Entire solutions of completely coercive quasilinear elliptic equations

Alberto Farina and **James Serrin**

Abstract. *A famous theorem of Sergei Bernstein says that every entire solution $u = u(x)$, $x \in \mathbf{R}^2$ of the minimal surface equation*

$$\operatorname{div}\left\{\frac{Du}{\sqrt{1+Du|^2}}\right\} = 0$$

is an affine function; no conditions being placed on the behavior of the solution u.

Bernstein's Theorem continue to hold up to dimension $n = 7$ while it fails to be true in higher dimensions, in fact if $x \in \mathbf{R}^n$, with $n \geq 8$, there exist entire non-affine minimal graphs (Bombieri, De Giorgi and Giusti).

Our purpose is to consider an extensive family of quasilinear elliptic-type equations which has the following strong Bernstein-Liouville property, that $u \equiv 0$ for any entire solution u, no conditions whatsoever being placed on the behavior of the solution (outside of appropriate regularity assumptions). In many cases, moreover, no conditions need be placed even on the dimension n. We also study the behavior of solutions when the parameters of the problem do not allow the Bernstein–Liouville property, and give a number of counterexamples showing that the results of the paper are in many cases best possible.

Table of Contents

1. Introduction.

We shall study entire solutions of quasilinear elliptic equations of the form

$$\operatorname{div} \mathcal{A}(x,u,Du) = \mathcal{B}(x,u,Du) \tag{1}$$

and also of the corresponding inequality

$$\operatorname{div} \mathcal{A}(x,u,Du) \geq \mathcal{B}(x,u,Du) \tag{1'}$$

under various coercive conditions on the vector-valued function $\mathcal{A}$ and the scalar function $\mathcal{B}$. The simplest typical example of equation (1), though far from the more general ones which we shall consider later, is the equation

$$\Delta_p u = |u|^{q-1}u \tag{2}$$

with $p > 1$, $q \geq 0$. For the semilinear case $p = 2$, $q > 1$, H. Brezis [1] showed in 1984 that if $u = u(x)$, $x \in \mathbf{R}^n$, is an entire C^1 distribution solution of (2), then $u \equiv 0$. The remarkable nature of this result is that *no boundedness conditions of any sort are imposed on the solution.* Moreover the condition $q > 1$ is best possible, for if $q \leq 1$ there exist non-trivial (even positive) entire solutions, see results A, B below. As we shall see, *Brezis's result continues to hold for equation* (2) *provided* $q > p-1$*, where again this condition is best possible.*

Throughout the paper we shall assume the general coercive (weak ellipticity) conditions

$$\begin{aligned} &\mathcal{A}(x,z,\rho)\cdot\rho \geq 0, \qquad \mathcal{B}(x,z,\rho)z \geq 0,\\ &\mathcal{A}(x,z,0) = 0, \qquad \mathcal{B}(x,0,0) = 0, \end{aligned} \tag{3}$$

for all $x \in \mathbf{R}^n$, $z \in \mathbf{R}$ and $\rho \in \mathbf{R}^n$, together with the property that

$$\begin{cases} \mathcal{A}(x,z,\rho)\cdot\rho + \mathcal{B}(x,z,\rho)z = 0 \\ \text{implies either } z = 0 \text{ or } \rho = 0. \end{cases} \tag{4}$$

Further conditions on the quantities $\mathcal{A}$ and $\mathcal{B}$ will be needed only for large values of x, say $|x| \geq R_0 >> 1$. Before stating these conditions it is worth pointing out several further model examples of our conclusions.

Example 1. Let $u = u(x)$ be an entire C^1 distribution solution of the equation

$$\operatorname{div}\,[A(x,u,Du)\,|Du|^{p-2}Du] = b(x,u,Du)|u|^{q-1}u \tag{5}$$

where $p > 1$, $q \geq 0$, and $A(x,z,\rho)$, $b(x,z,\rho)$ are non-negative measurable functions such that

$$A(x,z,\rho) \leq \text{Const.}\,|x|^s|z|^r, \quad b(x,z,\rho) \geq \text{Pos. Const.}\,|x|^{-t} \tag{5'}$$

for $|x| \geq R_0$, $z \neq 0$, $\rho \in \mathbf{R}^n$, with $r \geq 0$, s, $t \in \mathbf{R}$. In writing (5), and in later work, we define $|u|^{q-1}u$ to vanish at all points where $u = 0$.

It is clear that (3) is satisfied, with moreover

$$\mathcal{A}(x, z, \rho) \cdot \rho + \mathcal{B}(x, z, \rho)z = A(x, z, \rho)|\rho|^p + b(x, z, \rho)|z|^{q+1}.$$

The first line of the coercivity condition (4) then implies that

$$A(x, z, \rho)|\rho|^p = b(x, z, \rho)|z|^{q+1} = 0;$$

in turn the second line applies provided that

$$(5'') \qquad A(x, z, \rho) + b(x, z, \rho) > 0 \quad \text{for } z \neq 0,\ \rho \in \mathbf{R}^n,\ \text{and almost all } x \in \mathbf{R}^n.$$

Under these conditions, the conclusion for Example 1 is that if $q > p + r - 1$ and either

$$(6') \qquad s + t \leq p$$

or

$$(6'') \qquad s + t > p, \qquad q(s + t - p) - (q - p - r + 1)(t - n) < 0,$$

then $u \equiv 0$ in $\mathbf{R}^n$; see below.

Brezis' result is the special case $p = 2$, $A \equiv b \equiv 1$, $r = s = t = 0$ (see the appendix, Section 13, for a fuller discussion of Brezis' theorem and its relation to our work).

Example 2. Let $u = u(x)$ be an entire C^1 solution of the equation

$$(7) \qquad \operatorname{div} \left\{ A(x) \frac{Du}{\sqrt{1 + |Du|^2}} \right\} = b(x)f(u),$$

where A, b are non-negative measurable functions such that

$$(7') \qquad A(x) \leq \text{Const.}\, |x|^s, \quad b(x) \geq \text{Pos. Const.}\, |x|^{-t},$$

for almost all $|x| \geq R_0$, with s, t in $\mathbf{R}$, and $f(z)$ is a non-decreasing function with a single zero, $z = 0$. Here the coercivity condition (4) is valid when $A(x) + b(x) > 0$ (a.e.) in $\mathbf{R}^n$.

The conclusion is that if $s+t < 1$ then $u \equiv 0$; see Section 5. Moreover, if $f(z) = |z|^{q-1}z$ the conclusion continues to hold if $s + t < 1 + q$ (when $0 < q \leq 1$), and if $s + t \leq 2$ (when $q > 1$); see Section 6. Note that the last result is the same as that for the case $p = 2$, $r = 0$ of Example 1. For a complete statement of results concerning equation (7), see the comments at the end of Section 6.

When $A \equiv b \equiv 1$, the first of these results is due to Farina [3] and Tkachev [15]; the first result in the form stated is due to Serrin [14].

For the main results of the paper we shall require the following "large radii conditions", namely that for almost all $|x| \geq R_0$ and all $z \in \mathbf{R}\backslash\{0\}$, $\rho \in \mathbf{R}^n$, there exists an exponent $p \geq 1$ such that

$$|\mathcal{A}(x, z, \rho)|^p \leq C_{\mathcal{A}}|x|^s|z|^r[\mathcal{A}(x, z, \rho) \cdot \rho]^{p-1} \tag{8}$$

and also

$$\mathcal{B}(x, z, \rho) \operatorname{sign} z \geq C_{\mathcal{B}}|x|^{-t}|z|^q \tag{9}$$

where $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ are positive constants, and $q \geq 0$, $r \geq 0$, s, $t \in \mathbf{R}$.

Remark. The structural condition (8) is very general and the authors have not encountered it before. Indeed, to the best of our knowledge existing results related to the problem under consideration (see [6] and the literature cited therein) deal almost exclusively with (particular cases of) equations of the form (5) or (7), which, in turn, are special cases of the equations we consider here. To see this, observe that, *e.g.,* equation (5) is obtained from (1) by taking

$$\mathcal{A}(x, z, \rho) = A(x, z, \rho)|\rho|^{p-2}\rho, \qquad 0 \leq A(x, z, \rho) \leq C_{\mathcal{A}}|x|^s|z|^r, \qquad p > 1, \tag{5'''}$$

a typical p-ellipticity-type condition, which obeys (3), (8) but at the same time is obviously more restrictive than (8).

Remark. Despite the large number of defining parameters (n, p, q, r, s, t) in conditions (8)-(9), it turns out, most surprisingly, that just two combinations, *e.g.*

$$q - p - r + 1, \quad \text{and} \quad s + t - p$$

(and in particular their signs), are relevant in the formulations of almost all our conclusions.

With the help of conditions (8), (9), and assuming the coercive relations (3) and (4), Theorem 1 (Section 3) shows that *any entire C^1 distribution solution of equation* (1) *must vanish identically, provided that $p > 1$, $q > p + r - 1$ and either*

$$s + t \leq p, \tag{10'}$$

or

$$s + t > p, \qquad q\nu < t - n \quad (\textit{so } t > n). \tag{10''}$$

where $\nu = (s + t - p)/(q - p - r + 1)$.

On the other hand, Theorem 3 (Section 5) shows that *any entire C^1 distribution solution of equation* (1) *must vanish identically provided, that $p = 1$, $q > r\ (= p + r - 1)$ and either*

$$s + t < 1$$

or

$$s + t \geq 1, \qquad\qquad q\nu < t - n, \quad (\textit{so } t > n).$$

or

$$s + t = 1, \qquad\qquad s + n - 1 = 0, \quad (\textit{so } t = n).$$

where $\nu = (s + t - 1)/(q - r)$.

The inequality case ($1'$) is treated in Theorem 2 of Section 3 (respectively, Theorem 3 of Section 5), the corresponding conclusion being that $u \leq 0$. For the opposite inequality one finds similarly that $u \geq 0$.

The conclusion $u \equiv 0$ of Example 1 follows at once as a special case of Theorem 1, when one notes that ($6''$) is equivalent to ($10''$).

We note that in the case ($10'$) of Theorem 1 and in the first case of Theorem 3 the conclusion $u \equiv 0$ holds *independently* of the dimension n, as in Brezis' theorem. In the remaining cases of these results, on the other hand, the conclusion holds only under explicit restriction of the dimension, as in Bernstein's theorem.

It is interesting to inquire whether the parameter conditions of Theorem 1 are necessary for the conclusion $u \equiv 0$ to be valid. This is indeed the case (with one exception noted as an open problem in Section 12). This also leads us to ask more generally *what happens when any one of these conditions is not satisfied, but otherwise the remaining conditions, of course including* (3), (4), (8), (9), *continue to hold*. The results for these cases are perhaps unexpected, and not entirely simple:

A. If $q = p + r - 1 \geq 0$ and $s + t < p$, then any entire C^1 distribution solution of (1) which has at most algebraic growth at infinity must vanish identically. This result is essentially sharp since in this case there exist exponentially growing solutions of the form $e^{\kappa x_1}$, $\kappa > 0$, *e.g.,* for the equation (5) with $A = 1$ and $b = (p - 1)\kappa^2$.

B. If $0 \leq q < p + r - 1$ and $s + t < p$, then any entire C^1 distribution solution of (1) which satisfies

$$u(x) = o(|x|^\nu) \qquad \nu = \frac{p - s - t}{p + r - q - 1} \tag{11}$$

must vanish identically.

The exponent ν is best possible, in the sense that for all $n \geq 1$, $p > 1$, q, $r \geq 0$ with

$$0 \leq q < p + r - 1, \qquad s + t < p, \qquad q\nu > t - n,$$

there are non-negative functions $A = A(x, z)$ and $b = b(x)$ satisfying the large radii conditions ($5'$), ($5''$) and such that the corresponding equation (5) admits the explicit entire positive smooth solution

$$u(x) = (1 + |x|^2)^{\nu/2}, \tag{12}$$

see Example 6 of Section 11. This example equally shows that the conditions of Theorem 1 are sharp.

Theorems A and B for the special case of Example 1 with $A \equiv 1$ were given in [14], Section 3. In this paper, moreover, equations are treated which are not necessarily of divergence form.

C. If $q > p + r - 1$, $s + t > p$, and

$$\nu = \frac{s+t-p}{q-p-r+1} \geq \frac{t-n}{q},$$

then any entire C^1 distribution solution of (1) satisfies

$$\text{liminf}_{|x|\to\infty} \frac{|u(x)|}{|x|^\nu} \leq C, \tag{13}$$

where the constant C is universal, that is, depends only on n, p, q, r, s, t and the constants in (8) and (9).

The exponent ν in (13) is best possible, in the sense that the function u in (12) is again a solution of (5). Figure 1 shows the values of s, t for which Theorems 1 and C hold, corresponding to fixed values of n, p, q, r.

D. If $p > 1$, $0 \leq q \leq p + r - 1$ and $s + t = p$, then any entire C^1 distribution solution of (1) which is bounded must vanish identically.

We note that a related and overlapping result is included in Theorem 1 of [7].

The following result covers the remaining possibility when $0 \leq q \leq p + r - 1$, and indeed requires only that conditions (3) and (4) be valid.

E. Any entire C^1 distribution solution of (1) which is $o(1)$ as $|x| \to \infty$ must vanish.

The condition $o(1)$ as $|x| \to \infty$ is sharp, as shown by Example 8 in Section 11. A further result in the same direction is given in Section 4, see Theorem F.

The results A, B, C are proved in Section 3, the results D and E in Section 4.

In Section 5 we consider in more detail the special case $p = 1$ of (8), see Theorem 3. Moreover, in Theorem 4 we partially generalize Theorem 3 by replacing the large radii condition (9) by the more general condition (5.4), which involves a general function f with a single zero at the origin (and not necessarily non-decreasing).

The mean curvature equation of Example 2 is treated in Section 6, along with the possibility that condition (8) is satisfied for *multiple values* of the parameter p.

It is possible to weaken the hypotheses of the main results in a number of important and significant ways.

1. First, the "large radii conditions" can be relaxed, by making, in essence, the change of variables $v = g(u)$. More precisely, let $g \in C^0(\mathbf{R}) \cap C^1(\mathbf{R} \setminus \{0\})$ be such that $g(0) = 0$ and $g'(z) > 0$ for all $z \neq 0$. Then (8)–(9) can be generalized, namely to the form that, for almost all $|x| \geq R_0$ and all $z \in \mathbf{R}\setminus\{0\}$, $\rho \in \mathbf{R}^n$, there exists an exponent $p \geq 1$ such that

$$|\mathcal{A}(x,z,\rho)|^p \leq C_{\mathcal{A}} |x|^s \, |g(z)|^r \, [g'(z)\, \mathcal{A}(x,z,\rho)\cdot\rho]^{p-1} \tag{8$'$}$$

and

$$(9') \qquad \mathcal{B}(x,z,\rho)\ \mathrm{sign}\, z \ge C_{\mathcal{B}}|x|^{-t}\,|g(z)|^q$$

where $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ are positive constants, and $q \ge 0$, $r \ge 0$, s, $t \in \mathbf{R}$.

Conditions (8)-(9) are recovered by taking $g(z) = z$, while the case $g(z) = \tanh(z)$ and $g(z) = e^z - 1$ show the generality of the modified conditions. We emphasize also that g need not be differentiable at $z = 0$, so that a straightforward change of variables $v = g(u)$ cannot be applied.

Under the modified conditions (8'),(9') all the results of Theorems 1-9 hold in unchanged form, while Theorems A-E remain valid with obvious modifications. We note explicitly the remarkable case in which the function g is bounded. Under this assumption we can extend the conclusions of Theorems 1-4 to cover the *full range* $0 \le q < \infty$ of values of the exponent q, rather than the previously restricted set $q > p + r - 1$. See Section 10 for the relevant proofs.

2. Conditions (8)-(9) (and equally $(8')$-$(9')$) can also be weakened in another, and somewhat surprising direction, so as to apply only for a disjoint sequence of shells

$$T_i = B_{\kappa R_i} \backslash B_{R_i}, \quad i = 1, 2, 3, \cdots,$$

where κ is a constant greater than 1 and R_i, $i = 1, 2, 3, \cdots$, is an arbitrary sequence of radii tending to infinity. We take this up in Section 7.

3. In Section 8 we show that the condition that the solution u be of class C^1 can be replaced by the weak assumption that $u \in W^{1,\sigma}_{loc}(\mathbf{R}^n)$ for some $\sigma \ge 1$, this requiring however a more technically delicate discussion.

4. Finally, in Section 9 we consider the possibility that the exponent r is negative. This entails first of all that one must restrict consideration to solutions which avoid the value zero, that is, to solutions which are either everywhere positive or everywhere negative.With this proviso, the previous considerations carry over without difficulty. See Theorems 8 and 9 for a precise statement of the results for this case; it is worth noting that, in some circumstances, the results apply even when the parameter q is negative!

