TROPICAL OPTIMAL TRANSPORTATION AND LARGE DEVIATION PRINCIPLE

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Abstract. We consider the general tropical (max-plus) version of the classical Monge–Kantorovich optimal transportation problem. For this problem we show the existence of solutions, provide the formula for the optimal cost in terms of the data, and give explicit formulae for some solutions satisfying additional useful properties (e.g. maximal solutions). We further establish a connection between this problem and classical optimal transportation via the large deviation principle, which can be considered a version of the Maslov dequantization tailored for such a problem. Finally, we provide some explicit examples of solutions and study the metric properties of the optimal cost.

1. Introduction

The Monge–Kantorovich optimal mass transportation problem (see e.g. [4] for a comprehensive introduction to this classical subject) in its most general setting is that of finding the optimal plan of transportation of a given Borel measure μ over a metric space X to the given Borel measure ν over a metric space Y (both measures having finite total mass $\mu(X) = \nu(Y)$), given the cost $c(x, y)$ of transporting unit mass from point $x \in X$ to point $y \in Y$. Respectively, a transportation plan is a Borel measure π over $X \times Y$ with marginals μ and ν, i.e. $\pi(B_X \times Y) = \mu(B_X)$ for all Borel $B_X \subset X$ and $\pi(X \times B_Y) = \nu(B_Y)$ for all Borel $B_Y \subset Y$. The set of such plans being further denoted by $\Pi(\mu, \nu)$, and optimality is understood as minimization of the total transportation cost, that is, finding

$$\inf \left\{ \int_{X\times Y} c(x, y)\, d\pi(x, y) \colon \pi \in \Pi(\mu, \nu) \right\}.$$

In the current paper we consider a version of this problem in the realm of *idempotent analysis*, namely, analysis over the *tropical (max-plus) semiring* $\overline{\mathbb{R}}_- := \mathbb{R} \cup \{-\infty\}$ endowed with the operations

$$a \oplus b := \max\{a, b\}, \quad a \otimes b := a + b,$$

which substitute the usual addition and multiplication of real numbers respectively. In particular, the roles of 0 and 1 on the usual real line are played here by $-\infty$ and 0 respectively. Specifically, the value $-\infty$ is an identity with respect to $\oplus$, and 0 is

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an identity with respect to $\otimes$. Both operations are commutative, associative, and $a \otimes (b \oplus c) = a \otimes b \oplus a \otimes c$.

For a general overview of idempotent analysis we refer the reader to the classical book [3]. We only recall here that a *tropical (max-plus) measure* over a set X is just a function $I\colon X \to \overline{\mathbb{R}}_-$. The set of such measures we further denote by $\mathcal{M}^t(X)$, the integral over a set is substituted by a supremum, and in particular a tropical measure I is a *tropical probability measure* (i.e. the max-plus version of the usual probability measure), if $\sup_X I = 0$. The set of the latter being further denoted by $\mathcal{P}^t(X)$. Thus, given two tropical measures $I \in \mathcal{M}^t(X)$ and $J \in \mathcal{M}^t(Y)$, where X and Y are nonempty sets without any further structure, we may call a *tropical transportation plan* a function $Q\colon X \times Y \to \overline{\mathbb{R}}_-$ having marginals I and J, respectively, in the following max-plus sense:

$$\begin{aligned} &\sup_{y\in Y} Q(x,y) = I(x) \quad \text{for all } y \in Y,\\ &\sup_{x\in X} Q(x,y) = J(y) \quad \text{for all } x \in X. \end{aligned} \tag{1}$$

Clearly, for (1) to hold it is necessary that

$$\sup_{x\in X} I(x) = \sup_{y\in Y} J(y), \tag{2}$$

which is exactly the max-plus version of the equal total mass condition for classical measures. The set of tropical transportation plans with marginals (of course, in the sense of (1)) $I \in \mathcal{M}^t(X)$ and $J \in \mathcal{M}^t(Y)$, respectively, will be further denoted by $\Pi^t(I,J)$. Translating then the classical Monge–Kantorovich optimal transportation problem into the introduced above idempotent language yields the problem of minimizing, given $I \in \mathcal{M}^t(X)$ and $J \in \mathcal{M}^t(Y)$ satisfying (2), and the cost function $c\colon X \times Y \to \overline{\mathbb{R}}_-$, the supremal (rather than integral) functional F defined by the formula

$$F(Q) := \sup_{(x,y)\in X\times Y} (c(x,y) + Q(x,y)) \tag{3}$$

over $Q \in \Pi^t(I,J)$, referred to in the sequel as *tropical* (or *max-plus*) *optimal transportation problem*, that is,

$$\inf \left\{ F(Q)\colon Q \in \Pi^t(I,J) \right\}. \tag{4}$$

A possible "information-theoretic" interpretation is as follows. The elements of X are transmitters and those of Y receivers of information. Then $I(x)$ (resp. $J(y)$) is the bound on the amount of information the transmitter $x \in X$ may transmit (resp. the receiver $y \in Y$ may receive) at a time. The quantity $Q(x,y)$ for a $Q \in \Pi^t(I,J)$ says then how much information to send from a transmitter x to a receiver y, the condition (1) making this plan to conform with the capacities of each transmitter and receiver. The goal is to find a tropical transportation plan $Q \in \Pi^t(I,J)$ so as to minimize the maximum cost of information transmission between all transmitters and receivers, the cost for a single pair transmitter-receiver $(x,y) \in X \times Y$ being linear in $Q(x,y)$, i.e. given by the formula $g(x,y) + \gamma Q(x,y)$, where $\gamma > 0$ and $g(x,y)$ represents the fixed cost of using the transmission channel, $\gamma Q(x,y)$ being the part of the cost proportional to the amount of information transmitted. This leads to minimization of

$$Q \in \Pi^t(I,J) \mapsto \sup_{(x,y)\in X\times Y} (g(x,y) + \gamma Q(x,y)),$$

which, up to division by γ, is exactly the functional F.

The discrete version of the optimal tropical transportation problem (4), i.e. where X and Y are finite sets, has been studied in [1], and an explicit algorithm to solve such a problem is provided. In this paper we

(i) show that, as opposed to the classical Monge–Kantorovich problem, the tropical mass transportation problem always admits solutions without any assumptions on the data,
(ii) find some explicit optimal solutions to the latter problem with extra properties,
(iii) find an explicit formula for the optimal cost in terms of the data,
(iv) show a Γ-convergence type result which connects the optimal mass transportation problems and the classical Monge–Kantorovich problems through the large deviation principle,
(v) find some properties of the optimal cost (namely, when the latter is a distance between tropical measures).

2. Notation and preliminaries

For two real numbers a and b we denote $a \wedge b$ the minimum and $a \vee b$ the maximum between them. For a metric space X let $B_r(x)$ stand for the open ball of X of radius $r > 0$ centered at $x \in X$. We call the tropical characteristic function of a set $A \subset X$ the function

$$I_A(x) := \begin{cases} -\infty, & x \notin A \\ 0, & x \in A. \end{cases}$$

For the classical Monge–Kantorovich problem of optimal mass transportation between two finite Borel measures μ and ν of equal total mass over some metric space, with the transportation cost of unit mass being given by the function c we denote by

$$MK_c(\mu, \nu) := \inf \left\{ \int_{X \times Y} c(x, y)\, d\pi(x, y) : \pi \in \Pi(\mu, \nu) \right\}$$

the respective optimal cost (where X and Y stand for the supports of the measures μ and ν respectively). For the tropical mass transportation problem between two given tropical measures I and J satisfying (2) the respective optimal cost will be denoted by

$$\mathcal{J}_c(I, J) := \inf \left\{ F(Q) : Q \in \Pi^t(I, J) \right\}.$$

Note that $\Pi^t(I, J)$ is nonempty. Indeed, both $Q(x, y) := I(x) + J(y)$ ("direct product measure") and $Q(x, y) := I(x) \wedge J(y)$ (maximal measure) belong to $\Pi^t(I, J)$. Moreover, $\Pi^t(I, J)$ is closed under supremum, i.e. for any nonempty subset $\mathcal{Q} \subset \Pi^t(I, J)$ the point-wise supremum

$$Q^*(x, y) := \sup_{Q \in \mathcal{Q}} Q(x, y)$$

belongs to $\Pi^t(I, J)$.

3. Basic properties

3.1. **Existence of solutions.** The following general existence theorem holds true. Note that as opposed to the existence theorem for the Monge–Kantorovich optimal mass transportation problem it requires no continuity property of the cost function.

