DERIVATION OF LINEAR ELASTICITY FROM ENERGY FUNCTIONALS WITH INFINITELY MANY WELLS

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Abstract. In this paper we derive a linear elastic model starting from a nonlinear energy functional with infinitely many wells, including a second gradient perturbation that penalises transitions from one well to another. This is achieved by employing a suitable weak convergence for deformations which allows one to prove compactness by applying a rigidity estimate for incompatible fields.

Keywords: Nonlinear elasticity, Linearised elasticity, Gamma-convergence, Crystal symmetry.
MSC2020: 74B20, 74E15, 74G65, 49J45.

1. Introduction

This paper investigates the passage from nonlinear to linear elastic models for solid materials with infinitely many equilibrium states. We perform, through variational convergence, the linearisation of an energy functional given by the sum of a nonlinear elastic term and a second-order perturbation that controls the surface of transition layers.

The prototypical example of the nonlinear stored elastic energy under consideration is the integral functional

$$\int_\Omega W(\nabla v)\,\mathrm{d}x,$$

defined on a regular domain $\Omega \subset \mathbb{R}^d$ representing the reference configuration of an elastic body, where $v\colon \Omega \to \mathbb{R}^d$ is the deformation. Assuming that the reference configuration is an equilibrium, the energy density $W\colon \Omega \times \mathbb{R}^{d\times d} \to \mathbb{R}$ is minimised at the identity matrix. According to the assumption of frame indifference, one has $W(RF) = W(F)$ for every $R \in SO(d)$ and $F \in \mathbb{R}^{d\times d}$. Besides the invariance with respect to rotations in the deformed configuration, the energy may enjoy invariance with respect to transformations in the reference configuration, which are related to symmetries of the underlying crystal structure. Specifically, one may require that

$$W(FP) = W(F) \quad \text{for every } P \in \mathcal{P} \text{ and } F \in \mathbb{R}^d,$$

where $\mathcal{P}$ is the isotropy group of the crystal lattice, which may be a finite subgroup of $SO(d)$ or a larger noncompact subgroup of lattice invariant transformations. For a cubic lattice, $\mathcal{P}$ may be given by the group $GL(d,\mathbb{Z})$ of linear bijections of $\mathbb{Z}^d$ (with determinant ± 1) or by the subgroup $SL(d,\mathbb{Z})$ of orientation-preserving transformations. For a more general Bravais lattice parametrised by $H\mathbb{Z}^d$ with $H \in GL(d)$, one may have $\mathcal{P} = H\,GL(d,\mathbb{Z})\,H^{-1}$, i.e., $\mathcal{P}$ is conjugate to $GL(d,\mathbb{Z})$. For more details on this model see e.g. [19] and references therein.

In broad terms, in the present paper we consider a nonlinear energy with infinitely many wells $(SO(d)U_\ell)_{\ell\in\mathbb{N}}$, whose mutual distance is not controlled from above. This leads to a lack of coercivity at infinity which is overcome by choosing a suitable notion of convergence of the deformations. Moreover, as it is customary when dealing with multiwell problems [11, 3], in order to ensure

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February 27, 2026.

compactness we augment the stored elastic energy by a regularising second-order term which penalises transitions from one well to another. Specifically, for $p > 1$ and $\eta > 0$ we consider the functional

$$E^\eta(v) := \int_\Omega W(\nabla v)\,\mathrm{d}x + \eta^p \int_\Omega |\nabla^2 v|^p\,\mathrm{d}x,$$

defined for $v \in W^{2,p}(\Omega;\mathbb{R}^d)$, subject to suitable assumptions on the regularity and the growth of W, along with appropriate boundary conditions. In Proposition 2.4 it is shown that, if $E^\eta(v)/\eta$ vanishes as η tends to zero, then in most part of the domain the deformation gradient is close to the well $SO(d)U_1 = SO(d)$. More precisely, if $v(x) = x + \varepsilon g(x)$ on a portion of $\partial\Omega$ for g in a suitable class of boundary data and ε sufficiently small, we expect the deformation to be a perturbation of the identity. Accordingly, we introduce the (rescaled) displacements $u_\varepsilon(x) := (v(x)-x)/\varepsilon$ and study their asymptotics as ε tends to zero. Following, by now, a standard approach [14], one ought to rescale the energy by the factor ε^{-2} (which stands from the first nontrivial term in the Taylor expansion) and consider

$$\mathcal{F}_\varepsilon(u) := \frac{1}{\varepsilon^2} E^\eta(\mathrm{id}_{\mathbb{R}^d} + \varepsilon u),$$

which converges pointwise to the standard energy of linear elasticity

$$\mathcal{F}_{\mathrm{lin}}(u) := \frac{1}{2}\int_\Omega D^2W(x,I)e(u) : e(u)\,\mathrm{d}x.$$

The convergence of $\mathcal{F}_\varepsilon$ to $\mathcal{F}_{\mathrm{lin}}$ was proved in terms of Γ-convergence in [14, 2] in the case of a single well, in [25, 1] for a finite number of wells at infinitesimal distance, and in [8, 26] starting from a discrete setting; in all these cases $\eta = 0$. The case of a finite number of wells at fixed distance was studied in [3, 4] assuming that the small parameter $\eta = \eta(\varepsilon)$ satisfies a certain decay condition as ε tends to zero. In [16], in the setting of two wells with one rank-one connection, the authors combine linearisation with a diffuse-to-sharp interface limit.

The main difference between the present framework and the previous accounts is that, since the ground states of the elastic energy are unbounded, even a sequence of deformations with very small energy may still display very large deformation gradients. In effect, this may spark compactness issues in the weak topology of Sobolev spaces, which is a typical choice for problems of the kind. Hereby, the remedy is to resort to a weak notion of convergence for ∇u_ε, namely the weak L^r-convergence of $(\nabla u_\varepsilon)\chi_{\Omega_\varepsilon}$, with Ω_ε being the set where the deformation gradient ∇v_ε is close to $SO(d)$ (Definition 1.1). Since the standard rigidity estimate [21] can no longer be applied to $(\nabla v_\varepsilon)\chi_{\Omega_\varepsilon}$, the compactness result relies instead on a rigidity estimate for incompatible fields [24, 23, 10], which may be employed provided $r \geq 1^*$. The latter result yields that $(\nabla u_\varepsilon)\chi_{\Omega_\varepsilon}$ still converges to ∇u weakly in L^r, where u is the limit displacement (Theorem 1.2, proved in Section 2). The choice of this weak convergence is justified by Example 4.1, where, for $d > 2$ and $1^* < r < 2$, we show a sequence of displacements with equibounded energy that is unbounded in the norm of $W^{1,r}(\Omega;\mathbb{R}^d)$. In contrast, in the case of finitely many wells, compactness holds in $W^{1,r}(\Omega;\mathbb{R}^d)$ for such values of r, as proved in [3]. Equipped with the property of compactness of $(\nabla u_\varepsilon)\chi_{\Omega_\varepsilon}$, in Theorem 1.4 (proved in Section 3) we show that the Γ-convergence of $\mathcal{F}_\varepsilon$ to $\mathcal{F}_{\mathrm{lin}}$ persists in the setting of infinitely many wells. This is carried out under a scaling condition on $\eta = \eta(\varepsilon)$, which in some cases provides a larger range of allowed values for η with respect to the the assumptions of [3]. Note that in [3] the scaling was shown to be optimal; in the present context, the optimality is still an open problem. Nonetheless, when $\eta \ll \varepsilon^2$ we provide some examples of sequences of deformations with equibounded energy converging to deformations with jumps, see Section 4. Physically this may be interpreted as formation of plastic slips, which is a consistent occurrence in a vast body of literature in the subject. We refer to, for instance, [6, 18, 22], demonstrating that deformation gradients in $GL(d,\mathbb{Z})$ may describe plasticity and dislocations.

1.1. **Setup.** Let us give a couple of comments on the notation and conventions present in the paper:

(a) $d \geq 2$ is an integer;
(b) $\Omega \subset \mathbb{R}^d$ is an open, simply-connected and bounded set with Lipschitz boundary;
(c) $\mathscr{L}^d$ and $\mathscr{H}^{d-1}$ indicate the d-dimensional Lebesgue measure and the $(d-1)$-dimensional Hausdorff measure respectively;
(d) for U open in $\mathbb{R}^d$, $w\colon U \to \mathbb{R}^d$, $\xi\colon U \to \mathbb{R}^{d\times d}$ and $p \in [1,+\infty]$ we set $\|w\|_{L^p(U)} := \|w\|_{L^p(U;\mathbb{R}^d)}$ and $\|\xi\|_{L^p(U)} := \|\xi\|_{L^p(U;\mathbb{R}^{d\times d})}$;
(e) given $s \geq 1$ the associated Sobolev exponent is $s^* := ds/d - s$ if $s < d$, $s^* := +\infty$ if $s \geq d$;
(f) the symmetrised gradient of $w\colon U \to \mathbb{R}^d$ is denoted by $e(w) = (\nabla w^{\mathrm{T}} + \nabla w)/2$;
(g) $B_R(x)$ denotes the open ball of radius R centred at x in $\mathbb{R}^d$;
(h) I signifies the identity matrix in $\mathbb{R}^{d\times d}$ while id_U is the function $\mathrm{id}_{\mathbb{R}^d}(x) = x$ defined for $x \in U$;
(i) $SO(d) \subset \mathbb{R}^{d\times d}$ is the set of all proper rotations;
(j) $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ is the collection of symmetric matrices in $\mathbb{R}^{d\times d}$;
(k) for two matrices $M, N \in \mathbb{R}^{d\times d}$ we denote by $M : N = \mathrm{tr}(M^{\mathrm{T}}N)$ the Frobenius scalar product and by $|M|$ the Frobenius norm of M;
(l) $\{e_1, e_2, \dots, e_d\}$ is the standard Euclidean basis in $\mathbb{R}^d$;
(m) the usual tensor product of two vectors $x, y \in \mathbb{R}^d$ is the matrix $x \otimes y \in \mathbb{R}^{d\times d}$ such that $(x \otimes y)_{ij} = x_i y_j$; if $M \in \mathbb{R}^{d\times d}$ and $x \in \mathbb{R}^d$ then $M \otimes x \in \mathbb{R}^{d\times d\times d}$ is the 3-order tensor with $(M \otimes x)_{kij} = M_{ki} x_j$;
(n) $c > 0$ denotes a generic constant whose value may change from place to place and where every relevant dependence is indicated in the subscript;
(o) when we work with sequences indexed by ε, it is understood that ε denotes the element of a sequence $\varepsilon_n \to 0$.

