

GROMOV-HAUSDORFF AND INTRINSIC FLAT CONVERGENCE OF $\text{RCD}(K, N)$ AND KATO SPACES

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ABSTRACT. We consider metric measure spaces (X, d, \mathcal{H}^N) satisfying the curvature-free properties (ETR), (LDB), and with an almost everywhere connected regular set. In particular, these assumptions are satisfied by non-collapsed $\text{RCD}(K, N)$ spaces without boundary, as well as by non-collapsed strong Kato limit spaces without boundary. For both classes, we study orientability in the sense of metric currents, establish stability of orientation under pointed Gromov–Hausdorff convergence, and show that the pointed Gromov–Hausdorff limit coincides with the local flat limit.

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1. INTRODUCTION

Orientability of Ricci limit spaces was first investigated by Honda [31], who studied it via the existence of a top-dimensional differential form with Sobolev regularity. More recently, Brena–Brué–Pigati [8] carried out a comprehensive study of orientability for non-collapsed $\text{RCD}(K, N)$ spaces without boundary [21], providing several equivalent characterizations. These include: (i) Honda’s formulation in terms of Sobolev top-degree differential forms; (ii) topological orientability of the topological manifold part, whose complement is known to have Hausdorff codimension 2, [16, 34];

A. Mondino was funded by the European Research Council (ERC), under the European Union Horizon 2020 research and innovation programme, via the ERC Starting Grant “CURVATURE”, grant agreement No. 802689.

R. Perales was funded by the Austrian Science Fund (FWF) [Grant DOI: 10.55776/EFP6].

and (iii) the existence of a non-zero metric current with bounded density and zero boundary. They also proved stability of orientability within the non-collapsed RCD class, as well as stability of non-orientability for Ricci limit spaces. Their arguments primarily rely on the second characterization, namely topological orientability of the manifold part. As a notable application, they showed that any open 4-manifold with Euclidean volume growth and non-negative Ricci curvature must be orientable. Let us also mention [33] for a study of orientability in the setting of Alexandrov spaces.

In this paper, we consider metric measure spaces $(X, \mathbf{d}, \mathcal{H}^N)$ belonging to one of the following two classes:

- (1) non-collapsed $\text{RCD}(K, N)$ spaces without boundary;
- (2) non-collapsed strong Kato limit spaces without boundary.

For both classes, we study orientability in the sense of metric currents, establish stability of orientability under pointed Gromov–Hausdorff convergence, and deduce that the pointed Gromov–Hausdorff and local flat limits coincide. The last assertion has been established by Sormani–Wenger [46] in the framework of compact non-collapsed limits of smooth manifolds with no boundary and nonnegative Ricci curvature, and then extended to any uniform lower bound on the Ricci curvature by Matveev–Portegies [40].

In comparison with the earlier works [46, 40, 31, 8], our results extend the analysis from pointwise lower Ricci curvature bounds to lower Ricci bounds in the Kato sense (see below for precise definitions). Moreover, we provide more direct proofs than [31, 8] tailored to the metric current approach, avoiding the need to pass through alternative equivalent characterizations of orientability.

Metric currents were first introduced in the pioneering work of Ambrosio–Kirchheim [5] and further studied by Lang [35] and Lang–Wenger [36]. These are generalizations of currents in Euclidean space and smooth manifolds (see, e.g., [23]).

The starting point for this paper is the observation that, given an N -dimensional orientable Riemannian manifold (M, g) endowed with a fixed orientation, one can canonically associate an N -dimensional integral current T on M whose mass measure $\|T\|$ coincides with the Riemannian volume measure $\text{dvol}_g = \mathcal{H}^N$. Moreover, thanks to Stokes’ theorem, $\partial M = 0$ if and only if $\partial T = 0$. The following definition is motivated by this observation; we refer the reader to Section 2.2 for the notation and terminology.

Definition 1.1. Let $(X, \mathbf{d}, \mathcal{H}^N)$ be a complete and separable metric measure space. We say that $(X, \mathbf{d}, \mathcal{H}^N)$ is *orientable and with no boundary in the sense of currents* if there is a locally integral current $T \in \mathbf{I}_{\text{loc}, N}(X)$ such that $X = \text{set}(T)$, $\|T\| = \mathcal{H}^N$ and $\partial T = 0$.

The next proposition establishes a useful rigidity result for locally integral currents on non-collapsed RCD spaces with no boundary.

Proposition 1.2. *Let $(X, \mathbf{d}, \mathcal{H}^N)$ be an $\text{RCD}(K, N)$ space without boundary for some $K \in \mathbb{R}$ and $N \in \mathbb{N}$, that is orientable in the sense of currents by $T \in \mathbf{I}_{\text{loc}, N}(X)$, $\partial T = 0$. Then, for any $S \in \mathbf{I}_{\text{loc}, N}(X)$, $x \in X$ and $s > 0$ such that $\|\partial S\|(B_s(x)) = 0$, there exists $k \in \mathbb{Z}$ such that $S \llcorner B_s(x) = kT \llcorner B_s(x)$.*

From the previous proposition we deduce uniqueness of orientability in the sense of currents as in Definition 1.1, up to a sign.

Corollary 1.3. *Let $(X, \mathbf{d}, \mathcal{H}^N)$ be an $\text{RCD}(K, N)$ space without boundary, for some $K \in \mathbb{R}$ and $N \in \mathbb{N}$, that is orientable in the sense of currents by $T \in \mathbf{I}_{\text{loc}, N}(X)$, $\partial T = 0$. Then the only two integral currents without boundary orienting $(X, \mathbf{d}, \mathcal{H}^N)$ are either T or $-T$.*

For simplicity of presentation, we decided to state the two result above for RCD spaces. However, in the paper we will prove Proposition 1.2 and Corollary 1.3 in the more general setting of metric measure spaces satisfying the structural – but curvature free – assumption (ETR) (for *essential topological regularity*), see Definition 3.5. Roughly, a metric measure space $(X, \mathbf{d}, \mathcal{H}^N)$ satisfies (ETR) if it contains a set \mathcal{R} of full measure, such that each point $x \in \mathcal{R}$ has a neighbourhood which is rectifiable, homeomorphic to \mathbb{R}^n , and with almost Euclidean volume. In Proposition 3.9, we will prove that that (ETR) holds for

- Locally non-collapsed RCD spaces without boundary, i.e. $(X, \mathbf{d}, \mathcal{H}^N)$ with the following property: for every $x \in X$ there exists $K_x \in \mathbb{R}$ and a closed neighbourhood \bar{U}_x of x such that $(\bar{U}_x, \mathbf{d}|_{\bar{U}_x}, \mathcal{H}^N)$ is an $\text{RCD}(K_x, N)$ space without boundary. In particular, non-collapsed RCD spaces with no boundary.
- Non-collapsed strong Kato limit spaces, in the sense of Carron-Mondello-Tewodrose, see below.

We briefly describe the notion of a Kato limit. Non-collapsed strong Kato limit spaces, as well as variants under weaker assumptions, have been thoroughly studied by Carron–Mondello–Tewodrose [12, 13, 14]. In what follows all the manifolds are assumed to have empty boundary. For a complete Riemannian manifold (M, g) of dimension $N \geq 2$, define

$$k_t(M, g) = \sup_{x \in M} \int_0^t \int_M H(s, x, y) \text{Ric}_-(y) \text{dvol}_g(y) \, ds,$$

where H is the heat kernel of M and $\text{Ric}_- : M \rightarrow \mathbb{R}_+$ is the negative part of the smallest eigenvalue of the Ricci tensor. Following [13], we say that a family Λ of complete N -dimensional Riemannian manifolds satisfies a *strong Kato bound* if there exists $t_0 > 0$ and a non-decreasing function $f : (0, t_0] \rightarrow \mathbb{R}_+$ with

$$\lim_{t \rightarrow 0} \int_0^{t_0} \frac{f(t)}{t} \, dt < \infty \quad \text{and} \quad f(t_0) \leq \frac{1}{N-2}, \quad (1.1)$$

such that each $(M, g) \in \Lambda$ satisfies

$$k_t(M, g) \leq f(t), \quad \forall t \in (0, t_0]. \quad (1.2)$$

Kato bounds are implied by uniform lower bounds on the Ricci curvature or by suitable uniform L^p estimates on Ric_- .

Definition 1.4 (Carron–Mondello–Tewodrose). A metric measure space $(X, \mathbf{d}, \mathcal{H}^N)$ is a *non-collapsed strong Kato limit space* if there exists a sequence $\Lambda = \{(M_i, g_i)\}_{i \in \mathbb{N}}$ of complete N -dimensional Riemannian manifolds satisfying

- a strong Kato bound (in the aforementioned sense);
- volume non-collapsing, i.e, there exists $v > 0$ such that $\text{vol}_{g_i}(B_{\sqrt{t_0}}(x_i)) > v$, where $x_i \in M_i$ is a marked point;

such that (M_i, g_i, x_i) converges to (X, \mathbf{d}, x) in the pointed Gromov-Hausdorff sense.

Next, we compare the pGH limits and local flat limits of sequences of RCD spaces and strong Kato limits oriented in the sense of currents. For details about the local flat topology see Definition 2.20 and Theorem 2.21.

Theorem 1.5. *Let $(X_i, \mathbf{d}_i, \mathcal{H}^N, x_i)$ be a sequence of $\text{RCD}(K, N)$ spaces, for some $K \in \mathbb{R}$ and $N \in \mathbb{N}$, converging in the pGH topology to (X, \mathbf{d}, x) . Assume that each X_i has no boundary and is oriented by the N -dimensional local integral current $T_i \in \mathbf{I}_{\text{loc}, N}(X_i)$, $\partial T_i = 0$. Then, either one of the following holds:*

- *In the collapsing case, where $\lim_{i \rightarrow \infty} \mathcal{H}^N(X_i) = 0$, the sequence T_i converges in the local flat topology to the zero current $T = 0 \in \mathbf{I}_{\text{loc}, N}(X)$.*
- *In the non-collapsing case, where $\lim_{i \rightarrow \infty} \mathcal{H}^N(X_i) > 0$, it holds that the currents T_i subconverge in the local flat topology to a non zero current $T \in \mathbf{I}_{\text{loc}, N}(X)$ with $\partial T = 0$, $\|T\| = \mathcal{H}^N$, $X = \text{set}(T)$. In particular, $(X, \mathbf{d}, \mathcal{H}^N)$ is orientable and has no boundary in the sense of currents.*

An analogous result holds for non-collapsed strong Kato limits:

Theorem 1.6. *Let (M_i, g_i, x_i) be a non-collapsing sequence of complete, N -dimensional oriented Riemannian manifolds satisfying a strong Kato bound, converging in the pGH topology to (X, \mathbf{d}, x) . Then $(X, \mathbf{d}, \mathcal{H}^N)$ is orientable and has no boundary in the sense of currents.*

More precisely, denoting by $T_i \in \mathbf{I}_{\text{loc}, N}(M_i)$, $\partial T_i = 0$, $\|T_i\| = \text{dvol}_{g_i}$, a local integral current orienting M_i , it holds that T_i subconverge in the local flat topology to a non zero current $T \in \mathbf{I}_{\text{loc}, N}(X)$, with $\partial T = 0$, $\|T\| = \mathcal{H}^N$, and $X = \text{set}(T)$.

Remark 1.7. The statements in Theorems 1.5 and 1.6, asserting that non-collapsed limits of spaces without boundary still have no boundary (in the sense of currents), should be compared with the analogous property — proved by Cheeger–Colding [16] and later extended to non-collapsed RCD spaces [34, Thm. 5.1] — that non-collapsed Ricci limits of manifolds without boundary also have no boundary (in the sense of tangent cones).

Remark 1.8. Under the hypotheses of Theorem 1.5 (resp. Theorem 1.6), if additionally the metric spaces (X_i, \mathbf{d}_i) (resp. (M_i, g_i)) have uniformly bounded diameters, then it is not necessary to fix points x_i , and pGH convergence can be replaced with GH convergence. Furthermore, following the terminology of integral current spaces introduced by Sormani and Wenger [47], one can add that

- In the collapsing case, (X, \mathbf{d}, T) is the zero N -dimensional integral current space.
- In the non-collapsing case, (X, \mathbf{d}, T) is an N -dimensional integral current space. Moreover, the Gromov-Hausdorff and intrinsic flat limits agree.

Concerning the second statement, recall that the question of the equivalence $\text{GH}=\text{IF}$ was originally motivated by Wenger’s compactness theorem [52] and by the introduction of intrinsic flat (IF) convergence by Sormani–Wenger [47]. This problem has since been addressed in several works under a variety of assumptions; see, for instance, [46, 40, 37, 43, 44, 42].

Organization of the paper and ideas of the proofs. For the proofs, we have been inspired by the approach that Matveev–Portegies [40] used for proving $\text{GH}=\text{IF}$ in the setting of compact, non-collapsed Ricci limit spaces. However, the more

general framework considered in the present paper requires several new ideas; below we briefly summarize these together with the main steps of the proofs.

Rectifiable currents admit two representations. On the one hand, they can be written as sums of pushforwards of currents defined on Euclidean spaces. On the other hand, in close analogy with the classical setting, they can be represented as integrals over a rectifiable set involving the weight function of T and the pairing between an orientation and test forms. The latter representation requires the ambient space to be a w^* -separable dual space and was used extensively in [40]. In contrast, throughout this paper we work exclusively with the representation of rectifiable currents as sums of pushforward currents. We refer to Remark 4.5 for further discussion.

A key tool in our work is the use of splitting maps around regular points, as developed by Brué–Naber–Semola [10] (building on [16, 19]). In Section 3.1 we establish the following result, which is of independent interest and may be useful in other contexts: if $u : B_1(x) \rightarrow \mathbb{R}^N$ is a δ -splitting map, then $B_1(x)$ can be covered, up to a set of measure zero, by a countable family of disjoint Borel sets U_n , such that each restriction $u|_{U_n} : U_n \rightarrow \mathbb{R}^N$ is biLipschitz onto its image; see Proposition 3.4.

We conclude Section 3 by abstracting the properties required for the proofs of Proposition 1.2 and Corollary 1.3 to hold, and by showing that these assumptions are satisfied by locally non-collapsed RCD spaces and by non-collapsed strong Kato limits. The motivation for this is to make our results applicable to a wider range of potential settings.

In the first part of Section 4 we present the proofs of Proposition 1.2 and Corollary 1.3. The argument proceeds as follows. Using the atlas given by the restrictions $u|_{U_n \cap \text{set}(T)} : U_n \rightarrow \mathbb{R}^N$, we write the orientations as sums of pushforwards of currents in Euclidean spaces. Relying on the properties of u encoded in (ETR), together with a local constancy theorem for currents in Euclidean spaces, Corollary 2.5, we obtain a local constancy theorem for currents in metric spaces around regular points. As a consequence, assuming a.e.-connectedness of the regular set (see Definition 3.6) and (LDB), we deduce that the weight functions of the currents T and S appearing in Proposition 1.2 are locally constant. To show that the local multiplicity function of T takes values only in $\{1, -1\}$ we use the fact that the weight of T coincides, up to sign, with the weight of its 0-dimensional slices under u , see (2.20).

