delta)^{-1} f\| \le L^2 \|f\| \tag{4.19}$$

and using Young's inequality with ϵ and η, we can estimate,

$$\begin{aligned}
|\langle(-\Delta)^{-1}u_x, u\rangle| &= |\langle(-\Delta)^{-1}u_x, u+c_0\rangle - \langle(-\Delta)^{-1}u_x, c_0\rangle| \\
&\le \frac{\epsilon}{2}\|(-\Delta)^{-1}u_x\|^2 + \frac{1}{2\epsilon}\|u+c_0\|^2 + \frac{\epsilon}{2}\|(-\Delta)^{-1}u_x\|^2 + \frac{1}{2\epsilon}\|c_0\|^2 \\
&\le L^4\epsilon\|u_x\|^2 + \frac{1}{2\epsilon}\frac{\eta}{2}\|u+c_0\|_{L^4}^4 + \frac{1}{2\epsilon}\frac{1}{2\eta}L + \frac{1}{2\epsilon}L \\
&= \frac{1}{2}\|u_x\|^2 + \frac{1}{32}\|u+c_0\|_{L^4}^4 + 8L^9 + L^5,
\end{aligned} \tag{4.20}$$

after choosing $\epsilon = 1/2L^4$ and $\eta = 1/16L^4$. By using inequalities (4.18) and (4.20) in (4.17) and rearranging terms, we get

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(-\Delta)^{-1/2}u\|^2 + \frac{1}{2}\|u_x\|^2 + \frac{1}{16}\|u+c_0\|_{L^4}^4 &\le \frac{231}{4}L + L^5 + 8L^9 + |\langle\zeta, c_0\rangle| \\
&\le \frac{235}{4}L + L^5 + 8L^9 =: K_1(L),
\end{aligned} \tag{4.21}$$

which proves (4.14).

Step 3. Now we are able to prove the existence of an absorbing set. To do so, we define the "energy"

$$\mathcal{E}_1(t) := \int_\Omega W_0(u+c_0)\,\mathrm{d}x + \frac{1}{2}\|u_x\|^2 + \|(-\Delta)^{-1/2}u\|^2. \tag{4.22}$$

We will prove that $\mathcal{E}_1(t)$ is bounded if t is larger than a certain $t_B > 0$ that only depends on the data of the problem.

By adding estimate (4.14) to estimate (4.10), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t) + \theta\mathcal{E}_1(t) - \theta\left(\int_\Omega W_0(u+c_0)\,\mathrm{d}x + \frac{1}{2}\|u_x\|^2 + \|(-\Delta)^{-1/2}u\|^2\right) \\
&\qquad + \frac{1}{8}\|u+c_0\|_{L^4}^4 + \|u_x\|^2 + \frac{1}{2}\|(-\Delta)^{-1/2}u_t\|^2 \le \beta\|u\|^2 + \|\zeta_x\|^2 + 2K_1(L).
\end{aligned}$$

Here we added and subtracted a small fraction of $\mathcal{E}_1$ ($\theta > 0$). A rearrangement yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t) + \theta\mathcal{E}_1(t) + \left(\frac{1}{8} - \frac{\theta}{4}\right)\|u+c_0\|_{L^4}^4 + \left(1 - \frac{\theta}{2}\right)\|u_x\|^2 \\
&\le \theta\|(-\Delta)^{-1}u\|^2 + \frac{L\theta}{4} + \beta\|u\|^2 + \|\zeta_x\|^2 + 2K_1(L) \le (\theta L^4 + 1)\|u\|^2 + \frac{L\theta}{4} + \|\zeta_x\|^2 + 2K_1(L),
\end{aligned}$$

where we used (4.19) and $\beta \le 1$. By Young's inequality with $\epsilon = 1/16$, we obtain

$$\|u\|^2 \le \frac{1}{16}\|u+c_0\|_{L^4}^4 + 18L.$$

Thus, recalling (4.19), we estimate

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t) + \theta\mathcal{E}_1(t) + \left(\frac{1}{8} - \frac{\theta}{4}\right)\|u+c_0\|_{L^4}^4 + \left(1 - \frac{\theta}{2}\right)\|u_x\|^2 \\
&\le \frac{\theta L^4 + 1}{16}\|u+c_0\|_{L^4}^4 + \left[18(\theta L^4 + 1) + \frac{\theta}{4}\right]L + \|\zeta_x\|^2 + 2K_1(L)
\end{aligned}$$

By choosing $\bar\theta = (L^4+4)^{-1}$, the term with $\|u+c_0\|_{L^4}^4$ vanishes and we can neglect the term containing $\|u_x\|^2$, since $\bar\theta < 2$; we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t) + \frac{1}{L^4+4}\mathcal{E}_1(t) \le \frac{L(144(L^4+2)+1)}{4(L^4+4)} + \|\zeta_x\|^2 + 2K_1(L) =: K_2(L,\zeta),$$

so that Gronwall inequality leads to

$$\mathcal{E}_1(t) \le \big(\mathcal{E}_1(0) - (L^4+4)K_2(L,\zeta)\big)e^{-t/(L^4+4)} + (L^4+4)K_2(L,\zeta) \tag{4.23}$$

and this is enough to conclude. Indeed, it is easy to see that there exists $t_B > 0$ depending on $\mathcal{E}_1(0)$, L, and ζ such that $\mathcal{E}_1(t) < (L^4+4)K_2(L,\zeta) + 1 =: M_1(L,\zeta)$ for all $t \ge t_B$. ☐

The proof we presented above shows a uniform-in-$\beta \le 1$ bound on the absorbing set and time t_B . However, we need uniform estimates on u in higher Sobolev norm. The constant variation formula (4.8) yields bounds on $w = u - \eta$, hence we have them for u too.