Case (i) of Theorem 8 extends earlier work of Usami [16], Naito and Usami [12], Mitidieri and Pohozaev[10], and Filippucci [5], [6].

In related work [4] we treat a further generalization of condition (9), in which a factor $|Du|^\ell$, $\ell > 0$ is added to the right side, a possibility first introduced by Martio and Porru [8] and studied in detail by Filippucci [6].

Other important and closely related work corresponding to Examples 1 and 2 is due to Mitidieri and Pohozaev [10] (see also [15],[5],[6],[7],[9],[11]); all of these works are, however, restricted to positive (or non-negative) solutions u. For further references and discussions of the literature, see [6].

2. Preliminaries.

We begin with several preliminary lemmas which will be of importance throughout the paper. First we make precise the meaning of a C^1 distribution solution $u = u(x)$ of (1), namely that

$$\int \{\mathcal{A}(x,u,Du)\cdot D\eta + \mathcal{B}(x,u,Du)\eta\} = 0 \tag{2.1}$$

for all functions $\eta \in C^1(\mathbf{R}^n)$ having compact support in $\mathbf{R}^n$. Naturally one must require further that the functions $\mathcal{A}(\cdot,u,Du)$, $\mathcal{B}(\cdot,u,Du)$ in (2.1) are locally integrable in $\mathbf{R}^n$. It is worth adding that, under these integrability conditions, if $u \in C^2$ is an almost everywhere ($\mathbf{R}^n$) classical solution of (1), then u is a distribution solution as well.

For the inequality (1′) the meaning of solution is the same, with the exception that equality in (2.1) is now replaced by $\le$ and the test function η must also be non-negative.

We suppose throughout the rest of the paper, with the exception of Section 10, that conditions (8)-(9) are in force. Everything stands or falls, depending on the following lemma.

Lemma 1. *Let $u = u(x)$ be an entire C^1 distribution solution of the inequality* (1′)*. Then for every $\alpha > 0$, $\beta \ge p \ge 1$, $R_1 \ge R_0 > 0$, and for every compactly supported non-negative locally Lipschitz continuous test function φ we have*

(2.2)

$$\alpha \int_{B_{R_1}\cap\{u>0\}} \mathcal{A}(x,u,Du)\cdot Du\, u^{\alpha-1}\varphi^\beta + \beta\int_{B_{R_1}} \mathcal{A}(x,u,Du)\cdot D\varphi\,[u^+]^\alpha\varphi^{\beta-1}$$

$$+\int_{\mathbf{R}^n} \mathcal{B}(x,u,Du)\,[u^+]^\alpha\varphi^\beta \;\le\; C_1\int_{\mathbf{R}^n\setminus B_{R_1}} |x|^s[u^+]^{\alpha+p+r-1}\varphi^{\beta-p}\,|D\varphi|^p,$$

where

$$C_1 = \alpha^{1-p}\,\beta^p\,C_{\mathcal{A}}$$

and $C_{\mathcal{A}}$ is the constant appearing in (8).

Proof. For (1′) we use the non-negative test function

$$\eta_\varepsilon = [u^+ + \varepsilon]^\alpha\varphi^\beta.$$

where $0 < \varepsilon < 1$. This is Lipschitz continuous in $\mathbf{R}^n$ so that, as is clear (trivial mollification), it can be used in the corresponding inequality version of (2.1). This gives

$$\int \mathcal{B}(x,u,Du)\eta_\varepsilon$$
$$\le -\,\alpha\int \mathcal{A}(x,u,Du)\cdot Du^+\,[u^+ + \varepsilon]^{\alpha-1}\varphi^\beta - \beta\int \mathcal{A}(x,u,Du)\cdot D\varphi\,[u^+ + \varepsilon]^\alpha\varphi^{\beta-1}.$$

Since $Du^+ = 0$ a.e. in the set $\{u \le 0\}$ we can rewrite this as

$$0 \le \alpha \int_{\{u>0\}} \mathcal{A}(x,u,Du) \cdot Du\ [u^+ + \varepsilon]^{\alpha-1}\varphi^\beta$$

$$\le -\int \mathcal{B}(x,u,Du)\eta_\varepsilon - \beta \int \mathcal{A}(x,u,Du) \cdot D\varphi\ [u^+ + \varepsilon]^\alpha \varphi^{\beta-1}.$$

By letting $\varepsilon \to 0$ we obtain (using Fatou's Lemma and (3) for the first integral, and Lebesgue's dominated theorem for the others)

(2.3)

$$\int \mathcal{B}(x,u,Du)[u^+]^\alpha \varphi^\beta$$
$$\le -\alpha \int_{\{u>0\}} \mathcal{A}(x,u,Du) \cdot Du\, u^{\alpha-1}\varphi^\beta - \beta \int \mathcal{A}(x,u,Du) \cdot D\varphi\, [u^+]^\alpha \varphi^{\beta-1},$$

all the integrals being finite.

We now rewrite the last inequality as follows:

$$\int \mathcal{B}(x,u,Du)[u^+]^\alpha \varphi^\beta$$
$$\le -\alpha \int_{B_{R_1}\cap\{u>0\}} \mathcal{A}(x,u,Du) \cdot Du\ u^{\alpha-1}\varphi^\beta - \beta \int_{B_{R_1}} \mathcal{A}(x,u,Du) \cdot D\varphi\, [u^+]^\alpha \varphi^{\beta-1}$$
$$-\alpha \int_{\{\mathbf{R}^n\setminus B_{R_1}\}\cap\{u>0\}} \mathcal{A}(x,u,Du) \cdot Du\ u^{\alpha-1}\varphi^\beta - \beta \int_{\mathbf{R}^n\setminus B_{R_1}} \mathcal{A}(x,u,Du) \cdot D\varphi\, [u^+]^\alpha \varphi^{\beta-1}.$$

Using the large radii condition (8), we can estimate the second integrand on the line above (since $\alpha > 0$ it is enough to do so on the set $\{u > 0\}$).

First, when $p > 1$, by virtue of the (weighted) Young inequality

$$xy \le \frac{\varepsilon^p}{p}x^p + \frac{1}{p'\varepsilon^{p'}}y^{p'} \qquad \text{with} \quad \varepsilon^p = \left(\frac{p-1}{p\alpha}\right)^{p-1}, \quad p' = \frac{p}{p-1},$$

we obtain (use (8))

(2.4)

$$-\beta\mathcal{A}(x,u,Du) \cdot D\varphi\, u^\alpha \varphi^{\beta-1} \le \beta C_{\mathcal{A}}^{1/p}|x|^{s/p}\, u^{\alpha+r/p}(\mathcal{A}(x,u,Du)\cdot Du)^{1/p'}|D\varphi|\,\varphi^{\beta-1}$$

$$= [\beta C_{\mathcal{A}}^{1/p}|x|^{s/p}\, u^{(\alpha+p+r-1)/p}|D\varphi|\,\varphi^{-1+\beta/p}] \cdot [(\mathcal{A}(x,u,Du)\cdot Du)^{1/p'}\, u^{(\alpha-1)/p'}\varphi^{\beta/p'}]$$

$$\le C_1|x|^s\, u^{\alpha+p+r-1}\,|D\varphi|^p\varphi^{\beta-p} + \alpha\mathcal{A}(x,u,Du)\cdot Du\, u^{\alpha-1}\varphi^\beta.$$

When $p = 1$ condition (8) reduces to $\mathcal{A}(x, u, Du) \le C_{\mathcal{A}} |x|^s |u|^r$, so the above inequality is immediate, even without the last term.

The proof is now completed by collecting the above inequalities. ■

In what follows we fix the test function φ as follows:

$$\varphi(x) = \varphi_R(x) = \psi\left(\frac{|x|}{R}\right), \tag{2.5}$$

where

$$\psi(\tau) = \begin{cases} 1, & 0 \le \tau \le 1, \\ 2 - \tau, & 1 < \tau < 2, \\ 0, & \tau \ge 2. \end{cases}$$

Lemma 2. *Let $u = u(x)$ be an entire C^1 distribution solution of the inequality $(1')$. Then for every $\alpha > 0$, $\beta \ge p \ge 1$, $R \ge R_1 \ge R_0$ we have*

(2.6)

$$\begin{aligned} &\min\{\alpha, 1\} \int_{B_{R_1} \cap \{u>0\}} [\mathcal{A}(x, u, Du) \cdot Du + \mathcal{B}(x, u, Du)\, u]\, u^{\alpha-1} \\ &+ C_{\mathcal{B}} \int_{\mathbf{R}^n \setminus B_{R_1}} |x|^{-t}\, [u^+]^{\alpha+q} \varphi^\beta \le C_1 \int_{B_{2R} \setminus B_R} |x|^s [u^+]^{\alpha+p+r-1} |D\varphi|^p\, \varphi^{\beta-p} \end{aligned}$$

where C_1 is the constant appearing in Lemma 1.

Proof. Using $\varphi = \varphi_R$ in Lemma 1, we find (with $R \ge R_1 \ge R_0$)

$$\begin{aligned} &\alpha \int_{B_{R_1} \cap \{u>0\}} \mathcal{A}(x, u, Du) \cdot Du\, u^{\alpha-1} + 0 + \int_{B_{R_1}} \mathcal{B}(x, u, Du)\, [u^+]^\alpha \\ &+ \int_{\mathbf{R}^n \setminus B_{R_1}} \mathcal{B}(x, u, Du) [u^+]^\alpha \varphi^\beta \le C_1 \int_{B_{2R} \setminus B_R} |x|^s [u^+]^{\alpha+p+r-1} |D\varphi|^p\, \varphi^{\beta-p}, \end{aligned}$$

since $\varphi \equiv 1$, $D\varphi = 0$ in $B_R \supset B_{R_1}$, while $D\varphi = 0$ outside B_{2R}.

Next, using (9) to estimate (from below) the first integral on the last line, we obtain (2.6). ■

Lemma 3. *Let $u = u(x)$ be an entire C^1 distribution solution of the inequality $(1')$. Then for every $\alpha > 0$, $p \ge 1$ and $R \ge 2R_0$, we have*

$$\int_{B_R \setminus B_{R/2}} |x|^{-t}\, [u^+]^{\alpha+q} \le C_2 R^{s+t-p} \int_{B_{2R} \setminus B_R} |x|^{-t}\, [u^+]^{\alpha+p+r-1},$$

where

$$C_2 = \frac{p^p}{\alpha^{p-1}} \frac{C_{\mathcal{A}}}{C_{\mathcal{B}}}\, 2^{[s+t]^+}.$$

Proof. Take $R_1 = R/2 \geq R_0$ and $\beta = p$. In the first integral of the second line of (2.6) we have $\varphi = 1$ in B_R, while $D\varphi = 1/R$ in the shell $B_{2R}\backslash B_R$. Hence using (3) we obtain

$$\int_{B_R\backslash B_{R/2}} |x|^{-t}\,[u^+]^{\alpha+q} \leq 2^{[s+t]^+} C_1 C_{\mathcal{B}}^{-1} R^{s+t-p} \int_{B_{2R}\backslash B_R} |x|^{-t}[u^+]^{\alpha+p+r-1}. \qquad \blacksquare$$

Lemma 4. *Let $u = u(x)$ be an entire C^1 distribution solution of the inequality $(1')$ and assume $q > p + r - 1$. For $\alpha > 0$, $p \geq 1$ and $R \geq R_1 \geq R_0$ we have*

(2.7)

$$C_{\mathcal{B}}^{-1}\,\min\{\alpha,1\} \int_{B_{R_1}\cap\{u>0\}} [\mathcal{A}(x,u,Du)\cdot Du + \mathcal{B}(x,u,Du)u]\,u^{\alpha-1}$$
$$+ \int_{B_R\backslash B_{R_1}} |x|^{-t}\,[u^+]^{q+\alpha} \leq C_3 R^{(q+\alpha)\nu+n-t},$$

where $\nu = (s+t-p)/(q+1-p-r)$ and

$$(2.8) \qquad C_3 = \frac{(p\mu)^{p\mu}}{\alpha^{(p-1)\mu}}\left(\frac{C_{\mathcal{A}}}{C_{\mathcal{B}}}\right)^{\mu} 2^{[\mu(s+t)-t]^+ + n}\omega_n, \qquad \mu = (q+\alpha)/(q-p-r+1),$$

with ω_n the measure of the unit ball in $\mathbf{R}^n$.

Proof. Since $\alpha > 0$, $p \geq 1$, $r \geq 0$, we see that μ is greater than 1 and its conjugate exponent is $\mu' = (q+\alpha)/(\alpha+p+r-1)$.

Choose $\beta = p\mu > p$ in (2.6). Applying Young's inequality with the exponents μ and μ' to the last integrand in (2.6) gives

$$C_1 C_{\mathcal{B}}^{-1}|x|^s[u^+]^{\alpha+r+p-1}\varphi^{\beta-p}\,|D\varphi|^p$$
$$\leq \frac{1}{\mu'}\left[|x|^{-t/\mu'}[u^+]^{\alpha+r+p-1}\varphi^{\beta-p}\right]^{\mu'} + \frac{1}{\mu}\left[C_1 C_{\mathcal{B}}^{-1}|x|^{t/\mu'+s}|D\varphi|^p\right]^{\mu}.$$

Inserting this inequality into (2.6), using $R \geq R_1$ and observing that $(\beta-p)\mu' = \beta$, (2.6) then yields

$$C_{\mathcal{B}}^{-1}\,\min\{\alpha,1\} \int_{B_{R_1}\cap\{u>0\}} [\mathcal{A}(x,u,Du)\cdot Du + \mathcal{B}(x,u,Du)\,u]u^{\alpha-1}$$
$$+\frac{1}{\mu}\int_{\mathbf{R}^n\backslash B_{R_1}} |x|^{-t}\,[u^+]^{q+\alpha}\varphi^{\beta} \leq \frac{1}{\mu}\left[C_1 C_{\mathcal{B}}^{-1}\right]^{\mu} \int_{B_{2R}\backslash B_R} |x|^{t\mu/\mu'+s\mu}|D\varphi|^{p\mu}.$$

The desired conclusion then follows (a) by multiplying the last inequality by μ, and then using the relations $\mu/\mu' = \mu - 1$ and $(s+t/\mu')\mu - p\mu = (q+\alpha)\nu - t$ together with the fact that $D\varphi = 1/R$ in the shell $B_{2R}\backslash B_R$, to estimate the last integral, and then (b) by estimating the preceding integral by noting as in Lemma 3 that $\varphi \equiv 1$ in B_R. $\blacksquare$

Lemma 5. *Let $h(R)$ be a non-negative function such that for all $R \geq R_2 > 0$ there holds*

$$h(R) \leq \theta h(2R) \tag{2.9}$$

and

$$h(R) \leq CR^{\delta}, \tag{2.10}$$

where C, δ, θ are constants, $C > 0$, $\delta \in \mathbf{R}$ and $0 < \theta < 2^{-\delta}$. Then $h(R) = 0$ for $R \geq R_2$.

Proof. By an ℓ times iteration of (2.9) we find that, for every $R \geq R_2$ and every positive integer l,

$$h(R) \leq \theta^{\ell} h(2^{\ell} R).$$

Therefore, by (2.10) we obtain

$$h(R) \leq C(2^{\delta}\theta)^{\ell} R^{\delta} = C\xi^{\ell} R^{\delta},$$

where $\xi = 2^{\delta}\theta \in (0, 1)$ by assumption. The desired result then follows by letting $\ell \to \infty$ in the last inequality. ∎

3. Main results, I.

In this section we shall prove Theorems 1, 2, A, B and C, assuming throughout that conditions (3), (4), (8), (9) are valid.

Theorem 1. *Assume $p > 1$, $q > p + r - 1$ and either*

$$(i) \qquad s + t \leq p$$

or

$$(ii) \qquad s + t > p, \qquad \nu = \frac{s + t - p}{q - p - r + 1} < \frac{t - n}{q}.$$

Then any entire C^1 distribution solution u of the equation (1) *must vanish everywhere.*

Theorem 1 is a consequence of the following results for the inequality version of equation (1).

Theorem 2 (i). *Assume $p > 1$, $q > p + r - 1$ and*

$$s + t \leq p.$$

Then any entire C^1 distribution solution u of the inequality $(1')$ *must be non-positive.*

Proof. Consider first the case $s + t < p$. Since $\nu < 0$ we can choose α so large (say, $\alpha = 1 + [(t - n)/\nu]^+$) that the exponent $(q + \alpha)\nu + n - t < 0$ in (2.7) is negative. Then letting $R \to \infty$ in (2.7) one obtains

$$\int_{B_{R_1} \cap \{u>0\}} [\mathcal{A}(x, u, Du) \cdot Du + \mathcal{B}(x, u, Du)u]\, u^{\alpha - 1} = 0. \tag{3.1}$$

Hence, because R_1 can now be allowed arbitrarily large, and because of (3), we get

$$\mathcal{A}(x,u,Du)\cdot Du+\mathcal{B}(x,u,Du)u=0 \quad \text{a.e. in the set } \{u>0\}. \tag{3.2}$$

From (4) it follows that $Du=0$ almost everywhere in $\{u>0\}$. Therefore $Du^+=0$ almost everywhere in $\mathbf{R}^n$, and thus $u^+\equiv\gamma\geq 0$ in $\mathbf{R}^n$.

