CRITICAL POINTS OF THE TWO-DIMENSIONAL AMBROSIO-TORTORELLI FUNCTIONAL WITH CONVERGENCE OF THE PHASE-FIELD ENERGY

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Abstract. We consider a family $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ of critical points of the Ambrosio-Tortorelli functional. Assuming a uniform energy bound, the sequence $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ converges in $L^2(\Omega)$ to a limit $(u, 1)$ as $\varepsilon \to 0$, where u is in $SBV^2(\Omega)$. It was previously shown that if the full Ambrosio-Tortorelli energy associated to $(u_\varepsilon, v_\varepsilon)$ converges to the Mumford-Shah energy of u, then the first inner variation converges as well. In particular, u is a critical point of the Mumford-Shah functional in the sense of inner variations. In this work, focusing on the two-dimensional setting, we extend this result under the sole convergence of the phase-field energy to the length energy term in the Mumford-Shah functional.

1. Introduction

The Mumford-Shah (MS) functional is a prototypical energy used in free-discontinuity problems where the competition between a volume and a surface energy leads to concentration on a co-dimension one discontinuity set. It has been historically introduced in [28] in the context of image segmentation, and has also been used in fracture mechanics in [20] to describe the propagation of brittle cracks.

In order to present this energy, we need to introduce some notation. Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^N$ with $N \geq 1$, and let $g \in H^{\frac{1}{2}}(\partial\Omega)$ be a Dirichlet boundary data. The energy space, denoted by $SBV^2(\Omega)$, is made of all functions $u \in BV(\Omega)$ with vanishing Cantor part such that the approximate gradient ∇u belongs to $L^2(\Omega; \mathbb{R}^N)$ and $\mathcal{H}^{N-1}(J_u) < \infty$, where $J_u \subset \Omega$ is the jump set of u and $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. In order to account for possible boundary discontinuities, we introduce the extended jump set $\widehat{J_u}$ of u defined by

$$\widehat{J_u} := J_u \cup (\partial\Omega \cap \{u \neq g\}), \tag{1.1}$$

where, with a slight abuse of notation, we still denote by u the inner trace of u on $\partial\Omega$. We define the Mumford-Shah functional $MS : SBV^2(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} MS(u) &= \int_\Omega |\nabla u|^2\, dx + \mathcal{H}^{N-1}(J_u) + \mathcal{H}^{N-1}(\partial\Omega \cap \{u \neq g\}) \\ &= \int_\Omega |\nabla u|^2\, dx + \mathcal{H}^{N-1}(\widehat{J_u}) \quad \text{for all } u \in SBV^2(\Omega). \end{aligned} \tag{1.2}$$

Using the direct method in the calculus of variations, it can be proven that this functional admits minimizers as a consequence of Ambrosio's compactness and lower semicontinuity results, see [1, 15, 13]. The description of the jump set is of particular interest in connexion with the Mumford-Shah conjecture which stipulates that, in dimension $N = 2$, the jump set J_u is made of finitely many $\mathcal{C}^1$ curves intersecting at triple junction type points. In particular, the regularity (or lack of regularity) of the jump set represents a very interesting topic of research also from the application point of view for the description of edges in image segmentation and cracks in mechanics. We refer to the monographs [1, 14, 16] for related results in that direction.

If one is interested into the numerical approximation of minimizers of the Mumford-Shah energy, we face a difficulty related to the fact that the discontinuity set is not a priori known. It would require a high mesh size precision in the spatial region where u is discontinuous which might lead to too costly numerical schemes. This is why it is convenient to approximate the MS energy by a more regular functional. In the spirit of the Allen-Cahn model for phase transitions, see e.g. [27, 32], Ambrosio and Tortorelli proposed in [2] the following variational phase-field regularization of the MS energy. In our context, we set

$$\mathcal{A}_g := \{(u,v) \in [H^1(\Omega) \times (H^1(\Omega) \cap L^\infty(\Omega))] : \ u = g \text{ and } v = 1 \text{ on } \partial\Omega\} \tag{1.3}$$

and we define the Ambrosio-Tortorelli (AT) energy $AT_\varepsilon : \mathcal{A}_g \to \mathbb{R}$ by

$$AT_\varepsilon(u,v) = \int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2 dx + \int_\Omega \left(\varepsilon |\nabla v|^2 + \frac{(1-v)^2}{4\varepsilon} \right) dx \quad \text{for all } (u,v) \in \mathcal{A}_g, \tag{1.4}$$

where $0 < \eta_\varepsilon \ll \varepsilon$ is a small parameter that ensures ellipticity. This regularization has a mechanical interpretation. The phase-field variable v can be seen as a damage variable taking values in the interval $[0,1]$. The region $\{v = 1\}$ corresponds to completely sane material, whereas where $\{v = 0\}$ the material is totally damaged. In [2], in a slightly different context (i.e., in absence of Dirichlet boundary conditions), a Γ-convergence result of AT_ε to $\widetilde{MS}$ is proved by suitably extending MS as a two variables functional :

$$\widetilde{MS}(u,v) = \begin{cases} MS(u) & \text{if } u \in SBV^2(\Omega) \text{ and } v \equiv 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.5}$$

The existence of minimizers in $\mathcal{A}_g$ for AT_ε for a fixed value of ε is obtained via the direct method in the calculus of variations. The fundamental theorem of Γ-convergence then ensures that a subsequence $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ of minimizers of AT_ε converges in $[L^2(\Omega)]^2$ to $(u,1)$ where u a minimizer of MS. This motivates the use of AT_ε, with $\varepsilon > 0$ small, to numerically approximate minimizers of the MS functional. This regularized formulation is indeed at the basis of numerical simulations, see e.g. [11].

However, concerning the numerical implementation, we observe that the term $v^2|\nabla u|^2$ makes the functional AT_ε non-convex. Consequently, numerical methods might fail to converge to a minimizer of AT_ε. For example, using the fact that AT_ε remains separately strictly convex, it was proposed in [11] to perform an alternate minimization algorithm. It is proven in [10, Theorem 1] that the sequence of iterates converges to a critical point of AT_ε, but nothing guarantees this critical point to be a minimizer. It motivates the study of the convergence of critical points of the AT functional as $\varepsilon \to 0$, and it raises the question to know if their limits correspond to critical points of the MS functional in some sense.

We start by recalling that a critical point for the AT functional is a pair $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g$ such that

$$\frac{d}{dt}\Big|_{t=0} AT_\varepsilon(u_\varepsilon + t\psi, v_\varepsilon + t\varphi) = 0 \quad \text{for all } (\psi,\varphi) \in \ H^1_0(\Omega) \times (H^1_0(\Omega) \cap L^\infty(\Omega)). \tag{1.6}$$

Tools of Γ-convergence provide little help to study the convergence of critical points. The extension of the fundamental theorem of Γ-convergence to the convergence of critical points has been studied in various settings, see e.g. [21, 33, 34, 6] for the Allen-Cahn functional and [7, 8, 26, 29] for the Ginzburg-Landau functional, see also [12]. Concerning the AT functional, the general convergence of critical points to critical points of the MS energy is established in dimension $N = 1$ in [19, 22, 4]. We also refer to [9] for the convergence of critical points of a phase-field approximation of cohesive fracture energies in 1D. In dimension $N \geq 2$, the convergence of critical points has been established in [5], with the additional assumption of convergence of the energy

$$AT_\varepsilon(u_\varepsilon, v_\varepsilon) \to MS(u).$$

A similar assumption of convergence of energy was made in [23, 24, 25] to study the convergence of critical points of the Allen-Cahn functional to minimal surfaces, and to understand how stability passes to the limit. However this is not always true that convergence of the energy holds. For example, in [21, Section 6.3] it is proved that interfaces with multiplicities can be limits of a sequence of critical points of the Allen-Cahn functional. Consequently these critical points do not satisfy the energy convergence assumption.

