

# Uniform rectifiability of brittle fractures in linear elasticity

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## Abstract

We prove the uniform rectifiability of brittle fractures in arbitrary dimension. The existing approach for the Mumford-Shah functional, which relies on separation-type properties of the singular set, faces serious obstacles in the Griffith setting due to the lack of coarea formula for the symmetric gradient. We present an alternative route to uniform rectifiability for free-discontinuity problems by proving that cracks have “plenty of big projections”.

**Keywords:** Griffith functional; free discontinuity problems; uniform rectifiability; brittle fractures; linear elasticity.

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## 1 Introduction

Variational models of fracture describe the formation of cracks as the outcome of an energy minimization principle. Let  $\Omega \subset \mathbb{R}^N$  be a bounded open set representing the reference configuration of a linearly elastic body. When the body  $\Omega$  is subjected to a prescribed boundary deformation, it may deform and store energy, or release this energy by breaking. The equilibrium configuration is found by minimizing a functional that balances the bulk elastic energy with the energy required to create a crack, namely

$$\int_{\Omega} f(x, e(u)) \, dx + \int_{J_u} \phi(x, u^+, u^-, \nu_u) \, d\mathcal{H}^{N-1},$$

where  $e(u) = (\nabla u + \nabla u^T)/2$  denotes the symmetrized gradient of  $u$ , the set  $J_u$  is the discontinuity set of  $u$ ,  $u^\pm$  are the traces of  $u$  on each side of  $J_u$  and  $\nu_u$  is a unit normal to the crack. In the context of Griffith’s brittle fracture, the energy required to produce a crack is proportional to the surface measure of the crack, yielding the Griffith functional

$$\int_{\Omega} f(x, e(u)) \, dx + \mathcal{H}^{N-1}(J_u).$$

The existence of minimizers for such functionals is established in the space  $GSBD(\Omega)$  of generalized functions with bounded deformations [4, 2]. A noteworthy feature of this framework is that the rectifiability of the crack is directly provided by the general theory of  $GSBD$  functions.

Let us recall that a set  $E \subset \mathbb{R}^N$  is said to be rectifiable of dimension  $N - 1$  if it can be covered by countably many Lipschitz images of  $\mathbb{R}^{N-1}$ , except for a set of  $\mathcal{H}^{N-1}$  measure zero. This is equivalently characterized by the fact that  $E$  blows-up as a hyperplane at  $\mathcal{H}^{N-1}$  almost-every point. Rectifiability is a flexible and useful notion in geometric measure theory, but it is purely qualitative, which is not effective for describing fractures. As cracks arise as

minimizers of a functional, one expects them to have scale-invariant quantitative properties, rather than merely good asymptotic behavior at generic points. Uniform rectifiability is a quantitative form of rectifiability which is more natural in this setting.

As a first definition, a uniformly rectifiable set is a closed, Ahlfors-regular set which is contained in a reasonable parametrization of  $\mathbb{R}^{N-1}$ .

**Definition.** We say that a closed set  $E \subset \mathbb{R}^N$  is a uniformly rectifiable set if there is a constant  $C \geq 1$  such that

$$C^{-1}r^{N-1} \leq \mathcal{H}^{N-1}(E \cap B(x, r)) \leq Cr^{N-1} \quad \text{for all } x \in E, 0 < r < \text{diam}(E)$$

and if there is a weight  $\omega \in \mathcal{A}_1$  and an  $\omega$ -regular parametrization  $z : \mathbb{R}^{N-1} \rightarrow \mathbb{R}^N$  such that  $E \subset z(\mathbb{R}^{N-1})$ .

The Muckenhoupt classes  $\mathcal{A}_p$  arose in the 1970s and 1980s as the appropriate conditions for the weighted boundedness of many classical operators in harmonic analysis. The condition  $\omega \in \mathcal{A}_1$  means that  $\omega$  is a positive function in  $L^1_{\text{loc}}(\mathbb{R}^N)$  such that

$$\int_{B(x, r)} \omega(y) \, dy \leq C \text{ess.inf}_{y \in B(x, r)} \omega(y) \quad \forall \text{ ball } B(x, r) \subset \mathbb{R}^N.$$

A mapping  $z : \mathbb{R}^{N-1} \rightarrow \mathbb{R}^N$  is said to be  $\omega$ -regular if there is  $C \geq 1$  such that

$$|z(x) - z(y)| \leq C \left( \int_{B((x+y)/2, |x-y|)} \omega(u) \, du \right)^{1/(N-1)} \quad \forall x, y \in \mathbb{R}^{N-1}$$

and

$$\int_{\{y \in \mathbb{R}^d \mid z(y) \in B(x, r)\}} \omega(y) \, dy \leq Cr^{N-1} \quad \forall \text{ ball } B(x, r) \subset \mathbb{R}^N.$$

One can think of  $z$  as being Lipschitz in a reasonable weighted space  $(\mathbb{R}^{N-1}, \omega \, dx)$ . The difference with plain rectifiability is that a uniformly rectifiable set  $E$  is contained in a single surface (instead of countable many) and we have a uniform control on  $E$  at all scales and locations through the constant  $C$ . This prevents the set from ever being too scattered for instance, but still allows cusps and self-intersections to some extent.

An alternative way of characterizing uniformly rectifiable sets is by “big pieces of biLipschitz images” (BPBI), [8, Theorem 1.57].

**Proposition.** A closed set  $E \subset \mathbb{R}^N$  is uniformly rectifiable if and only if there exists a constant  $C \geq 1$  such that for all  $x \in E$  and  $0 < r < \text{diam}(E)$ ,

$$C^{-1}r^{N-1} \leq \mathcal{H}^{N-1}(E \cap B(x, r)) \leq Cr^{N-1}$$

and there exists a compact subset  $A \subset \mathbb{R}^{N-1}$  and a mapping  $f : A \rightarrow \mathbb{R}^N$  such that

$$C^{-1}|z - w| \leq |f(z) - f(w)| \leq C|z - w| \quad \forall z, w \in A$$

and

$$\mathcal{H}^{N-1}(E \cap f(A) \cap B(x, r)) \geq C^{-1}r^{N-1}.$$

It is not difficult to see that this is a rectifiability property<sup>1</sup>. There is also a variant of this property called “big pieces of Lipschitz graphs” (BPLG), where one replaces biLipschitz

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<sup>1</sup>Decompose  $E = F \cup G$  as the disjoint union of a rectifiable part  $F$  and a purely unrectifiable part  $G$ . As Lipschitz images are rectifiable, BPBI implies that for all  $x \in E$ ,  $\liminf_{r \rightarrow 0} r^{1-N} \mathcal{H}^{N-1}(F \cap B(x, r)) \geq C^{-1}$ . One deduces that  $G = E \setminus F$  has zero  $\mathcal{H}^{N-1}$  measure by standard density theorems.

images by Lipschitz graphs: for all  $x \in E$  and  $0 < r < \text{diam}(E)$ , there exists a  $C$ -Lipschitz graph  $\Gamma$  of dimension  $N - 1$  such that

$$\mathcal{H}^{N-1}(E \cap \Gamma \cap B(x, r)) \geq C^{-1}r^{N-1}.$$

This condition is strictly stronger than uniform rectifiability, as shown by an example of Hrycak (the “venetian blind”, see [24, Subsection 1.2]).

In this paper, we shall prove that brittle fractures are locally contained in a uniformly rectifiable set with big pieces of Lipschitz graphs. We refer to Section 2 for various notations and the definition of topological quasiminimizers.

**Theorem 1.1** (Uniform rectifiability of brittle fractures). *There exists constants  $\varepsilon_0 > 0$  and  $C \geq 1$  (which depends on  $N, M, \mathbb{A}$ ) such that the following holds. If  $(u, K)$  is a Griffith topological quasiminimizer with gauge  $h$  in a domain  $\Omega$ , then for all  $x \in K$  and all  $r > 0$  with  $B(x, 2r) \subset \Omega$  and  $h(2r) \leq \varepsilon_0$ ,*

*$K \cap B(x, r)$  is contained in a uniformly rectifiable set  $E$  (with BPLG) of constant  $C$ .*

The Griffith energy presents substantial new challenges compared to its scalar analogue, the Mumford-Shah functional. We begin by briefly recalling how David and Semmes [10, 5, 6] established uniform rectifiability for Mumford-Shah minimizers, in order to highlight these difficulties. Let  $(u, K)$  be a local Mumford-Shah minimizer in a domain  $\Omega$  (without fidelity term, for simplicity). An easy case is when  $\nabla u = 0$  on  $\Omega \setminus K$ . In this situation,  $K$  induces a minimal partition of  $\Omega$  and locally satisfies the so-called “Condition B”: for all  $x \in K$  and  $r > 0$  such that  $B(x, 2r) \subset \Omega$ ,

$$\begin{aligned} &\text{there exists two } B_1, B_2 \subset B(x, r) \text{ of radius } \geq C^{-1}r \\ &\text{which lie in distinct connected components of } \Omega \setminus K. \end{aligned}$$

This is a well-known sufficient condition for uniform rectifiability. Heuristically, if  $K$  were not rectifiable, it would be too scattered to separate points properly. In general however, Mumford-Shah minimizers do not satisfy Condition B because of crack-tips. Nevertheless, in a ball  $B$  where the elastic energy  $\int_B |\nabla u|^2 dx$  is small compared to the surface term  $\mathcal{H}^{N-1}(K \cap B)$ , one can expect that  $K$  should be close to a minimal partition. In such a ball, David and Semmes proved that  $K$  satisfies Condition B up to “filling the holes” by a level set  $\{x \in B \setminus K \mid u(x) = t\}$  of small, controlled measure. Their argument relies on a clever use of the co-area formula to control the size of a level set. Moreover, a Carleson estimate provides shows quantitatively that this occurs in most balls centred on  $K$ . Thus, the singular set  $K$  still attempts to separate its complement at most locations and scales, and this weaker form of separation is sufficient to deduce uniform rectifiability. Adapting this approach to the Griffith setting is highly challenging, since no counterpart of the coarea formula exists for the symmetrized gradient.

In a series of papers [16, 17, 19], Friedrich established a piecewise Korn inequality which provides fine quantitative information on *GSBD* functions and their jump set. This powerful tool was used in [22] to bypass the coarea formula when controlling the size of holes, thereby making it possible to adapt the proof of David and Semmes to the Griffith setting. Unfortunately, the piecewise Korn inequality is, as of now, only available in the plane.