Theorem 1. *If $\mathcal{J}_c(I,J) > -\infty$ (which is satisfied, for instance, if the cost function c is bounded from below), then the tropical mass transportation problem admits a solution.*

Proof. If $\mathcal{J}_c(I,J) = +\infty$, the claim is trivial. Otherwise, under the assumption of the theorem being proven $\mathcal{J}_c(I,J)$ is a finite number, and we let $\{Q_n\} \subset \Pi^t(I,J)$ be a sequence of almost minimizers of the optimal tropical mass transportation problem, that is,

$$\mathcal{J}_c(I,J) \le \sup_{(x,y)\in X\times Y} (c(x,y) + Q_n(x,y)) \le \mathcal{J}_c(I,J) + \frac{1}{n}.$$

Define

$$Q(x,y) := \sup_n \left(Q_n(x,y) - \frac{1}{n} \right).$$

It is easy to see that for any $x \in X$

$$\sup_{y\in Y} Q(x,y) = \sup_n \sup_{y\in Y} \left(Q_n(x,y) - \frac{1}{n} \right) = \sup_n \left(I(x) - \frac{1}{n} \right) = I(x),$$

and analogously, $\sup_{x\in X} Q(x,y) = J(y)$, i.e. $Q \in \Pi^t(I,J)$. Since

$$\begin{aligned}
\sup_{(x,y)\in X\times Y} (c(x,y) + Q(x,y)) &= \sup \left(c(x,y) + \sup_n \left(Q_n(x,y) - \frac{1}{n} \right) \right) \\
&= \sup \sup_n \left(c(x,y) + Q_n(x,y) - \frac{1}{n} \right) \\
&= \sup_n \left(\sup_{(x,y)\in X\times Y} (c(x,y) + Q_n(x,y)) - \frac{1}{n} \right) \\
&= \mathcal{J}_c(I,J),
\end{aligned}$$

we conclude that Q is a solution to (4). The fact that boundedness from below of c implies $\mathcal{J}_c(I,J) > -\infty$ follows from the estimate

$$F(Q) \ge \inf_{(x,y)\in X\times Y} c(x,y) + \sup_{(x,y)\in X\times Y} Q(x,y) = \inf_{(x,y)\in X\times Y} c(x,y),$$

concluding the proof. □

It is important to observe that the assumption $\mathcal{J}_c(I,J) > -\infty$ in the above existence Theorem 1 is essential as the following example shows.

Example 1. Let $X = Y := \mathbb{N}$, $c(x,y) := -(x+y)$, $I = J = 0$ (i.e. $I(x) = J(y) = 0$ for all $x \in \mathbb{N}$, $y \in \mathbb{N}$). Then the tropical plans

$$Q_n(x,y) := \begin{cases} 0, & x = n \text{ or } y = n, \\ -\infty, & \text{otherwise}, \end{cases}$$

of course, belong to $\Pi^t(I,J)$, and $F(Q_n) = -n$, so that we get $\mathcal{J}_c(I,J) = -\infty$ since n can be chosen arbitrarily. On the other hand, if $Q \in \Pi^t(I,J)$ and $(x,y) \in X \times Y$ be such that $Q(x,y)$ is finite, then we have

$$F(Q) \ge c(x,y) + Q(x,y) = Q(x,y) - (x+y) > -\infty,$$

and hence among the admissible tropical transportation plans no one is optimal.

3.2. **Optimal tropical transportation cost.** Now we are going to obtain an explicit formula for $\mathcal{J}_c(I,J)$. Let us define auxiliary sets

$$\begin{aligned}\Pi^t(I,\leq J) &:= \left\{Q\in\mathcal{M}^t(X\times Y)\colon \sup_{y\in Y} Q(x,y)=I(x),\ \sup_{x\in X} Q(x,y)\leq J(y)\right\},\\ \Pi^t(\leq I,J) &:= \left\{Q\in\mathcal{M}^t(X\times Y)\colon \sup_{y\in Y} Q(x,y)\leq I(x),\ \sup_{x\in X} Q(x,y)= J(y)\right\}.\end{aligned} \tag{5}$$

First, we show that the tropical transportation problem can be decomposed into two "semimarginal" problems as follows.

Proposition 2. *If $Q_1\in\Pi^t(I,\leq J)$ and $Q_2\in\Pi^t(\leq I,J)$, then $Q_1\vee Q_2\in\Pi^t(I,J)$. Moreover, if $\mathcal{J}_c(I,J)>-\infty$, then there exist $Q_1\in\Pi^t(I,\leq J)$ minimizing the functional F defined by (3) over $\Pi^t(I,\leq J)$ and $Q_2\in\Pi^t(\leq I,J)$ minimizing F over $\Pi^t(\leq I,J)$. For every such Q_1 and Q_2 one has that $Q:=Q_1\vee Q_2$ minimizes F over $\Pi^t(I,J)$ i.e. is an optimal tropical transportation plan, and $F(Q)=F(Q_1)\vee F(Q_2)$.*

Remark 3. The above Proposition 2 is only specific to the tropical optimal mass transportation problem. In the classical Monge–Kantorovich problem, one cannot decrease only one of the marginals because doing so this would violate the balance of their total masses. In the tropical setting this is possible once the new (decreased) marginal has the same supremum, (e.g. is still a tropical probability measure like the old one).

Proof. The existence of Q_1 and Q_2 can be proven by exactly the same reasoning as in the proof of Theorem 1. If $Q=Q_1\vee Q_2$, then $Q\in\Pi^t(I,J)$ and $F(Q)=F(Q_1)\vee F(Q_2)$. If Q is not an optimal transportation plan, then there is some $\widetilde{Q}\in\Pi^t(I,J)$ such that $F(\widetilde{Q})<F(Q)$. Since

$$\Pi^t(I,J)=\Pi^t(\leq I,J)\cap\Pi^t(I,\leq J),$$

then if $F(Q)=F(Q_1)$, we would have $F(\widetilde{Q})<F(Q_1)$ contradicting the optimality of Q_1, and if $F(Q)=F(Q_2)$, we would have $F(\widetilde{Q})<F(Q_2)$ contradicting the optimality of Q_2, hence proving the last claim. □

Now we are ready to prove the main result of this section, i.e. to obtain an explicit formula for $\mathcal{J}_c(I,J)$.

Theorem 4. *One has*

$$\begin{aligned}\mathcal{J}_c(I,J)=\sup_{x\in X}&\left(I(x)+\sup_{\varepsilon>0}\inf\left\{c(x,y)\colon y\in Y,\ J(y)\geq I(x)-\varepsilon\right\}\right)\\ &\vee\sup_{y\in Y}\left(J(y)+\sup_{\varepsilon>0}\inf\left\{c(x,y)\colon x\in X,\ I(x)\geq J(y)-\varepsilon\right\}\right).\end{aligned} \tag{6}$$

Proof. We split the proof into four steps.

Step 1. We first show that

$$\sup_{y\in Y}(Q_1(x,y)+c(x,y))\geq I(x)+\sup_{\varepsilon>0}\inf\left\{c(x,y)\colon y\in Y,\ J(y)\geq I(x)-\varepsilon\right\} \tag{7}$$

for all $x\in X$ and $Q_1\in\Pi^t(I,\leq J)$, i.e. satisfying

$$Q_1(x,y)\leq J(y)\quad\text{and}\ \sup_{y\in Y}Q_1(x,y)=I(x)\quad\text{for all } x\in X.$$

Clearly, it suffices to prove (7) for all $x \in X$ such that $I(x) > -\infty$. For every $\varepsilon > 0$ define

$$Q_1^\varepsilon(x,y) := \begin{cases} Q_1(x,y), & J(y) \geq I(x) - \varepsilon, \\ -\infty, & J(y) < I(x) - \varepsilon. \end{cases}$$

Clearly, $Q_1^\varepsilon \leq Q_1$, and

$$\begin{aligned}\sup_{y\in Y} Q_1^\varepsilon(x,y) &\geq \sup\{Q_1^\varepsilon(x,y)\colon y \in Y,\ Q_1(x,y) \geq I(x) - \varepsilon\} \\ &= \sup\{Q_1(x,y)\colon y \in Y,\ Q_1(x,y) \geq I(x) - \varepsilon\} = I(x)\end{aligned}$$

(so that, in particular, the inequality above is an equality). Thus

$$\begin{aligned}\sup_{y\in Y}(Q_1^\varepsilon(x,y) + c(x,y)) &\geq \sup_{y\in Y} Q_1^\varepsilon(x,y) + \inf\{c(x,y)\colon y \in Y,\ J(y) \geq I(x) - \varepsilon\} \\ &= I(x) + \inf\{c(x,y)\colon y \in Y,\ J(y) \geq I(x) - \varepsilon\},\end{aligned}$$

which implies

$$\begin{aligned}\sup_{y\in Y}(Q_1(x,y) + c(x,y)) &\geq \sup_{\varepsilon>0}\sup_{y\in Y}(Q_1^\varepsilon(x,y) + c(x,y)) \\ &\geq I(x) + \sup_{\varepsilon>0}\inf\{c(x,y)\colon y \in Y,\ J(y) \geq I(x) - \varepsilon\}.\end{aligned}$$

proving (7).

STEP 2. We claim that

$$F(Q_1) = \sup_{x\in X}\left(I(x) + \sup_{\varepsilon>0}\inf\{c(x,y)\colon y \in Y,\ J(y) \geq I(x) - \varepsilon\}\right) \tag{8}$$

for every $Q_1 \in \Pi^t(I, \leq J)$ minimizing the functional F over $\Pi^t(I, \leq J)$, if the right-hand side of (8) is not $-\infty$. Consider to this aim an arbitrary $x \in X$ for which the right-hand side of (7) is not $-\infty$, and an arbitrary $\varepsilon > 0$. Set then

$$\begin{aligned} u_x^\varepsilon &:= \inf\{c(x,y)\colon y \in Y, J(y) \geq I(x) - \varepsilon\}, \\ A_x^\varepsilon &:= \{y \in Y\colon J(y) \geq I(x) - \varepsilon, c(x,y) \leq u_x^\varepsilon + \varepsilon\}.\end{aligned}$$

Since $\sup_{y\in Y} J(y) = 0 \geq I(x)$, then $\{y \in Y\colon J(y) \geq I(x) - \varepsilon\} \neq \emptyset$, so that u_x^ε is well defined. Moreover, in view of the assumption on $x \in X$ we have $u_x^\varepsilon > -\infty$, and hence $A_x^\varepsilon \neq \emptyset$. Now, define

$$Q_1(x,y) := \sup_{\varepsilon>0}\begin{cases} I(x) - \varepsilon, & y \in A_x^\varepsilon, \\ -\infty, & \text{otherwise.}\end{cases}$$

Clearly, $Q_1(x,y) \leq J(y)$, $\sup_{y\in Y} Q_1(x,y) = I(x)$, and

$$\begin{aligned}\sup_{y\in Y}(Q_1(x,y) + c(x,y)) &= \sup_{\varepsilon>0}\sup_{y\in A_x^\varepsilon}(I(x) - \varepsilon + c(x,y)) \\ &\leq \sup_{\varepsilon>0}(I(x) + u_x^\varepsilon) \\ &= I(x) + \sup_{\varepsilon>0}\inf\{c(x,y)\colon y \in Y,\ J(y) \geq I(x) - \varepsilon\}.\end{aligned}$$

Together with (7), this yields the optimality of Q_1, which also implies (8) both for this particular plan Q_1 and for every $Q_1 \in \Pi^t(I, \leq J)$ minimizing the functional F over $\Pi^t(I, \leq J)$.