Lemma 4.5. *Let us suppose that $w_0 \in B \subset V$, where B is bounded, and that $t \ge t_2 := t_B + 1$, with t_B provided by Proposition 4.4. Then there exists a positive constant $M_2(B,L,\eta,\zeta)$, depending only on B, L, η, and ζ, such that $S_\beta(t)w_0 \in H^2(0,L)$ and $\|w(t)\|_{H^2(0,L)} \le M_2(B,L,\eta,\zeta)$.*

Proof. It is enough to apply Δ to both sides of (4.8) to estimate the second-order derivative of w. If we take $\lambda \in (0, \inf\sigma(\Delta^2))$, then estimate (4.5) yields

$$\begin{aligned}\|\Delta w(t)\| \le{}& \|\Delta(e^{-\Delta^2(t-t_B)}(w-\eta)+\eta)\|\\ &+ \int_{t_B}^{t} \big\|\Delta\big(e^{-\Delta^2(t-\sigma)}\big(\Delta\partial_s W(x, w+\eta+c_0) + \beta w_x - \beta\eta_x\big)\big)\big\|\,\mathrm{d}\sigma =: I_1(t) + I_2(t).\end{aligned}$$

It is easy to see that due to (4.23) we have the following estimate, where we used (4.5) again,

$$\begin{aligned}I_1(t) &\le C_{1/4}(t-t_B)^{-1/4}e^{-\lambda(t-t_B)}\|w_0\|_{H^1} + C_{1/2}(t-t_B)^{-1/2}e^{-\lambda(t-t_B)}\|\eta\| + \|\eta_{xx}\|\\ &\le C_{1/4}\|w_0\|_{H^1} + C_{1/2}\|\eta\| + \|\eta_{xx}\| =: c_0(B,L,\eta),\end{aligned}$$

for every $t \ge t_2 = t_B + 1$. Moreover,

$$\begin{aligned}I_2(t) \le{}& \int_{t_B}^{\infty} \|\Delta(e^{-\Delta^2(t-\sigma)}\Delta\partial_s W(x,w+\eta+c_0))\|\,\mathrm{d}\sigma + \int_{t_B}^{\infty} \|\Delta\big(e^{-\Delta^2(t-\sigma)}(\beta w_x - \beta\eta_x)\big)\|\,\mathrm{d}\sigma\\ =:{}& J_1(t) + J_2(t).\end{aligned}$$

Again, since for $t \ge t_B$ the solution lies in the absorbing set, it is easy to see that

$$J_2(t) \le \beta \int_{t_B}^{\infty} C_{1/2}\frac{e^{-\lambda(t-s)}}{(t-s)^{1/2}}(M_1(L,\zeta) + \|\eta_x\|)\,\mathrm{d}\sigma =: c_1(B,L,\eta,\zeta).$$

Finally, we estimate J_1 ; for this purpose we notice that

$$\big\|\big((w+\eta+c_0)^3\big)_x\big\| \le 3\big(LM_1(L,\zeta) + (\|\eta\|_{L^\infty}+1)^2\big)(\|w_x\| + \|\eta\|) =: c_2(B,L,\eta,\zeta),$$

where we also used that $\|w\|_{L^\infty} \le L^{1/2}\|w_x\|$ for $w \in V$.

Our claim follows by combining these estimates with the bound in V, *i.e.*,

$$M_2(B,L,\eta,\zeta) := \sqrt{M_1(L,\zeta)^2 + c_0(B,L,\eta)^2 + c_1(B,L,\eta,\zeta)^2 + c_2(B,L,\eta,\zeta)^2}.$$

The proof is concluded. ☐

The above lemma implies that the absorbing set in V is compact. We combine it with the following well-knows fact.

Proposition 4.6. *(see* [16, Theorem 1]*) Let us suppose that a strongly continuous semigroup $S(\cdot)$ on Z has a compact attracting set K. Then there is a compact global attractor for $S(\cdot)$ and $\mathcal{A} = \omega(K)$.*

In this way we deduce the following corollary.

Corollary 4.7. *For all $\beta \in (0,1)$ the semigroup S_β has a global attractor, $\mathcal{A}_\beta$, which is compact in H^1-topology. Moreover, there exists $R > 0$ such that $\mathcal{A}_\beta$ is contained in the ball $B_V(0;R)$ in the H^1-topology.*

Proof. Proposition 4.4 combined with Lemma 4.5 imply that the absorbing set is compact in the H^1-topology. The fact that S_β is a strongly continuous semigroup in V was established in the course of proof of [3, Theorem 6.5]. Hence, Proposition 4.6 yields existence of a global attractor for each $\beta \in (0,1]$. The uniform bound on the attractor in H^1-topology follows from Proposition 4.4. □

We finally note the following uniform bounds, away from initial time.