Prior to the work of David and Semmes, a first quantitative rectifiability property was introduced for Mumford-Shah minimizers in the plane by Dibos and Koepfler [13, 12], and later extended to all dimensions by Solimini [25]; this is the so-called “property of projections”. We state here a slightly weaker variant. We say that a closed Ahlfors-regular set  $E \subset \mathbb{R}^N$  has the property of projection if, for all  $x \in E$  and all  $0 < r < \text{diam}(E)$ ,

$$\int_{G(N-1, N)} \mathcal{H}^{N-1}(p_V(E \cap B(x, r))) dV \geq C^{-1}r^{N-1},$$

where  $G(N-1, N)$  denotes the Grassmanian space of hyperplanes equipped with its canonical measure, and  $p_V$  is the orthogonal projection onto  $V$ . This implies rectifiability by the Besicovitch-Federer projection theorem and, as observed by Dibos and Koepfler, is stable under Hausdorff convergence. It is conjectured that the property of projection implies uniform rectifiability (see [6, Open question 24.33] and [8, 9]); however, to the best of the author’s knowledge, this remains a difficult open problem. Two major recent contributions are due to Dabrowski [3], who proved the conjecture for one-dimensional sets, and Orponen [24], who showed that BPLG is equivalent to a stronger condition called “plenty of big projections” (PBP): for all  $x \in E$  and  $0 < r < \text{diam}(E)$ , there exists  $V \in G(N-1, N)$  such that for all  $W \in G(N-1, N)$  with  $\text{dist}(V, W) \leq C^{-1}$ ,

$$\mathcal{H}^{N-1}(p_W(K \cap B(x, r))) \geq C^{-1}r^{N-1}.$$

The main contribution of this paper is to prove that cracks have plenty of big projections. This provides a new route to uniform rectifiability for free-discontinuity functionals, alternative to the 1993 work of David and Semmes.

We begin by proving a first rectifiability theorem for Griffith minimizers, namely the property of projections (Theorem 3.1). As the excision method of Dibos–Koepfler [13, 12] and Solimini [25] does not extend readily to the symmetric gradient, this poses the first challenge in our paper. Our argument relies on a Federer-Fleming type projection estimate, yielding many rays that do not meet the crack and along which the directional variation of  $u$  can be controlled. While the property of projections does not yet give uniform rectifiability, it has the key feature of being stable under Hausdorff limits. We use this in a contradiction argument showing that every ball contains a smaller shifted ball where the crack has a small flatness (Proposition 3.2). In such a ball, we can then apply the slicing theorem of [20] to control the size of holes through a projection.

This line of approach produces the “flat in many places” lemma in a self-contained way, instead of relying on the theory of uniformly rectifiable sets, and yields at once both the uniform concentration property and plenty of big projections.

## 2 Definitions

Our working space is an open set  $\Omega \subset \mathbb{R}^N$ , where  $N \geq 2$ . We say that a constant is *universal* when it depends only on  $N$ . Given a set  $A$ , the notation  $A \subset\subset \Omega$  stands for  $\bar{A} \subset \Omega$ . We define a *rigid motion* as an affine map  $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$  of the form  $a(x) = b + Ax$ , where  $b \in \mathbb{R}^N$  and  $A \in \mathbb{R}^{N \times N}$  is a skew-symmetric matrix.

**Elasticity tensor.** Given two matrices  $\xi, \eta \in \mathbb{R}^{N \times N}$ , the notation  $\xi : \eta$  denotes the Frobenius inner product of  $\xi$  and  $\eta$ ,

$$\xi : \eta := \sum_{ij} \xi_{ij} \eta_{ij}$$

and  $|\xi| = \sqrt{\xi : \xi}$  the Frobenius norm. We fix for the whole paper a fourth-order elastic tensor  $\mathbb{A}$  such that for which there exists a constant  $\lambda \geq 1$  such that

$$\mathbb{A}(\xi - \xi^T) = 0 \quad \text{and} \quad \mathbb{A}\xi : \xi \geq \lambda|\xi + \xi^T|^2 \quad \text{for all } \xi \in \mathbb{R}^{N \times N}.$$

**Coral pairs.** A *pair*  $(u, K)$  consists of a relatively closed subset  $K \subset \Omega$  and a Sobolev function  $u \in W_{\text{loc}}^{1,2}(\Omega \setminus K; \mathbb{R}^N)$ . We say that a pair  $(u, K)$  has a *locally finite energy* if for all ball  $B \subset\subset \Omega$ ,

$$\int_{B \setminus K} |e(u)|^2 dx + \mathcal{H}^{N-1}(K \cap B) < +\infty,$$

where  $e(u) = (\nabla u + \nabla u^T)/2$ . We say that a relatively closed set  $K \subset \Omega$  is *coral* if for all  $x \in K$ , for all  $r > 0$ ,

$$\mathcal{H}^{N-1}(K \cap B(x, r)) > 0,$$

where  $\mathcal{H}^{N-1}$  is the Hausdorff measure of dimension  $N - 1$ . A pair  $(u, K)$  is also called coral if  $K$  is coral in the above sense.

**Competitors.** Let  $(u, K)$  be a pair. Let  $B$  be an open ball such that  $B \subset\subset \Omega$ . A *competitor* of  $(u, K)$  in  $B$  is a pair  $(\tilde{u}, \tilde{K})$  such that

$$\tilde{K} \setminus B = K \setminus B \quad \text{and} \quad \tilde{u} = u \quad \text{a.e. in} \quad \Omega \setminus (K \cup B).$$

A *topological competitor* of  $(u, K)$  in  $B$  is a competitor  $(\tilde{u}, \tilde{K})$  such that

all points  $x, y \in \Omega \setminus (K \cup B)$  which are separated by  $K$  are also separated by  $\tilde{K}$ .

This means that if  $x, y \in \Omega \setminus (K \cup B)$  belongs to different connected component of  $\Omega \setminus K$ , they also belong to different connected components of  $\Omega \setminus \tilde{K}$ . This notion was introduced by Bonnet [1] and appears naturally when one look at the minimality properties of limits of minimizers, such as blow-up limits for instance.

**Quasiminimizers.** We are interested in quasiminimizers in the sense of David and Semmes [6, 11], which are meant to represent minimizers of functionals with possibly highly irregular coefficients. One could be interested in two conditions of such type. First, assume that  $u \in GSDB^2(\Omega)$  is a local minimizer of a functional of the form

$$\int_{\Omega} f(x, e(u)) \, dx + \int_{J_u} \psi(x, u^+, u^-, \nu_u) \, d\mathcal{H}^{N-1},$$

where  $\psi: \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1} \rightarrow \mathbb{R}_+$  and  $f: \Omega \times \mathbb{R}_{\text{sym}}^{N \times N} \rightarrow \mathbb{R}_+$  are Borel functions satisfying  $\psi(x, a, b, \nu) = \psi(x, b, a, -\nu)$ ,

$$M^{-1} \leq \psi \leq M,$$

and

$$|f(x, \xi) - \mathbb{A}\xi : \xi| \leq g(|\xi|^2)$$

for some non-decreasing function  $g: [0, +\infty) \rightarrow [0, +\infty)$  such that  $\int_1^{+\infty} s^{-2}g(s) \, ds < +\infty$  (for instance  $g(t) = M(1 + t^{q/2})$  with  $0 \leq q < 2$ ). These assumptions allow for non heterogeneous bulk density that are perturbations of  $\mathbb{A}\xi : \xi$  up to a lower-order term, as well as very general surfaces terms accounting for anisotropy and (non-degenerate) cohesive material properties. The minimality of  $u$  entails (see the [18, proof of Theorem 2.7]) that for all ball  $B(x, r) \subset\subset \Omega$  and all *GSBD* competitors  $v$  of  $u$  in  $B(x, r)$ ,

$$\begin{aligned} \int_{B(x,r)} \mathbb{A}e(u) : e(u) \, dx + M^{-1}\mathcal{H}^{N-1}(J_u \cap B(x, r)) \\ \leq \int_{B(x,r)} \mathbb{A}e(v) : e(v) \, dx + M\mathcal{H}^{N-1}(J_v \cap B(x, r)) + h(r)r^{N-1}, \end{aligned} \quad (1)$$

where  $h: (0, +\infty) \rightarrow [0, +\infty[$  is some non-decreasing function such that  $\lim_{r \rightarrow 0} h(r) = 0$ . If instead  $u \in GSDB^2(\Omega)$  is a local minimizer of a functional of the form

$$\int_{\Omega} f(x, e(u)) \, dx + \mu(J_u),$$

where  $\mu$  is a measure such that  $M^{-1}\mathcal{H}^{N-1} \leq \mu \leq M\mathcal{H}^{N-1}$ , one can remove  $\mu(J_u \cap J_v)$  on both sides and finds a stronger condition: for all ball  $B(x, r) \subset\subset \Omega$  and all *GSBD* competitors  $v$  of  $u$  in  $B(x, r)$ ,

$$\begin{aligned} \int_{B(x,r)} \mathbb{A}e(u) : e(u) \, dx + M^{-1}\mathcal{H}^{N-1}(J_u \setminus J_v) \\ \leq \int_{B(x,r)} \mathbb{A}e(v) : e(v) \, dx + M\mathcal{H}^{N-1}(J_v \setminus J_u) + h(r)r^{N-1}, \end{aligned} \quad (2)$$

where  $h : (0, +\infty) \rightarrow [0, +\infty[$  is some non-decreasing function such that  $\lim_{r \rightarrow 0} h(r) = 0$ .

Let us emphasize that condition (1) is genuinely weak. It is primarily effective with competitors that try to eliminate  $J_u$  completely from a given ball. On the other hand, condition (2) is more subtle and interesting. The present author expects that it may prevent components of  $\Omega \setminus K$  from being isolated, from accumulating at a point, and from having cusps (at least in low dimensions, see [21] for the case where there is no elastic energy). In this work, we adopt for (1) as our definition of quasiminimality, since our technique is flexible enough to deal with this broad class. While the existence of Griffith minimizers is established in the space *GSBD* [2], we shall directly work with classical pairs  $(u, K)$ , since the jump set of a *GSBD* quasiminimizer is essentially closed [18, Theorem 2.7]. We shall also take into account quasiminimizers whose competitors are required to satisfy a topological constraint. We define a *gauge* as a nondecreasing function  $h : (0, +\infty) \rightarrow [0, +\infty[$  satisfying  $\limsup_{r \rightarrow 0} h(r) < \infty$ . It need not vanish as  $r \rightarrow 0$ ; for instance  $h$  may be a small constant.