Throughout we fix a set of countably many wells labelled as

$$K = \bigcup_{\ell=1}^{\infty} K_\ell$$

whereby

(K1) for every $\ell \in \mathbb{N}$ there exists $U_\ell \in GL(d,\mathbb{R})$ such that $K_\ell := SO(d)U_\ell$;
(K2) $K_1 = SO(d)$;
(K3) there exists $\delta_{\min} > 0$ such that $\min_{\ell \neq 1} \mathrm{dist}(K_1, K_\ell) \geq \delta_{\min}$.

Prominent examples of sets satisfying assumptions (K1)-(K3) are the linear group $GL(d,\mathbb{Z})$, i.e. all matrices in $\mathbb{Z}^{d\times d}$ with determinant ± 1, and the special linear group $SL(d,\mathbb{Z}) = GL^+(d,\mathbb{Z})$, i.e. all matrices in $GL(d,\mathbb{Z})$ with determinant 1.

The energy of the considered hyperelastic body is modelled by means of an infinite-well energy density $W\colon \mathbb{R}^d \times \mathbb{R}^{d\times d} \to [0,+\infty)$ which satisfies the following properties:

(W1) (Measurability) W is $\mathcal{L}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^{d\times d})$-measurable where $\mathcal{L}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R}^{d\times d})$ are the σ-algebras of Lebesgue measurable sets in $\mathbb{R}^d$ and Borel sets in $\mathbb{R}^{d\times d}$ respectively;
(W2) (Frame indifference) for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$, any $F \in \mathbb{R}^{d\times d}$ and $R \in SO(d)$ it holds $W(x, RF) = W(x, F)$;
(W3) (Equilibrium) for any $F \in K$ the equality $W(\cdot, F) = 0$ holds $\mathscr{L}^d$-a.e in $\mathbb{R}^d$;
(W4) (Coercivity) there exists a constant $c_W > 0$ such that for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$ and any $F \in \mathbb{R}^{d\times d}$ we have $W(x, F) \geq c_W \min\{\mathrm{dist}(F, K)^2, 1\}$;

(W5) (Regularity) there exists $\sigma > 0$ such that for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$ the function $W(x, \cdot)$ is C^2 in $\mathbb{B}_\sigma$ where $\mathbb{B}_\sigma := \{F \in \mathbb{R}^{d\times d} : \operatorname{dist}(F, K) < \sigma\}$ and

$$\sum_{i,j=1}^{d} |\partial_i \partial_j W(x, \cdot)| \leq c_\sigma$$

for some constant $c_\sigma > 0$.

We fix $r \in [1^*, 2]$, $p > 1$ and a function $\eta = \eta(\varepsilon) > 0$ such that

(H1) $\lim_{\varepsilon\to 0} \eta\, \varepsilon^{1-\frac{2}{p}} = 0$;

(H2) for $d = 2$, $\lim_{\varepsilon\to 0} \eta/\varepsilon = +\infty$; for $d \geq 3$, there exists $c > 0$ such that $\eta \geq c\, \varepsilon^{2-\frac{r}{1^*}}$.

For an open set $U \subset \mathbb{R}^d$ and $v \in W^{2,p}(U; \mathbb{R}^d)$ we consider the rescaled, singularly perturbed energy functionals defined by

$$F_\varepsilon(v, U) := \frac{1}{\varepsilon^2} \int_U W(x, \nabla v)\, \mathrm{d}x + \frac{\eta^p}{\varepsilon^2} \int_U |\nabla^2 v|^p\, \mathrm{d}x$$

Assumption (H1) is needed to ensure that the singular perturbation vanishes in the limit as $\varepsilon \to 0$. Assumption (H2) will be essential in the proof of the compactness result, Theorem 1.2. Note that (H1) and (H2) are consistent when $r \leq \min\{2, (3p - 2)1^*/p\}$ (where $(3p - 2)1^*/p > 1^*$ if and only if $p > 1$, while $(3p - 2)1^*/p \geq 2$ if and only if $p^* \geq 2$). Moreover, if $d = 2$, then $r = 2$ is the only possibility allowed by (H2).

The variational problem which we are going to analyse shall be posed on deformations $v \colon \Omega \to \mathbb{R}^d$ satisfying a prescribed boundary condition on a subset $\Gamma \subset \partial\Omega$. We assume that Γ is open in the relative topology of $\partial\Omega$, $\mathscr{H}^{d-1}(\Gamma) > 0$ and $\operatorname{cap}(\overline{\Gamma}\backslash\Gamma) = 0$, where the notion of capacity is as in [3]. The boundary condition takes the form $v(x) = x + \varepsilon\, g$ on Γ in the sense of traces, with

$$g \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d) \cap W^{2,p}(\mathbb{R}^d; \mathbb{R}^d). \tag{1}$$

It is always possible to define an open set $\Omega_D \subset \mathbb{R}^d$, bounded, simply-connected, Lipschitz and such that $\Omega \subset \Omega_D$, $\mathscr{L}^d(\Omega_D\backslash\Omega) > 0$ and $\Gamma = \Omega_D \cap \partial\Omega$. Deformations defined in Ω and complying with the boundary condition are extended to Ω_D by setting $v(x) = x + \varepsilon\, g$ in $\Omega_D\backslash\Omega$.

For $s \geq 1$ let us define the admissible set of competitors by

$$\begin{aligned} W^{1,s}_{g,\Gamma}(\Omega_D; \mathbb{R}^d) &:= \{u \in W^{1,s}(\Omega_D; \mathbb{R}^d) : u = g \text{ in } \Omega_D\backslash\Omega\}, \\ \mathcal{W}_g(\Omega_D; \mathbb{R}^d) &:= W^{1,r}_{g,\Gamma}(\Omega_D; \mathbb{R}^d) \cap W^{2,p}(\Omega_D; \mathbb{R}^d). \end{aligned} \tag{2}$$

We also employ the notation $H^1_{g,\Gamma}(\Omega_D; \mathbb{R}^d) := W^{1,2}_{g,\Gamma}(\Omega_D; \mathbb{R}^d)$. In the context of the forthcoming analysis we shall regard the energy as a functional of the displacement and we set:

$$\mathcal{F}_\varepsilon(u) := \begin{cases} F_\varepsilon\big(\mathrm{id}_{\mathbb{R}^d} + \varepsilon u, \Omega_D\big) & \text{if } u \in \mathcal{W}_g(\Omega_D; \mathbb{R}^d), \\ +\infty & \text{otherwise in } W^{1,r}(\Omega_D; \mathbb{R}^d). \end{cases} \tag{3}$$

In the sequel we will use the convention $v := \mathrm{id}_{\mathbb{R}^d} + \varepsilon u$.

1.2. **The main statements.** The overarching theme of our discourse is the derivation of the linearised elasticity model through the asymptotic analysis of the functionals $\mathcal{F}_\varepsilon$. To complete the introductory section let us state the main results to which this paper is devoted. Before embarking on the statement of the Γ-convergence result, we introduce the specific notion of convergence that will be used in the sequel. Recall the exponent r defined in (H2).

Definition 1.1 (ω-convergence)**.** Let $U \subset \mathbb{R}^d$ be an open set. We say that a sequence $(u_\varepsilon) \subset W^{1,r}_{g,\Gamma}(U; \mathbb{R}^d)$ ω-converges to some $u \in W^{1,r}_{g,\Gamma}(U; \mathbb{R}^d)$ in U, written $u_\varepsilon \xrightarrow{\omega} u$, provided there exist measurable sets $U_\varepsilon \subset U$ such that $\mathscr{L}^d(U\backslash U_\varepsilon) \to 0$ and $(\nabla u_\varepsilon)\chi_{U_\varepsilon} \rightharpoonup \nabla u$ weakly in $L^r(U; \mathbb{R}^{d\times d})$.

We may now state the compactness result, proof of which is provided in Section 2.

Theorem 1.2 (Compactness). *Let η and r satisfy* (H2) *and g as in* (1) *with $p>1$. Furthermore let $W\colon \mathbb{R}^d \times \mathbb{R}^{d\times d} \to [0,+\infty)$ be a density satisfying* (W1)-(W4). *Let $(u_\varepsilon) \subset \mathcal{W}_g(\Omega_D;\mathbb{R}^d)$ be a sequence satisfying*

$$\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty. \tag{4}$$

Then there exists $u \in W^{1,r}_{g,\Gamma}(\Omega_D;\mathbb{R}^d)$ such that, up to subsequences, $u_\varepsilon \xrightarrow{\omega} u$ in Ω_D. Moreover there exists a constant $C>0$ and a sequence $\sigma_\varepsilon > 0$ with $1/C \le \sigma_\varepsilon \le C$ for all $\varepsilon > 0$ such that $(\nabla u_\varepsilon)\chi_{\Omega_\varepsilon} \rightharpoonup \nabla u$ weakly in $L^r(\Omega_D;\mathbb{R}^{d\times d})$, where $\Omega_\varepsilon := \{\mathrm{dist}(\nabla v_\varepsilon; SO(d)) < \sigma_\varepsilon\} \subset \Omega_D$.

Remark 1.3. In view of the following theorem, a posteriori one has $u \in H^1_{g,\Gamma}(\Omega_D;\mathbb{R}^d)$. △

The following Γ-convergence result will be proved in Section 3.

Theorem 1.4 (Γ-convergence). *Let η, r and p satisfy* (H1)-(H2) *and g as in* (1). *Furthermore let $W\colon \mathbb{R}^d \times \mathbb{R}^{d\times d} \to [0,+\infty)$ be a density satisfying* (W1)-(W5). *Then $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}_{\mathrm{lin}}$ as $\varepsilon \to 0$ with respect to the ω-convergence in Ω_D, where*

$$\mathcal{F}_{\mathrm{lin}}(u) := \begin{cases} \dfrac{1}{2}\displaystyle\int_{\Omega_D} D^2W(x,I)e(u) : e(u)\,\mathrm{d}x & \text{if } u \in H^1_{g,\Gamma}(\Omega_D;\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

Let us provide a further convergence property of the recovery sequences, which is verified in Section 3.

