Γ-convergence of free discontinuity problems for circle-valued maps in the linear regime

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Abstract

We investigate the Γ-convergence of Ambrosio–Tortorelli type-functionals for circle-valued functions, in the case of volume terms with linear growth. We show the emergence of a non-local Γ-limit, which is due to the topological structure of the target space, and discuss compactness of minimal liftings. Our results extend the analysis of a previous work on the quadratic case.

Key words: Minimizing liftings, Γ-convergence, $\mathbb{S}^1$-valued maps, free boundary problems.

AMS (MOS) subject classification: 49Q15, 49Q20, 49J45, 58E12.

1 Introduction

Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a connected and simply connected bounded open set with Lipschitz boundary, and let $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$ denote the unit circle. For $u \colon \Omega \to \mathbb{S}^1$ and $v \colon \Omega \to [0,1]$, consider the family of functionals defined, for $\varepsilon > 0$, by

$$\mathrm{AT}_\varepsilon^{\mathbb{S}^1}(u,v) := \int_\Omega \left(v^2 |\nabla u|^2 + \varepsilon |\nabla v|^2 + \frac{(1-v)^2}{\varepsilon} \right) \mathrm{d}x \,.$$

As in the classical Ambrosio-Tortorelli approximation [5, 6], one observes a decoupling, as $\varepsilon \to 0^+$, between the bulk term $\int_\Omega v^2 |\nabla u|^2 \, \mathrm{d}x$ and the phase-field term $\int_\Omega (\varepsilon |\nabla v|^2 + (1-v)^2/\varepsilon) \, \mathrm{d}x$. However, in sharp contrast with the classical setting, [12] shows that for $\mathbb{S}^1$-valued maps the Γ-limit of $\mathrm{AT}_\varepsilon^{\mathbb{S}^1}$ depends on the choice of the functional domain. More precisely, if one considers

$$(u,v) \in W^{1,1}(\Omega; \mathbb{S}^1) \times W^{1,2}(\Omega) \quad \text{with} \quad v|\nabla u| \in L^2(\Omega) \,,$$

then $\mathrm{AT}_\varepsilon^{\mathbb{S}^1}$ Γ-(L^1)-converges to the Mumford-Shah functional for circle-valued maps, namely to

$$\mathrm{MS}_{\mathbb{S}^1}(u) := \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \mathcal{H}^{n-1}(S_u) \,, \quad u \in SBV^2(\Omega; \mathbb{S}^1) \,.$$

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If, instead, one restricts to the smaller domain

$$(u,v) \in W^{1,2}(\Omega;\mathbb{S}^1) \times W^{1,2}(\Omega)\,,$$

then $\mathrm{AT}_\varepsilon^{\mathbb{S}^1}$ Γ-(L^1)-converges to the nonlocal functional

$$\mathrm{MS}_{\mathrm{lift}}(u) := \int_\Omega |\nabla u|^2 \,\mathrm{d}x + m_2[u]\,, \quad u \in SBV^2(\Omega;\mathbb{S}^1)\,,$$

where, denoting by S_φ the jump set of a function φ,

$$m_2[u] := \inf\{\mathcal{H}^{n-1}(S_\varphi) \colon \varphi \in GSBV^2(\Omega)\,,\ e^{i\varphi} = u \text{ a.e. in } \Omega\}\,. \tag{1.1}$$

Since $S_u \subseteq S_\varphi$ for any lifting φ of u, it follows

$$\mathrm{MS}_{\mathrm{lift}}(u) \geq \mathrm{MS}_{\mathbb{S}^1}(u)\,.$$

This dichotomy reflects the influence of the topological structure of $\mathbb{S}^1$, in particular the possibility of a nonzero degree for circle-valued maps. In the first case, one has more freedom in the construction of the recovery sequence, and it is possible to preserve the degree throughout the approximation procedure. In contrast, in the second case, since a map $u \in W^{1,2}(\Omega;\mathbb{S}^1)$ has zero topological degree whenever Ω is simply connected (see, e.g., [17,18]), one is forced to approximate maps with nonzero degree by functions with zero degree, which entails an additional energetic cost. This phenomenon is clearly illustrated by the example of the vortex map described in [12] and is observed also in other variational problems concerning the relaxation of the area functional in codimension two [8] and the study of ferromagnetic spin systems [22].

The minimisation problem in (1.1) can be formulated for any $u \in SBV^p(\Omega;\mathbb{S}^1)$ with $p > 1$ by minimizing over all liftings $\varphi \in GSBV^p(\Omega)$; we denote the corresponding value by $m_p[u]$. In [12] it is proven that $m_p[u]$ admits a minimizer which in general does not belong to $SBV^p(\Omega)$, and that such minimisers are related either to suitably defined optimal transport problems (when $n = 2$) or to certain solutions of the Plateau problem (when $n > 2$).

Linear growth setting. In the present paper we consider a variant where the quadratic term $|\nabla u|^2$ is replaced by a term of the form $f(|\nabla u|)$, with f having linear growth at infinity (see Section 3 for the precise assumptions). Similarly to the unconstrained case [1,2], the linear growth introduces a genuine interaction between the bulk term and the phase-field term which is absent in the quadratic case. In addition, the resulting Γ-limits, which again depend on the choice of the functional domain, are defined on the larger space $BV(\Omega;\mathbb{S}^1)$ and thus have a Cantor part.

More precisely, by allowing also for more general phase-field terms (cf. Section 3), our first main result Theorem 3.1 states that if $(u,v) \in W^{1,1}(\Omega;\mathbb{S}^1) \times W^{1,2}(\Omega)$ then the Γ-limit is a local free-discontinuity functional of the form

$$\int_\Omega f(|\nabla u|)\,\mathrm{d}x + |D^c u|(\Omega) + \int_{S_u} g(|u^+ - u^-|)\,\mathrm{d}\mathcal{H}^{n-1}\,, \quad u \in BV(\Omega;\mathbb{S}^1)\,,$$

where g is the truncation function introduced in (2.3) which encodes the interaction between the volume term and the phase-field term and already appears in the unconstrained linear-growth theory [1]. Our second main result Theorem 3.2 states that if one restricts to the smaller domain $(u,v) \in W^{1,2}(\Omega;\mathbb{S}^1) \times W^{1,2}(\Omega)$, then the Γ-limit is the nonlocal functional

$$\int_\Omega f(|\nabla u|)\,\mathrm{d}x + |D^c u|(\Omega) + m_g[u]\,, \quad u \in BV(\Omega;\mathbb{S}^1)\,,$$

where $m_g[u]$ is a minimization problem associated to the cost function g, that is

$$m_g[u] := \inf\Big\{ \int_{S_\varphi} g(|\varphi^+ - \varphi^-|)\,\mathrm{d}\mathcal{H}^{n-1} \colon \varphi \in GBV(\Omega)\,,\ e^{i\varphi} = u \text{ a.e. in } \Omega \Big\}\,. \tag{1.2}$$

We also prove that $m_g[u]$ is actually a minimum, i.e., we prove the existence of a minimal lifting for $m_g[u]$. As a byproduct of this, we establish compactness results for sequences of liftings in the linear-growth setting (Theorems 4.2 and 4.5), extending the results of [12] to the full BV-framework.

The linear growth case entails substantial differences in the proof strategy. First, compactness results for liftings in the class GBV are required, which rely on slicing arguments and measure-theoretic localization techniques. Moreover, the genuine interaction between the bulk and the phase-field terms prevents a direct adaptation of the quadratic-growth analysis and necessitates a localized approach to Γ-convergence. These tools allow us to simplify several arguments with respect to the quadratic case, while at the same time extending the theory to a broader and more flexible variational framework. In addition, our analysis reveals natural connections with optimal transport problems and with the relaxation of the area functional in $\mathbb{S}^1$. We briefly discuss these relations below and defer a complete investigation to future work.

Connections with optimal transport. The quantity $m_g[u]$ admits a natural interpretation as an optimal transport problem with cost g. Assume for simplicity $\Omega \subset \mathbb{R}^2$ and let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ have distributional Jacobian of the form

$$\mathrm{Det}(\nabla u) = \pi \sum_{i=1}^{N} (\delta_{x_i} - \delta_{y_i})\,, \quad x_i, y_i \in \Omega\,.$$

Setting $\mu := \sum_{i=1}^N \delta_{x_i}$ and $\nu := \sum_{i=1}^N \delta_{y_i}$ one can show that

$$m_g[u] = \inf \int_R g(\theta)\,\mathrm{d}\mathcal{H}^1\,,$$

where the infimum is taken over all integer multiplicity 1-currents $T = (R,\theta,\tau)$ with $\partial T = \mu - \nu$. We refer to [12] for the discussion on the equivalence between the minimization problem $m_g[u]$ and the optimal transport problem with cost function $g \equiv 1$. For optimal transport problems in this setting we refer to [19,26,28,29] and references therein.

Connections with the relaxation of the area functional in $\mathbb{S}^1$. The relaxed area in $\mathbb{S}^1$ of the graphs of L^1-functions $u\colon \Omega \to \mathbb{S}^1$ is defined by

$$\mathcal{A}^{\mathbb{S}^1}(u,\Omega) := \inf\left\{ \liminf_{k\to+\infty} \int_\Omega \sqrt{1+|\nabla u_k|^2}\,\mathrm{d}x \colon u_k \in C^1(\Omega;\mathbb{S}^1)\,,\ u_k \to u \text{ in } L^1 \right\}\,.$$

This functional was characterised in terms of Cartesian currents (cf. [27]). Alternatively one could propose a characterisation using lifting theory. For instance, if Ω is simply connected, it is not difficult to prove that

$$\mathcal{A}^{\mathbb{S}^1}(u,\Omega) = \int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x + |D^c u|(\Omega) + m[u]\,,$$

where

$$m[u] := \min\left\{\int_{S_\varphi} |\varphi^+ - \varphi^-| \mathrm{d}\mathcal{H}^{n-1} \colon \varphi \in BV(\Omega)\,,\, e^{i\varphi} = u \text{ a.e. in } \Omega\right\}.$$

The similarity with (1.2) (when $f(t) = \sqrt{1+t^2}$) suggests the possibility of approximating $\mathcal{A}^{\mathbb{S}^1}$ through suitably chosen linear-growth Ambrosio–Tortorelli type functionals. A diagonal argument indeed provides such an approximation, opening interesting perspectives for future research related to the relaxation of the area functional in codimension two (see [8–11, 13, 13, 14, 20, 30, 31] and references therein for related results).