**Definition 2.1.** *Let  $M \geq 1$  and let  $h$  be a gauge. A Griffith topological  $M$ -quasiminimizer with gauge  $h$  in  $\Omega$  is a coral pair  $(u, K)$  with locally finite energy such that for all  $x \in \Omega$ , for all  $r > 0$  with  $\overline{B}(x, r) \subset \Omega$  and for all topological competitor  $(v, F)$  of  $(u, K)$  in  $B(x, r)$ , we have*

$$\begin{aligned} \int_{B(x,r) \setminus K} \mathbb{A}e(u) : e(u) \, dx + M^{-1}\mathcal{H}^{N-1}(K \cap B(x, r)) \\ \leq \int_{B(x,r) \setminus F} \mathbb{A}e(v) : e(v) \, dx + M\mathcal{H}^{N-1}(F \cap B(x, r)) + h(r)r^{N-1}. \end{aligned}$$

*Remark 2.2.* We recall that if a pair  $(u, K)$  is a topological  $M$ -quasiminimizer in  $B(x_0, r_0)$  with gauge  $h$ , then the pair  $(u_0, K_0)$  defined by

$$u_0(x) := r_0^{-1/2}u(x_0 + xr_0) \quad \text{and} \quad K_0 := r_0^{-1}(K - x_0)$$

is a topological  $M$ -quasiminimizer in  $B(0, 1)$ , with gauge  $h_0(t) := h(r_0t)$ . We will always work with scale invariant quantities.

**Flatness.** Given a pair  $(u, K)$  in a ball  $B(x, r) \subset \mathbb{R}^N$ , we define the *flatness*  $\beta_K(x, r)$  of  $K$  in  $B(x, r)$  by

$$\beta_K(x, r) := \inf_P \sup_{y \in K \cap B(x, r)} \text{dist}(y, P),$$

where  $P$  ranges among affine hyperplanes passing through  $x_0$  (the infimum is always attained by compactness of the Grassmann space). When there is no ambiguity, we simply write  $\beta(x, r)$ . It is straightforward to verify that for  $0 < t \leq r$ ,

$$\beta_K(x, t) \leq \left(\frac{r}{t}\right)\beta_K(x, r).$$

**Normalized elastic energy.** Given a pair  $(u, K)$  in a ball  $B(x, r) \subset \mathbb{R}^N$ , we define the *normalized elastic energy* of  $u$  in  $B(x, r)$  by

$$\omega(x, r) := r^{1-N} \int_{B(x,r) \setminus K} |e(u)|^2 \, dx.$$

More generally, for  $p \geq 1$ , we set

$$\omega_p(x, r) := r^{1-2N/p} \left( \int_{B(x,r) \setminus K} |e(u)|^p dx \right)^{\frac{2}{p}}.$$

The exponent of  $r$  is chosen so that  $\omega_p$  is invariant under rescaling; see Remark 2.2. Note that  $\omega_2 = \omega$ , and for  $p \in [1, 2]$ , we have  $\omega_p \leq \omega$  by Hölder's inequality.

For the rest of the paper, we fix a constant  $M \geq 1$ . Every quasiminimizer considered in the following will be understood to satisfy the definition with this choice of  $M$ , which we omit to recall for simplicity. We recall two useful properties of topological quasiminimizers. The first one is Ahlfors-regularity [18, Theorem 2.3 (2)].

**Proposition 2.3** (Ahlfors-regularity). *There exist constants  $\varepsilon_0 > 0$  and  $C \geq 1$  (depending on  $N, M, \mathbb{A}$ ) such that the following holds. Let  $(u, K)$  be a topological quasiminimizer with any gauge  $h$  in  $\Omega$ . For all  $x \in K$ , for all  $r > 0$  such that  $B(x, r) \subset \Omega$  and  $h(r) \leq \varepsilon_0$ , we have*

$$\int_{B(x,r)} |e(u)|^2 dx + \mathcal{H}^{N-1}(K \cap B(x, r)) \leq Cr^{N-1} \quad (3)$$

and

$$\mathcal{H}^{N-1}(K \cap B(x, r)) \geq C^{-1}r^{N-1}. \quad (4)$$

Our next property is a Carleson estimate, which shows that at most scales and locations, the elastic energy is small compared to the surface energy.

**Proposition 2.4** (Carleson estimate). *There exist a constant  $\varepsilon_0 > 0$  (depending on  $N, M, \mathbb{A}$ ) and for all  $p \in (1, 2]$ , there exists  $C_p \geq 1$  (depending on  $N, M, \mathbb{A}, p$ ) such that the following holds. Let  $(u, K)$  be a topological quasiminimizer with any gauge  $h$  in  $\Omega$ . For all  $x \in K$  and  $r > 0$  such that  $B(x, 2r) \subset \Omega$  and  $h(2r) \leq \varepsilon_0$ , we have*

$$\int_{y \in K \cap B(x,r)} \int_0^r \omega_p(y, t) \frac{dt}{t} d\mathcal{H}^{N-1}(y) \leq C_p r^{N-1}. \quad (5)$$

*Proof.* The argument for (5) is the same as in Theorem 23.8 in [6], with  $|\nabla u|^2$  replaced by  $|e(u)|^2$ . It only uses the fact that, if  $h(2r) \leq \varepsilon_0$  with  $\varepsilon_0$  smaller than in Proposition 2.3, the energy estimates (3) and (4) hold in every ball  $B \subset B(x, 2r)$  centred on  $K$ .  $\square$

**Corollary 2.5.** *There exist a constant  $\varepsilon_0 > 0$  (depending on  $N, M, \mathbb{A}$ ) and for all  $p \in (1, 2]$  and  $\varepsilon > 0$ , there exists  $C(\varepsilon) \geq 1$  (depending on  $N, M, \mathbb{A}, p, \varepsilon$ ) such that the following holds. Let  $(u, K)$  be a topological quasiminimizer with any gauge  $h$  in  $\Omega$ . For all  $x \in K$  and  $r > 0$  with  $B(x, r) \subset \Omega$  and  $h(r) \leq \varepsilon_0$ , there exist  $y \in B(x, r/2)$  and  $t \in [C(\varepsilon)^{-1}r, r/2]$  such that*

$$\omega_p(y, t) \leq \varepsilon.$$

*Proof.* This corollary is standard; we reproduce the proof for completeness. Fix  $\varepsilon_0$  smaller than in Proposition 2.3 and Lemma 2.4. Let  $x \in K$  and  $r > 0$  be such that  $B(x, r) \subset \Omega$  and  $h(r) \leq \varepsilon_0$ . Assume that there exists some constant  $C_0 \geq 2$  such that  $\omega_p(y, t) > \varepsilon$  for every  $y \in B(x, r/2)$  and every  $t \in [C_0^{-1}r, r/2]$ . By (5), we have

$$\int_{y \in K \cap B(x, r/2)} \int_0^{r/2} \omega_p(y, t) \frac{dt}{t} d\mathcal{H}^{N-1}(y) \leq C_p r^{N-1}.$$

On the other hand, by assumption on  $C_0$ ,

$$\begin{aligned} \int_{y \in K \cap B(x, r/2)} \int_0^{r/2} \omega_p(y, t) \frac{dt}{t} d\mathcal{H}^{N-1}(y) &\geq \int_{y \in K \cap B(x, r/2)} \int_{C_0^{-1}r}^{r/2} \omega_p(y, t) \frac{dt}{t} d\mathcal{H}^{N-1}(y) \\ &\geq \varepsilon \mathcal{H}^{N-1}(K \cap B(x, r/2)) \ln(C_0/2), \end{aligned}$$

which yields an upper bound on  $C_0$ . Taking  $C_0$  large enough therefore ensures the existence of pair  $(y, t)$  with  $\omega_p(y, t) \leq \varepsilon$ .  $\square$

### 3 The property of projections

**Theorem 3.1** (Property of projections). *There exists constants  $\varepsilon_0 > 0$  and  $C \geq 1$  (depending on  $N, M, \mathbb{A}$ ) such that the following holds. Let  $(u, K)$  be a Griffith topological quasiminimizer in  $\Omega$ . For all  $x \in K$  and all  $r > 0$  such that  $B(x, r) \subset \Omega$  and  $h(r) \leq \varepsilon_0$ , we have*

$$\int_{G(N-1, N)} \mathcal{H}^{N-1}(p_V(K \cap B(x, r))) \, dV \geq C^{-1} r^{N-1}.$$

*Proof.* Choose an exponent  $p \in (2(N-1)/N, 2)$ , for instance  $p = (2N-1)/N$ . To prove Theorem 3.1, it suffices to show that there exists constants  $\varepsilon_1 > 0$  and  $C \geq 1$  (depending on  $N, M, \mathbb{A}$ ) such that for all  $x \in K$  and  $r > 0$  with  $B(x, r) \subset \Omega$ ,  $h(r) \leq \varepsilon_1$  and  $\omega_p(x, r) \leq \varepsilon_1$ , we have

$$\int_{G(N-1, N)} \mathcal{H}^{N-1}(p_V(K \cap B(x, r))) \, dV \geq C^{-1} r^{N-1}. \quad (6)$$

Indeed, assume that such constants  $\varepsilon_1$  and  $C$  have been fixed, and let  $\varepsilon_0$  be the constant of Corollary 2.5. Let any  $x \in K$  and  $r > 0$  be such that  $B(x, r) \subset \Omega$  and  $h(r) \leq \min(\varepsilon_1, \varepsilon_0)$ . By Corollary 2.5, there exists a point  $y \in K \cap B(x, r/2)$  and a radius  $t$  with  $C(\varepsilon_1)^{-1}r \leq t \leq r/2$  and  $\omega_p(y, t) \leq \varepsilon_1$ , where  $C(\varepsilon_1) \geq 2$  depends on  $N, M, \mathbb{A}, \varepsilon_1$ . By definition of  $\varepsilon_1$ , the estimate (6) then holds in the ball  $B(y, t)$ :

$$\int_{G(N-1, N)} \mathcal{H}^{N-1}(p_V(K \cap B(y, t))) \, dV \geq C^{-1} t^{N-1}.$$

Since  $t \geq C(\varepsilon_1)^{-1}r$  and  $K \cap B(y, t) \subset K \cap B(x, r)$ , this yields the property of projection in  $B(x, r)$ , with a larger constant  $C$ .

We now show that (6) holds when  $\omega_p$  is sufficiently small. After rescaling to the unit ball, we argue by contradiction and assume that there exists  $\varepsilon_0, \varepsilon_1 \in (0, 1)$  and a Griffith minimizer  $(u, K)$  in  $B(0, 1)$  such that

$$\left( \int_{B(0, 1)} |e(u)|^p \, dx \right)^{1/p} \leq \varepsilon_0, \quad h(1) \leq \varepsilon_0 \quad (7)$$

and

$$\int_{G(N-1, N)} \mathcal{H}^{N-1}(p_V(K \cap B(0, 1))) \, dV \leq \varepsilon_1. \quad (8)$$

We shall obtain a contradiction for a suitable choice of  $\varepsilon_0, \varepsilon_1$  which depends only on  $N, M, \mathbb{A}$  (we can already assume  $\varepsilon_0$  to be less than in Proposition 2.3). Throughout the proof,  $C$  denotes a generic constant  $\geq 1$  which depends only on  $N, M, \mathbb{A}$ .