In this article we weaken the assumption of convergence of energy made in [5]. Our main result Theorem 2.1, establishes the convergence of critical points under the assumption that only the phase-field term converges, i.e.,

$$\int_\Omega \left(\varepsilon|\nabla v|^2 + \frac{(1-v)^2}{4\varepsilon}\right) dx \to \mathcal{H}^{N-1}(\widehat{J_u}).$$

This result is restricted to dimension $N = 2$, which is meaningful from the application point of view either in image segmentation, or in the anti-plane brittle fracture model.

Let us close this introduction by some notations and definitions used throughout the paper.

Linear algebra. Let $\mathbb{M}^{N\times N}$ be the set of real $N \times N$ matrices, and $\mathbb{M}^{N\times N}_{\rm sym}$ the set of real symmetric $N \times N$ matrices. Given two vectors a and b in $\mathbb{R}^N$, we denote by $a \cdot b \in \mathbb{R}$ their inner product and by $a \otimes b = a^T b \in \mathbb{M}^{N\times N}$ their tensor product. We will use the Frobenius inner product of matrices A and $B \in \mathbb{M}^{N\times N}$, defined by $A : B = \mathrm{Tr}(A^T B)$ and the associated norm $|A| = \mathrm{Tr}(A^T A)^{1/2}$.

Measures. The Lebesgue measure in $\mathbb{R}^N$ is denoted by $\mathcal{L}^N$, and the k-dimensional Hausdorff measure by $\mathcal{H}^k$.

If $X \subset \mathbb{R}^N$ is a locally compact set and Y an Euclidean space, we denote by $\mathcal{M}(X;Y)$ the space of Y-valued bounded Radon measures in X endowed with the norm $\|\mu\| = |\mu|(X)$, where $|\mu|$ is the variation of the measure μ. If $Y = \mathbb{R}$, we simply write $\mathcal{M}(X)$ instead of $\mathcal{M}(X;\mathbb{R})$. By Riesz representation theorem, $\mathcal{M}(X;Y)$ can be identified with the topological dual of $\mathcal{C}_0(X;Y)$, the space of continuous functions $f : X \to Y$ such that $\{|f| \geq \varepsilon\}$ is compact for all $\varepsilon > 0$. The weak* topology of $\mathcal{M}(X;Y)$ is defined using this duality.

Functional spaces. We use standard notation for Lebesgue, Sobolev and Hölder spaces. Given a bounded open set $\Omega \subset \mathbb{R}^N$, the space of functions of bounded variation is defined by

$$BV(\Omega) = \{u \in L^1(\Omega) : \ Du \in \mathcal{M}(\Omega;\mathbb{R}^N)\}.$$

We shall also consider the subspace $SBV(\Omega)$ of special functions of bounded variation made of functions $u \in BV(\Omega)$ whose distributional derivative can be decomposed as

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

In the previous expression, ∇u is the Radon-Nikodým derivative of Du with respect to $\mathcal{L}^N$, and it is called the approximate gradient of u. The Borel set J_u is the (approximate) jump set of u. It is a countably $\mathcal{H}^{N-1}$-rectifiable subset of Ω oriented by the (approximate) normal direction of jump $\nu_u : J_u \to \mathbf{S}^{N-1}$, and $u^\pm$ are the one-sided approximate limits of u on J_u according to ν_u. Finally we define

$$SBV^2(\Omega) = \{u \in SBV(\Omega) : \ \nabla u \in L^2(\Omega;\mathbb{R}^N) \text{ and } \mathcal{H}^{N-1}(J_u) < \infty\}.$$

We say that a Lebesgue measurable set $E \subset \Omega$ has finite perimeter in Ω if its characteristic function $\mathbf{1}_E \in BV(\Omega)$. We denote by $\partial^\star E$ its reduced boundary. We refer to [1] for a detailed description of the space BV.

Varifolds. Let us recall several basic ingredients of the theory of varifolds (see e.g. [31]). We denote by $\mathbf{G}_{N-1}$ the Grassmannian manifold of all $(N-1)$-dimensional linear subspaces of $\mathbb{R}^N$. The set $\mathbf{G}_{N-1}$ is as usual identified with the set of all orthogonal projection matrices onto $(N-1)$-dimensional linear subspaces of $\mathbb{R}^N$, i.e., $N \times N$ symmetric matrices A such that $A^2 = A$ and $\mathrm{tr}(A) = N - 1$, in other words, matrices of the form

$$A = \mathrm{Id} - e \otimes e$$

for some $e \in \mathbf{S}^{N-1}$.

A $(N-1)$-varifold in X (a locally compact subset of $\mathbb{R}^N$) is a bounded Radon measure on $X \times \mathbf{G}_{N-1}$. The class of $(N-1)$-varifold in X is denoted by $\mathbf{V}_{N-1}(X)$. The mass of $V \in \mathbf{V}_{N-1}(X)$ is simply the measure $\|V\| \in \mathcal{M}(X)$ defined by $\|V\|(B) = V(B \times \mathbf{G}_{N-1})$ for all Borel sets $B \subset X$. We define the first variation of an $(N-1)$-varifold in V in an open set $U \subset \mathbb{R}^N$ by

$$\delta V(\varphi) = \int_{U \times \mathbf{G}_{N-1}} D\varphi(x) : A \, \mathrm{d}V(x, A) \quad \text{for all } \varphi \in \mathcal{C}^1_c(U; \mathbb{R}^N).$$

We say that an $(N-1)$-varifold is stationary in U if $\delta V(\varphi) = 0$ for all φ in $\mathcal{C}^1_c(U; \mathbb{R}^N)$. We recall that such a varifold satisfies the monotonicity formula

$$\frac{\|V\|(B_\varrho(x_0))}{\varrho^{N-1}} = \frac{\|V\|(B_r(x_0))}{r^{N-1}} + \int_{(B_\varrho(x_0)\setminus B_r(x_0)) \times \mathbf{G}_{N-1}} \frac{|P_{A^\perp}(x - x_0)|^2}{|x - x_0|^{N+1}} \, \mathrm{d}V(x, A) \tag{1.7}$$

for all $x_0 \in U$ and $0 < r < \varrho$ with $B_\varrho(x_0) \subset U$, where $P_{A^\perp}$ is the orthogonal projection onto the one-dimensional space $A^\perp$ (see [31, paragraph 40]).

2. Formulation of the problem and statements of the main result

Throughout the paper, Ω is a bounded open subdomain of $\mathbb{R}^2$ with $\mathcal{C}^{2,1}$ boundary. We denote by ν the outward unit normal field on $\partial\Omega$.