Proposition 4.8 (Uniform higher-order bounds after an arbitrary delay)**.** *Let $B \subset V$ be bounded and let t_B be the entering time of the absorbing ball provided by Proposition 4.4. Then, for every $\tau > 0$ and every $m \geq 2$, there exists a positive constant $M_{m,\tau} = M_{m,\tau}(B, L, \eta, \zeta)$ depending on B, L, η, and ζ, such that, for all $\beta \in [0,1]$, all $u_0 \in B$, and all $t \geq t_B + \tau$,*

$$\|S_\beta(t)u_0\|_{H^m(0,L)} \leq M_{m,\tau}.$$

Proof. Let $w(t) = S_\beta(t)u_0 - \eta$. Fix $\tau > 0$ and set

$$\tau_j := 2^{-j}\tau, \qquad s_n := \sum_{j=1}^{n} \tau_j.$$

Then $s_n < \tau$ for every $n \in \mathbb{N}$ and $s_n \uparrow \tau$.

By Proposition 4.4, there exists $M_1(L,\eta,\zeta) > 0$ such that

$$\sup_{\beta\in[0,1]} \sup_{u_0\in B} \sup_{t\geq t_B} \|w(t)\|_{H^1(0,L)} \leq M_1(L,\eta,\zeta).$$

We start with the case $m = 2$. Arguing exactly as in the proof of Lemma 4.5, but writing (4.8) on the interval $[t-\tau_1, t]$ instead of $[t_B, t]$, we find a positive constant $M_{2,\tau} = M_{2,\tau}(B, L, \eta, \zeta)$ such that

$$\sup_{\beta\in[0,1]} \sup_{u_0\in B} \sup_{t\geq t_B+s_1} \|w(t)\|_{H^2(0,L)} \leq M_{2,\tau}\,.$$

Since we have an absorbing set in H^1, the constant $M_{2,\tau}$ depends on B not on an individual $u_0 \in B$. Neither $M_{2,\tau}$ depends on t, because the function $s \mapsto e^{-\lambda s}s^{-\alpha}$ is integrable over $(0,\infty)$, provided that $\lambda > 0$ and $\alpha \in (0,1)$.

We now bootstrap. Assume that for some $m \geq 2$ we already know that there exists a positive constant $M_{m,\tau} = M_{m,\tau}(u_0, L, \eta, \zeta)$ such that

$$\sup_{\beta\in[0,1]} \sup_{u_0\in B} \sup_{t\geq t_B+s_{m-1}} \|w(t)\|_{H^m(0,L)} \leq M_{m,\tau}\,.$$

Fix $t \geq t_B + s_m$. Then $t - \tau_m \geq t_B + s_{m-1}$, so the induction hypothesis is available on the whole interval $[t-\tau_m, t]$. Arguing as in (4.9) we have the existence of a positive constant

$$\sup_{s\in[t-\tau_m,t]} \|F_\beta(w(s))\|_{H^{m-2}(0,L)} \leq \widetilde{C}_{m,\tau},$$

where $C_{m,\tau}$ is independent of t, β, and $u_0 \in B$. Writing (4.8) with initial time $t - \tau_m$, applying $A^{(m+1)/4}$, and using (4.5), we obtain

$$\begin{aligned}\|w(t)\|_{H^{m+1}(0,L)} \leq\, & C\big\|A^{1/4}e^{-A\tau_m}A^{m/4}w(t-\tau_m)\big\| \\ & + C\int_{t-\tau_m}^{t} \big\|A^{3/4}e^{-A(t-s)}A^{(m-2)/4}F_\beta(w(s))\big\|\,\mathrm{d}s \\ \leq\, & C\tau_m^{-1/4}\|w(t-\tau_m)\|_{H^m(0,L)} + C\int_{t-\tau_m}^{t} (t-s)^{-3/4}\|F_\beta(w(s))\|_{H^{m-2}(0,L)}\,\mathrm{d}s \\ \leq\, & M_{m+1,\tau}.\end{aligned}$$

The constant $M_{m+1,\tau}$ is independent of t, β, and $u_0 \in B$.

This proves by induction that for every $m \geq 2$,

$$\sup_{\beta \in [0,1]} \sup_{u_0 \in B} \sup_{t \geq t_B + s_{m-1}} \|w(t)\|_{H^m(0,L)} \leq M_{m,\tau}.$$

Since $s_{m-1} < \tau$, the same bound holds a fortiori for all $t \geq t_B + \tau$. Finally, since η is smooth, the estimate transfers from w to $u = w + \eta$. □

5. STABILIZATION OF SOLUTIONS

5.1. **Tools of the dynamical systems.** Our aim is to recall the weak notion of the gradient flow studied in [5]. It is based on the notion of global compact attractor.

Definition 5.1. ([5, Definition 1.5]) A set $\mathcal{A} \subset Z$ is the global attractor for a semigroup $S(\cdot)\colon Z \to Z$ if (i) $\mathcal{A}$ is compact; (ii) $\mathcal{A}$ is invariant; (iii) $\mathcal{A}$ attracts each bounded subset of Z.