STEP 3. In a completely symmetric way one proves

$$F(Q_2) = \sup_{y\in Y}\left(J(y) + \sup_{\varepsilon>0}\inf\{c(x,y)\colon x \in X,\ I(x) \geq J(y) - \varepsilon\}\right) \tag{9}$$

for every $Q_2 \in \Pi^t(\leq I, J)$ minimizing F over $\Pi^t(\leq I, J)$, if the right-hand side of (9) is not $-\infty$. By Proposition 2, one has $\mathcal{J}_c(I, J) = F(Q_1) \vee F(Q_2)$, where $Q_1 \in \Pi^t(I, \leq J)$ minimizes F over $\Pi^t(I, \leq J)$ and $Q_2 \in \Pi^t(\leq I, J)$ minimizes F over $\Pi^t(\leq I, J)$. This, combined with (8) and (9), concludes the proof of the Theorem for the case the right-hand side of (6) is not $-\infty$, since the latter occurs exactly when either the right-hand side of (8) or that of (9) is not $-\infty$.

STEP 4. It remains to consider the case when the right-hand side of (6) is $-\infty$, that is, the right-hand sides of both (8) and (9) is $-\infty$. Consider an $x \in X$ such that $I(x) > -\infty$. Since the right-hand side of (9) is $-\infty$, then

$$\inf \{c(x, y) \colon y \in Y, J(y) \geq I(x) - \varepsilon\} = -\infty.$$

We define then

$$A_x^{n,\varepsilon} \coloneqq \{y \in Y \colon J(y) \geq I(x) - \varepsilon, c(x, y) \leq -n + \varepsilon\}.$$

for all $\varepsilon > 0$, $n \in \mathbb{N}$, and

$$Q_1^n(x, y) \coloneqq \sup_{\varepsilon > 0} \begin{cases} I(x) - \varepsilon, & y \in A_x^{n,\varepsilon}, \\ -\infty, & \text{otherwise.} \end{cases}$$

Clearly, $Q_1^n(x, y) \leq J(y)$, $\sup_{y \in Y} Q_1^n(x, y) = I(x)$, that is, $Q_1^n \in \Pi^t(I, \leq J)$ for all $n \in \mathbb{N}$, and

$$\begin{aligned} \sup_{y \in Y}(Q_1^n(x, y) + c(x, y)) &= \sup_{\varepsilon > 0} \sup_{y \in A_x^{m,\varepsilon}} (I(x) - \varepsilon + c(x, y)) \\ &\leq \sup_{\varepsilon > 0} (I(x) - n) = I(x) - n, \end{aligned}$$

which implies

$$F(Q_1^n) = \sup_{(x,y) \in X \times Y} (Q_1^n(x, y) + c(x, y)) \leq \sup_{x \in X} I(x) - n = -n.$$

Symmetrically, one finds an $y \in Y$ with $J(y) > -\infty$ and a sequence $Q_1^n \in \Pi^t(\leq I, J)$ such that $F(Q_2^n) \leq -n$. But then

$$F(Q_1^n \vee Q_2^n) = F(Q_1^n) \vee F(Q_2^n) \leq -n,$$

and since $Q_1^n \vee Q_2^n \in \Pi^t(I, J)$, we get that $\mathcal{J}_c(I, J) = -\infty$, concluding the proof. $\square$

Remark 5. It is worth observing that for the particular case of a tropical optimal transportation cost between the tropical characteristic functions of two sets $A \subset X$ and $B \subset Y$ respectively we get from Theorem 4 the formula

$$\mathcal{J}_c(I_A, I_B) = \sup_{x \in A} \inf_{y \in B} c(x, y) \bigvee \sup_{y \in B} \inf_{x \in A} c(x, y). \tag{10}$$

In particular, if c is a distance on $X = Y$ and A and B are compact, this is the classical Hausdorff distance between A and B.

The next corollary simplifies a bit the formula for the cost given by Theorem 4 in the case of tropical measures with compact superlevel sets (in particular, such tropical measures are u.s.c. functions) and an l.s.c. (or even just l.s.c. separately in each variable) cost function.

Corollary 6. *Let there exist topologies in X and in Y such that the cost function c is l.s.c. separately in each variable, i.e. $c(x,\cdot)$ and $c(\cdot,y)$ are l.s.c. for all $x\in X$ and $y\in Y$, and for any $a\le 0$ the superlevel sets*

$$\{x\in X : I(x)\ge a\},\quad \{y\in Y : J(y)\ge a\}$$

are compact. Then

$$\begin{aligned}\mathcal{J}_c(I,J) = \sup_{x\in X}&\big(I(x)+\inf\big\{c(x,y): y\in Y,\ J(y)\ge I(x)\big\}\big)\\ &\vee \sup_{y\in Y}\big(J(y)+\inf\big\{c(x,y): x\in X,\ I(x)\ge J(y)\big\}\big).\end{aligned}$$

Proof. Fix an arbitrary $x\in X$. We are going to show that

(11)
$$\sup_{\varepsilon>0}\inf\left\{c(x,y)\colon y\in Y,\ J(y)\ge I(x)-\varepsilon\right\} = \inf\left\{c(x,y)\colon y\in Y,\ J(y)\ge I(x)\right\}.$$

To this aim take an arbitrary monotone sequence $\varepsilon_n\to 0$. Clearly,

$$K\coloneqq\{y\in Y : J(y)\ge I(x)\} = \bigcap_{n\in\mathbb{N}} K_n,\quad K_n\coloneqq\{y\in Y : J(y)\ge I(x)-\varepsilon_n\}.$$

Fix $\delta>0$. Since $c(x,\cdot)$ is l.s.c., there is an open set $U\supset K$ such that

$$\inf_{y\in U} c(x,y)\ge \inf_{y\in K} c(x,y)-\delta.$$

Now notice that

$$K_1\subset U\cup\bigcup_{n\in\mathbb{N}}(Y\setminus K_n) = Y,$$

thus, due to the compactness of K_1, there are indices $n_1<n_2<\cdots<n_k$ such that

$$K_1\subset U\cup(Y\setminus K_{n_1})\cup\cdots\cup(Y\setminus K_{n_k}) = U\cup(Y\setminus K_{n_k}).$$

Therefore, for any $n\ge n_k$ $K_n\subset K_{n_k}\subset U$, hence

$$\lim_{n\to\infty}\inf_{y\in K_n} c(x,y)\ge \inf_{y\in U} c(x,y)\ge \inf_{y\in K} c(x,y)-\delta.$$

Since

$$\sup_{\varepsilon>0}\inf\left\{c(x,y)\colon y\in Y,\ J(y)\ge I(x)-\varepsilon\right\} = \lim_{n\to\infty}\inf_{y\in K_n} c(x,y),$$

we get (11).

Symmetrically, we prove

(12)
$$\sup_{\varepsilon>0}\inf\left\{c(x,y)\colon x\in X,\ I(x)\ge J(y)-\varepsilon\right\} = \inf\left\{c(x,y)\colon x\in X,\ I(x)\ge J(y)\right\}.$$

for every $y\in Y$. Then (11) and (12) together give the desired statement. ☐

3.3. **Special optimal tropical transportation plans.** The following results give explicit formulae for some particular optimal tropical transportation plans.

Theorem 7. *If $\mathcal{J}_c(I,J)>-\infty$, then*

$$Q^*(x,y)\coloneqq I(x)\wedge J(y)\wedge(\mathcal{J}_c(I,J)-c(x,y))$$

is an optimal tropical transportation plan which is maximum among all the optimal tropical transportation plans in the sense that

$$Q(x,y)\le Q^*(x,y) \tag{13}$$

for every optimal tropical transportation plan Q and for all $(x,y)\in X\times Y$.

Proof. For an arbitrary $Q \in \Pi^t(I,J)$ and $(x,y) \in X \times Y$ one has

$$Q(x,y) \leq I(x) \quad \text{and} \quad Q(x,y) \leq J(y). \tag{14}$$

Further, if Q is optimal, then

$$\sup_{(x,y)\in X\times Y} (Q(x,y) + c(x,y)) = \mathcal{J}_c(I,J),$$

and hence $Q(x,y) + c(x,y) \leq \mathcal{J}_c(I,J)$, which implies

$$Q(x,y) \leq \mathcal{J}_c(I,J) - c(x,y). \tag{15}$$

The relations (14) and (15) together give (13).