If $\gamma>0$, then by (3.2) again and the fact that $Du\equiv 0$ a.e. on $\{u>0\}$, we obtain $\gamma\mathcal{B}(x,\gamma,0)=0$ for almost all x such that $u(x)>0$. In view of (9) the latter entails $\{u>0\}\subset B_{R_0}$ (up to a set of Lebesgue measure zero) which clearly contradicts $u^+\equiv\gamma>0$ in all of $\mathbf{R}^n$. Hence necessarily $\gamma=0$, and so in turn $u\leq 0$ in $\mathbf{R}^n$, which completes the proof when $s+t<p$. The case $s+t=p$ will be treated in Section 4. ∎

Theorem 2 (*ii*). *If $p>1$, $q>p+r-1$ and $s+t>p$, then any entire C^1 distribution solution of* (1′) *must be non-positive provided the exponents n, p, q, r, s, t are such that*

$$\nu=\frac{s+t-p}{q-p-r+1}<\frac{t-n}{q} \qquad (t>n).$$

Proof. Since $\nu q<t-n$ there exists $\alpha>0$ such that $(q+\alpha)\nu+n-t<0$. Therefore, by letting $R\to\infty$ in (2.7) we are led to the conclusion $u\leq 0$, exactly as in the proof of Theorem 2 (*i*). ∎

Proof of Theorem 1, *Case* (*i*). By Theorem 2 (*i*) we have $u\leq 0$. On the other hand the function $v=-u$ solves an equation of the form (1) with, in an obvious notation, $\tilde{\mathcal{A}}(x,z,\rho)=-\mathcal{A}(x,-z,-\rho)$ and $\tilde{\mathcal{B}}(x,z,\rho)=-\mathcal{B}(x,-z,-\rho)$. Since $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ also satisfy the large radii conditions (8) and (9), it follows that $v\leq 0$ by another application of Theorem 2 (*i*). Thus $u\equiv 0$.

The proof of Theorem 1, Case (*ii*), is the same, only using Theorem 2 (*ii*) instead of Theorem 2 (*i*). ∎

Theorem A. *If $q=p+r-1\geq 0$ and $s+t<p$, then any entire C^1 distribution solution of* (1) (*respectively,* (1′)) *which has at most algebraic growth at infinity must vanish identically* (*solution of* (1)) *or satisfy $u\leq 0$* (*solution of* (1′)).

Proof. From Lemma 3, with $\alpha=2$, $p+r-(1+q)=0$ and $R\geq 2R_0$, we obtain

$$h(R)\equiv\int_{B_R\setminus B_{R/2}}[u^+]^{q+2}\leq 4^{|t|}C_2R^{s+t-p}\int_{B_{2R}\setminus B_R}[u^+]^{q+2}=4^{|t|}C_2R^{s+t-p}h(2R).$$

Also by direct estimation,

$$h(R)\leq CR^{\delta}, \qquad \delta=(q+2)d+n$$

for some constant C and for all R suitably large, say $R\geq R_2$, where $d>0$ is the algebraic growth rate of u at infinity. Since $s+t<p$ we can choose R_2 even larger if necessary so that also

$$4^{|t|}C_2R^{s+t-p}\leq 2^{-\delta-1}\equiv\theta$$

for $R \geq R_2$.

Applying Lemma 5 gives $h(R) = 0$ for all $R \geq R_2$, whence $u \leq 0$ for all $|x| \geq R_2/2$. In turn, if $R \geq R_2/2$ then $u \leq 0$ in $B_{2R}\backslash B_R$.

Let $R_1 \geq R_0$ be given, and suppose (as we can) that $R_2 \geq 2R_1$. Therefore (2.6) holds when $R \geq R_2/2 \geq R_1$. But in this case the right side of (2.6) vanishes, and we obtain (3.1). Then letting $R_1 \to \infty$ gives (3.2). It now follows as in the proof of Theorem 2 that $u \leq 0$ in $\mathbf{R}^n$. Finally if u is a solution of (1) then $u \equiv 0$. ∎

Theorem B. *If $0 \leq q < p + r - 1$ and $s + t < p$, then any entire C^1 distribution solution of* (1) (*respectively,* (1′)) *which satisfies*

$$u(x) = o(|x|^\nu) \qquad \nu = \frac{s+t-p}{q-p-r+1} \tag{3.3}$$

must vanish identically (*solution of* (1)) *or satisfy $u \leq 0$* (*solution of* (1′)).

Proof. From Lemma 3, with $\alpha = 2$, $p + r - q - 1 > 0$, and $R \geq 2R_0$, we obtain

$$h(R) \equiv \int_{B_R\backslash B_{R/2}} [u^+]^{q+2} \leq 4^{|t|} C_2 R^{s+t-p} \int_{B_{2R}\backslash B_R} [u^+]^{q+2+(p+r-1-q)} = o(1)h(2R)$$

by (3.3). Also, as above,

$$h(R) \leq CR^\delta, \qquad \delta = (q+2)\nu + n$$

for all R suitably large. Applying Lemma 5 as in Theorem A, we conclude that $h(R) = 0$ for $R \geq R_2$, for a suitably large value R_2. The rest of the proof is the same as for Theorem A. ∎

Remark. A more careful use of Lemma 5 shows that we can replace the condition (3.3) in Theorem B by the relation $u(x) \leq \overline{C}|x|^\nu$, where $\overline{C}$ is a (small) constant depending only on the given parameters of the problem.

Theorem C. *If $q > p + r - 1$ and $s + t > p$, and*

$$\nu = \frac{s+t-p}{q-p-r+1} \geq \frac{t-n}{q},$$

then any entire C^1 distribution solution of (1) *satisfies*

$$\operatorname{liminf}_{|x|\to\infty} \frac{|u(x)|}{|x|^\nu} \leq \overline{C}, \tag{3.4}$$

where $\overline{C}$ is the universal constant

$$\overline{C} = [cp^p(q-p-r+2)^{1-p}\, C_{\mathcal{A}}/C_{\mathcal{B}}]^{1/(q-r-p+1)}. \tag{3.5}$$

and $c = 2^{2(s+t)-p+(q+1)\nu+n-t+1}$.

Proof. The proof is by contradiction. Let us suppose that (3.4) is not satisfied; then by the continuity of u (and up to changing u to $-u$, if necessary) there are constants $C' > \overline{C}$ and $R_2 \geq 2R_0$ such that

$$u^+(x) \geq C'|x|^\nu, \qquad |x| \geq R_2. \tag{3.6}$$

From (3.6) and Lemma 3 we obtain, for every $\alpha > 0$ and for every $R \geq R_2$,

$$\begin{aligned}
&(C'(R/2)^\nu)^{q-p-r+1} \int_{B_R \setminus B_{R/2}} |x|^{-t}\,[u^+]^{\alpha+p+r-1} \\
&\quad \leq \int_{B_R \setminus B_{R/2}} |x|^{-t}\,[u^+]^{\alpha+q} \leq C_2 R^{s+t-p} \int_{B_{2R} \setminus B_R} |x|^{-t}\,[u^+]^{\alpha+p+r-1}.
\end{aligned}$$

Therefore

$$\int_{B_R \setminus B_{R/2}} |x|^{-t}\,[u^+]^{\alpha+p+r-1} \leq 2^{s+t-p}\,C_2 (C')^{-(q-p-r+1)} \int_{B_{2R} \setminus B_R} |x|^{-t}\,[u^+]^{\alpha+p+r-1}; \tag{3.7}$$

this inequality of course continues to hold in the limit $C' \to \overline{C}$.

Fix $\alpha = q - p - r + 2 > 1$ so that $\alpha + p + r - 1 = q + 1$, and consider the function $h = h(R)$ given by

$$h(R) \equiv \int_{B_R \setminus B_{R/2}} |x|^{-t}[u^+]^{q+1}$$

for $R \geq R_2$. From Lemma 4 (with $\alpha = 1$ and replacing R by $R/2$) we have

$$h(R) \leq C_3 R^\delta$$

where $\delta = (q+1)\nu + n - t$. On the other hand, inequality (3.7) together with the explicit choice (3.5) for $\overline{C}$ shows that, for all $R \geq R_2$,

$$h(R) \leq 2^{-\delta-1} h(2R),$$

whence applying Lemma 5 with $\theta = 2^{-\delta-1}$ and $C = C_3$ we find that $h(R) = 0$ for all $R \geq R_2$. This implies $u^+ = 0$ on $\mathbf{R}^n \setminus B_{R_2}$, which contradicts (3.6). ■

4. Main results, II.

We continue to assume conditions (3), (4), (8), (9), and shall prove Theorems D and E and complete the proof of Theorem 2 (i). To this purpose we note that, by using the modified test function

$$\eta = [(u-\gamma)^+]^\alpha \varphi^\beta, \qquad \gamma > 0, \qquad \alpha > 1,$$

in (2.1) and proceeding as in the proofs of Lemmas 1–3 we easily obtain the following modified version of Lemma 3.

Lemma 3_γ**.** *Let $u = u(x)$ be an entire C^1 distribution solution of the inequality* $(1')$*. Then for every $\alpha > 1$, $\gamma > 0$, $p \geq 1$ and $R \geq 2R_0$, we have*

$$\int_{B_R \setminus B_{R/2}} |x|^{-t}\, [u^+]^q [(u-\gamma)^+]^\alpha \leq C_2 R^{s+t-p} \int_{B_{2R}\setminus B_R} |x|^{-t}\, [u^+]^r [(u-\gamma)^+]^{\alpha+p-1},$$

where C_2 is the constant appearing in Lemma 3*, that is*

$$C_2 = \frac{p^p}{\alpha^{p-1}}\, 2^{[s+t]^+} \frac{C_{\mathcal{A}}}{C_{\mathcal{B}}}. \tag{4.1}$$

We are now ready to complete the proof of Theorem 2.

Completion of proof of Theorem 2*: the case $p > 1$, $q > p + r - 1$, $s + t = p$.*
Let $\gamma > 0$ and consider the (slightly modified) function $h(R)$ given by

$$h(R) \equiv \int_{B_R\setminus B_{R/2}} |x|^{-t} [u^+]^r [(u-\gamma)^+]^{\alpha+p-1} \tag{4.2}$$

where $\gamma > 0$, $R \geq 2R_0$. Noting that

$$[u^+]^q = [u^+]^{q-p-r+1} [u^+]^r\, [u^+]^{p-1} \geq \gamma^{q-p-r+1} [u^+]^r\, [(u-\gamma)^+]^{p-1},$$

Lemma 3_γ now yields, after a crucial use of the assumption $s + t = p$,

$$\gamma^{q-p-r+1} h(R) \leq C_2\, h(2R), \qquad R \geq 2R_0, \tag{4.3}$$

Also by Lemma 4, for any $\alpha > 0$,

$$\int_{B_R\setminus B_{R/2}} |x|^{-t} [u^+]^{\alpha+q} \leq C_3 R^\delta, \qquad \delta = n - t \tag{4.4}$$

since $\nu = 0$ because $s + t = p$ (once again, this assumption is crucial).

Since $p > 1$ there exists $\alpha = \alpha(\gamma)$ so large that (4.1), (4.3) give $h(R) \le 2^{-\delta-1} h(2R)$. Furthermore, for this value of α, condition (4.4) and the assumption $q > p + r - 1$, imply

$$h(R) \le (2^{t^+}\omega_n + C_3)R^\delta$$

(consider separately the case when $0 \le u^+(x) \le 1$ and when $u^+(x) \ge 1$). Applying Lemma 5 then yields $h(R) = 0$ for $R \ge 2R_0$, that is $u(x) \le \gamma$ for $|x| \ge R_0$. Letting $\gamma \to 0$ now implies $u(x) \le 0$ for $|x| \ge R_0$. Hence fixing $R_1 \ge 2R_0$ and using Lemma 2 as in the proof of Theorem A, we obtain $u \le 0$ everywhere, and the proof is complete.■

Remark. Note that this proof fails when $p = 1$, showing why the conclusions of Theorem 1 and those of Theorem 3 are different. The reason for this discrepancy is that the above proof is crucially based on the fact that the constant $C_2 = C_2(\alpha, p)$ in (4.1) satisfies $C_2 \searrow 0$ as $\alpha \to \infty$ whenever $p > 1$. Clearly, this is no longer true for $p = 1$. For the same reason the following result holds only for $p > 1$.

Theorem D. *If $p > 1$, $0 \le q \le p + r - 1$ and $s + t = p$, then any entire C^1 distribution solution of* (1) (*respectively* (1′))*) which is bounded must vanish identically* (*solution of* (1)) *or satisfy $u \le 0$* (*solution of* (1′))*.*

Proof. Set

$$h(R) = \int_{B_R \setminus B_{R/2}} |x|^{-t}[u^+]^{q+\alpha}, \qquad R \ge 2R_0,$$

with $\alpha > 1$ still to be determined. By Lemma 3 and the fact that $s + t = p$ we then get

$$h(R) \le C_2 \int_{B_{2R}\setminus B_R} [u^+]^{p+r-q-1}|x|^{-t}[u^+]^{q+\alpha} \le C_2 ||u||_\infty^{p+r-1-q}\, h(2R) \equiv \hat{C}_2 h(2R).$$

Moreover

$$h(R) \le 2^{t^+}\omega_n ||u||_\infty^{q+\alpha}\, R^{n-t}.$$

Fix α so large that $\hat{C}_2 \le 2^{-n+t-1}$, which is possible in view of (4.1) and the assumptions that $p > 1$ and u is bounded. By Lemma 5 we then get $h(R) = 0$ for $R \ge 2R_0$. The rest of the proof is essentially the same as for Theorem A. ■

Theorem E. *Let conditions* (3), (4) *hold. Then any entire C^1 distribution solution of* (1) *which is $o(1)$ as $|x| \to \infty$ must vanish.*

Proof. Suppose u is $o(1)$ as $|x| \to \infty$. It follows that for every $\gamma > 0$ the function $\eta = (u - \gamma)^+$ is locally Lipschitz continuous and hence can be used as a test function in (2.1). This gives

$$\int_{\mathbf{R}^n \cap \{u > \gamma\}} [\mathcal{A}(x, u, Du) \cdot Du + \mathcal{B}(x, u, Du)\,(u - \gamma)^+] = 0.$$

By (3) we have $\mathcal{B} \ge 0$ when $u > 0$. Thus

$$\mathcal{A}(x, u, Du) \cdot Du + \mathcal{B}(x, u, Du)\,(u - \gamma) = 0 \quad \text{a.e. in the set } \{u > \gamma\}.$$

In turn, by letting $\gamma \to 0$ we have

$$\mathcal{A}(x, u, Du) \cdot Du + \mathcal{B}(x, u, Du)\, u = 0 \quad \text{a.e. in the set } \{u > 0\}.$$

From (4) it follows that $Du = 0$ almost everywhere in $\{u > 0\}$ and therefore $Du^+ = 0$ a.e. on $\mathbf{R}^n$. This yields $u^+ = \text{Const.} = 0$, since u is $o(1)$ as $|x| \to \infty$, and thus $u \le 0$ on $\mathbf{R}^n$. That $u \equiv 0$ on $\mathbf{R}^n$ then follows as in the case of Theorem 1. ∎

Remark. It is interesting to observe that Theorem E has been obtained without using the large radii conditions (8), (9), but only conditions (3) and (4).

There is a final result of interest in the same direction, extending Theorems B–E under the additional restriction $s + n < p$. As for Theorem E, not all the condition (3), (4), (8), (9) are required for this result, only (3), (4) and (8).

Theorem F. *Suppose $s + n < p$, $p \ge 1$. Then any entire C^1 distribution solution of* (1) *which is bounded must be identically constant.*

Proof. Without the help of condition (9), the result of Lemma 2 still continues to hold, with however the first integral on the second line of (2.6) omitted. Moreover, the right side of inequality (2.6) in the present case is less than

$$\text{Const.} R^{s+n-p}\, ||u||_\infty^{\alpha+q+r-1}.$$

Using $s + n < p$ and letting $R \to \infty$ we immediately obtain (3.1). The proof is then completed as before, noting however that the case of constant solutions is no longer ruled out by (9). ∎

5. The case $p = 1$.

This section is devoted to the special case $p = 1$. The first conclusion is an extension of Theorem 1 and Theorem 2.