In the second part of Section 4 we complete the proofs of Theorem 1.5 and Theorem 1.6. We begin with the proof of Theorem 1.5 in the compact case; see Remark 1.8. By Sormani–Wenger’s compactness theorem [47], we may assume that both the Gromov–Hausdorff and intrinsic flat convergences are realized within a single metric space, and that the intrinsic flat limit is a subset of the Gromov–Hausdorff limit. Our first step is to show that the multiplicity function θ_T is constant and equal to either 1 or -1 in neighborhoods of regular points of X . This is achieved by combining the results of Matveev–Portegies [40], which relate the multiplicity of a current T to the degrees of its Lipschitz charts with respect to T , together with an L^1 estimate for these degrees. Using the properties of δ -splitting maps encoded in (ETR), we deduce that the multiplicity functions θ_{T_i} subconverge to θ_T in neighborhoods of regular points of X . Once this is established, we argue as in the proof of Proposition 1.2. Using the relation between the mass of T and the integral of the masses of its slices, we conclude that the regular set of X is contained in $\text{set}(T)$ and that $\|T\| = \mathcal{H}^N$. Bishop–Gromov volume monotonicity will imply that $X = \text{set}(T)$. In the non-compact case, we apply Lang–Wenger’s convergence

theorem [36] to obtain a limit current $T \in \mathbf{I}_{\text{loc}, N}(W)$ in a complete metric space W . By restricting the sequence of currents to bounded regions, we reduce the argument to the compact case.

The proof of Theorem 1.6 follows along the same lines as Theorem 1.5, using the structural properties for strong Kato limits established in [13]. We emphasize, however, that the results in the strong Kato class *do not* follow directly from the statements in the framework of non-collapsed RCD spaces; indeed, by [13], strong Kato limits are biLipshitz equivalent to *collapsed* RCD spaces.

Acknowledgements: The authors gratefully acknowledge support from the Simons Center for Geometry and Physics where they had the opportunity to meet as part of the Program *Geometry and Convergence in Mathematical General Relativity* in September 2025. The authors wish to thank J. Portegies and D. Semola for helpful discussions.

2. PRELIMINARIES

2.1. RCD spaces. Throughout the paper, (X, \mathbf{d}) is a complete and separable metric space, endowed with a Borel non-negative measure \mathbf{m} . The triple $(X, \mathbf{d}, \mathbf{m})$ is called a *metric measure space*. When a reference point \bar{x} is fixed, the tuple $(X, \mathbf{d}, \mathbf{m}, \bar{x})$ is called a *pointed metric measure space*.

On metric measure spaces $(X, \mathbf{d}, \mathbf{m})$, Sturm [48, 49] and Lott-Villani [39] introduced the $\text{CD}(K, N)$ condition, encoding that the Ricci curvature is bounded below by $K \in \mathbb{R}$ and the dimension is bounded above by $N \in [1, \infty]$ in a synthetic sense, via optimal transport. In order to isolate the Riemannian-like structures from the Finslerian ones, Ambrosio-Gigli-Savaré [3] and Gigli [25] (see also [2]), reinforced the Lott-Sturm-Villani CD condition by asking the linearity of the heat flow or, equivalently, that the Sobolev space $W^{1,2}(X, \mathbf{d}, \mathbf{m})$ is a Hilbert space. The resulting "Riemannian-like" curvature-dimension condition is known as $\text{RCD}(K, N)$. The $\text{RCD}(K, N)$ condition can be equivalently characterized in terms of the Bochner inequality [4, 22, 7] and it satisfies a local-to-global property [15] (see also [38]). Inspired by Cheeger-Colding theory of Ricci-limit spaces [16], an $\text{RCD}(K, N)$ is said [21] to be *non-collapsed* if $\mathbf{m} = \mathcal{H}^N$, the N -dimensional Hausdorff measure (see also [32, 9]). In this case, N must be an integer. For more about CD and RCD spaces we refer to the surveys [51, 1, 50, 27].

2.1.1. Singular strata and boundaries. We say that a pointed metric space (Y, \mathbf{d}_Y, y) is *tangent* to (X, \mathbf{d}) at $x \in X$ if there exists a sequence $r_i \downarrow 0$ such that

$$(X, r_i^{-1}\mathbf{d}, x) \rightarrow (Y, \mathbf{d}_Y, y)$$

in the pGH-topology. The collection of all the tangent spaces of (X, \mathbf{d}) at x is denoted by $\text{Tan}_x(X, \mathbf{d})$. A compactness argument originally due to Gromov [29], yields that $\text{Tan}_x(X, \mathbf{d})$ is non-empty for every $x \in X$, if $(X, \mathbf{d}, \mathcal{H}^N)$ is a non-collapsed $\text{RCD}(K, N)$ space. The set of *regular* points is defined as

$$\mathcal{R} := \left\{ x \in X : \text{Tan}_x(X, \mathbf{d}) = \{(\mathbb{R}^N, \mathbf{d}_{\text{eucl}})\} \right\} \quad (2.1)$$

and, for any $0 \leq k \leq N - 1$,

$$\mathcal{S}^k := \left\{ x \in X : \text{no tangent space at } x \text{ splits off } \mathbb{R}^{k+1} \right\}. \quad (2.2)$$

The following *stratification of the singular set* $\mathcal{S} := X \setminus \mathcal{R}$ holds:

$$\mathcal{S}^0 \subset \mathcal{S}^1 \subset \dots \subset \mathcal{S}^{N-1} = \mathcal{S}. \quad (2.3)$$

A quantitative analysis of the regular and singular strata leads to the definition of a (k, δ) -symmetric ball and a δ -splitting map [20]. A ball $B_s(x) \subset X$ is said to be (k, δ) -symmetric provided

$$d_{GH}(B_s(x), B_s(z)) \leq \delta s,$$

where $z \in Z \times \mathbb{R}^k$ is a tip of the metric cone $C(Z) \times \mathbb{R}^k$, for some metric space (Z, d_Z) . The notion of δ -splitting map recalled below has been extremely powerful in developing a structure theory of Ricci limits [16, 17, 18] and RCD(K, N) spaces [41, 11].

Definition 2.1. Let $N \in [1, \infty)$ and $k \in \mathbb{N}, k \leq N$. Let (X, d, m) be an RCD($-(N-1), N$) space, $x \in X$ and $\delta > 0$ be fixed. We say that $u = (u_1, \dots, u_k) : B_r(x) \rightarrow \mathbb{R}^k$ is a δ -splitting map provided

- (i) u_i is harmonic and $|\nabla u_i| < C(N)$, for all $i = 1, \dots, k$;
- (ii) $r^2 \int_{B_r(x)} |\text{Hess } u_i|^2 dm < \delta$, for all $i = 1, \dots, k$;
- (iii) $\int_{B_r(x)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}| dm < \delta$, for all $i, j = 1, \dots, k$.

For the proof of the next theorem, we refer to [10, Theorem 3.8]; see also [11] for similar results.

Theorem 2.2 (δ -splitting vs ε -GH isometry). *Let $1 \leq N < \infty$ be fixed.*

- (i) *For every $0 < \delta < 1/2$ and $\varepsilon \leq \varepsilon(N, \delta)$, the following holds. If (X, d, m) is an RCD($-\varepsilon(N-1), N$) space satisfying*

$$d_{mGH}(B_2(x), B_2^{\mathbb{R}^k \times Z}(0, z)) \leq \varepsilon \quad (2.4)$$

for some integer k , some $x \in X$ and some pointed metric space (Z, d_Z, z) , then there exists a δ -splitting map

$$u = (u_1, \dots, u_k) : B_1(x) \rightarrow \mathbb{R}^k.$$

- (ii) *For every $\varepsilon > 0$ and $\delta < \delta(N, \varepsilon)$ the following holds. If (X, d, m) is a normalized RCD($-\delta(N-1), N$) space and there exists a δ -splitting map $u : B_6(x) \rightarrow \mathbb{R}^k$ for a given $x \in X$, then*

$$d_{GH}(B_{1/k}(x), B_{1/k}^{\mathbb{R}^k \times Z}(0, z)) < \varepsilon \quad (2.5)$$

for some pointed metric space (Z, d_Z, z) . Moreover, there exists $f : B_1(x) \rightarrow Z$ such that

$$(u - u(x), f) : B_{1/k}(x) \rightarrow B_{1/k}^{\mathbb{R}^k \times Z}(0, z) \quad \text{is an } \varepsilon\text{-GH isometry.} \quad (2.6)$$

- (iii) *If we additionally assume that (X, d, \mathcal{H}^N) is a RCD($-\delta(N-1), N$) non-collapsed space with $\mathcal{H}^N(B_1(x)) > v > 0$, $k = N-1$, and $\delta < \delta(N, v, \varepsilon)$, then (Z, d_Z, z) in (ii) can be chosen to be the ball of a one dimensional Riemannian manifold, possibly with boundary.*

Two different notions of boundary for a non-collapsed RCD(K, N) space (X, d, \mathcal{H}^N) have been proposed [21, 34]; however, it follows from [10, Theorem 6.6.(i)] that the fact of X having empty boundary is equivalent in the two formulations. Throughout

this note, we will assume that X has empty boundary, i.e., $\mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2} = \emptyset$. By [21], this is equivalent to requiring that

$$\text{Codim}_{\text{Haus}}(X \setminus \mathcal{R}) > 1, \quad (2.7)$$

where $\text{Codim}_{\text{Haus}}$ denotes the Hausdorff codimension.

The following theorem gives topological regularity for a neighbourhood of a regular point and useful regularity properties of δ -splitting maps. In the RCD setting, the former was proved in [34] employing Cheeger-Colding's Reifenberg theorem for metric spaces [16], while the latter was proved in [10]. Both were previously established for Ricci limits, respectively by Cheeger-Colding [16] and by Cheeger-Jiang-Naber [19].

Theorem 2.3. *Let $N \in \mathbb{N}$, $N \geq 1$. For each $\varepsilon \in (0, 1/5)$ there exists $\delta = \delta(\varepsilon, N) > 0$ such that for any $\text{RCD}(-\delta(N-1), N)$ space $(X, \mathbf{d}, \mathcal{H}^N)$ and for any (N, δ) -symmetric ball $B_{16}(p) \subset X$, the following holds:*

- *The ball $B_8(p)$ is homeomorphic to a smooth N -dimensional manifold without boundary.*
- *There exists a map $u : B_8(p) \rightarrow \mathbb{R}^N$ verifying the following properties:*
 - i) *$u : B_8(p) \rightarrow \mathbb{R}^N$ is an ε -splitting map;*
 - ii) *there exists a closed set $U \subset B_1(p)$ such that*

$$\mathcal{H}^N(B_1(p) \setminus U) \leq \varepsilon, \quad (2.8)$$

and, for all $x, y \in U$:

$$(1 - \varepsilon)\mathbf{d}(x, y) \leq |u(x) - u(y)| \leq (1 + \varepsilon)\mathbf{d}(x, y); \quad (2.9)$$

- iii) *for all $x, y \in B_1(p)$ it holds that*

$$(1 - \varepsilon)\mathbf{d}(x, y)^{1+\varepsilon} \leq |u(x) - u(y)| \leq (1 + \varepsilon)\mathbf{d}(x, y); \quad (2.10)$$

- iv) *$u(B_1(p)) \supset B_{1-2\varepsilon}^{\mathbb{R}^N}(0)$;*

- v) *the following volume estimate holds:*

$$(1 - \varepsilon)\omega_N \leq \mathcal{H}^N(B_1(p)) \leq (1 + \varepsilon)\omega_N. \quad (2.11)$$

2.2. Ambrosio-Kirchheim's metric currents. Currents with finite mass in complete metric spaces were introduced by Ambrosio-Kirchheim [5]. Here we review, in a non-extensive way, some of their results that we will need.

2.2.1. Definition and basic properties. Integral currents with finite mass in complete metric spaces were introduced by Ambrosio-Kirchheim [5]. The theory was extended to currents with non-finite mass and in non necessarily complete spaces by Lang [35] and Lang-Wenger [36]. Here we review, in a non-extensive way, some of their results.

Let (X, \mathbf{d}) be a complete metric space. We denote by $\text{Lip}(X)$ the set of Lipschitz functions on X and by $\text{Lip}_b(X)$ the bounded ones. For $n \geq 0$ an integer, let

$$\mathcal{D}^n(X) = \text{Lip}_b(X) \times [\text{Lip}(X)]^n.$$

Elements in $\mathcal{D}^n(X)$ will usually be denoted as (f, π) , where the first entry is an element in $\text{Lip}_b(X)$ and $\pi = (\pi_1, \dots, \pi_n) \in [\text{Lip}(X)]^n$. Sometimes (f, π) will be denoted as $f d\pi$ (to resemble an n -differential form).

An n -dimensional current T on X is a multilinear map $T : \mathcal{D}^n(X) \rightarrow \mathbb{R}$ that satisfies:

- T is continuous: for any sequence $(f, \pi^i) \in \mathcal{D}^n(X)$ such that π^i converge pointwise to $\pi \in [\text{Lip}(X)]^n$ and such that $\sup_{i,j} \text{Lip}(\pi_i^j) < \infty$, it holds

$$\lim_{i \rightarrow \infty} T(f, \pi^i) = T(f, \pi);$$

- T is a local map: $T(f, \pi) = 0$, whenever there exists j such that π_j is constant on a neighborhood of $\{f \neq 0\}$;
- T is bounded: there exists a finite Borel measure μ on X such that

$$|T(f, \pi)| \leq \text{Lip}(\pi_1) \cdots \text{Lip}(\pi_n) \int_X |f| \, d\mu, \quad (2.12)$$

for all $(f, \pi) \in \mathcal{D}^n(X)$.

The vector space of all n -dimensional currents in X is denoted by $\mathbf{M}_n(X)$.

The smallest measure μ that satisfies (2.12) is denoted by $\|T\|$. It is called the mass measure of T . The mass and the support of T are defined as $\mathbb{M}(T) = \|T\|(X)$ and $\text{spt}(T) = \text{spt}(\|T\|)$.

When (X, \mathbf{d}) is the standard Euclidean space of dimension n , a prime example of an n -dimensional current is

$$[[\theta]](f, \pi) = \int_{\mathbb{R}^n} \theta(x) f(x) \det(D_x \pi) \, d\mathcal{L}^n(x),$$

where $\theta \in L^1(\mathbb{R}^n, \mathbb{R})$ and $D_x \pi$ denotes the derivative of π at x , which exists \mathcal{L}^n -a.e. by Rademacher's theorem.

Given another complete metric space Y and a Lipschitz function $\Psi : X \rightarrow Y$, the *push-forward* of T under Ψ is the current $\Psi_{\#} T : \mathcal{D}^n(Y) \rightarrow \mathbb{R}$ defined as

$$\Psi_{\#} T(f, \pi) = T(f \circ \Psi, \pi \circ \Psi), \quad \forall (f, \pi) \in \mathcal{D}^n(Y).$$

It holds that

$$\|\Psi_{\#} T\| \leq \text{Lip}(\Psi)^n \Psi_{\#} \|T\|. \quad (2.13)$$

For $n \geq 1$, the boundary of T is the multilinear functional

$$\partial T : \text{Lip}_b(X) \times [\text{Lip}(X)]^{n-1} \rightarrow \mathbb{R},$$

defined as

$$\partial T(f, \pi_1, \dots, \pi_{n-1}) = T(1, f, \pi_1, \dots, \pi_{n-1}),$$

where $1 : X \rightarrow \mathbb{R}$ denotes the constant function equal to 1. We say that $T \in \mathbf{M}_n(X)$ is a *normal* current if $\partial T \in \mathbf{M}_{n-1}(X)$. The vector space of all normal n -currents on X is denoted by $\mathbf{N}_n(X)$. For $n = 0$, we set $\mathbf{N}_0(X) = \mathbf{M}_0(X)$.