We recall that the Ambrosio-Tortorelli functional is defined for $(u, v) \in \mathcal{A}_g$ by (1.4), where $\mathcal{A}_g$ is given by (1.3) and $\eta_\varepsilon > 0$ is a positive small parameter such that $\eta_\varepsilon/\varepsilon \to 0$. We also recall that the Mumford-Shah functional is defined by (1.2). A critical point $(u_\varepsilon, v_\varepsilon)$ of AT_ε with a prescribed Dirichlet boundary condition is defined in (1.6). It corresponds to a zero of the outer variations of AT_ε. It is immediate to check that $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g$ is a weak solution of the following system of elliptic partial differential equations:

$$\begin{cases} -\varepsilon \Delta v_\varepsilon + \dfrac{v_\varepsilon - 1}{4\varepsilon} + v_\varepsilon |\nabla u_\varepsilon|^2 = 0 & \text{in } \Omega, \\ \operatorname{div}\left((\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon\right) = 0 & \text{in } \Omega, \\ u = g, \quad v = 1 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Whereas the notion of critical points for the AT functional is fairly standard, the notion of critical points for the MS energy is more involved, see [1, Section 7.4]. We first observe that considering the outer variations of MS is not sufficient to obtain a proper notion of critical point for MS. Indeed, outer variations of u with respect to a smooth direction leaves the surface energy $\mathcal{H}^1(J_u)$ unchanged, so that such variations do not bring any information on the discontinuity set J_u. In order to complete the equations derived from the outer variations, we consider *inner variations* of the MS energy. Such variations correspond to deformations of the domain Ω. To define the inner variations of MS up to the boundary, in addition to requiring $\partial\Omega$ to be of class $\mathcal{C}^{2,1}$ we also ask the boundary data g to belong to the Hölder space $\mathcal{C}^{2,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$. These assumptions are needed to invoke the boundary regularity results of [5, Theorem 3.2] which ensures that the weak solutions $(u_\varepsilon, v_\varepsilon)$ of (2.1) are actually classical solutions and belong to $[\mathcal{C}^{2,\alpha}(\overline{\Omega})]^2$. Note that, it is necessary, in this type of variational problems to have some regularity results at our disposal in order to relate inner and outer variations, cf. e.g. [25, Section 2].

Definition 2.1. *Let $X \in \mathcal{C}^1_c(\mathbb{R}^2; \mathbb{R}^2)$ be a vector field satisfying $X \cdot \nu = 0$ on $\partial\Omega$. The* flow map $\Phi_t(x)$ *of X, defined for every x in $\mathbb{R}^2$, is the unique solution of the Cauchy problem*

$$\begin{cases} \dfrac{d}{dt}\Phi_t(x) = X(\Phi_t(x)) & \text{for all } t \in \mathbb{R}, \\ \Phi_0(x) = x. \end{cases}$$

Thanks to the Cauchy-Lipschitz Theorem, it can be shown that for all $x \in \mathbb{R}^2$, the map $t \in \mathbb{R} \mapsto \Phi_t(x)$ is globally well-defined, that $(t, x) \mapsto \Phi_t(x)$ belongs to $\mathcal{C}^1(\mathbb{R} \times \mathbb{R}^2; \mathbb{R}^2)$ and that $\{\Phi_t\}_{t\in\mathbb{R}}$ is a one-parameter group of $\mathcal{C}^1$-diffeomorphisms of $\mathbb{R}^2$ with $\Phi_0 = \mathrm{Id}$. Furthermore, for each $t > 0$, we can check that Φ_t is a $\mathcal{C}^1$-diffeomorphism of $\overline{\Omega}$ which preserves both Ω and $\partial\Omega$.

In the sequel, we consider an arbitrary extension G of $g \in \mathcal{C}^{2,\alpha}(\partial\Omega)$, which we assume to belong to $\mathcal{C}^{2,\alpha}_{\rm loc}(\mathbb{R}^2)$. As already observed in [5], the notion of inner variations introduced below a priori depends on the choice of the extension G. However, we do not explicitly highlight this dependence in order not to overburden notation.

Definition 2.2. *Let $X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$ be such that $X\cdot\nu = 0$ on $\partial\Omega$ and $\{\Phi_t\}_{t\in\mathbb{R}}$ be its flow map. For all $u \in SBV^2(\Omega)$, the* first inner variation *of MS at u in the direction X is defined by*

$$\delta MS(u)[X] := \lim_{t\to 0}\frac{MS(u\circ\Phi_t^{-1} - G\circ\Phi_t + G) - MS(u)}{t}.$$

In order to define outer variations, we introduce the following notation. Given $\varphi \in SBV^2(\Omega)$, we set $\widehat{J_\varphi} = J_\varphi \cup (\partial\Omega\cap\{\varphi\neq g\})$ where we still denote by φ the inner trace of φ on $\partial\Omega$. In other words, setting $\hat\varphi := \varphi\mathbf{1}_\Omega + G\mathbf{1}_{\mathbb{R}^2\setminus\Omega} \in SBV^2_{\rm loc}(\mathbb{R}^2)$, then $\widehat{J_\varphi} = J_{\hat\varphi}$.

Definition 2.3. *Let $\varphi \in SBV^2(\Omega)$ be such that $\widehat{J_\varphi}\subset\widehat{J_u}$. The* first outer variation *of MS is defined by*

$$dMS(u)[\varphi] := \lim_{t\to 0}\frac{MS(u+t\varphi) - MS(u)}{t}. \tag{2.2}$$

Observe that tests function are taken in $SBV^2(\Omega)$ and not only in $\mathcal{C}^\infty_c(\Omega)$. The assumption $\widehat{J_\varphi}\subset\widehat{J_u}$ ensures the existence of the limit in (2.2).

Definition 2.4. *A function $u \in SBV^2(\Omega)$ is a* critical point *of MS if it satisfies the following two conditions:*

$$\begin{cases} dMS(u)[\varphi] = 0 & \text{for all } \varphi\in SBV^2(\Omega) \text{ such that } \widehat{J_\varphi}\subset\widehat{J_u},\\ \delta MS(u)[X] = 0 & \text{for all } X\in\mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2) \text{ such that } X\cdot\nu = 0 \text{ on } \partial\Omega.\end{cases}$$

Computing the inner and the outer variations of MS, cf. e.g. [5, Appendix A], we find that a critical point of MS satisfies

$$\int_\Omega \nabla u\cdot\nabla\varphi\,dx = 0 \quad \text{for all } \varphi\in SBV^2(\Omega) \text{ such that } \widehat{J_\varphi}\subset\widehat{J_u} \tag{2.3}$$

and

$$\int_\Omega (|\nabla u|^2\mathrm{Id} - 2\nabla u\otimes\nabla u) : DX\,dx + \int_{\widehat{J_u}}(\mathrm{Id} - \nu_u\otimes\nu_u) : DX\,d\mathcal{H}^1$$
$$= -2\int_{\partial\Omega}(\nabla u\cdot\nu)(X\cdot\nabla g)\,d\mathcal{H}^1 \text{ for all } X\in\mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2) \text{ with } X\cdot\nu = 0 \text{ on } \partial\Omega.$$

Note that if $\widehat{J_u}$ is smooth enough, (2.3) translates into $\Delta u = 0$ in $\Omega\setminus J_u$ and $\partial_{\nu_u}u = 0$ on $\widehat{J_u}$ (see [1, Section 7.4]).

One can also define the inner variations of AT_ε.