We show now that Q^* is optimal. To this aim we verify first that $Q^* \in \Pi^t(I,J)$. From Theorem 4 we get that when $J(y) \geq I(x)$, then

$$\mathcal{J}_c(I,J) \geq I(x) + \inf\left\{c(x,y)\colon y \in Y,\ J(y) \geq I(x) - \varepsilon\right\}.$$

Hence there is a sequence $y_n \in Y$, $y_n = y_n(x)$, such that

$$\mathcal{J}_c(I,J) \geq I(x) + c(x,y_n) - \frac{1}{n},$$

and therefore

$$\sup_{y\in Y} (\mathcal{J}_c(I,J) - c(x,y)) \geq \mathcal{J}_c(I,J) - c(x,y_n) \geq I(x) - \frac{1}{n}.$$

Passing to the limit as $n \to \infty$ in the above estimate we get

$$\sup_{y\in Y} (\mathcal{J}_c(I,J) - c(x,y)) \geq I(x). \tag{16}$$

Further, from

$$\sup_{x\in X} I(x) = \sup_{y\in Y} J(y)$$

it follows that

$$\sup_{y\in Y} J(y) \geq I(x). \tag{17}$$

Combining (16) and (17) we get $\sup_{y\in Y} Q^*(x,y) = I(x)$, and in a completely symmetric way one proves $\sup_{x\in X} Q^*(x,y) = J(y)$, so that $Q^* \in \Pi^t(I,J)$ as claimed. To finish the proof it remains thus to show the optimality of Q^*. The latter follows from the fact that $Q^*(x,y) \leq \mathcal{J}_c(I,J) - c(x,y)$, hence

$$F(Q^*) = \sup_{(x,y)\in X\times Y} (Q^*(x,y) + c(x,y)) \leq \mathcal{J}_c(I,J)$$

showing the claim and hence concluding the proof. □

To formulate the next result we call, following [1], a tropical transportation plan $Q \in \Pi^t(I,J)$ is *reduced*, if $Q(x,y) = -\infty$ unless either $Q(x,y) = I(x)$ (i.e. $Q(x,y)$ is maximum over all $y \in Y$) or $Q(x,y) = J(y)$ (i.e. $Q(x,y)$ is maximum over all $x \in X$). Denote by $\Pi^t_R(I,J)$ the set of all reduced tropical transportation plans in $\Pi^t(I,J)$. In [1], the authors prove that among optimal tropical transportation plans for a discrete problem (i.e. with both X and Y finite) there always are reduced ones. The following theorem shows that this in fact is true in a much more general situation.

Theorem 8. *Let I and J have compact superlevel sets $\{x \in X\colon I(x) \geq \alpha\}$ and $\{y \in Y\colon J(y) \geq \alpha\}$ for all $\alpha \in \mathbb{R}$, hence in particular, I and J are u.s.c., and let c be l.s.c. (the assumption on I and J is automatically satisfied e.g. when they are both u.s.c. and X and Y are compact). If $\mathcal{J}_c(I,J) > -\infty$, then*

$$Q_*(x,y) := \begin{cases} I(x) \wedge J(y), & \mathcal{J}_c(I,J) - c(x,y) \geq I(x) \wedge J(y), \\ -\infty, & \textit{otherwise}, \end{cases}$$

is a reduced optimal tropical transportation plan, and hence, in particular,

$$\min_{Q \in \Pi^t(I,J)} F(Q) = \min_{Q \in \Pi^t_R(I,J)} F(Q).$$

Proof. The proof is divided into two steps.

Step 1. Fix an arbitrary $\varepsilon > 0$. Let

$$Q_\varepsilon(x,y) := \begin{cases} I(x) \wedge J(y), & \mathcal{J}_c(I,J) - c(x,y) \geq I(x) \wedge J(y) - \varepsilon, \\ -\infty, & \text{otherwise}. \end{cases}$$

We claim that $Q_\varepsilon \in \Pi^t(I,J)$.

For the sake of brevity we denote

$$D_\varepsilon := \{(x,y) \in X \times Y : \mathcal{J}_c(I,J) - c(x,y) \geq I(x) \wedge J(y) - \varepsilon\}.$$

We first show

$$\sup_{y \in Y} Q^*(x,y) = \sup_{y \in Y\colon (x,y) \in D_\varepsilon} Q^*(x,y) \tag{18}$$

for all $x \in X$, where Q^* is the maximum optimal tropical transportation plan defined in Theorem 7. In fact, denoting

$$D_\varepsilon(x) := \{y \in Y\colon (x,y) \in D_\varepsilon\},$$

we get that

$$I(x) = \sup_{y \in Y} Q^*(x,y) = \sup_{y \in D_\varepsilon(x)} Q^*(x,y) \bigvee \sup_{y \in D_\varepsilon(x)^c} Q^*(x,y).$$

Since

$$\sup_{y \in D_\varepsilon(x)^c} Q^*(x,y) \leq \sup_{y \in D_\varepsilon(x)^c} (\mathcal{J}_c(I,J) - c(x,y)) \leq \sup_{y \in D_\varepsilon(x)^c} (I(x) \wedge J(y) - \varepsilon) \leq I(x) - \varepsilon.$$

we get

$$\sup_{y \in D_\varepsilon(x)} Q^*(x,y) = I(x) = \sup_{y \in Y} Q^*(x,y).$$

In a completely symmetric way one shows

$$\sup_{x \in X} Q^*(x,y) = \sup_{x \in X\colon (x,y) \in D_\varepsilon} Q^*(x,y). \tag{19}$$

Since $Q_\varepsilon \geq Q^*$ on D_ε by construction,

$$\begin{aligned} \sup_{y \in Y} Q_\varepsilon(x,y) &\geq \sup_{y \in D_\varepsilon(x)} Q^*(x,y) \\ &= \sup_{y \in Y} Q^*(x,y) \quad \text{by (18)} \\ &= I(x), \end{aligned}$$

From $Q_\varepsilon(x,y) \leq I(x) \wedge J(y)$ we get $\sup_{y \in Y} Q_\varepsilon(x,y) = I(x)$. Similarly,

$$\sup_{x \in X} Q_\varepsilon(x,y) = J(y),$$

which means $Q_\varepsilon \in \Pi^t(I,J)$ as claimed.

STEP 2. Now, we prove the claim of the theorem, i.e. that $Q_* \in \Pi^t(I,J)$ and is optimal. Consider an arbitrary $x \in X$ with $I(x) > -\infty$ and the sequence of tropical transport plans $\{Q_{1/k}\}$. Since $I(x) \le 0 = \sup_{y\in Y} J(y)$, by the assumption on J we get that the superlevel set

$$G_{1/k}(x) \coloneqq \left\{ y \in Y \colon J(y) \ge I(x) - \frac{1}{k} \right\}$$

is non-empty and compact.

We claim that $G_{1/k}(x) \cap D_{1/k}(x) \neq \emptyset$. In fact, otherwise

$$\begin{aligned} I(x) &= \sup_{y\in Y} Q_{1/k}(x,y) = \sup_{y\in D_{1/k}(x)} \big(I(x) \wedge J(y)\big) \\ &= \sup_{y\in D_{1/k}(x)\setminus G_{1/k}(x)} \big(I(x) \wedge J(y)\big) \le I(x) - \frac{1}{k}, \end{aligned}$$

a contradiction.

Now we observe that

$$\begin{aligned} &G_{1/k}(x) \cap D_{1/k}(x) \subset \\ &\qquad H_{1/k}(x) \coloneqq \left\{ y \in Y : J(y) \ge I(x) - \frac{1}{k},\ \mathcal{J}_c(I,J) - c(x,y) \ge I(x) - \frac{2}{k} \right\}. \end{aligned}$$

Thus, $H_{1/k}(x)$ is not empty. Moreover, since c is l.s.c., then $H_{1/k}(x)$ is also compact. Therefore, the intersection

$$\begin{aligned} H(x) \coloneqq \bigcap_k H_{1/k}(x) &= \{y \in Y \colon J(y) \ge I(x),\ \mathcal{J}_c(I,J) - c(x,y) \ge I(x)\} \\ &= \{y \in Y \colon J(y) \ge I(x),\ \mathcal{J}_c(I,J) - c(x,y) \ge I(x) \wedge J(y)\} \end{aligned}$$

is a non-empty compact set as well. Then

$$\sup_{y\in Y} Q_*(x,y) = \sup_{y\in D_0(x)} (I(x) \wedge J(y)) \ge \sup_{y\in H(x)} (I(x) \wedge J(y)) = I(x).$$

Symmetrically, we get

$$\sup_{x\in X} Q_*(x,y) \ge J(y).$$

Since $Q_*(x,y) \le Q^*(x,y)$ for all $(x,y) \in X \times Y$, and $Q^* \in \Pi^t(I,J)$, we get $Q_* \in \Pi^t(I,J)$. But this also gives $F(Q_*) \le F(Q^*)$, concluding the proof. □

4. Examples

We provide here a couple of examples of optimal tropical transportation costs for particular marginal tropical measures.

Example 2. Let I and J be "max-plus delta-measures", i.e.

$$I(x) = \begin{cases} 0, & x = x_0, \\ -\infty, & x \neq x_0, \end{cases} \qquad J(y) = \begin{cases} 0, & y = y_0, \\ -\infty, & y \neq y_0 \end{cases}$$

for some $x_0 \in X$ and $y_0 \in Y$. Then

$$\sup_{\varepsilon>0} \inf \{c(x_0,y) \colon y \in Y,\ J(y) \ge I(x_0) - \varepsilon\} = c(x_0,y_0),$$

hence

$$\sup_{x\in X}\left(I(x)+\sup_{\varepsilon>0}\inf\{c(x,y)\colon y\in Y,\ J(y)\geq I(x)-\varepsilon\}\right)=c(x_0,y_0).$$

Repeating the same lines for x and y swapped, we conclude that in this case $\mathcal{J}_c(I,J)=c(x_0,y_0)$, as expected.