For any integer $k \leq n$ and $(g, \tau) \in \mathcal{D}^k(X)$, the *restriction* of T to (g, τ) is the current

$$T \llcorner (g, \tau) : \mathcal{D}^{n-k}(X) \rightarrow \mathbb{R}$$

given by $T \llcorner (g, \tau)(f, \pi) = T(fg, \tau, \pi)$. It holds that

$$\|T \llcorner (g, \tau_1, \dots, \tau_k)\| \leq \sup |g| \text{Lip}(\tau_1) \cdots \text{Lip}(\tau_k) \|T\|. \quad (2.14)$$

Currents $T \in \mathbf{M}_n(X)$ can be naturally extended to $\mathcal{B}^\infty(X) \times [\text{Lip}(X)]^n$, where $\mathcal{B}^\infty(X)$ denotes the class of bounded Borel functions on X . This extension is denoted in the same way and satisfies natural product and chain rules for derivatives, as well as continuity, locality and boundedness properties, see [5, Theorem 3.5]. In particular, for a Borel set $A \subset X$, the restriction of T to A , is the current $T \llcorner A : \mathcal{D}^n(X) \rightarrow \mathbb{R}$ given by

$$T \llcorner A(f, \pi) = T(1_A f, \pi), \quad \forall (f, \pi) \in \mathcal{D}^n(X),$$

where $1_A : X \rightarrow \mathbb{R}$ denotes the indicator function of A . In this case $\|T \llcorner A\| = \|T\| \llcorner A$.

2.2.2. *Integral currents.* Let $n \geq 1$ be an integer, $T \in \mathbf{M}_n(X)$ is called *rectifiable* if:

- $\|T\|$ is concentrated on an \mathcal{H}^n -rectifiable subset of X ;
- $\|T\|$ vanishes on \mathcal{H}^n -negligible sets.

If additionally, for all Lipschitz maps $\Psi : X \rightarrow \mathbb{R}^n$ and all open sets $A \subset X$, it holds

$$\Psi_{\#}(T \llcorner A) = \llbracket \theta \rrbracket \quad \text{for some } \theta \in L^1(\mathbb{R}^n, \mathbb{Z}),$$

we say that T is *integer rectifiable*. The collection of n -rectifiable currents is denoted by $\mathcal{R}_n(X)$. The collection of n -integer rectifiable currents is denoted by $\mathcal{I}_n(X)$. Furthermore, the class of n -dimensional *integral currents* is defined as

$$\mathbf{I}_n(X) = \mathcal{I}_n(X) \cap \mathbf{N}_n(X).$$

The (*canonical*) *set* of $T \in \mathcal{R}_n(X)$ defined by

$$\text{set}(T) = \left\{ x \in X \mid \liminf_{r \rightarrow 0} \frac{\|T\|(B_r(x))}{\omega_n r^n} > 0 \right\} \quad (2.15)$$

is countably \mathcal{H}^n -rectifiable and $\|T\|$ is concentrated on $\text{set}(T)$. Moreover, any Borel set on which $\|T\|$ is concentrated must contain $\text{set}(T)$, up to an \mathcal{H}^n -negligible set, see [5, Theorem 4.6].

We next recall a slight variant of the classical constancy theorem for currents in Euclidean spaces [23, 4.1.7], whose corollary below will be crucial in the proof of the main results of this paper.

Theorem 2.4. *Let $S \in \mathbf{N}_n(\mathbb{R}^n)$. Then there exists a unique function $g \in \text{BV}(\mathbb{R}^n; \mathbb{R}^n)$ that satisfies*

$$S = \llbracket [g] \rrbracket.$$

Moreover,

$$\|\partial S\| = |Dg|,$$

where Dg is the derivative in the sense of distributions of g and $|Dg|$ denotes its total variation. If $S \in \mathbf{I}_n(\mathbb{R}^n)$, then $g \in \text{BV}(\mathbb{R}^n)$ takes values in \mathbb{Z} .

If, in addition, $\|\partial S\|(U) = 0$ for some connected open set $U \subset \mathbb{R}^n$, then g is constant on U .

Proof. The existence and uniqueness of $g \in \text{BV}(\mathbb{R}^n)$ such that $S = \llbracket [g] \rrbracket$, together with the identity $\|\partial S\| = |Dg|$, are established in [5, Theorem 3.7]. If $S \in \mathbf{I}_n(\mathbb{R}^n)$, by definition of integer rectifiable currents g must take integer values almost everywhere.

The last statement holds as follows. Let $\mu := g, \mathcal{L}^n$. In the proof of [5, Theorem 3.7] it is shown that

$$\left| \int_{\mathbb{R}^n} \frac{\partial \varphi}{\partial x_i} d\mu \right| \leq \int_{\mathbb{R}^n} |\varphi| d\|\partial S\| \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n), \quad i = 1, \dots, n. \quad (2.16)$$

Hence, the distributional derivative $\partial_i g$ is a finite Radon measure whose total variation is controlled by $\|\partial S\|$. Assuming that $\|\partial S\|(U) = 0$ with U open and connected implies that the right-hand side of (2.16) vanishes, and hence g must be constant almost everywhere on U . By redefining g on a negligible set we may assume that g is constant everywhere on U . \square

Corollary 2.5. *Let X be a complete metric space, $T \in \mathbf{I}_n(X)$ and $\Psi : X \rightarrow \mathbb{R}^n$ be a Lipschitz map. Assume that $\|\partial T\|(V) = 0$ for some connected open set $V \subset X$, that Ψ restricted to V is bijective into its image and $\Psi(V)$ is open. Then $\Psi_{\#}T = [[g]]$, where $g \in \text{BV}(\mathbb{R}^n)$ is integer valued and constant on $\Psi(V)$.*

Proof. We know that $\Psi_{\#}T \in \mathbf{I}_n(\mathbb{R}^n)$, thus, we can write $\Psi_{\#}T$ as $\Psi_{\#}T = [[g]]$, where $g \in \text{BV}(\mathbb{R}^n; \mathbb{Z})$ is as in Theorem 2.4. From the fact that the push-forward commutes with the boundary operator, we get that

$$\|\partial(\Psi_{\#}T)\| = \|\Psi_{\#}(\partial T)\| \leq \text{Lip}(\Psi)^{n-1} \Psi_{\#} \|\partial T\|.$$

Hence, $\|\partial(\Psi_{\#}T)\|(\Psi(V)) \leq \text{Lip}(\Psi)^{n-1} \|\partial T\|(V) = 0$. By the last part of Theorem 2.4, it follows that g is a constant function on $\Psi(V)$. \square

2.2.3. *Parametric representation of rectifiable currents.* In this section, we define *charts, atlases and weights* for rectifiable currents. We represent a rectifiable current and its mass measure in terms of these objects and note the relation between the degree as in Definition 2.16 and weight of a current.

Definition 2.6. Let $T \in \mathbf{M}_n(X)$. Let $f_i : K_i \rightarrow X$, $i \in \mathbb{N}$, be a countable collection of maps that are biLipschitz to its images with the following properties:

- Each subset $K_i \subset \mathbb{R}^n$ is compact;
- $f_i(K_i)$, $i \in \mathbb{N}$, are pairwise disjoint;
- $\mathcal{H}^n(\text{set}(T) \setminus \bigcup_i f_i(K_i)) = 0$.

We say that each (f_i, K_i) is a *chart* for $\text{set}(T)$ and the collection of charts is called an *atlas* for $\text{set}(T)$.

We note that by [5, Lemma 4.1], for any $T \in \mathcal{R}_n(X)$, there always exists an atlas for $\text{set}(T)$.

Theorem 2.7 ([5, Theorem 4.5]). *Let $T \in \mathcal{R}_n(X)$ and let $(f_i, K_i)_i$ be an atlas for $\text{set}(T)$. Then there exist functions $\theta_i \in L^1(\mathbb{R}^n)$ with $\text{spt}(\theta_i) \subset K_i$ such that*

$$T = \sum_i f_{i\#}[[\theta_i]] \quad \text{and} \quad \mathbb{M}(T) = \sum_i \mathbb{M}(f_{i\#}[[\theta_i]]). \quad (2.17)$$

Conversely, any current $T \in \mathbf{M}_n(X)$ that can be written as in (2.17) is an element of $\mathcal{R}_n(X)$. Additionally, $T \in \mathcal{I}_n(X)$ if and only if the θ_i 's are integer valued.

Definition 2.8. Let $T \in \mathcal{R}_n(X)$. Using the notation of Theorem 2.7, the function

$$\theta_T : \text{set}(T) \rightarrow \mathbb{R} \setminus \{0\} \quad \text{given by} \quad \theta_T = \sum_i \theta_i \circ f_i^{-1}$$

is a Borel function such that $\int_{\text{set}(T)} |\theta| \, d\mathcal{H}^n < \infty$, and is called the *weight* (or multiplicity) function of T .

More explicitly, given an atlas for $\text{set}(T)$, the θ_i 's that appear in (2.17) are obtained using the definition of rectifiable current; that is, for each $i \in \mathbb{N}$ there exists $\theta_i \in L^1(\mathbb{R}^n)$ such that $(f_i^{-1})_{\#}T = [[\theta_i]]$.

Remark 2.9. Within the proof of Theorem 2.7, as a consequence of the change of variables formula, it was shown that for any open set $V \subset X$ and $\varphi : X \rightarrow \mathbb{R}^n$ Lipschitz:

$$\varphi_{\#}[(f_{i\#}[[\theta_i]]) \llcorner V] = \left[\left[\sum_{b \in h^{-1}(a) \cap f_i^{-1}(V)} \theta_i(b) \text{sgn}(\det D_b h) \right] \right],$$

where $h = \varphi \circ f_i : K_i \cap f_i^{-1}(V) \rightarrow \varphi(f_i(K_i) \cap V)$. This implies that for θ_i integer valued, $\varphi_{\#}[(f_{i\#}[\theta_i]) \llcorner V]$ is an integer rectifiable current.

If φ is bijective then the formula above reduces to

$$\varphi_{\#}[(f_{i\#}[\theta_i]) \llcorner V] = [[\theta_i \circ (\varphi \circ f_i)^{-1} \text{sgn}(\det D(\varphi \circ f_i))]]. \quad (2.18)$$

Remark 2.10. Notice that by Remark 2.9, Definition 2.8 is well posed in the following sense: For any other collection of functions $g_i : Q_i \subset \mathbb{R}^n \rightarrow X$ satisfying the same conditions as the f_i 's with corresponding weight functions $\tilde{\theta}_i$, it holds

$$|\tilde{\theta}_j(g_j^{-1}(x))| = |\theta_i(f_i^{-1}(x))|, \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in f_i(K_i) \cap g_j(Q_j),$$

in other words, the multiplicity function is well defined up to a sign.

The mass measure of a rectifiable current $T \in \mathcal{R}_n(X)$ has the following representation formula

$$\|T\| = \lambda \theta_T \mathcal{H}^n \llcorner \text{set}(T), \quad (2.19)$$

where $\lambda : \text{set}(T) \rightarrow [n^{-n/2}, 2^n/\omega_n]$ is a function such that $\lambda(x)$ equals the area factor of the vector space $\text{Tan}^{(n)}(\text{set}(T), x)$. See Section 9 in [5], in particular, [5, Theorem 9.5 and (9.11)].

2.2.4. 0-dimensional slices and weight functions. We now state some results concerning the relationship between the mass of 0-slices of integral currents and its weight function. This relationship – see in particular equations (2.21) and (2.22) below – will be useful when proving that the weight function of a given orientation of an RCD space, in the sense of currents, is constant with absolute value equal to 1 (see the proof of Proposition 1.2).

First recall that a 0-dimensional integral current $T : \mathcal{D}^0(X) \rightarrow \mathbb{R}$ can be written as a finite linear combination of Dirac masses,

$$T = \sum_{i=1}^N a_{p_i} \delta_{p_i}, \quad (2.20)$$

where $p_1, \dots, p_N \in X$ is a finite family of points and a_{p_1}, \dots, a_{p_N} are non-zero integers; we set $\delta_p(f) = f(p)$, for any $f \in \mathcal{D}^0(X)$. With a slight abuse of notation, below we will also denote with δ_p the corresponding Radon measure, given by duality:

$$\delta_p(E) = \begin{cases} 0 & \text{if } p \notin E \\ 1 & \text{if } p \in E \end{cases} \quad \text{for all } E \subset X \text{ Borel.}$$

Definition 2.11. For $T \in \mathbf{I}_n(X)$ and any Lipschitz map $\Phi : X \rightarrow \mathbb{R}^n$, the 0-slices of T under Φ , denoted by

$$\langle T, \Phi, a \rangle \in I_0(X), \quad \mathcal{L}^n\text{-a.e. } a \in \mathbb{R}^n,$$

are characterized by the equality

$$\int_{\mathbb{R}^n} \psi(a) \langle T, \Phi, a \rangle \, d\mathcal{L}^n(a) = T \llcorner (\psi \circ \Phi) \, d\Phi, \quad \text{for all } \psi \in C_c(\mathbb{R}^n).$$

We now collect some of the properties satisfied by 0-slices.

Proposition 2.12. *Let $T \in \mathbf{I}_n(X)$, L be a σ -compact set where T and ∂T are concentrated and $\Phi : X \rightarrow \mathbb{R}^n$ a Lipschitz map. Then*

- (1) $\langle T, \Phi, a \rangle$ is concentrated on $\Phi^{-1}(a) \cap L$;
- (2) $\langle T \llcorner A, \Phi, a \rangle = \langle T, \Phi, a \rangle \llcorner A$ for any Borel set $A \subset X$;

- (3) For \mathcal{L}^n -a.e. $a \in \mathbb{R}^n$, we have $\langle \Phi_{\#}T, \text{Id}, a \rangle = \Phi_{\#}\langle T, \Phi, a \rangle$;
(4) We have

$$\|T \llcorner d\Phi\| = \int_{\mathbb{R}^n} \|\langle T, \Phi, a \rangle\| \, d\mathcal{L}^n(a). \quad (2.21)$$

- (5) If $\Phi^{-1}(a) \cap \text{set}(T)$ contains at most finitely many points, then

$$\langle T, \Phi, a \rangle = \sum_{p \in \Phi^{-1}(a) \cap \text{set}(T)} \sigma_a(p) \theta_T(p) \delta_p,$$

where $\theta_T : \text{set}(T) \rightarrow \mathbb{R}$ is the weight function of T , as in Definition 2.8, and $\sigma_a(p) \in \{-1, 1\}$. In particular, if $\Phi^{-1}(a) \cap \text{set}(T) = \{p\}$, then

$$\begin{aligned} \langle T, \Phi, a \rangle(1) &= \sigma_a(p) \theta_T(p), \\ \|\langle T, \Phi, a \rangle\| &= |\langle T, \Phi, a \rangle(1)| \delta_p = |\theta_T(p)| \delta_p, \end{aligned} \quad (2.22)$$

where 1 denotes the constant function equal to 1.