Theorem 3. *Assume $p = 1$, $q > r$. Let $u = u(x)$ be an entire C^1 distribution solution of equation* (1) (*respectively* (1′)) *with either*

$$s + t < 1,$$

or

$$s + t \ge 1, \qquad \nu = \frac{s + t - 1}{q - r} < \frac{t - n}{q},$$

or

$$s + t = 1, \qquad s + n - 1 = 0.$$

Then $u \equiv 0$ in $\mathbf{R}^n$ if u is a solution of (1)*, or $u \le 0$ if u is a solution of* (1′)*.*

Proof. Clearly, it is enough to consider the case of inequality (1′). For the first case, that is $s + t < 1$, we have $\nu < 0$. Hence if α is chosen sufficiently large, then by letting $R \to \infty$ in (2.7) followed by $R_1 \to \infty$, we get

$$[\mathcal{A}(x, u, Du) \cdot Du + \mathcal{B}(x, u, Du)\, u] = 0 \qquad \text{a.e. in the set } \{u > 0\}.$$

By (3), (4) we now conclude in the usual way that $u \le 0$ in $\mathbf{R}^n$.

For the second case, $q\nu + n - t < 0$. Therefore we can choose α suitably near 0 so that $(q+\alpha)\nu + n - t < 0$ in (2.7), and then conclude as in the first case that $u \le 0$. The remaining possibility is more involved. Here we have $s + t = 1$ and $t = n$; hence by letting $R \to \infty$ in (2.7) there results, for every $\alpha > 0$,

$$\int_{R^n \setminus B_{R_1}} |x|^{-n}[u^+]^{\alpha+q} \le C_4.$$

where C_4 is independent of α, see (2.8) with $p = 1$. This clearly entails the limit condition

$$\lim_{R\to\infty} \int_{B_{2R}\setminus B_R} |x|^{-n}[u^+]^{\alpha+q} = 0. \tag{5.1}$$

From Lemma 2 with $\beta = 1 = p$, however,

(5.2)
$$\begin{aligned}
&C_{\mathcal{B}}^{-1} \min\{\alpha, 1\} \int_{B_{R_1}\cap\{u>0\}} [\mathcal{A}(x,u,Du)\cdot Du \ + \mathcal{B}(x,u,Du)u]\, u^{\alpha-1} \\
&+ \int_{\mathbf{R}^n\setminus B_{R_1}} |x|^{-n}\, [u^+]^{\alpha+q}\varphi \le C_1 C_{\mathcal{B}}^{-1} \int_{B_{2R}\setminus B_R} |x|^{1-n}[u^+]^{\alpha+r}|D\varphi| \\
&\qquad \le C_5 \int_{B_{2R}\setminus B_R} |x|^{-n}[u^+]^{\alpha+r},
\end{aligned}$$

where C_5 does not depend on R since $|x|\,|D\varphi|$ is bounded (≤ 2).

Take $\alpha = 1 + q - r$ in (5.2). Then letting $R \to \infty$ in (5.2) and applying (5.1) with $\alpha = 1$, we obtain finally (!)

$$[\mathcal{A}(x,u,Du)\cdot Du + \mathcal{B}(x,u,Du)\, u]\, u = 0 \qquad \text{a.e. in the set } \{u > 0\}$$

and the required conclusion $u \equiv 0$ follows as before. ∎

As already noted in Sections 3 and 4, the conclusions of Theorem A, B, C and E, F continue to hold when $p = 1$, while Theorem D requires the additional restriction $p > 1$ (see the Remark before Theorem D in Section 4).

The case $p = 1$ is special also for another reason. In particular, consider equation (1) (and inequality (1')) when

$$|\mathcal{A}(x,z,\rho)| \le C_{\mathcal{A}}|x|^s, \tag{5.3}$$

with also (9) replaced by

$$\mathcal{B}(x,z,\rho)\,\mathrm{sign} z \ge C_{\mathcal{B}}|x|^{-t}|f(z)|, \tag{5.4}$$

where $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ are positive constants, s, $t \in \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ is a measurable function such that

$$(5.5) \qquad f(0) = 0, \quad \text{liminf}_{z\to t} f(z)\,\text{sign}z > 0 \quad \text{when} \quad t \in [-\infty, 0) \cup (0, +\infty].$$

Condition (5.5) is obviously satisfied by any non-decreasing function f with a single zero at 0, or by a lower semicontinuous function f such that

$$f(0) = 0; \quad f(z)\,\text{sign}z > 0 \quad \text{when} \quad z \neq 0; \quad \text{liminf}_{z\to\pm\infty} f(z)\,\text{sign}z > 0.$$

Under the conditions (5.3)–(5.5), and assuming also that conditions (3), (4) continue to hold, the following result is valid, corresponding to the case $r = 0$ of Theorem 3. Surprisingly, even under these weaker assumptions, the following result is sharp (see Example 9 in Section 11).

Theorem 4. *Assume that conditions* (5.3) − (5.5) *hold. Let $u = u(x)$ be an entire C^1 distribution solution of equation* (1) (*respectively,* (1′)), *with either*

$$s + t < 1$$

or

$$s + t \geq 1, \qquad s + n - 1 < 0$$

or

$$s + t = 1, \qquad s + n - 1 = 0.$$

Then $u \equiv 0$ in $\mathbf{R}^n$ if u is a solution of (1), *or $u \leq 0$ if u is a solution of* (1′).

The proof is based on the following lemma.

Lemma 6. *Let f satisfy condition* (5.5). *Then there exists $\hat{f} \in C^0(\mathbf{R}) \cap C^1(\mathbf{R} \setminus \{0\})$ such that $\hat{f}(0) = 0$, $\hat{f}'(z) > 0$ for all $z \neq 0$ and*

$$\hat{f}(z) > f(z) \text{ when } z < 0, \qquad \hat{f}(z) < f(z) \text{ when } z > 0.$$

We omit the proof, which can be carried out by a straight hands-on construction.

Returning to the proof of Theorem 4, it is enough to consider the case of inequality (1′). By virtue of Lemma 6, the proof can be reduced at once to the case where $f \in C^0(\mathbf{R}) \cap C^1(\mathbf{R} \setminus \{0\})$ and is such that $f(0) = 0$ and $f'(z) > 0$ for all $z \neq 0$. These are however exactly the conditions on the function g in conditions (8′) and (9′). But then Theorem 4 becomes a corollary of the later results in Section 10; we give the full proof in Section 10. ∎

Example 2 is an obvious special case of Theorem 4.

6. Operators allowing multiple values of p.

In this section we study the case where the function $\mathcal{A}$ satisfies the large radii condition (8) for multiple values of the exponent p. Thus, when condition (9) is in force, we can use this information to improve the previous results.

More precisely, we consider the case of functions $\mathcal{A}$ which satisfy the large radii condition (8) for all p such that

$$p_1 \leq p \leq p_2$$

for given constants p_1, $p_2 \geq 1$. [If (8) is satisfied for $p = p_1$ and for $p = p_2 > p_1$, then it is easy to check that it also holds for all p in $[p_1, p_2]$.]

In particular, note that the classical case of Example 2 satisfies (8) with $1 \leq p \leq 2$, see below. More generally, consider the operator

$$\mathcal{A}(x,z,\rho) = A(x,z,\rho)\, \frac{|\rho|^{\xi-2}\rho}{(1+|\rho|^2)^{\sigma/2}}, \qquad \xi > 1, \quad \sigma \geq 0,$$

with

$$0 \leq A(x,z,\rho) \leq C_{\mathcal{A}}|x|^s|z|^r.$$

A straightforward calculation shows that condition (8) then reduces to

$$1 \leq (|\rho|^{2/\sigma})^{p-\xi} + (|\rho|^{2/\sigma})^{p-\xi+\sigma},$$

which is valid exactly when $\xi - \sigma \leq p \leq \xi$. Example 2 is then the special case $\xi = 2$, $\sigma = 1$ and $A(x,z,\rho) = A(x)$.

Observe also that when $p_1 = p_2\ (= p)$, we are led exactly to the situation studied in Sections 3, 4 and 5. Therefore, throughout this section we always suppose that $p_2 > p_1$.

The main results of the present section are :

Theorem 5. *Assume $1 \leq p_1 < p_2 < \infty$ and $r \geq 0$, $s, t \in \mathbf{R}$. Let condition (8) hold for all $p \in [p_1, p_2]$. Also suppose that (9) is in force.*

(a) Assume $q > p_2 + r - 1$ and either

$$(i) \qquad s + t \leq p_2$$

or

$$(ii) \qquad s+t > p_2, \qquad \min\left\{ \frac{s+t-p_1}{q-p_1-r+1} - \frac{t-n}{q},\ \frac{s+t-p_2}{q-p_2-r+1} - \frac{t-n}{q} \right\} < 0.$$

Then any entire C^1 distribution solution of equation (1) must vanish everywhere.

(b) Assume $p_1 + r - 1 < q \leq p_2 + r - 1$ and either

$$(i) \qquad s + t < 1 + q - r$$

or

$$(ii) \qquad s+t \geq 1+q-r, \qquad \frac{s+t-p_1}{q-p_1-r+1} < \frac{t-n}{q}.$$

Then any entire C^1 distribution solution of equation (1) *must vanish everywhere.*

Theorem 6. *Assume $1 \leq p_1 < p_2 < \infty$ and $r \geq 0$, $s, t \in \mathbf{R}$. Let condition* (8) *hold for all $p \in [p_1, p_2]$. Also suppose that* (9) *is in force.*

Under the assumptions (a) *or* (b) *of Theorem* 5 *any entire C^1 distribution solution of the inequality* ($1'$) *must be non-positive.*

Note that in the conditions of Theorems 5 and 6 there is no appearance of the parameter p, since the main exponents are now p_1 and p_2.

A simple calculation shows that conditions (i), (ii) of Theorem 5 (a) can be written alternatively as

$$s < \max\left\{p_2 - t,\ \frac{p_2+r-1}{q}\,(n-t) + p_2 - n,\ \frac{p_1+r-1}{q}\,(n-t) + p_1 - n\right\} \tag{6.1}$$

(the equality in (i) being unstated). The graph of the borderline condition has two corners, at $s = p_2 - n$, $t = n$ and at $s = 1-n-r$, $t = n+q$, see Figure 2.

Similarly, conditions (i), (ii) of Theorem 5 (b) can be written

$$s < \max\left\{1+q-r-t,\ \frac{p_1+r-1}{q}\,(n-t) + p_1 - n\right\}. \tag{6.2}$$

Here there is a (single) corner at $s = 1-n-r$, $t = n+q$.

Clearly Theorem 5 is a consequence of Theorem 6.

Case (a), (i) of Theorem 6 follows at once from Theorem 2(i) by taking $p = p_2$. Case (a), (ii) is a consequence of Theorem 2(ii) by taking first $p = p_1$ and then $p = p_2$.

To obtain case (b), (i) we take $p = q-r+1-\varepsilon$, with ε so small that $p_1 < p \leq p_2$. Then $q > p+r-1$ and the result again follows from Theorem 2(i), after letting $\varepsilon \to 0$.

Case (b), (ii) is a consequence of Theorem 2(ii) by taking $p = p_1$. ∎

Corollary. *Consider the equation*

$$\operatorname{div}\left\{A(x,u,Du)\,\frac{Du}{\sqrt{1+|Du|^2}}\right\} = b(x,u,Du)\,|u|^q \operatorname{sign} u \tag{6.3}$$

with $0 \leq A(x,z,\rho) \leq C_{\mathcal{A}}\,|x|^s\,|z|^r$ and $b(x,z,\rho) \geq C_{\mathcal{B}}\,|x|^{-t}$.

(a) *Assume $q > r+1$ and either*

$$s+t \leq 2$$

or

$$s < \max\left\{\frac{r+1}{q}\,(n-t) + 2 - n,\ \frac{r}{q}\,(n-t) + 1 - n\right\}.$$

(b) *Assume* $r < q \le r+1$ *and*

$$s < \max\left\{1+q-r-t,\ \ \frac{r}{q}(n-t)+1-n\right\}.$$

Then any entire C^1 *distribution solution of* (6.3) *must vanish everywhere and any entire* C^1 *distribution solution of the corresponding inequality must be non-positive.*

The corollary is just the special case $p_1 = 1$, $p_2 = 2$ of Theorem 5 and Theorem 6.

Remark. The case $r = 0$ of the corollary is particularly simple: Part (a) states that if $q > 1$ and either

$$s+t \le 2$$

or

$$s < \max\left\{\frac{1}{q}(n-t)+2-n,\ \ 1-n\right\}.$$

then $u \equiv 0$, while part (b) shows that if $0 < q \le 1$ and

$$s < \max\{1+q-t,\ 1-n\}.$$

then again $u \equiv 0$. The result of Example 2 in the introduction is an immediate consequence.

The conclusion of the corollary can also be compared with the result of Theorem 1 in the case $p = 2$. First suppose that $q \ge r+1$. Then *both* results equally assert that $u \equiv 0$ if $s+t \le 2$ or if

$$s < \frac{r+1}{q}(n-t)+2-n \tag{6.4}$$

(equivalent to $\nu q + n - t < 0$). However when $t > n+q$ the corollary gives a better result than (6.4), namely that $u \equiv 0$ even when

$$s < \frac{r}{q}(n-t)+1-n;$$

in other words, very surprisingly, a *better* result than Theorem 1 !

Equally surprising, in case $p = 2$, $q = 1$, $r = s = t = 0$ equation (5) has exponentially growing solutions, while in the case of equation (7) with $s = t = 0$, $f(u) = |u|^q \mathrm{sign}\, u$, if $0 < q \le 1$ then again $u \equiv 0$.

7. Shell conditions.

The large radii conditions (8),(9) (equally, (5.3), (5.4)) are in fact much stronger than necessary for the conclusions of the paper. In particular, let R_i, $i = 1, 2, 3, \cdots$, be a given sequence of radii *tending to infinity*, and consider the corresponding sequence of disjoint shells T_i in $\mathbf{R}^n$, defined by

$$T_i = B_{\kappa R_i} \backslash B_{R_i}, \qquad i = 1, 2, 3, \cdots,$$

where $\kappa > 1$ is a (fixed) constant and $R_{i+1} > \kappa R_i$. The conditions (8),(9) then in fact need to hold *only for* $x \in T_i$, $i = 1, 2, 3, \cdots$, with the exception that for Theorems A - D we also require the special radii restriction (exponential growth)

$$R_i \leq \lambda L^i, \qquad i \geq i_0, \tag{7.1}$$

for some fixed $\lambda > 0$, $L > 1$ and $i_0 \geq 1$.

To this end, observe that when the test function (2.5) in Section 2 is replaced by

$$\varphi = \varphi_i = \psi \left(\frac{|x|}{R_{i+1}} \right),$$

where ψ is now given by

$$\psi(\tau) = \frac{1}{\kappa - 1} \begin{cases} \kappa - 1, & 0 \leq \tau \leq 1, \\ \kappa - \tau, & 1 < \tau < \kappa, \\ 0, & \tau \geq \kappa, \end{cases}$$

then Lemma 2 takes essentially the same form with only the exception that R_1 can be taken as R_i (since $D\varphi = 0$ in $B_{R_{i+1}}$), that is

$$\begin{aligned} &C_{\mathcal{B}}^{-1} \min\{\alpha, 1\} \int_{B_{R_i} \cap \{u>0\}} [\mathcal{A}(x, u, Du) \cdot Du + \mathcal{B}(x, u, Du)\, u]\, u^{\alpha-1} \\ &\quad + \int_{\mathbf{R}^n \backslash B_{R_i}} |x|^{-t}\, [u^+]^{\alpha+q} \varphi^\beta \leq C_1 C_{\mathcal{B}}^{-1} \int_{T_{i+1}} |x|^s [u^+]^{\alpha+r+p-1} |D\varphi|^p\, \varphi^{\beta-p}. \end{aligned} \tag{7.2}$$

Similarly Lemma 3 has the same form except the integral on the left can now be taken over T_i (since $\varphi = 1$ on this set), that is

$$\int_{T_i} |x|^{-t}\, [u^+]^{\alpha+q} \leq C_2 R_{i+1}^{s+t-p} \int_{T_{i+1}} |x|^{-t}\, [u^+]^{\alpha+p+r-1}, \tag{7.3}$$

where C_2 now depends also on κ since $D\varphi = 1/R_{i+1}(\kappa - 1)$ in T_{i+1}.

Finally in Lemma 4 the first integral on the left is to be taken over B_{R_i} the second over T_i, and the value R on the right side is replaced by R_{i+1}, that is for $\alpha > 0$,

$$\begin{aligned} &C_{\mathcal{B}}^{-1} \min\{\alpha, 1\} \int_{B_{R_i} \cap \{u>0\}} [\mathcal{A}(x, u, Du) \cdot Du + \mathcal{B}(x, u, Du) u]\, u^{\alpha-1} \\ &\qquad\qquad + \int_{T_i} |x|^{-t}\, [u^+]^{q+\alpha} \leq C_3 R_{i+1}^{(q+\alpha)\nu+n-t}, \end{aligned} \tag{7.4}$$

where C_3 now depends also on κ. Note that up to this point we have not used the restriction (7.1).