Example 3. Let $X=Y=\mathbb{R}$, $c(x,y)=(x-y)^2$, and $I(x)=-a^2x^2$, $J(y)=-b^2(y-y_0)^2$ for some $a,b>0$ and $y_0\in\mathbb{R}$. Clearly, without loss of generality we can assume that $y_0\geq 0$. Then for any fixed $x\in\mathbb{R}$

$$\begin{aligned}\inf\{c(x,y)\colon y\in Y,\ J(y)\geq I(x)\}&=\inf\left\{(x-y)^2\colon y\in\mathbb{R},\ b^2(y-y_0)^2\leq a^2x^2\right\}\\&=\inf\left\{(x-y)^2\colon y\in\mathbb{R},\ |y-y_0|\leq\frac{a}{b}|x|\right\}\\&=\begin{cases}0, & |x-y_0|\leq\frac{a}{b}|x|,\\ \left(|x-y_0|-\frac{a}{b}|x|\right)^2, & |x-y_0|>\frac{a}{b}|x|\end{cases}\\&=\left(|x-y_0|-\frac{a}{b}|x|\right)_+^2.\end{aligned}$$

Thus

$$\begin{aligned}\sup_{x\in X}\big(I(x)+\inf\{c(x,y)\colon y\in Y,\ J(y)\geq I(x)\}\big)&=\sup_{x\in\mathbb{R}}\left(|x-y_0|-\frac{a}{b}|x|\right)_+^2-a^2x^2\\&=\sup_{x\leq 0}\left(y_0-x+\frac{a}{b}x\right)_+^2-a^2x^2\\&=\sup_{z\geq 0}\left(y_0+\frac{b-a}{b}z\right)_+^2-a^2z^2.\end{aligned}$$

Assume that $y_0>0$. First, we consider the case $a\geq b$: $\left(y_0+\frac{b-a}{b}z\right)_+$ is decreasing in z, thus

$$\sup_{z\geq 0}\left(y_0+\frac{b-a}{b}z\right)_+^2-a^2z^2=y_0^2.$$

Now suppose $a<b$:

$$\begin{aligned}\sup_{z\geq 0}\left(y_0+\frac{b-a}{b}z\right)_+^2-a^2z^2&=\sup_{z\geq 0}\left(\left(\frac{b-a}{b}\right)^2-a^2\right)z^2+2\frac{b-a}{b}y_0z+y_0^2\\&=\begin{cases}\frac{(ab)^2}{(ab)^2-(a-b)^2}y_0^2, & b-a<ab,\\ +\infty, & b-a\geq ab.\end{cases}\end{aligned}$$

Clearly, by the symmetry one can obtain that

$$\begin{aligned}\sup_{y\in Y}\big(J(y)+\inf\{c(x,y)\colon x\in X,\ I(x)\geq J(y)\}\big)&=\sup_{z\geq 0}\left(y_0+\frac{a-b}{a}z\right)_+^2-b^2z^2\\&=\begin{cases}y_0^2, & a-b\leq 0,\\ \frac{(ab)^2}{(ab)^2-(a-b)^2}y_0^2, & 0<a-b<ab,\\ +\infty, & a-b\geq ab.\end{cases}\end{aligned}$$

Therefore,

$$\mathcal{J}_c(I,J)=\begin{cases}\frac{(ab)^2}{(ab)^2-(a-b)^2}y_0^2, & |a-b|<ab,\\ +\infty, & |a-b|\geq ab.\end{cases}$$

Now consider $y_0 = 0$. It is easy to see that

$$\sup_{z\geq 0}\left(\frac{b-a}{b}z\right)_+^2 - a^2z^2 = \begin{cases} 0, & b-a\leq ab,\\ +\infty, & b-a>ab,\end{cases}$$

and thus

$$\mathcal{J}_c(I,J) = \begin{cases} 0, & |a-b|\leq ab,\\ +\infty, & |a-b|>ab.\end{cases}$$

Moreover, a possible optimal transportation plan is given by

$$Q(x,y) = \begin{cases} -a^2x^2, & y=\frac{a}{b}x\\ -\infty, & \text{otherwise.}\end{cases}$$

5. Maslov dequantization and large deviation principle

Let us recall the following definition from the large deviations theory.

Definition 1. *The sequence of Borel probability measures $\{\mu_k\}$ over a Polish space X is said to satisfy the large deviation principle (LDP) with the rate function $I\colon X\to\mathbb{R}_+\cup\{+\infty\}$ and speed α_k, where $\{\alpha_k\}_k\subset(0,\infty)$ converge to ∞, if*

$$\liminf_k \frac{1}{\alpha_k}\log\mu_k(U)\geq \sup_{x\in U}(-I(x)), \tag{20}$$

$$\limsup_k \frac{1}{\alpha_k}\log\mu_k(C)\leq \sup_{x\in C}(-I(x)) \tag{21}$$

for all open sets $U\subset X$ and closed sets $C\subset X$. We will always assume by default that the rate function I has compact sublevel sets $\{x\in X\colon I(x)\leq c\}$ for all $c\in\mathbb{R}$, hence in particular, it is l.s.c.

Let us mention that in different books on large deviations slightly different terminology is used. For instance, in [2], the rate functions with compact sublevel sets are called *good*. Further, the definition may be generalized to measures over rather general topological spaces (see also [2]), but here we prefer to remain within the more classical setting of Polish spaces.

Note that (20) and (21) are equivalent to the estimate

$$\lim_k \frac{1}{\alpha_k}\log\int_X e^{\alpha_k f(x)}\,d\mu_k(x) = \sup_{x\in X}(f(x)-I(x)) \tag{22}$$

for all $f\colon X\to\mathbb{R}$ bounded and continuous (Varadhan theorem 4.3.1 from [2]), or to the estimates

$$\liminf_k \frac{1}{\alpha_k}\log\int_X e^{\alpha_k f(x)}\,d\mu_k(x) \quad \geq \sup\nolimits_{x\in X}(f(x)-I(x)), \tag{23}$$

$$\limsup_k \frac{1}{\alpha_k}\log\int_X e^{\alpha_k g(x)}\,d\mu_k(x) \quad \leq \sup\nolimits_{x\in X}(g(x)-I(x)), \tag{24}$$

for all $f\colon X\to\mathbb{R}$ l.s.c. and $g\colon X\to\mathbb{R}$ bounded from above and u.s.c. (lemmata 4.3.4 and 4.3.6 from [2] respectively). In particular, if μ is a Borel probability measure over X with $\operatorname{supp}\mu = X$, then the constant sequence $\mu_k := \mu$ satisfies LDP with rate function zero and any speed, hence the sequence of measures $\nu_k := e^{-\alpha_k I}\mu$ satisfies LDP with rate function I and speed α_k when $-I\in\mathcal{P}^t(X)$. Note also that (24) is valid for g not necessarily bounded: it suffices that g be u.s.c. and the so-called tail condition

$$\lim_{M\to+\infty}\limsup_k \frac{1}{\alpha_k}\log\int_X \mathbf{1}_{\{g(x)\geq M\}}e^{\alpha_k g(x)}\,d\mu_k(x) = -\infty, \tag{25}$$

hold (see lemma 4.3.6 from [2]).

The following "Γ-limit" style result relating the optimal tropical transportation cost to limits of classical Monge–Kantorovich mass transportation costs can be viewed as a kind of Maslov dequantization of the latter problem.

Theorem 9. *Let X and Y be metric spaces. If $c\colon X\times Y\to\mathbb{R}$ is l.s.c., $\{\mu_k\}$ and $\{\nu_k\}$ are two sequences of Borel probability measures over X and Y respectively, satisfying LDP with the rate function $-I$ and speed α_k and LDP with the rate function $-J$ and speed α_k respectively. Then*

$$\mathcal{J}_c(I,J)\leq \liminf_k \frac{1}{\alpha_k}\log MK_{e^{\alpha_k c}}(\mu_k,\nu_k). \tag{26}$$

Furthermore, there exist two sequences of Borel probability measures $\{\overline{\mu}_k\}$ over X and $\{\overline{\nu}_k\}$ over Y satisfying LDP with the rate function $-I$ and speed α_k with the rate function $-J$ and speed α_k, respectively, and such that

$$\mathcal{J}_c(I,J)\geq \limsup_k \frac{1}{\alpha_k}\log MK_{e^{\alpha_k c}}(\overline{\mu}_k,\overline{\nu}_k). \tag{27}$$

for every $c\colon X\times Y\to\mathbb{R}$ u.s.c. and bounded from above. Moreover, (27) holds for every sequence of $\overline{\mu}_k$, $\overline{\nu}_k$ such that some $\gamma_k\in\Pi(\overline{\mu}_k,\overline{\nu}_k)$ satisfy the LDP with rate function $-Q\in\Pi^t(I,J)$ and speed α_k (in particular, for $\overline{\mu}_k=\pi_{X\#}\gamma_k$, $\overline{\nu}_k=\pi_{Y\#}\gamma_k$, where $\gamma_k:=e^{-\alpha_k Q}\gamma$ and γ a probability measure over $X\times Y$ having $\operatorname{supp}\gamma\supset \operatorname{supp} I\times\operatorname{supp} J$) once c is u.s.c. and the tail condition

$$\lim_{M\to+\infty}\limsup_k \frac{1}{\alpha_k}\log\int_{X\times Y} \mathbf{1}_{\{c(x,y)\geq M\}}e^{\alpha_k c(x,y)}\,d\gamma_k(x,y) = -\infty, \tag{28}$$

is satisfied (in particular, when c is bounded from above).

Proof. The proof will be divided into several steps.