Proof. All of the properties are proven in [5]. More precisely:

- (1) is proven in Theorem 5.6, Eq. (5.8).
- (2) is proven in Theorem 5.7.
- (3) is proven in Eq. (5.18).
- (4) is proven in Theorem 5.6, Eq. (5.9).
- (5) follows from the Kuratowski embedding theorem, the expression for 0-integral currents (2.20) combined with [5, Theorem 9.7], where it is shown that for a w^* -separable dual space X , for \mathcal{L}^n -a.e. $a \in \mathbb{R}^n$, there exists an orientation σ_a of $\text{set}(T) \cap \Phi^{-1}(a)$ such that the 0-slices $\langle T, \Phi, a \rangle$ satisfy the identity

$$\langle T, \Phi, a \rangle = [[\text{set}(T) \cap \Phi^{-1}(a), \theta_T, \sigma_a]].$$

□

2.3. Intrinsic Flat and Gromov-Hausdorff convergence. Let (X, \mathbf{d}) be a complete metric space and $T_1, T_2 \in \mathbf{I}_n(X)$. The *flat distance* between T_1 and T_2 is defined as

$$\mathbf{d}_F^X(T_1, T_2) = \inf \{ \mathbb{M}(U) + \mathbb{M}(V) : U \in \mathbf{I}_n(X), V \in \mathbf{I}_{n+1}(X), T_2 - T_1 = U + \partial V \}.$$

If $T_i \rightarrow T$ in flat sense to $T \in \mathbf{I}_n(X)$, then

- $\lim_{i \rightarrow \infty} \mathbf{d}_F^X(\partial T_i, \partial T) = 0$;
- $\mathbb{M}(T) \leq \liminf_{i \rightarrow \infty} \mathbb{M}(T_i)$;
- If $\lim_{i \rightarrow 0} \mathbb{M}(T_i) = 0$, then $T = 0$.

Theorem 2.13 (Wenger [52]). *Let (X_i, \mathbf{d}_i) be a sequence of complete metric spaces and $T_i \in \mathbf{I}_n(X_i)$. If*

$$\sup_{i \in \mathbb{N}} \left\{ \|T_i\|(X_i) + \|\partial T_i\|(X_i) \right\} < \infty,$$

then there exist a complete metric space W , a subsequence $(i_j)_{j \in \mathbb{N}}$ and isometric embeddings

$$\varphi_j : X_{i_j} \rightarrow W, \quad j \in \mathbb{N},$$

such that

$$\mathbf{d}_F^W(\varphi_{j\#}T_{i_j}, T) \rightarrow 0,$$

for some $T \in \mathbf{I}_n(W)$.

Sormani and Wenger introduced the notion of an integral current space and defined the intrinsic flat distance between them [47]. An n -dimensional integral current space (X, \mathbf{d}, T) consists of a metric space (X, \mathbf{d}) and an n -dimensional integral current, $T \in \mathbf{I}_n(\bar{X})$, where \bar{X} denotes the metric completion of (X, \mathbf{d}) , such that $\text{set}(T) = X$. The zero n -dimensional integral current space denoted as $\mathbf{0} = (X, \mathbf{d}, T)$, is defined by $T = 0$ and $\text{set}(T) = \emptyset$. We will use the notation $\mathbb{M}(X, \mathbf{d}, T) = \mathbb{M}(T)$ and $\text{set}((X, \mathbf{d}, T)) = \text{set}(T)$. We say that (X, \mathbf{d}, T) is precompact if (X, \mathbf{d}) is precompact. The definition of intrinsic flat distance is as follows.

Definition 2.14 ([47, Definition 1.1]). Given two n -dimensional precompact integral current spaces, (X_1, \mathbf{d}_1, T_1) and (X_2, \mathbf{d}_2, T_2) , the *intrinsic flat distance* between them is defined as

$$\mathbf{d}_{\mathcal{F}}((X_1, \mathbf{d}_1, T_1), (X_2, \mathbf{d}_2, T_2)) = \inf \left\{ \mathbf{d}_F^X(\varphi_{1\#}T_1, \varphi_{2\#}T_2) : \varphi_j : X_j \rightarrow X \right\},$$

where the infimum is taken over all complete metric spaces X and all isometric embeddings φ_j .

Now we recall that if a sequence admits both a Gromov–Hausdorff limit and an intrinsic flat limit, then the two convergences can be realized within a single ambient metric space, and the intrinsic flat limit space is contained in the Gromov–Hausdorff limit space.

Theorem 2.15 (Sormani-Wenger [47]). *Let (X_i, \mathbf{d}_i) be a sequence of compact metric spaces and $T_i \in \mathbf{I}_n(X_i)$. If (X_i, \mathbf{d}_i, T_i) are n -dimensional integral current spaces such that*

$$\sup_{i \in \mathbb{N}} \left\{ \|T_i\|(X_i) + \|\partial T_i\|(X_i) \right\} < \infty,$$

and (X_i, \mathbf{d}_i) converges in the GH topology to (X, \mathbf{d}) , then there exists a complete metric space W , a subsequence $(i_j)_{j \in \mathbb{N}}$, isometric embeddings

$$\varphi_j : X_{i_j} \rightarrow W, \quad j \in \mathbb{N},$$

$\varphi : X \rightarrow W$ and $T \in \mathbf{I}_n(W)$ such that

$$\mathbf{d}_F^W(\varphi_{j\#}T_{i_j}, T) \rightarrow 0,$$

$$\mathbf{d}_H^W(\varphi_j(X_{i_j}), \text{spt}(T)) \rightarrow 0,$$

and $\text{set}(T) \subset X$.

2.3.1. *The degree of a Lipschitz map with respect to a current and the L^1 -convergence of the degree.* Following the notation of Matveev-Portegies [40], we define the degree of a Lipschitz map with respect to a current. We also refer the reader to the classical monograph [24, Ch. 4, Sect. 3], for a related notion of degree in the framework of cartesian currents.

Definition 2.16. Let X be a complete metric space, $T \in \mathbf{I}_n(X)$ and $\Psi : X \rightarrow \mathbb{R}^n$ be a Lipschitz map. The *degree of Ψ with respect to T* is the unique function

$$\text{deg}(T, \Psi, \cdot) \in \text{BV}(\mathbb{R}^n)$$

taking values in \mathbb{Z} such that

$$\Psi_{\#}T = [[\text{deg}(T, \Psi, \cdot)]] \in \mathbf{I}_n(\mathbb{R}^n).$$

Using the notation of Corollary 2.5, it holds that $\deg(T, \Psi, \cdot) = g$. Hence, given $T \in \mathbf{I}_n(X)$ with $(f_i, K_i)_i$ an atlas for $\text{set}(T)$, the weight function of T can be written as:

$$\theta_T(x) = \sum_i \deg(T \llcorner f_i(K_i), f_i^{-1}, f_i^{-1}(x)) \quad \mathcal{H}^n\text{-a.e. } x \in \text{set}(T). \quad (2.23)$$

Moreover, by Proposition 2.12 for \mathcal{L}^n -a.e. $a \in \mathbb{R}^n$, we have

$$\deg(T, \Psi, a) = \langle \Psi_{\#}T, \text{Id}, a \rangle(1) = \Psi_{\#} \langle T, \Psi, a \rangle(1) = \langle T, \Psi, a \rangle(1).$$

This equation, together with the fact that the flat distance between two currents provides a bound for the integrals of the flat distances between their slices, was combined by Matveev–Portegies to obtain the following estimate.

Lemma 2.17 (Matveev-Portegies [40], c.f. Portegies-Sormani [44]). *Let X be a complete metric space, $T, S \in \mathbf{I}_n(X)$ and $\Phi, \Psi : X \rightarrow \mathbb{R}^n$, be such that all components of $\Phi = (\varphi_1, \dots, \varphi_n)$ and $\Psi = (\psi_1, \dots, \psi_n)$ are L -Lipschitz functions. Then*

$$\begin{aligned} \|\deg(T, \Phi, \cdot) - \deg(S, \Psi, \cdot)\|_{L^1(\mathbb{R}^n)} \leq \\ L^n \mathbf{d}_F^X(T, S) + 2L^{n-1} \sum_{j=1}^n \|\varphi_j - \psi_j\|_{\infty} (\mathbb{M}(S) + \mathbb{M}(\partial S)). \end{aligned}$$

We note that Lemma 2.17 is the key result to show that the weight functions of a converging sequence of currents converge to the weight function of the limit current (see Section 4.2).

2.4. Lang–Wenger’s currents. The theory of Ambrosio–Kirchheim’s currents was extended to currents with non-finite mass and in non necessarily complete metric spaces by Lang [35] and Lang-Wenger [36]. We now quickly review Lang-Wenger’s currents, discuss their relation to Ambrosio–Kirchheim’s currents and state some convergence results in this setting.

Let (X, \mathbf{d}) be a non-necessarily complete metric space, let us denote by $\text{Lip}_{\text{bs}}(X)$ the space of Lipschitz functions with bounded support in X , and $\text{Lip}_{\text{loc}}(X)$ the space of functions which are Lipschitz on bounded sets of X .

Recall that an n -dimensional metric functional in the sense of [36] is a multilinear function $T : \text{Lip}_{\text{bs}}(X) \times [\text{Lip}_{\text{loc}}(X)]^n \rightarrow \mathbb{R}$ such that

- T is continuous: for any sequence $\pi^i \in [\text{Lip}_{\text{loc}}(X)]^n$ converging pointwisely to $\pi \in [\text{Lip}_{\text{loc}}(X)]^n$ with $\sup_{i,j} \text{Lip}(\pi_i^j|_A) < \infty$ for any bounded $A \subset X$, it holds

$$T(f, \pi^i) \rightarrow T(f, \pi);$$

- T is local: if there exists j such that π_j is constant on a neighborhood of $\{f \neq 0\}$, then $T(f, \pi) = 0$.

For every open set $V \subset X$, the mass of T in V is defined as the possibly infinite quantity

$$\mathbb{M}_V(T) := \sup \sum_{\lambda \in \Lambda} T(f_{\lambda}, \pi_{\lambda}),$$

where the supremum is taken over all finite families $((f_{\lambda}, \pi_{\lambda}))_{\lambda \in \Lambda}$ such that $(f_{\lambda}, \pi_{\lambda}) = (f_{\lambda}, \pi_{\lambda,1}, \dots, \pi_{\lambda,n}) \in \text{Lip}_{\text{bs}}(X) \times [\text{Lip}_1(X)]^n$, $\text{spt}(f_{\lambda}) \subset V$, and $\sum_{\lambda \in \Lambda} |f_{\lambda}| \leq 1$. Here $\text{Lip}_1(X) \subset \text{Lip}(X)$ denotes the subset of all 1-Lipschitz functions.

Definition 2.18 ([36]). For $n \geq 0$, an n -dimensional metric current with locally finite mass is an n -dimensional metric functional T as defined above, satisfying that for every bounded open set $V \subset X$ and every $\epsilon > 0$ there exists a compact set $C \subset V$ such that $\mathbb{M}_V(T) < \infty$ and $\mathbb{M}_{V \setminus C}(T) < \epsilon$. We denote by $\mathbf{M}_{\text{loc},n}(X)$ the vector space of all n -dimensional metric currents with locally finite mass.

If $T \in \mathbf{M}_{\text{loc},n}(X)$, the set function $\|T\|: 2^X \rightarrow [0, \infty]$ given by

$$\|T\|(A) := \inf \{ \mathbb{M}_V(T) : V \subset X \text{ is open, } A \subset V \}$$

is an outer measure and, if V is open, then $\|T\|(V) = \mathbb{M}_V(T)$. As before, $\mathbb{M}(T) := \|T\|(X)$, $\text{spt}(T) := \text{spt}\|T\|$, and $\|T\|(X \setminus \text{spt}(T)) = 0$.

Every $T \in \mathbf{M}_{\text{loc},n}(X)$ can be extended to a multilinear function

$$T: \mathcal{B}_{\text{bs}}^\infty(X) \times [\text{Lip}_{\text{loc}}(X)]^n \rightarrow \mathbb{R},$$

where $\mathcal{B}_{\text{bs}}^\infty(X)$ denotes the set of all bounded Borel functions with bounded support defined on X . This extension is continuous and satisfies

$$|T(f, \pi)| \leq \text{Lip}(\pi_1|_{\text{spt}(f)}) \cdots \text{Lip}(\pi_n|_{\text{spt}(f)}) \int_X |f| \, d\|T\|.$$

Given $T \in \mathbf{M}_{\text{loc},n}(X)$ and $(g, \tau) \in \mathcal{B}_{\text{bs}}^\infty(X) \times [\text{Lip}_{\text{loc}}(X)]^k$ with $0 \leq k \leq n$ and any map $\varphi \in \text{Lip}_{\text{loc}}(X, X')$, the restriction $T \llcorner (g, \tau) \in \mathbf{M}_{\text{loc},n-k}(X)$ and the push-forward of T by φ , $\varphi_\# T \in \mathbf{M}_{\text{loc},n}(X')$, are defined in the same way as for Ambrosio-Kirchheim currents. For $n \geq 1$, the functional ∂T is defined by

$$\partial T(f, \pi_1, \dots, \pi_{n-1}) = T(\sigma, f, \pi_1, \dots, \pi_{n-1}),$$

where $\sigma \in \text{Lip}_{\text{bs}}(X)$ is any function satisfying $\sigma|_{\text{spt}(f)} = 1$, and n -dimensional local normal currents are defined as

$$\mathbf{N}_{\text{loc},n}(X) = \{ T \in \mathbf{M}_{\text{loc},n}(X) : \partial T \in \mathbf{M}_{\text{loc},n-1}(X) \}.$$

For $n = 0$, we set $\mathbf{N}_{\text{loc},0}(X) = \mathbf{M}_{\text{loc},0}(X)$.

Locally integer rectifiable currents are defined to be currents $T \in \mathbf{M}_{\text{loc},n}(X)$ that additionally satisfy that for every bounded Borel set $B \subset X$ and every $\Psi \in \text{Lip}(X, \mathbb{R}^n)$ there exists $\theta \in L^1(\mathbb{R}^n, \mathbb{Z})$ such that $\Psi_\#(T \llcorner B) = \llbracket \theta \rrbracket$. The space of n -dimensional integer rectifiable currents is denoted by $\mathcal{I}_{\text{loc},n}(X)$.

For $n \geq 1$, the abelian group of all $T \in \mathcal{I}_{\text{loc},n}(X)$ such that $\partial T \in \mathcal{I}_{\text{loc},n-1}(X)$ is denoted by $\mathbf{I}_{\text{loc},n}(X)$, and for $n = 0$ we set $\mathbf{I}_{\text{loc},0}(X) = \mathcal{I}_{\text{loc},0}(X)$. Elements in $\mathbf{I}_{\text{loc},n}(X)$ are called locally integral currents. Similar to Ambrosio-Kirchheim's currents, one has

$$\mathbf{I}_{\text{loc},n}(X) = \mathcal{I}_{\text{loc},n}(X) \cap \mathbf{N}_{\text{loc},n}(X).$$

2.4.1. *Relation between Ambrosio-Kirchheim and Lang-Wenger's currents.* The proof of the next proposition is contained in [36, Sect. 2.5].