By Ahlfors-regularity of  $K$ , it is standard (see [6, Lemma 23.25]) that

$$|\{x \in B(0, 1/2) \mid \text{dist}(x, K) \leq \rho\}| \leq C\rho \quad \forall \rho > 0. \quad (9)$$

Choose a radius  $r_0 \in (0, 1/10)$  (depending on  $N, M, \mathbb{A}$ ) so small that, for  $\rho = 2r_0$ , the left-hand side has a volume strictly less than  $|B(0, 1/100)|$ . Hence, there exists a point  $x_0 \in B(0, 1/100)$  with  $\text{dist}(x_0, K) \geq 2r_0$ . Note in particular that  $B(x_0, 2r_0) \subset B(0, 1/10)$ .

We also shall use the following observation. Let  $(e_1, \dots, e_{2N})$  denotes the family of vectors such that  $(e_1, \dots, e_N)$  is the canonical basis of  $\mathbb{R}^N$  and  $(e_{N+1}, \dots, e_{2N}) = -(e_1, \dots, e_N)$ . The family of points  $(x_0 + r_0 e_k)_k$  is being well distributed along  $\partial B(x_0, r_0)$  in the following sense. For every family of points  $x_1, \dots, x_{2N} \in \mathbb{R}^N$  with  $x_k \in B(x_0 + r_0 e_k, r_1)$ , where  $r_1 = r_0/(2N)$ , we have

$$\sup_k |y \cdot (x - x_k)| \geq \frac{r_0}{2N} |y| \quad \forall x, y \in \mathbb{R}^N. \quad (10)$$

Indeed, for every  $k$ ,

$$|y \cdot (x - x_k)| \geq |y \cdot (x - x_0 - r_0 e_k)| - \frac{r_0}{2N} |y|$$

and

$$\begin{aligned} \sup_k |y \cdot (x - x_0 - r_0 e_k)| &\geq \frac{1}{2N} \sum_{k=1}^{2N} |y \cdot (x - x_0 - r_0 e_k)| \\ &\geq \frac{1}{2N} \sum_{k=1}^N |y \cdot (x - x_0 - r_0 e_k)| + |y \cdot (x - x_0 + r_0 e_k)| \\ &\geq \frac{1}{2N} \sum_{k=1}^N |y \cdot (x - x_0 - r_0 e_k) - y \cdot (x - x_0 + r_0 e_k)| = \frac{r_0}{N} |y|. \end{aligned}$$

We now record three estimates that hold in average. Our first one is the Korn-Poincaré inequality applied to  $u \in W_{\text{loc}}^{1,2}(B(x_0, 2r_0); \mathbb{R}^N)$ : there exists a universal constant  $C_0 \geq 1$  and rigid motion  $a$  such that

$$\int_{B(x_0, 2r_0)} |u - a|^p dx \leq C_0 r_0^p \int_{B(x_0, 2r_0)} |e(u)|^p dx.$$

We may assume without loss of generality that  $a = 0$  since a (quasi-)minimizer remains (quasi-)minimizer after subtracting a rigid motion. We will only remember that

$$\int_{B(x_0, 2r_0)} |u|^p dx \leq C_0 \int_{B(0,1)} |e(u)|^p dx. \quad (11)$$

(Because of the gauge  $h$  in the definition of quasiminimizers, we cannot use elliptic regularity to control the  $L^\infty$  norm of  $u$  in  $B(x_0, r_0)$ .)

Next, we claim that, up to considering a bigger universal constant  $C_0$ ,

$$\int_{B(0,1/2)} \vartheta(x) dx \leq C_0 \int_{B(0,1)} |e(u)|^p dx, \quad (12)$$

where, for  $x \in B(0, 1/2)$ ,

$$\vartheta(x) := \int_{z \in \partial B(0,1)} \int_{[x,z]} |e(u)|^p d\mathcal{H}^1(s) d\mathcal{H}^{N-1}(z).$$

For  $z \in \partial B(0, 1)$ , we have

$$\int_{t \in [x,z]} |e(u)|^p d\mathcal{H}^1(t) = \int_0^{+\infty} |e(u)(x + t(z-x))|^p \mathbf{1}_{\{|x+t(z-x)| < 1\}}(z) d\mathcal{H}^1(t),$$

so by Fubini, the change of variable  $y = x + t(z-x)$ , and the co-area formula in polar coordinates,

$$\begin{aligned} \vartheta(x) &\leq C \int_0^{+\infty} \frac{1}{t^{N-1}} \int_{\partial B(x,t) \cap B(0,1)} |e(u)|^p d\mathcal{H}^{N-1}(y) dt \\ &\leq C \int_{B(0,1)} \frac{|e(u)|^p}{|z-x|^{N-1}} dz. \end{aligned}$$

Applying Fubini once more

$$\int_{B(0,1/2)} \vartheta(x) dx \leq C \int_{B(0,1)} |e(u)|^p \left( \int_{B(0,1/2)} \frac{dx}{|z-x|^{N-1}} \right) dz$$

and the result follows by integrability of  $x \mapsto |x|^{1-N}$  near the origin.

We now turn to our last estimate. We claim that up to increasing the universal constant  $C_0$ , we have

$$\int_{B(0,1/2)} \mathcal{H}^{N-1}(\Sigma(x)) \, dx \leq C_0 \int_{G(N-1,N)} \mathcal{H}^{N-1}(p_V(K \cap B(0,1))) \, dV, \quad (13)$$

where, for  $x \in B(0,1/2)$ ,

$$\Sigma(x) := \{z \in \partial B(0,1) \mid [x,z] \cap K \neq \emptyset\}.$$

For  $d = 1, \dots, N-1$ , let  $\gamma_{N,d}$  denote the canonical measure of  $G(N,d)$  (see [23, Chapter 3, §3.9]); we will be only use  $\gamma_{1,N}$  and  $\gamma_{N-1,N}$ . We recall that for all Borel set  $A \subset G(1,N)$ ,

$$\gamma_{1,N}(A) = \gamma_{N-1,N}(\{V^\perp \mid V \in A\}),$$

and that for all Borel set  $S \subset \partial B(0,1)$ , we have

$$\mathcal{H}^{N-1}(S) \leq C\gamma_{1,N}(\{L \mid L \cap S \neq \emptyset\}).$$

The same estimate remains valid if we slightly shift the centers: for  $x \in B(0,1/2)$  and for all Borel set  $S \subset \partial B(0,1)$ ,

$$\mathcal{H}^{N-1}(S) \leq C\gamma_{1,N}(\{L \mid (x+L) \cap S \neq \emptyset\}). \quad (14)$$

The point is that  $S$  is biLipschitz equivalent to its radial projection of center  $x_0$  onto  $\partial B(x_0,1)$ , whereas this radial projection does not affect the right-hand side of (14). Applying (14) with  $S = \Sigma(x)$ , we obtain that for all  $x \in B(0,1/2)$ ,

$$\mathcal{H}^{N-1}(\Sigma(x)) \leq C\gamma_{1,N}(\{L \mid (L+x) \cap K \cap B(0,1) \neq \emptyset\}).$$

By Fubini's theorem,

$$\begin{aligned} \int_{B(0,1/2)} \mathcal{H}^{N-1}(\Sigma(x)) \, dx &\leq C \int_{G(1,N)} |\{x \in B(0,1/2) \mid (x+L) \cap K \cap B(0,1) \neq \emptyset\}| \, dL \\ &\leq C \int_{G(N-1,N)} |\{x \in B(0,1/2) \mid (x+V^\perp) \cap K \cap B(0,1) \neq \emptyset\}| \, dV \\ &\leq C \int_{G(N-1,N)} \mathcal{H}^{N-1}(p_V(K \cap B(0,1))) \, dV, \end{aligned}$$

which is exactly (13).

Now, we show that these three estimates hold pointwise at most points. Let  $C_1 > 0$  (to be fixed shortly), and let  $A$  be the set of points  $x \in B(x_0, 2r_0)$  such that

$$|u(x)|^p > C_1 \int_{B(0,1)} |e(u)|^p \, dx. \quad (15)$$

If  $\int_{B(0,1)} |e(u)| \, dx > 0$ , we use (11) and Chebyshev's inequality,

$$|A| \left( C_1 \int_{B(0,1)} |e(u)|^p \, dx \right) \leq \int_{B(x_0, 2r_0)} |u(x)|^p \, dx \leq C_0 \int_{B(0,1)} |e(u)|^p \, dx,$$

so  $|A| \leq C_0 C_1^{-1}$ . If  $\int_{B(0,1)} |e(u)|^p \, dx = 0$ , then (11) shows that  $|u| = 0$  a.e. so that we have  $|A| = 0$  anyways. Similarly, the set of points  $x \in B(0, 1/10)$  such that

$$\vartheta(x) > C_1 \int_{B(0,1)} |e(u)|^p \, dx \quad (16)$$

has measure at most  $C_1^{-1}C_0$  by (12), and the set of point of points  $x \in B(0, 1/10)$  such that

$$\int_{B(0,1/2)} \mathcal{H}^{N-1}(\Sigma(x)) > C_1 \int_{G(N-1,N)} \mathcal{H}^{N-1}(p_V(K \cap B(0, 1))) dV, \quad (17)$$

also has measure at most  $C_1^{-1}C_0$ . We may therefore choose  $C_1$  so large (depending on  $N$ ,  $M$ ,  $\mathbb{A}$ ) that the bad set where either (15), (16) or (17) holds has a strictly smaller volume than any ball of the form  $B(x_0 + r_0 e_k, r_1)$ , where we recall  $r_1 = r_0/(2N)$ . Consequently, for each  $k = 1, \dots, 2N$ , we can find a point  $x_k \in B(x_0 + r_0 e_k, r_1)$  such that

$$|u(x_k)|^p \leq C_1 \int_{B(0,1)} |e(u)|^p dx, \quad (18)$$

$$\vartheta(x_k) \leq C_1 \int_{B(0,1)} |e(u)|^p dx, \quad (19)$$

and

$$\mathcal{H}^{N-1}(\Sigma(x_k)) dx \geq C_1 \int_{G(N-1,N)} \mathcal{H}^{N-1}(p_V(K \cap B(0, 1))) dV. \quad (20)$$

From this point on, the constant  $C_1$  is fixed and universal, so rename it  $C$  as usual.