Step 1. We show (27) first. To this aim, for any $Q\colon X\times Y\to\mathbb{R}\cup\{\pm\infty\}$ with

$$\inf\{y\in Y\colon Q(x,y)\}=-I(x),\quad \inf\{x\in X\colon Q(x,y)\}=-J(y)\}$$

(i.e., $-Q\in\Pi^t(I,J)$), let $\{\gamma_k\}$ be a sequence of finite Borel measures over $X\times Y$ satisfying the large deviation principle with the rate function Q and speed α_k. By contraction principle (theorem 4.2.1 from [2]) their marginals $\overline{\mu}_k:=\pi_{X\#}\gamma_k$ and $\overline{\nu}_k:=\pi_{Y\#}\gamma_k$ satisfy the large deviation principle with the rate functions $-I$ and $-J$, respectively, and speed α_k. In particular, when c is bounded from above (or, more generally, satisfies (28)) and u.s.c., using (24), we get

$$\begin{aligned}\sup_{(x,y)\in X\times Y}(c(x,y)-Q(x,y)) &\geq \limsup_k\frac{1}{\alpha_k}\log\int_X e^{\alpha_k c(x,y)}\,d\gamma_k(x)\\ &\geq \limsup_k\frac{1}{\alpha_k}\log MK_{e^{\alpha_k c}}(\overline{\mu}_k,\overline{\nu}_k).\end{aligned} \tag{29}$$

Taking the optimal $-Q\in\Pi^t(I,J)$ in (29), we get (27).

Step 2. To prove (26), fix first an arbitrary $x_0\in X$, and denote for the sake of brevity

$$A_\varepsilon(x_0):=\{y\in Y\colon J(y)\geq I(x_0)-\varepsilon\}\subset Y.$$

For an $r>0$ consider the r-neighborhood of $(A_\varepsilon(x_0))_r$ of $A_\varepsilon(x_0)$ defined by

$$(A_\varepsilon(x_0))_r := \{y \in Y \colon \operatorname{dist}_Y(y, A_\varepsilon(x_0)) < r\}.$$

where dist_Y stands for the distance between a point and a set in Y, that is, $\operatorname{dist}_Y(y,A) := \inf\{d_Y(y,a) \colon a \in A\}$ for $A \subset Y$, $y \in Y$, with d_Y standing for the distance in Y. Observe that its complement $(A_\varepsilon(x_0))_r^c$ is a closed set satisfying

$$(A_\varepsilon(x_0))_r^c \subset A_\varepsilon^c(x_0) = \{y \in Y \colon J(y) < I(x_0) - \varepsilon\}.$$

Thus one has

(30)

$$\begin{aligned}\limsup_k \frac{1}{\alpha_k}\log \nu_k\left((A_\varepsilon(x_0))_r^c\right) &\le \sup\left\{J(y)\colon y \in (A_\varepsilon(x_0))_r^c\right\}\\ &\le \sup\left\{J(y)\colon y\in Y, J(y) < I(x_0)-\varepsilon\right\} \le I(x_0)-\varepsilon,\\ \liminf_k \frac{1}{\alpha_k}\log\mu_k\left(B_\delta(x_0)\right) &\ge \sup\left\{I(x)\colon x \in B_\delta(x_0)\right\} \ge I(x_0).\end{aligned}$$

Let now γ_k be a Borel measure over $X \times Y$ with marginals μ_k and ν_k respectively which is an optimal transportation plan for the Kantorovich cost $MK_{e^{\alpha_k c}}(\mu_k,\nu_k)$. We get from (30) the following estimate:

$$\begin{aligned}&\limsup_k \frac{1}{\alpha_k}\log\frac{\gamma_k\left(B_\delta(x_0)\times(A_\varepsilon(x_0))_r^c\right)}{\mu_k(B_\delta(x_0))}\\ &\quad\le \limsup_k \frac{1}{\alpha_k}\log\frac{\gamma_k\left(X\times(A_\varepsilon(x_0))_r^c\right)}{\mu_k(B_\delta(x_0))}\\ &\quad= \limsup_k \frac{1}{\alpha_k}\log\frac{\nu_k\left((A_\varepsilon(x_0))_r^c\right)}{\mu_k(B_\delta(x_0))}\\ &\quad\le \limsup_k \frac{1}{\alpha_k}\log\nu_k\left((A_\varepsilon(x_0))_r^c\right) - \liminf_k \frac{1}{\alpha_k}\log\mu_k(B_\delta(x_0))\\ &\quad\le I(x_0)-\varepsilon - I(x_0) = -\varepsilon.\end{aligned}$$

This implies

$$\frac{\gamma_k\left(B_\delta(x_0)\times(A_\varepsilon(x_0))_r^c\right)}{\mu_k(B_\delta(x_0))} \le e^{-\frac{\varepsilon}{2}\alpha_k}$$

for all sufficiently large k, and hence

(31) $$\lim_k \frac{\gamma_k\left(B_\delta(x_0)\times(A_\varepsilon(x_0))_r\right)}{\mu_k(B_\delta(x_0))} = 1 - \lim_k \frac{\gamma_k\left(B_\delta(x_0)\times(A_\varepsilon(x_0))_r^c\right)}{\mu_k(B_\delta(x_0))} = 1.$$

From (30) and (31) we get

(32)

$$\begin{aligned}&\liminf_k \frac{1}{\alpha_k}\log\gamma_k\left(B_\delta(x_0)\times(A_\varepsilon(x_0))_r\right)\\ &\qquad= \liminf_k \frac{1}{\alpha_k}\log\frac{\gamma_k\left(B_\delta(x_0)\times(A_\varepsilon(x_0))_r\right)}{\mu_k(B_\delta(x_0))}\mu_k\left(B_\delta(x_0)\right)\\ &\qquad\ge \liminf_k \frac{1}{\alpha_k}\log\frac{\gamma_k\left(B_\delta(x_0)\times(A_\varepsilon(x_0))_r\right)}{\mu_k(B_\delta(x_0))} + \liminf_k \frac{1}{\alpha_k}\log\mu_k\left(B_\delta(x_0)\right)\\ &\qquad= \liminf_k \frac{1}{\alpha_k}\log\mu_k\left(B_\delta(x_0)\right) \quad \text{by (31)}\\ &\qquad\ge I(x_0) \quad \text{by (30)}.\end{aligned}$$

We estimate now

$$\begin{aligned} MK_{e^{\alpha_k c}}(\mu_k,\nu_k) &= \int_{X\times Y} e^{\alpha_k c(x,y)}\, d\gamma_k(x,y) \\ &\geq \int_{B_\delta(x_0)\times (A_\varepsilon(x_0))_r} e^{\alpha_k c(x,y)}\, d\gamma_k(x,y) \\ &\geq \exp\bigl(\alpha_k \inf\{c(x,y)\colon (x,y)\in B_\delta(x_0)\times (A_\varepsilon(x_0))_r\}\bigr)\gamma_k\left(B_\delta(x_0)\times (A_\varepsilon(x_0))_r\right), \end{aligned}$$

and hence

$$\begin{aligned} \liminf_k \frac{1}{\alpha_k} \log MK_{e^{\alpha_k c}}(\mu_k,\nu_k) & \\ \geq \inf\{c(x,y)\colon (x,y)\in B_\delta(x_0)\times (A_\varepsilon(x_0))_r\}+ & \\ \liminf_k \frac{1}{\alpha_k}\log \gamma_k\left(B_\delta(x_0)\times (A_\varepsilon(x_0))_r\right) & \\ \geq \inf\{c(x,y)\colon (x,y)\in B_\delta(x_0)\times (A_\varepsilon(x_0))_r\} + I(x_0) \quad \text{by (32)}. & \end{aligned} \tag{33}$$

Letting $\delta\to 0^+$ and $r\to 0^+$ in (33) and recalling that c is assumed to be l.s.c., while $A_\varepsilon(x_0)$ is compact (as a sublevel of a rate function,) we get

$$\liminf_k \frac{1}{\alpha_k} \log MK_{e^{\alpha_k c}}(\mu_k,\nu_k) \geq \inf\{c(x_0,y) : J(y)\geq I(x_0)-\varepsilon\} + I(x_0),$$

and since $x_0\in X$ is arbitrary, then

$$\liminf_k \frac{1}{\alpha_k} \log MK_{e^{\alpha_k c}}(\mu_k,\nu_k) \geq \sup_{x\in X} \left\{I(x) + \inf\left\{c(x,y)\colon J(y)\geq I(x)-\varepsilon\right\}\right\}. \tag{34}$$

The estimate

$$\liminf_k \frac{1}{\alpha_k} \log MK_{e^{\alpha_k c}}(\mu_k,\nu_k) \geq \sup_{y\in Y} \left\{J(y) + \inf\left\{c(x,y)\colon I(x)\geq J(y)-\varepsilon\right\}\right\} \tag{35}$$

is obtained in a completely symmetric way. Now, (34) and (35) together show the claim (26) in view of Theorem 4, thus concluding the proof. ☐

It is worth remarking that the inequality in (26) may be strict even if the cost function c is continuous, as the following simple example (even for finite metric spaces) shows.

Example 4. Let $X=Y=\{1,2\}$. We associate the measures and functions over $X=Y$ with vectors and over $X\times Y$ with matrices. The sequences of measures $\mu_k := (1-e^{-k}, e^{-k})$ and $\nu_k := (1-2e^{-k}, 2e^{-k})$ both satisfy the LDP with $-I = -J = (0,-1)$ and speed k. For the cost function c represented by the matrix

$$c := \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

one has the estimate for the mass transportation cost

$$MK_{e^{kc}}(\mu_k,\nu_k) \geq e^{2k}e^{-k} = e^k,$$

since at least the mass of e^{-k} has to be moved at cost e^{2k}. Hence

$$\liminf_k \frac{1}{k}\log MK_{e^{kc}}(\mu_k,\nu_k)\geq 1.$$

On the other hand, clearly, $\mathcal{J}_c(I,J)=0$, the latter optimal tropical transportation cost being attained, for instance, with the tropical transportation plan

$$Q := \begin{pmatrix} 0 & -\infty \\ -\infty & -1 \end{pmatrix}$$

The following easy corollary for Kantorovich distances[1] W_p between measures is also worth stating.