Let us also mention that, due to its property of approximation of the Mumford-Shah type energy of circle valued maps, as observed in [12], our Ambrosio-Tortorelli functional has a strict connection with analysis of Ginzburg-Landau energies and dislocations mechanics (we refer to [15] for the former topic). Specifically, the Mumford-Shah functional for $\mathbb{S}^1$-valued maps has been successfully used to model the appearence of dislocation in 2-dimensional domains (see [23, 25] for more details; we also refer to the introduction of [12] for a more exhaustive discussion on the link with vortices singularities).

Content of the paper. In Section 2 we collect some notation and recall some useful results needed to prove our main results. In Section 3 we introduce the problem and state our main results Theorem 3.1 and Theorem 3.2. In Section 4 we state two compactness results for sequences of liftings (Theorem 4.2 and Theorem 4.5). We also state and prove existence of solution to (1.2). Section 5.2 is instead devoted to the proof of Theorems 4.2 and 4.5. Eventually in Section 6 we provide the proofs of Theorem 3.1 and Theorem 3.2.

2 Notation and preliminaries

We start to recall some notions concerning BV and GBV functions [4] and lifting theory [17]. In what follows: $\partial^* A$ denotes the reduced boundary of a finite perimeter set $A \subset \mathbb{R}^n$, $|\cdot|$ and $\mathcal{H}^{n-1}$ denote the Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, respectively, and χ_A denotes the characteristic function of the set $A \subset \mathbb{R}^n$.

We recall the following lemma [16, Proposition 1.16]:

Lemma 2.1 (Localisation)**.** *Let $\Omega \subset \mathbb{R}^n$ be open bounded with Lipschitz boundary and let $\mathcal{A}(\Omega)$ be the family of the open subsets of Ω. Let $\mu \colon \mathcal{A}(\Omega) \to [0, +\infty)$ be a superadditive function on disjoint open sets, λ be a positive measure on Ω and, for $k \in \mathbb{N}$, $h_k \colon \Omega \to [0, +\infty]$ be a Borel function such that*

$$\mu(A) \geq \int_A h_k \,\mathrm{d}\lambda \qquad \textit{for every } A \in \mathcal{A}(\Omega) \textit{ and } k \in \mathbb{N}.$$

Then

$$\mu(A) \geq \int_A \sup_{k\geq 1} h_k \,\mathrm{d}\lambda \qquad \textit{for every } A \in \mathcal{A}(\Omega)\,.$$

2.1 BV and GBV functions

Let $U \subset \mathbb{R}^n$ be open and bounded, and $m \geq 1$ be an integer. We denote by $BV(U; \mathbb{R}^m)$ the space of vector-valued *functions with bounded variation in* U, and with $|\cdot|_{BV}$ and $\|\cdot\|_{BV}$ the BV seminorm and norm, respectively, i.e.

$$|u|_{BV} := |Du|(U), \qquad \|u\|_{BV} := \|u\|_{L^1} + |u|_{BV},$$

see [4]. We recall that if $u \in BV(U;\mathbb{R}^m)$ then $u \in L^1(U;\mathbb{R}^m)$ and its distributional gradient is a finite $\mathbb{R}^{m\times n}$- valued Radon measure which can be decomposed as

$$Du = \nabla u \mathcal{L}^n + D^c u + [\![u]\!] \otimes \nu_u \mathcal{H}^{n-1} \llcorner S_u\,,$$

where ∇u is the approximate gradient of u, $D^c u$ the Cantor part, S_u is the approximate jump set of u, $[\![u]\!] = u^+ - u^-$ is the jump opening, ν_u is a unit normal to S_u and $\otimes$ stands for the tensor product, see [4, Def. 3.67]. We say that $u \in BV(U;\mathbb{R}^m)$ is a special function of bounded variation in U if $D^c u = 0$. We denote by $SBV(U;\mathbb{R}^m)$ the space of vector-valued *special functions with bounded variation in* U. For $p > 1$ we define also the space

$$SBV^p(\Omega;\mathbb{R}^m) = \{u \in SBV(\Omega;\mathbb{R}^m) \colon |\nabla u| \in L^p(\Omega)\,,\ \mathcal{H}^{n-1}(S_u) < +\infty\}\,.$$

A measurable function $u : U \to \mathbb{R}^m$ belongs to the space of *generalised functions with bounded variation in* U, that is, $u \in GBV(U;\mathbb{R}^m)$, if $\phi \circ u \in BV_{\mathrm{loc}}(U)$ for any $\phi \in C^1(\mathbb{R}^m)$ with $\nabla\phi$ compactly supported.[1] Note that also in GBV we can define the approximate gradient, the Cantor part and the jump set. Moreover $GBV(\Omega,\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m) = BV(\Omega,\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$. Analogously one can define $GSBV(U;\mathbb{R}^m)$ and $GSBV^p(U;\mathbb{R}^m)$. If $m = 1$ we write $BV(U) = BV(U;\mathbb{R})$, and $GBV(U) = GBV(U;\mathbb{R})$.

Remark 2.2 (Equivalent definition of GBV for $m = 1$). $GBV(U)$ can be equivalently defined as the space of measurable functions $u \colon U \to \mathbb{R}$ such that $u \wedge M \vee (-M) \in BV_{\mathrm{loc}}(U)$ for any $M > 0$.

We set

$$BV(U;\mathbb{S}^1) = \{u \in BV(U;\mathbb{R}^2) \colon |u| = 1 \ \text{ a.e. in } U\}\,.$$

Eventually, a (finite or countable) family $\{E_i\}$ of finite perimeter subsets of a finite perimeter set F is called a Caccioppoli partition of F if the sets E_i are pairwise disjoint, their union is F, and the sum of their perimeters is finite.

2.2 Approximation and compactness for the vector Mumford-Shah functional

We recall a result by Alicandro-Braides-Shah [1], then extended to the vector case by Alicandro-Focardi [2]. Let

$$\mathcal{D} := \{(u,v) \in W^{1,2}(\Omega;\mathbb{R}^m) \times W^{1,2}(\Omega) \colon 0 \le v \le 1\}.$$

Consider the family of functionals $\mathrm{F}_\varepsilon \colon L^1(\Omega;\mathbb{R}^m) \times L^1(\Omega) \to [0,+\infty]$ given by

$$\mathrm{F}_\varepsilon(u,v) := \begin{cases} \displaystyle\int_\Omega \left(\psi(v) f(|\nabla u|) + \varepsilon|\nabla v|^2 + \frac{W(v)}{\varepsilon}\right) \mathrm{d}x & \text{if } (u,v) \in \mathcal{D}\,, \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^m) \times L^1(\Omega)\,, \end{cases} \tag{2.1}$$

where

(a) $\psi \colon [0,1] \to [0,1]$ is an increasing lower semicontinuous function with $\psi(0) = 0$, $\psi(1) = 1$ and $\psi(t) > 0$ if $t \neq 0$;

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[1] Recall that $f \in BV_{\mathrm{loc}}(U)$ if $f \in BV(U')$ for every $U' \subset U$ open with $\overline{U}'$ compact in U.

(b) $f\colon [0,+\infty) \to [0,+\infty)$ is a convex increasing function such that

$$\lim_{t\to\infty} \frac{f(t)}{t} = 1\,;$$

(c) $W\colon [0,1] \to [0,+\infty)$ is a continuous function such that $W(s) = 0$ if and only if $s = 1$.

Set also

$$c_W(t) := 2\int_t^1 \sqrt{W(s)}\,\mathrm{d}s\,, \tag{2.2}$$

and

$$g(z) := \min\{\psi(t)z + 2c_W(t)\colon 0 \leq t \leq 1\}\,. \tag{2.3}$$

Then the following theorem holds.

Theorem 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set with Lipschitz boundary. Then F_ε $\Gamma - L^1$-converge to the functional F defined as*

$$\mathrm{F}(u,v) := \int_\Omega f(|\nabla u|)\,\mathrm{d}x + |D^c u|(\Omega) + \int_{S_u} g(|u^+ - u^-|)\,\mathrm{d}\mathcal{H}^{n-1}$$

if $u \in GBV(\Omega;\mathbb{R}^m)$, $v = 1$ a.e., and $+\infty$ otherwise in $L^1(\Omega;\mathbb{R}^m) \times L^1(\Omega)$, as $\varepsilon \to 0^+$.

Remark 2.4. By inspecting the proof of Theorem 2.3 one actually deduces the following properties:

(i) **Γ-convergence on a larger domain.** The result still holds if one replaces $\mathcal{D}$ with the larger class

$$\widehat{\mathcal{D}} := \{(u,v) \in W^{1,1}(\Omega;\mathbb{R}^m) \times W^{1,2}(\Omega)\colon 0 \leq v \leq 1\}.$$

(ii) **Lower bound on open sets.** The lower bound inequality holds on every $A \in \mathcal{A}(\Omega)$. Precisely, let $\mathrm{F}_\varepsilon(\cdot,\cdot,A)$ and $\mathrm{F}(\cdot,\cdot,A)$ denote the localised functionals on A and let $(u_\varepsilon, v_\varepsilon) \to (u,v)$ in $L^1(\Omega;\mathbb{R}^m) \times L^1(\Omega)$. Then

$$\liminf_{\varepsilon\to 0^+} \mathrm{F}_\varepsilon(u_\varepsilon, v_\varepsilon, A) \geq \mathrm{F}(u,v,A);$$

(iii) **g is a truncating function.** The function g in (2.3) satisfies the following properties:

1. g is increasing, $g(0) = 0$ and $\lim_{z\to+\infty} g(z) = 2c_W(0)$;
2. g is subadditive, i.e., $g(z_1 + z_2) \leq g(z_1) + g(z_2)$ for all $z_1, z_2 \in [0,+\infty)$;
3. g is Lipschitz-continuous with Lipschitz constant 1;
4. $g(z) \leq z$ for all $z \in [0,+\infty)$ and $\lim_{z\to 0^+} g(z)/z = 1$;
5. For any $T > 0$ there exists $c_T > 0$ such that $z \leq c_T g(z)$ for all $z \in [0,T]$;
6. For $\sigma > 0$ there exists $C_\sigma > 0$ such that $g(z) \geq C_\sigma$ for all $z \geq \sigma > 0$.

Properties 1. – 5. follow by [1, Remark 4.2]. Property 6. instead follows by observing that for any $z \geq \sigma > 0$ it holds

$$\psi(t)z + 2c_W(t) \geq \psi(t)\sigma + 2c_W(t) > 0 \qquad \text{for all } 0 \leq t \leq 1\,.$$

Thus $g(z) \geq C_\sigma$ for all $z \geq \sigma > 0$.