We also shall require a more abstract (discrete) version of Lemma 5.

Lemma 5′. *Let h_i, $i \geq i_0 \geq 1$ be a sequence of non-negative numbers satisfying*

$$h_i \leq \theta h_{i+1}, \tag{7.5}$$

and

$$h_i \leq CK^i, \tag{7.6}$$

where C, K, θ are positive constants with $\theta K < 1$ Then $h_i = 0$ for $i \geq i_0$.

Proof. Fix $i \geq i_0$. By an ℓ times iteration of (7.5) we find that

$$h_i \leq \theta^\ell h_{i+\ell} \leq C\theta^\ell K^{i+\ell},$$

by (7.6). Thus in turn

$$h_i \leq CK^i(\theta K)^\ell.$$

The desired result follows by letting $\ell \to \infty$ in the last inequality and using the condition $\theta K < 1$. ■

The main results of this section can now be stated.

Theorem 1′. *Let conditions* (8), (9) *hold on the sequence of shells T_i, $i = 1, 2, 3, \cdots$. Assume $q > p + r - 1$ and either*

$$s + t < p,$$

or

$$s + t \geq p, \qquad \nu < \frac{t-n}{q},$$

Then any entire C^1 distribution solution u of the equation (1) *must vanish everywhere.*

Theorem 2′. *Let conditions* (8), (9) *hold on the sequence of shells T_i, $i = 1, 2, 3, \cdots$. Assume $q > p + r - 1$ and either*

$$s + t < p,$$

or

$$s + t \geq p, \qquad \nu < \frac{t-n}{q},$$

Then any entire C^1 distribution solution u of the equation (1′) *must be everywhere non-positive.*

Theorem 1′ is a consequence of Theorem 2′, so it is enough to prove the latter. But this is obtained exactly as for Theorem 2 (resp. Theorem 3, for $p = 1$), by letting $i \to \infty$ in (7.4), rather than $R \to \infty$ in (2.7) . ■

Since the sequence R_i is essentially arbitrary, *the shells T_i can have arbitrarily large gaps separating them!* With respect to Theorems 1 and 2, however, this is partially paid for by *not allowing* the limit case $s+t=p$ *for all values of* n *and* t. To obtain this final case of Theorem 1′, as well as the corresponding extensions of Theorems A–D, it is enough to add the restriction (7.1).

Theorem A′. *Let conditions* (8), (9) *hold on the sequence of shells* T_i, $i=1,2,3,\cdots$, *where the radii* R_i *satisfy condition* (7.1). *If* $q=p+r-1\geq 0$ *and* $s+t<p$, *then any entire* C^1 *distribution solution of* (1) *which has at most algebraic growth at infinity must vanish identically.*

Proof of Theorem A′. As in the proof of Theorem A, we take $\alpha=2$, $p+r-q-1=0$. Define

$$h_i \equiv \int_{T_i} |u^+|^{q+2} \leq L^{2|t|} C_2 R_{i+1}^{s+t-p} \int_{T_{i+1}} |u|^{q+2} = L^{2|t|} C_2 R_{i+1}^{s+t-p} h_{i+1} \tag{7.7}$$

in view of (7.3). Also by direct estimation, for all i suitably large, say $i\geq i_0$,

$$h_i \leq \text{Const.}\, R_i^{\delta}, \qquad \delta=(q+2)d+n, \tag{7.8}$$

where $d>0$ is the algebraic growth rate of u at infinity. On the other hand, by (7.1) we have

$$h_i \leq \text{Const.}\, \lambda^{\delta} L^{\delta i} = CK^i, \tag{7.9}$$

with $K=L^{\delta}$ and an appropriate constant C.

Now choose i_0 even larger if necessary, so that for any $i\geq i_0$ we have also

$$L^{2|t|} C_2 R_{i+1}^{s+t-p} \leq 1/2K \equiv \theta.$$

Thus (7.7) yields $h_i\leq \theta h_{i+1}$, whence applying Lemma 5′ gives $h_i=0$ for all $i\geq i_0$, that is $u(x)\leq 0$ for $x\in T_i$, $i\geq i_0$.

Now, using (7.2) rather than (2.6), the required conclusion follows exactly as in the proof of Theorem A.∎

Theorem B′. *Let conditions* (8), (9) *hold on the sequence of shells* T_i, $i=1,2,3,\cdots$, *where the radii* R_i *satisfy condition* (7.1). *If* $0\leq q<p+r-1$ *and* $s+t<p$, *then any entire* C^1 *distribution solution of* (1) *which satisfies*

$$u(x)=o(|x|^{\nu}) \qquad \nu=\frac{p-s-t}{p+r-q-1}$$

must vanish identically.

Theorem D′. *Let conditions* (8), (9) *hold on the sequence of shells* T_i, $i=1,2,3,\cdots$, *where the radii* R_i *satisfy condition* (7.1). *If* $p>1$, $0\leq q\leq p+r-1$ *and* $s+t=p$, *then any entire* C^1 *distribution solution of* (1) *which is bounded must vanish identically.*

Theorem 1′′. *Let conditions* (8), (9) *hold on the sequence of shells* T_i, $i = 1, 2, 3, \cdots$, *where the radii* R_i *satisfy condition* (7.1). *Assume* $p > 1$ *and* $q > p + r - 1$ *and*

$$s + t = p.$$

Then any entire C^1 *distribution solution* u *of the equation* (1) *must vanish everywhere.*

Taking account of the argument used above to obtain (7.9), the proof of Theorems B′, D' and 1′′ are essentially the same as for the proof of Theorems B, D and 1 (where for Theorem 1′′ we use Lemma 3_γ instead of Lemmas 1-3).

8. Entire solutions in Sobolev spaces.

The assumption that u is a distribution solution *of class* C^1 of (1), or (1′), is stronger than necessary, though it has the advantage of avoiding technical difficulties. In particular, we may equally consider entire solutions of (1) of class $W^{1,1}_{\rm loc}(\mathbf{R}^n)$, with the definition (2.1) continuing to apply for functions $\eta \in C^1(\mathbf{R}^n)$ having compact support in $\mathbf{R}^n$.

To state the main results of the section, it is convenient first to introduce the following (*c.f.* [13], Section 3.1)

Definition. *A distribution solution* $u \in W^{1,\sigma}_{loc}(\Omega)$, $1 \le \sigma \le \infty$, *of* (1) *in a domain* Ω *is called* σ*-regular if* $\mathcal{A}(\cdot, u, Du) \in L^{\sigma'}_{loc}(\Omega)$ *and* $\mathcal{B}(\cdot, u, Du) \in L^1_{loc}(\Omega)$.

The following results, exactly corresponding to Theorems 1, 2 and 3, (Sections 3, 5) are then valid under the assumptions (3),(4),(8),(9).

Theorem 7. *Assume* $p > 1$, $q > p + r - 1$. *Let* u *be an entire* σ*-regular distribution solution of equation* (1) (*respectively,* (1′)) *with either*

$$(i) \qquad s + t \le p$$

or

$$(ii) \qquad s + t > p, \qquad \nu = \frac{s + t - p}{q - p - r + 1} < \frac{t - n}{q}.$$

Then $u \equiv 0$ *in* $\mathbf{R}^n$ *if* u *is a solution of* (1), *or* $u \le 0$ *if* u *is a solution of* (1′).

Theorem 7′. *Assume* $p = 1$, $q > r$. *Let* $u = u(x)$ *be an entire* σ*-regular distribution solution of equation* (1) (*respectively,* (1′)) *with either*

$$s + t < 1,$$

or

$$s + t \ge 1, \qquad \nu = \frac{s + t - 1}{q - r} < \frac{t - n}{q},$$

or

$$s + t = 1, \qquad s + n - 1 = 0.$$

Then $u \equiv 0$ *in* $\mathbf{R}^n$ *if* u *is a solution of* (1), *or* $u \le 0$ *if* u *is a solution of* (1′).

The requirement in Theorems 7 and 7′ that u be σ-regular is an important condition which can significantly constrain the class of a solution. See *e.g.* Example 3 below.

Proof of Theorem 7 and Theorem 7′. It is enough (*cfr.* the proofs of Theorems 1-3) to prove Lemmas 2-4 (and Lemma 3_γ) for the case when u is an entire σ-regular solution of inequality (1′) in the space $W^{1,\sigma}_{\rm loc}(\mathbf{R}^n)$.

To this end we distinguish between the cases $\sigma = \infty$ and $\sigma \in [1,\infty)$. The first case is immediate by observing that members of $W^{1,\infty}_{\rm loc}(\mathbf{R}^n)$ always have a locally Lipschitz continuous representative, and that σ-regularity is then simply the natural condition that $\mathcal{A}$ and $\mathcal{B}$ are locally integrable, and that the desired Lemmata hold true with the same proofs if we consider locally Lipschitz continuous solutions u instead of C^1 solutions (indeed, all that we used in proving these lemmata was the property that $u \in C^1$ implies $u^+ \in C^{0,1}_{loc}$).

The case $\sigma \in [1,\infty)$ is more involved, requiring a more delicate test function than before, that is

$$\eta = \eta_{N,\varepsilon,h} = [\{u_N^+ + \varepsilon\}^\alpha]_h\, \varphi^\beta$$

where $\alpha > 0$, $\varepsilon > 0$, $\beta \ge p$, f_N is the function f truncated above at the value N, namely $f_N = f$ when $f < N$ and $f_N = N$ when $f \ge N$, and f_h denotes the mollification (regularization) of the function f with mollification radius h. Using the relation $Df_h = [Df]_h$ we then have

$$D\eta_{N,\varepsilon,h} = \alpha\left[\{u_N^+ + \varepsilon\}^{\alpha-1} Du_N^+]\right]_h \varphi^\beta + \beta\left[\{u_N^+ + \varepsilon\}^\alpha\right]_h \varphi^{\beta-1} D\varphi.$$

Therefore the inequality version of (2.1) becomes

$$\begin{aligned}\int \mathcal{B}(x,u,Du)\eta_{N,\varepsilon,h} \le &-\alpha \int \mathcal{A}(x,u,Du)\cdot\,\left[\{u_N^+ + \varepsilon\}^{\alpha-1} Du_N^+\right]_h \varphi^\beta \\ &-\beta \int \mathcal{A}(x,u,Du)\cdot\, D\varphi\,\left[\{u_N^+ + \varepsilon\}^\alpha\right]_h \varphi^{\beta-1}.\end{aligned}$$

The integrals are finite because, by σ-regularity, $\mathcal{A}(\cdot,u,Du) \in L^{\sigma'}_{loc}(\mathbf{R}^n)$ and $\mathcal{B}(\cdot,u,Du) \in L^1_{loc}(\mathbf{R}^n)$; while the remaining factors are bounded and φ is compactly supported in $\mathbf{R}^n$.

We can let $h \to 0$ in the last inequality. We note first that the factors $\{u_N^+ + \varepsilon\}^\alpha$ and $\{u_N^+ + \varepsilon\}^{\alpha-1}$ are positive and uniformly bounded (depending on ε and N). Since $Du_N^+ \in L^\sigma_{loc}(\mathbf{R}^n)$ it follows that, for both integrals on the right-hand side, one has $[\cdot]_h \to [\cdot]$ in the norm of L^σ_{loc} according to standard mollification theory. For the integral on the left we have $[\cdot]_h \to [\cdot]$ pointwise almost everywhere (see *e.g.,* [2], Theorem 6 of Appendix C). Consequently, using Hölder's inequality (for the integrals on the right) and dominated convergence (for the integral on the left), there results

$$\begin{aligned}\int \mathcal{B}(x,u,Du)\eta_{N,\varepsilon} \le &-\alpha \int_{\{0<u<N\}} \mathcal{A}(x,u,Du)\,\cdot\, Du\{u_N^+ + \varepsilon\}^{\alpha-1}\varphi^\beta \\ &-\beta \int \mathcal{A}(x,u,Du)\,\cdot\, D\varphi\,\{u_N^+ + \varepsilon\}^\alpha \varphi^{\beta-1}\end{aligned}$$

where *all the integrals are well-defined.* Next, as in the proof of Lemma 1 one can let $\varepsilon \to 0$, to obtain finally, by Fatou's Lemma and dominated convergence,

$$\int \mathcal{B}(x,u,Du)[u_N^+]^\alpha \varphi^\beta \le -\,\alpha \int_{\{0<u<N\}} \mathcal{A}(x,u,Du)\,\cdot\, Du\, u^{\alpha-1}\varphi^\beta - \beta \int \mathcal{A}(x,u,Du)\,\cdot\, D\varphi\,[u_N^+]^\alpha \varphi^{\beta-1}, \tag{8.2}$$

all the integrals being finite. Inequality (8.2) exactly corresponds to (2.3), with the only exceptions being that u^+ is replaced by u_N^+ and the integration set $\{u>0\}$ is replaced by $\{0<u<N\}$.

This being the case, by keeping N fixed we can now proceed *exactly as in Section 2* to obtain (cfr. (2.6))

(8.3)

$$C_{\mathcal{B}}^{-1}\min\{\alpha,1\}\int_{B_{R_1}\cap\{0<u<N\}}[\mathcal{A}(x,u,Du)\cdot Du+\mathcal{B}(x,u,Du)\,u]\,u^{\alpha-1} + \int_{\mathbf{R}^n\setminus B_{R_1}}|x|^{-t}\,[u_N^+]^{\alpha+q}\varphi^\beta \le C_1C_{\mathcal{B}}^{-1}\int_{B_{2R}\setminus B_R}|x|^s[u_N^+]^{\alpha+p+r-1}|D\varphi|^p\,\varphi^{\beta-p}$$

where C_1 is the constant appearing in Lemma 1 and then (*cfr. the last display line in the proof of Lemma 4*)

(8.4)

$$C_{\mathcal{B}}^{-1}\ \min\{\alpha,1\}\int_{B_{R_1}\cap\{0<u<N\}}[\mathcal{A}(x,u,Du)\,\cdot\,Du+\mathcal{B}(x,u,Du)\,u]u^{\alpha-1} + \frac{1}{\mu}\int_{\mathbf{R}^n\setminus B_{R_1}}|x|^{-t}\,[u_N^+]^{q+\alpha} \le \frac{1}{\mu}\left[C_1C_{\mathcal{B}}^{-1}\right]^\mu\int_{B_{2R}\setminus B_R}|x|^{t\mu/\mu'+s\mu}|D\varphi|^{p\mu}.$$

The last integral can be estimated by $C_3R^{\nu q+n-t}$ as in the proof of Lemma 4, with C_3 independent of N. Thus, for fixed $R\ge R_1\ge R_0$ we can let $N\to\infty$ in (8.4) and use monotone convergence for the first two integrals to obtain the inequality (2.7) (that is, Lemma 4) for the present case.

To get Lemma 2, observe that the last integral in (8.3) has the estimate

$$\begin{aligned}\int_{B_{2R}\setminus B_R}|x|^s[u_N^+]^{\alpha+p+r-1}|D\varphi|^p\,\varphi^{\beta-p} &\le 2^{(s+t)^+}\,R^{s+t-p}\int_{B_{2R}\setminus B_R}|x|^{-t}[u_N^+]^{\alpha+p+r-1}\\ &\le 2^{(s+t)^+}\,R^{s+t-p}\left[\int_{B_{2R}\setminus B_R}|x|^{-t}[u_N^+]^{\alpha+q}+\int_{B_{2R}\setminus B_R}|x|^{-t}\right]\end{aligned}$$

since $q>p+r-1$ (consider separately the case when $0\le u_N^+(x)\le 1$ and when $u_N^+(x)>1$). But then by (8.4) the right side is bounded by

$$2^{(s+t)^+}\left[C_3\,R^{\nu(q+\alpha)+s+n-p}+2^{|t|}\omega_n\,R^{s+n-p}\right].$$

Therefore, for fixed $R \geq R_1 \geq R_0$, all the integrals in (8.3) are uniformly bounded independently of N, so we can let $N \to \infty$ in (8.3) and use monotone convergence to obtain inequality (2.6) (that is, Lemma 2) for the present case. Finally, Lemma 3 is a direct consequence of Lemma 2 (from the above arguments, it is also clear that Lemma 3_γ holds true for the present case). ∎

The proofs of Theorems A, B, D use only Lemmas 2-4, so these results also hold equally when u is a σ-regular entire solution of (1), while Theorem E obviously remains true in this case.