We may now recall notion of the gradient flow used in [5].

Definition 5.2. [5, Definition 5.3, Definition 5.4, Theorem 5.5] We say that a semigroup S with a global attractor $\mathcal{A}$ is a gradient flow with respect to the family $\mathcal{S} = \{\mathcal{E}^0, \dots, \mathcal{E}^k\}$ of invariant sets provided that:
1) For any global (eternal) solution $\xi\colon \mathbb{R} \to Z$ taking values in $\mathcal{A}$, there exist $i, j \in \{0, \dots, k\}$ such that

$$\lim_{t \to -\infty} \operatorname{dist}(\xi(t), \mathcal{E}^i) = 0 \quad \text{and} \quad \lim_{t \to +\infty} \operatorname{dist}(\xi(t), \mathcal{E}^j) = 0.$$

2) The collection $\mathcal{S}$ contains no homoclinic structures.

Before we state our main tool we recall that $\operatorname{dist}_{\mathrm{H}}(A, B)$ denotes the Hausdorff distance between compact sets A and B, which is defined with the help of the metric of the ambient space Z.

The theorem below calls for checking the collective assymptotic compactness. We say that strongly continuous semigroups $\{S_\beta(\cdot)\}_\beta$ on Z are *collectively asymptotically compact* provided that for any sequence $\{\beta_n\}_n$ for any sequence $t_n \to +\infty$, and any bounded sequence $\{x_n\} \subset Z$ such that $\{S_{\beta_n}(t_n)x_n\}_n$ is also bounded, we can show that $\{S_{\beta_n}(t_n)x_n\}_n$ contains a convergent subsequence.

Our main tool is the following theorem.

Theorem 5.3. ([5, Theorem 5.26]: Stability of gradient semigroups). *Let $S_0(\cdot)$ be a semigroup on a Banach space $(Z, \|\cdot\|_Z)$ that has a global attractor $\mathcal{A}_0$ and that is a gradient flow with respect to the finite collection $\mathcal{S}^0$ of isolated invariant sets $\{\mathcal{E}_0^0, \mathcal{E}_0^1, \dots, \mathcal{E}_0^k\}$. Assume that:*

(a) *for each $\beta \in (0,1]$, $S_\beta(\cdot)$ is a semigroup on Z with a global attractor $\mathcal{A}_\beta$;*

(b) *$\{S_\beta(\cdot)\}_{\beta \in [0,1]}$ is collectively asymptotically compact and $\overline{\bigcup_{\beta \in [0,1]} \mathcal{A}_\beta}$ is bounded;*

(c) *$S_\beta(\cdot)$ converges to $S_0(\cdot)$, in the sense that*

$$\|S_\beta(t)u - S_0(t)u\|_Z \to 0 \qquad \text{as} \qquad \beta \to 0$$

uniformly for u in compact subsets of Z; and

(d) *for $\beta \in (0,1]$, the attractors $\mathcal{A}_\beta$ contains a finite collection of isolated invariant sets $\mathcal{S}_\beta = \{\mathcal{E}_\beta^0, \mathcal{E}_\beta^1, \dots, \mathcal{E}_\beta^k\}$ such that*

$$\lim_{\beta \to 0} \operatorname{dist}_{\mathrm{H}}(\mathcal{E}_\beta^j, \mathcal{E}_0^j) = 0$$

and there exist $\eta > 0$ and $\beta_1 \in (0,1)$ such that for all $\beta \in (0, \beta_1)$, if $\xi_\beta\colon \mathbb{R} \to \mathcal{A}_\beta$ is a global (or eternal) solution, then

$$\sup_{t \in \mathbb{R}} \operatorname{dist}(\xi_\beta(t), \mathcal{E}_\beta^j) \leq \eta \quad \Rightarrow \quad \xi_\beta(t) \in \mathcal{E}_\beta^j \text{ for all } t \in \mathbb{R}.$$

Then there exists a $\beta_2 \in (0, \beta_1)$ such that $\{S_\beta(\cdot)\}_{\beta \in (0, \beta_2)}$ is a gradient semigroup with respect to $\mathcal{S}_\beta$. In particular, for $\beta \in (0, \beta_2)$,

$$\mathcal{A}_\beta = \bigcup_{i=1}^{k} \mathcal{W}^{\mathrm{u}}(\mathcal{E}_\beta^i)\,.$$

One of the thing we have to do before we apply Theorem 5.3 is to choose the Banach space Z. Corollary 4.7 tells us that the right choice for Z is $Z = V$, which is equal to $\mathscr{H}^{1/4}$.

5.2. **Convergence of solutions.** In this subsection we check that conditions of Theorem 5.3 are satisfied.