2.3 Liftings of $\mathbb{S}^1$-valued maps

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected and simply connected open set with Lipschitz boundary, and $u\colon \Omega \to \mathbb{S}^1$ be a measurable function. A *lifting* of u is a measurable function $\varphi\colon \Omega \to \mathbb{R}$ such that

$$u(x) = e^{i\varphi(x)} \quad \text{for a.e. } x \in \Omega\,.$$

Given a Borel set $B \subseteq \Omega$, we say that $\varphi : B \to \mathbb{R}$ is a lifting of u on B if the previous equality holds a.e. on B. Clearly, if φ is a lifting of u, then so is $\varphi + 2\pi m$ for all $m \in \mathbb{Z}$.

If u has some regularity, a natural question is whether φ can be chosen with the same regularity. The answer is partially positive, see [17, 24] for more details: we recall in particular that if $n \geq 2$ and $u \in W^{1,p}(\Omega;\mathbb{S}^1)$ for some $p \in [2,+\infty)$, then u has a lifting $\varphi \in W^{1,p}(\Omega)$, and φ is unique (mod 2π). Moreover, for $n = 2$ and $p \in [1,2)$ there exists $u \in W^{1,p}(\Omega;\mathbb{S}^1)$ for which there are no liftings $\varphi \in W^{1,p}(\Omega)$, see [17, Theorem 1.2, Remark 1.9]. Indeed it can be shown [17, Pages 17-19] that there are no liftings of the vortex map $u_V(x) = x/|x|$ in $W^{1,1}(B_1)$ (and thus there are no liftings in $W^{1,p}(B_1)$ for all $p \in [1,2)$).

Next we recall the

Theorem 2.5 (Davila-Ignat). *Let $u \in BV(\Omega;\mathbb{S}^1)$. Then there exists a lifting $\varphi \in BV(\Omega)$ such that $\|\varphi\|_{L^\infty} \leq 2\pi$ and $|\varphi|_{BV} \leq 2|u|_{BV}$.*

Proof. See [24, Theorem 1.1], and also [17, Theorem 1.4]. ☐

As usual, for any lifting $\varphi \in SBV(\Omega)$ of u we write[2] $S_\varphi = S_\varphi^I \cup S_\varphi^f$ where

$$S_\varphi^I := \{x \in S_\varphi : [\![\varphi]\!](x) \in 2\pi\mathbb{Z}\}, \qquad S_\varphi^f := S_\varphi \setminus S_\varphi^I.$$

In particular $S_\varphi^f = S_u$; moreover $|\nabla u| = |\nabla \varphi|$ a.e. in Ω and $|D^c u|(\Omega) = |D^c\varphi|(\Omega)$.

Let $u \in BV(\Omega;\mathbb{S}^1)$ and consider the minimum problem

$$\inf\{|\varphi|_{BV} \colon \varphi \in BV(\Omega),\ e^{i\varphi} = u \text{ a.e. in } \Omega\}.$$

Then there exists a minimizer $\varphi \in BV(\Omega)$ such that $|\varphi|_{BV} \leq 2|u|_{BV}$ and $0 \leq \int_\Omega \varphi \, dx \leq 2\pi|\Omega|$ [17, page 25]. Moreover, as already mentioned in the introduction, in [12] it was proven that for any $p > 1$ and any $u \in SBV^p(\Omega;\mathbb{S}^1)$ the minimisation problem

$$m_p[u] := \{\mathcal{H}^{n-1}(S_\varphi) \colon \varphi \in GSBV^p(\Omega),\ e^{i\varphi} = u \text{ a.e. in } \Omega\}$$

admits a solution. Here, we are instead concerned with the existence of minimal lifting for a minimisation problem which depends on the cost function g in (2.3) (cf. (1.2) for its precise definition).

3 Setting of the problem and main results

In this section we introduce the variational model under consideration and state our main results. Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a connected bounded and open set with Lipschitz boundary. To the pair $u\colon \Omega \to \mathbb{S}^1$, $v\colon \Omega \to [0,1]$ we associate the functional

$$\mathrm{F}_\varepsilon(u,v) := \int_\Omega \left(\psi(v) f(|\nabla u|) + \varepsilon|\nabla v|^2 + \frac{W(v)}{\varepsilon}\right) dx\,,$$

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[2] S_φ^I stands for the “integer” part of the jump, and S_φ^f for the “fractional” part.

where $\varepsilon \in (0,1]$, ψ, f and W satisfy properties (a), (b) and (c) in Section 2.2.

We consider the functional domains $\widehat{\mathcal{D}}_{\mathbb{S}^1} \supset \mathcal{D}_{\mathbb{S}^1}$ given by

$$\begin{aligned}
&\widehat{\mathcal{D}}_{\mathbb{S}^1} := \big\{(u,v) \in W^{1,1}(\Omega;\mathbb{S}^1) \times W^{1,2}(\Omega)\colon\ 0 \le v \le 1\big\} \subset L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega)\,,\\
&\mathcal{D}_{\mathbb{S}^1} := \big\{(u,v) \in W^{1,2}(\Omega;\mathbb{S}^1) \times W^{1,2}(\Omega)\colon\ 0 \le v \le 1\big\} \subset L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega)\,.
\end{aligned} \tag{3.1}$$

For $\varepsilon \in (0,1]$ let us consider the corresponding families of functionals

$$\widehat{\mathrm{F}_\varepsilon^{\mathbb{S}^1}}\,,\ \mathrm{F}_\varepsilon^{\mathbb{S}^1} : L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega) \to [0,+\infty]$$

given by

$$\widehat{\mathrm{F}_\varepsilon^{\mathbb{S}^1}}(u,v) := \begin{cases} \mathrm{F}_\varepsilon(u,v) & \text{if } (u,v) \in \widehat{\mathcal{D}}_{\mathbb{S}^1}\,,\\ \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega)\,, \end{cases} \tag{3.2}$$

$$\mathrm{F}_\varepsilon^{\mathbb{S}^1}(u,v) := \begin{cases} \mathrm{F}_\varepsilon(u,v) & \text{if } (u,v) \in \mathcal{D}_{\mathbb{S}^1}\,,\\ \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega)\,. \end{cases} \tag{3.3}$$

The two main results of the following paper read as follows.

Theorem 3.1 (Γ-convergence of $\widehat{\mathrm{F}_\varepsilon^{\mathbb{S}^1}}$)**.** *We have*

$$\Gamma - L^1 \lim_{\varepsilon \to 0^+} \widehat{\mathrm{F}_\varepsilon^{\mathbb{S}^1}} = \mathrm{F}^{\mathbb{S}^1},$$

where $\mathrm{F}^{\mathbb{S}^1} : L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega) \to [0,+\infty]$ *is given by*

$$\mathrm{F}^{\mathbb{S}^1}(u,v) := \int_\Omega f(|\nabla u|)\,\mathrm{d}x + |D^c u|(\Omega) + \int_{S_u} g(|u^+ - u^-|)\,\mathrm{d}\mathcal{H}^{n-1} \tag{3.4}$$

if $u \in BV(\Omega;\mathbb{S}^1)$, $v = 1$ *a.e., and* $+\infty$ *otherwise in* $L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega)$*, with* g *as in* (2.3)*.*

Theorem 3.2 (Γ-convergence of $\mathrm{F}_\varepsilon^{\mathbb{S}^1}$)**.** *Provided* Ω *is also simply-connected, we have*

$$\Gamma - L^1 \lim_{\varepsilon \to 0^+} \mathrm{F}_\varepsilon^{\mathbb{S}^1} = \mathrm{F}_{\mathrm{lift}},$$

where $\mathrm{F}_{\mathrm{lift}}\colon L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega) \to [0,+\infty]$ *is given by*

$$\mathrm{F}_{\mathrm{lift}}(u,v) := \int_\Omega f(|\nabla u|)\,\mathrm{d}x + |D^c u|(\Omega) + m_g[u] \tag{3.5}$$

if $u \in BV(\Omega;\mathbb{S}^1)$, $v = 1$ *a.e., and* $+\infty$ *otherwise in* $L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega)$*. In addition,* $m_g[u]$ *is a minimum.*

4 On g-minimizing liftings

In this section we establish the existence of solutions to the minimization problem (1.2), for g as in (2.3) and $u \in BV(\Omega;\mathbb{S}^1)$, as stated in Corollary 4.3. We first note that the admissible set in (1.2) is nonempty, thanks to Theorem 2.5. As a further outcome of our analysis, we derive compactness results for sequences of liftings (see Theorems 4.2 and 4.5). In particular, Theorem 4.5 will play a key role in establishing the lower bound in Theorem 3.2. For later use, we introduce the following notion of convergence.

Definition 4.1 (Local convergence modulo 2π). *Let $(\varphi_k)_{k\geq 1}\subset GBV(\Omega)$ and $\varphi_\infty \in GBV(\Omega)$. We say that (φ_k) converges locally modulo 2π to φ_∞ if the following holds: there exist a Caccioppoli partition $\{E_i\}_{i\geq 1}$ of Ω and sequences $(d_k^{(i)})_{k\geq 1}\subset \mathbb{Z}$ for $i\in\mathbb{N}$ such that*

$$\begin{aligned} &\lim_{k\to+\infty}(\varphi_k(x)-2\pi d_k^{(i)}) = \varphi_\infty(x) \qquad \forall i\in\mathbb{N}, \quad \textit{for a.e. } x\in E_i\,, \\ &\lim_{k\to+\infty}|\varphi_k(x)-2\pi d_k^{(i)}| = +\infty \qquad \forall i\in\mathbb{N}, \quad \textit{for a.e. } x\in \Omega\setminus E_i\,. \end{aligned} \tag{4.1}$$

Theorem 4.2 (Compactness). *Let $0\leq\sigma\leq\pi$. Let $u\in BV(\Omega;\mathbb{S}^1)$ and $(\varphi_k)_{k\geq 1}\subset GBV(\Omega)$ be a sequence of liftings of u with*

$$H := \sup_{k\in\mathbb{N}} \mathcal{H}^{n-1}(S^\sigma_{\varphi_k}) < +\infty\,, \tag{4.2}$$

where

$$S^\sigma_{\varphi_k} := \{x\in S_{\varphi_k} : |[\![\varphi_k(x)]\!]| \geq \sigma\}.$$

Then there exist a not-relabelled subsequence of indices k and a lifting $\varphi_\infty\in GBV(\Omega)$ of u in Ω such that (φ_k) converges locally modulo 2π to φ_∞.