Given a constant  $C_2 > 0$ , we define for each  $k$ , the set  $\Sigma'(x_k)$  of points  $z \in \partial B(0, 1)$  such that

$$\int_{[x_k, z]} |e(u)|^p dx > C_2 \int_{B(0,1)} |e(u)|^p dx.$$

By (19) and a now usual Chebyshev argument, we obtain  $\mathcal{H}^{N-1}(\Sigma'(x_k)) \leq CC_2^{-1}$ . We now choose  $C_2 = C(\varepsilon_1)$  so large that  $\mathcal{H}^{N-1}(\Sigma'(x_k)) \leq \varepsilon_1$ . In the rest of the proof,  $C(\varepsilon_1)$  for a generic constant  $\geq 1$  depending on  $N$ ,  $M$ ,  $\mathbb{A}$  and  $\varepsilon_1$ . In particular, for each  $k$  and every  $z \in \partial B(0, 1) \setminus \Sigma'(x_k)$ ,

$$\int_{[x_k, z]} |e(u)|^p dx \leq C(\varepsilon_1) \int_{B(0,1)} |e(u)|^p dx. \quad (21)$$

We finally introduce for  $k = 1, \dots, N$ , the union of (bad) rays

$$R_k = \{ tx \mid x \in \Sigma(x_k) \cup \Sigma'(x_k), t \in [0, 1] \}$$

and set  $R = \bigcup_{k=1}^N R_k$ . By (20) and (8), we have  $\mathcal{H}^{N-1}(\Sigma(x_k)) \leq C\varepsilon_1$ , and by the previous paragraph  $\mathcal{H}^{N-1}(\Sigma'(x_k)) \leq \varepsilon_1$ . It follows that  $|R| \leq C\varepsilon_1$ . This bound means that for most points  $x \in B(0, 1)$ , the segment  $[x_k, x]$  does not meet  $K$  and moreover

$$\int_{[x_k, x]} |e(u)|^p d\mathcal{H}^1 \leq C(\varepsilon_1) \int_{B(0,1)} |e(u)|^p dx$$

This property will play a key role in controlling the elastic energy of our competitor, which we construct in the next step.

Let  $D$  denote the annulus

$$D = \{ x \in \mathbb{R}^N \mid 1/4 < |x| < 1/2 \}.$$

We claim that for  $\delta \in (0, 1/10)$ ,

$$\int_{\{x \in D \mid d(x, K) \geq \delta\}} |u|^p \leq C\delta^{p-N} C(\varepsilon_1) \varepsilon_0^p, \quad (22)$$

where  $C(\delta)$  is a constant depending on  $N$  and  $\delta$ . Choose a maximal family of points  $y_i \in D$  such that  $\text{dist}(y_i, K) \geq \delta$  and  $|y_i - y_j| \geq \delta/10$ . Then the balls  $B_i = B(y_i, \delta/100)$  are pairwise

disjoint, and since they are contained in  $B(0, 1)$ , their number is at most  $C\delta^{-N}$ . Moreover,  $D$  is covered by the balls  $B(y_i, \delta/10)$ , so to prove (22), it suffices to show that for all  $i$ ,

$$\int_{B_i} |u|^p dx \leq C\delta^p C(\varepsilon_1)\varepsilon_0^p. \quad (23)$$

Since  $B_i \cap K = \emptyset$ , we may apply Korn-Poincaré inequality in  $B_i$  and obtain a rigid motion  $a_i$  such that

$$\int_{B_i} |u - a_i|^p dx \leq C\delta^p \int_{B_i} |e(u)|^p dx. \quad (24)$$

We now show that for  $x \in B_i \setminus R$ , we have  $|u(x)| \leq C(\varepsilon_1)\varepsilon_0$ . By definition of  $R$ , together (21) and (7), we know that for every such  $x$  and for all  $k$ , the ray  $[x_k, x]$  does not intersect  $K$  and

$$\int_{[x_k, x]} |e(u)|^p d\mathcal{H}^1 \leq C(\varepsilon_1) \int_{B(0,1)} |e(u)|^p dx \leq C(\varepsilon_1)\varepsilon_0^p.$$

Since

$$\frac{d}{dt} [u(x_k + t(x - x_k)) \cdot (x - x_k)] = (e(u)(x_k + t(x - x_k))(x - x_k)) \cdot (x - x_k),$$

we find

$$|(u(x) - u(x_k)) \cdot (x - x_k)| \leq C \int_{[x_k, x]} |e(u)| d\mathcal{H}^1 \leq C(\varepsilon_1)\varepsilon_0.$$

Using (10), we deduce that  $|u(x) - u(x_k)| \leq C(\varepsilon_1)\varepsilon_0$ , and by (18), this implies  $|u(x)| \leq C(\varepsilon_1)\varepsilon_0$ . It follows that the rigid motion  $a_i$  has a small  $L^1$  measure in  $B_i \setminus R$  since

$$\int_{B_i \setminus R} |a_i|^p dx \leq C \int_{B_i \setminus R} |u - a_i|^p dx + C \int_{B_i \setminus R} |u|^p dx \leq \delta^p C(\varepsilon_1)\varepsilon_0^p.$$

As  $|R| \leq C\varepsilon_1$ , we may choose  $\varepsilon_1$  sufficiently small (depending on  $N, \delta$ ) so that  $|B_i \setminus R| \geq |B_i|/2$ . Lemma A.1 in the appendix then yields

$$|a_i|^p \leq \delta^p C(\varepsilon_1)\varepsilon_0^p \quad \text{uniformly on } B_i.$$

Inserting this bound into (24) gives (23), and in turn (22).

Consider a cut-off function  $\varphi \in C_c^1(D; [0, 1])$  such that  $\varphi(x) = 1$  when  $\text{dist}(x, K) \geq 2\delta$ ,  $\varphi(x) = 0$  when  $\text{dist}(x, K) \leq \delta$  and  $|\nabla\varphi| \leq C\delta^{-1}$  everywhere. We extend  $u$  on the whole annulus  $D$  by  $v(x) := \varphi(x)u(x)$ . One estimates

$$|e(v)| \leq |\nabla\varphi||u| + \varphi|e(u)|,$$

so

$$\begin{aligned} \int_D |e(v)|^p dx &\leq C\delta^{-p} \int_{\{x \in D | \text{dist}(x, K) \geq \delta\}} |u|^p dx + C \int_D |e(u)|^p dx \\ &\leq C\delta^{-N} C(\varepsilon_1)\varepsilon_0^p. \end{aligned}$$

As  $v$  is Sobolev in the annulus  $D$ , Korn's inequality yields a rigid motion  $a$  such that

$$\int_D |\nabla v - \nabla a|^p dx \leq C\delta^{-N} C(\varepsilon_1)\varepsilon_0^p.$$

and hence, there exists  $\rho \in (1/4, 1/2)$  such that

$$\int_{\partial B(0, \rho)} |\nabla v - \nabla a|^p dx \leq C\delta^{-N} C(\varepsilon_1)\varepsilon_0^p.$$

Moreover, (9) gives

$$|\{x \in B(0, 1/2) \mid \text{dist}(x, K) \leq 2\delta\}| \leq C\delta$$

so we may choose  $\rho \in (1/4, 1/2)$  such that the set

$$Z := \{x \in \partial B(0, \rho) \mid \text{dist}(x, K) \leq 2\delta\}$$

satisfies  $\mathcal{H}^{N-1}(Z) \leq C\delta$  as well. By [6, Lemma 22.32], the function  $w := v - a$  defined on  $\partial B(0, \rho)$  admits a  $C^1$  extension to  $B(0, \rho)$  such that

$$\int_{B(0, \rho)} |\nabla w|^2 dx \leq C \left( \int_{\partial B(0, \rho)} |\nabla v - \nabla a|^p dx \right)^{2/p} \leq C\delta^{-2N/p} C(\varepsilon_1) \varepsilon_0^2.$$

Assume for the moment that  $(u, K)$  is a plain quasiminimizer, we will treat later the topological case. Define  $(\tilde{u}, \tilde{K})$  by

$$\tilde{K} = (K \setminus B(0, \rho)) \cup Z$$

and

$$\tilde{u} = \begin{cases} u & \text{in } B(0, 1) \setminus (B(0, \rho) \cup K) \\ w + a & \text{in } \bar{B}(0, \rho) \setminus Z. \end{cases}$$

Then  $(\tilde{u}, \tilde{K})$  is a competitor of  $(u, K)$  in every ball  $B(0, r)$  with  $\rho < r < 1$ , and hence

$$\begin{aligned} \int_{B(0, \rho)} \mathbb{A}e(u) : e(u) dx + M^{-1} \mathcal{H}^{N-1}(K \cap B(0, \rho)) \\ \leq \int_{B(0, \rho)} \mathbb{A}e(w) : e(w) dx + M \mathcal{H}^{N-1}(Z) + h(1). \end{aligned}$$

By Ahlfors-regularity (Proposition 2.3), there exists a constant  $c_0 > 0$  which only depends on  $N, M, \mathbb{A}$  such that

$$\int_{B(0, \rho)} \mathbb{A}e(u) : e(u) dx + M^{-1} \mathcal{H}^{N-1}(K \cap B(0, \rho)) \geq c_0.$$

On the other hand, we can bound the right-hand side using

$$\int_{B(0, \rho)} |e(w)|^2 dx \leq C\delta^{-2N/p} C(\varepsilon_1) \varepsilon_0^2 \quad \text{and} \quad M \mathcal{H}^{N-1}(Z) \leq CM\delta.$$

We first fix  $\delta > 0$  small enough (depending on  $N, M, \mathbb{A}$ ) so that  $M \mathcal{H}^{N-1}(Z) \leq c_0/10$ , and then choose  $\varepsilon_0$  sufficiently small (again depending on  $N, M, \mathbb{A}$ ) so that  $\int_{B(0, \rho)} |e(w)|^2 dx \leq c_0/10$  and  $h(1) \leq c_0/10$ . This contradicts the previous lower bound.

In case where  $(u, K)$  is a topological quasiminimizers, choose a point  $x_0 \in B(0, 1/10)$  such that the set

$$\Sigma = \{z \in \partial B(0, \rho) \mid [z, x_0] \text{ meets } K\}$$

satisfies  $\mathcal{H}^{N-1}(\Sigma) \leq C_1 \varepsilon_1$ , where  $C_1 \geq 1$  is a universal constant. We then redefine  $\tilde{K}$  by

$$\tilde{K} = (K \setminus B(0, \rho)) \cup Z \cup \Sigma.$$

The set  $\Sigma$  is compact, since it is the image of  $K \cap B(0, 1)$  under the radial projection  $\mathbb{R}^N \setminus \{x_0\} \rightarrow \partial B(0, 1)$ , hence  $\tilde{K}$  remains relatively closed in  $\Omega$ . Moreover,  $\mathcal{H}^{N-1}(\Sigma) \leq C_1 \varepsilon_1$ , so the previous contradiction argument still applies provided we also assume  $\varepsilon_1 \leq c_0/(10C_1)$ .