Theorem 1.5 (Stronger convergence of symmetric gradients). *Under the hypotheses of Theorem 1.4, let $u \in H^1_{g,\Gamma}(\Omega_D;\mathbb{R}^d)$ and suppose that $(u_\varepsilon) \subset \mathcal{W}_g(\Omega_D;\mathbb{R}^d)$ is a recovery sequence for u, i.e., $u_\varepsilon \xrightarrow{\omega} u$ in Ω_D and $\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon) = \mathcal{F}_{\mathrm{lin}}(u)$. Then, up to subsequences, there exist compact sets $U_\varepsilon \subset \Omega_D$ such that $\mathscr{L}^d(\Omega_D \backslash U_\varepsilon) \to 0$ and $e(u_\varepsilon)\chi_{U_\varepsilon} \to e(u)$ strongly in $L^r(\Omega_D;\mathbb{R}^{d\times d}_{\mathrm{sym}})$.*

Remark 1.6. It is apparent that as a consequence of Theorems 1.2, 1.4 and 1.5, setting

$$m_\varepsilon := \inf\{\mathcal{F}_\varepsilon(u)\colon u \in \mathcal{W}_g(\Omega_D;\mathbb{R}^d)\},$$
$$m_{\mathrm{lin}} := \min\{\mathcal{F}_{\mathrm{lin}}(u)\colon u \in H^1_{g,\Gamma}(\Omega_D;\mathbb{R}^d)\},$$

we have $m_\varepsilon \to m_{\mathrm{lin}}$ as $\varepsilon \to 0$. Also for any sequence $(u_\varepsilon) \subset \mathcal{W}_g(\Omega_D;\mathbb{R}^d)$ such that $\mathcal{F}_\varepsilon(u_\varepsilon) = m_\varepsilon + o(1)$, the maps u_ε ω-converge to u where $u \in H^1_{g,\Gamma}(\Omega_D;\mathbb{R}^d)$ is the unique solution to the minimum problem for m_{lin}. Furthermore there exist measurable sets $U_\varepsilon \subset \Omega_D$ such that $\mathscr{L}^d(\Omega_D\backslash U_\varepsilon) \to 0$ and $e(u_\varepsilon)\chi_{U_\varepsilon} \to e(u)$ strongly in $L^r(\Omega_D;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. △

Remark 1.7. In the case of finitely many wells analysed in [3], a certain scaling regime on η allowed one to obtain a stronger notion of convergence for u_ε than the one in Definition 1.1, i.e., the weak convergence in $W^{1,r}(\Omega)$. Let us stress that in the present setting compactness cannot be retained in the regime of weak topology of $W^{1,r}(\Omega;\mathbb{R}^d)$ if $1^* < r < 2$ and $d > 2$ as showed in Example 4.1. For $r = 2$ we do not have any counterexample and the question of compactness in all of Ω is still open.

The results in [3] were proved under the additional assumption that $W(x,F) \ge c\,\mathrm{dist}(F,K)^q$ for some $q \in [0,2]$ when F is large enough. Let us now compare our assumptions on the range of r and η with those in [3] in the case of $q \in [0,1]$ for simplicity. The hypotheses (H1)-(H2) above permit greater values of r when $p^* \le 2$. Moreover, they allow a larger range for η when $1^* < r < 2$; in fact, in the latter case the range for η provided in [3] was not proved to be optimal. △

Remark 1.8 (The case d=1). We comment on the special instance of d=1 which is not included in the hypotheses of the results displayed above. In [3, Theorem 1.7] it is shown that, already at the level of finitely many wells, any transitions between the wells are ruled out. Thus if K is given by a countable set of isolated points in $\mathbb{R}$, the methodology of tackling linearisation readily translates to the aforementioned result. △

2. Compactness result

Firstly let us give some prerequisite measure theoretic aspects, a thorough explanation of which can be found, e.g. in the monographs [5, 20]. We recall that a measurable set $E \subset \mathbb{R}^d$ is said to be of finite perimeter in an open set $U \subset \mathbb{R}^d$ provided

$$\mathrm{Per}\,(E,U) := \sup\left\{ \int_E \operatorname{div}\varphi \,\mathrm{d}x : \varphi \in C^1_c(U;\mathbb{R}^d),\ \|\varphi\|_{L^\infty(U)} \le 1 \right\} < +\infty.$$

Any set $E \subset \mathbb{R}^d$ of finite perimeter in U satisfies the equality $\mathrm{Per}\,(E,U) = \mathscr{H}^{d-1}(U \cap \partial^* E)$ where $\partial^* E$ denotes the reduced boundary of E. In addition let us recall a generalisation of the coarea formula, cf. [20, Section 4.5.9].

Lemma 2.1 (Fleming-Rishel). *Let $U \subset \mathbb{R}^d$ be an open set. Given a Borel function $\phi\colon U \to [0,+\infty)$ and $\psi \in W^{1,1}(U)$ there holds*

$$\int_U \phi |\nabla \psi| \,\mathrm{d}x = \int_{\mathbb{R}} \int_{U \cap \partial^* \{\psi < s\}} \phi \,\mathrm{d}\mathscr{H}^{d-1} \,\mathrm{d}s.$$

In the interest of proving the compactness theorem, it is crucial to invoke some relevant properties of the curl operator on matrix-valued special functions of bounded variation. In particular we will state a curl-type rigidity estimate and an estimate on the curl of the approximate gradient. Let us recall that given $\varphi \in C^1(\Omega_D;\mathbb{R}^{d\times d})$, by canonical identification $\mathbb{R}^{d\times d\times d} \cong \mathbb{R}^d \times \mathbb{R}^{d\times d}$, the object $\operatorname{curl}\varphi$ is an antisymmetric tensor in $\mathbb{R}^{d\times d\times d}$ whose entries are $(\operatorname{curl}\varphi)_{kij} = \partial_i \varphi_{kj} - \partial_j \varphi_{ki}$. Thereby for a mapping $\Phi \in L^1(\Omega_D;\mathbb{R}^{d\times d})$, the curl of Φ is defined in the sense of distributions by

$$\langle \operatorname{curl}\Phi, \varphi \rangle := \sum_{i,j,k=1}^{d} \int_{\Omega_D} \Phi_{ki} \partial_j (\varphi_{kij} - \varphi_{kji}) \,\mathrm{d}x \tag{5}$$

with $\varphi \in C^\infty_c(\Omega_D;\mathbb{R}^{d\times d\times d})$. Moreover, given β such that $\operatorname{curl}\beta$ is a measure, we denote by $|\operatorname{curl}\beta|$ its total variation.

We will employ the following rigidity estimate with a curl term which was proved in [24, Theorem 3.3] for $d = 2$. For the extensions to $d \ge 3$ see [23, Theorem 3], [15, Lemma 3.3, Remark 3.4] and also [10]. In order to apply Lemma 2.2 below, we shall need the assumption that $r \ge 1^*$; for $r < 1^*$, one may argue locally on a subdomain and get a rigidity estimate with a constant depending on that subdomain. However, this would not be sufficient to obtain a compactness result with the convergence of Definition 1.1(3).

Lemma 2.2 (Rigidity). *Let $r \in [1^*, 2]$. Given a set $U \subset \mathbb{R}^d$ which is bounded, open, Lipschitz and simply-connected, there exists a constant $c_1 = c_1(r,U) > 0$ such that for any $\beta \in L^r(U;\mathbb{R}^{d\times d})$ with* $\operatorname{curl}\beta$ *a bounded measure on U, there exists a rotation $R = R(\beta) \in SO(d)$ such that*

$$\|\beta - R\|_{L^r(U)} \le c_1 \Big(\|\operatorname{dist}(\beta, SO(d))\|_{L^r(U)} + |\operatorname{curl}\beta|(U)^{\frac{1^*}{r}} \Big).$$

Secondly, we will use the following fact.

Remark 2.3. We note that there exists a constant $c_2 = c_2(d) > 0$ such that for any measurable $E \subset U$ and any $v \in W^{2,1}(U;\mathbb{R}^d)$ satisfying $\operatorname{Per}(E;U) < +\infty$ and $\nabla v \in L^\infty(E;\mathbb{R}^{d\times d})$, $\operatorname{curl}((\nabla v)\chi_E)$ is a measure and the following curl-type estimate holds:

$$|\operatorname{curl}((\nabla v)\chi_E)|(U) \leq c_2 \, \|\nabla v\|_{L^\infty(E)} \operatorname{Per}(E;U). \tag{6}$$

The estimate (6) is obtained directly from the definition of the curl in (5) along with Schwartz's theorem. Concretely, we argue as follows: using (5) and the fact that $(\nabla v)\chi_E \in SBV(\Omega_D;\mathbb{R}^{d\times d})$ with $D((\nabla v)\chi_E) = \nabla^2 v \, \mathscr{L}^d \llcorner E + [(\nabla v)\chi_E] \otimes \nu_E \, \mathscr{H}^{d-1} \llcorner \partial^* E$, for $\varphi \in C_c^\infty(\Omega_D;\mathbb{R}^{d\times d\times d})$ we compute

$$\begin{aligned}
\langle \operatorname{curl}((\nabla v)\chi_E), \varphi\rangle &:= \sum_{i,j,k=1}^d \int_{\Omega_D} (\partial_i v^k)\chi_E \, \partial_j(\varphi_{kij} - \varphi_{kji}) \,\mathrm{d}x \\
&= \sum_{i,j,k=1}^d \int_{\partial^* E} ([\nabla v] \otimes \nu_E)_{kij} (\varphi_{kij} - \varphi_{kji}) \,\mathrm{d}\mathscr{H}^{d-1} - \sum_{i,j,k=1}^d \int_E \partial_j \partial_i v^k (\varphi_{kij} - \varphi_{kji}) \,\mathrm{d}x \\
&= \sum_{i,j,k=1}^d \int_{\partial^* E} ([\nabla v] \otimes \nu_E)_{kij} (\varphi_{kij} - \varphi_{kji}) \,\mathrm{d}\mathscr{H}^{d-1} - \sum_{i,j,k=1}^d \int_E \varphi_{kij} (\partial_i \partial_j v^k - \partial_j \partial_i v^k) \,\mathrm{d}x.
\end{aligned}$$

Now by Schwarz's theorem, the second summand in the last equality above vanishes.

See also [9, Theorem 3.1] for a more general result for SBV-vector fields. △

The objective of the next result is to verify that, given a sequence v_ε of deformations with a boundary condition and a suitable control on the energy, ∇v_ε lies around $SO(d)$ in most part of the domain.