Lemma 5.4 (Hypothesis (b) of Theorem 5.3)**.** *Let $Z = V = \mathscr{H}^{1/4}$. Then the family $\{S_\beta(\cdot)\}_{\beta\in[0,1]}$ is collectively asymptotically compact on Z. Moreover, $\overline{\bigcup_{\beta\in[0,1]} \mathcal{A}_\beta}$ is bounded in Z.*

Proof. Boundedness of the union of attractors. By Corollary 4.7, there exists $R > 0$, independent of β, such that

$$\mathcal{A}_\beta \subset B_V(0; R) \qquad \text{for all } \beta \in (0, 1].$$

The same uniform dissipative estimate used to prove Corollary 4.7 (and ultimately Proposition 4.4) does not depend on β (the transport term is lower order), hence the same R also bounds $\mathcal{A}_0$ in $H^1(0, L)$. Therefore, $\bigcup_{\beta\in[0,1]} \mathcal{A}_\beta$ is bounded in V, hence so is its closure.

Collective asymptotic compactness. Let $\{\beta_n\} \subset [0, 1]$, let $t_n \to +\infty$, and let $B := \{x_n : n \in \mathbb{N}\} \subset V$ be bounded. By Lemma 4.5, there exist $t_2 > 0$ and $M_2 = M_2(B, L, \eta, \zeta) > 0$ (depending only on the bound of B in V, but not on β) such that for every $\beta \in [0, 1]$ and every $x \in B$,

$$S_\beta(t)x \in H^2(0, L) \quad \text{and} \quad \|S_\beta(t)x\|_{H^2(0,L)} \le M_2 \qquad \text{for all } t \ge t_2\,.$$

Choose $N \in \mathbb{N}$ such that $t_n \ge t_2$ for all $n \ge N$. Since, in addition, $S_{\beta_n}(t_n)x_n \in D(A)$, then $\{S_{\beta_n}(t_n)x_n\}_{n\ge N}$ is bounded in $\mathscr{H}^{1/2} \subset H^2(0, L)$. Since the embedding $\mathscr{H}^{1/2} \hookrightarrow \mathscr{H}^{1/4} = V$ is compact by the Rellich Theorem, the sequence $\{S_{\beta_n}(t_n)x_n\}_{n\ge N}$ has a convergent subsequence in V. This is precisely collective asymptotic compactness in $Z = V$. □

Now, we will check that hypothesis (c) holds.

Lemma 5.5 (Hypothesis (c) of Theorem 5.3)**.** *Let $Z = V = \{u \in H^1(0, L) : u(0) = 0\}$ and let $K \subset V$ be compact. Then for every $T > 0$ we have*

$$\sup_{t\in[0,T]} \sup_{u_0\in K} \|S_\beta(t)u_0 - S_0(t)u_0\|_Z \longrightarrow 0 \qquad \text{as } \beta \to 0.$$

In particular, for each fixed $t \ge 0$,

$$\sup_{u_0\in K} \|S_\beta(t)u_0 - S_0(t)u_0\|_Z \longrightarrow 0 \qquad \text{as } \beta \to 0,$$

which is exactly hypothesis (c) *of Theorem 5.3.*

Proof. Fix $T > 0$. Since $K \subset V$ is compact, it is bounded, hence there exists $M > 0$ such that

$$\|u_0\|_V \le M \qquad \text{for all } u_0 \in K.$$

For $u_0 \in K$ and $\beta \in (0, 1)$, set $u_\beta(t) := S_\beta(t)u_0$ and $\bar{u}(t) := S_0(t)u_0$. By [3, Theorem 6.6] (applied with $\beta_0 = 1$ and with our potential W, see (1.2)), there exists a constant $C > 0$, depending only on L, W, T, and the bound M (but not on the particular choice of $u_0 \in K$), such that

$$\|u_\beta(t) - \bar{u}(t)\|_V \le C\,\beta \qquad \text{for all } t \in [0, T] \text{ and all } \beta \in (0, 1).$$

Taking the supremum over $u_0 \in K$ and $t \in [0, T]$ gives

$$\sup_{t\in[0,T]} \sup_{u_0\in K} \|S_\beta(t)u_0 - S_0(t)u_0\|_V \le C\,\beta \xrightarrow[\beta\to 0]{} 0.$$

Since $Z = V$, this yields the desired convergence in Z and proves hypothesis (c). □

We are going to check that part (d) of the hypothesis of Theorem 5.3 holds. In the present case the invariant sets $\mathcal{E}^j_\beta$ are points in Z, *i.e.*, u^j_β, hence $\mathrm{dist}_{\mathrm{H}}(\mathcal{E}^j_\beta, \mathcal{E}^j_0) = \|u^j_\beta - u^j_0\|_Z\,$.

Lemma 5.6 (hypothesis (d) of Theorem 5.3 is satisfied)**.** *Let us suppose that* $L > 0$ *does not belong to* E*. Then,*

$$\lim_{\beta\to 0} \|u^j_\beta - u^j_0\|_Z = 0. \tag{5.1}$$

Moreover, there exist $\gamma > 0$ *and* $\beta_1 > 0$ *such that if* $\xi_\beta\colon \mathbb{R} \to \mathcal{A}_\beta$ *, for all* $\beta < \beta_1$ *, is an eternal solution, then*

$$\sup_{t\in\mathbb{R}} \|\xi_\beta(t) - u^j_\beta\|_Z \le \gamma \quad\Longrightarrow\quad \xi_\beta(t) = u^j_\beta \quad \textit{for all } t \in \mathbb{R}.$$

Proof. Step 1. The limit in (5.1) is an immediate consequence of (3.2) from Theorem 3.1. We are left with the latter statement. Since we deal with couples $(L, \tilde{z}^j)$ which are nondegenerate, as an immediate consequence, we have that the linearized operator $\mathscr{L}^\beta_L$ associated with (1.3), namely

$$\mathscr{L}^\beta_L h = \Delta(-\Delta h + \partial^2_{s^2} W_0(u^j_\beta)h) - \beta h_x\,,$$

is non-singular when $\beta = 0$, due to Theorem 2.10.