Proposition 2.19. *Assume that (X, \mathbf{d}) is complete.*

- *If $T \in \mathbf{M}_{\text{loc},n}(X)$ and $g \in \mathcal{B}_{\text{bs}}^\infty(X)$, then $T_g := T \llcorner g : \mathcal{D}^n(X) \rightarrow \mathbb{R}$ given by*

$$T_g(f, \pi) = T(fg, \pi)$$

is an element of $\mathbf{M}_n(X)$. If $n \geq 1$, $g \in \text{Lip}_{\text{bs}}(X)$ and $T \llcorner g \in \mathbf{N}_{\text{loc},n}(X)$, then $T_g \in \mathbf{N}_n(X)$ and $\|\partial T_g\| = \|\partial(T \llcorner g)\|$.

- Conversely, given an Ambrosio–Kirchheim current $T' \in \mathbf{M}_n(X)$, by setting

$$T(f, \pi) := T'(f, \pi'),$$

for $(f, \pi) \in \mathcal{B}_{bs}^\infty(X) \times [\text{Lip}_{\text{loc}}(X)]^n$ and any $\pi' \in [\text{Lip}(X)]^n$ with $\pi'_i|_{\text{spt}f} = \pi_i|_{\text{spt}f}$, one obtains a local current $T \in \mathbf{M}_{\text{loc},n}(X)$ for which $\|T\| = \|T'\|$. In case $n \geq 1$, it follows that

$$\partial T(f, \pi) = \partial T'(f, \pi'),$$

for $(f, \pi) \in \mathcal{B}_{bs}^\infty(X) \times [\text{Lip}_{\text{loc}}(X)]^n$ and $\pi' \in [\text{Lip}(X)]^{n-1}$ with $\pi'_i|_{\text{spt}f} = \pi_i|_{\text{spt}f}$. In particular, if $T' \in \mathbf{N}_n(X)$, then $T \in \mathbf{N}_{\text{loc},n}(X)$ and $\|\partial T\| = \|\partial T'\|$.

2.4.2. Local flat topology and p GH convergence. Analogously to the previous section, one can define a notion of convergence for locally integral currents, as well as a result that relates the pointed Gromov–Hausdorff limit of a sequence to its local flat limit whenever both limits exist.

Definition 2.20. Let W be a metric space, $T_i \in \mathbf{I}_{\text{loc},n}(W)$, $i \in \mathbb{N}$, converges to $T \in \mathbf{I}_{\text{loc},n}(W)$ in the local flat topology if for any bounded closed set $B \subset W$ there exist $S_i \in \mathbf{I}_{\text{loc},n+1}(W)$ such that

$$\|T - T_i - \partial S_i\|(B) + \|S_i\|(B) \rightarrow 0.$$

Note that whenever $T_i \in \mathbf{I}_{\text{loc},n}(W)$ converges in the local flat topology to some $T \in \mathbf{I}_{\text{loc},n}(W)$, then $\partial T_i \in \mathbf{I}_{\text{loc},n-1}(W)$ converges in the local flat topology to $\partial T \in \mathbf{I}_{\text{loc},n-1}(W)$. Indeed, we have $\|\partial(T - T_i - \partial S_i)\|(B) + \|\partial S_i\|(B) \rightarrow 0$.

Theorem 2.21 (Theorem 1.2 and Proposition 1.3 in [36]). *Let (X_i, \mathbf{d}_i, x_i) be complete pointed metric spaces and $T_i \in \mathbf{I}_{\text{loc},n}(X_i)$ such that*

$$\sup_{i \in \mathbb{N}} \left\{ \|T_i\|(B_r(x_i)) + \|\partial T_i\|(B_r(x_i)) \right\} < \infty$$

for all $r > 0$. Then there exist a complete pointed metric space (W, \mathbf{d}_W, w_0) , a subsequence $(i_j)_{j \in \mathbb{N}}$ and isometric embeddings

$$\varphi_j : X_{i_j} \rightarrow W \quad \forall j \in \mathbb{N}, \quad \varphi_j(x_{i_j}) \rightarrow w_0,$$

such that $\varphi_{j\#} T_{i_j} \rightarrow T$ in the local flat topology to some $T \in \mathbf{I}_{\text{loc},n}(W)$.

Furthermore, if (X_i, \mathbf{d}_i, x_i) converges in the p GH topology to (X, \mathbf{d}, x) and (X, \mathbf{d}) is proper, there exists an isometric embedding

$$\psi : \text{spt}(T) \cup \{w_0\} \rightarrow X \quad \text{such that} \quad \psi(w_0) = x.$$

Remark 2.22. By the proof of Theorem 2.21, T is given as a sum

$$T = \sum_{r=1}^{\infty} \bar{T}_r \in \mathbf{I}_{\text{loc},n}(W),$$

where each $\bar{T}_r \in \mathbf{I}_n(W)$ is supported in the annulus

$$\text{spt}(\bar{T}_r) = \{w \in W \mid R_{r-1} \leq \mathbf{d}_W(w_0, w) \leq R_r\}$$

for a sequence $R_0 = 0 < R_1 < R_2 < \dots$ with $R_i \rightarrow \infty$.

For each $r \in \mathbb{N}$, the annulus and currents,

$$T_{r,i} = T_i \llcorner A_{r,i} \in \mathbf{I}_n(X_i), \quad A_{r,i} = \bar{B}_{R_r}(x_i) \setminus B_{R_{r-1}}(x_i),$$

are made to satisfy, after passing to a subsequence that by abusing notation we do not relabel,

$$d_F^W(\varphi_{i\sharp}(T_{r,i}), \bar{T}_r) \rightarrow 0. \quad (2.24)$$

Hence, summing over $r = 1, \dots, n$, it follows that

$$d_F^W(\varphi_{i\sharp}(T_i \llcorner B_{R_r}(x_i)), T^r) \rightarrow 0 \quad (2.25)$$

where

$$T^r = \sum_{i=1}^r \bar{T}_i \in \mathbf{I}_n(W), \quad \text{spt}(T^r) \subset \bar{B}_{R_r}(w_0).$$

3. BILIPSCHITZ CHARTS AND REGULARITY PROPERTIES

3.1. Regular points of harmonic functions and bilipschitz charts in non-collapsed RCD spaces. The aim of this section is to prove Proposition 3.4. From this result, we will derive the existence of biLipschitz charts for the set of a current orienting a non-collapsed RCD space. We start by introducing and studying regular points of harmonic functions.

Definition 3.1. Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}(K, N)$ metric measure space, let $B_r(x) \subset X$ and $u : B_r(x) \rightarrow \mathbb{R}$ be a harmonic function. We say that $y \in B_r(x)$ is a *regular point* for u if y is a Lebesgue point for $|\nabla u|^2$ and

$$\lim_{s \downarrow 0} \int_{B_s(y)} |\nabla u|^2 \, \mathbf{d}\mathbf{m} \neq 0. \quad (3.1)$$

If $u : B_r(x) \rightarrow \mathbb{R}^k$ is a vector valued harmonic function, we say that $y \in B_r(x)$ is a regular point for u if it is regular for each u_i , $i = 1, \dots, k$. We denote by $\mathcal{R}(u)$ the set of regular points for u .

Lemma 3.2. Let $(X, \mathbf{d}, \mathcal{H}^N)$ be a non-collapsed $\text{RCD}(K, N)$ space and let $u : B_r(x) \rightarrow \mathbb{R}^N$ be a δ -splitting map, for some $\delta \in (0, \delta_{K,N})$. Then $\mathcal{H}^N(B_r(x) \setminus \mathcal{R}(u)) = 0$, i.e., \mathcal{H}^N -a.e. $y \in B_r(x)$ is a regular point for u .

Proof. First of all, since $|\nabla u_i| \in L^2(B_r(x))$, then the classical Lebesgue theorem yields that \mathcal{H}^N -a.e. $y \in B_r(x)$ is a Lebesgue point for u_i , for all $i = 1, \dots, N$. Denote

$$\mathcal{C}(u) := \left\{ y \in B_r(x) \mid \lim_{s \downarrow 0} \int_{B_s(y)} |\nabla u_i|^2 \, \mathbf{d}\mathbf{m} = 0, \quad \text{for some } i = 1, \dots, N \right\}. \quad (3.2)$$

Assume by contradiction that $\mathcal{H}^N(\mathcal{C}(u)) > 0$. Pick $y_0 \in B_r(x)$ a Lebesgue density point for $\mathcal{C}(u)$. Thanks to the transformation theorem (see [10, Proposition 3.13] for a proof in the RCD setting, after [19]), we know that for every $s > 0$, there exists an invertible matrix $T_{y_0,s}$ such that the function $v = v_{y_0,s} := T_{y_0,s}u$ is an ε -splitting map, for $\varepsilon = 1/(2N)$. Since $T_{y_0,s}$ is invertible, then $\mathcal{C}(u) = \mathcal{C}(v)$. Therefore

$$\begin{aligned} \frac{1}{2} = N\varepsilon &\geq \frac{1}{N} \sum_{i,j=1}^N \int_{B_s(y_0)} |\nabla v_i \cdot \nabla v_j - \delta_{ij}| \, \mathbf{d}\mathcal{H}^N \\ &\geq \frac{\mathcal{H}^N(\mathcal{C}(v))}{\mathcal{H}^N(B_s(y_0))} = \frac{\mathcal{H}^N(\mathcal{C}(u))}{\mathcal{H}^N(B_s(y_0))}. \end{aligned}$$

For $s > 0$ small, this contradicts that y_0 is a Lebesgue density point for $\mathcal{C}(u)$. \square

Lemma 3.3. *Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}(-\delta(N-1), N)$ metric measure space and let $u : B_1(x) \rightarrow \mathbb{R}^k$ be a harmonic map. Assume that x is a regular point for u . Then for any $\varepsilon > 0$ there exist $r_0 = r_0(\varepsilon) > 0$ and an invertible $k \times k$ matrix A such that $A \circ u : B_r(x) \rightarrow \mathbb{R}^k$ is an ε -splitting map for any $r \in (0, r_0)$.*

Proof. The assumption that x is a regular point for u combined with standard linear algebra ensures that there exists an invertible $k \times k$ -matrix $A \in \text{GL}(k)$ with the following property: for every $\varepsilon > 0$ there exists $r_0 = r_0(x, \varepsilon) > 0$ such that

$$\int_{B_r(x)} |\nabla(Au)_i \cdot \nabla(Au)_j - \delta_{ij}| \, \mathbf{d}\mathbf{m} < \varepsilon, \quad \text{for all } i, j = 1, \dots, k \text{ and all } r \in (0, r_0), \quad (3.3)$$

yielding condition (iii) in [Definition 2.1](#). Condition (i) in [Definition 2.1](#) holds as $\Delta(Au) = A(\Delta u) \equiv 0$, since by assumption u is harmonic; moreover, the Cheng-Yau gradient estimate (see [\[53, Theorem 1.6\]](#) for the RCD setting) gives $|\nabla(Au)_i| \leq C(N)$ on $B_{r_0}(x)$, for all $i = 1, \dots, k$. Condition (ii) in [Definition 2.1](#) follows from conditions (i) and (iii) thanks to the *improved Bochner inequality* (see [\[30, Theorem 3.3\]](#), after [\[45, 26\]](#)). Let us briefly recall the argument for the reader's convenience. From [\[6, 41\]](#), we know that there exists a regular cut-off function $\varphi : X \rightarrow [0, 1]$, with

$$\varphi \equiv 1 \text{ on } B_r(x), \quad \varphi \equiv 0 \text{ on } X \setminus B_{2r}(x), \quad |\Delta\varphi| \leq \frac{C(N)}{r^2}. \quad (3.4)$$

From the improved Bochner inequality applied to $(Au)_i$, with test φ , we obtain

$$\begin{aligned} \frac{1}{2} \int_{B_{2r}(x)} |\nabla(Au)_i|^2 \Delta\varphi \, \mathbf{d}\mathbf{m} &\geq \int_{B_{2r}(x)} |\text{Hess}(Au)_i|^2 \varphi \, \mathbf{d}\mathbf{m} \\ &\quad - \delta(N-1) \int_{B_{2r}(x)} |\nabla(Au)_i|^2 \varphi \, \mathbf{d}\mathbf{m}. \end{aligned}$$

Using the local doubling of the measure, [\(3.3\)](#) and [\(3.4\)](#), we get

$$\begin{aligned} \int_{B_r(x)} |\text{Hess}(Au)_i|^2 \, \mathbf{d}\mathbf{m} &\leq C(N)\delta + \frac{1}{2} \int_{B_{2r}(x)} (|\nabla(Au)_i|^2 - 1) \Delta\varphi \, \mathbf{d}\mathbf{m} \\ &\leq C(N)\delta + \varepsilon \frac{C(N)}{r^2}, \end{aligned}$$

yielding condition (ii) in [Definition 2.1](#). □

We are now ready to state and prove the main result of this section.

Proposition 3.4. *For every $N \in \mathbb{N}, N \geq 1$, there exists $\delta(N) > 0$ with the following property. Let $(X, \mathbf{d}, \mathcal{H}^N)$ be a non-collapsed $\text{RCD}(-(N-1), N)$ metric measure space. Let $u : B_1(x) \rightarrow \mathbb{R}^N$ be a δ -splitting map, for some $\delta < \delta(N)$. Then there exists a countable family of disjoint Borel sets $U_n \subset B_1(x)$ with the following properties:*

- i) $\mathcal{H}^N(B_1(x) \setminus \bigsqcup_{n \in \mathbb{N}} U_n) = 0$;
- ii) the restriction $u|_{U_n} : U_n \rightarrow \mathbb{R}^N$ is biLipschitz to its image, for every $n \in \mathbb{N}$.

Proof. Let us start with an elementary observation. If A is an invertible $N \times N$ matrix and $A \circ u$ is biLipschitz on $K \subset B_1(x)$, then also u is biLipschitz on K (where of course the biLipschitz constants might be worsened).

A second key observation is that if a map v is an ε -GH approximation at all locations and scales when restricted to a set K , then for ε sufficiently small the map

is biLipschitz. This is the key remark to obtain rectifiability in [41] and subsequent works about rectifiability on RCD spaces.

By [Lemma 3.2](#), we know that there exists a set $\mathcal{N} \subset B_1(x)$ with $\mathcal{H}^N(\mathcal{N}) = 0$ such that every $y \in B_1(x) \setminus \mathcal{N}$ is regular for u . From [Lemma 3.3](#) it follows that for every $y \in B_1(x) \setminus \mathcal{N}$, there exists $r_0 > 0$ and an invertible matrix A_y such that $A_y \circ u : B_s(y) \rightarrow \mathbb{R}^N$ is a δ -splitting map for every $s \in (0, r_0]$. From the proof of [Lemma 3.3](#), the map $B_1(x) \setminus \mathcal{N} \ni y \mapsto A_y \in \text{GL}(N)$ can be selected to be measurable. By Lusin Theorem, we know that for every $k > 0$, there exists a compact subset $E_k \subset B_1(x)$ with

$$\mathcal{H}^N(E_k) \geq \left(1 - \frac{1}{k}\right) \mathcal{H}^N(B_1(x)), \quad (3.5)$$

such that the map $y \mapsto A_y \in \text{GL}(N)$ is uniformly continuous. Thus there exist $s_k = s_k(\delta) > 0$, $I_k = I_k(\delta) \in \mathbb{N}$, and finitely many points $y_1, \dots, y_{I_k} \in E_k$, such that

- $E_k \subset \bigcup_{i=1, \dots, I_k} B_{s_k}(y_i)$;
- $A_{y_i} \circ u : B_s(y) \rightarrow \mathbb{R}^N$ is a 2δ -splitting map for every $s \in (0, s_k)$ and $y \in B_{s_k}(y_i) \cap E_k$, $i = 1, \dots, I_k$.