Proposition 2.4 (One-well bound). *Let $g \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$. Furthermore let $W\colon \mathbb{R}^d \times \mathbb{R}^{d\times d} \to [0,+\infty)$ be a density satisfying* (W1)-(W4). *Let $(v_\varepsilon) \subset W^{2,p}(\Omega_D;\mathbb{R}^d)$ be a sequence satisfying $v_\varepsilon = \mathrm{id}_{\mathbb{R}^d} + \varepsilon g$ in $\Omega_D \backslash \Omega$ and*

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^2}{\eta} F_\varepsilon(v_\varepsilon, \Omega_D) = 0.$$

Then for every $\kappa \in (0, \min\{1, \delta_{\min}\}/2)$ there exists $c_\kappa > 0$ such that for any $\varepsilon > 0$ sufficiently small, there exists $\sigma_\varepsilon \in (\kappa/2, \kappa)$ such that for

$$\Omega_\varepsilon := \{\operatorname{dist}(\nabla v_\varepsilon, SO(d)) < \sigma_\varepsilon\},$$

the following are satisfied:

- *(i)* $\operatorname{Per}(\Omega_\varepsilon, \Omega_D) \leq c_\kappa \frac{\varepsilon^2}{\eta} F_\varepsilon(v_\varepsilon, \Omega_D)$;
- *(ii)* $\mathscr{L}^d(\Omega_D \backslash \Omega_\varepsilon) \leq c_\kappa \Big(\frac{\varepsilon^2}{\eta} F_\varepsilon(v_\varepsilon, \Omega_D)\Big)^{1^*}$;
- *(iii)* $\|\operatorname{dist}(\nabla v_\varepsilon; SO(d))\|_{L^r(\Omega_\varepsilon)} \leq c_\kappa \Big(\varepsilon^2 F_\varepsilon(v_\varepsilon, \Omega_\varepsilon)\Big)^{\frac{1}{2}}$.

Proof. The main idea of the proof is to select, with the aid of Lemma 2.1, a sublevel set of $\operatorname{dist}(\nabla v_\varepsilon, SO(d))$ whose perimeter decays at an appropriate rate with respect to ε. The required properties will follow by means of an isoperimetric-type argument and the coercivity of the density W. Firstly let us set, for any $s > 0$,

$$\Omega_\varepsilon^s := \{\operatorname{dist}(\nabla v_\varepsilon, SO(d)) < s\}.$$

We define also, for notational brevity,

$$f_\varepsilon(x) = \operatorname{dist}(\nabla v_\varepsilon(x), SO(d))^{\frac{2}{p'}} |\nabla^2 v_\varepsilon(x)|,$$

where $p' := p/(p-1)$. Fix $\kappa \in (0, \min\{1, \delta_{\min}\}/2)$. Using the coercivity (W4) of W and Young's inequality gives

$$\|f_\varepsilon\|_{L^1(A)} \leq \frac{c}{\eta}\int_A W(x,\nabla v_\varepsilon)^{\frac{1}{p'}}\,\eta|\nabla^2 v_\varepsilon|\,\mathrm{d}x \leq c\frac{\varepsilon^2}{\eta}F_\varepsilon(v_\varepsilon, A) \quad \text{for any open set } A \subset \Omega_\varepsilon^\kappa. \tag{7}$$

Appealing to the Fleming-Rishel formula of Lemma 2.1 applied to $\phi_\varepsilon = \mathrm{dist}(\nabla v_\varepsilon, SO(d))^{\frac{2}{p'}}\chi_{\Omega_\varepsilon^\kappa}$ and $\psi_\varepsilon = \mathrm{dist}(\nabla v_\varepsilon, SO(d))$, we may write:

$$\int_{\Omega_D}\phi_\varepsilon|\nabla\psi_\varepsilon|\,\mathrm{d}x \geq \int_0^\kappa\int_{\Omega_D\cap\partial^*\Omega_\varepsilon^s}\mathrm{dist}(\nabla v_\varepsilon, SO(d))^{\frac{2}{p'}}\,\mathrm{d}\mathscr{H}^{d-1}\,\mathrm{d}s = \int_0^\kappa s^{\frac{2}{p'}}\,\mathrm{Per}\,(\Omega_\varepsilon^s, \Omega_D)\,\mathrm{d}s. \tag{8}$$

By the mean value theorem we find $\sigma_\varepsilon \in (\kappa/2, \kappa)$ such that

$$\int_0^\kappa s^{\frac{2}{p'}}\,\mathrm{Per}\,(\Omega_\varepsilon^s,\Omega_D)\,\mathrm{d}s \geq \int_{\kappa/2}^\kappa s^{\frac{2}{p'}}\,\mathrm{Per}\,(\Omega_\varepsilon^s,\Omega_D)\,\mathrm{d}s \geq \frac{\kappa}{2}\sigma_\varepsilon^{\frac{2}{p'}}\,\mathrm{Per}\,(\Omega_\varepsilon^{\sigma_\varepsilon},\Omega_D) \geq c\,\mathrm{Per}\,(\Omega_\varepsilon^{\sigma_\varepsilon},\Omega_D), \tag{9}$$

where the constant c depends on κ. Observing that $|\nabla\psi_\varepsilon(x)| \leq |\nabla^2 v_\varepsilon(x)|$ for $\mathscr{L}^d$-a.e. $x \in \Omega_D$ in conjunction with (8) and (9) leads to

$$\|f_\varepsilon\|_{L^1(\Omega_\varepsilon^\kappa)} \geq \int_0^\kappa s^{\frac{2}{p'}}\,\mathrm{Per}\,(\Omega_\varepsilon^s,\Omega_D)\,\mathrm{d}s \geq c\,\mathrm{Per}\,(\Omega_\varepsilon^{\sigma_\varepsilon},\Omega_D). \tag{10}$$

Combining (7) and (10) yields *(i)*. Note that $\mathrm{Per}\,(\Omega_\varepsilon^{\sigma_\varepsilon},\Omega_D) \to 0$. Moreover, since $v_\varepsilon = \mathrm{id}_{\mathbb{R}^d} + \varepsilon g$ in $\Omega_D\backslash\Omega$, for $\varepsilon > 0$ small enough we have $\Omega_D\backslash\Omega \subset \{\mathrm{dist}(\nabla v_\varepsilon, SO(d)) < \sigma_\varepsilon\}$ and thus an application of the isoperimetric inequality yields *(ii)*. From the coercivity (W4) of W this leads to:

$$\int_{\Omega_\varepsilon^{\sigma_\varepsilon}}\mathrm{dist}^r(\nabla v_\varepsilon, SO(d))\,\mathrm{d}x \leq c\,\|\mathrm{dist}(\nabla v_\varepsilon, SO(d))\|^r_{L^2(\Omega_\varepsilon^{\sigma_\varepsilon})} \leq c\,\varepsilon^r(F_\varepsilon(v_\varepsilon,\Omega_\varepsilon^{\sigma_\varepsilon}))^{\frac{r}{2}}.$$

Therefore *(iii)* follows and the proof is completed. □

Remark 2.5. Comparing the estimate in Proposition 2.4 (iii) to the case of finitely many wells considered in [3], there it was possible to show an estimate on $\|\mathrm{dist}(\nabla u_\varepsilon, SO(d)\|_{L^r(\Omega)}$, i.e. a control in L^r across the entire domain Ω. This is mainly due to the possibility of bounding $\mathrm{dist}(\cdot, SO(d))$ from above by an auxiliary geodesic distance $d_W(\cdot, SO(d))$ associated to W, see the proof of [3, Theorem 2.3] for a precise definition. However in the setting of infinitely many wells an analogous geodesic-type bound does no longer hold since there is no apriori bound from above on the mutual distance between the wells, see also Example 4.1. △

Remark 2.6. The statement of the proposition above could be phrased more generally without the imposition of boundary conditions, but at the cost of assuming a strong separation requirement on the set wells, namely,

(K3') there exists $\delta_{\min} > 0$ such that $\min_{\ell\neq m}\mathrm{dist}(K_\ell, K_m) \geq \delta_{\min}$.

An example satisfying such an assumption is provided in (25) below. Under (K3') let us prove the existence of a *majority well* K_{ℓ_ε} around which the deformation gradient ∇u_ε lies in most of the domain. This would be trivial if the number of energy wells were finite. In contrast, in the present setting we may establish the existence of a majority phase by employing an isoperimetric argument.

Firstly let us set, for any $\ell \in \mathbb{N}$ and any $s > 0$, $U_\varepsilon^s := \{\mathrm{dist}(\nabla v_\varepsilon, K) < s\}$. Arguing as in the previous proof with $\phi_\varepsilon = \mathrm{dist}(\nabla v_\varepsilon, K)^{\frac{2}{p'}}\chi_{U_\varepsilon^\kappa}$ and $\psi_\varepsilon = \mathrm{dist}(\nabla v_\varepsilon, K)$, we obtain

$$\mathrm{Per}\,(U_\varepsilon^{\sigma_\varepsilon},\Omega) \leq c\frac{\varepsilon^2}{\eta}F_\varepsilon(v_\varepsilon,\Omega).$$

Now setting $U_\varepsilon^{\ell,s} := \{\mathrm{dist}(\nabla v_\varepsilon, K_\ell) < s\}$ let us prove that for any $\varepsilon > 0$ sufficiently small there exists a unique $\ell_\varepsilon \in \mathbb{N}$ such that $U_\varepsilon^{\ell_\varepsilon,\sigma_\varepsilon}$ occupies majority of volume, namely:

$$\mathscr{L}^d\big(\Omega\backslash U_\varepsilon^{\ell_\varepsilon,\sigma_\varepsilon}\big) \le \mathscr{L}^d\big(U_\varepsilon^{\ell_\varepsilon,\sigma_\varepsilon}\big).$$