Corollary 10. *Let d be a distance in $X=Y$, $\{\mu_k\}$ and $\{\nu_k\}$ be sequences of Borel probability measures over X satisfying LDP with the rate function $-I$ and speed α_k and with the rate function $-J$ and speed α_k, respectively. Then*

$$\exp\left(\frac{1}{p}\mathcal{J}_{\log d^p}(I,J)\right) \leq \liminf_k W_{\alpha_k p}(\mu_k,\nu_k). \tag{36}$$

for every $p>0$, where the Kantorovich distance W_p between Borel probability measures μ and ν over X is defined by the formula

$$W_p^p(\mu,\nu) := \inf\left\{\int_{X\times X} d^p(x,y)\,d\gamma(x,y)\colon \gamma\in\Pi(\mu,\nu)\right\}.$$

In particular, for $p=1$ this gives

$$\mathcal{J}_{\log d}(I,J) \leq \liminf_k \log W_{\alpha_k}(\mu_k,\nu_k).$$

If d is bounded over $\operatorname{supp} I\times\operatorname{supp} J$, there also exist two sequences of Borel probability measures $\{\overline{\mu}_k\}$ and $\{\overline{\nu}_k\}$ over X satisfying LDP with the rate function $-I$ and speed α_k with the rate function $-J$ and speed α_k, respectively, such that

$$\exp\left(\frac{1}{p}\mathcal{J}_{\log d^p}(I,J)\right) = \lim_k W_{\alpha_k p}(\overline{\mu}_k,\overline{\nu}_k). \tag{37}$$

for every $p>0$. For $p=1$ this gives

$$\mathcal{J}_{\log d}(I,J) \geq \limsup_k \log W_{\alpha_k}(\overline{\mu}_k,\overline{\nu}_k).$$

Proof. Apply Theorem 9 with $c(x,y):=\log d^p(x,y)$ (so that $e^{\alpha_k c}=d^{\alpha_k p}$) to get

$$\mathcal{J}_c(I,J) \leq \liminf_k \frac{1}{\alpha_k}\log MK_{d^{\alpha_k p}}(\mu_k,\nu_k) = p\liminf_k \log W_{\alpha_k p}(\mu_k,\nu_k),$$
$$\mathcal{J}_c(I,J) \geq \limsup_k \frac{1}{\alpha_k}\log MK_{d^{\alpha_k p}}(\overline{\mu}_k,\overline{\nu}_k) = p\limsup_k \log W_{\alpha_k p}(\overline{\mu}_k,\overline{\nu}_k),$$

for μ_k, ν_k, $\overline{\mu}_k$, $\overline{\nu}_k$ as in the statement, so that (36) and (37) hold (to obtain the latter from Theorem 9, recall that the boundedness of d over $\operatorname{supp} I\times\operatorname{supp} J$ implies boundedness of c on the same set from above). □

6. Metric properties of optimal costs

The following example shows that the cost $\mathcal{J}_c$ normally is not a distance (nor even a pseudo-distance) between tropical measures even when c is a distance over $X=Y$.

[1] usually called Wasserstein distances. However, this name is historically incorrect, so we prefer to avoid it.

Example 5. Let $X = Y := \{1, 2, 3\}$ and $c(i, j) := 2|i - j|$. The functions over $X = Y$ are then naturally identified with $3D$ vectors. Let then $I_1 := (0, -2, -5)$, $I_2 := (0, -3, -4)$, and $I_3 := (0, -1, -2)$. Then $\mathcal{J}_c(I_1, I_2) = 0$, the latter cost being attained, for instance, at the plan

$$Q_{12} := \begin{pmatrix} 0 & -\infty & -4 \\ -2 & -3 & -\infty \\ -\infty & -\infty & -5 \end{pmatrix}.$$

Further, $\mathcal{J}_c(I_1, I_3) = 1$ and $\mathcal{J}_c(I_2, I_3) = 2$ are attained, for instance, at the plans

$$Q_{13} := \begin{pmatrix} 0 & -1 & -\infty \\ -\infty & -\infty & -2 \\ -\infty & -\infty & -5 \end{pmatrix} \quad \text{and } Q_{23} := \begin{pmatrix} 0 & -1 & -2 \\ -\infty & -3 & -\infty \\ -\infty & -\infty & -4 \end{pmatrix}$$

respectively. Hence $\mathcal{J}_c(I_1, I_2) + \mathcal{J}_c(I_1, I_3) < \mathcal{J}_c(I_2, I_3)$ and thus $\mathcal{J}_c$ does not satisfy the triangle inequality even though c is a distance.

On the other hand, Remark 5 suggests that $\mathcal{J}_c$ has some flavor of a distance. We first mention a simpler cost between tropical measures which somewhat resembles the dual cost in the classical optimal mass transportation.

6.1. **A "dual" cost.** Similarly to the Kantorovich dual problem we can consider the following one in terms of the tropical algebra:

(38)
$$\sup\{G(f, g)\colon f\colon X \to \mathbb{R} \cup \{-\infty\}, g\colon Y \to \mathbb{R} \cup \{-\infty\}, f(x) \vee g(y) \le c(x, y)\}.$$
$$\text{where } G(f, g) := \sup_{x\in X}(f(x) + I(x)) \vee \sup_{y\in Y}(g(y) + J(y)).$$

We denote the respective cost by $\mathcal{D}_c(I, J)$. In the assertion below we collect some basic properties of the cost $\mathcal{D}_c$.

Proposition 11. *The following assertions are valid.*

(i) *One has*

$$\mathcal{D}_c(I, J) = \sup_{x\in X}(I(x) + \inf_{y\in Y} c(x, y)) \vee \sup_{y\in Y}(J(y) + \inf_{x\in X} c(x, y)), \tag{39}$$

(ii) $\mathcal{D}_c(I, J) \le \mathcal{J}_c(I, J)$,

(iii) $\mathcal{D}_c$ *satisfies the strong triangle inequality*

$$\mathcal{D}_c(I_1, I_3) \le \mathcal{D}_c(I_1, I_2) \vee \mathcal{D}_c(I_2, I_3), \tag{40}$$

(iv) *if* $X = Y$ *and* c *is symmetric, then so is* $\mathcal{D}_c$,

(v) *if* c *is non negative, then* $\mathcal{D}_c(I, J) \ge 0$ *for all* $\{I, J\} \in \mathcal{P}^t(X) \times \mathcal{P}^t(Y)$,

(vi) *if* $\inf_{x\in X} c(x, \cdot) = \inf_{y\in Y} c(\cdot, y) = 0$, *then*

$$\mathcal{D}_c(I, J) = 0 \quad \textit{for all } \{I, J\} \in \mathcal{P}^t(X) \times \mathcal{P}^t(Y).$$

Proof. The claim (i) follows from the chain of equalities

$$\begin{aligned}\mathcal{D}_c(I, J) &= \sup\{G(f, g)\colon f(x) \le c(x, y), g(y) \le c(x, y) \quad \text{for all } (x, y) \in X \times Y\}\\ &= \sup\left\{G(f, g)\colon f(x) \le \inf_{y\in Y} c(x, y), g(y) \le \inf_{x\in X} c(x, y) \quad \text{for all } (x, y) \in X \times Y\right\}\\ &= G\left(\inf_{y\in Y} c(\cdot, y), \inf_{x\in X} c(x, \cdot)\right),\end{aligned}$$

Then claim (ii) follows from (i) and the representation formula for the tropical transportation cost $\mathcal{J}_c$ provided by Theorem 4. Claims (iii) and (iv) are straightforward from (i). If $c \geq 0$, then from (39) we get

$$\begin{aligned} \mathcal{D}_c(I,J) &= \sup_{x\in X}(I(x) + \inf_{x\in X} c(x,y)) \vee \sup_{y\in Y}(J(y) + \inf_{x\in X} c(x,y)), \\ &\geq \sup_{x\in X} I(x) \vee \sup_{y\in Y} J(y) = 0, \end{aligned} \tag{41}$$

when $\{I,J\} \in \mathcal{P}^t(X)\times\mathcal{P}^t(Y)$, showing (v). Since the inequality in (41) becomes an equality when $\inf_{x\in X} c(x,\cdot) = \inf_{y\in Y} c(\cdot,y) = 0$, then we have also (vi). □

Remark 12. Of course, one only has $\mathcal{D}_c(I,J) \leq \mathcal{J}_c(I,J)$, with possibly strict inequality, e.g. when c is a distance over $X = Y$, then $\mathcal{D}_c(I,J) = 0$ for all $\{I,J\} \in \mathcal{P}^t(X)\times\mathcal{P}^t(Y)$ by the above Proposition 11(vi), while $\mathcal{J}_c(I,J) > 0$ already when I and J are tropical delta-measures, i.e. $I = \delta^t_{x_0}$, $J = \delta^t_{y_0}$ with $x_0 \neq y_0$, and hence $\mathcal{J}_c(I,J) = c(x_0,y_0) > 0$ according to Example 2.