A first consequence of Theorem 4.2 is the following result.

Corollary 4.3 (Existence of a minimizer to $m_g[u]$). *Let g be as in (2.3). Let $u\in BV(\Omega;\mathbb{S}^1)$ and let $(\varphi_k)_{k\geq 1}\subset GBV(\Omega)$ be a sequence of liftings of u in Ω with*

$$C := \sup_{k\in\mathbb{N}}\int_{S_{\varphi_k}} g(|\varphi_k^+-\varphi_k^-|)\,\mathrm{d}\mathcal{H}^{n-1} < +\infty\,. \tag{4.3}$$

Then the following holds:

(i) Compactness: There exist a not-relabelled subsequence of indices k, and a lifting $\varphi_\infty\in GBV(\Omega)$ of u in Ω such that (φ_k) converges locally modulo 2π to φ_∞;

(ii) Lower semicontinuity:

$$\liminf_{k\to+\infty}\int_{S_{\varphi_k}} g(|\varphi_k^+-\varphi_k^-|)\,\mathrm{d}\mathcal{H}^{n-1} \geq \int_{S_{\varphi_\infty}} g(|\varphi_\infty^+-\varphi_\infty^-|)\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

As a consequence, there exists a minimizer $\varphi\in GBV(\Omega)$ of $m_g[u]$.

Proof. (i). Let $0<\sigma\leq\pi$ be fixed. By Remark 2.4-(iii) and (4.3) it follows

$$C \geq \int_{S_{\varphi_k}} g(|\varphi_k^+-\varphi_k^-|)\,\mathrm{d}\mathcal{H}^{n-1} \geq C_\sigma \mathcal{H}^{n-1}(S^\sigma_{\varphi_k})\,.$$

Hence from Theorem 4.2 there exists a lifting $\varphi_\infty\in GBV(\Omega)$ of u such that, up to a not-relabelled subsequence, (φ_k) converges locally modulo 2π to φ_∞. More precisely, there exist a Caccioppoli partition $\{E_i\}_{i\geq 1}$ of Ω and sequences $(d_k^{(i)})_{k\geq 1}\subset\mathbb{Z}$ such that (4.1) holds for every $i\in\mathbb{N}$.

(ii). We exploit the fact that g arises from the Γ-limit of the functionals in (2.1). Fix $N, i\in\mathbb{N}$ and set

$$\varphi_k^{i,N} := ((\varphi_k - 2\pi d_k^{(i)})\wedge N)\vee(-N)\,.$$

Then $\varphi_k^{i,N} \to \varphi_\infty^{i,N}$ in $L^1(\Omega)$ with $\varphi_\infty^{i,N} := (\varphi_\infty \wedge N) \vee (-N)$ in E_i and $\varphi_\infty^{i,N} = \pm N$ in two suitable finite perimeter subsets $F_i^\pm$ of Ω with $\Omega \setminus E_i = F_i^+ \cup F_i^-$.

Define

$$G(\phi, A) := \mathrm{F}(\phi, 1, A) \quad \text{ for } \phi \in GBV(\Omega)\,,\ A \in \mathcal{A}(\Omega)$$

where $\mathrm{F}(\cdot,\cdot,A)$ is as in Remark 2.4-(ii) with $f(t) = t$. From Theorem 2.3 and properties of Γ-convergence it follows that G is lower semicontinuous in the L^1-topology, so that

$$\liminf_{k\to+\infty} G(\varphi_k, A) \geq \liminf_{k\to+\infty} G(\varphi_k^{i,N}, A) \geq G(\varphi_\infty^{i,N}, A)\,. \tag{4.4}$$

Next we define the superadditive set function

$$\mu(A) := \liminf_{k\to+\infty} G(\varphi_k, A) \qquad \forall A \in \mathcal{A}(\Omega), \tag{4.5}$$

and the positive measure

$$\begin{aligned}\lambda(B) := G(\varphi_\infty, B) = \int_B |\nabla\varphi_\infty|\,\mathrm{d}x + |D^c\varphi_\infty|(B) + \int_{(S_{\varphi_\infty}\setminus\Sigma)\cap B} g(|\varphi_\infty^+ - \varphi_\infty^-|)\,\mathrm{d}\mathcal{H}^{n-1}\\ + 2c_W(0)\mathcal{H}^{n-1}(\Sigma\cap B),\end{aligned}$$

for every Borel set $B \subseteq \Omega$, where we have denoted

$$\Sigma := \Big(\bigcup_{i=1}^{+\infty} \partial^* E_i\Big) \setminus \partial\Omega. \tag{4.6}$$

We identify E_i with the set of density 1 for χ_{E_i}. Define also, for $i \geq 1$,

$$h_1^i := \begin{cases} 1 & \text{in } E_i \setminus (\Sigma \cup S_{\varphi_\infty}) \\ 0 & \text{otherwise in } \Omega \end{cases},$$

and

$$h_2^i := \begin{cases} 1 & \text{in } (S_{\varphi_\infty} \cap E_i) \setminus \partial^* E_i \\ 0 & \text{otherwise in } \Omega \end{cases}, \quad h_3^i := \begin{cases} 1 & \text{in } (\partial^* E_i) \setminus \partial\Omega \\ 0 & \text{otherwise in } \Omega. \end{cases}.$$

By definition of local convergence modulo 2π, there holds

$$\lim_{N\to+\infty} |(\varphi_\infty^{i,N})^+ - (\varphi_\infty^{i,N})^-| = |\varphi_\infty^+ - \varphi_\infty^-| \qquad \mathcal{H}^{n-1} \text{ a.e. in } (S_{\varphi_\infty} \cap E_i) \setminus \partial^* E_i,$$

and

$$\lim_{N\to\infty} |(\varphi_\infty^{i,N})^+ - (\varphi_\infty^{i,N})^-| = +\infty \qquad \mathcal{H}^{n-1} \text{ a.e. in } \partial^* E_i,$$

hence by monotonicity of g and the monotone convergence theorem we arrive at

$$\lim_{N\to+\infty} \int_{(S_{\varphi_\infty^{i,N}}\setminus\partial^* E_i)\cap A} g(|(\varphi_\infty^{i,N})^+ - (\varphi_\infty^{i,N})^-|)\,\mathrm{d}\mathcal{H}^{n-1} \geq \int_A h_2^i \mathrm{d}\lambda$$

and

$$2c_W(0)\mathcal{H}^{n-1}(A \cap \partial^* E_i) = \lim_{N\to\infty} \int_{A\cap\partial^* E_i} g(|(\varphi_\infty^{i,N})^+ - (\varphi_\infty^{i,N})^-|)\,\mathrm{d}\mathcal{H}^{n-1} \geq \int_A h_3^i \mathrm{d}\lambda.$$

Whence, by (4.4) we conclude

$$\mu(A) \geq \int_A h_2^i \mathrm{d}\lambda \qquad\qquad \mu(A) \geq \int_A h_3^i \mathrm{d}\lambda.$$

On the other hand, still by (4.4) it is easily seen that

$$\mu(A) \geq \int_A h_1^i \mathrm{d}\lambda \qquad \forall i \geq 1,$$

and so observing that

$$\sup_{j=1,2,3,\ i\in\mathbb{N}} h_j^i \equiv 1\,,$$

by invoking Lemma 2.1 we infer

$$\mu(A) \geq \lambda(A) \quad \text{for all } A \in \mathcal{A}(\Omega)\,.$$

As a consequence we get

$$\liminf_{k\to+\infty} G(\varphi_k, \Omega) \geq G(\varphi_\infty, \Omega)\,. \tag{4.7}$$

Since $\varphi_k, \varphi_\infty$ are liftings of u it holds $|\nabla\varphi_k| = |\nabla\varphi_\infty| = |\nabla u|$ and $|D^c\varphi_k|(\Omega) = |D^c\varphi_\infty|(\Omega) = |D^c u|(\Omega)$, hence (4.7) in turn implies

$$\liminf_{k\to+\infty} \int_{S_{\varphi_k}} g(|\varphi_k^+ - \varphi_k^-|)\,\mathrm{d}\mathcal{H}^{n-1} \geq \int_{S_{\varphi_\infty}} g(|\varphi_\infty^+ - \varphi_\infty^-|)\,\mathrm{d}\mathcal{H}^{n-1}\,,$$

and the proof of (ii) is concluded. □

From the proof above we also obtain the following:

Corollary 4.4. *In the same hypotheses and notations of Corollary 4.3, we have*

$$\liminf_{k\to+\infty} \int_{S_{\varphi_k}} g(|\varphi_k^+ - \varphi_k^-|)\,\mathrm{d}\mathcal{H}^{n-1} \geq \int_{S_{\varphi_\infty}\setminus\Sigma} g(|\varphi_\infty^+ - \varphi_\infty^-|)\,\mathrm{d}\mathcal{H}^{n-1} + 2c_W(0)\mathcal{H}^{n-1}(\Sigma),$$

where we recall $2c_W(0) = \sup g$.

Now we state a generalization of Theorem 4.2 to a sequence (u_k), needed to show Theorem 3.2; the proof will be given in the next section.

Theorem 4.5 (Generalised compactness). *Let $u, u_k \in BV(\Omega; \mathbb{S}^1)$ be such that*

$$u_k \overset{*}{\rightharpoonup} u \quad \text{in } BV(\Omega; \mathbb{S}^1), \tag{4.8}$$

and let $\varphi_k \in GBV(\Omega)$ be a lifting of u_k in Ω, for all $k \geq 1$. Suppose that for some $0 \leq \sigma \leq \pi$

$$H := \sup_{k\in\mathbb{N}} \mathcal{H}^{n-1}(S_{\varphi_k}^\sigma) < +\infty. \tag{4.9}$$

Then there exist a not-relabelled subsequence of indices k and a lifting $\varphi_\infty \in GBV(\Omega)$ of u in Ω such that φ_k converges locally modulo 2π to φ_∞.

5 Proofs of Theorems 4.2 and 4.5

We split the proof of Theorems 4.2 and 4.5 into a number of intermediate lemmas.