Let us stress that, in the last reduction, we gained that the matrix A_{y_i} is *constant* for all $y \in B_{s_k}(y_i) \cap E_k$. [Theorem 2.2\(ii\)](#) yields that $A_{y_i} \circ u : B_s(y) \rightarrow \mathbb{R}^N$ is a $s\varepsilon = s\varepsilon(\delta)$ -GH isometry for every $s \in (0, s_k)$ and for every $y \in B_{s_k}(y_i) \cap E_k$, with $\lim_{\delta \rightarrow 0} \varepsilon(\delta) = 0$. By the second key observation at the beginning of the proof, we infer that $A_{y_i} \circ u : B_{s_k}(y_i) \cap E_k \rightarrow \mathbb{R}^N$ is biLipschitz onto its image, for every $i = 1, \dots, I_k$. Since A_{y_i} is invertible, then $u : B_{s_k}(y_i) \cap E_k \rightarrow \mathbb{R}^N$ is biLipschitz onto its image, for every $i = 1, \dots, I_k$. Recalling that $E_k \subset \bigcup_{i=1, \dots, I_k} B_{s_k}(y_i)$, we get that $u : E_k \rightarrow \mathbb{R}^N$ is biLipschitz onto its image. We conclude by setting $U_1 = E_1$ and $U_n = E_n \setminus \bigcup_{k=1, \dots, n-1} E_k$, and by recalling [\(3.5\)](#). \square

3.2. Some regularity properties for metric measure spaces. We now state the properties, [Definitions 3.5-3.7](#), that a metric measure space has to satisfy for the proofs of [Proposition 1.2](#) and [Corollary 1.3](#) to hold. Then in [Proposition 3.9](#) we establish that non-collapsed RCD spaces and non-collapsed strong Kato limits satisfy these properties.

Definition 3.5. (ETR) Let (X, d, \mathcal{H}^N) be a metric measure space, where $1 \leq N < \infty$ is a natural number, such that every metric ball $B_r(p)$ is connected. Fix $\varepsilon > 0$. We say that $p \in X$ is *regular* if for every $\varepsilon > 0$ there exists $r > 0$ such that the following holds.

- There exists a homeomorphism $u : B_r(p) \rightarrow u(B_r(p)) \subset \mathbb{R}^N$ verifying the following properties:
 - i) for any $x, y \in B_r(p)$ it holds that

$$|u(x) - u(y)| \leq (1 + \varepsilon)r d(x, y); \quad (3.6)$$

- ii) $u(B_r(p)) \supset B_{(1-2\varepsilon)r}^{\mathbb{R}^N}(0)$;
- iii) the following volume estimate holds:

$$(1 - \varepsilon)\omega_N r^N \leq \mathcal{H}^N(B_r(p)) \leq (1 + \varepsilon)\omega_N r^N. \quad (3.7)$$

- There exists a countable family of disjoint Borel sets $U_i \subset B_r(p)$ such that:
 - i) $\mathcal{H}^N(B_r(p) \setminus \bigcup_{i \in \mathbb{N}} U_i) = 0$;

- ii) the restriction map $u|_{U_i}$ is biLipschitz from U_i to $u(U_i) \subset \mathbb{R}^N$, for all $i \in \mathbb{N}$.

We say that $(X, \mathbf{d}, \mathcal{H}^N)$ satisfies (ETR) (for *essential topological regularity*) if there exists a subset $\mathcal{R} \subset X$ such that $\mathcal{H}^N(X \setminus \mathcal{R}) = 0$ and each point $p \in \mathcal{R}$ is regular.

Let $(X, \mathbf{d}, \mathcal{H}^N)$ satisfy (ETR). We say that $(X, \mathbf{d}, \mathcal{H}^N)$ has no boundary in the (ETR) sense if

$$\text{Codim}_{\text{Haus}}(X \setminus \mathcal{R}) > 1, \quad (3.8)$$

where $\text{Codim}_{\text{Haus}}$ denotes the Hausdorff codimension.

Notice that the notion of having no boundary for a space satisfying (ETR) is compatible with the analogous notion for non-collapsed RCD spaces, see (2.7).

If $p \in X$ is regular, we can assume that $B_r(p)$ has an atlas $(f_i, K_i)_{i \in \mathbb{N}}$ as in Definition 2.6, where $f_i : K_i \rightarrow X$ equals $u^{-1}|_{u(U_i)}$ and these restrictions are biLipschitz maps.

Definition 3.6 (a.e.-connected regular set). Let $(X, \mathbf{d}, \mathcal{H}^N)$ satisfy (ETR). We say that the regular set $\mathcal{R} \subset X$ is *a.e.-connected* if for all $p \in \mathcal{R}$ and \mathcal{H}^N -a.e. $q \in \mathcal{R}$, with exceptional set possibly depending on p , there exists a continuous curve joining p to q and which is entirely contained in \mathcal{R} .

Definition 3.7. (LDB) Let $(X, \mathbf{d}, \mathcal{H}^N)$ be a metric measure space, where $1 \leq N < \infty$ is a natural number. We say that $(X, \mathbf{d}, \mathcal{H}^N)$ satisfies (LDB) (for *lower density bound*) if

$$\liminf_{r \rightarrow 0^+} \frac{\mathcal{H}^N(B_r(x))}{r^N} > 0, \quad \text{for all } x \in X. \quad (3.9)$$

Lemma 3.8. Let $(X, \mathbf{d}, \mathcal{H}^N)$ be a metric measure space. Assume that

- there exists a dense set $E \subset X$,
- for every point $x_0 \in X$ there exist positive constants $r_{x_0}, c_{x_0} > 0$ and a function $o_{x_0}(\cdot) : (0, r_{x_0}) \rightarrow \mathbb{R}$ with $\limsup_{t \rightarrow 0^+} \frac{o_{x_0}(t)}{t} = 0$,

satisfying the following properties:

$$\mathcal{H}^N(B_r(p)) \geq c_{x_0} r^N, \quad \text{for all } p \in E \cap B_{r_{x_0}}(x_0) \text{ and all } r \in (0, r_{x_0}); \quad (3.10)$$

$$(0, r_{x_0}) \ni r \rightarrow \frac{\mathcal{H}^N(B_r(x_0))}{r^N + o_{x_0}(r^N)} \quad \text{is monotone non-increasing.} \quad (3.11)$$

Then $(X, \mathbf{d}, \mathcal{H}^N)$ satisfies (LDB).

Proof. It is clear that (3.9) is satisfied for all $x \in E$. Let $x_0 \in X \setminus E$. We now prove that (3.9) holds at x_0 . By assumption, there exists a sequence $(x_j) \subset E \cap B_{r_{x_0}/2}(x_0)$. By the triangle inequality, it holds that $B_{r_{x_0}/2}(x_j) \subset B_{r_{x_0}}(x_0)$ and thus

$$\mathcal{H}^N(B_{r_{x_0}}(x_0)) \geq \mathcal{H}^N(B_{r_{x_0}/2}(x_j)) \geq \frac{c_{x_0}}{2^N} r_{x_0}^N. \quad (3.12)$$

The lower bound (3.12) combined with the monotonicity (3.11) gives the claimed (3.9) at x_0 . \square

Proposition 3.9. The following classes of spaces satisfy both (ETR) and (LDB):

- Locally non-collapsed RCD spaces, i.e. $(X, \mathbf{d}, \mathcal{H}^N)$ with the following property: for every $x \in X$ there exists $K_x \in \mathbb{R}$ and a closed neighbourhood \bar{U}_x of x such that $(\bar{U}_x, \mathbf{d}|_{\bar{U}_x}, \mathcal{H}^N)$ is an $\text{RCD}(K_x, N)$ space. In particular, non-collapsed RCD spaces, i.e. $(X, \mathbf{d}, \mathcal{H}^N)$ is an $\text{RCD}(K, N)$ space for some $K \in \mathbb{R}$ and $N \in \mathbb{N}$.

- *Non-collapsed strong Kato limit spaces, in the sense of Carron-Mondello-Tewodrose (see Definition 1.4).*

Moreover, if (X, d, \mathcal{H}^N) belongs to one of two classes above and has no boundary in the sense of (3.8), then X has a.e.-connected regular set (see Definition 3.6).

Proof. For *Locally non-collapsed RCD spaces*: (ETR) follows by the structure theory of RCD spaces, see Theorem 2.3 and Proposition 3.4. From now on, we will take the regular set \mathcal{R} given in (2.1) to be the regular set in (ETR). (LDB) follows from Lemma 3.8: one can take $E = \mathcal{R}$, and the monotonicity (3.11) is a consequence of Bishop-Gromov theorem (see [49, Thm 2.3]). It remains to show that \mathcal{R} is a.e.-connected. Since concatenation of continuous curves gives a continuous curve, by the definition of locally non-collapsed RCD space without boundary, it suffices to prove that the regular set of a non-collapsed RCD space without boundary is a.e.-connected. So, let (X, d, \mathcal{H}^N) be an RCD(K, N) space without boundary. By assumption (3.8), $\mathcal{H}^{N-1}(X \setminus \mathcal{R}) = 0$. The desired property (actually, in this case, the continuous curve can be taken as a geodesic) follows from [34, Cor. A.8].

For *Non-collapsed strong Kato limit spaces* (X, d, \mathcal{H}^N) : (LDB) is proved in [13, Prop. 5.9]. (ETR) follows from passing to the limit the uniform estimates obtained in [13, Thm. 5.11 and Rem. 5.12], taking the regular set \mathcal{R} as in (2.1). It remains to show that, if X has no boundary, then \mathcal{R} is a.e.-connected. From [13], we know that (X, d, \mathcal{H}^N) is biLipschitz – via the identity map – to an RCD(K, N') structure over X , say (X, d', \mathbf{m}') , for some $K \in \mathbb{R}$, $N' \in [N, \infty)$, with $\mathbf{m}' \ll \mathcal{H}^N$. Assumption (3.8) implies that $\mathbf{m}'_{-1}(X \setminus \mathcal{R}) = 0$, where \mathbf{m}'_{-1} denotes the codimension-one measure made out of \mathbf{m}' (see, for instance, [34, App. A.3] and references therein). Therefore, [34, Prop. A.6] implies that \mathcal{R} is a.e.-connected in (X, d', \mathbf{m}') ; actually, the continuous curve can be taken as a geodesic in (X, d') . Since $\mathbf{m}' \ll \mathcal{H}^N$, we conclude that \mathcal{R} is a.e.-connected in (X, d, \mathcal{H}^N) . \square

4. PROOFS OF MAIN RESULTS

4.1. Proof of Proposition 1.2 and Corollary 1.3. Inspired by Matveev-Portegies [40, Theorem 4.1 ii-iv)], we state and prove the following results, which lead to the proofs of Proposition 1.2 and Corollary 1.3.

We start establishing a useful consequence of the (ETR) and (LDB) conditions.

Lemma 4.1. *Let (X, d, \mathcal{H}^N) be a complete metric measure space satisfying (ETR) and (LDB) for the same set \mathcal{R} . Assume there exists $T \in \mathbf{I}_{\text{loc}, N}(X)$ with $\|T\| = \mathcal{H}^N$. Then $\text{set}(T) = X$.*

Proof. Since the definition of $\text{set}(T)$ is local and recalling Proposition 2.19, we can assume that $T \in \mathbf{I}_N(X)$, without loss of generality; i.e., for each point $x \in X$ one could restrict T to some $g \in \text{Lip}_{\text{bs}}(X)$ which is equal to 1 on a neighborhood of x . Combining the assumption that $\|T\| = \mathcal{H}^N$ with (3.7), we deduce that $\mathcal{R} \subset \text{set}(T)$. Recall that $\|T\| = \lambda_{\text{set}(T)} \theta_T \mathcal{H}^N \llcorner \text{set}(T)$, see (2.19). Furthermore, by (LDB), the closure of \mathcal{R} must be contained in $\text{set}(T)$. Hence, $X \subset \text{set}(T)$ which implies the claim. \square

Lemma 4.2. *Let (X, d, \mathcal{H}^N) be a complete metric space and let p be a regular point in the sense of Definition 3.5. For $\varepsilon > 0$, let $u : B_r(p) \subset X \rightarrow \mathbb{R}^N$ be a map satisfying the properties of Definition 3.5.*

Let $S \in \mathbf{I}_N(X)$ satisfy $\|\partial S\|(B_r(p)) = 0$. Then there exists $k \in \mathbb{Z}$ such that:

- $\theta_S = k$, \mathcal{H}^N - a.e. on $B_r(p)$;
- If additionally $\|S\|(B_r(p)) > 0$, then $k \neq 0$ and

$$S \llcorner B_r(p) = k \sum_i f_{i\#}[[1_{K_i}]],$$

for charts of the form (f_i, K_i) , where $f_i : K_i \rightarrow X$ equals $u^{-1}|_{u(U_i)}$;

- If additionally $\|S\| \llcorner B_r(p) = \mathcal{H}^N \llcorner B_r(p)$, then

$$|\theta_S| \leq (1 + \varepsilon) \left(\frac{1 + \varepsilon}{1 - 2\varepsilon} \right)^N, \quad \mathcal{H}^N \text{ - a.e. on } B_r(p).$$

Proof. If $\|S\|(B_r(p)) = 0$ then $S \llcorner B_r(p) = 0$, and thus the first item holds. Hence, we can assume that $\|S\|(B_r(p)) > 0$.

Step 1. We prove that $\mathcal{H}^N(B_r(p) \setminus \text{set}(S)) = 0$.

Up to rescaling the distance \mathbf{d} by the constant factor $1/r$, we can assume that $r = 1$. Since $\|\partial S\|(B_1(p)) = 0$ and u is a Lipschitz homeomorphism from $B_1(p)$ into its image, by Corollary 2.5, we obtain that

$$u_{\#}S = [[k]] \quad \text{on } u(B_1(p)), \quad (4.1)$$

for some $k \in \mathbb{Z}$. That is, the degree of u with respect to S satisfies $\deg(S, u, \cdot) = k$ on $u(B_1(p))$.

By assumption $\|S\|(B_1(p)) > 0$, hence $k \neq 0$, and since u is a Lipschitz homeomorphism from $B_1(p)$ into its image, we have that $\text{spt}(S) \cap B_1(p) = B_1(p)$. This implies that $\text{set}(S) \neq \emptyset$.