Definition 2.5. *Let $X\in\mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$ be such that $X\cdot\nu = 0$ on $\partial\Omega$ and let Φ_t be the flow map associated to X. The first inner variation of AT_ε at (u,v) is defined by*

$$\delta AT_\varepsilon(u,v)[X] = \lim_{t\to 0}\frac{AT_\varepsilon(u\circ\Phi_t^{-1} - G\circ\Phi_t^{-1} + G, v\circ\Phi_t^{-1}) - AT_\varepsilon(u,v)}{t}.$$

According to [5, Theorem 3.1], if $(u_\varepsilon, v_\varepsilon)\in\mathcal{A}_g$ is a critical point of AT_ε in the sense of (2.1), then $(u_\varepsilon, v_\varepsilon)\in[\mathcal{C}^\infty(\Omega)]^2$. Moreover, since we assumed that the boundary data g belongs to $\mathcal{C}^{2,\alpha}(\partial\Omega)$ and $\partial\Omega$ to be of class $\mathcal{C}^{2,1}$, we deduce from [5, Theorem 3.2] that $(u_\varepsilon, v_\varepsilon)\in[\mathcal{C}^{2,\alpha}(\overline{\Omega})]^2$. This regularity property ensures that $(u_\varepsilon, v_\varepsilon)$ is also a zero of the inner variations, i.e., $\delta AT_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$, cf. e.g. [25, Corollary

2.3] or [5, Proposition 4.2]. Computing explicitly the inner variations of AT_ε (see [5, Lemma A.3]), we find that a critical point $(u_\varepsilon, v_\varepsilon)$ of AT_ε also satisfies

$$\int_\Omega (\eta_\varepsilon + v_\varepsilon^2)\left(2\nabla u_\varepsilon \otimes \nabla u_\varepsilon - |\nabla u_\varepsilon|^2 \mathrm{Id}\right) : DX\, dx + \int_\Omega \left[2\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon - \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon |\nabla v_\varepsilon|^2\right) \mathrm{Id}\right] : DX\, dx = 2(\eta_\varepsilon + 1)\int_{\partial\Omega} (\partial_\nu u_\varepsilon)(X\cdot \nabla g)\, d\mathcal{H}^1 \quad (2.4)$$

for all $X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$ such that $X \cdot \nu = 0$ on $\partial\Omega$.

We are now in position to state the main result of this work.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with boundary of class $\mathcal{C}^{2,1}$ and $g \in \mathcal{C}^{2,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$. Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a sequence of critical points of AT_ε in the sense of (2.1). Suppose that the sequence $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ satisfies the energy bound*

$$\sup_{\varepsilon>0} AT_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty. \quad (2.5)$$

Then, up to extraction, we have that

$$(u_\varepsilon, v_\varepsilon) \xrightarrow[\varepsilon\to 0]{[L^2(\Omega)]^2} (u, 1) \text{ with } u \in SBV^2(\Omega) \text{ and } \mathrm{div}(\nabla u) = 0 \text{ in } \mathcal{D}'(\Omega). \quad (2.6)$$

If we further assume the following phase-field energy convergence

$$\int_\Omega \left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon|\nabla v_\varepsilon|^2\right) dx \to \mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega \cap \{u \neq g\}) = \mathcal{H}^1(\widehat{J_u}), \quad (2.7)$$

then

$$\int_\Omega (|\nabla u|^2 \mathrm{Id} - 2\nabla u \otimes \nabla u) : DX\, dx + \int_{\widehat{J_u}} (\mathrm{Id} - \nu_u \otimes \nu_u) : DX\, d\mathcal{H}^1 = -2\int_{\partial\Omega} (\nabla u \cdot \nu)(X \cdot \nabla g)\, d\mathcal{H}^1 \text{ for all } X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2) \text{ with } X\cdot \nu = 0 \text{ on } \partial\Omega. \quad (2.8)$$

Note that the first point of (2.6) follows from the compactness part of the Γ-convergence theory for AT_ε, see [2, 5]. The second point of (2.6) is also a rather direct consequence of the Γ-convergence theory for AT_ε. Indeed from this theory we obtain the weak convergence $(\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon \rightharpoonup \nabla u$ in $L^2(\Omega;\mathbb{R}^2)$, and we can pass to the limit in the second equation of (2.1) to obtain $\mathrm{div}(\nabla u) = 0$ in $\mathcal{D}'(\Omega)$. Note that this condition differs from (2.3) because test functions are smooth, and not discontinuous across $\widehat{J_u}$. Hence, this condition does not suffice to claim that u is a critical point of MS for the outer variations because the Neumann condition on $\widehat{J_u}$, even in a weak sense, is not satisfied. We leave as an open problem to determine if (2.3) is satisfied by limits of critical points of the AT energy satisfying the assumptions of Theorem 2.1, see also Remark 4.2. Finally, the validity of condition (2.8) actually shows that u a critical point of MS for the inner variations.

We point out that, whereas the main assumption in [5, Theorem 1.2] was

$$AT_\varepsilon(u_\varepsilon, v_\varepsilon) \xrightarrow[\varepsilon\to 0]{} MS(u)$$

we are able to obtain the same conclusion with the weaker assumption (2.7). However our result is restricted to the two-dimensional case. We would like to stress that this result does not amount to proving the convergence of the whole energy from the only convergence of the phase-field variable. In fact, we will see in Proposition 4.1 that the elastic term $(\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2$ of AT_ε does not necessarily converge to the elastic term $|\nabla u|^2$ of MS in the sense of measures, and that a singular defect measure might arise. In particular, the lack of strong $L^2(\Omega;\mathbb{R}^2)$-convergence of $\{(\eta_\varepsilon+v_\varepsilon^2)\nabla u_\varepsilon\}_{\varepsilon>0}$ to ∇u is an obstacle to pass to the limit in the second inner variation as in [5, Theorem 1.3].

To complete this section, let us explain the strategy of proof of Theorem 2.1 and the organization of the paper. As expected, one attempts to pass to the limit in (2.4) as $\varepsilon\to 0$. To do so, we first use some consequences of the compactness in the Γ-convergence theory for AT_ε to MS, and of the convergence of the phase-field energy (2.7) in Section 3. They lead to the so-called equi-partition of the (phase-field) energy principle and the identification of the limiting measure of the phase-field density as the one-dimensional Hausdorff measure restricted to the jump set $\widehat{J_u}$. We also explain how to associate a natural varifold to the phase-field variable v_ε following the ideas of [33]. In Section 4, we present one of our new ingredients which consists in a finer analysis of the sequence $\{(\eta_\varepsilon+v_\varepsilon^2)\nabla u_\varepsilon\otimes\nabla u_\varepsilon\}_{\varepsilon>0}$. Assumption (2.5) implies that this sequence is bounded, so that, up to an extraction, there exists a matrix-valued measure μ such that

$$(\eta_\varepsilon+v_\varepsilon^2)\nabla u_\varepsilon\otimes\nabla u_\varepsilon\mathcal{L}^2\llcorner\Omega\overset{\star}{\rightharpoonup}\mu\quad\text{in }\mathcal{M}(\Omega;\mathbb{M}^{2\times 2}_{\rm sym}).\tag{2.9}$$

In Propositions 4.1 and 4.2, we perform a fine analysis of the defect measure μ by showing that its Lebesgue-absolutely continuous part is given by $\nabla u\otimes\nabla u\mathcal{L}^2\llcorner\Omega$. This is achieved by means of a blow-up argument around convenient approximate differentiability points of u, which allows us to reduce to the case where the limit function u is actually an affine function. Then the convexity of AT_ε with respect to the phase-field variable enables one to use the equivalence between minimality and criticality with respect to v and then, improve in that specific case the weak L^2-convergence of the term $\{\sqrt{\eta_\varepsilon+v_\varepsilon^2}\nabla u_\varepsilon\}_{\varepsilon>0}$ to ∇u into a strong L^2-convergence. We observe that the blow-up argument requires the L^2-approximate differentiability of BV functions which holds only in dimension 2. This is one of the reason why our proof is restricted to the dimension $N=2$. Finally Section 5 completes the proof of the inner variations convergence thanks to a result of De Phillipis-Rindler [17] on the singular part of measures satisfying a linear PDE. To this aim, we introduce the stress-energy tensor

$$T_\varepsilon:=(\eta_\varepsilon+v_\varepsilon^2)(2\nabla u_\varepsilon\otimes\nabla u_\varepsilon-|\nabla u_\varepsilon|^2\mathrm{Id})+2\varepsilon\nabla v_\varepsilon\otimes\nabla v_\varepsilon-\left(\frac{(1-v_\varepsilon)^2}{\varepsilon}+\varepsilon|\nabla v_\varepsilon|^2\right)\mathrm{Id}.$$

According to (2.4), T_ε is a divergence measure matrix-valued function in Ω. The energy bound (2.5) ensures that $T_\varepsilon\overset{\star}{\rightharpoonup}T$ weakly* as measures in Ω, for some $T\in\mathcal{M}(\Omega;\mathbb{M}^{2\times 2}_{\rm sym})$ which remains divergence free. Applying the result of [17] implies that the singular part of T has a polar taking values into the set of singular matrices. In particular, it ensures that the singular part of T with respect to the Lebesgue measure must be absolutely continuous with respect to $\mathcal{H}^1\llcorner\widehat{J_u}$. The density of T with respect to $\mathcal{H}^1\llcorner\widehat{J_u}$ is then characterized by means of a blow-up argument. We then employ the previous information on the absolutely continuous part of μ with respect to the Lebesgue measure to deduce that the limit map u in Theorem 2.1 does satisfy (2.8).