Lemma 5.1 (**Localized compactness**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and $u \in BV(\Omega; \mathbb{S}^1)$. Let $0 \le \sigma \le \pi$ and $(\varphi_k)_{k\ge1} \subset GBV(\Omega)$ be a sequence of liftings of u satisfying* (4.2). *Let $F \subset \Omega$ be a nonempty finite perimeter set in Ω. Then, for a not-relabelled subsequence, there exist a sequence $(d_k)_{k\ge1} \subset \mathbb{Z}$, a finite perimeter set*

$$E \subseteq F$$

in Ω, and a function $\varphi_\infty \in GBV(\Omega)$, such that:

$$\varphi_\infty \text{ is a lifting of } u \text{ in } E\,, \quad \varphi_\infty = 0 \quad \text{in } \Omega \setminus E\,, \tag{5.1}$$

and

$$\begin{aligned} &\lim_{k\to+\infty} (\varphi_k(x) - 2\pi d_k) = \varphi_\infty(x) && \text{for a.e. } x \in E\,,\\ &\lim_{k\to+\infty} |\varphi_k(x) - 2\pi d_k| = +\infty && \text{for a.e. } x \in F \setminus E, \end{aligned} \tag{5.2}$$

$$|E| \ge \frac{n^n \omega_n |F|^n}{2^n(2H + \mathcal{H}^{n-1}(\partial^* F))^n} > 0, \tag{5.3}$$

If furthermore

$$|\varphi_k(x) - 2\pi d_k| \to +\infty \qquad \text{for a.e. } x \in \Omega \setminus E, \tag{5.4}$$

and $\{x \in \Omega : \varphi_k(x) - 2\pi d_k \to \pm\infty\}$ are finite perimeter sets in Ω, then

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}\big(S^\sigma_{\varphi_k} \cap A\big) \ge \mathcal{H}^{n-1}\big(A \cap \partial^* E\big) \quad \text{for any } A \in \mathcal{A}(\Omega), \tag{5.5}$$

and

$$\mathcal{H}^{n-1}(F \cap \partial^* E) = \mathcal{H}^{n-1}(F \cap \partial^*(F \setminus E)) \le C. \tag{5.6}$$

Proof. Following the lines of the proof of [12, Lemma 4.1] and noticing that [12, equation 4.8] still holds if we replace S_{φ_k} and S_{φ_1} with $S^\sigma_{\varphi_k}$ and $S^\sigma_{\varphi_1}$ we can prove that there exist a sequence $(d_k)_{k\ge1} \subset \mathbb{Z}$, a measurable set $E \subset F$, and a measurable function $\varphi_\infty \colon \Omega \to \mathbb{R}$ such that (5.1)–(5.3) hold true for a not-relabelled subsequence.

We next show that E has finite perimeter and $\varphi_\infty \in GBV(\Omega)$. For each $k \ge 1$ we define the auxiliary function

$$\widehat{\varphi}_k := \begin{cases} \varphi_k - 2\pi d_k & \text{in } F \\ \\ k & \text{in } \Omega \setminus F. \end{cases}$$

Since F has finite perimeter we have $(\widehat{\varphi}_k) \subset GBV(\Omega)$. Moreover

$$\sup_{k\in\mathbb{N}} \mathcal{H}^{n-1}(S^\sigma_{\widehat{\varphi}_k}) \le H + \mathcal{H}^{n-1}(\partial^* F)\,,$$

and

$$\begin{aligned} &\lim_{k\to+\infty} \widehat{\varphi}_k(x) = \varphi_\infty(x) && \text{for a.e. } x \in E\,,\\ &\lim_{k\to+\infty} |\widehat{\varphi}_k(x)| = +\infty && \text{for a.e. } x \in \Omega \setminus E\,. \end{aligned}$$

Now we proceed by adapting a slicing argument in [3] (see also [21]). For $\xi \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $y \in \Pi^\xi := \{y \in \mathbb{R}^n \colon y \cdot \xi = 0\}$ we introduce the following notation:

$$A^\xi_y := \{t \in \mathbb{R} \colon y + t\xi \in A\}\,, \quad f^\xi_y(t) := f(y+t\xi)\,,$$

for any $A \subset \mathbb{R}^n$ and $f \colon A \to \mathbb{R}^m$. In particular $(\widehat{\varphi}_k)^\xi_y$ is a map of one variable in $GBV(\Omega^\xi_y)$ for a.e. $y \in \Pi^\xi$.

Recalling that $S^\sigma_{(\widehat{\varphi}_k)^\xi_y} := \{t \in S_{(\widehat{\varphi}_k)^\xi_y} \colon |[\![(\widehat{\varphi}_k)^\xi_y(t)]\!]| \geq \sigma\}$ and $S^\sigma_{\widehat{\varphi}_k} := \{x \in S_{\widehat{\varphi}_k} \colon |[\![\widehat{\varphi}_k(x)]\!]| \geq \sigma\}$, we have

$$S^\sigma_{(\widehat{\varphi}_k)^\xi_y} \subset (S^\sigma_{\widehat{\varphi}_k})^\xi_y\,.$$

This together with Fatou's lemma imply

$$\begin{aligned}
&\int_{\Pi_\xi} \liminf_{k\to+\infty} \Big(|D(\widehat{\varphi}_k)^\xi_y|(\Omega^\xi_y \setminus S^\sigma_{(\widehat{\varphi}_k)^\xi_y}) + \mathcal{H}^0(\Omega^\xi_y \cap S^\sigma_{(\widehat{\varphi}_k)^\xi_y}) \Big)\, \mathrm{d}\mathcal{H}^{n-1}(y) \\
&\leq \int_{\Pi_\xi} \liminf_{k\to+\infty} \Big(|D(\widehat{\varphi}_k)^\xi_y|(\Omega^\xi_y \setminus S^\sigma_{(\widehat{\varphi}_k)^\xi_y}) + \mathcal{H}^0\left(\Omega^\xi_y \cap (S^\sigma_{\widehat{\varphi}_k})^\xi_y \right) \Big)\, \mathrm{d}\mathcal{H}^{n-1}(y) \\
&\leq \liminf_{k\to+\infty} \int_{\Pi_\xi} |D(\widehat{\varphi}_k)^\xi_y|(\Omega^\xi_y \setminus S^\sigma_{(\widehat{\varphi}_k)^\xi_y}) + \mathcal{H}^0\left(\Omega^\xi_y \cap (S^\sigma_{\widehat{\varphi}_k})^\xi_y \right) \mathrm{d}\mathcal{H}^{n-1}(y) \\
&\leq \liminf_{k\to+\infty} \left(|D\widehat{\varphi}_k|(\Omega \setminus S^\sigma_{\widehat{\varphi}_k}) + \int_{\Omega \cap S^\sigma_{\widehat{\varphi}_k}} |\nu_{\widehat{\varphi}_k} \cdot \xi| \mathrm{d}\mathcal{H}^{n-1} \right) \\
&\leq c|Du|(\Omega) + \sigma \mathcal{H}^{n-1}(\partial^* F) + c \liminf_{k\to+\infty} \mathcal{H}^{n-1}(S^\sigma_{\widehat{\varphi}_k}) < +\infty\,,
\end{aligned}$$

where the last inequality follows from the fact that $(S_{\widehat{\varphi}_k} \setminus S^\sigma_{\widehat{\varphi}_k}) \subset S_u \cup \partial^* F$, and since $\sigma \leq \pi$, one has $|[\![\widehat{\varphi}(x)]\!]| \leq c|[\![u(x)]\!]|$ for a.e. $x \in S_{\widehat{\varphi}_k} \setminus S^\sigma_{\widehat{\varphi}_k}$ and some $c > 0$. Thus for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ and $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we can consider a not relabelled subsequence (depending on ξ and y) such that

$$\sup_k \left[|D(\widehat{\varphi}_k)^\xi_y|(\Omega^\xi_y \setminus S^\sigma_{(\widehat{\varphi}_k)^\xi_y}) + \mathcal{H}^0(\Omega^\xi_y \cap S^\sigma_{(\widehat{\varphi}_k)^\xi_y}) \right] < +\infty\,.$$

Hence for not relabelled subsequence (depending on ξ and y) there exists $n \in \mathbb{N}$ depending on ξ and y such that $\mathcal{H}^0\left(\Omega^\xi_y \cap S^\sigma_{(\widehat{\varphi}_k)^\xi_y} \right) = n$ for all $k \geq 1$, and the jump points converge as well in $\overline{\Omega^\xi_y}$ as $k \to +\infty$ for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ and $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. Hence, using the fact that $\widehat{\varphi}_k \to +\infty$ in $\Omega \setminus E$, and therefore on $\Omega^\xi_y \setminus E^\xi_y$, it is possible to show that

$$\liminf_{k\to+\infty} \mathcal{H}^0\left(A^\xi_y \cap S^\sigma_{(\widehat{\varphi}_k)^\xi_y} \right) \geq \mathcal{H}^0\left(A^\xi_y \cap \partial^* E^\xi_y \right)$$

(see [3] for details). By integrating over Π^ξ and using once more the Fatou's lemma we conclude

$$\begin{aligned}
\liminf_{k\to+\infty} \int_{A \cap S^\sigma_{\widehat{\varphi}_k}} |\nu_{\widehat{\varphi}_k} \cdot \xi| \mathrm{d}\mathcal{H}^{n-1} &\geq \int_{\Pi^\xi} \liminf_{k\to+\infty} \mathcal{H}^0\left(A^\xi_y \cap S^\sigma_{(\widehat{\varphi}_k)^\xi_y} \right) \mathrm{d}\mathcal{H}^{n-1}(y) \\
&\geq \int_{\Pi^\xi} \mathcal{H}^0\left(\partial^* E^\xi_y \right) \mathrm{d}\mathcal{H}^{n-1}(y)\,,
\end{aligned}$$

for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ and every open set $A \subseteq \Omega$. As a consequence we deduce that E has finite perimeter in Ω. Moreover by integrating over $\mathbb{S}^{n-1}$ we get

$$\alpha_n \liminf_{k\to+\infty} \mathcal{H}^{n-1}(A \cap S^{\sigma}_{\widehat{\varphi}_k}) \geq \alpha_n \mathcal{H}^{n-1}(A \cap \partial^* E)\,,$$

where $\alpha_n := \int_{\mathbb{S}^{n-1}} |\nu \cdot \xi| \mathrm{d}\mathcal{H}^{n-1}(\nu)$, which in turn implies

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(A \cap S^{\sigma}_{\widehat{\varphi}_k}) \geq \mathcal{H}^{n-1}(A \cap \partial^* E)\,. \tag{5.7}$$

To prove that $\varphi_\infty \in GBV(\Omega)$ we argue as follows. For $N \in \mathbb{N}$ we set $\widehat{\varphi}^N_k := (\widehat{\varphi}_k \wedge N) \vee (-N)$, so that

$$|D\widehat{\varphi}^N_k|(\Omega) \leq |Du|(\Omega) + 2N\mathcal{H}^{n-1}(S_{\widehat{\varphi}^N_k}) \leq c\,.$$