By Definition 3.5 there exist disjoint Borel sets U_i covering $B_1(p)$ up to a \mathcal{H}^N -negligible set, and the restriction of the map u to U_i is biLipschitz to its image in \mathbb{R}^N . Consider the atlas of $\text{set}(S \llcorner B_1(p))$ given by

$$f_i : u(U_i \cap \text{set}(S)) \rightarrow U_i \cap \text{set}(S)$$

with

$$f_i(a) = u^{-1}(a), \quad \forall a \in u(U_i \cap \text{set}(S)).$$

We will replace each $U_i \subset B_1(p)$ with a Borel subset $\tilde{U}_i \subset U_i$ such that $\mathcal{H}^N(U_i \setminus \tilde{U}_i) = 0$ and

$$\bigsqcup_{i \in \mathbb{N}} \tilde{U}_i \subset \text{set}(S). \quad (4.2)$$

Consider the Lebesgue density points of $u(U_i)$:

$$\mathcal{L}(u(U_i)) = \left\{ a \in u(U_i) : \lim_{r \rightarrow 0} \frac{\mathcal{L}^N(B_r(a) \cap u(U_i))}{\mathcal{L}^N(B_r(a))} > 0 \right\}. \quad (4.3)$$

We define $\tilde{U}_i = u^{-1}(\mathcal{L}(u(U_i)))$. Since $f_i = (u|_{U_i})^{-1}$ is Lipschitz, we get that

$$\mathcal{H}^N(U_i \setminus \tilde{U}_i) = \mathcal{H}^N(f_i(u(U_i) \setminus \mathcal{L}(u(U_i)))) \leq \text{Lip}(f_i)^N \mathcal{L}^N(u(U_i) \setminus \mathcal{L}(u(U_i))) = 0.$$

For $x \in U_i$, we can estimate

$$\begin{aligned} \|S\|(B_r(x)) &\geq \|S\|(B_r(x) \cap U_i) = \|S \llcorner U_i\|(B_r(x)) = u_{\#}\|S \llcorner U_i\|(u(B_r(x))) \\ &\geq (\text{Lip}(u))^{-N} \|u_{\#}(S \llcorner U_i)\|(u(B_r(x))) \\ &\stackrel{(4.1)}{=} (\text{Lip}(u))^{-N} k \mathcal{L}^N(u(B_r(x)) \cap u(U_i)). \end{aligned} \quad (4.4)$$

We now claim that $\tilde{U}_i \setminus \text{set}(S) = \emptyset$. Indeed, if $x \in \tilde{U}_i \setminus \text{set}(S)$ then, combining (2.15), (4.3), the fact that $u|_{U_i}$ is biLipschitz to its image, and (4.4), we reach the contradiction $0 \neq 0$:

$$0 \stackrel{(2.15)}{=} \lim_{r \rightarrow 0} \frac{\|S\|(B_r(x))}{\omega_N r^N} \stackrel{(4.3)}{=} \lim_{r \rightarrow 0} \frac{\|S\|(B_r(x))}{\mathcal{L}^N(B_r(u(x)) \cap u(U_i))} \stackrel{(4.4)}{\geq} \frac{k}{\text{Lip}(u)^N} \neq 0.$$

Thus the claim (4.2) holds. From this we infer that $\mathcal{H}^N(B_1(p) \setminus \text{set}(S)) = 0$, proving the claim.

Step 2. We prove that $\theta_S = k$, \mathcal{H}^{N-} a.e. on $B_1(p)$.

Let $\tilde{U}_i \subset \text{set}(S) \cap B_1(p)$ be given by the proof of step 1. By Theorem 2.7, there exist functions $\theta_i \in L^1(\mathbb{R}^N, \mathbb{Z})$ with $\text{spt}(\theta_i) \subset u(\tilde{U}_i)$ such that

$$S \llcorner B_1(p) = \sum_i f_{i\#}[\theta_i]. \quad (4.5)$$

Hence, the weight function θ_S can be written as

$$\theta_S = \sum_i \theta_i \circ f_i^{-1}, \quad \mathcal{H}^N\text{-a.e. on } \bigsqcup_i U_i = \text{set}(S). \quad (4.6)$$

Taking the pushforward of $S \llcorner B_1(p)$ with respect to u gives

$$u_{\#}(S \llcorner B_1(p)) = (u_{\#}S) \llcorner u(B_1(p)) = \left[\sum_i \theta_i \right], \quad (4.7)$$

where we used that $f_i = (u|_{U_i})^{-1}$ and the formula (2.18) provided in Remark 2.9. Since u is a homeomorphism from $B_1(p)$ into its image, the images $u(U_i)$ are pairwise disjoint; thus, the combination of (4.1) and (4.7) yields

$$\theta_i = k \quad \mathcal{L}^N\text{-a.e. on } u(\tilde{U}_i), \quad \forall i \in \mathbb{N}. \quad (4.8)$$

Since from step 1 we know that $\mathcal{H}^N(B_1(p) \setminus \bigsqcup_i U_i) = 0$, combining (4.6) and (4.8) we conclude that $\theta_S = k$, \mathcal{H}^{N-} a.e. on $B_1(p)$.

Step 3. Conclusion.

The second item follows by the combination of (4.5), (4.8) and step 1.

We finally prove the last item. We first apply (2.14) to S and $du = (1, u)$, and use the expression (2.21) for $\|S \llcorner du\|$ in terms of the mass of the slices $\|\langle S, u, a \rangle\|$,

$$\begin{aligned} \text{Lip}(u_1) \cdots \text{Lip}(u_N) \|S\|(B_1(p)) &\geq \|S \llcorner du\|(B_1(p)) \\ &= \int_{\mathbb{R}^N} \|\langle S, u, a \rangle\|(B_1(p)) d\mathcal{L}^N(a) \\ &= |\theta_S| \mathcal{L}^N(u(B_1(p))), \end{aligned} \quad (4.9)$$

where we used that θ_S is constant \mathcal{H}^N - a.e. on $B_1(p)$ and $\mathcal{H}^N(B_1(p) \setminus \text{set}(S)) = 0$. Hence,

$$|\theta_S| \leq \frac{\text{Lip}(u_1) \cdots \text{Lip}(u_N) \|S\|(B_1(p))}{\mathcal{L}^N(u(B_1(p)))}, \quad \mathcal{H}^{N-} \text{ a.e. on } B_1(p).$$

Recalling Definition 3.5, i.e. that each component of u is a $(1 + \varepsilon)$ -Lipschitz function, that $B_{1-2\varepsilon}(0) \subset u(B_1(p))$ and that $\mathcal{H}^N(B_1(p)) \leq (1 + \varepsilon)\omega_N$, together with the assumption of this item, we obtain

$$|\theta_S| \leq (1 + \varepsilon) \left(\frac{1 + \varepsilon}{1 - 2\varepsilon} \right)^N, \quad \mathcal{H}^{N-} \text{ a.e. on } B_1(p).$$

□

Proposition 4.3. *Let $N \geq 1$ be a natural number and let $(X, \mathbf{d}, \mathcal{H}^N)$ be a metric measure space satisfying (ETR). Assume there exists $T \in \mathbf{I}_{\text{loc}, N}(X)$, with $\|T\| = \mathcal{H}^N$ and $\partial T = 0$. Then:*

- *For any $p \in \mathcal{R}$ there exist $r = r(p) > 0$ and $\sigma = \sigma(p) \in \{1, -1\}$ such that $\theta_T = \sigma$, \mathcal{H}^N -a.e. on $B_r(p)$.*
- *Assume in addition that the regular set \mathcal{R} of X is a.e.-connected (see Definition 3.6). Then there exists $\sigma \in \{1, -1\}$ such that $\theta_T = \sigma$, \mathcal{H}^N -a.e. on X .*

In particular, the second item holds if $(X, \mathbf{d}, \mathcal{H}^N)$ is a locally non-collapsed RCD space without boundary, or a non-collapsed strong Kato limit space without boundary (in the sense of (3.8)).

Proof. Fix $p \in \mathcal{R}$ a regular point. By restricting T to some $g \in \text{Lip}_{\text{bs}}(X)$ which is equal to 1 on a bounded neighborhood of p , we can assume with no loss of generality, by Proposition 2.19, that $T \in \mathbf{I}_N(X)$.

Let $\varepsilon > 0$ be sufficiently small such that, applying Lemma 4.2 we obtain that θ_T is a.e. a constant function on $B_r(p)$ that satisfies

$$|\theta_T| \leq (1 + \varepsilon) \left(\frac{1 + \varepsilon}{1 - 2\varepsilon} \right)^N < 2 \quad \text{on } B_r(p).$$

Since θ_T is a non-zero integer, then it is either the constant function 1 or -1 a.e. on $B_r(p)$. This proves the first item.

We prove the second item by contradiction. If it fails then, thanks to the first item, we can assume that there exist $p, q \in \mathcal{R}$, and $r > 0$, such that

$$\begin{aligned} \theta_T &= 1, & \mathcal{H}^N\text{-a.e. on } B_r(p) \\ \theta_T &= -1, & \mathcal{H}^N\text{-a.e. on } B_r(q). \end{aligned} \tag{4.10}$$

Thanks to the a.e.-connectedness of the regular set, up to slightly perturbing q and replacing r by $r/2$, we can also assume that there exists a continuous curve $\gamma : [0, 1] \rightarrow \mathcal{R} \subset X$ from p to q . Since $\gamma([0, 1])$ is compact and connected, we can form a finite chain of metric balls $\{B_{s_i}(x_i)\}_{i=1, \dots, n}$ with the following properties:

- (1) for each i , the radius $s_i > 0$ is small enough such that $\theta_T = k_i$, \mathcal{H}^N -a.e. on $B_{s_i}(x_i)$, for some $k_i \in \{-1, 1\}$. This holds by the first item of the proposition.
- (2) $B_{s_i}(x_i) \cap B_{s_{i+1}}(x_{i+1}) \neq \emptyset$, for all $i = 1, \dots, n-1$;
- (3) $\gamma([0, 1]) \subset \bigcup_{i=1}^n B_{s_i}(x_i)$.

By property (2), the non-empty intersection of consecutive balls is a non-empty open set and thus has positive \mathcal{H}^N -measure; therefore, using (1), we infer that $k_{i+1} = k_i$, for all i . Combining this fact with (3) yields a contradiction with (4.10).

The last assertion follows from Proposition 3.9 and the previous items. \square

Proposition 1.2 and Corollary 1.3 follow from the following result.

Proposition 4.4. *Let $N \geq 1$ be a natural number and let $(X, \mathbf{d}, \mathcal{H}^N)$ be a metric measure space satisfying (ETR) and such that the regular set \mathcal{R} of X is a.e.-connected (in particular, the assumptions are satisfied if $(X, \mathbf{d}, \mathcal{H}^N)$ is a locally non-collapsed RCD space without boundary, or a non-collapsed strong Kato limit space without boundary, in the sense of (3.8)). Let $S \in \mathbf{I}_{\text{loc}, N}(X)$, $x \in X$ and $s > 0$ such that $\|\partial S\|(B_s(x)) = 0$. Then:*

- *There exists an integer $k \in \mathbb{Z}$ such that $\theta_S = k$ a.e. on $B_s(x)$;*

- If additionally there exists $T \in \mathbf{I}_{\text{loc}, N}(X)$ with $\|T\| = \mathcal{H}^N$ and $\partial T = 0$, then $S \llcorner B_s(x) = \text{sgn}(\sigma)kT \llcorner B_s(x)$, where $\sigma \in \{-1, 1\}$, and $\theta_T = \sigma$, \mathcal{H}^N -a.e..

Proof. We start by proving the first item. The argument is analogous to the proof of the second item of Proposition 4.3. Fix $p \in B_s(x) \cap \mathcal{R}$ a regular point and define

$$E = \{q \in B_s(x) \cap \mathcal{R} \mid \text{there exists a continuous curve from } p \text{ to } q \text{ contained in } \mathcal{R}\}$$

Since by assumption \mathcal{R} is a.e.-connected, E is a subset of $B_s(x)$ of full \mathcal{H}^N -measure. For any $q \in E$ let $\gamma : [0, 1] \rightarrow X$ be a continuous curve from p to q contained in the regular set \mathcal{R} . Since $\gamma([0, 1])$ is compact and connected, we can form a finite chain of metric balls $\{B_{s_i}(x_i)\}_{i=1, \dots, n}$ with the following properties:

- (1) for each i , the radius $s_i > 0$ is small enough such that $\theta_S = k_i$ a.e. on $B_{s_i}(x_i)$ for some $k_i \in \mathbb{Z}$. This holds by Lemma 4.2;
- (2) $B_{s_i}(x_i) \cap B_{s_{i+1}}(x_{i+1}) \neq \emptyset$ for all $i = 1, \dots, n-1$;
- (3) $\gamma([0, 1]) \subset \bigcup_{i=1}^n B_{s_i}(x_i)$.

By property (2), consecutive balls have intersection of positive \mathcal{H}^N -measure; therefore $k_{i+1} = k_i$, for all i . Thus $\theta_S = k$ \mathcal{H}^N -a.e. on a neighbourhood of both p and q . Since the choice of such two points was arbitrary, we deduce that $\theta_S = k$ a.e. on $B_s(x)$

The second assertion follows by combining the first item with Proposition 4.3. \square

Remark 4.5. We note that a different way of proving the results in this subsection could have been to argue as in [40] where, by further assuming that $(X, \mathbf{d}, \mathcal{H}^N)$ is isometrically embedded in a w^* -separable dual space Z , the authors write $T \llcorner B_r(p)$ and $S \llcorner B_r(p)$ as

$$T \llcorner B_r(p)(g, \pi) = \int_{\text{set}(T) \cap B_r(p)} \theta_T(y) g(y) \langle \tau, \wedge_N d_y^{\text{set}(T)} \pi \rangle d\mathcal{H}^N(y),$$

for any $(g, \pi) \in \mathcal{D}^N(Z)$ and where τ is the orientation given by

$$\tau(x) = \frac{w d_a f_i(e_1) \wedge \dots \wedge w d_a f_i(e_n)}{J_n(w d_a f_i)} \in \wedge_n \text{Tan}^{(n)}(f_i(K_i), x),$$

for $x \in f_i(K_i)$, $a = f_i^{-1}(x)$ and e_1, \dots, e_n the canonical basis of \mathbb{R}^n , for charts given as explained below Definition 3.5. Furthermore, for \mathcal{H}^N -a.e. $y \in \text{set}(S) \cap B_r(p)$

$$\begin{aligned} S \llcorner B_r(p)(g, \pi) &= \int_{\text{set}(S) \cap B_r(p)} \theta_S(y) g(y) \langle \tau, \wedge_N d_y^{\text{set}(S)} \pi \rangle d\mathcal{H}^N(y) \\ &= \frac{\theta_S(y)}{\theta_T(y)} T \llcorner B_r(p)(g, \pi). \end{aligned}$$

We omit the details and refer to [40] for this approach.

4.2. Proof of Theorem 1.5 and Theorem 1.6. The next result is inspired by [40, Theorem 4.1, i)] and will be used to prove Theorem 1.5 at the end of the section.

Theorem 4.6. *Let $(X_i, \mathbf{d}_i, \mathcal{H}^N)$ be a sequence of non collapsed $\text{RCD}(K, N)$ spaces, for some $K \in \mathbb{R}$ and $N \in \mathbb{N}$, without boundary. Assume that*

- $\sup_i \text{diam}(X_i) < \infty$;
- $(X_i, \mathbf{d}_i) \rightarrow (X, \mathbf{d})$ in GH sense;
- Each X_i is oriented by an N -dimensional integral current $T_i \in \mathbf{I}_N(X_i)$, $\partial T_i = 0$, $\|T_i\| = \mathcal{H}^N$.