3. Preliminary results

We first show that the energy bound assumption (2.5) provides compactness properties leading to (2.6).

Lemma 3.1. *Let $\{(u_\varepsilon,v_\varepsilon)\}_{\varepsilon>0}\subset\mathcal{A}_g$ be a sequence of critical points of AT_ε satisfying* (2.5)*. Then, up to a subsequence (not relabeled), there exists $u\in SBV^2(\Omega)$ such that $(u_\varepsilon,v_\varepsilon)\to(u,1)$ in $[L^2(\Omega)]^2$,*

$$\sqrt{v_\varepsilon^2+\eta_\varepsilon}\nabla u_\varepsilon\rightharpoonup\nabla u\quad\text{in } L^2(\Omega;\mathbb{R}^2).\tag{3.1}$$

and

$$\mathrm{div}(\nabla u)=0\qquad\text{in }\mathcal{D}'(\Omega).$$

Proof. First of all, the energy bound immediately implies that $v_\varepsilon \to 1$ in $L^2(\Omega)$.

Next according to the maximum principle (see [5, Lemma 3.1]), we have $0 \le v_\varepsilon \le 1$ in Ω, and $\|u_\varepsilon\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\partial\Omega)}$. In particular, the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(\Omega)$.

Let $0 < a < b < 1$, we use Young's inequality and the coarea formula to get that

$$AT_\varepsilon(u_\varepsilon, v_\varepsilon) \ge \int_\Omega v_\varepsilon^2 |\nabla u_\varepsilon|^2\, dx + \int_\Omega (1-v_\varepsilon)|\nabla v_\varepsilon|\, dx \tag{3.2}$$

$$\ge \int_\Omega v_\varepsilon^2 |\nabla u_\varepsilon|^2\, dx + \int_a^b (1-t)\mathcal{H}^1(\partial^\star\{v_\varepsilon > t\})\, dt. \tag{3.3}$$

By the mean value Theorem, there exists $t_\varepsilon \in (a,b)$ such that the set $A_\varepsilon = \{v_\varepsilon > t_\varepsilon\}$ has finite perimeter in Ω and

$$AT_\varepsilon(u_\varepsilon, v_\varepsilon) \ge t_\varepsilon^2 \int_{A_\varepsilon} |\nabla u_\varepsilon|^2\, dx + \left[\frac{(1-a)^2}{2} - \frac{(1-b)^2}{2}\right] \mathcal{H}^1(\partial^\star A_\varepsilon).$$

Let $\hat u_\varepsilon := u_\varepsilon \mathbf{1}_{A_\varepsilon}$. According to [1, Theorem 3.84], $\hat u_\varepsilon \in SBV^2(\Omega)$ with $\nabla \hat u_\varepsilon = \nabla u_\varepsilon \mathbf{1}_{A_\varepsilon}$ and $J_{\hat u_\varepsilon} \subset \partial^\star A_\varepsilon$ so that, using $t_\varepsilon > a$,

$$\int_\Omega |\nabla \hat u_\varepsilon|^2\, dx + \mathcal{H}^1(J_{\hat u_\varepsilon}) \le C_{a,b},$$

for some constant $C_{a,b} > 0$. Moreover, recalling that $\|u_\varepsilon\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\partial\Omega)}$, it follows that the sequence $\{\hat u_\varepsilon\}_{\varepsilon>0}$ is also bounded in $L^\infty(\Omega)$. We are thus in position to apply Ambrosio's compactness Theorem (Theorem 4.8 in [1]) which ensures, up to the extraction of a subsequence (not relabelled), the existence of $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$ such that $\hat u_\varepsilon \to u$ in $L^2(\Omega)$, $\|u\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\partial\Omega)}$ and $\nabla \hat u_\varepsilon \rightharpoonup \nabla u$ in $L^2(\Omega;\mathbb{R}^2)$.

According to the Chebychev inequality, we have

$$\mathcal{L}^2(\Omega \setminus A_\varepsilon) \le \frac{1}{(1-t_\varepsilon)^2}\int_\Omega (1-v_\varepsilon)^2\, dx \le \frac{1}{(1-a)^2}\int_\Omega (1-v_\varepsilon)^2\, dx \to 0,$$

where we used the energy bound (2.5). As a consequence,

$$\|u_\varepsilon - u\|_{L^2(\Omega)} \le \|u_\varepsilon - \hat u_\varepsilon\|_{L^2(\Omega)} + \|\hat u_\varepsilon - u\|_{L^2(\Omega)} \le 2\|g\|_{L^\infty(\partial\Omega)}\mathcal{L}^2(\Omega\setminus A_\varepsilon) + \|\hat u_\varepsilon - u\|_{L^2(\Omega)} \to 0,$$

hence $u_\varepsilon \to u$ in $L^2(\Omega)$. Similarly, for all $\varphi \in \mathcal{C}_c^\infty(\Omega;\mathbb{R}^2)$, we have

$$\int_\Omega (\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon - \nabla u)\cdot\varphi\, dx = \int_\Omega (\nabla \hat u_\varepsilon - \nabla u)\cdot \varphi\, dx + \int_{\Omega\setminus A_\varepsilon} \sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon \cdot \varphi\, dx + \int_{A_\varepsilon} (\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon - \nabla u_\varepsilon)\cdot\varphi\, dx. \tag{3.4}$$

On the one hand, we have

$$\int_\Omega (\nabla \hat u_\varepsilon - \nabla u)\cdot\varphi\, dx \to 0. \tag{3.5}$$

On the other hand, thanks to the Cauchy-Schwarz inequality,

$$\left|\int_{\Omega\setminus A_\varepsilon} \sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon\cdot\varphi\, dx\right| \le \|\varphi\|_{L^\infty(\Omega)} \left\|\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon\right\|_{L^2(\Omega)} \mathcal{L}^2(\Omega\setminus A_\varepsilon)^{1/2} \to 0, \tag{3.6}$$

where we used once more the energy bound. Finally, since $v_\varepsilon > t_\varepsilon > a > 0$ in A_ε and $\{\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(\Omega;\mathbb{R}^2)$, we infer that $\{\|\nabla u_\varepsilon\|_{L^2(A_\varepsilon)}\}_{\varepsilon>0}$ is bounded. Thus, using again the Cauchy-Schwarz inequality and the energy bound, we get that

$$\left|\int_{A_\varepsilon} (\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon - \nabla u_\varepsilon)\cdot\varphi\, dx\right| \le \|\varphi\|_{L^\infty(\Omega)}\|\nabla u_\varepsilon\|_{L^2(A_\varepsilon)} \left\|\sqrt{\eta_\varepsilon + v_\varepsilon^2} - 1\right\|_{L^2(\Omega)} \to 0. \tag{3.7}$$

Gathering (3.4)–(3.7) together with the boundedness of $\{\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega;\mathbb{R}^2)$, we conclude that $\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^2(\Omega;\mathbb{R}^2)$.