Remark. Inequalities (2.6) and (2.7) for σ-regular solutions imply in particular that the integrand on the left hand side is locally integrable on the set $\{u > 0\}$. Since neither $\mathcal{A} \cdot Du\ u^{\alpha-1}$ nor $\mathcal{B}\, u^\alpha$ is *a priori* integrable on this set, the conclusion is deeper lying than might have seemed at first sight. Indeed, the proof of Theorems 7 and 7′ crucially uses all the major convergence theorems of Lebesgue theory, Dominated Convergence, Monotone Convergence, and Fatou's Lemma.

Remark. A slight weakening of the hypotheses of Theorem 7 is obtained by replacing the set of σ-regular solutions in $W^{1,\sigma}_{\mathrm{loc}}(\mathbf{R}^n)$ by the set $W^{1,\sigma(tr)}_{\mathrm{loc}}(\mathbf{R}^n)$, $1 \leq \sigma \leq \infty$, where the latter consists of functions $u \in W^{1,1}_{loc}(\mathbf{R}^n)$ such that :

i) $\mathcal{A}(\cdot, u, Du) \in L^{\sigma'}_{loc}(\mathbf{R}^n)$ and $\mathcal{B}(\cdot, u, Du) \in L^1_{loc}(\mathbf{R}^n)$.

ii) for any $N > 0$ the truncated functions u_N (= truncation of u above by N) and u_{-N} (= truncation of u below by $-N$) are in $W^{1,\sigma}_{\mathrm{loc}}(\mathbf{R}^n)$.

This is allowable since the proof remains word-for-word unchanged.

Example 3. Consider equation (5) with

$$A = |z|^r, \qquad b(x) = 1,$$

where $p > 1$, $r \geq 0$. We are interested in what restrictions, if any, are placed on the Sobolev class $W^{1,\sigma}_{\mathrm{loc}}(\mathbf{R}^n)$ of a solution by the condition that u be σ-regular, that is (since $|\mathcal{A}| = |u|^r\, |Du|^{p-1}$ and $|\mathcal{B}| = |u|^q$), that $|u|^r |Du|^{p-1} \in L^{\sigma'}_{loc}(\mathbf{R}^n)$ and $|u|^q \in L^1_{loc}(\mathbf{R}^n)$.

First, we emphasize, as an independent condition, that $|u|^q \in L^1_{loc}(\mathbf{R}^n)$. It is also necessary that $\sigma \geq p$, for otherwise we have $\sigma' > p'$ and $(p-1)\sigma' > p$, which means that u would again need to be in $L^p_{loc}(\mathbf{R}^n)$, at least.

This being shown, if $r = 0$ it is easy to see that we can take for σ any value $\geq p$. If next $p > n$, $r > 0$, then in view of Morrey's lemma we can again take for σ any value $\geq p$ while if $p = n$ and $r > 0$, we can similarly use any $\sigma > p$. To treat the remaining case $p < n$, $r > 0$, we have by Hölder's inequality (provided $\sigma > p$, which is surely necessary in this case)

$$\int_{B_R} |u|^{r\sigma'} |Du|^{(p-1)\sigma'} \leq \left(\int_{B_R} |u|^{r\sigma/(\sigma-p)} \right)^{1-(p-1)/(\sigma-1)} \cdot \left(\int_{B_R} |Du|^\sigma \right)^{(p-1)/(\sigma-1)} .$$

The first integral on the right can be estimated (a) by using the Sobolev inequality, that is $u \in L^{n\sigma/(n-\sigma)}_{loc}$ when $\sigma < n$ and $u \in W^{1,\sigma}_{loc}$, and (b) by applying the condition $|u|^q \in$

$L^1_{loc}(\mathbf{R}^n)$. Therefore it is not hard to see that $\mathcal{A}$ is locally in $L^{\sigma'}$ if

$$\sigma \geq \sigma_0 = \text{Min}\left\{(p+r)\,\frac{n}{n+r},\, p\,\frac{q}{q-r}\right\};$$

here obviously $p < \sigma_0 < n$, as needed. ■

Note that if $\sigma_0 \geq nq/(n+q)$, or if one simply asks that $\sigma_0 \geq nq/(n+q)$, then $q \leq n\sigma/(n-\sigma)$ and the condition $|u|^q \in L^1_{loc}(\mathbf{R}^n)$ becomes redundant.

9. The case r < 0. Nonexistence of positive solutions.

Rather than seeking conditions under which all entire solutions of (1) must vanish identically, one may instead ask the related question, *whether there can exist everywhere positive solutions.* For this end, the large radii condition (9) can be be significantly weakened, specifically so as to apply for the full range of parameters q, r, s, $t \in \mathbf{R}$, and moreover to hold only for $|x| \geq R_0$, $z > 0$, $\rho \in \mathbf{R}^n$. We get the following result.

Theorem 8. *Suppose $q > p + r - 1$ and either*

$$(i) \qquad s + t \leq p, \qquad (s + t < 1 \text{ when } p = 1),$$

or

$$(ii) \qquad s + t > p, \qquad p + r - 1 \geq 0, \quad q\nu < t - n,$$

where $\nu = (s + t - p)/(q - p - r + 1)$, (so $q > 0$, $t > n$), or

$$(iii) \qquad s + t > p, \quad p + r - 1 < 0, \quad s + n - p < 0.$$

Then the inequality $(1')$ has no everywhere positive entire C^1 distribution solutions.

Theorem 8 in the case $r \geq 0$ is just Theorem 2; it seems preferable however not to restrict the statement only to the case $r < 0$. Surprisingly, q may be negative in cases (i) and (iii).

Proof. One checks that Lemmas 1-3 hold when $\alpha > 0$; and that Lemma 4 is valid if additionally $\alpha+p+r-1 \geq 0$ and $q > p+r-1$. First consider case (i) with $s+t < p$. Then $\nu < 0$ in (2.7), and by choosing $\alpha > 0$ sufficiently large we can secure that $\alpha+p+r-1 \geq 0$ as well as $(q + \alpha)\nu + n - t < 0$. Then as in the proof of Theorem 2 (i) one finds that the set $\{u > 0\}$ is empty; that is there can be no everywhere positive solutions. The case $s + t = p > 1$ can be treated separately as in Section 4.

For case (ii) we have $\nu > 0$. Since $p + r - 1 \geq 0$ we can choose $\alpha > 0$ so small that again $(q + \alpha)\nu + n - t < 0$.

For case (iii) again $\nu > 0$. The main conditions

$$\alpha > 0, \qquad \alpha + p + r - 1 \geq 0, \qquad (q + \alpha)\nu + n - t < 0$$

can be simultaneously met by choosing α so that

$$0 < -(p+r-1) \le \alpha < \frac{1}{\nu}(t-n-\nu q).$$

To see that this is possible, it is enough to check that

$$t-n-\nu q+\nu(p+r-1) > 0.$$

But by direct calculation this is equivalent to $s+n-p<0$. ∎

Theorem 8 can be extended without difficulty to operators allowing multiple values of p.

Theorem 9. *Assume* $1 \le p_1 < p_2 < \infty$ *and* $q, r, s, t \in \mathbf{R}$. *Let condition* (8) *hold for all* $p \in [p_1, p_2]$, *and define*

$$\theta_1 = (p_1+r-1)^+, \qquad \theta_2 = (p_2+r-1)^+.$$

Finally suppose that (9) *is in force.*

(a) *Assume* $q > p_2+r-1$ *and either*

$$(i) \qquad s+t \le p_2$$

or

$$(ii) \quad s < \max\left\{\frac{\theta_1}{q}(n-t)+p_1-n,\ \frac{\theta_2}{q}(n-t)+p_2-n\right\}$$

(note that if either $\theta_1 > 0$ *or* $\theta_2 > 0$ *then* $q > 0$*, so* (ii) *is well-defined).*

Then the inequality ($1'$) *has no everywhere positive entire* C^1 *distribution solutions.*

(b) *Assume* $p_1+r-1 < q \le p_2+r-1$ *and either*

$$(i) \qquad s+t < 1+q-r$$

or

$$(ii) \quad s < \frac{\theta_1}{q}(n-t)+p_1-n.$$

Then the inequality ($1'$) *has no everywhere positive entire* C^1 *distribution solutions.*

Proof. A simple calculation shows that cases (ii) and (iii) in Theorem 8 can be combined and rewritten in the form

$$(ii) \quad s+t > p, \quad s < \frac{\theta}{q}(n-t)+p-n,$$

where $\theta = (p+r-1)^+$. Theorem 9 is now a direct consequence of the proof technique of Theorem 6; see particularly the reformulations (6.1) and (6.2) of the conditions of Theorems 5 and 6. ∎

Remark. In the same way, one can easily show under corresponding conditions that there are no everywhere negative solutions of the inequality reverse to ($1'$). Similarly, for the case of non-positive or non-negative solutions u of ($1'$) or of its reverse inequality, the conclusion is that $u \equiv 0$.

An interesting special case of Theorem 9 occurs when $p_2+r-1 \le 0$. Then $\theta_1 = \theta_2 = 0$ and the conditions of Theorem 9 (a) become $q > p_2 + r - 1$ and either $s + t \le p_2$ or $s + n - p_2 < 0$, while for Theorem 9 (b) they are

$$p_1 + r - 1 < q \le p_2 + r - 1$$

and either $s + t < 1 + q - r$ or $s + n - p_1 < 0$.

On the other hand, in the special (but important) case $p_1 = 1$, $p_2 = 2$, $q > 1$, $r = 0$, one finds that there can be no positive entire solutions whenever either

$$s + t \le 2$$

or

$$s < \max \left\{ \frac{1}{q}\,(n-t) + 2 - n,\ \ 1 - n \right\}.$$

10. Conditions ($8'$)–($9'$)

In this section we consider a generalization of the large radii conditions (8)$-$(9). More precisely we will assume that for almost all $|x| \ge R_0$ and all $z \in \mathbf{R}\backslash\{0\}$, $\rho \in \mathbf{R}^n$, there exists an exponent $p \ge 1$ such that

$$(8') \qquad |\mathcal{A}(x,z,\rho)|^p \le C_{\mathcal{A}}|x|^s|g(z)|^r[g'(z)\,\mathcal{A}(x,z,\rho)\cdot\rho]^{p-1},$$

and

$$(9') \qquad \mathcal{B}(x,z,\rho)\ \mathrm{sign}\, z \ge C_{\mathcal{B}}|x|^{-t}|g(z)|^q$$

where $g \in C^0(\mathbf{R}) \cap C^1(\mathbf{R}\setminus\{0\})$ is a function satisfying $g(0) = 0$ and $g'(z) > 0$ for all $z \ne 0$, $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ are positive constants, and $q \ge 0$, $r \ge 0$, s, $t \in \mathbf{R}$.

As already observed, conditions $(8) - (9)$ are recovered by choosing $g(z) = z$.

Under the new conditions ($8'$)$-$($9'$) *the conclusions of Theorems 1-9 hold in unchanged form, while Theorems A-E remain valid with suitable modifications*, see below. Except for Theorem 4, proved below, these results are direct consequence of the fact that Lemmas 1-4 continue to hold with natural modifications due to the presence of the new function g. More precisely, the modified lemmas take the following form.

Lemma 1$'$. *Let $u = u(x)$ be an entire C^1 distribution solution of the inequality* ($1'$). *Then for every $\alpha > 0$, $\beta \ge p \ge 1$, $R_1 \ge R_0 > 0$, and for every compactly supported non-negative locally Lipschitz continuous test function φ we have*

(10.1)

$$\alpha \int_{B_{R_1}\cap\{u>0\}} \mathcal{A}(x,u,Du)\cdot Du\, g^{'}(u)\,[g(u)]^{\alpha-1}\varphi^{\beta} + \beta \int_{B_{R_1}} \mathcal{A}(x,u,Du)\cdot D\varphi\,[g(u^+)]^{\alpha}\varphi^{\beta-1}$$

$$+ \int_{\mathbf{R}^n} \mathcal{B}(x,u,Du)\,[g(u^+)]^{\alpha}\varphi^{\beta} \;\le\; C_1 \int_{\mathbf{R}^n\setminus B_{R_1}} |x|^s [g(u^+)]^{\alpha+p+r-1}\varphi^{\beta-p}\,|D\varphi|^p,$$

where

$$C_1 = \alpha^{1-p}\beta^p C_{\mathcal{A}}$$

and $C_{\mathcal{A}}$ *is the constant appearing in* $(8')$.

The proof of Lemma $1'$ uses the new test function

$$\eta_\varepsilon = [g(u^+ + \varepsilon)]^{\alpha}\varphi^{\beta},$$

(instead of $[u^+ + \varepsilon]^{\alpha}\varphi^{\beta}$ as in Lemma 1). Here

$$D\eta_\varepsilon = \alpha\,[g(u^+ + \varepsilon)]^{\alpha-1} g'(u^+ + \varepsilon) Du^+\,\varphi^{\beta} + \beta\, g(u^+ + \varepsilon)]^{\alpha}\varphi^{\beta-1}D\varphi.$$

Then from (2.1), after letting $\varepsilon \to 0$ we obtain, corresponding to (2.3),

$$\int \mathcal{B}(x{,}u,Du) g(u^+)^{\alpha}\varphi^{\beta}$$
$$\le -\,\alpha \int_{\{u>0\}} \mathcal{A}(x,u,Du)\cdot Du\, g'(u)\, g(u)^{\alpha-1}\varphi^{\beta} - \beta \int \mathcal{A}(x,u,Du)\cdot D\varphi\, g(u^+)^{\alpha}\varphi^{\beta-1}.$$

We can now proceed exactly as in the proof of Lemma 1: here the presence of the extra terms $||g(z)||^r$, $g'(z)$ on the right hand side of $(8')$ are just what one needs for the final stage of the proof, that is, to derive the crucial inequality

$$-\beta\mathcal{A}(x,u,Du)\cdot D\varphi\, g(u)^{\alpha}\varphi^{\beta-1} \le \beta C_{\mathcal{A}}^{1/p}|x|^{s/p}\, g(u)^{\alpha+r/p}[g'(u)\mathcal{A}(x,u,Du)\cdot Du]^{1/p'}|D\varphi|\,\varphi^{\beta-1}$$

$$\le C_1|x|^s\, g(u)^{\alpha+p+r-1}\,|D\varphi|^p\varphi^{\beta-p} + \alpha\mathcal{A}(x,u,Du)\cdot Du\, g'(u)\, g(u)^{\alpha-1}\varphi^{\beta},$$

c.f. (2.4).■

With Lemma $1'$ in hand, it is straightforward to obtain the corresponding versions of Lemmas 2-4, (as well as Lemma 3_γ), which we state for the reader's convenience.

Lemma 2′. *Let* $u = u(x)$ *be an entire* C^1 *distribution solution of the inequality* $(1')$. *Then for every* $\alpha > 0$, $\beta \ge p \ge 1$, $R \ge R_1 \ge R_0$ *we have*

(10.2)

$$C_{\mathcal{B}}^{-1}\min\{\alpha,1\}\int_{B_{R_1}\cap\{u>0\}}[\mathcal{A}(x,u,Du)\cdot Du\,g'(u)+\mathcal{B}(x,u,Du)\,g(u)]\,[g(u)]^{\alpha-1}$$
$$+\int_{\mathbf{R}^n\setminus B_{R_1}}|x|^{-t}\,[g(u^+)]^{\alpha+q}\varphi^{\beta}\le C_1C_{\mathcal{B}}^{-1}\int_{B_{2R}\setminus B_R}|x|^{s}[g(u^+)]^{\alpha+p+r-1}|D\varphi|^p\,\varphi^{\beta-p}$$

where C_1 is the constant appearing in Lemma $1'$.

Lemma $3'$. *Let $u=u(x)$ be an entire C^1 distribution solution of the inequality $(1')$. Then for every $\alpha>0$, $p\ge 1$ and $R\ge 2R_0$, we have*

$$\int_{B_R\setminus B_{R/2}}|x|^{-t}\,[g(u^+)]^{\alpha+q}\le C_2R^{s+t-p}\int_{B_{2R}\setminus B_R}|x|^{-t}\,[g(u^+)]^{\alpha+p+r-1},$$

where

$$C_2=\frac{p^p}{\alpha^{p-1}}\,\frac{C_{\mathcal{A}}}{C_{\mathcal{B}}}\,2^{[s+t]^+}.$$

Lemma $4'$. *Let $u=u(x)$ be an entire C^1 distribution solution of the inequality $(1')$ and assume $q\ge p+r-1$. For $\alpha>0$, $p\ge 1$ and $R\ge R_1\ge R_0$ we have*

(10.3)

$$C_{\mathcal{B}}^{-1}\min\{\alpha,1\}\int_{B_{R_1}\cap\{u>0\}}[\mathcal{A}(x,u,Du)\cdot Du\,g'(u)+\mathcal{B}(x,u,Du)g(u)]\,[g(u)]^{\alpha-1}$$
$$+\int_{B_R\setminus B_{R_1}}|x|^{-t}\,[g(u^+)]^{q+\alpha}\le C_3R^{(q+\alpha)\nu+n-t},$$

where $\nu=(s+t-p)/(q+1-p-r)$ and

$$C_3=\frac{(p\mu)^{p\mu}}{\alpha^{(p-1)\mu}}\left(\frac{C_{\mathcal{A}}}{C_{\mathcal{B}}}\right)^{\mu}2^{[\mu(s+t)-t]^++n}\omega_n,\qquad \mu=(q+\alpha)/(q-p-r+1),$$

with ω_n the measure of the unit ball in $\mathbf{R}^n$.■

For the sake of completeness we give here the proof of Theorem 4, Section 5.