Hence $(\widehat{\varphi}^N_k)$ converges weakly* in $BV(\Omega)$ to $\widehat{\varphi}^N_\infty$ where

$$\widehat{\varphi}^N_\infty := \begin{cases} (\varphi_\infty \wedge N) \vee (-N) & \text{in } E, \\ \pm N & \text{in } \Omega \setminus E, \end{cases} \quad \text{so that} \quad \widehat{\varphi}^N_\infty \in BV(\Omega)\,.$$

Hence observing that in the whole of Ω we have $(\varphi_\infty \wedge N) \vee (-N) = \widehat{\varphi}^N_\infty \chi_E \in BV(\Omega)$ we conclude $\varphi_\infty \in GBV(\Omega)$. Eventually if in addition (5.4) holds then (5.5) can be deduced in the same way as (5.7) with $\varphi_k - 2\pi d_k$ in place of $\widehat{\varphi}_k$. This additionally implies

$$P(E, F) \leq H,$$

which shows (5.6). $\square$

Lemma 5.2. *Let $0 \leq \sigma \leq \pi$. Let $(\varphi_k)_{k\geq 1} \subset GBV(\Omega)$ be a sequence of functions satisfying*

$$C := \sup_{k\in\mathbb{N}} \mathcal{H}^{n-1}(S^{\sigma}_{\varphi_k}) < +\infty. \tag{5.8}$$

Let $N \geq 1$ be an integer, $E_1, \ldots, E_N \subset \Omega$ be pairwise disjoint nonempty finite perimeter sets with the following property: for any $i = 1, \ldots, N$,

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S^{\sigma}_{\varphi_k} \cap A) \geq \mathcal{H}^{n-1}(A \cap \partial^* E_i) \text{ for any } A \in \mathcal{A}(\Omega)\,.$$

Then

$$H \geq \liminf_{k\to+\infty} \mathcal{H}^{n-1}(S^{\sigma}_{\varphi_k}) \geq \mathcal{H}^{n-1}\big(\Omega \cap (\cup_{i=1}^N \partial^* E_i)\big)\,.$$

Proof. We define the superadditive set function $\mu\colon \mathcal{A}(\Omega) \to [0, +\infty)$ as

$$\mu(A) := \liminf_{k\to+\infty} \mathcal{H}^{n-1}(A \cap S^{\sigma}_{\varphi_k})\,,$$

and the positive measure

$$\lambda(B) := \mathcal{H}^{n-1}\big(B \cap (\cup_{i=1}^N \partial^* E_i)\big) \quad \forall \text{ Borel set } B \subseteq \Omega\,.$$

Moreover for $i = 1, \ldots, N$ we set

$$h_i := \begin{cases} 1 & \text{in } \partial^* E_i \\ \\ 0 & \text{otherwise in } \Omega \end{cases}\,.$$

Clearly we have

$$\mu(A) \geq \int_A h_i \, \mathrm{d}\lambda \quad \forall i = 1, \ldots, N\,.$$

Therefore, since

$$h := \sup_{i=1,\ldots,N} h_i = \begin{cases} 1 & \text{in } \cup_{i=1}^N \partial^* E_i \\ \\ 0 & \text{otherwise in } \Omega \end{cases} ,$$

by Lemma 2.1 it follows

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S^\sigma_{\varphi_k} \cap A) = \mu(A) \geq \int_A h \, \mathrm{d}\lambda = \mathcal{H}^{n-1}(A \cap (\cup_{i=1}^N \partial^* E_i))\,,$$

which implies the thesis by taking $A = \Omega$.

□

5.1 Proof of Theorem 4.2

We adapt the arguments of [12, Theorem 3.1].

Base step $N = 1$. We set

$$F_1 := \Omega.$$

By Lemma 5.1 we find, after extracting a not-relabelled subsequence, a finite perimeter set $E_1 \subseteq F_1$, a sequence $(d_k^{(1)})_{k\geq 1} \subset \mathbb{Z}$ and a function $\varphi_\infty^{(1)} \in GBV(\Omega)$, such that

$$\varphi_\infty^{(1)} \text{ is a lifting of } u \text{ in } E_1, \qquad \varphi_\infty^{(1)} = 0 \quad \text{in } F_1 \setminus E_1\,,$$

and

$$|E_1| \geq \frac{n^n \omega_n |F_1|^n}{2^n (2H + \mathcal{H}^{n-1}(\partial\Omega))^n}\,.$$

Moreover

$$(\varphi_k(x) - 2\pi d_k^{(1)}) \to \varphi_\infty^{(1)}(x) \qquad \text{for a.e. } x \in E_1\,,$$

$$|\varphi_k(x) - 2\pi d_k^{(1)}| \to +\infty \qquad \text{for a.e. } x \in F_1 \setminus E_1\,.$$

Since $F_1 = \Omega$, estimate (5.5) yields

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S^\sigma_{\varphi_k} \cap A) \geq \mathcal{H}^{n-1}(A \cap \partial^* E_1) \text{ for any open set } A \subseteq F_1\,.$$

Inductive step $N \rightsquigarrow N + 1$. Let $N \geq 2$. Suppose we have pairwise disjoint nonempty finite perimeter sets, $E_1, \ldots, E_N \subset \Omega$, and define inductively

$$F_1 := \Omega\,, \quad F_i := \Omega \setminus \bigcup_{j=1}^{i-1} E_j \quad \text{for } i = 2, \ldots, N,$$

so that $E_i \subset F_i$ for all $i = 1, \ldots, N$. Assume that, still for all $i = 1, \ldots, N$, the following holds:

(i) There exists a function $\varphi_\infty^{(i)} \in GBV(\Omega)$ which is a lifting of u in E_i, and $\varphi_\infty^{(i)} = 0$ in $\Omega \setminus E_i$;

(ii) The set E_i satisfies

$$|E_i| \geq \frac{n^n \omega_n |F_i|^n}{2^n(2H + \mathcal{H}^{n-1}(\partial^* F_i))^n} \geq \frac{n^n \omega_n |F_i|^n}{2^n(3H + \mathcal{H}^{n-1}(\partial\Omega))^n};$$

(iii) There exists a sequence $(d_k^{(i)})_k \subset \mathbb{Z}$, such that

$$\begin{aligned}
&(\varphi_k(x) - 2\pi d_k^{(i)}) \to \varphi_\infty^{(i)}(x) \qquad \text{for a.e. } x \in E_i\,, \\
&|\varphi_k(x) - 2\pi d_k^{(i)}| \to +\infty \qquad \text{for a.e. } x \in \Omega \setminus E_i\,, \\
&\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S^\sigma_{\varphi_k} \cap A) \geq \mathcal{H}^{n-1}(A \cap \partial^* E_i) \quad \text{for any } A \in \mathcal{A}(\Omega)\,.
\end{aligned} \tag{5.9}$$

Set, for $N \geq 1$,

$$F_{N+1} := \Omega \setminus \Big(\bigcup_{i=1}^N E_i\Big),$$

If $F_{N+1} = \emptyset$ nothing remains to prove. Therefore, we may suppose $F_{N+1} \neq \emptyset$. From Lemma 5.2

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S^\sigma_{\varphi_k}) \geq \mathcal{H}^{n-1}(\Omega \cap (\cup_{i=1}^N \partial^* E_i))\,, \tag{5.10}$$

and thus, since $\partial^*(\cup_{i=1}^N E_i) \subseteq \cup_{i=1}^N \partial^* E_i$,

$$\mathcal{H}^{n-1}\Big(\Omega \cap \partial^*(\cup_{i=1}^N E_i)\Big) = \mathcal{H}^{n-1}(\Omega \cap \partial^* F_{N+1}) \leq H\,. \tag{5.11}$$

Applying Lemma 5.1 to the set $F = F_{N+1}$, we obtain a finite perimeter set

$$E_{N+1} \subseteq F_{N+1}\,,$$

a function $\varphi_\infty^{(N+1)} \in GBV(\Omega)$ and a sequence $(d_k^{(N+1)})_{k\geq1} \subset \mathbb{Z}$, such that

$$\varphi_\infty^{(N+1)} = 0 \quad \text{in } \Omega \setminus E_{N+1}\,, \quad \varphi_\infty^{(N+1)} \text{ is a lifting of } u \text{ in } E_{N+1}\,,$$

and

$$|E_{N+1}| \geq \frac{n^n \omega_n |F_{N+1}|^n}{2^n(2H + \mathcal{H}^{n-1}(\partial^* F_{N+1}))^n}\,, \tag{5.12}$$

together with

$$\begin{aligned}
&(\varphi_k(x) - 2\pi d_k^{(N+1)}) \to \varphi_\infty^{(N+1)}(x) \qquad \text{for a.e. } x \in E_{N+1}\,, \\
&|\varphi_k(x) - 2\pi d_k^{(N+1)}| \to +\infty \qquad \text{for a.e. } x \in F_{N+1} \setminus E_{N+1}\,, \\
&\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S^\sigma_{\varphi_k} \cap B) \geq \mathcal{H}^{n-1}(B \cap \partial^* E_{N+1}) \text{ for any open set } B \subseteq F_{N+1}\,.
\end{aligned}$$

Gathering (5.11) and (5.12) we have

$$|E_{N+1}| \geq \frac{n^n \omega_n |F_{N+1}|^n}{2^n(2H + \mathcal{H}^{n-1}(\partial^* F_{N+1}))^n} \geq \frac{n^n \omega_n |F_{N+1}|^n}{2^n(3H + \mathcal{H}^{n-1}(\partial\Omega))^n} > 0\,.$$

Moreover by (5.9), also

$$|\varphi_k(x) - 2\pi d_k^{(N+1)}| \to +\infty \qquad \text{for a.e. } x \in \Omega \setminus E_{N+1}\,.$$

Thus properties (i)-(iii) are preserved at level $N+1$.