Since $\sqrt{\eta_\varepsilon + v_\varepsilon^2} \to 1$ in $L^2(\Omega)$ and $\operatorname{div}((\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon) = 0$ in $\mathcal{D}'(\Omega)$, we deduce that $\operatorname{div}(\nabla u) = 0$ in $\mathcal{D}'(\Omega)$. □

The phase-field energy convergence assumption (2.7) has several consequences, the main one being the so called equi-partition of the energy (see [5, Proposition 5.1]).

Proposition 3.1. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a sequence of critical points of AT_ε satisfying the assumptions of Theorem 2.1. We define the discrepancy*

$$\xi_\varepsilon = \varepsilon|\nabla v_\varepsilon|^2 - \frac{(1-v_\varepsilon)^2}{4\varepsilon}.$$

Then $\xi_\varepsilon \to 0$ in $L^1(\Omega)$ and setting

$$w_\varepsilon = \Phi(v_\varepsilon) \quad \text{with } \Phi(t) = t - t^2/2, \tag{3.8}$$

then

$$|\nabla w_\varepsilon|\mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} \mathcal{H}^1 \llcorner \widehat{J_u} \quad \text{weakly}^\star \text{ in } \mathcal{M}(\overline{\Omega}).$$

The equi-partition of the energy is crucial in [5] to pass to the limit in (2.4). Combining [1, Proposition 1.80] and (2.7), we find the following convergence.

Proposition 3.2. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a sequence of critical points of AT_ε satisfying the assumptions of Theorem 2.1. Then*

$$\left(\frac{(1-v_\varepsilon)^2}{4\varepsilon} + \varepsilon|\nabla v_\varepsilon|^2\right)\mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} \mathcal{H}^1 \llcorner \widehat{J_u} \quad \text{weakly}^\star \text{ in } \mathcal{M}(\overline{\Omega}).$$

The energy bound also implies a weak-convergence result for the boundary terms, which corresponds to [5, Lemma 4.1].

Lemma 3.2. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a sequence of critical points of AT_ε satisfying the assumptions of Theorem 2.1. Up to a further subsequence (not relabeled), $\partial_\nu u_\varepsilon \rightharpoonup \nabla u \cdot \nu$ in $L^2(\partial\Omega)$. Moreover, there exists a non-negative boundary Radon measure $m \in \mathcal{M}(\partial\Omega)$ such that*

$$\left[|\partial_\nu u_\varepsilon|^2 + \varepsilon|\partial_\nu v_\varepsilon|^2\right]\mathcal{H}^1 \llcorner \partial\Omega \overset{\star}{\rightharpoonup} m \quad \text{weakly}^\star \text{ in } \mathcal{M}(\partial\Omega).$$

The results concerning the measure μ and the passage to the limit in the inner variations of AT_ε find a better formulation in terms of varifolds. We now recall the expression of the varifold associated to the phase-field function v_ε, and some of the convergence results established in [5].

Definition 3.1. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a sequence satisfying $\sup_\varepsilon AT_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$ and let w_ε be defined by (3.8). We introduce the varifold $V_\varepsilon \in \mathcal{M}(\overline{\Omega} \times \mathbf{G}_1)$ by*

$$\langle V_\varepsilon, \varphi\rangle = \int_{\Omega\cap\{\nabla w_\varepsilon \neq 0\}} \varphi\left(x, \mathrm{Id} - \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} \otimes \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|}\right)|\nabla w_\varepsilon|\,dx \quad \text{for all } \varphi \in \mathcal{C}(\overline{\Omega}\times \mathbf{G}_1). \tag{3.9}$$

Definition 3.9 implies that the weight measure of V_ε is given by $\|V_\varepsilon\| = |\nabla w_\varepsilon|\mathcal{L}^2 \llcorner \Omega$. In [5, Section 5] the following convergence result of Proposition 3.3 is established. It is an immediate consequence of the energy bound (2.5), the weak* compactness of bounded sequences of Radon measures and the Disintegration Theorem (see e.g. [1, Theorem 2.28])

Proposition 3.3. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a sequence satisfying $\sup_\varepsilon AT_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$, and let V_ε be the varifold defined in* (3.9). *Up to a subsequence (not relabeled), there exists a varifold $V \in \mathcal{M}(\overline{\Omega} \times \mathbf{G}_1)$ such that*

$$V_\varepsilon \xrightharpoonup{\star} V \quad \text{weakly}^\star \text{ in } \mathcal{M}(\overline{\Omega} \times \mathbf{G}_1). \tag{3.10}$$

Moreover, there exists a weak* $(\mathcal{H}^1 \llcorner \widehat{J_u})$-measurable mapping $x \mapsto V_x \in \mathcal{M}(\mathbf{G}_1)$ of probability measures such that $V = (\mathcal{H}^1 \llcorner \widehat{J_u}) \otimes V_x$, i.e.

$$\int_{\overline{\Omega}\times\mathbf{G}_1} \varphi(x, A)\, dV(x, A) = \int_{\widehat{J_u}} \left(\int_{\mathbf{G}_1} \varphi(x, A)\, dV_x(A) \right) d\mathcal{H}^1(x) \quad \text{for all } \varphi \in \mathcal{C}(\overline{\Omega} \times \mathbf{G}_1).$$

Using the disintegrated structure of V, we have an alternative expression of the first variation of the varifold V.

Proposition 3.4. *For $\mathcal{H}^1$-a.e $x \in \widehat{J_u}$, we define the first moment of V_x as*

$$\overline{A}(x) := \int_{\mathbf{G}_1} A\, dV_x(A). \tag{3.11}$$

Then, the matrix $\overline{A}(x) \in \mathbb{M}^{2\times2}_{\rm sym}$ is an orthogonal projector for $\mathcal{H}^1$-a.e x in $\widehat{J_u}$ and[1]

$$\delta V(X) = \int_{\widehat{J_u}} \overline{A}(x) : DX(x)\, d\mathcal{H}^1(x) \quad \text{for all } X \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^2). \tag{3.12}$$

Moreover, for $\mathcal{H}^1$-a.e $x \in \widehat{J_u}$, we have $V_x = \delta_{\overline{A}(x)}$.

Proof. According to [5, Lemma 5.2], for $\mathcal{H}^1$-a.e $x \in \widehat{J_u}$, the matrix $\overline{A}(x) \in \mathbb{M}^{2\times2}_{\rm sym}$ satisfies $\mathrm{Tr}(\overline{A}(x)) = 1$ and $\rho(\overline{A}(x)) = 1$, where ρ denotes the spectral radius. In dimension two, those conditions imply that the eigenvalues of $\overline{A}(x)$ are exactly 0 and 1, which ensures that $\overline{A}(x)$ is an orthogonal projector.

Concerning the representation of the first variation of V, we consider a test function $X \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^2)$. According to Proposition 3.3 with $\varphi(x, A) = DX(x) : A$ together with Fubini's Theorem, we infer that

$$\begin{aligned}\delta V(X) = \int_{\overline{\Omega}\times\mathbf{G}_1} DX(x) : A\, dV(x, A) &= \int_{\widehat{J_u}} \left(\int_{\mathbf{G}_1} DX(x) : A\, dV_x(A) \right) d\mathcal{H}^1(x) \\ &= \int_{\widehat{J_u}} \left(\int_{\mathbf{G}_1} A\, dV_x(A) \right) : DX(x)\, d\mathcal{H}^1(x),\end{aligned}$$

which corresponds to (3.12) by definition (3.11) of $\overline{A}$.