Proof of Theorem 4. In view of Lemma 6 and the hypotheses (5.3)–(5.5) of Theorem 4, the proof can be reduced to the case when $r=0$, $q=1$ and $g=f=\hat{f}$ in $(8')$–$(9')$. Then, if $s+t<1$ and if α is chosen sufficiently large, we find by letting $R\to\infty$ in (10.3) followed by $R_1\to\infty$, that

$$[\mathcal{A}(x,u,Du)\cdot Du\;g'(u)+\mathcal{B}(x,u,Du)g(u)]=0\qquad\text{a.e. in the set }\{u>0\}.$$

By (3), (4) we now conclude in the usual way that $u\le 0$ in $\mathbf{R}^n$.

For the second case, we have $\nu + n - t = (s + t - 1) + n - t < 0$. Here we can choose α suitably near 0 so that $(q + \alpha)\nu + n - t = (1 + \alpha)\nu + n - t < 0$ in (10.3), and then conclude as in the first case that $u \le 0$.

The remaining possibility is more involved. Here we have $\nu = 0$ and $t = n$; hence by letting $R \to \infty$ in (10.3), there results, for every $\alpha > 0$,

$$\int_{R^n \setminus B_{R_1}} |x|^{-t} [g(u^+)]^{\alpha+q} \le C_4.$$

This clearly entails the limit condition

$$\lim_{R\to\infty} \int_{B_{2R}\setminus B_R} |x|^{-t} [g(u^+)]^{\alpha+q} = 0. \tag{10.4}$$

From Lemma $2'$, however,

(10.5)

$$C_{\mathcal{B}}^{-1} \min\{\alpha, 1\} \int_{B_{R_1} \cap \{u>0\}} [\mathcal{A}(x,u,Du) \cdot Du\, g'(u) + \mathcal{B}(x,u,Du)\, g(u)]\, [g(u)]^{\alpha-1}$$
$$+ \int_{\mathbf{R}^n \setminus B_{R_1}} |x|^{-t}\, [g(u^+)]^{\alpha+q} \varphi^\beta \le C_1 C_{\mathcal{B}}^{-1} \int_{B_{2R}\setminus B_R} |x|^s [g(u^+)]^{\alpha+r} |D\varphi|\, \varphi^{\beta-1}$$
$$\le C_5 \int_{B_{2R}\setminus B_R} |x|^{-t} [g(u^+)]^{\alpha+r},$$

where C_5 does not depend on R since $|x|\,|D\varphi|$ is bounded (recall $\varphi \le 1$ and $s + t = 1$).

Take $\alpha = 1 + q - r$ in (10.5). Then letting $R \to \infty$ in (10.5) and applying (10.4) with $\alpha = 1$, we obtain finally (!)

$$[\mathcal{A}(x,u,Du) \cdot Du\ g'(u) + \mathcal{B}(x,u,Du) g(u)]\, g(u) = 0 \qquad \text{a.e. in the set } \{u > 0\}$$

and the required conclusions follow as before. ∎

Finally, Theorems A–E continue to hold in the present case, *provided that the growth conditions on the solution u in their statements are imposed* **instead on the function** $g(u)$. The proofs remain essentially unchanged (Lemmas 1-4 being replaced by Lemmas $1'$-$4'$), with the single exception that in the proof of Theorem E we use the alternative test function $g((u - \gamma)^+)$.

We conclude the present section with the remarkable case in which the function g is bounded. Under this assumption we can extend the conclusions of Theorems 1, 2, 3 and 4 to cover the *full range* $0 \le q < \infty$ of values for the exponent q, rather than the previously restricted set $q > p + r - 1$. More precisely we have

Theorem 10. *Assume $p > 1$, $q \geq 0$, $r \geq 0$, s, $t \in \mathbf{R}$ and let conditions $(8') - (9')$ hold with a* **bounded** *function g. Let $u = u(x)$ be an entire C^1 distribution solution of equation* (1) *(respectively,* $(1')$*) with either*

$$s + t \leq p,$$

or

$$s + t > p, \qquad s + n - p < 0.$$

Then $u \equiv 0$ in $\mathbf{R}^n$ if u is a solution of (1)*, or $u \leq 0$ if u is a solution of* $(1')$*.*

Note that Theorem 10 is false when g is unbounded (see Example 5 and 6 in section 11).

Proof of Theorem 10. There are five cases.

1. $q > p + r - 1$, $s + t \leq p$. The conclusion follows from Theorems 1 and 2 for the case $(8')$, $(9')$.

2. $q = p + r - 1$, $s + t < p$. Since $g(u(x))$ has algebraic growth as $|x| \to \infty$, the conclusion follows from Theorem A for the case $(8')$, $(9')$.

3. $0 \leq q < p + r - 1$, $s + t < p$. Since $g(u(x)) = o(|x|^\nu)$ as $|x| \to \infty$ and $\nu > 0$, the conclusion follows from Theorem B for the case $(8')$, $(9')$.

4. $0 \leq q \leq p + r - 1$, $s + t = p$. Since $g(u(x))$ is bounded the conclusion follows from Theorem D for the case $(8')$, $(9')$; recall here that $p > 1$.

5. $s + n - p < 0$. The right hand integral in (10.2), with the choices $\alpha = 1$, $\beta = p$, is easily seen to be bounded by Const. R^{s+n-p}. Thus letting $R \to \infty$ in (10.2) gives

$$\int_{B_{R_1} \cap \{u>0\}} \Big[\mathcal{A}(x, u, Du) \cdot Du\, g'(u) + \mathcal{B}(x, u, Du)\, g(u)\Big] = 0.$$

We then conclude as usual that $u \leq 0$ in $\mathbf{R}^n$ when u is a solution of $(1')$, and that $u \equiv 0$ when u is a solution of (1). ∎

The case $p = 1$, g bounded, becomes a corollary of Theorem 4, once it is observed that $(8')$, $(9')$ then reduce to (5.3)–(5.5), with the new constant $C_{\mathcal{A}}\, ||g||_\infty^r$ in (5.3) and with the function $f(z) = |g(z)|^q \operatorname{sign} z$ in (5.4), (5.5). As a result, we then get exactly the three conclusions of Theorem 4 :

Theorem 11. *Assume $p = 1$, $q \geq 0$, $r \geq 0$, s, $t \in \mathbf{R}$ and let conditions $(8') - (9')$ hold with a* **bounded** *function g. Let $u = u(x)$ be an entire C^1 distribution solution of equation* (1) *(respectively,* $(1')$*), with either*

$$s + t < 1$$

or

$$s + t \geq 1, \qquad s + n - 1 < 0$$

or

$$s + t = 1, \qquad s + n - 1 = 0.$$

Then $u \equiv 0$ in $\mathbf{R}^n$ if u is a solution of (1)*, or $u \leq 0$ if u is a solution of* $(1')$*.* ∎

11. Counterexamples.

This section is devoted to the sharpness of the various theorems above.

Example 4 - Sharpness of Theorems 1 *and* 3 (Sections 3 and 5). We shall show first that when $p > 1$ the conditions $(10')$, $(10'')$ of Theorem 1 are best possible, in the sense that for all $n \geq 1$, $p > 1$, $q \geq 0$, $r \geq 0$ with

$$(11.1) \qquad q > p + r - 1, \qquad s + t > p, \qquad \nu = \frac{s + t - p}{q - p - r + 1} > \frac{t - n}{q}.$$

there are non-negative functions $A = A(x, z)$ and $b = b(x)$ satisfying the large radii conditions $(5')$ and such that the corresponding equation (5) admits an explicit non-negative, unbounded $C^1(\mathbf{R}^n)$ entire solution.

In fact, the solution which we shall construct will also be of class C^∞.

We assert that the positive smooth function

$$u(x) = (1 + |x|^2)^{\nu/2}$$

solves equation (5) in $\mathbf{R}^n$ with

$$A(x, z) = C(x)^{2-p-s}|x|^s|z|^r,$$

$$b(x, z) = b(x) = \nu^{p-1}\left\{n + (\nu q + n - t)|x|^2\right\}(1 + |x|^2)^{-1-t/2},$$

where we have set $C(x) = |x|/\sqrt{1 + |x|^2}$.

When $|x| \geq 1$ we have $1/\sqrt{2} \leq C(x) < 1$ so the large radii conditions are satisfied with $R_0 = 1$, as are also (3) and (4). Moreover the functions

$$\mathcal{A}(x, u(x), Du(x)) = \nu^{p-1}(1 + |x|^2)^{[s+\nu r+\nu(p-1)-p]/2}x = \nu^{p-1}(1 + |x|^2)^{(\nu q-t)/2}x$$

and

$$\mathcal{B}(x, u(x), Du(x)) = \nu^{p-1}\left\{n + (\nu q + n - t)|x|^2\right\}(1 + |x|^2)^{-1+\frac{\nu q-t}{2}} = b(x)u^q,$$

are smooth in $\mathbf{R}^n$. In turn we find easily that

$$\mathrm{div}\mathcal{A}(x, u(x), Du(x)) = \nu^{p-1}\{(\nu q - t)C(x)^2 + n\}(1 + |x|^2)^{(\nu q-t)/2} = \mathcal{B}(x, u(x), Du(x))$$

as required. That u is a distribution solution is obvious, so u is the desired example.

When $p = 1$ a more delicate counterexample is required, namely

$$\mathcal{A}(x, z, \rho) = C(x)^{1-\varepsilon-s}|x|^s|z|^r\frac{|\rho|^{\varepsilon-1}\,\rho}{(1 + |\rho|^2)^{\varepsilon/2}},$$

for which (8) holds for $1 \le p \le 1+\varepsilon$. Using the ideas of Example 10 below then shows that there are explicit non-negative entire solutions when

$$q > r+\varepsilon, \qquad s+t > 1+\varepsilon \qquad \nu = \frac{s+t-1-\varepsilon}{q-r} > \frac{t-n}{q}.$$

Letting $\varepsilon \to 0$ then completes the counterexample. ■

The borderline relation $s+t-p>0$, $\nu q+n-t=0$ in $(10'')$ is not covered by the above example. The situation is illustrated in Figure 1, where one observes that the borderline relation can equally be written in the form

$$\frac{p-n-s}{p+r-1} = \frac{t-n}{q}, \quad t > n,$$

yielding the dashed line in the figure, with slope $= -(p+r-1)/q-1$.

Remark. When $q-p-r+1>0$, $s+t>p$ and $\nu q+n-t>0$ the function $u = c|x|^\nu$ is, for an appropriate constant $c \ne 0$, a classical solution of equation (5) in $\mathbf{R}^n\backslash\{0\}$, corresponding to the explicit functions $A \equiv |x|^s\,|z|^r$ and $b \equiv |x|^{-t}$, see $(5')$, $(5'')$. Here also

$$\mathcal{A}(x,u(x),Du(x)) = c^{p+r-1}\nu^{p-1}x|x|^{\nu q-t}, \qquad \mathcal{B}(x,u(x),Du(x)) = c^q|x|^{\nu q-t}.$$

Since we have $\mathcal{A}(x,u(x),Du(x))\,|x|^{n-1} = \text{Const.}\,|x|^{\nu q+n-t} \to 0$ as $|x|\to 0$, and similarly $\mathcal{B}(x,u(x),Du(x))\,|x|^n \to 0$ as $|x| \to 0$, it follows from a simple limiting argument that u is an entire distribution solution of (5).

When $\nu \ge 1$ then u is of class C^1 (locally Lipschitz continuous if $\nu = 1$). On the other hand, When $0 < \nu \le 1$ the situation is more delicate. A calculation shows that $u \in W^{1,\sigma}_{loc}(\mathbf{R}^n)$ with $\sigma = n/(1-\nu)$. At the same time $\mathcal{A} \in L^{n/(n-1)}_{loc}(\mathbf{R}^n)$ since $\nu q+n-t>0$. Hence u is a σ-regular entire solution of (1).

In particular, when $\nu \ge 1$ the non-trivial solution $u = c|x|^\nu$ shows that the conclusion of Theorem 7 (just as the conclusion of Theorem 1) is sharp. However when $0<\nu<1$ this function is in no better space than $W^{1,n/(1-\nu)}_{loc}(\mathbf{R}^n)$, and so cannot serve as a counterexample for Theorem 1.

The function $c|x|^\nu$ was introduced earlier in [7] in a similar context.

Two other solutions of equation (5) will be useful in what follows, namely (with the condition $p>1$)

$$u = E_\alpha(x) = e^{k|x|^\alpha}, \qquad u = E_\beta(x) = e^{-k/|x|^\beta},$$

where k, α, β are positive constants. In particular, the function $E_\alpha(x)$ satisfies (5) in the classical sense (except possibly at $x=0$), with $A(x,z) = |x|^s|z|^r$ and

$$\begin{aligned} b(x,z) = (\alpha k)^{p-1}\{&s+n-p+\alpha(p-1)+(p+r-1)\alpha k|x|^\alpha\} \\ &\cdot |x|^{s+t-p+\alpha(p-1)}\,E_\alpha^{-(q-p-r+1)}\,|x|^{-t}, \end{aligned}$$

while the (bounded) function $E_\beta(x)$ is a solution with $A(x,z) = |x|^s|z|^r$ and

$$b(x,z) = (\beta k)^{p-1}\{s+n-p-\beta(p-1)+(p+r-1)\beta k|x|^{-\beta}\}$$
$$\cdot\, |x|^{s+t-p-\beta(p-1)}\, E_\beta^{-(q-p-r+1)}\, |x|^{-t}.$$

If $q-p-r+1<0$ and $s+n-p+\alpha(p-1)\geq 0$, then when $u = E_\alpha(x)$ we have $b(x)\geq 0$ for all x and $b(x)\geq$ Pos. Const.$|x|^{-t}$ for $|x|\geq 1$. It is also not hard to see that the functions $\mathcal{A}(\cdot,u,Du)$ and $\mathcal{B}(\cdot,u,Du)$ are locally integrable (smooth except possibly at $x=0$), as required for E_α to be a distribution solution.

Similarly, if $q-p-r+1\leq 0$, with $s+n-p-\beta(p-1)\geq 0$ and $s+t-p-\beta p\geq 0$, then again when $u=E_\beta$ we find $b(x)\geq 0$ for all x and $b(x)\geq$ Pos. Const. $|x|^{-t}$ for $|x|\geq 1$. Similarly, the functions $\mathcal{A}(\cdot,u,Du)$ and $\mathcal{B}(\cdot,u,Du)$ are smooth.

Example 5 - Theorem A is best possible, in the sense that the algebraic growth condition cannot be weakened to exponential growth. In fact, there exist exponential solutions $E_\alpha(x)$ whenever $q-p-r+1=0$, $s+t<p$; see Figure 3. In particular, when $s=t=0$ we have $\alpha=1$, as noted in the introduction.

The figure also shows that in the previously undiscussed case when $q-p-r+1=0$, $s+t\geq p$, there exist bounded solutions $E_\beta(x)$ when $s+t>p$, $s>p-n$, and exponential solutions $E_\alpha(x)$ when $s\leq p-n$.

Example 6 - Sharpness of condition (11) *in Theorem* B. The function $u(x)=(1+|x|^2)^{\nu/2}$ constructed in Example 4 is also a solution of the corresponding equation (5) when

$$q<p+r-1, \qquad s+t<p, \qquad \nu q+n-t>0,$$

and
$$\nu = \frac{s+t-p}{q-p-r+1}.$$

This clearly implies the sharpness of (11), *provided however that* $\nu q+n-t>0$.

In the part of the set $\{s+t<p\}$ where $\nu q+n-t\leq 0$ we can no longer assert that the condition (11) is best possible. On the other hand, in this set the exponential solution $u=E_\alpha=e^{k|x|^\alpha}$ is valid with

$$\alpha = \frac{p-n-s}{p-1}, \quad k>0,$$

thus at least yielding an upper bound beyond which the conclusion $u\equiv 0$ cannot hold; see Figure 4. ■.

Example 7 - Sharpness of condition (13) *in Theorem* C. Example 4 shows that there are solutions for which the limit in (13) is a non-zero constant, and in particular that the exponent ν cannot be reduced.