Conclusion. Iterating the above construction and extracting a diagonal subsequence, we obtain a sequence $(E_i)_{i\geq 1}$ of mutually disjoint finite perimeter sets in Ω such that for every $N \geq 1$, $E_1, \dots, E_N$ satisfy properties (i)-(iii). From (5.10),

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S^\sigma_{\varphi_k}) \geq \mathcal{H}^{n-1}(\Omega \cap (\cup_{i=1}^m \partial^* E_i)) \qquad \forall m \geq 1\,. \tag{5.13}$$

To conclude, we show that

$$|\Omega \setminus (\cup_{i=1}^{+\infty} E_i)| = 0.$$

Since $\sum_{m=1}^{+\infty} |E_m| < +\infty$, the sequence $(|E_m|)$ tends to zero as $m \to +\infty$, and using

$$|E_m| \geq \frac{n^n \omega_n |F_m|^2}{2^n(3H + \mathcal{H}^{n-1}(\partial\Omega))^2}$$

we deduce that $|F_m| \to 0$. In particular,

$$|\Omega \setminus (\cup_{i=1}^{+\infty} E_i)| = \lim_{m\to+\infty} |\Omega \setminus (\cup_{i=1}^{m-1} E_i)| = \lim_{m\to+\infty} |F_m| = 0\,.$$

Define

$$\varphi_\infty(x) := \varphi_\infty^{(i)}(x) \qquad \text{if } x \in E_i \text{ for some } i \in \mathbb{N}.$$

Then φ_∞ is a lifting of u in Ω. We now show that

$$\varphi_\infty \in GBV(\Omega). \tag{5.14}$$

First observe that by letting $m \to \infty$ in (5.13) we deduce that

$$M \geq \liminf_{k\to+\infty} \mathcal{H}^{n-1}(S^\sigma_{\varphi_k}) \geq \mathcal{H}^{n-1}(\Omega \cap (\cup_{i=1}^\infty \partial^* E_i))\,. \tag{5.15}$$

Now, for each $N \in \mathbb{N}$ we define the auxiliary sequence

$$\Phi_k^N := ((\varphi_k - 2\pi d_k^{(i)}) \wedge N) \vee (-N) \quad \text{in} \quad E_i \quad \text{for all } i \geq 1\,.$$

Thus, noticing that $S_{\Phi_k^N} \subset S_u \cup S^\sigma_{\varphi_k} \cup (\cup_{i=1}^\infty \partial^* E_i)$ we have from (5.15)

$$|D\Phi_k^N|(\Omega) \leq \int_\Omega |\nabla u|\,\mathrm{d}x + |D^c u|(\Omega) + \int_{S_u} |u^+ - u^-|\mathrm{d}\mathcal{H}^{n-1} + 2N\mathcal{H}^{n-1}(S^\sigma_{\varphi_k}) \leq C\,,$$

with $C > 0$ depending on N. Therefore, up to a subsequence, (Φ_k^N) converges to $(\varphi_\infty \wedge N) \vee (-N)$ weakly* in $BV(\Omega)$. By the arbitrariness of N we deduce (5.14).

□

5.2 Proof of Theorem 4.5

Let $\overline{\varphi}, \overline{\varphi}_k \in BV(\Omega) \cap L^\infty(\Omega)$ be liftings of u and of u_k in Ω, respectively with

$$|\overline{\varphi}|_{BV} \leq 2|u|_{BV} \leq C, \qquad |\overline{\varphi}_k|_{BV} \leq 2|u_k|_{BV} \leq C,$$

(see Theorem 2.5). The constant $C > 0$ can be found thanks to the hypothesis (4.8) and so u_k are uniformly bounded in $BV(\Omega)$. In particular, for $\sigma \in [0, \pi]$ as in the statement, there is a positive constant $\overline{C}$ such that

$$\sup_{k \geq 1} \{\mathcal{H}^{n-1}(S^\sigma_{\overline{\varphi}_k}) + \mathcal{H}^{n-1}(S^\sigma_{\overline{\varphi}})\} \leq \overline{C}.$$

Furthermore, $\overline{\varphi}_k$ have equibounded BV-norm, and so there is a (not relabelled) subsequence such that

$$\overline{\varphi}_k \to \overline{\varphi}_\infty \text{ weakly}^* \text{ in } BV(\Omega); \tag{5.16}$$

since up to a subsequence the same convergence holds pointwise a.e., and also $u_k \to u$ a.e. on Ω, we deduce that $\overline{\varphi}_\infty$ must be a lifting of u.

Thanks to the fact that $\sigma \leq \pi$, the functions

$$v_k := \varphi_k - \overline{\varphi}_k, \qquad k \geq 1$$

take values in $2\pi\mathbb{Z}$, and belong to $GSBV(\Omega; 2\pi\mathbb{Z})$, since the jump sets satisfy $S_{v_k} \subseteq S^\sigma_{\overline{\varphi}_k} \cup S^\sigma_{\varphi_k}$, and so

$$\mathcal{H}^{n-1}(S_{v_k}) \leq \mathcal{H}^{n-1}(S^\sigma_{\overline{\varphi}_k}) + \mathcal{H}^{n-1}(S^\sigma_{\varphi_k}) < \overline{C} < +\infty,$$

for all $k \geq 1$. Trivially v_k are liftings of the same function $f \equiv (1, 0) \in \mathbb{S}^1$, and so we can apply Theorem 4.2 to obtain that, for a non-relabelled subsequence, there exist a Caccioppoli partition $\{E_i\}_{i\in\mathbb{N}}$ of Ω, sequences of integers $(d^{(i)}_k)_{k\geq1}$, and a function $v_\infty \in GSBV(\Omega; 2\pi\mathbb{Z})$ such that, for all $i \in \mathbb{N}$,

$$\begin{aligned}
&(\varphi_k - \overline{\varphi}_k - 2\pi d^{(i)}_k) \to v_\infty \text{ pointwise a.e. on } E_i, \\
&|\varphi_k - \overline{\varphi}_k - 2\pi d^{(i)}_k| \to +\infty \text{ pointwise a.e. on } \Omega \setminus E_i.
\end{aligned}$$

From this and (5.16) we conclude

$$\begin{aligned}
&(\varphi_k - 2\pi d^{(i)}_k) \to v_\infty + \overline{\varphi}_\infty \text{ pointwise a.e. on } E_i, \\
&|\varphi_k - 2\pi d^{(i)}_k| \to +\infty \text{ pointwise a.e. on } \Omega \setminus E_i.
\end{aligned}$$

This is the thesis, just by setting $\varphi_\infty := \overline{\varphi}_\infty + v_\infty$, which is a lifting of u since v_∞ takes values in $2\pi\mathbb{Z}$. □

6 Γ-convergence of functionals on $\mathbb{S}^1$- valued maps

The proof of Theorem 3.1 can be achieved by suitably adapting the proof of [12, Theorem 1.1] to the case with linear growth. For this reason we omit here the details. The proof of Theorem 3.2 is more delicate. In particular the lower bound inequality requires a local argument which relies on the compactness result for liftings (Theorem 4.5). For convenience we introduce the localised Modica-Mortola-type (or Allen-Cahn type) functionals

$$\mathrm{MM}_\varepsilon(v, A) := \int_A \left(\varepsilon|\nabla v|^2 + \frac{W(v)}{\varepsilon}\right) \mathrm{d}x \qquad \forall v \in W^{1,2}(\Omega),$$

for every open set $A \subseteq \Omega$, where W is defined as in (c) of Section 2.2.

6.1 Proof of Theorem 3.2

Step 1: Lower bound. Let $\varepsilon_k \searrow 0$ as $k \to +\infty$. We have to show that, for every sequence $((u_k, v_k))_{k\geq 1} \subset L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$ converging to (u, v) in $L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$,

$$\liminf_{k\to+\infty} \mathrm{F}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k) \geq \mathrm{F}_{\mathrm{lift}}(u, v)\,. \tag{6.1}$$

We may assume

$$\sup_{k\in\mathbb{N}} \mathrm{F}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k) \leq C < +\infty$$

so that $(u_k, v_k) \in \mathcal{D}_{\mathbb{S}^1}$, $v = 1$ a.e. in Ω and, up to a not relabelled subsequence,

$$\liminf_{k\to+\infty} \mathrm{F}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k) = \lim_{k\to+\infty} \mathrm{F}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k),$$

From Theorem 2.3 it follows that

$$\liminf_{k\to+\infty} \mathrm{F}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k) \geq \int_\Omega f(|\nabla u|)\,\mathrm{d}x + |D^c u|(\Omega) + \int_{S_u} g(|u^+ - u^-|)\mathrm{d}\mathcal{H}^{n-1}\,.$$

In particular, from Remark 2.4-(iii)-5 and the fact that $|u^+ - u^-| \leq 2$ a.e., since u is $\mathbb{S}^1$-valued we deduce $u \in BV(\Omega; \mathbb{S}^1)$. For every $k \geq 1$ we choose a lifting $\varphi_k \in W^{1,2}(\Omega)$ of u_k in Ω. Using that $|\nabla u_k| = |\nabla \varphi_k|$ we have

$$+\infty > C \geq \mathrm{F}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k) = \int_\Omega \left(\psi(v_k) f(|\nabla\varphi_k|) + \varepsilon_k |\nabla v_k|^2 + \frac{W(v_k)}{\varepsilon_k}\right) \mathrm{d}x\,.$$

Thus, from the coarea formula,

$$C \geq \mathrm{MM}_{\varepsilon_k}(v_k, \Omega) \geq \int_\Omega \sqrt{W(v_k)} |\nabla v_k|\,\mathrm{d}x = \int_0^1 \sqrt{W(t)} \mathcal{H}^{n-1}(\partial^* F^t_k)\,\mathrm{d}t \tag{6.2}$$

for any $k \geq 1$, where $F^t_k := \{x \in \Omega : v_k(x) \leq t\}$. Let $\eta', \eta'' \in (0, 1)$, $\eta' < \eta''$ be fixed. By (6.2) and the mean value theorem there exists $t(k) \in (\eta', \eta'')$ such that

$$C \geq C(\eta', \eta'')(\eta'' - \eta')\mathcal{H}^{n-1}(\partial^* F^{t(k)}_k)\,. \tag{6.3}$$

Moreover, using also that ψ is increasing,

$$C \geq \int_\Omega \psi(v_k) f(|\nabla\varphi_k|)\,\mathrm{d}x \geq C\psi(\eta') \int_\Omega \chi_{\Omega\setminus F^{t(k)}_k} |\nabla\varphi_k|\,\mathrm{d}x\,. \tag{6.4}$$