Step 2. We claim that $\mathscr{L}^\beta_L$ is not only non-singular for $\beta \in (0, \beta_1)$, for a certain $\beta_1 > 0$ (that will be defined at the end of this step), but also that there exists $\delta_0 > 0$ such that for all $\delta \in (-\delta_0, \delta_0)$ and all $a \in \mathbb{R}$ we have

$$\delta + ai \in \rho(\mathscr{L}^\beta_L) \qquad \text{for } \beta \in (0, \beta_1). \tag{5.2}$$

For this purpose we recall a well-known fact that $\mathscr{L}^0_L$ is equivalent to a self-adjoint operator $T\colon D(T) \subset L^2(0, L) \to L^2(0, L)$, *i.e.*,

$$(-\Delta)^{1/2}(-\Delta + \partial^2_{s^2} W_0(u^j_\beta))(-\Delta)^{1/2} =: T = (-\Delta)^{-1/2}\mathscr{L}^0_L(-\Delta)^{1/2},$$

where

$$D(T) = \{u \in H^4 : (-\Delta)^{1/2}u(0) = ((-\Delta)^{1/2}u)_x(L) = \mu((-\Delta)^{1/2}u)(0) = \mu((-\Delta)^{1/2}u)_x(L) = 0\}.$$

The operators $-\Delta$ and $-\Delta + \partial^2_{s^2} W_0(u^j_0 + c_0)$ are self-adjoint. Hence,

$$\begin{aligned}(Tu, w) &= \big((-\Delta + \partial^2_{s^2} W_0(x, u^j_\beta + c_0))(-\Delta)^{1/2}u, (-\Delta)^{1/2}w\big)\\ &= \big((-\Delta)^{1/2}u, (-\Delta + \partial^2_{s^2} W_0(u^j_\beta))(-\Delta)^{1/2}w\big) = \big(u, (-\Delta)^{1/2}(-\Delta + \partial^2_{s^2} W_0(u^j_\beta))(-\Delta)^{1/2}w\big)\\ &= (u, Tw).\end{aligned}$$

Due to this equivalence we conclude that the spectrum of $\mathscr{L}^0_L$ is real.

We define $\delta_0 >$ as follows. Since $\sigma(\mathscr{L}^0_L)$ is real discrete and it does not contain zero there is $\delta_0 > 0$ such that if z belongs to the strip $\Sigma_{\delta_0} = \{z \in \mathbb{C} : |\Re z| \le \delta_0\}$, then the distance from z to $\sigma(\mathscr{L}^0_L)$ is at least δ_0. Now, we shall see that there is $\beta_1 > 0$ such that $\Sigma_{\delta_0} \subset \rho(\mathscr{L}^\beta_L)$ for all $\beta \in (0, \beta_1)$. We compute

$$\begin{aligned}(\mathscr{L}^\beta_L - (\delta + ai)Id)^{-1} &= (\mathscr{L}^0_L - (\delta + ai)Id + \mathscr{L}^\beta_L - \mathscr{L}^0_L)^{-1}\\ &= (\mathscr{L}^0_L - (\delta + ai)Id)^{-1}(Id + (\mathscr{L}^0_L - (\delta + ai)Id)^{-1}(\mathscr{L}^\beta_L - \mathscr{L}^0_L))^{-1}.\end{aligned}$$

Our claim will follow if we show that

$$\|(\mathscr{L}^0_L - (\delta + ai)Id)^{-1}(\mathscr{L}^\beta_L - \mathscr{L}^0_L)\| \le \frac{1}{2}$$

for all $\beta \in (0, \beta_1)$ independently of a and δ_0 . We notice that

$$\|(\mathscr{L}^0_L - (\delta + ai)Id)^{-1}(\mathscr{L}^\beta_L - \mathscr{L}^0_L)\| \le \|(\mathscr{L}^0_L - (\delta + ai)Id)^{-1}\mathscr{L}^0_L\| \cdot \|(\mathscr{L}^0_L)^{-1}(\mathscr{L}^\beta_L - \mathscr{L}^0_L)\|.$$

Moreover, since

$$(\mathscr{L}^0_L - (\delta + ai)Id)^{-1}\mathscr{L}^0_L = Id + (\delta + ai)(\mathscr{L}^0_L - aiId)^{-1}$$

and $\|(\mathscr{L}^0_L - (\delta + ai)Id)^{-1}\|$ is bounded by the inverse of the distance from $\delta + ai$ to $\sigma(\mathscr{L}^0_L)$, we infer that

$$\|(\mathscr{L}^0_L - (\delta + ai)Id)^{-1}\mathscr{L}^0_L\| \le 1 + \frac{\sqrt{\delta_0^2 + a^2}}{\sqrt{\delta_0^2 + a^2}} = 2.$$