To this end let us suppose for contradiction for all $\ell \in \mathbb{N}$ one has $\mathscr{L}^d\big(\Omega\backslash U_\varepsilon^{\ell,\sigma_\varepsilon}\big) \ge \mathscr{L}^d\big(U_\varepsilon^{\ell,\sigma_\varepsilon}\big)$. Since $\sigma_\varepsilon < \min\{1, \delta_{\min}/2\}$, by (K3') the sets $\overline{U_\varepsilon^{\ell,\sigma_\varepsilon}}$ are pairwise disjoint. Hence, we have

$$\begin{aligned}\mathrm{Per}\,(U_\varepsilon^{\sigma_\varepsilon},\Omega) &= \sum_{\ell=1}^{\infty}\mathrm{Per}\,(U_\varepsilon^{\ell,\sigma_\varepsilon},\Omega) \ge c\sum_{\ell=1}^{\infty}\min\big\{\mathscr{L}^d\big(\Omega\backslash U_\varepsilon^{\ell,\sigma_\varepsilon}\big),\mathscr{L}^d\big(U_\varepsilon^{\ell,\sigma_\varepsilon}\big)\big\}^{\frac{d-1}{d}}\\ &\ge c\sum_{\ell=1}^{\infty}\mathscr{L}^d\big(U_\varepsilon^{\ell,\sigma_\varepsilon}\big)^{\frac{d-1}{d}} \ge c\sum_{\ell=1}^{\infty}\mathscr{L}^d\big(U_\varepsilon^{\ell,\sigma_\varepsilon}\big)\mathscr{L}^d\big(\Omega\big)^{-\frac{1}{d}} \ge c\sum_{\ell=1}^{\infty}\mathscr{L}^d\big(U_\varepsilon^{\ell,\sigma_\varepsilon}\big) = c\,\mathscr{L}^d(U_\varepsilon^{\sigma_\varepsilon}).\end{aligned}$$

Also using coercivity (W4) of W and the fact that $\sigma_\varepsilon > \kappa/2$ we have

$$\mathscr{L}^d\big(\Omega\backslash U_\varepsilon^{\sigma_\varepsilon}\big) \le \frac{1}{\sigma_\varepsilon^2 c_W}\int_{U\backslash U_\varepsilon^{\sigma_\varepsilon}} W(x,\nabla v_\varepsilon)\,\mathrm{d}x \le \frac{4}{\kappa^2 c_W}\varepsilon^2 F_\varepsilon(v_\varepsilon,\Omega).$$

In other words $\mathscr{L}^d\big(\Omega\backslash U_\varepsilon^{\sigma_\varepsilon}\big) \to 0$ as $\varepsilon \to 0$. Thus for $\varepsilon > 0$ small enough, combining the previous inequalities yields

$$\mathscr{L}^d(\Omega) = \mathscr{L}^d\big(\Omega\backslash U_\varepsilon^{\sigma_\varepsilon}\big) + \mathscr{L}^d(U_\varepsilon^{\sigma_\varepsilon}) \le \mathscr{L}^d\big(\Omega\backslash U_\varepsilon^{\sigma_\varepsilon}\big) + c\frac{\varepsilon^2}{\eta}F_\varepsilon(v_\varepsilon, U_\varepsilon^{\kappa})$$

which is a contradiction since the quantity on the right hand side tends to zero as $\varepsilon \to 0$. Therefore there exists $\ell_\varepsilon \in \mathbb{N}$ such that $\mathscr{L}^d\big(\Omega\backslash U_\varepsilon^{\ell_\varepsilon,\sigma_\varepsilon}\big) \le \mathscr{L}^d(U_\varepsilon^{\ell_\varepsilon,\sigma_\varepsilon})$.

Finally, by the same reasoning as in the previous proof, we obtain $\mathrm{Per}\,(U_\varepsilon^{\ell_\varepsilon,\sigma_\varepsilon},\Omega) \le c\frac{\varepsilon^2}{\eta}F_\varepsilon(v_\varepsilon,\Omega)$, $\mathscr{L}^d\big(\Omega\backslash U_\varepsilon^{\ell_\varepsilon,\sigma_\varepsilon}\big) \le c\Big(\frac{\varepsilon^2}{\eta}F_\varepsilon(v_\varepsilon,\Omega)\Big)^{1^*}$ and $\|\mathrm{dist}(\nabla v_\varepsilon; K_{\ell_\varepsilon})\|_{L^r(U_\varepsilon^{\ell_\varepsilon,\sigma_\varepsilon})} \le c\Big(\varepsilon^2 F_\varepsilon(v_\varepsilon, U_\varepsilon^{\ell_\varepsilon,\sigma_\varepsilon})\Big)^{\frac{1}{2}}$. △

We now contemplate the proof of compactness property of equi-bounded energies stated in Theorem 1.2. In particular we will show that a sequence of displacements with uniformly bounded energies admits a convergent subsequence with respect to the topological framework of Definition 1.1.

Proof of Theorem 1.2. We notice that the sequence (v_ε) satisfies the assertions of Proposition 2.4. We proceed with the proof in several steps.

Step 1: Curl-estimate in the majority well. In view of the definition of Ω_ε and specifically the choice of σ_ε, $|\nabla v_\varepsilon(x)| < \sqrt{n} + \delta_{\min}$ for all $x \in \Omega_\varepsilon$, thus $(\nabla v_\varepsilon)\chi_{\Omega_\varepsilon}$ is uniformly bounded in Ω_D. In particular, by (6) the sequence $(\nabla v_\varepsilon)\chi_{\Omega_\varepsilon}$ satisfies the hypotheses of Lemma 2.2. Thus we may find a rotation $R_\varepsilon \in SO(d)$ such that

$$\|(\nabla v_\varepsilon)\chi_{\Omega_\varepsilon} - R_\varepsilon\|_{L^r(\Omega_D)} \le c\Big(\|\mathrm{dist}((\nabla v_\varepsilon)\chi_{\Omega_\varepsilon}, SO(d))\|_{L^r(\Omega_D)} + |\mathrm{curl}\,((\nabla v_\varepsilon)\chi_{\Omega_\varepsilon})|(\Omega_D)^{\frac{1^*}{r}}\Big). \tag{11}$$

Subsequently using Proposition 2.4 parts (ii) and (iii) we obtain:

$$\begin{aligned}\|\mathrm{dist}((\nabla v_\varepsilon)\chi_{\Omega_\varepsilon}, SO(d))\|_{L^r(\Omega_D)} &\le \|\mathrm{dist}(\nabla v_\varepsilon, SO(d))\|_{L^r(\Omega_\varepsilon)} + c\,\mathscr{L}^d\big(\Omega_D\backslash\Omega_\varepsilon\big)^{\frac{1}{r}}\\ &\le c\Big(\varepsilon^2 F_\varepsilon(v_\varepsilon,\Omega_D)\Big)^{\frac{1}{2}} + c\Big(\frac{\varepsilon^2}{\eta}F_\varepsilon(v_\varepsilon,\Omega_D)\Big)^{\frac{1^*}{r}}.\end{aligned} \tag{12}$$

On the other hand invoking the estimate (6) yields

$$\begin{aligned}|\mathrm{curl}\,((\nabla v_\varepsilon)\chi_{\Omega_\varepsilon})|(\Omega_D) &\le c\|\nabla v_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\,\mathrm{Per}\,(\Omega_\varepsilon,\Omega_D)\\ &\le c\,\mathrm{Per}\,(\Omega_\varepsilon,\Omega_D).\end{aligned} \tag{13}$$

Inserting the estimates (12)-(13) into (11), in conjunction with Proposition 2.4 (i), we arrive at

$$\|(\nabla v_\varepsilon)\chi_{\Omega_\varepsilon} - R_\varepsilon\|_{L^r(\Omega_D)} \le c\Big(\Big(\varepsilon^2 F_\varepsilon(v_\varepsilon,\Omega_D)\Big)^{\frac{1}{2}} + \Big(\frac{\varepsilon^2}{\eta}F_\varepsilon(v_\varepsilon,\Omega_D)\Big)^{\frac{1^*}{r}}\Big).$$

Now notice that by the boundary condition (2) one has $|R_\varepsilon - I| \le c\,\varepsilon$, where c also depends on g and $\Omega_D\backslash\Omega$. Therefore, for $\varepsilon > 0$ small enough we may actually write

$$\|(\nabla v_\varepsilon)\chi_{\Omega_\varepsilon} - I\|_{L^r(\Omega_D)} \le c\Big(\Big(\varepsilon^2 F_\varepsilon(v_\varepsilon,\Omega_D)\Big)^{\frac{1}{2}} + \Big(\frac{\varepsilon^2}{\eta}F_\varepsilon(v_\varepsilon,\Omega_D)\Big)^{\frac{1^*}{r}}\Big). \tag{14}$$

By (4) and the convergence $\varepsilon^2/\eta \to 0$, cf. (H2), we have $(\nabla v_\varepsilon)\chi_{\Omega_\varepsilon} \to I$ strongly in $L^r(\Omega_D;\mathbb{R}^{d\times d})$ as $\varepsilon \to 0$.

Step 2: Convergence of $(\nabla u_\varepsilon)\chi_{\Omega_\varepsilon}$. Since $\nabla v_\varepsilon = I + \varepsilon\nabla u_\varepsilon$, the bound in (14) can be rewritten as

$$\|(\nabla u_\varepsilon)\chi_{\Omega_\varepsilon}\|_{L^r(\Omega_D)} \le c\left(\mathcal{F}_\varepsilon(u_\varepsilon)^{\frac{1}{2}} + \left(\frac{\varepsilon^{2-\frac{1^*}{r}}}{\eta}\mathcal{F}_\varepsilon(u_\varepsilon)\right)^{\frac{r}{1^*}}\right) \le c, \tag{15}$$

where the last conclusion follows from (4) and the bound $\varepsilon^{2-\frac{r}{1^*}} \le c\,\eta$, cf. (H2). Up to subsequences $(\nabla u_\varepsilon)\chi_{\Omega_\varepsilon}$ converges weakly in $L^r(\Omega_D;\mathbb{R}^{d\times d})$ as $\varepsilon \to 0$ to some $\xi \in L^r(\Omega_D;\mathbb{R}^{d\times d})$. Moreover, the bounds in (6) and (13) imply convergence of the curl of $(\nabla u_\varepsilon)\chi_{\Omega_\varepsilon}$ in total variation:

$$\begin{aligned}|\mathrm{curl}\,((\nabla u_\varepsilon)\chi_{\Omega_\varepsilon})|(\Omega_D) &= \frac{1}{\varepsilon}\,|\mathrm{curl}\,((\nabla v_\varepsilon - I)\chi_{\Omega_\varepsilon})|(\Omega_D)\\ &\le \frac{c}{\varepsilon}\,\|\nabla v_\varepsilon - I\|_{L^\infty(\Omega_\varepsilon)}\,\mathrm{Per}\,(\Omega_\varepsilon,\Omega_D) \le c\,\frac{\varepsilon}{\eta} \longrightarrow 0\end{aligned} \tag{16}$$

as $\varepsilon \to 0$. By standard decomposition results [7] the limit ξ is a gradient; since $\nabla u_\varepsilon = \nabla g$ in $\Omega_D \setminus \Omega$, there exists $u \in W^{1,r}_{g,\Gamma}(\Omega_D;\mathbb{R}^d)$ such that $\xi = \nabla u$. □