Setting

$$\phi_k := \varphi_k \chi_{\Omega\setminus F^{t(k)}_k} \in SBV^2(\Omega),$$

we have $S_{\phi_k} \subset \partial^* F^{t(k)}_k$ and, concerning the approximate gradients, $\nabla\phi_k = \nabla\varphi_k \chi_{\Omega\setminus F^{t(k)}_k}$. Therefore, from (6.4) and (6.3),

$$\int_\Omega |\nabla\phi_k|\,\mathrm{d}x + \mathcal{H}^{n-1}(S_{\phi_k}) \leq C(\eta', \eta'') \tag{6.5}$$

for some $C(\eta', \eta'') > 0$ depending on η', η'' and independent of k. Define

$$\overline{u}_k := e^{i\phi_k} = u_k \chi_{\Omega\setminus F^{t(k)}_k} + (1, 0)\chi_{F^{t(k)}_k} \qquad \forall k \in \mathbb{N},$$

which, by (6.3) and (6.4), are uniformly bounded in $BV(\Omega;\mathbb{S}^1)$. This together with

$$|F_k^{t(k)}| \to 0, \tag{6.6}$$

imply that the sequence $(\overline{u}_k)$ weakly* converges to u in $BV(\Omega;\mathbb{S}^1)$. Hence, using (6.5), we can apply Theorem 4.5 to the sequence $(\phi_k)_{k\ge 1}$ and get, for a not-relabelled subsequence, a lifting $\varphi_\infty \in GBV(\Omega)$ of u, such that (ϕ_k) converges locally modulo 2π to φ_∞. Namely, there exists a Caccioppoli partition $\{E_i\}_{i\in\mathbb{N}}$ of Ω, sequences $(d_k^{(i)})_{k\ge 1}\subset\mathbb{Z}$ for any integer $i\ge 1$ with the following properties:

$$\begin{aligned}
&\lim_{k\to+\infty}(\phi_k(x)-2\pi d_k^{(i)}) = \varphi_\infty(x) \qquad \forall i\in\mathbb{N},\ \text{for a.e. } x\in E_i\,,\\
&\lim_{k\to+\infty}|\phi_k(x)-2\pi d_k^{(i)}| = +\infty \qquad \forall i\in\mathbb{N},\ \text{for a.e. } x\in \Omega\setminus E_i.
\end{aligned}$$

Again, using (6.6), the same holds for φ_k, i.e.,

$$\begin{aligned}
&\lim_{k\to+\infty}(\varphi_k(x)-2\pi d_k^{(i)}) = \varphi_\infty(x) \qquad \forall i\in\mathbb{N},\ \text{for a.e. } x\in E_i\,,\\
&\lim_{k\to+\infty}|\varphi_k(x)-2\pi d_k^{(i)}| = +\infty \qquad \forall i\in\mathbb{N},\ \text{for a.e. } x\in \Omega\setminus E_i.
\end{aligned} \tag{6.7}$$

In particular, to prove the validity of (6.1) it is sufficient to show

$$\liminf_{k\to+\infty} \mathrm{F}^{\mathbb{S}^1}_{\varepsilon_k}(u_k,v_k) \ge \int_\Omega f(|\nabla\varphi_\infty|)\,\mathrm{d}x + |D^c\varphi_\infty|(\Omega) + \int_{S_u} g(|\varphi_\infty^+ - \varphi_\infty^-|)\mathrm{d}\mathcal{H}^{n-1}\,.$$

since, being φ_∞ a lifting of u, we have

$$\int_\Omega f(|\nabla\varphi_\infty|)\,\mathrm{d}x + |D^c\varphi_\infty|(\Omega) = \int_\Omega f(|\nabla u|)\,\mathrm{d}x + |D^c u|(\Omega)\,,$$

and

$$\int_{S_u} g(|\varphi_\infty^+ - \varphi_\infty^-|)\mathrm{d}\mathcal{H}^{n-1} \ge m_g[u]\,.$$

For any integer $K\ge 1$ we consider the truncated function

$$\varphi_k^{i,K} := ((\varphi_k - 2\pi d_k^{(i)})\wedge K)\vee(-K) \in W^{1,2}(\Omega)\,,$$

and

$$\phi_k^{i,K} := \varphi_k^{i,K}\chi_{\Omega\setminus F_k^{t(k)}}.$$

According to $S_{\phi_k^{i,K}} \subset \partial^* F_k^{t(k)}$ and $[\![\phi_k^{i,K}]\!] \le K$, we have

$$\begin{aligned}
|D\phi_k^{i,K}|(\Omega) &= \int_\Omega |\nabla\phi_k|\chi_{\{|\varphi_k - 2\pi d_k^{(i)}|<K\}}\,\mathrm{d}x + \int_{S_{\phi_k^{i,K}}} [\![\phi_k^{i,K}]\!]\,\mathrm{d}\mathcal{H}^{n-1}\\
&\le \int_\Omega |\nabla\phi_k|\,\mathrm{d}x + K\mathcal{H}^{n-1}(\partial^* F_k^{t(k)}) \le C\,.
\end{aligned}$$

Hence, up to a subsequence (depending on K), $(\phi_k^{i,K})$ converges to some $\phi_\infty^{i,K}$ in $L^1(\Omega)$ as $k\to+\infty$. Moreover from (6.7) it holds $\phi_\infty^{i,K} := (\varphi_\infty\wedge K)\vee(-K)$ in E_i. Let also $F_i^\pm$ be such that $\Omega\setminus E_i = F_i^+\cup F_i^-$ and $\phi_\infty^{i,K} = \pm K$ in $F_i^\pm$. As $|F_k^{t(k)}|\to 0$ it follows that

$(\varphi_k^{i,K})$ converges to $\phi_\infty^{i,K}$ in $L^1(\Omega)$ as $k \to +\infty$. Hence the sequence $((\varphi_k^{i,K}, v_k))$ converges to $(\phi_\infty^{i,K}, 1)$ in $L^1(\Omega) \times L^1(\Omega)$ and

$$\int_A \left(\psi(v_k) f(|\nabla \varphi_k^{i,K}|) + \varepsilon_k |\nabla v_k|^2 + \frac{W(v_k)}{\varepsilon_k} \right) \mathrm{d}x \leq C\,,$$

for any open set $A \subset \Omega$, for some $C > 0$ independent of k. Thus, from Theorem 2.3 and Remark 2.4-(ii)

$$\begin{aligned} &\liminf_{k\to+\infty} \int_A \left(\psi(v_k) f(|\nabla \varphi_k|) + \varepsilon_k |\nabla v_k|^2 + \frac{W(v_k)}{\varepsilon_k} \right) \mathrm{d}x \\ &\quad \geq \liminf_{k\to+\infty} \int_A \left(\psi(v_k) f(|\nabla \varphi_k^{i,K}|) + \varepsilon_k |\nabla v_k|^2 + \frac{W(v_k)}{\varepsilon_k} \right) \mathrm{d}x \\ &\quad \geq \int_A f(|\nabla \phi_\infty^{i,K}|)\,\mathrm{d}x + |D^c \phi_\infty^{i,K}|(A) + \int_{S_{\phi_\infty^{i,K}} \cap A} g(|(\phi_\infty^{i,K})^+ - (\phi_\infty^{i,K})^-|) \mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned} \tag{6.8}$$

In particular Remark 2.4 together with $|(\phi_\infty^{i,K})^+ - (\phi_\infty^{i,K})^-| \leq 2K$ imply $\phi_\infty^{i,K} \in BV(\Omega)$.

Consider the bounded positive measure given by

$$\lambda(B) := \int_B f(|\nabla \varphi_\infty|)\,\mathrm{d}x + |D^c \varphi_\infty|(B) + \mathcal{H}^{n-1}(S_{\varphi_\infty} \cap B) \quad \forall \text{ Borel set } B \subseteq \Omega\,.$$

For each $i, K \in \mathbb{N}$ we define the following functions

$$h^{i,K}(x) := \begin{cases} g(|(\phi_\infty^{i,K})^+ - (\phi_\infty^{i,K})^-|) & \text{if } x \in S_{\phi_\infty^{i,K}} \\ 1 & \text{otherwise in } \Omega. \end{cases}$$

Thus from (6.8) we have

$$\liminf_{k\to+\infty} \int_A \left(\psi(v_k) f(|\nabla \varphi_k|) + \varepsilon_k |\nabla v_k|^2 + \frac{W(v_k)}{\varepsilon_k} \right) \mathrm{d}x \geq \int_A h^{i,K} \mathrm{d}\lambda \quad \forall i, K \in \mathbb{N},$$

for all $A \in \mathcal{A}(\Omega)$. Next observing that

$$h(x) := \sup_{i,K\in\mathbb{N}} h^{i,K}(x) = \begin{cases} 1 & \text{if } x \in \Omega \setminus S_{\varphi_\infty} \\ g(|\varphi_\infty^+ - \varphi_\infty^-|) & \text{if } x \in S_{\varphi_\infty}, \end{cases}$$

by invoking Lemma 2.1 we conclude.

Step 2: Upper bound. Let $\varepsilon_k \searrow 0$ and $u \in BV(\Omega; \mathbb{S}^1)$. We have to find a sequence $((u_k, v_k)) \subset \mathcal{D}_{\mathbb{S}^1}$ converging to $(u, 1)$ in $L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$ and

$$\limsup_{k\to+\infty} \mathrm{F}_{\varepsilon_k}^{\mathbb{S}^1}(u_k, v_k) \leq \mathrm{F}_{\mathrm{lift}}(u, 1)\,.$$

By Corollary 4.3 we can select a jump minimizing lifting $\varphi \in GBV(\Omega)$ of u in Ω, and

$$\int_{S_\varphi} g(|\varphi^+ - \varphi^-|) \mathrm{d}\mathcal{H}^{n-1} = m_g[u]\,. \tag{6.9}$$

By Theorem 2.3 there exist $(\varphi_k, v_k) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ such that $(\varphi_k, v_k) \to (\varphi, 1)$ in $L^1(\Omega) \times L^1(\Omega)$ and

$$\limsup_{k\to+\infty} \mathrm{F}_{\varepsilon_k}(\varphi_k, v_k) \leq \mathrm{F}(\varphi, 1)\,. \tag{6.10}$$

Next we let $u_k := e^{i\varphi_k} \in W^{1,2}(\Omega;\mathbb{S}^1)$, so tat $(u_k, v_k) \to (u, 1)$ in $L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega)$. Moreover, from (6.9), (6.10), $|\nabla u_k| = |\nabla \varphi_k|$ and $|\nabla u| = |\nabla \varphi|$ we get

$$\mathrm{F}_{\varepsilon_k}(\varphi_k, v_k) = \mathrm{F}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k)\,, \quad \mathrm{F}(\varphi, 1) = \mathrm{F}_{\mathrm{lift}}(u, 1)$$

and so

$$\limsup_{k\to+\infty} \mathrm{F}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k) \leq \mathrm{F}_{\mathrm{lift}}(u, 1)\,.$$

□

Data availability statement

No new data was created or analysed in this work.

Conflict of interest statement

The authors have no conflict of interest to declare that are relevant to the content of this article.

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