Now, we investigate $(\mathscr{L}_L^0)^{-1}(\mathscr{L}_L^\beta - \mathscr{L}_L^0)$. We notice that

$$(\mathscr{L}_L^\beta - \mathscr{L}_L^0)h = 3\Delta[((u_\beta^j)^2 - (u_0^j)^2)h] - \beta h_x\,.$$

Now, Theorem 3.1 implies that

$$\|(\mathscr{L}_L^0)^{-1}(\mathscr{L}_L^\beta - \mathscr{L}_L^0)\| \le C_1\beta \qquad \text{for } \beta \in (0,\beta_0)$$

where C_1 depends on $\|u_0^j\|_{H^4}$ and the constant C appearing in (3.2). By combining these estimates, we reach

$$\|(\mathscr{L}_L^0 - aiId)^{-1}(\mathscr{L}_L^\beta - \mathscr{L}_L^0)\| \le 2C_1\beta \le \frac{\beta_0}{2} \le \frac{1}{2},$$

for $0 < \beta < \beta_1 := \frac{\beta_0}{4C_1}$.

Step 3. By (5.2), the equilibrium u_β^j is hyperbolic. Hence there exist local stable and unstable manifolds $\mathcal{W}^{\rm s}_{\rm loc}(u_\beta^j)$ and $\mathcal{W}^{\rm u}_{\rm loc}(u_\beta^j)$ (see [9, Theorem 2.3]). Moreover, there exists $\gamma > 0$ such that the following characterisation holds: if a solution $z(t)$ satisfies $z(0) \in B_Z(u_\beta^j,\gamma)$ and $z(t) \in B_Z(u_\beta^j,\gamma)$ for all $t \ge 0$, then $z(0) \in \mathcal{W}^{\rm s}_{\rm loc}(u_\beta^j)$; similarly, if $z(t) \in B_Z(u_\beta^j,\gamma)$ for all $t \le 0$, then $z(0) \in \mathcal{W}^{\rm u}_{\rm loc}(u_\beta^j)$.

Now let $\xi_\beta\colon \mathbb{R} \to \mathcal{A}_\beta$ be an eternal solution with $\sup_{t\in\mathbb{R}} \|\xi_\beta(t) - u_\beta^j\|_Z \le \gamma$. Then $\xi_\beta(0) \in \mathcal{W}^{\rm s}_{\rm loc}(u_\beta^j)$ (by the $t \ge 0$ condition) and $\xi_\beta(0) \in \mathcal{W}^{\rm u}_{\rm loc}(u_\beta^j)$ (by the $t \le 0$ condition), hence

$$\xi_\beta(0) \in \mathcal{W}^{\rm s}_{\rm loc}(u_\beta^j) \cap \mathcal{W}^{\rm u}_{\rm loc}(u_\beta^j) = \{u_\beta^j\}.$$

Therefore $\xi_\beta(t) \equiv u_\beta^j$ for all $t \in \mathbb{R}$, which is exactly the implication required in hypothesis (d). ☐

After checking that the assumptions of Theorem 5.3 are satisfied we may state our main result.

Theorem 5.7. *Let us suppose that $L > 0$ does not belong to E and let $\beta \in (0,\beta_2)$, where β_2 is provided by Theorem 5.3 for this L. Then for any $u_0 \in V$ there exists $\psi \in C^\infty(0,L)$, such that the unique solution u to* (1.3) *corresponding to u_0 converges to ψ in $H^m(0,L)$ for all $m \in \mathbb{N}$.*

Proof. We divide the proof into three steps.

Step 1. We are going to check that we may invoke Theorem 5.3 while taking $Z = V = \mathscr{H}^{1/4}$. In the course of proof of Corollary 4.7 we recalled that semigroup S_β generated by (1.3) is strongly continuous in V. Above all, this corollary shows existence of global attractors and a uniform bound, hence the hypothesis (a) of Theorem 5.3 holds.

Step 2. We showed in Lemmas 5.4, 5.5, and 5.6 that the assumptions (b), (c), and (d) of Theorem 5.3 are satisfied. Thus, we deduce from this result existence of positive β_2 such that each semigroup S_β for $\beta \in (0,\beta_2)$ is a gradient flow in the sense of Definition 5.2. This implies that for any $u_0 \in V$ the ω-limit set $\omega(u_0)$ may consist only of the steady states, because there are no homoclinic orbits. Since the set $\mathcal{S}_\beta$ of steady states is finite and $\omega(u_0)$ is connected, we see that it is a singleton, thus there exists $\psi \in \mathscr{H}^{1/4}$ such that

$$\lim_{t\to+\infty} \|u(t) - \psi\|_V = 0.$$

Step 3. By Lemma 4.3, we have that $u(t) \in C^\infty(0,L)$ for every $t > 0$. Moreover, Proposition 4.8 applied with $B = \{u_0\}$ and $\tau = 1$ shows that, for every $N \ge 2$,

$$\sup_{t\ge t_B+1} \|u(t)\|_{H^N(0,L)} < \infty.$$

Since ψ is a steady state of (1.3), elliptic bootstrapping applied to the stationary equation yields $\psi \in C^\infty(0,L)$.