Then, either one of the following happens:

- Collapsing case, *i.e.* $\lim_{i \rightarrow \infty} \mathcal{H}^N(X_i) = 0$: the sequence T_i converges in the flat topology to the zero current $T = 0 \in \mathbf{I}_N(X)$;
- Non-collapsing case, *i.e.* $\lim_{i \rightarrow \infty} \mathcal{H}^N(X_i) > 0$: the limit $(X, \mathbf{d}, \mathcal{H}^N)$ is a non-collapsed RCD(K, N) space without boundary, the currents T_i subconverge in the flat topology to a non zero current $T \in \mathbf{I}_N(X)$, with $\partial T = 0$, $X = \text{set}(T)$, $\|T\| = \mathcal{H}^N$; *i.e.*, T is an orientation in the sense of currents for $(X, \mathbf{d}, \mathcal{H}^N)$.

Proof. Given that $\|T_i\| = \mathcal{H}^N$ and that $\sup_i \text{diam}(X_i) < \infty$, we can apply Bishop-Gromov's volume comparison (see [49, Thm 2.3]) to infer that

$$\sup_{i \in \mathbb{N}} \left\{ \|T_i\|(X_i) \right\} < \infty.$$

Thus, Wenger's compactness theorem 2.13 implies that there exists a subsequence of the (X_i, \mathbf{d}_i, T_i) , that we do not relabel, that converges in intrinsic flat sense to an N -dimensional integral current space (Y, \mathbf{d}_Y, T) .

If $\lim_{i \rightarrow \infty} \mathcal{H}^N(X_i) = 0$, since $\|T_i\| = \mathcal{H}^N$ and by the lower semi-continuity of the mass under flat convergence, we obtain that $T = 0$ and thus (Y, \mathbf{d}_Y, T) is the zero integral current space.

Otherwise, $\lim_{i \rightarrow \infty} \mathcal{H}^N(X_i) = \mathcal{H}^N(X) > 0$ and the sequence $(X_i, \mathbf{d}_i, \mathcal{H}^N)$ converges in mGH sense to $(X, \mathbf{d}, \mathcal{H}^N)$ (see [21]). By the stability of the RCD condition under mGH convergence (see for instance [28]), it follows that $(X, \mathbf{d}, \mathcal{H}^N)$ is an RCD(K, N) space. Moreover, since $\partial X_i = \emptyset$ then the limit space X has no boundary, thanks to [34, Thm. 5.1]. We will show that T is an orientation for $(X, \mathbf{d}, \mathcal{H}^N)$.

By Theorem 2.15, there exist a complete metric space W , and isometric embeddings

$$\varphi_i : X_i \rightarrow W \quad \forall i \in \mathbb{N},$$

and $\varphi : X \rightarrow W$, such that

$$\varphi_{i\#} T_i \rightarrow \varphi_{\#} T \quad \text{in flat sense} \quad (4.11)$$

$$\varphi_i(X_i) \rightarrow \varphi(X) \quad \text{in Hausdorff sense in } W \quad (4.12)$$

$$\varphi(Y) = \text{set}(\varphi_{\#}(T)) \subset \varphi(X). \quad (4.13)$$

By the definition of flat convergence, $\partial(\varphi_{i\#} T_i) = 0$ converges to $\partial(\varphi_{\#} T)$. Hence, $\partial(\varphi_{\#} T) = 0$. Since the pushforward and boundary operators commute, we have $\partial T = 0$.

In order to show that $\|T\| = \mathcal{H}^N$, we will first establish that the multiplicity function of T , θ_T , is the constant function either 1 or -1 on the regular set of X . Recall that by Corollary 1.3, for each i there exists $\sigma_i \in \{-1, 1\}$ so that

$$\theta_{T_i} \equiv \sigma_i \quad \text{on } X_i \cap \text{set}(T_i).$$

Passing to a subsequence, that we do not relabel, assume that the sequence σ_i is constant and equal to 1.

For any $p \in \mathcal{R} \subset X$ there exist $p_i \in \mathcal{R}_i \subset X_i$ such that $\varphi_i(p_i) \rightarrow \varphi(p)$ in W . Fix $\varepsilon > 0$. Combining Theorem 2.3 with a standard rescaling argument, we can assume that there exist functions

$$u_i : B_8(p_i) \subset X_i \rightarrow \mathbb{R}^N \quad \forall i \in \mathbb{N}, \quad u : B_8(p) \subset X \rightarrow \mathbb{R}^N$$

that are $(1 + \varepsilon)$ -Lipschitz on $B_1(p_i) \subset X_i$ and $B_1(p) \subset X$, respectively. We define in W the measure given by

$$\mathbf{m}_W = \mathcal{H}^N \llcorner \left(\varphi(B_8(p)) \cup \bigcup_{i \in \mathbb{N}} \varphi_i(B_8(p_i)) \right).$$

By the construction of the $\delta(N, \varepsilon)$ -splitting maps u_i and u , one can extend $\varphi_i \circ u_i$ and $\varphi \circ u$ to maps

$$\hat{u}_i, \hat{u} : W \rightarrow \mathbb{R}^N$$

such that each component is still $(1 + \varepsilon)$ -Lipschitz and

$$\hat{u}_i \rightarrow \hat{u} \quad \text{in } L^\infty(B_5(p)).$$

Now, by (4.11), there exists $r \in (4, 5)$ such that $T_i \llcorner B_r(p) \rightarrow T \llcorner B_r(p)$ in flat sense in W . Lemma 2.17 implies that

$$\begin{aligned} \|\deg(T_i \llcorner B_r(p), \hat{u}_i, \cdot) - \deg(T \llcorner B_r(p), \hat{u}, \cdot)\|_{L^1(\mathbb{R}^n)} &\leq (1 + \varepsilon)^N d_F^W(T_i \llcorner B_r(p), T \llcorner B_r(p)) \\ &+ 2(1 + \varepsilon)^{N-1} \sum_{j=1}^N \|\hat{u}_i^j - \hat{u}^j\|_\infty (\mathbb{M}(T \llcorner B_r(p)) + \mathbb{M}(\partial(T \llcorner B_r(p)))). \end{aligned}$$

Hence, we conclude that

$$\deg(T_i \llcorner B_r(p), \hat{u}_i, \cdot) \rightarrow \deg(T \llcorner B_r(p), \hat{u}, \cdot) \quad \mathcal{L}^N - \text{a.e. on } \mathbb{R}^N.$$

By (2.23), we get

$$\theta_{\varphi_i \sharp T_i} \rightarrow \theta_{\varphi \sharp T} \quad \text{in } L^1(B_r(p), \mathcal{H}^N) \text{ sense.}$$

Recalling that, by assumption, $\theta_{T_i} = 1$ \mathcal{H}^N -a.e. on $B_1(p_i)$, we infer that $\theta_T = 1$ \mathcal{H}^N -a.e. on a neighbourhood of $p \in \mathcal{R}$. Furthermore, since $\|T_i\| = \mathcal{H}^N$, the volume estimate (2.11) implies

$$\|T_i\|(B_r(p_i)) \leq (1 + \varepsilon_r) \omega_N r^N, \quad \text{for all } r \in (0, 1),$$

with $\varepsilon_r \rightarrow 0$ as $r \rightarrow 0$. The lower semi-continuity of the mass measure under flat convergence yields

$$\|T\|(B_r(p)) \leq (1 + \varepsilon_r) \omega_N r^N,$$

and thus

$$\Theta_*^N(\|T\|, p) := \liminf_{r \rightarrow 0} \frac{\|T\|(B_r(p))}{\omega_N r^N} \leq 1.$$

Since we have shown that $|\theta_T| = 1$ on \mathcal{R} , proceeding as in (4.9), for $p \in \mathcal{R}$ we get,

$$\|T\|(B_1(p)) \geq (1 + \varepsilon)^{-N} \omega_N (1 - 2\varepsilon)^N.$$

By Theorem 2.3-v), we get that

$$\Theta_*^N(\|T\|, p) \geq 1.$$

Thus, the regular set \mathcal{R} of X is contained in $Y = \text{set}(T)$, and $\Theta_*^N(\|T\|, \cdot) = 1$ on \mathcal{R} . We get $\|T\| \llcorner \mathcal{R} = \mathcal{H}^N \llcorner \mathcal{R}$. Recalling that the singular set of X , $\mathcal{S} = X \setminus \mathcal{R}$, satisfies $\mathcal{H}^N(\mathcal{S}) = 0$ and that $\|T\| \ll \mathcal{H}^N$ (see (2.19)), we conclude that $\|T\| = \mathcal{H}^N$ and $Y = \text{set}(T) = X$, concluding the proof. \square

We finally prove Theorem 1.5.

Proof of Theorem 1.5. Given that $\|T_i\| = \mathcal{H}^N$, we can apply Bishop-Gromov's volume comparison result to see that for all $r > 0$,

$$\sup_{i \in \mathbb{N}} \left\{ \|T_i\|(B_r(x_i)) \right\} < \infty. \quad (4.14)$$

Since $\partial T_i = 0$, it is trivial that $\sup_{i \in \mathbb{N}} \|\partial T_i\|(B_r(x_i)) < \infty$. Thus, by Theorem 2.21, there exist a subsequence of the T_i 's (that abusing notation we do not relabel), a

pointed complete metric space (W, \mathbf{d}_W, w_0) , a current $\tilde{T} \in \mathbf{I}_{\text{loc}, N}(W)$ and isometric embeddings

$$\varphi_i : X_i \rightarrow W \quad \forall i \in \mathbb{N},$$

such that $\varphi_{i\#}T_i \rightarrow \tilde{T}$ in the local flat sense and $\varphi_i(x_i) \rightarrow w_0$ in W .

If $\lim_{i \rightarrow \infty} \mathcal{H}^N(B_1(x_i)) = 0$, then $\lim_{i \rightarrow \infty} \mathcal{H}^N(X_i) = 0$ and the lower semi-continuity of the mass under flat convergence implies that $\tilde{T} = 0$. Otherwise, $\lim_{i \rightarrow \infty} \mathcal{H}^N(B_1(x_i)) = \mathcal{H}^N(B_1(x)) > 0$ and the sequence of pointed metric measured spaces $(X_i, \mathbf{d}_i, \mathcal{H}^N, x_i)$ converges in pmGH sense to $(X, \mathbf{d}, \mathcal{H}^N, x)$ (see [21]). By the stability of the RCD condition under pmGH convergence (see for instance [28]), it follows that $(X, \mathbf{d}, \mathcal{H}^N, x)$ is an RCD(K, N) space. In particular, (X, \mathbf{d}) is proper and by Theorem 2.21 there exists an isometric embedding

$$\psi : \text{spt}(\tilde{T}) \cup \{w_0\} \rightarrow X, \quad \psi(w_0) = x.$$

Denote by T the current $\psi_{\#}\tilde{T} \in \mathbf{I}_{\text{loc}, N}(X)$. We will show that T is an orientation for $(X, \mathbf{d}, \mathcal{H}^N)$.

Since the push-forward and boundary operators commute, we have $\varphi_{i\#}(\partial T_i) = 0$. By the convergence of $\varphi_{i\#}T_i \rightarrow \tilde{T}$ in the local flat sense, we obtain that $0 = \partial(\varphi_{i\#}T_i) \rightarrow \partial\tilde{T}$ in the local flat sense. Hence, $\partial\tilde{T} = 0$, which implies $\partial T = 0$.

Now recall that by Remark 2.22, there exists an increasing sequence of real numbers $(R_r)_{r=0}^{\infty}$, with $R_0 = 0$, such that

$$\tilde{T} = \sum_{r=1}^{\infty} \bar{T}_r \in \mathbf{I}_{\text{loc}, N}(W),$$

where each $\bar{T}_r \in \mathbf{I}_n(W)$ is supported in the annulus

$$\text{spt}(\bar{T}_r) = \{w \in W \mid R_{r-1} \leq \mathbf{d}_W(w_0, w) \leq R_r\},$$

and that for all $r \in \mathbb{N}$,

$$d_F^W(\varphi_{i\#}(T_i \llcorner B_{R_r}(x_i)), T^r) \rightarrow 0$$

where

$$T^r = \sum_{i=1}^r \bar{T}_i \in \mathbf{I}_n(W), \quad \text{spt}(T^r) \subset \bar{B}_{R_r}(w_0).$$

From the properties of ψ , $T \llcorner \psi(\text{spt}(T^r)) \subset \bar{B}_{R_r}(x)$. Hence, for each $r \in \mathbb{N}$, we can apply Theorem 2.15, taking $X_i = \bar{B}_{R_r}(x_i)$ and $T_i = T_i \llcorner \bar{B}_{R_r}(x_i)$ and conclude, as in the proof of Theorem 4.6, that $B_{R_r}(x) \subset \text{set}(T)$ and $\|T\| = \mathcal{H}^N$ on $B_{R_r}(x)$. \square

Proof of Theorem 1.6. The proof of Theorem 1.6 is analogous to the one of Theorem 1.5, thanks to [13, Thm. 5.11 and Rem. 5.12]. We briefly sketch the main adaptations.

Since each (M_i, g_i) is an orientable smooth N -dimensional Riemannian manifold without boundary, we can endow it with an orienting current $T_i \in \mathbf{I}_{\text{loc}, N}(M_i)$ with $\|T_i\| = \text{vol}_{g_i}$ and $\partial T_i = \emptyset$.

By [13, Thm. A], for each (M_i, g_i) there exists $h_i \in C^2(M_i)$ such that

- $0 \leq h_i \leq C(N)$;
- The weighted manifold $(M_i, e^{2h_i}g, e^{2h_i}\text{vol}_{g_i})$ is an RCD($-K/t_0, N'$) space, where $K > 0, t_0 > 0$ and $N' \geq N$ are uniform along the sequence.

Such two assertions combined with the Bishop-Gromov inequality on RCD($-K/t_0, N'$) spaces imply the mass bound (4.14).

Applying Theorem 2.21, there exist a (non relabeled) subsequence of the T_i 's, a pointed complete metric space (W, w_0) , a current $\tilde{T} \in \mathbf{I}_{\text{loc}, N}(W)$ and isometric embeddings

$$\varphi_i : X_i \rightarrow W \quad \forall i \in \mathbb{N},$$

such that $\varphi_{i\#}T_i \rightarrow \tilde{T}$ in the local flat sense and $\varphi_i(x_i) \rightarrow w_0$ in W . The volume non-collapsing assumption implies that $\lim_{i \rightarrow \infty} \mathcal{H}^N(B_1(x_i)) = \mathcal{H}^N(B_1(x)) > 0$ and that $(M_i, g_i, \text{vol}_{g_i}, x_i)$ converges in pmGH sense to $(X, \mathbf{d}, \mathcal{H}^N, x)$, we refer the reader to the proof of [13, Prop. 5.9]. By construction, $(X, \mathbf{d}, \mathcal{H}^N)$ is a strong Kato limit space, in particular it is proper. By Theorem 2.21 there exists an isometric embedding

$$\psi : \text{spt}(\tilde{T}) \cup \{w_0\} \rightarrow X, \quad \psi(w_0) = x.$$

Denote by T the current $\psi_{\#}\tilde{T} \in \mathbf{I}_{\text{loc}, N}(X)$. Now, repeating verbatim the last part in the proof of Theorem 1.5, one can show that $\partial T = 0$, $\|T\| = \mathcal{H}^N$ and $\text{set}(T) = X$, i.e. that T is an orientation for $(X, \mathbf{d}, \mathcal{H}^N)$. \square

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