We now check that  $\tilde{K}$  is a topological competitor of  $K$ . Let  $x, y \in B(0, 1) \setminus \tilde{K}$  be two points that are connected by a path  $\gamma$  in  $B(0, 1) \setminus \tilde{K}$ . We claim that  $x$  and  $y$  are connected

in  $B(0,1) \setminus K$ . If  $\gamma$  never meets  $\overline{B}(0,\rho)$ , then  $x$  and  $y$  are also connected in  $B(0,1) \setminus K$  because  $\widetilde{K} = K$  outside  $\overline{B}(0,\rho)$ . Otherwise, let  $x_1$  be the first point where  $\gamma$  meets  $\partial B(0,\rho)$ , and  $y_1$  be the last such point. Then  $x$  is connected to  $x_1$  by a path in  $B(0,1) \setminus K$  and  $y$  to  $y_1$  by a path in  $B(0,1) \setminus K$ . Since  $x_1, y_1 \notin \Sigma$ , the segments  $[z, x_1]$  and  $[z, y_1]$  don't intersect  $K$ . Concatenating these four paths shows that  $x$  and  $y$  are connected in  $B(0,1) \setminus K$ , as required.  $\square$

The property of projection already ensures that the crack is rectifiable (assuming  $\limsup_{r \rightarrow 0} h(r) < \varepsilon_0$ ). To see this, decompose as usual  $K = F \cup G$  as the disjoint union of a rectifiable part  $F$  and a purely unrectifiable part  $G$ . According to the Federer-Besicovitch projection theorem, we have

$$\int_{G(N-1,N)} \mathcal{H}^{N-1}(p_V(G)) \, dV = 0.$$

Therefore, Theorem 3.1 shows that for all  $x \in K$  and  $r > 0$  such that  $B(x,r) \subset \Omega$  and  $h(r) \leq \varepsilon_0$ ,

$$C^{-1}r^{N-1} \leq \int_{G(N-1,N)} \mathcal{H}^{N-1}(p_V(F \cap B(x,r))) \, dV \leq \mathcal{H}^{N-1}(F \cap B(x,r)).$$

One deduces that  $\mathcal{H}^{N-1}(G) = 0$  by standard density theorems [23, Theorem 6.2].

The next results states that, given any ball, one can find a smaller shifted ball (with comparable radius) where  $K$  has a small flatness. This is a standard property of uniformly rectifiable sets, typically derived from the ‘‘Weak Geometric Lemma’’ (WGL); see [6, Section 73]. Here we obtain it instead by a contradiction argument and the stability of the projection property under Hausdorff convergence. This proof is inspired by works on the Plateau problem, such as those of Fang [15, 14] and David [7, Lemma 10.21], where the rectifiability of limits is obtained by different methods but used in the same spirit.

**Proposition 3.2** (Flat in many places). *There exists  $\varepsilon_0 > 0$  (depending on  $N, M, \mathbb{A}$ ) and for all  $\varepsilon > 0$ , there exists  $C(\varepsilon) \geq 1$  (depending on  $N, M, \mathbb{A}, \varepsilon$ ) such that the following holds. If  $(u, K)$  is a topological quasiminimizer in a domain  $\Omega$ , then for all  $x \in K$  and all  $r > 0$  with  $B(x,r) \subset \Omega$  and  $h(r) \leq \varepsilon_0$ , there exists  $y \in B(x, r/2)$  and  $t \in [C(\varepsilon)^{-1}r, r/2]$  such that*

$$\beta(y, t) \leq \varepsilon.$$

*Proof.* Fix  $\varepsilon_0$  smaller than in Proposition 2.3 and Theorem 3.1. Up to rescaling, it suffices to prove for the statement for quasiminimizers in the unit ball. We proceed by contradiction for a given  $\varepsilon > 0$ . Suppose there exists a sequence of topological quasiminimizers  $(u_i, K_i)_i$  with gauges  $h_i$  in  $B(0,1)$  such that, for all  $i$ ,  $h_i(1) \leq \varepsilon_0$  and, for all  $y \in B(0, 1/2)$ , for all  $t \in [2^{-i}, 1/2]$ , one has  $\beta_{K_i}(x, r) > \varepsilon$ .

We extract a subsequence (not relabeled) such that  $(K_i)_i$  converges to a relatively closed subset  $K \subset B(0,1)$  in local Hausdorff distance (see [6, Proposition 34.6]). Since each  $K_i$  contains 0, the limit set  $K$  also contains 0. It is standard that limit set  $K$  inherits Ahlfors-regularity from the sequence  $(K_i)$ ; that is, there exists  $C_0 \geq 0$  such that for all  $x \in K$  and all  $r > 0$  with  $B(x,r) \subset B(0,1)$ ,

$$C_0^{-1}r^{N-1} \leq \mathcal{H}^{N-1}(K \cap B(x,r)) \leq C_0r^{N-1}.$$

Next, fix  $x \in K \cap B(0,1)$  and  $r > 0$  with  $B(x,r) \subset B(0,1)$ . We claim that  $K$  has the property of projection in  $B(x,r)$ . Without loss of generality, we may assume that

$\overline{B}(x, r) \subset B(0, 1)$ . Let  $0 < \rho < r$  and  $\varepsilon \in (0, r - \rho)$ . For all sufficiently large  $i$ , every point of  $K_i \cap B(x, \rho)$  lies at distance at most  $\varepsilon$  from  $K$ ; hence

$$\begin{aligned} \int_{G(N-1, N)} \mathcal{H}^{N-1}(p_V(K_i \cap B(x, \rho))) \, dV \\ \leq \int_{G(N-1, N)} \mathcal{H}^{N-1}(\{z \in V \mid \text{dist}(z, p_V(K \cap \overline{B}(x, r))) \leq \varepsilon\}) \, dV. \end{aligned}$$

Since  $x \in K$ , there exists a sequence of points  $(x_i)_i$  with  $x_i \in K_i \cap B(0, 1)$  and  $x_i \rightarrow x$ . In particular, for  $i$  large enough we have

$$\int_{G(N-1, N)} \mathcal{H}^{N-1}(p_V(K_i \cap B(x_i, \rho/2))) \, dV \leq \int_{G(N-1, N)} \mathcal{H}^{N-1}(p_V(K_i \cap B(x, \rho))) \, dV.$$

By applying Theorem 3.1 to  $(u_i, K_i)$  in the ball  $B(x_i, \rho/2)$ , we obtain

$$\int_{G(N-1, N)} \mathcal{H}^{N-1}(\{z \in V \mid \text{dist}(z, p_V(K \cap \overline{B}(x, r))) \leq \varepsilon\}) \, dV \geq C^{-1} \rho^{N-1}.$$

Letting  $\rho \rightarrow r$  and  $\varepsilon \rightarrow 0$ , and using dominated convergence, we deduce that

$$\int_{G(N-1, N)} \mathcal{H}^{N-1}(p_V(K \cap \overline{B}(x, r))) \, dV \geq C^{-1} r^{N-1}.$$

This proves the claim, and therefore  $K$  is rectifiable.

As a standard consequence from Ahlfors-regularity and rectifiability, we have

$$\lim_{r \rightarrow 0} \beta(x, r) = 0 \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in K.$$

Since  $0 \in K$  and  $K$  is Ahlfors-regular, we also have  $\mathcal{H}^{N-1}(K \cap B(0, 1/10)) > 0$ , so we can find such a point  $x \in K \cap B(0, 1/10)$ . Then, there exists  $r \in (0, 1/10)$  and a hyperplane  $P$  through  $x$  such that  $B(x, r) \subset B(0, 1)$  and

$$K \cap B(x, r) \subset \{y \in B(x, r) \mid \text{dist}(y, V) \leq \varepsilon r/200\}.$$

Let  $(x_i) \in K_i$  be a sequence such that  $x_i \rightarrow x$ . For  $i$  sufficiently large,

$$K_i \cap B(x_i, r/10) \subset \{y \in B(x_i, r/10) \mid \text{dist}(y, V) \leq \varepsilon r/100\},$$

and hence  $\beta_{K_i}(x_i, r/10) \leq \varepsilon/10$ . Choosing  $i$  large enough so that  $2^{-i} < r/10$ , this contradicts our assumption that  $\beta_{K_i}(y, t) \geq \varepsilon$  for all  $y \in K_i \cap B(0, 1/2)$  and all  $t \geq [2^{-i}, 1/2]$ .  $\square$

Combining the previous Proposition 3.2 (flat in many places) with Corollary 2.5 (small elastic energy in many places), we obtain both conditions simultaneously.

**Corollary 3.3.** *There exists a constant  $\varepsilon_0 > 0$  (depending on  $N, M, \mathbb{A}$ ) and for every  $p \in (1, 2]$  and every  $\varepsilon > 0$ , there exists a constant  $C(\varepsilon) \geq 1$  (depending on  $N, \mathbb{A}, p, \varepsilon$ ) such that the following holds. If  $(u, K)$  is a topological quasiminimizer in a domain  $\Omega$ , then for all  $x \in K$  and  $r > 0$  with  $B(x, r) \subset \Omega$  and  $h(r) \leq \varepsilon_0$ , there exists  $y \in B(x, r/2)$  and  $t \in [C(\varepsilon)^{-1}r, r/2]$  such that*

$$\beta(y, t) + \omega_p(y, t) \leq \varepsilon.$$

*Proof.* This is standard. Fix  $\varepsilon_0$  smaller than in Corollary 2.5 and Proposition 3.2. Let  $\varepsilon_1 > 0$  be a small constant to be chosen later. Assuming that  $h(r) \leq \varepsilon_0$ , Proposition 3.2 yields a point  $y \in B(x, r/2)$  and a radius  $t \in [C(\varepsilon_1)^{-1}r, r/2]$  such that  $\beta(y, t) \leq \varepsilon_1$ . Applying Corollary 2.5 in  $B(y, t)$ , we then obtain a point  $z \in B(y, t/2)$  and a radius  $s \in [C(\varepsilon)^{-1}t, t/2]$  such that  $\omega_p(z, s) \leq \varepsilon/2$ . Since  $s \geq C(\varepsilon)^{-1}t$ , we still control  $\beta(z, s) \leq C(\varepsilon)\beta(y, t) \leq C(\varepsilon)\varepsilon_1$ . Finally, we choose  $\varepsilon_1$  (depending on  $N, M, \mathbb{A}, p$  and  $\varepsilon$ ) so that  $\beta(z, s) \leq \varepsilon/2$ . This choice gives