3. Γ-CONVERGENCE

This section is devoted to the proof of the Γ-convergence stated in Theorem 1.4. We first deal with the establishment of the Γ-limit of $(\mathcal{F}_\varepsilon)$. Referring to [12] for details, let us begin by defining the lower and upper Γ-envelopes of $(\mathcal{F}_\varepsilon)$: for any $u \in H^1_{g,\Gamma}(\Omega_D;\mathbb{R}^d)$ we set

$$\begin{aligned}\mathcal{F}'(u) &:= \inf\big\{\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon)\colon (u_\varepsilon) \subset W^{1,r}(\Omega_D;\mathbb{R}^d),\ u_\varepsilon \xrightarrow{\omega} u\big\}\\ \mathcal{F}''(u) &:= \inf\big\{\limsup_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon)\colon (u_\varepsilon) \subset W^{1,r}(\Omega_D;\mathbb{R}^d),\ u_\varepsilon \xrightarrow{\omega} u\big\}.\end{aligned} \tag{17}$$

For notational brevity we also rephrase the functional $\mathcal{F}_{\mathrm{lin}}(u)\colon H^1_{g,\Gamma}(\Omega_D;\mathbb{R}^d) \to [0,+\infty)$ from Theorem 1.4 as

$$\mathcal{F}_{\mathrm{lin}}(u) := \int_{\Omega_D}\mathbb{C}(x)e(u) : e(u)\,\mathrm{d}x \tag{18}$$

whereby the symbol $\mathbb{C}$ customarily stands for the fourth order tensor defined through the relation

$$\mathbb{C}(x) := \frac{1}{2}\,D^2W(x,I)$$

for $\mathscr{L}^d$-a.e. $x \in \Omega_D$.

In view of establishing the Γ-lim sup, we recount the following density result which can be found in the supplementary section of [3].

Proposition 3.1. [3, Proposition A.3] *For any* $u \in H^1_{g,\Gamma}(\Omega_D;\mathbb{R}^d)$ *there exists a sequence* $(u_\varepsilon) \subset W^{1,\infty}(\Omega_D;\mathbb{R}^d) \cap W^{2,p}(\Omega_D;\mathbb{R}^d)$ *satisfying* $u_\varepsilon = g$ *in* $\Omega_D\backslash\Omega$ *and* $u_\varepsilon \to u$ *strongly in* $H^1(\Omega_D;\mathbb{R}^d)$ *as* $\varepsilon \to 0$.

We are now ready to proceed with the proof of Theorem 1.4.

Proof of Theorem 1.4. It suffices to prove that for any $u \in H^1_{g,\Gamma}(\Omega_D;\mathbb{R}^d)$ there holds

$$\mathcal{F}'(u) = \mathcal{F}''(u) = \mathcal{F}_{\text{lin}}(u).$$

Firstly we notice that by definition $\mathcal{F}' \leq \mathcal{F}''$. We now divide the proof into two steps.

Step 1: $\mathcal{F}'' \leq \mathcal{F}_{\text{lin}}$. Since $\mathcal{F}_{\text{lin}}$ is continuous in the strong topology of $H^1(\Omega_D;\mathbb{R}^d)$, in view of the density result Proposition 3.1, it is sufficient to verify the claim for $u \in W^{1,\infty}(\Omega_D;\mathbb{R}^d) \cap W^{2,p}(\Omega_D;\mathbb{R}^d)$ such that $u = g$ in $\Omega_D \backslash \Omega$. In this case it is easily seen that the functionals $\mathcal{F}_\varepsilon$ pointwise converge to $\mathcal{F}_{\text{lin}}$, see[3] for details.

Step 2: $\mathcal{F}' \geq \mathcal{F}_{\text{lin}}$. Let $(u_\varepsilon) \subset W^{1,r}_{g,\Gamma}(\Omega_D;\mathbb{R}^d)$ be a sequence such that $u_\varepsilon \xrightarrow{\omega} u$. Without loss of generality we may suppose that $\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$ so in particular $(u_\varepsilon) \subset \mathcal{W}_g(\Omega_D;\mathbb{R}^d)$. Let $\sigma_\varepsilon > 0$ and $\Omega_\varepsilon \subset \Omega_D$ be as in Theorem 1.2, so $\mathscr{L}^d(\Omega_D \backslash \Omega_\varepsilon) \to 0$. Then invoking Theorem 1.2 there exists $\hat{u} \in W^{1,r}_{g,\Gamma}(\Omega_D;\mathbb{R}^d)$ such that $(\nabla u_\varepsilon)\chi_{\Omega_\varepsilon} \rightharpoonup \nabla \hat{u}$ weakly in $L^r(\Omega_D;\mathbb{R}^{d\times d})$ and $u_\varepsilon \xrightarrow{\omega} \hat{u}$. Combining these facts along with the hypotheses leads to the equality $\hat{u} = u$. Now let us define the sequence $(w_\varepsilon) \subset L^r(\Omega_D;\mathbb{R}^{d\times d})$ by

$$w_\varepsilon := e(u_\varepsilon)\chi_{B_\varepsilon}, \quad B_\varepsilon := \big\{|\nabla u_\varepsilon| \leq 1/\varepsilon^{\frac{1}{3}}\big\}.$$

We notice that for $\varepsilon > 0$ small enough $B_\varepsilon \subset \Omega_\varepsilon$. Also, by the definition of B_ε and by (15),

$$\mathscr{L}^d(\Omega_\varepsilon \backslash B_\varepsilon) \leq \varepsilon^{\frac{r}{3}} \|\nabla u_\varepsilon\|^r_{L^r(\Omega_\varepsilon)} \leq c\,\varepsilon^{\frac{r}{3}},$$

which in conjunction with the convergence $(\nabla u_\varepsilon)\chi_{\Omega_\varepsilon} \rightharpoonup \nabla u$ weakly in $L^r(\Omega_D;\mathbb{R}^{d\times d})$ implies that $w_\varepsilon \rightharpoonup e(u)$ weakly in $L^r(\Omega_D;\mathbb{R}^{d\times d})$. From here onwards, we mimic the argumentation in the proof of [3, Theorem 1.9] with slight adjustments. By property (W2) it is possible to express the energy density W in terms of the function $V \colon \Omega_D \times \mathbb{R}^{d\times d}_{\text{sym}} \to [0,+\infty)$ in the following manner

$$V\big(x, \tfrac{1}{2}(CF - I)\big) = W(x,F).$$

where $CF := F^{\mathrm{T}}F$ is the Cauchy-Green stress tensor. The properties (W2)-(W4) of W imply existence of $\lambda > 0$ such that for all $M \in \mathbb{R}^{d\times d}_{\text{sym}}$

$$D^2V(x,0)M : M = D^2W(x,I)M : M \geq \lambda |M|^2. \tag{19}$$

Let us define two auxiliary quantities associated to V: for $x \in \Omega_D$ and $\varrho, \tau > 0$

$$\omega_\varrho(x) := \sup\{|D^2V(x,S) - D^2V(x,0)| : |S| \leq \varrho\} \quad \text{and} \quad C^{\varrho,\tau} := \{x \in \Omega_D : \omega_\varrho(x) \leq \tau\}. \tag{20}$$

Clearly for any $\tau > 0$ there holds $\mathscr{L}^d(\Omega_D \backslash C^{\varrho,\tau}) \to 0$ as $\varrho \to 0$. Performing a Taylor expansion of $V(x,\cdot)$ around 0 yields $t = t(\varepsilon) \in (0,1)$ such that the equality

$$\begin{aligned} &V(\,\cdot\,, \varepsilon\, e(u_\varepsilon) + \varepsilon^2 C\nabla u_\varepsilon) \\ &\qquad = \frac{1}{2} D^2V\big(\,\cdot\,, t(\varepsilon\, e(u_\varepsilon) + \varepsilon^2 C\nabla u_\varepsilon)\big)(\varepsilon\, e(u_\varepsilon) + \varepsilon^2 C\nabla u_\varepsilon) : (\varepsilon\, e(u_\varepsilon) + \varepsilon^2 C\nabla u_\varepsilon) \end{aligned} \tag{21}$$

holds $\mathscr{L}^d$-a.e. in $B_\varepsilon \cap C^{\varrho,\tau}$. Fix parameters $\alpha, \varrho \in (0,1)$. Let us observe that for $\varepsilon > 0$ small enough $|\varepsilon\, e(u_\varepsilon) + \varepsilon^2 C\nabla u_\varepsilon| \leq \varrho$ in B_ε. Thus in view of (20) and (21) there exists $\tau = \tau(\alpha,\lambda) > 0$ where $\lambda > 0$ is the one of (19), such that the bound

$$W(x, I + \varepsilon\nabla u_\varepsilon) \geq \frac{\alpha}{2} D^2V(x,0)(\varepsilon\, e(u_\varepsilon) + \varepsilon^2 C\nabla u_\varepsilon) : (\varepsilon\, e(u_\varepsilon) + \varepsilon^2 C\nabla u_\varepsilon) \tag{22}$$

holds for $\mathscr{L}^d$-a.e. $x \in B_\varepsilon \cap C^{\varrho,\tau}$. Henceforth integrating (22) yields

$$\begin{aligned}&\frac{1}{\varepsilon^2}\int_{B_\varepsilon \cap C^{\varrho,\tau}} W(x, I+\varepsilon\nabla u_\varepsilon)\,\mathrm{d}x \\ &\geq \frac{\alpha}{2}\int_{B_\varepsilon \cap C^{\varrho,\tau}} D^2V(x,0)(e(u_\varepsilon)+\varepsilon C\nabla u_\varepsilon) : (e(u_\varepsilon)+\varepsilon C\nabla u_\varepsilon)\,\mathrm{d}x.\end{aligned} \tag{23}$$

Further we recount that from the properties of W and (18), the relation $D^2V(\cdot,0)F : F = 2\mathbb{C}(\cdot)F : F$, holds for all $M \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $\mathscr{L}^d$-a.e. in Ω_D. Since $\varepsilon(C\nabla u_\varepsilon)\chi_{B_\varepsilon} \to 0$ uniformly in Ω_D, we have $w_\varepsilon + \varepsilon(C\nabla u_\varepsilon)\chi_{B_\varepsilon} \rightharpoonup e(u)$ weakly in $L^r(\Omega_D;\mathbb{R}^{d\times d})$. In turn the lower semicontinuity of the functional on the right hand side of the inequality in (23) implies

$$\begin{aligned}\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\Omega_D} W(x,I+\varepsilon\nabla u_\varepsilon)\,\mathrm{d}x &\geq \frac{\alpha}{2}\int_{C^{\varrho,\tau}} D^2V(x,0)e(u):e(u)\,\mathrm{d}x \\ &= \alpha\int_{C^{\varrho,\tau}} \mathbb{C}(x)e(u):e(u)\,\mathrm{d}x.\end{aligned} \tag{24}$$

Passing to the limit as $\varrho \searrow 0$ followed by $\alpha \nearrow 1$ in (24) gives

$$\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\Omega_D} W(x,I+\varepsilon\nabla u_\varepsilon)\,\mathrm{d}x \geq \mathcal{F}_{\mathrm{lin}}(u)$$

and this concludes the proof. □

To conclude this section we provide a proof of strong convergence property of recovery sequences asserted in Theorem 1.5.