Remark 13. Theorem 4 can be interpreted as an adjustment to the above translation into the "tropical language" of the Kantorovich duality for the classical optimal mass transportation. In fact, it gives

$$\begin{aligned} &\mathcal{J}_c(I,J) = G(f,g), \\ &\qquad \text{where } G(f,g) := \sup_{x\in X}(f(x)+I(x)) \vee \sup_{y\in Y}(g(y)+J(y)), \\ &\qquad f(x) := \sup_{\varepsilon>0}\inf\{c(x,y)\colon y\in Y,\ J(y)\geq I(x)-\varepsilon\}, \\ &\qquad g(y) := \sup_{\varepsilon>0}\inf\{c(x,y)\colon x\in X,\ I(x)\geq J(y)-\varepsilon\}. \end{aligned}$$

6.2. **Metric properties of the tropical transportation cost.** Similarly to $\Pi(I,J)$, we can define the set of multimarginal plans

$$\Pi^t(I_1,\dots,I_n) \subset \mathcal{P}^t(X_1\times\cdots\times X_n).$$

The following assertion is valid.

Lemma 14 ("Gluing lemma")**.** *Let $I_i \in \mathcal{P}^t(X_i)$, $i=1,2,3$, and $Q_{1,2}\in\Pi^t(I_1,I_2)$, $Q_{2,3}\in\Pi^t(I_2,I_3)$. Then there exists a multimarginal plan $Q\in\Pi^t(I_1,I_2,I_3)$ such that*

$$\sup_{x_3} Q(x_1,x_2,x_3) = Q_{1,2}(x_1,x_2), \quad \sup_{x_1} Q(x_1,x_2,x_3) = Q_{2,3}(x_2,x_3).$$

Proof. Define

$$Q := Q_{1,2}(x_1,x_2) + Q_{2,3}(x_2,x_3) - I_2(x_2),$$

with the convention $\infty - \infty = -\infty$. Then

$$\begin{aligned} \sup_{x_1} Q(x_1,x_2,x_3) &= \sup_{x_1\in X} Q_{1,2}(x_1,x_2) + Q_{2,3}(x_2,x_3) - I_2(x_2) \\ &= I_2(x_2) + Q_{2,3}(x_2,x_3) - I_2(x_2) = Q_{2,3}(x_2,x_3). \end{aligned}$$

Here we used that if $I_2(x_2) = -\infty$, then $Q_{2,3}(x_2,x_3) = -\infty$. Similarly, one has $\sup_{x_3} Q(x_1,x_2,x_3) = Q_{1,2}(x_1,x_2)$. □

The assertion below provides the tropical (max-plus) counterparts of the Kantorovich p-distances W^p for $p>0$ between probability measures.

Proposition 15. *Let $c(x,y) := \log d^p(x,y)$ for some $p>0$, where $d\colon X\times X\to [0,+\infty)$ is a distance. Then $e^{\mathcal{J}_c/p}$ is also a distance between tropical measures. In particular, for $p := 1$ we get that $e^{\mathcal{J}_{\log d}}$ is a distance.*

Proof. Clearly, $\mathcal{J}_c$ is symmetric. Further, consider $I\in\mathcal{P}^t(X)$ and the trivial plan

$$Q_I(x,y) := \begin{cases} I(x), & x=y,\\ -\infty, & \text{otherwise.}\end{cases} \tag{42}$$

Then, since $c(x,x)=-\infty$ for all $x\in X$,

$$\mathcal{J}_c(I,I) = \sup(c+Q_I) = \sup_{x\in X}(c(x,x)+I(x)) = -\infty,$$

i.e. $e^{\mathcal{J}_c(I,I)}=0$. Now, consider two tropical measures $I\neq J$ and an optimal plan $Q_{I,J}\in\Pi^t(I,J)$. Suppose that $Q_{I,J}(x,y)=-\infty$ for all $x\neq y$. Then $I(x)=J(x)=Q(x,x)$ for all $x\in X$, and we get a contradiction. Thus, there are $x\neq y$ such that $Q(x,y)>-\infty$, hence

$$\mathcal{J}_c(I,J) = \sup(c+Q_{I,J}) \ge c(x,y)+Q_{I,J}(x,y) > -\infty,$$

i.e. $e^{\mathcal{J}_c(I,J)}>0$.

Finally, consider $\{I_1,I_2,I_3\}\subset\mathcal{P}^t(X)$ and optimal plans $Q_{1,2}\in\Pi^t(I_1,I_2)$, $Q_{2,3}\in\Pi^t(I_2,I_3)$. Take a multimarginal plan $Q\in\Pi(I_1,I_2,I_3)$ defined by Lemma 14 and define

$$Q_{1,3} := \sup_{x_2\in X} Q(x_1,x_2,x_3).$$

Obviously, $Q_{1,3}\in\Pi^t(I_1,I_3)$. Furthermore,

$$\begin{aligned}
e^{\mathcal{J}_c(I_1,I_3)/p} &\le e^{\frac{1}{p}\sup_{\{x_2,x_3\}\subset X}(c+Q_{1,3})}\\
&= \sup_{\{x_1,x_2,x_3\}\subset X} e^{c(x_1,x_3)/p}e^{Q(x_1,x_2,x_3)/p}\\
&= \sup_{\{x_1,x_2,x_3\}\subset X} e^{\log d(x_1,x_3)}e^{Q(x_1,x_2,x_3)/p}\\
&= \sup_{\{x_1,x_2,x_3\}\subset X} d(x_1,x_3)e^{Q(x_1,x_2,x_3)/p}\\
&\le \sup_{\{x_1,x_2,x_3\}\subset X} \left(d(x_1,x_2)+d(x_2,x_3)\right)e^{Q(x_1,x_2,x_3)/p}\\
&= \sup_{\{x_1,x_2,x_3\}\subset X} \left(e^{c(x_1,x_2)/p}+e^{c(x_2,x_3)/p}\right)e^{Q(x_1,x_2,x_3)/p}\\
&\le \sup_{\{x_1,x_2\}\subset X} \exp\left\{\frac{1}{p}\left(c(x_1,x_2)+Q_{1,2}(x_1,x_2)\right)\right\}\\
&\quad + \sup_{\{x_2,x_3\}\subset X} \exp\left\{\frac{1}{p}\left(c(x_2,x_3)+Q_{2,3}(x_2,x_3)\right)\right\}\\
&= e^{\mathcal{J}_c(I_1,I_2)/p} + e^{\mathcal{J}_c(I_2,I_3)/p}.
\end{aligned}$$

Therefore, $e^{\mathcal{J}_c/p}$ satisfies the triangle inequality, which concludes the proof. □

6.3. **Ultrametric property of the cost.** As we have already shown, the cost $\mathcal{J}_c$ is in general not a pseudo-distance, because it does not satisfy the triangle inequality, even if the cost function c does so. Nevertheless the proposition below states that $\mathcal{J}_c$ is a pseudo-ultrametric whenever c is.

Proposition 16. *Let $c\colon X \times X \to \mathbb{R}_+$ be a pseudo-ultrametric, i.e. nonnegative, symmetric, $c(x,x) = 0$ for all $x \in X$, and*

$$c(x,z) \le c(x,y) \vee c(y,z) \quad \text{for all } x, y, z \in X.$$

Then $\mathcal{J}_c$ is a pseudo-ultrametric over $\mathcal{P}^t(X)$.

Proof. Clearly, $\mathcal{J}_c$ is symmetric. For $\{I, J\} \subset \mathcal{P}^t(X)$ the Theorem 4 implies in view of nonnegativity of the cost function c that

$$\mathcal{J}_c(I,J) \ge \sup_{x\in X} I(x) \wedge \sup_{y \in Y} J(y) = 0.$$

Further, $\mathcal{J}_c(I,I) = 0$ being attained on the trivial plan (42). Finally, as in the proof of Proposition 15, consider $\{I_1, I_2, I_3\} \subset \mathcal{P}^t(X)$ and optimal plans $Q_{1,2} \in \Pi^t(I_1, I_2)$, $Q_{2,3} \in \Pi^t(I_2, I_3)$. Take a multimarginal plan $Q \in \Pi(I_1, I_2, I_3)$ defined by Lemma 14 and define

$$Q_{1,3} \coloneqq \sup_{x_2 \in X} Q(x_1, x_2, x_3).$$

Since clearly $Q_{1,3} \in \Pi^t(I_1, I_3)$, one has

$$\begin{aligned}
\mathcal{J}_c(I_1, I_3) &\le \sup_{\{x_1,x_3\}\subset X} \big(c(x_1,x_3) + Q_{1,3}(x_1,x_3)\big) \\
&= \sup_{\{x_1,x_2,x_3\}\subset X} \big(c(x_1,x_3) + Q(x_1,x_2,x_3)\big) \\
&\le \sup_{\{x_1,x_2,x_3\}\subset X} \big(c(x_1,x_2) \vee c(x_2,x_3) + Q(x_1,x_2,x_3)\big) \\
&= \sup_{\{x_1,x_2,x_3\}\subset X} \big(c(x_1,x_2) + Q(x_1,x_2,x_3)\big) \vee \big(c(x_2,x_3) + Q(x_1,x_2,x_3)\big) \\
&= \sup_{\{x_1,x_2,x_3\}\subset X} \big(c(x_1,x_2) + Q(x_1,x_2,x_3)\big) \\
&\qquad \bigvee \sup_{\{x_1,x_2,x_3\}\subset X} \big(c(x_2,x_3) + Q(x_1,x_2,x_3)\big) \\
&= \sup_{\{x_1,x_2\}\subset X} \big(c(x_1,x_2) + Q_{12}(x_1,x_2)\big) \bigvee \sup_{\{x_2,x_3\}\subset X} \big(c(x_2,x_3) + Q_{23}(x_2,x_3)\big) \\
&= \mathcal{J}_c(I_1,I_2) \vee \mathcal{J}_c(I_2,I_3).
\end{aligned}$$

showing the claim. ☐

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