$$\beta(z, s) + \omega_p(z, s) \leq \varepsilon.$$

$\square$

## 4 Fine estimate on holes through a projection

We define a “normalized jump” in the spirit of [6]. Let  $(u, K)$  be a pair in a ball  $B(x, r) \subset \mathbb{R}^N$  such that  $\beta(x, r) \leq 1/2$ . Let  $\nu_0$  be a unit normal to a hyperplane realizing the infimum in the definition of  $\beta(x, r)$ . Let  $a_1, a_2$  be two rigid motions that approximate  $u$  in the lower and upper part of  $B(x, r)$  and write

$$a_i(y) = b_i + A_i(y - x),$$

where  $b_i \in \mathbb{R}^N$  and  $A_i \in \mathbb{R}_{\text{skew}}^{N \times N}$  are given by

$$A_i = \int_{D_i} \frac{\nabla u(y) - \nabla a_i(y)}{2} dy, \quad b_i = \int_{D_i} u(y) dy + A_i(x - x_i),$$

and where  $D_1 = B(x_1, r/8)$  and  $D_2 = B(x_2, r/8)$  are balls with centers  $x_1 = x + (3r/4)\nu_0$  and  $x_2 = x - (3r/4)\nu_0$ . The *normalized jump* of  $u$  in  $B(x, r)$  is then

$$J(x, r) := \frac{|b_1 - b_2| + r|A_1 - A_2|}{\sqrt{r}}.$$

This quantity is invariant under rescaling to the unit ball; see Remark 2.2. We now recall [20, Lemma 4.3] on the initialization of the jump (the assumption on  $D_1, D_2$  in the statement below is only required for *topological* quasiminimizers).

**Lemma 4.1** (Initialization of the jump). *There exists constants  $\varepsilon_0$  (depending on  $N, M, \mathbb{A}$ ) and for every  $p \in (2(N-1)/N, 2]$ , a constant  $\eta_0 \geq 1$  (depending on  $N, M, \mathbb{A}, p$ ) such that the following holds. Let  $(u, K)$  be a topological quasiminimizer with any gauge  $h$  in  $\Omega$ . For all  $x \in K$  and  $r > 0$  with  $B(x, r) \subset \Omega$  such that*

$$\beta(x, r) + \omega_p(x, r) \leq \eta_0, \quad h(r) \leq \varepsilon_0$$

and

$$D_1 \text{ and } D_2 \text{ lie in the same connected component of } \Omega \setminus K,$$

where  $D_1, D_2$  are as defined at the beginning of Section 4, we have

$$J(x, r) \geq \eta_0.$$

We will also need Lemma 4.4 from [20], whose statement we recall below.

**Lemma 4.2** (Holes through slicing). *Let  $(u, K)$  be a pair in  $B(0, 1)$ . Assume there exists  $\varepsilon \in (0, 1/4)$  such that  $\beta(x, r) \leq \varepsilon$ . Let  $\nu_0$  and  $a_1, a_2$  be as defined at the beginning of Section 4. Then, for all unit vector  $\nu \in \mathbf{S}^{N-1}$  satisfying  $|\nu - \nu_0| \leq \varepsilon$ , we have*

$$J(\nu) \mathcal{H}^{N-1}(S_\nu)^2 \leq C\varepsilon^{-1} \left( \int_{B(0,1)} |\nabla u| dx \right)^{1/2},$$

where  $C \geq 1$  is a universal constant,  $S_\nu$  are the holes through slicing in the direction  $\nu$ ,

$$S_\nu = V \cap B(0, (1 - 4\varepsilon)) \setminus p_V(K \cap B(0, 1)) \quad \text{with } V = \nu^\perp,$$

and  $J(\nu)$  denotes the component of the jump in the direction  $\nu$ ,

$$J(\nu) := |(b_1 - b_2) \cdot \nu| + |(A_1 - A_2)\nu|.$$

**Proposition 4.3.** *There exists a constant  $\varepsilon_0 > 0$  (which depends on  $N, M, \mathbb{A}$ ) and for all  $p \in (2(N-1)/N, 2]$ , for all  $\varepsilon > 0$ , there exists  $\varepsilon_1 > 0$  (which depends on  $N, M, \mathbb{A}, p$  and  $\varepsilon$ ) such that the following holds. Let  $(u, K)$  be a topological quasiminimizer with any gauge  $h$  in a domain  $\Omega$ . For every  $x \in K$  and  $r > 0$  such that  $B(x, r) \subset \Omega$  and*

$$\beta(x, r) + \omega_p(x, r) \leq \varepsilon_1 \quad \text{and} \quad h(r) \leq \varepsilon_0,$$

*there exists a hyperplane  $V \in G(N-1, N)$  such that, for all  $W \in G(N-1, N)$  with  $\text{dist}(V, W) \leq \varepsilon_1$ ,*

$$\mathcal{H}^{N-1}(W \cap B(x, r) \setminus p_W(K \cap B(x, r))) \leq \varepsilon r^{N-1}. \quad (25)$$

*Proof.* To simplify the notations, we may assume that  $B(x, r) = B(0, 1)$ . Fix  $\varepsilon_0$  smaller than in Lemma 4.1. Let  $p \in (2(N-1)/N, 2]$ , let  $\varepsilon > 0$ , which we may assume to satisfy  $\varepsilon < 1/4$  without loss of generality. Let  $(u, K)$  be a topological quasiminimizer  $(u, K)$  in  $B(0, 1)$  such that

$$\beta(0, 1) \leq \varepsilon_2, \quad \omega_p(0, 1) \leq \varepsilon_1 \quad \text{and} \quad h(1) \leq \varepsilon_0, \quad (26)$$

where  $\varepsilon_1, \varepsilon_2 \in (0, 1/100)$  are small constant to be chosen later (depending on  $N, M, \mathbb{A}, p$  and  $\varepsilon$ ). We denote by  $\nu_0$  a unit normal to a hyperplane realizing the infimum in the definition of  $\beta(0, 1)$ , and by  $D_1, D_2$  the balls defined at the beginning of Section 4. The letter  $C$  denotes a generic constant  $\geq 1$  which depends on  $N, M, \mathbb{A}$  and  $p$ .

If the balls  $D_1$  and  $D_2$  lie in distinct component of  $\Omega \setminus K$ , we check (25) directly. Indeed, recall that  $D_i = B(x_i, 1/8)$ , where  $x_1 = (3/4)\nu_0$  and  $x_2 = -(3/4)\nu_0$ , and that

$$K \cap B(0, 1) \subset \{x \in B(0, 1) \mid |x \cdot \nu_0| \leq \varepsilon_2\}.$$

Assume that  $\varepsilon_2 \leq \varepsilon$ . Then for any  $\nu \in \mathbf{S}^{N-1}$  with  $|\nu - \nu_0| \leq \varepsilon$ , we have

$$K \cap B(0, 1) \subset \{x \in B(0, 1) \mid |x \cdot \nu| \leq 2\varepsilon\}$$

and moreover  $x_1 \cdot \nu > 2\varepsilon$  and  $x_2 \cdot \nu < -2\varepsilon$ . Letting  $W = \nu^\perp$ , we deduce that for every  $x \in W \cap B(0, (1-4\varepsilon))$ , the segment  $x + [-2\varepsilon, 2\varepsilon]\nu$  intersects  $K$ ; otherwise it could be used to connect the centers of  $D_1$  and  $D_2$ . It follows that the projection  $p_W(K \cap B(0, 1))$  contains  $W \cap B(0, 1-4\varepsilon)$ . In this case we obtain

$$\mathcal{H}^{N-1}(W \cap B(0, 1) \setminus p_W(K \cap B(0, 1))) \leq C\varepsilon.$$

We now turn to the case where  $D_1$  and  $D_2$  are not separated by  $K$ . Assume that  $\varepsilon_1, \varepsilon_2$  are twice smaller than the constant  $\eta_0$  in Lemma 4.1. In this situation, we may apply 4.1 which yields

$$J = |b_1 - b_2| + |A_1 - A_2| \geq C^{-1}.$$

Assuming in addition  $\varepsilon_2 \leq \varepsilon$ , Lemma 4.2 further shows that, for every unit vector  $\nu \in \mathbf{S}^{N-1}$  with  $|\nu - \nu_0| \leq \varepsilon$ ,

$$J(\nu)\mathcal{H}^{N-1}(S_\nu)^2 \leq C\varepsilon^{-1}\sqrt{\omega_1(0, 1)}. \quad (27)$$

From now on, the constant  $\varepsilon_2$  is fixed. The next step is to find a direction  $\tau_0 \in \mathbf{S}^{N-1}$  such that  $J(\nu)$  admits a uniform lower bound for all  $\nu$  in a neighborhood of  $\tau_0$ .

For  $\delta > 0$ , the set of vectors  $\nu \in \mathbf{S}^{N-1}$  such that

$$|\nu \cdot (b_1 - b_2)| < \delta|b_1 - b_2|$$

is the intersection of the unit sphere with a  $\delta$ -neighborhood of hyperplane, hence  $\mathcal{H}^{N-1}$  measure is at most  $C\delta$ . Similarly, for each vector  $e_k$  of the canonical basis, the set of  $\nu \in \mathbf{S}^{N-1}$  such that

$$|\nu \cdot (A_1^T - A_2^T)e_k| < \delta|(A_1^T - A_2^T)e_k|$$

has  $\mathcal{H}^{N-1}$  measure at most  $C\delta$ . As a consequence, we may fix  $\delta > 0$  sufficiently small (depending only on  $N$  and  $\varepsilon$ ) so that there exists a vector  $\tau_0 \in \mathbf{S}^{N-1}$  with  $|\nu_0 - \tau_0| \leq \varepsilon/2$  and

$$|\tau_0 \cdot (b_1 - b_2)| \geq \delta|b_1 - b_2|$$

as well as

$$|\tau_0 \cdot (A_1^T - A_2^T)e_k| \geq \delta|(A_1^T - A_2^T)e_k| \quad \forall k.$$

(We fix  $\delta$  for the remainder of the proof.) From this, we deduce that

$$J(\tau_0) \geq C_0^{-1}\delta(|b_1 - b_2| + |A_1 - A_2|),$$

for some universal constant  $C_0 \geq 1$ . Observe that the map

$$\nu \mapsto \frac{|(b_1 - b_2) \cdot \nu| + |(A_1 - A_2)\nu|}{|b_1 - b_2| + |A_1 - A_2|}$$

is Lipschitz with a universal constant. Hence, we may assume that  $\varepsilon_1 \in (0, \varepsilon/2)$  is sufficiently small (depending only on  $N, \varepsilon$ ) so that for every  $\nu \in \mathbf{S}^{N-1}$  with  $|\nu - \tau_0| \leq \varepsilon_1$ ,

$$J(\nu) \geq (2C_0)^{-1}\delta(|b_1 - b_2| + |A_1 - A_2|),$$

and in particular  $J(\nu) \geq C^{-1}\delta$ .