Proof of Theorem 1.5. Let $s \in (1,r)$ be fixed. We have to show that there exist sets $U_\varepsilon \subset \Omega_D$ such that $\mathscr{L}^d(\Omega_D\backslash U_\varepsilon) \to 0$ and $e(u_\varepsilon)\chi_{U_\varepsilon} \to e(u)$ strongly in $L^r(\Omega_D;\mathbb{R}^{d\times d})$. To this end let us recount that in view of the proof of Theorem 1.4 the sets $B_\varepsilon := \{|\nabla u_\varepsilon| \leq 1/\varepsilon^{\frac{1}{3}}\}$ are such that $\mathscr{L}^d(\Omega_\varepsilon\backslash B_\varepsilon) \leq c\,\varepsilon^{\frac{r}{3}}$ and $e(u_\varepsilon)\chi_{B_\varepsilon} \rightharpoonup e(u)$ weakly in $L^r(\Omega_D;\mathbb{R}^{d\times d})$. Here the set $\Omega_\varepsilon \subset \Omega_D$ is as in Theorem 1.2. Furthermore from the estimates in (23) and (24) in the proof of Theorem 1.4 we have

$$\begin{aligned}\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{B_\varepsilon\cap C^{\varrho,\tau}} W(x,I+\varepsilon\nabla u_\varepsilon)\,\mathrm{d}x &\geq \liminf_{\varepsilon\to 0}\alpha\int_{B_\varepsilon\cap C^{\varrho,\tau}} \mathbb{C}(x)e(u_\varepsilon):e(u_\varepsilon)\,\mathrm{d}x \\ &\geq \alpha\int_{C^{\varrho,\tau}} \mathbb{C}(x)e(u):e(u)\,\mathrm{d}x\end{aligned}$$

where $\alpha, \varrho \in (0,1)$ and the set $C^{\varrho,\tau}$ is as in (20). Using the fact that $\mathcal{F}_\varepsilon(u_\varepsilon) \geq 1/\varepsilon^2\int_\Omega W(x,I+\varepsilon\nabla u_\varepsilon)\,\mathrm{d}x$ in conjunction with $\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon) = \mathcal{F}_{\mathrm{lin}}(u)$ and then resorting to a diagonal argument, one finds $\varrho_\varepsilon, \alpha_\varepsilon > 0$ with $\varrho_\varepsilon \to 0$ and $\alpha_\varepsilon \to 1$ as $\varepsilon \to 0$ such that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{B_\varepsilon\cap C^{\varrho_\varepsilon,\tau}} W(x,I+\varepsilon\nabla u_\varepsilon)\,\mathrm{d}x &= \int_{\Omega_D} \mathbb{C}(x)e(u):e(u)\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\int_{B_\varepsilon\cap C^{\varrho_\varepsilon,\tau}} \mathbb{C}(x)e(u_\varepsilon):e(u_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

We now set $D_\varepsilon := B_\varepsilon \cap C^{\varrho_\varepsilon,\tau}$. Since also $\mathscr{L}^d(\Omega_\varepsilon\backslash D_\varepsilon) \leq c\,\varepsilon^{\frac{r}{3}}$ by Chebyshev's inequality, the weak convergence of $e(u_\varepsilon)\chi_{D_\varepsilon} \rightharpoonup e(u)$ in $L^r(\Omega_D;\mathbb{R}^{d\times d})$ is preserved. Appealing to the positive definiteness of $\mathbb{C}(x)$ on $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ yields $e(u_\varepsilon)\chi_{D_\varepsilon} \to e(u)$ strongly in $L^r(\Omega_D;\mathbb{R}^{d\times d})$. □

4. Examples

In this section we provide constructions that aim to elucidate the preceding discussions and complete the narrative. In these examples we operate with the special linear group $SL(d,\mathbb{Z}) = GL^+(d,\mathbb{Z}) \subset \mathbb{Z}^{d\times d}$, i.e. all matrices in $GL(d,\mathbb{Z})$ with determinant 1. This type of wells is motivated by models of crystals and is considered for instance in [6, 18, 19].

In the interest of presentational clarity let us henceforth assume the set of wells to be given by

$$K := \bigcup_{k\in\mathbb{Z}} SO(d)L_k \quad \text{with } L_k := I + k\, e_{d-1} \otimes e_d. \tag{25}$$

Also let us assume $c_W \operatorname{dist}^2(F,K) \le W(x,F) \le C_W \operatorname{dist}^2(F,K)$ which implies a boundedness of the energy density in the set $\{I + t\, e_{d-1} \otimes e_d \colon t \in \mathbb{R}\}$. As above we use the convention $v_\varepsilon := \mathrm{id}_{\mathbb{R}^d} + \varepsilon u_\varepsilon$. For the due constructions in all the present section we employ a family of mollifiers ϑ_n such that $\operatorname{supp}(\vartheta_n) \subset B_{1/n}(0)$, $|\nabla \vartheta_n| \le c n^{d+1}$ and $\|\vartheta_n\|_{L^1(\mathbb{R}^d)} = 1$ for any $n \in \mathbb{N}$.

In the following example, we show a sequence of displacements with equibounded energy that is unbounded in the norm of $W^{1,r}(\Omega;\mathbb{R}^d)$, justifying the choice of the weaker convergence of Definition 1.1. We are only able to display such an example when $r \in (1^*, 2)$, which implies $d > 2$. Note that, in the case of finitely many wells, compactness holds in $W^{1,r}(\Omega;\mathbb{R}^d)$ for such values of r, as proved in [3].

Example 4.1 (No boundedness in $W^{1,r}(\Omega)$)**.** Here we suppose that $d > 2$ and the exponents r, p fulfill the assumptions as above, moreover $r \ne 1^*, 2$. Further for simplicity we assume that (H2) holds as equality, i.e., $\eta \simeq \varepsilon^{2-\frac{r}{1^*}}$. Since $d > 2$ and $r \in (1^*, 2)$, this implies $\varepsilon^{\frac{2}{d}}/\eta \to +\infty$. Consider now parameters $k_\varepsilon, n_\varepsilon \in \mathbb{N}$ and $\nu_\varepsilon \in (0,1)$ such that

(a) $k_\varepsilon \to +\infty$,
(b) $k_\varepsilon \le \varepsilon^{\frac{2}{d}}/2\eta \simeq \varepsilon^{(r-2)/1^*}$,
(c) $n_\varepsilon \simeq 1/(\eta\, k_\varepsilon) \simeq 1/(\varepsilon^{2-\frac{r}{1^*}}\, k_\varepsilon)$,
(d) $\nu_\varepsilon \simeq (\varepsilon^2 n_\varepsilon)^{\frac{1}{d-1}} \simeq (\varepsilon^{r/1^*}/k_\varepsilon)^{\frac{1}{d-1}}$.

We notice that (a) and (b) are compatible because of the assumption $\varepsilon^{\frac{2}{d}}/\eta \to +\infty$. Further, (b) and (c) imply $n_\varepsilon \to +\infty$, while (d) implies $\nu_\varepsilon \to 0$. Finally, (b) implies $(\eta k_\varepsilon)^{\frac{d}{d-1}} \le \varepsilon^{\frac{2}{d-1}}/2$, thus $1/n_\varepsilon \le \nu_\varepsilon/2$.

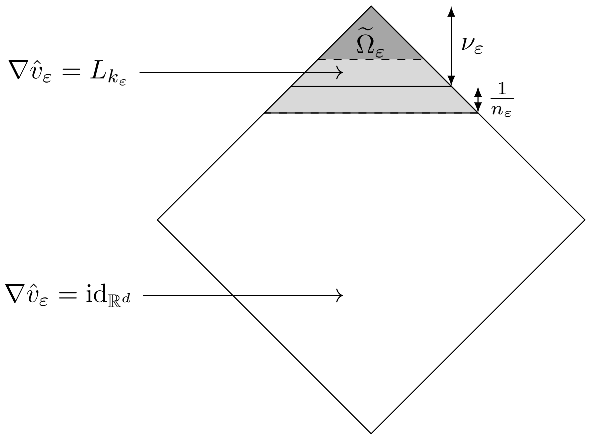

Figure 1. Construction of v_ε in Example 4.1. The picture displays a two-dimensional section of a three-dimensional cube, passing through opposite vertices of the cube.