Since any element $A \in \mathbf{G}_1$ satisfies $|A| = 1$ and V_x is a probability measure over $\mathbf{G}_1$ for $\mathcal{H}^1$-a.e. $x \in \widehat{J_u}$, we infer that the variance of V_x is zero, i.e.

$$\int_{\mathbf{G}_1} |\overline{A}(x) - A|^2\, dV_x(A) = 0,$$

and thus, $A = \overline{A}(x)$ for V_x-a.e $A \in \mathbf{G}_1$. As a consequence, V_x is a probability measure concentrated at $\{\overline{A}(x)\}$, hence the Dirac measure at $\overline{A}(x)$. □

In higher dimension, Proposition 3.4 remains true under the hypothesis (2.7). A direct consequence is that the probability measure V_x is actually a Dirac mass concentrated at $\overline{A}(x)$.

[1] the first variation of a varifold is usually defined on an open set. We extend this definition to treat the boundary here.

4. Analysis of the defect measure

The aim of this section is to prove the following result, which is independent of the phase-field energy convergence assumption.

Proposition 4.1. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be family of critical points of the Ambrosio-Tortorelli functional such that*

$$\sup_{\varepsilon>0} AT_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty.$$

Then, there exist a subsequence (not relabeled) and a non-negative measure $\sigma \in \mathcal{M}(\Omega)$ which is singular with respect to the Lebesgue measure, such that

$$(\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2 \mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} |\nabla u|^2 \mathcal{L}^2 \llcorner \Omega + \sigma \quad \textit{weakly}^\star \textit{ in } \mathcal{M}(\Omega). \tag{4.1}$$

Proof. Since the families $\{(\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2\}_{\varepsilon>0}$ and $\{\varepsilon|\nabla v_\varepsilon|^2 + (1-v_\varepsilon)^2/4\varepsilon\}_{\varepsilon>0}$ are bounded in $L^1(\Omega)$ by the energy bound (2.5), up to a subsequence (not relabelled), there exist nonnegative measures λ_1 and $\lambda_2 \in \mathcal{M}(\Omega)$ such that

$$(\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2 \mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} \lambda_1 \quad \text{weakly}^\star \text{ in } \mathcal{M}(\Omega)$$

and

$$\left(\varepsilon|\nabla v_\varepsilon|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon}\right) \mathcal{L}^2 \llcorner \Omega \overset{\star}{\rightharpoonup} \lambda_2 \quad \text{weakly}^\star \text{ in } \mathcal{M}(\Omega).$$

On the other hand, since from Lemma 3.1 we know that $\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^2(\Omega;\mathbb{R}^2)$, it results from the weak lower semi-continuity of the norm that $\lambda_1 \geqslant |\nabla u|^2 \mathcal{L}^2 \llcorner \Omega$. Thus the measure $\sigma = \lambda_1 - |\nabla u|^2 \mathcal{L}^2 \llcorner \Omega$ is nonnegative, and it remains to show that it is singular with respect to $\mathcal{L}^2$. To this aim, by Lebesgue's differentiation Theorem, it is enough to prove that

$$\frac{d\lambda_1}{d\mathcal{L}^2} = |\nabla u|^2 \quad \mathcal{L}^2\text{-a.e. in } \Omega. \tag{4.2}$$

We write $\lambda_1 = \frac{d\lambda_1}{d\mathcal{L}^2}\mathcal{L}^2 + \lambda_1^s$ with λ_1^s which is singular with respect to the Lebesgue measure. Let $x_0 \in \Omega$ be

(i) such that $\frac{d\lambda_1}{d\mathcal{L}^2}(x_0)$ and $\frac{d\lambda_2}{d\mathcal{L}^2}(x_0)$ exist and are finite;
(ii) a Lebesgue point of u, ∇u and $\frac{d\lambda_1}{d\mathcal{L}^2}$;
(iii) such that

$$\lim_{\rho\to 0} \frac{1}{\pi\rho^2} \int_{B_\rho(x_0)} \frac{|u(y) - u(x_0) - \nabla u(x_0)\cdot(y - x_0)|^2}{\rho^2}\, dy = 0; \tag{4.3}$$

(iv) such that $\frac{d\lambda_1^s}{d\mathcal{L}^2}(x_0) = 0$.

Note that $\mathcal{L}^2$ almost every points x_0 in Ω fullfil these properties by the Lebesgue differentiation Theorem, the L^2-differentiability a.e. of BV functions cf. [18, Theorem 1, Section 6.1] and the fact that λ_1^s is singular with respect to $\mathcal{L}^2$. We point out that the L^2-differentiability a.e. of BV functions holds only in dimension 2 which is one of the reasons of our dimensional restriction.

Let $R > 0$ be such that $B_R(x_0) \subset\subset \Omega$, and $\{\varrho_j\}_{j\in\mathbb{N}} \searrow 0^+$ be an infinitesimal sequence of radii such that $\rho_j < R$ for all $j \in \mathbb{N}$ and

$$\lambda_1(\partial B_{\rho_j}(x_0)) + \lambda_2(\partial B_{\rho_j}(x_0)) = 0. \tag{4.4}$$

For all y in B_1, we define the rescaled functions.

$$\hat{v}_{\varepsilon,j}(y) = v_\varepsilon(x_0 + \rho_j y), \quad \hat{u}_{\varepsilon,j}(y) = \frac{u_\varepsilon(x_0 + \rho_j y) - u(x_0)}{\rho_j}.$$

Then, since $(u_\varepsilon, v_\varepsilon) \to (u,1)$ in $[L^2(\Omega)]^2$,

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0}\int_{B_1}|\hat v_{\varepsilon,j}-1|^2\,dy = \lim_{j\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}|v_\varepsilon-1|^2\,dx = 0,$$

while

$$\begin{aligned}\lim_{j\to\infty}\lim_{\varepsilon\to 0}&\int_{B_1}|\hat u_{\varepsilon,j}(y)-\nabla u(x_0)\cdot y|^2\,dy\\ &= \lim_{j\to\infty}\int_{B_1}\left|\frac{u(x_0+\rho_j y)-u(x_0)}{\rho_j}-\nabla u(x_0)\cdot y\right|^2 dy\\ &= \lim_{j\to\infty}\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}\frac{|u(x)-u(x_0)-\nabla u(x_0)\cdot(x-x_0)|^2}{\rho_j^2}\,dx = 0,\end{aligned}$$

where we used (4.3). Moreover, for all $\zeta \in \mathcal{C}_c(B_1)$,

$$\begin{aligned}\lim_{j\to\infty}\lim_{\varepsilon\to 0}&\int_{B_1}(\eta_\varepsilon+\hat v_{\varepsilon,j}^2)|\nabla \hat u_{\varepsilon,j}|^2\zeta\,dy\\ &= \lim_{j\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}(\eta_\varepsilon+v_\varepsilon(x)^2)|\nabla u_\varepsilon(x)|^2\zeta\left(\frac{x-x_0}{\rho_j}\right)dx\\ &= \lim_{j\to\infty}\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}\zeta\left(\frac{x-x_0}{\rho_j}\right)d\lambda_1(x).\end{aligned}$$

Condition (iv) implies that

$$\frac{1}{\rho_j^2}\left|\int_{B_{\rho_j}(x_0)}\zeta\left(\frac{x-x_0}{\rho_j}\right)d\lambda_1^s\right| \leqslant \|\zeta\|_{L^\infty(B_1)}\frac{\lambda_1^s(B_{\rho_j}(x_0))}{\rho_j^2}\underset{j\to+\infty}{\longrightarrow} 0.$$