Now, (27) implies that for all  $\nu \in \mathbf{S}^{N-1}$  with  $|\nu - \tau_0| \leq \varepsilon_1$ ,

$$\mathcal{H}^{N-1}(S_\nu)^2 \leq C\delta^{-1}\varepsilon^{-1}\sqrt{\omega_1(0,1)}.$$

Letting  $W = \nu^\perp$ , we readily observe  $\mathcal{H}^{N-1}(W \cap B(0,1) \setminus B(0,(1-4\varepsilon))) \leq C\varepsilon$ , so that in fact

$$\mathcal{H}^{N-1}(W \cap B(0,1) \setminus p_V(K \cap B(0,1)))^2 \leq C\delta^{-1}\varepsilon^{-1}\sqrt{\omega_1(0,1)} + C\varepsilon^2.$$

As  $\omega_1(0,1) \leq \omega_p(0,1)$ , we can finally choose  $\varepsilon_1$  small enough in (26) so that

$$\mathcal{H}^{N-1}(W \cap B(0,1) \setminus p_V(K \cap B(0,1))) \leq C\varepsilon.$$

This completes the proof of Proposition 4.3.  $\square$

## 5 Plenty of big projections

The following two results are immediate consequences of Corollary 3.3 and Proposition 4.3, and their proof is omitted. The first one is the *uniform concentration* property; in this way, our argument provides a variant of the proof in [20] and leads to a stronger result, as it extends to quasiminimizers.

**Corollary 5.1** (Uniform concentration). *There exists a constant  $\varepsilon_0 > 0$  (which depends on  $N, M, \mathbb{A}$ ) and for all  $\varepsilon > 0$ , there exists  $C(\varepsilon) \geq 1$  (which depends on  $N, M, \mathbb{A}, \varepsilon$ ) such that the following holds. If  $(u, K)$  is a topological quasiminimizer in a domain  $\Omega$ , then for all  $x \in K$  and all  $r > 0$  with  $B(x, r) \subset \Omega$  and  $h(r) \leq \varepsilon_0$ , there exists  $y \in B(x, r/2)$  and  $t \in [C(\varepsilon)^{-1}r, r/2]$  such that*

$$\mathcal{H}^{N-1}(K \cap B(y, t)) \geq (1 - \varepsilon)\omega_{N-1}t^{N-1},$$

where  $\omega_{N-1}$  is the measure of  $N - 1$ -dimensional unit disk.

The second one is the *plenty of big projections* property, which we use to establish the uniform rectifiability of fractures.

**Corollary 5.2** (Plenty of big projections). *There exists constants  $\varepsilon_0 > 0$ ,  $\delta > 0$  and  $C \geq 1$  (which depend on  $N, M, \mathbb{A}$ ) such that the following holds. If  $(u, K)$  is a topological quasiminimizer in a domain  $\Omega$ , then for all  $x \in K$  and all  $r > 0$  with  $B(x, r) \subset \Omega$  and  $h(r) \leq \varepsilon_0$ , there exists a hyperplane  $V \in G(N-1, N)$  such that for all  $W \in G(N-1, N)$  with  $\text{dist}(V, W) \leq \delta$ ,*

$$\mathcal{H}^{N-1}(p_W(K \cap B(x, r))) \geq C^{-1}r^{N-1}$$

We finally deduce our main result.

**Corollary 5.3** (Uniform rectifiability). *There exists constants  $\varepsilon_0 > 0$  and  $C \geq 1$  (which depends on  $N, M, \mathbb{A}$ ) such that the following holds. If  $(u, K)$  is a topological quasiminimizer in a domain  $\Omega$ , then for all  $x \in K$  and all  $r > 0$  with  $B(x, 2r) \subset \Omega$  and  $h(2r) \leq \varepsilon_0$ ,*

*$K \cap B(x, r)$  is contained in a uniformly rectifiable set  $E$  (with BPLG) of constant  $C$ .*

*Proof.* This is essentially an application of [24, Theorem 1.6], which states that plenty of big projections (PBP) implies big pieces of Lipschitz graphs (BPLG). A minor adaptation is required since our PBP property (Corollary 5.2) is local, whereas [24, Theorem 1.6] is global. Fix  $x \in K$  and  $r > 0$  such that  $B(x, 2r) \subset \Omega$  and  $h(2r) \leq \varepsilon_0$ , where  $\varepsilon_0$  is smaller than in Proposition 2.3 and Corollary 5.2. Consider any hyperplane  $P$  passing through  $x$  and set

$$E = (K \cap B(x, r)) \cup \partial B(x, r) \cup (P \setminus B(x, r)).$$

The set  $E$  is closed and Ahlfors-regular (we omit the proof). Let us verify that  $E$  has plenty of big projections. The key point is that hyperplanes and spheres have plenty of big projections: there exists universal constants  $\delta > 0$  and  $C \geq 1$  such that for all  $y \in P$  and  $\rho > 0$ ,

$$\exists V \in G(N-1, N), \forall W \in B(V, \delta), \mathcal{H}^{N-1}(p_W(P \cap B(y, \rho))) \geq C^{-1}\rho^{N-1} \quad (28)$$

and for all  $y \in \partial B(x, r)$  and  $0 < \rho < 2r$ ,

$$\exists V \in G(N-1, N), \forall W \in B(V, \delta), \mathcal{H}^{N-1}(p_W(\partial B(x, r) \cap B(y, \rho))) \geq C^{-1}\rho^{N-1}. \quad (29)$$

Now, fix  $y \in E$  and  $\rho > 0$ . Our first case is when  $B(y, \rho/2) \subset B(x, r)$ . Then necessarily  $y \in K$ , and we may apply Corollary 5.2 in  $B(y, \rho/2)$ . Since  $K \cap B(y, \rho/2) \subset E \cap B(y, \rho)$ , it follows that  $E$  has plenty of big projections in  $B(y, \rho)$ .

Our second case is when  $B(y, 3\rho/4)$  contains a point  $z \in P \setminus B(x, r + \rho/100)$ . We apply (28) in  $B(z, \rho/100)$  and, because  $P \cap B(z, \rho/100) \subset E \cap B(y, \rho)$ , we again obtain that  $E$  has plenty of big projections in  $B(y, \rho)$ .

We are left with the case where  $B(y, \rho/2) \not\subset B(x, r)$  and  $B(y, 3\rho/4)$  does not meet  $V \setminus B(x, r + \rho/100)$ . From the second condition, we immediately obtain  $|x - y| < r + \rho/100$ . In particular  $B(y, \rho/2) \cap B(x, r) \neq \emptyset$ , and since  $B(y, \rho/2) \not\subset B(x, r)$ , there exists a point  $z \in B(y, \rho/2) \cap \partial B(x, r)$ . We also check that  $\rho < 4r$ : if this were not the case, then and using again  $|x - y| < r + \rho/100$ , we would have

$$\overline{B}(x, r + \rho/100) \subset B(y, 2r + \rho/50) \subset B(y, 3\rho/4)$$

and therefore any point of  $V \cap \partial B(x, r + \rho/100)$  would belong to both  $B(y, 3\rho/4)$  and  $V \setminus B(x, r + \rho/100)$ . Since  $\rho/2 < 2r$ , we may apply (29) in  $B(z, \rho/2)$ . As  $\partial B(x, r) \cap B(z, \rho/2) \subset E \cap B(y, \rho)$ , this shows once more that  $E$  has plenty of big projections in  $B(y, \rho)$   $\square$

# Appendices

## A An auxiliary lemma on affine maps

**Lemma A.1.** *For real number  $p \geq 1$ , for all affine map  $a : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , for all ball  $B \subset \mathbb{R}^n$  and for all Borel set  $E \subset B$  such that  $|E| > 0$ , we have*

$$\left( \int_E |a(x)|^p dx \right)^{1/p} \geq C^{-1} \|a\|_{L^\infty(B)} \left( \frac{|E|}{|B|} \right), \quad (30)$$

where  $C \geq 1$  depends on  $n, m, p$ .

*Proof.* By scale invariance of the inequality, we may assume that  $B = B(0, 1)$ . Writing  $a_1, \dots, a_m$  for the coordinates of  $a$ , we have for all  $x \in \mathbb{R}^n$ ,

$$C^{-1} \sum_{i=1}^m |a_i(x)|^p \leq |a(x)|^p \leq C \sum_{i=1}^m |a_i(x)|^p$$

so it suffices to prove (30) for each coordinate of  $a$ . Thus we may restrict to the case where  $a$  is an affine map  $a : \mathbb{R}^n \rightarrow \mathbb{R}$  of the form  $a(x) = c + v \cdot x$ , where  $c, v \in \mathbb{R}^n$ . We can also assume  $(c, v) \neq 0$ , otherwise (30) is trivial.

The main point is to show that there exists a universal constant  $C_0 \geq 1$  such that for all  $t > 0$ ,

$$|\{x \in B \mid |a(x)| \leq t\}| \leq \frac{C_0 t}{|c| + |v|}. \quad (31)$$

If  $|c| \geq 2|v|$ , then  $|a(x)| \geq |c|/2$  in  $B$ , and we distinguish two cases. Either  $|c|/2 > t$ , in which case

$$|\{x \in B \mid |a(x)| \leq t\}| = 0$$

or  $|c|/2 \leq t$ , and then we have the trivial bound

$$|\{x \in B \mid |a(x)| \leq t\}| \leq |B| \leq \frac{2|B|t}{|c|} \leq \frac{3|B|t}{|c| + |v|}.$$

Next, if  $|c| < 2|v|$ , then in particular  $v \neq 0$ , and the condition  $|a(x)| \leq t$  describes a  $(|v|^{-1}t)$ -neighborhood of a hyperplane, so

$$|\{x \in B \mid |a(x)| \leq t\}| \leq \frac{Ct}{|v|} \leq \frac{3Ct}{|c| + |v|},$$

for some universal constant  $C \geq 1$ . This proves (31).

For  $t > 0$ , we then estimate

$$\begin{aligned} \int_E |a(x)|^p dx &\geq t^p |\{x \in E \mid |a(x)| > t\}| \\ &\geq t^p \left( |E| - |\{x \in B \mid |a(x)| \leq t\}| \right) \\ &\geq t^p \left( |E| - \frac{C_0 t}{|c| + |v|} \right) \end{aligned}$$

and the desired inequality follows by choosing  $t$  so that

$$|E| - \frac{C_0 t}{|c| + |v|} = \frac{|E|}{2}.$$

□

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