We note also without discussion that there are non-trivial solutions for which the limit in (13) can be zero. ■

Example 8 - Sharpness of the growth condition in Theorem E. When

$$q-p-r+1<0, \quad s+t>p, \quad s+n-p>0$$

the function $u=E_\beta(x)=e^{-k/|x|^\beta}$ satisfies (5), (5′) provided that $\beta>0$ is suitably small. It is easy to see, therefore, at least when $s+n-p>0$, that there are non-trivial non-negative solutions u of (5), (5′) such that $u\neq o(1)$ as $|x|\to\infty$ and at the same time $u\leq C$ for any positive constant C. That is, the growth condition $u=o(1)$ is best possible in Theorem E when $s+n-p>0$.

In the part of the set $\{s+t>p\}$ where $s+n-p\leq 0$ we can of course no longer assert that this condition is best possible. On the other hand, in this set the exponential solution $u=E_\alpha=e^{k|x|^\alpha}$ is valid with

$$\alpha=\frac{p-n-s}{p-1}, \quad k>0,$$

thus again at least yielding an upper bound beyond which the conclusion $u\equiv 0$ cannot hold; see Figure 4. ■.

Example 9 - Necessity of the parameter conditions for Theorem 4 (Section 5). We shall construct the example for the special case of equation (7) (see Example 2).

Case I. We assert that for all n, s, t with

$$n\geq 1, \qquad s+t\geq 1, \qquad s+n-1>0,$$

there exist non-negative functions $A(x)$, $b(x)$ satisfying the large radii condition (7′), and a continuous function f satisfying (5.5), such that the corresponding equation (7) admits an explicit $C^2(\mathbf{R}^n)$ positive entire solution.

Indeed, the function $u(x)=1+|x|^2/2$ solves (7) in $\mathbf{R}^n$ when

$$A(x)=(1+|x|^2)^{s/2}, \qquad b(x)=(1+|x|^2)^{-t/2},$$

and

$$f(z)=(2z-1)^{(s+t-3)/2}\{n+2(s+n-1)(z-1)\}, \quad z\geq 1.$$

To see this, we have $Du=x$ and so

$$\text{div}\left\{A(x)\frac{Du}{\sqrt{1+|Du|^2}}\right\}=(1+|x|^2)^{(s-3)/2}\{n+(s+n-1)|x|^2\}, \tag{11.2}$$

and the assertion follows since $|x|^2=2(u-1)$. Here $f(z)$ satisfies (5.5) for $z\geq 1$, and clearly can be extended to an (odd) continuous function satisfying (5.5) for all z. Here we crucially use the fact that $f(1)>0$.

Note that the above function u cannot serve as a counterexample for the case $s+n-1=0$ whenever $1\leq s+t<3$, since in this case $\liminf_{z\to+\infty}f(z)=0$.

Case II. We shall show that for all n, s, t with

$$n \geq 1, \qquad s+t > 1, \qquad s+n-1 = 0$$

there exist functions $A(x)$, $b(x)$ satisfying the structural assumptions (3), (4), (5.3), (5.4) and a continuous function f satisfying (5.5), such that the corresponding equation (7) admits the explicit C^2 non-negative entire solution $u = |x|^2/2$.

We introduce a (*new*) function $A(x) = \hat{A}(x) = (1+|x|^2)^{s/2}\psi(x)$ with

$$\psi(x) = 1 - \frac{1}{(1+|x|^2)^{(s+t-1)/2}}.$$

Then for $u = |x|^2/2$ we have

$$\mathcal{A}(x, u(x), Du(x)) = \hat{A}(x)\frac{Du}{\sqrt{1+|Du|^2}} = \psi(x)(1+|x|^2)^{(s-1)/2}x.$$

In turn, recalling (11.2),

$$\begin{aligned}\mathrm{div}\mathcal{A}(x, u(x), Du(x)) &= [x\psi'(x)](1+|x|^2)^{(s-1)/2} + n\psi(x)(1+|x|^2)^{(s-3)/2} \\ &= \theta|x|^2(1+|x|^2)^{-1-t/2} + n\psi(x)(1+|x|^2)^{(s-3)/2}\end{aligned}$$

where $\theta = s+t-1$. Therefore $u = |x|^2/2$ is a solution of (7) with

$$b(x) = (1+|x|^2)^{-t/2} + \frac{n}{\theta}\psi(x)|x|^{-2}(1+|x|^2)^{(s-1)/2}$$

and

$$f(z) = 2\theta\,\frac{z}{2z+1} \qquad \text{(obeying (5.5))}.$$

Indeed, then

$$b(x)f(u) = b(x)f(|x|^2/2) = (1+|x|^2)^{-t/2}\,\frac{\theta|x|^2}{1+|x|^2} + n\psi(x)(1+|x|^2)^{(s-3)/2},$$

so $\mathrm{div}\,\mathcal{A} = b(x)f(u)$ as required.

Finally, $0 \leq A(x) \leq \mathrm{Const.}|x|^s$, $b(x) \geq 0$, and $b(x) \geq \mathrm{Pos.\ Const.}\,|x|^{-t}$ for $|x| \geq 1$, while also $\mathcal{A}$ and $\mathcal{B} = b(x)f(u)$ are smooth. This completes the counterexample.

Example 10 - Necessity of the parameter conditions for the corollary of Theorems 5, 6 *in Section* 6. Consider the equation

$$\mathrm{div}\left\{A(x,u,Du)\,\frac{|Du|^{\varepsilon-1}\,Du}{(1+|Du|^2)^{\varepsilon/2}}\right\} = b(x,u,Du)\,|u|^q\,\mathrm{sign}\,u \tag{11.3}$$

with

$$A(x,z,\rho) = C(x)^{1-\epsilon-s}|x|^s\,|z|^r \qquad b(x,z,\rho) = c(x)(1+|x|^2)^{-t}.$$

For this equation, condition (8) holds for $1 \le p \le 1+\varepsilon$; that is, $p_1 = 1$ and $p_2 = 1+\varepsilon$. Equation (6.3) is the particular case $\varepsilon = 1$.

Suppose that

$$q > r+\varepsilon, \qquad d = s+n-1-\varepsilon - \frac{r+\varepsilon}{q}\,(n-t) > 0.$$

Then one can verify that, for parameters s, t in the set $1+\varepsilon < s+t < 1+q-r$, the function

$$u(x) = (1+|x|^2)^{\nu/2}, \qquad \nu = \frac{s+t-1-\varepsilon}{q-r-\varepsilon} < 1,$$

is a solution of (11.3), with a corresponding positive function $c(x)$. Condition (9) moreover holds, since

$$\lim_{|x|\to\infty} c(x) = c' = \frac{q}{q-r-\varepsilon}\,\nu^\varepsilon d > 0.$$

Similarly, if

$$r < q \le r+\varepsilon, \qquad d = s+n-1-\frac{r}{q}\,(n-t) > 0,$$

then, for parameters s, t in the set $s+t \ge 1+q-r$, the function

$$u(x) = (1+|x|^2)^{\nu/2}, \qquad \nu = \frac{s+t-1}{q-r} \ge 1,$$

is again a solution of (11.3), with a corresponding positive function $c(x)$. Here, if $\nu > 1$, there holds

$$\lim_{|x|\to\infty} c(x) = c' = \frac{q}{q-r}\,d > 0,$$

while if $\nu = 1$ then

$$\lim_{|x|\to\infty} c(x) = c' = \frac{1}{2^{\varepsilon/2}}\frac{q}{q-r}\,d > 0,$$

whence condition (9) is satisfied in both cases.

We leave the somewhat lengthy calculations to the reader.

12. Open Questions

1. Does Theorem 1 (page 12) hold in the borderline case $\nu q+n-t = 0$, $t > n$? We have no proof but also no counterexample.

2. Can Theorem B be improved when $\nu q+n-t \le 0$? The counterexample 6 above does not apply for this case.

3. Can Theorems E and F be improved when $s \le p-n$? The counterexample 8 above does not apply for this case.

4. (H. Brezis) As a consequence of Brezis' Theorem (see the next section) there can be at most one entire solution of the equation

$$\Delta\, u = |u|^{q-1}\, u + g(x) \qquad \text{in} \quad \mathcal{D}'(\mathbf{R}^n)$$

when $q > 1$, $u \in L^q_{loc}(\mathbf{R}^n)$ and g is a given function in $L^1_{loc}(\mathbf{R}^n)$.

We include the proof (a slight modification of Brezis' original argument) for explicitness and for the convenience of the reader.

Proof. Let u_1 and u_2 be solutions, and set $v = u_2 - u_1$. Then

$$\Delta\, v = |u_2|^{q-1} u_2 - |u_1|^{q-1} u_1 \equiv \hat{f}.$$

By Lemma A 1 of [2] (see below) we then have

$$\begin{aligned} \Delta\, v^+ &\geq \operatorname{sign}^+(v)\, \hat{f} \geq \operatorname{sign}^+(v)\, \big||u_2|^{q-1} u_2 - |u_1|^{q-1} u_1\big| \\ &\geq \operatorname{sign}^+(u_2 - u_1)\, |u_2 - u_1|^q / 2^{q-1} = \frac{1}{2^{q-1}} |v^+|^{q-1}\, v^+. \end{aligned}$$

Then $v^+ \leq 0$ by Brezis' Theorem, the factor $1/2^{q-1}$ being unimportant by scaling, and so $v \leq 0$. Similarly $v \geq 0$, and we are done.∎

Does the same result hold for C^1 entire solutions of the equation

$$\Delta_p\, u = |u|^{q-1}\, u + g(x)$$

when $q > p - 1$? For other equations of the form (5)?

13. Appendix. Brezis' theorem

As noted in the introduction, H. Brezis showed that every entire C^1 distribution solution u of the equation $\Delta\, u = |u|^{q-1}\, u$ with $q > 1$ must vanish everywhere. In fact, Brezis proved a considerably more general result ([1], Lemma 2), namely

Let $u \in L^q_{loc}(\mathbf{R}^n)$, $q > 1$, satisfy

$$\Delta\, u \geq |u|^{q-1}\, u, \qquad \text{in} \quad \mathcal{D}'(\mathbf{R}^n). \tag{13.1}$$

Then $u \leq 0$ a.e. in $\mathbf{R}^n$.

That (13.1) holds in $\mathcal{D}'(\mathbf{R}^n)$ can be expressed explicitly in the form

$$\int u\, \Delta\, \eta \geq \int |u|^{q-1}\, u\, \eta \tag{13.2}$$

for all non-negative functions $\eta \in C^\infty(\mathbf{R}^n)$ with compact support in $\mathbf{R}^n$. In particular u need not be of class C^1 or even σ-regular in any Sobolev space $W^{1,\sigma}_{loc}(\mathbf{R}^n)$ (cfr. section 8).

Nevertheless, Brezis' Theorem can still be seen as a corollary of our Theorem 2 (i), Section 3. To this end we follow the proof of Theorem 4.7 in [3].

Indeed, let $f(t) = |t|^{q-1}\, t$, $t \in \mathbf{R}$. Then (13.1) takes the form

$$\Delta\, u \geq f(u(x)) \equiv \hat{f}(x) \qquad \text{in} \quad \mathcal{D}'(\mathbf{R}^n)$$

where

$$\hat{f} \in L^1_{loc}(\mathbf{R}^n)$$

by hypothesis. These are exactly the conditions of Lemma A.1 of [1], the conclusion of the lemma then being that

$$\Delta\, u^+ \geq \operatorname{sign}^+(u)\, \hat{f} \qquad \text{in} \quad \mathcal{D}'(\mathbf{R}^n). \tag{13.3}$$

But $\operatorname{sign}^+(u)\, \hat{f} = f(u^+)$, so

$$\int u^+\, \Delta\, \eta \geq \int f(u^+)\eta, \tag{13.4}$$

where η is a non-negative test function as above.

Take $\eta = \eta(y) = k_h(y-x)$, where k_h is a mollification kernel with mollification radius h, such that $\int k_h(y)\, dy = 1$. Let $v_h = v_h(x)$ be the mollification of u^+; from (13.4) we then get

$$\begin{aligned} \Delta v_h &= \Delta_x \int u^+(y)\, k_h(y-x)\, dy \\ &= \int u^+(y)\, \Delta_y\, [k_h(y-x)]\, dy \geq \int f(u^+(y))\, k_h(y-x)\, dy \end{aligned}.$$

Since $f(t)$ is convex for $t \geq 0$, it now follows from Jensen's inequality that

$$\Delta v_h \geq f\left\{ \int u^+(y)\, k_h(y-x)\, dy \right\} = f(v_h) = |v_h|^{q-1}\, v_h. \tag{13.5}$$

But $v_h \in C^\infty(\mathbf{R}^n)$, so applying Theorem 2 (i) yields $v_h \leq 0$ in $\mathbf{R}^n$ (*e.g.*, inequality (13.5) corresponds to the parameter values $p = 2$, $q > 1$, $r = s = t = 0$ in (8), (9); thus $q > p + r - 1 = 1$, $s + t \leq 2$ and Theorem 2 is applicable). On the other hand $v_h \geq 0$ by construction, so $v_h \equiv 0$. Letting $h \to 0$ gives $u^+ = 0$ a.e in $\mathbf{R}^n$, that is $u \leq 0$ a.e in $\mathbf{R}^n$. ∎

Remark. It is evident that the argument above cannot carry over to the general equation (1) or even to the special case (5).

Acknowledgements. We thank P. Pucci, R. Filippucci and E. Mitidieri for their useful comments.

Bibliograhic note. This paper was completed, and circulated in preprint form, in May, 2009.

References

[1] Brezis, H., Semilinear equations in $\mathbf{R}^n$ without condition at infinity. *Appl. Math. Optimization*, **12** (1984), 271–282.

[2] Evans, L. C., *Partial differential equations. Graduate Studies in Mathematics, 19.* Amer. Math. Soc., Providence (1998).

[3] Farina, A., Liouville–type Theorems for Elliptic Problems. In *Handbook of Differential Equations – Vol. 4, Stationary Partial Differential Equations*, ed. M. Chipot. Elsevier (2007), 60–116.

[4] Farina, A. and J. Serrin, Entire solutions of completely coercive quasilinear elliptic equations, II. To appear in *J. Diff. Equations* (2010).

[5] Filippucci, R., Existence of radial solutions of elliptic systems and inequalities of mean curvature type. *J. Math. Anal. Appl.*, **334** (2007), 604-620.

[6] Filippucci, R., Nonexistence of positive weak solutions of elliptic inequalities. *Nonlinear Analysis, TMA*, (2009), 2903–2916.

[7] Filippucci, R., P. Pucci and M. Rigoli, Non-existence of entire solutions of degenerate elliptic inequalities with weights. *Archive Rational Mech. Anal.*, **188** (2008), 155–179.

[8] Martio, O. and Porru,G., Large solutions of quasilinear elliptic equations in the degenerate case. Acta Univ. Uppsala (Skr. Uppsala Univ. C, Organ. Hist.), **64** (1999), 225-241.

[9] Mitidieri, E. and S. I. Pohozaev, Nonexistence of positive solutions for quasilinear elliptic problems on $\mathbf{R}^n$. *Trudy Mat. Inst. Steklov*, **227** (1999), 1–32. Translated in *Proc. Steklov Inst. Math.*, **227** (1999), 1–32.

[10] Mitidieri, E. and S. I. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. *Trudy Mat. Inst. Steklov*, **234** (2001), 1–384. Translated in *Proc. Steklov Inst. Math.*, **234** (2001), 1–382.

[11] D'Ambrosio, L. and E. Mitidieri, A priori estimates, positivity results and nonexistence theorems for quasilinear degenerate elliptic inequalities.*Adv. Math. 224 (2010), no.3, 967-1020.*

[12] Naito, Y. and H. Usami, Nonexistence results of positive entire solutions for quasilinear elliptic inequalities. *Canad. Math. Bull.*, **40** (1997), 244-253.

[13] Pucci, P. and J. Serrin, *The Maximum Principle.* Birkhauser Verlag, Basel (2007).

[14] Serrin, J., Entire solutions of quasilinear elliptic equations. *J. Math. Anal. Appl.*, **352** (2009), 3–14.

[15] Tkachev, V. G., Some estimates for the mean curvature of nonparametric surfaces defined over domains in $\mathbf{R}^n$. *Ukrain. Geom. Sb.*, **35** (1992), 135–150. Translated in *J. Math. Sciences* (New York), **72** (1994), 3250–3260.

[16] Usami, H., Nonexistence of positive entire solutions for elliptic inequalities of the mean curvature type. *J. Diff. Equations*, **111** (1994), 472-480.

Alberto Farina
LAMFA – CNRS UMR 6140
Université de Picardie Jules Verne
Faculté des sciences
33, rue Saint-Leu
80039 Amiens CEDEX 1, France
email: *alberto.farina@u-picardie.fr*

James Serrin
Department of Mathematics
University of Minnesota, Minneapolis, USA
e-mail: *serrin@math.umn.edu*