We already know from Step 2 that $u(t) \to \psi$ in $V = H^1(0,L)$. Fix $m \ge 2$ and choose $N > m$. Since both $u(t)$ and ψ are bounded in $H^N(0,L)$ for $t \ge t_B + 1$, interpolation gives

$$\|u(t) - \psi\|_{H^m(0,L)} \le C\,\|u(t) - \psi\|^{\theta}_{H^1(0,L)} \|u(t) - \psi\|^{1-\theta}_{H^N(0,L)} \to 0 \qquad \text{as } t \to \infty$$

for a suitable $\theta \in (0,1)$. This proves convergence in $H^m(0,L)$ for every $m \in \mathbb{N}$. ☐

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References

[1] N. D. Alikakos, G. Fusco, and P. Smyrnelis, *Elliptic systems of phase transition type*, vol. 91 of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser/Springer, Cham, 2018.

[2] K. B. Blodgett, *Films built by depositing successive monomolecular layers on a solid surface*, J. Am. Chem. Soc., 57 (1935), pp. 1007–1022.

[3] M. Bonacini, E. Davoli, and M. Morandotti, *Analysis of a perturbed Cahn-Hilliard model for Langmuir-Blodgett films*, NoDEA Nonlinear Differential Equations Appl., 26 (2019), pp. Paper No. 36, 40.

[4] J. W. Cahn and J. Hilliard, *Free energy of a nonuniform system. I. Interfacial free energy*, J. Chem. Phys., 28 (1958), pp. 258–267.

[5] A. Carvalho, J. Langa, and J. Robinson, *Attractors for infinite-dimensional non-autonomous dynamical systems*, Springer, New York, 2013.

[6] X. D. Chen, S. Lenhert, M. Hirtz, N. Lu, H. Fuchs, and L. F. Chi, *Langmuir-Blodgett patterning: A bottom-up way to build mesostructures over large areas*, Acc. Chem. Res., 40 (2007), pp. 393–401.

[7] A. Eden and V. K. Kalantarov, *The convective Cahn-Hilliard equation*, Appl. Math. Lett., 20 (2007), pp. 455–461.

[8] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[9] J. K. Hale and G. Raugel, *Convergence in gradient-like systems with applications to PDE*, Z. Angew. Math. Phys., 43 (1992), pp. 63–124.

[10] D. Henry, *Geometric theory of semilinear parabolic equations*, vol. 840 of Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1981.

[11] M. H. Köpf, S. V. Gurevich, R. Friedrich, and U. Thiele, *Substrate-mediated pattern formation in monolayer transfer: a reduced model*, New Journal of Physics, 14 (2012), p. 023016.

[12] M. D. Korzec, P. Nayar, and P. Rybka, *Global attractors of sixth order PDEs describing the faceting of growing surfaces*, J. Dynam. Differential Equations, 28 (2016), pp. 49–67.

[13] S. G. Krantz and H. R. Parks, *The implicit function theorem*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2013. History, theory, and applications, Reprint of the 2003 edition.

[14] I. Langmuir, *The constitution and fundamental properties of solids and liquids. II. Liquids. 1*, Journal of the American Chemical Society, 39 (1917), pp. 1848–1906.

[15] S. Łojasiewicz, *Introduction to complex analytic geometry*, Birkhäuser Verlag, Basel, 1991. Translated from the Polish by Maciej Klimek.

[16] G. Łukaszewicz, J. Real, and J. C. Robinson, *Invariant measures for dissipative systems and generalized Banach limits*, J. Dynam. Differential Equations, 23 (2011), pp. 225–250.

[17] A. Miranville, *The Cahn-Hilliard equation*, vol. 95 of CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2019. Recent advances and applications.

[18] L. Nirenberg, *Topics in nonlinear functional analysis*, vol. 6, American Mathematical Society, 1974.

[19] A. Novick-Cohen, *The Cahn-Hilliard equation*, in Handbook of differential equations: evolutionary equations. Vol. IV, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008, pp. 201–228.

[20] O. L. Oliveira, Jr., *Langmuir-Blodgett Films - Properties and Possible Applications*, Brazilian Journal of Physics, 22 (1992), pp. 60–69.

[21] O. Purrucker, A. Förtig, K. Lüdtke, R. Jordan, and M. Tanaka, *Confinement of Transmembrane Cell Receptors in Tunable Stripe Micropatterns*, J. Am. Chem. Soc., 127 (2005), pp. 1258–1264.

[22] G. Roberts (Ed.), *Langmuir-Blodgett Films*, Springer US, 1990.

[23] P. Rybka and G. Wheeler, *Convergence of solutions to a convective Cahn-Hilliard-type equation of the sixth order in case of small deposition rates*, SIAM J. Math. Anal., 55 (2023), pp. 5823–5861.

[24] M. Wilczek and S. V. Gurevich, *Locking of periodic patterns in Cahn-Hilliard models for Langmuir-Blodgett transfer*, Physical Review E, 90 (2014), p. 042926.

[25] J. A. Zasadzinski, R. Viswanathan, L. Madsen, J. Garnaes, and D. K. Schwartz, *Langmuir-blodgett films*, Science, 263 (1994), pp. 1726–1733.

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