Let $\Omega = \{(x_1,\dots,x_d) \in \mathbb{R}^d \colon |x_1| + \dots + |x_d| < 1\}$. Define $\widetilde{\Omega}_\varepsilon := \Omega \cap \{x_d > 1 - \nu_\varepsilon\}$, see Figure 1. Let $\hat{v}_\varepsilon \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ be such that $\nabla \hat{v}_\varepsilon := I\chi_{\Omega\setminus\widetilde{\Omega}_\varepsilon} + L_{k_\varepsilon}\chi_{\widetilde{\Omega}_\varepsilon}$ with L_k as in (25) and subsequently define $v_\varepsilon \in W^{2,p}(\Omega;\mathbb{R}^d)$ by the convolution product $v_\varepsilon := \vartheta_{n_\varepsilon} * \hat{v}_\varepsilon$. Note that $\mathscr{L}^d(\operatorname{supp}(W(\cdot,\nabla v_\varepsilon))) = \mathscr{L}^d(\operatorname{supp}(\nabla^2 v_\varepsilon)) \simeq \nu_\varepsilon^{d-1}/n_\varepsilon$ and $|\nabla^2 v_\varepsilon| \simeq k_\varepsilon n_\varepsilon \simeq 1/\eta$ when not zero. From the computation

$$F_\varepsilon(v_\varepsilon,\Omega) \simeq \frac{1}{\varepsilon^2}\frac{\nu_\varepsilon^{d-1}}{n_\varepsilon}\left(1 + \eta^p k_\varepsilon^p n_\varepsilon^p\right),$$

by (c) and (d) and by the boundedness of W, we deduce that $F_\varepsilon(v_\varepsilon,\Omega)$ is uniformly bounded. Simultaneously we observe that

$$\|\nabla u_\varepsilon\|_{L^r(\Omega)}^r \simeq \frac{k_\varepsilon^r}{\varepsilon^r}\nu_\varepsilon^d \gg \frac{k_\varepsilon^{1^*}}{\varepsilon^r}\nu_\varepsilon^d \simeq 1,$$

where we employed (a), (d) and the fact that $r > 1^*$. This shows that the Sobolev norm of u_ε is unbounded in Ω. However, easy computations show that there is $\bar{s} \in (1^*, r)$ such that $\|\nabla u_\varepsilon\|_{L^s(\Omega)}$ is unbounded for every $s > \bar{s}$ and bounded for $s = \bar{s}$.

Henceforth, we assume that $d = 2$ and $p = r = 2$ for simplicity. In the following example we see that when $\eta \ll \varepsilon^2$, which violates (H2), then the relative perimeter of the set Ω_ε introduced in Theorem 1.2 does not vanish. This produces jumps in the limit deformation gradient or in the limit deformation itself.

Example 4.2 (Scaling versus compactness)**.** Assume $d = p = r = 2$ and $\eta \ll \varepsilon^2$. Let $\Omega := \{(x_1,x_2) \in \mathbb{R}^2 : |x_1| + |x_2| < 1\}$.

i) (Plastic slip). Consider two diverging sequences $k_\varepsilon, n_\varepsilon \in \mathbb{N}$ whose scaling will be determined in the due course. Let $\nu_\varepsilon \in (0,1/2)$ with $\nu_\varepsilon \to 0$. Let $\widetilde{\Omega}_\varepsilon := \{(x_1,x_2) \in \Omega : 1/2 - \nu_\varepsilon < x_2 < 1/2 + \nu_\varepsilon\}$, see Figure 2. Further let $\hat{v}_\varepsilon \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ be such that $\nabla \hat{v}_\varepsilon = I\,\chi_{\Omega\setminus\widetilde{\Omega}_\varepsilon} + L_{k_\varepsilon}\chi_{\widetilde{\Omega}_\varepsilon}$, cf. (25), and define $v_\varepsilon \in H^2(\Omega;\mathbb{R}^2)$ by the convolution product $v_\varepsilon := \vartheta_{n_\varepsilon} * \hat{v}_\varepsilon$. Since $\mathscr{L}^2(\operatorname{supp}(W(\cdot,\nabla v_\varepsilon))) = \mathscr{L}^2(\operatorname{supp}(\nabla^2 v_\varepsilon)) \simeq 1/n_\varepsilon$ and $\|\nabla^2 v_\varepsilon\|_{L^2(\Omega)}^2 \simeq k_\varepsilon^2 n_\varepsilon$, setting $n_\varepsilon \simeq 1/\varepsilon^2$ it holds

$$F_\varepsilon(v_\varepsilon,\Omega) \simeq \frac{1}{\varepsilon^2\, n_\varepsilon} + \frac{\eta^2}{\varepsilon^2}\, k_\varepsilon^2 n_\varepsilon \simeq 1 + \left(\frac{\eta}{\varepsilon^2}\right)^2 k_\varepsilon^2.$$

We may choose $k_\varepsilon \to \infty$ such that $k_\varepsilon(\eta/\varepsilon^2) \simeq 1$ in which case it follows $F_\varepsilon(v_\varepsilon,\Omega) \le c$. On the other hand $\|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 \simeq k_\varepsilon^2/(\varepsilon^2\, n_\varepsilon) \simeq \varepsilon^4/\eta^2 \gg 1$ and one readily observes that (u_ε)

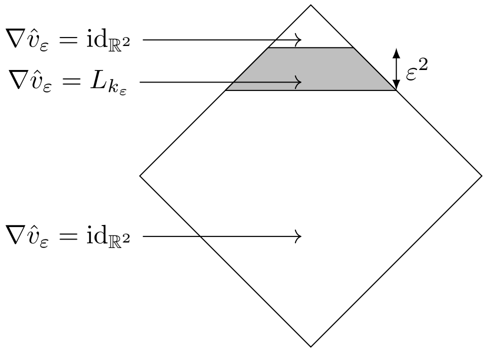

Figure 2. Construction of a deformation with a jump as in Example 4.2 (i).

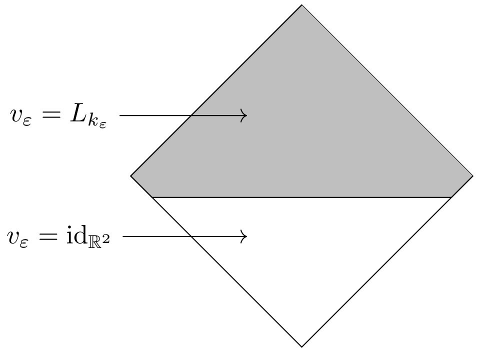

Figure 3. Construction of a deformation with no majority well as in Example 4.2 (ii).

create a jump discontinuity in the limit as $\varepsilon \to 0$, which may be interpreted as a plastic slip.

ii) (No majority well). Define $\Omega' := \{(x_1, x_2) \in \Omega : x_2 > -1/4\}$, $\hat{v}_\varepsilon \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ such that $\nabla \hat{v}_\varepsilon := I\,\chi_{\Omega\setminus\Omega'} + L_{k_\varepsilon}\chi_{\Omega'}$, and $v_\varepsilon := \vartheta_{n_\varepsilon} * \hat{v}_\varepsilon$, see Figure 3. Let k_ε and n_ε be as above. Reproducing the computations of part i) makes $F_\varepsilon(v_\varepsilon, \Omega)$ equibounded and $\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \gg 1$. Also in this way, there is no well occupying the majority of the domain.

Now we present an explicit construction exhibiting lack of exact rigidity for differential inclusions in the discrete group $SL(2,\mathbb{Z})$. The non-rigid maps that comply with the gradient constraint can be defined in such a way that they fulfill given boundary conditions on the whole boundary.

Example 4.3 (Non-rigidity in $SL(d,\mathbb{Z})$)**.** Let $\Omega = (0,1)^2$. For $k \in \mathbb{N}$ let $p_1^k, \dots, p_4^k$ be the vertices (enumerated anticlockwise) of the square $\Omega^k := (1/(2k), 1 - 1/(2k))^2$ and let $\Omega_1^k, \dots, \Omega_8^k$ be a triangulation of the annulus $\Omega\setminus\Omega^k$ as in Figure 4 below. Let us subsequently define $v_k \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ by setting $\nabla v_k := R^k$ in Ω^k where R^k is a rotation of Ω^k by $\pi/2$ anticlockwise such that $v_k(p_i^k) = p_{i+1}^k$ and $v_k(p_4^k) = p_1^k$; let v_k be given by the piecewise affine interpolation in $\Omega\setminus\Omega^k$ subject to the triangulation as in Figure 4. Then one can check that $v_k \in H_0^1(\Omega;\mathbb{R}^2) + \mathrm{id}_{\mathbb{R}^2}$ and $\nabla v_k \in SO(2)SL(2,\mathbb{Z})$.

Using such microstructure of Example 4.3 in conjunction with the scaling argument as in part i) of Example 4.2, we may come up sequences of deformations with equibounded energy which simultaneously are equal to the identity across the entire boundary and which give rise to jump discontinuity in the limit.

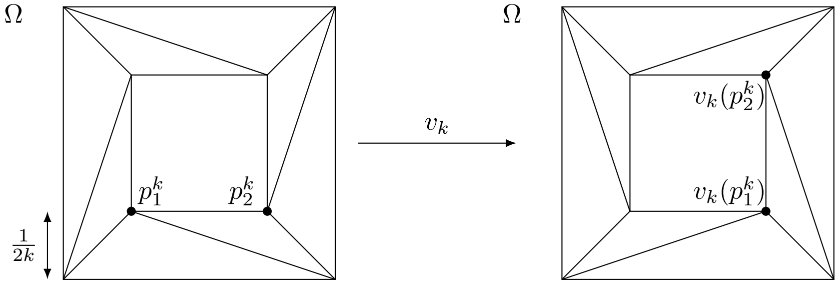

Figure 4. Transformation between reference and deformed configuration induced by the deformation of Example 4.3.

Let us relate such non-rigidity phenomenon of the latter example to the linearisation procedure carried out in Section 3. In particular, setting $K = SO(2)SL(2,\mathbb{Z})$ as the set of wells, the use of the second order regularisation term in the definition of $\mathcal{F}_\varepsilon$ is inevitable for the derivation of a linear model. Indeed the construction exhibited in Example 4.3 tells us that there exist nontrivial equilibrium states even after imposing boundary conditions across the entire boundary of the reference domain. Nevertheless, should we specify our considerations to the set of wells being laminates instead, as in (25), and set $\Gamma = \partial\Omega_D$, the linearisation procedure may be performed without employing the regularisation term. This can be deduced from the fact that there is only one rank-one connection between wells which in consequence forces the equilibrium states to be trivial.

Acknowledgments

The authors are members of INdAM-GNAMPA. GL, MP, and PW were partially supported by the INdAM-GNAMPA Project CUP:E5324001950001. The work of RA was funded by the Italian Ministry of University and Research through the PRIN project 2022 J4FYNJ “Variational methods for stationary and evolution problems with singularities and interfaces” CUP:B53D23009320006. The work of GL and PW was funded by the Italian Ministry of University and Research through the PRIN project 2022 HKBF5C “Variational Analysis of complex systems in Materials Science, Physics and Biology” CUP:B53D23009280006. PW was also supported by the Deutsche Forschungsgemeinschaft (DFG) within the CRC 1114 Scaling Cascades in Complex Systems, Project-Number 235221301, Project B09 Materials with Discontinuities on Many Scales.

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