Hence, using item (ii),

$$\begin{aligned}\lim_{j\to\infty}\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}\zeta\left(\frac{x-x_0}{\rho_j}\right)d\lambda_1(x) &= \lim_{j\to\infty}\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}\zeta\left(\frac{x-x_0}{\rho_j}\right)\frac{d\lambda_1}{d\mathcal{L}^2}(x)\,dx\\ &= \frac{d\lambda_1}{d\mathcal{L}^2}(x_0)\int_{B_1}\zeta\,dy.\end{aligned}$$

We next observe that, thanks to (4.4), (i) and (iv),

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\pi\rho_j^2}\int_{B_{\rho_j}(x_0)}(\eta_\varepsilon+v_\varepsilon^2)|\nabla u_\varepsilon|^2\,dx = \lim_{j\to\infty}\frac{\lambda_1(B_{\rho_j}(x_0))}{\pi\rho_j^2} = \frac{d\lambda_1}{d\mathcal{L}^2}(x_0)$$

and

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\pi\rho_j^2}\int_{B_{\rho_j}(x_0)}\left(\varepsilon|\nabla v_\varepsilon|^2+\frac{(v_\varepsilon-1)^2}{4\varepsilon}\right)dx = \lim_{j\to\infty}\frac{\lambda_2(B_{\rho_j}(x_0))}{\pi\rho_j^2} = \frac{d\lambda_2}{d\mathcal{L}^2}(x_0),$$

while, changing variables

$$\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}(\eta_\varepsilon+v_\varepsilon^2)|\nabla u_\varepsilon|^2\,dx = \int_{B_1}(\eta_\varepsilon+\hat v_{\varepsilon,j}^2)|\nabla \hat u_{\varepsilon,j}|^2\,dy,$$

and

$$\frac{1}{\rho_j^2}\int_{B_{\rho_j}(x_0)}\left(\varepsilon|\nabla v_\varepsilon|^2+\frac{(v_\varepsilon-1)^2}{4\varepsilon}\right)dx = \frac{1}{\rho_j}\int_{B_1}\left((\varepsilon/\rho_j)|\nabla \hat v_{\varepsilon,j}|^2+\frac{(\hat v_{\varepsilon,j}-1)^2}{4(\varepsilon/\rho_j)}\right)dy.$$

Using a diagonal extraction argument together with the separability of $\mathcal{C}_c(B_1)$, we can find an infinitesimal sequence $\varepsilon_j \to 0$ such that, setting $\hat\varepsilon_j := \varepsilon_j/\rho_j$, $\hat v_j := \hat v_{\varepsilon_j,j}$, $\hat u_j := \hat u_{\varepsilon_j,j}$ and $\hat u(y) = \nabla u(x_0)\cdot y$, then

$$\begin{cases} \hat\varepsilon_j \to 0,\\ (\hat u_j,\hat v_j)\to(\hat u,1) \text{ in } [L^2(B_1)]^2,\\ (\eta_{\varepsilon_j}+\hat v_j^2)|\nabla \hat u_j|^2\mathcal{L}^2 \llcorner \Omega \rightharpoonup \frac{d\lambda_1}{d\mathcal{L}^2}(x_0)\mathcal{L}^2\llcorner\Omega \quad \text{weakly}^\star \text{ in } \mathcal{M}(B_1), \end{cases} \tag{4.5}$$

and

$$\int_{B_1}(\eta_{\varepsilon_j}+\hat v_j^2)|\nabla\hat u_j|^2\,dy \to \pi\frac{d\lambda_1}{d\mathcal{L}^2}(x_0), \quad \frac{1}{\rho_j}\int_{B_1}\left(\hat\varepsilon_j|\nabla\hat v_j|^2+\frac{(\hat v_j-1)^2}{4\hat\varepsilon_j}\right)dy \to \pi\frac{d\lambda_2}{d\mathcal{L}^2}(x_0).$$

By the classical compactness argument of the Ambrosio-Tortorelli functional (see Lemma 3.1), the previous convergences imply that $(\eta_{\varepsilon_j}+\hat v_j^2)^{1/2}\nabla\hat u_j \rightharpoonup \nabla\hat u = \nabla u(x_0)$ weakly in $L^2(B_1;\mathbb{R}^2)$. Now using that $(u_\varepsilon, v_\varepsilon)$ is a critical point of the Ambrosio-Tortorelli functional, we infer that $\operatorname{div}((\eta_{\varepsilon_j}+\hat v_j^2)\nabla\hat u_j)=0$ in B_1. Note that this partial differential equation is equivalent to the minimality property

$$\int_{B_1}(\eta_{\varepsilon_j}+\hat v_j^2)|\nabla\hat u_j|^2\,dy \le \int_{B_1}(\eta_{\varepsilon_j}+\hat v_j^2)|\nabla z|^2\,dy \quad \text{for all } z\in\hat u_j + H^1_0(B_1).$$

Let $\varphi\in\mathcal{C}^\infty_c(B_1)$ be a cut-off function with $0\le\varphi\le1$ in Ω. Since $\hat u\in H^1(B_1)$, we are allowed to take $z=\varphi\hat u+(1-\varphi)\hat u_j$ as competitor which leads to

$$\int_{B_1}(\eta_{\varepsilon_j}+\hat v_j^2)|\nabla\hat u_j|^2\,dy \le \int_{B_1}(\eta_{\varepsilon_j}+\hat v_j^2)|\varphi\nabla\hat u+(1-\varphi)\nabla\hat u_j+\nabla\varphi(\hat u_j-\hat u)|^2\,dy.$$

Expanding the square and using that $0\le\hat v_j\le 1$, and that $0\le\varphi^2\le\varphi$, we find that

$$\begin{aligned}\int_{B_1}(\eta_{\varepsilon_j}+\hat v_j^2)|\nabla\hat u_j|^2\,dy \quad\le\quad & \int_{B_1}\big(\varphi(\eta_{\varepsilon_j}+1)|\nabla\hat u|^2+(1-\varphi)(\eta_{\varepsilon_j}+\hat v_j^2)|\nabla\hat u_j|^2\big)\,dy\\ &+\int_{B_1}|\nabla\varphi|^2(\hat u_j-\hat u)^2\,dy\\ &+2\int_{B_1}(\eta_{\varepsilon_j}+\hat v_j^2)(\hat u_j-\hat u)(\varphi\nabla\hat u+(1-\varphi)\nabla\hat u_j)\cdot\nabla\varphi\,dy.\end{aligned}$$

Reorganizing the terms in the right-hand side and recalling that $\nabla\hat u=\nabla u(x_0)$, we obtain that

$$\begin{aligned}\int_{B_1}(\eta_{\varepsilon_j}+\hat v_j^2)|\nabla\hat u_j|^2\varphi\,dy \quad\le\quad & (\eta_{\varepsilon_j}+1)|\nabla u(x_0)|^2\int_{B_1}\varphi\,dy+\int_{B_1}|\nabla\varphi|^2(\hat u_j-\hat u)^2\,dy\\ &+2\int_{B_1}(\eta_{\varepsilon_j}+\hat v_j^2)(\hat u_j-\hat u)(\varphi\nabla\hat u+(1-\varphi)\nabla\hat u_j)\cdot\nabla\varphi\,dy.\end{aligned}$$

Since $\hat v_j\to1$ in $L^2(B_1)$, $\hat u_j\to\hat u$ in $L^2(B_1)$ and $\{(\eta_{\varepsilon_j}+\hat v_j^2)\nabla\hat u_j\}_{j\in\mathbb{N}}$ is bounded in $L^2(B_1;\mathbb{R}^2)$, we can pass to the limit and get that

$$\limsup_{j\to\infty}\int_{B_1}(\eta_{\varepsilon_j}+\hat v_j^2)|\nabla\hat u_j|^2\varphi\,dy \le |\nabla u(x_0)|^2\int_{B_1}\varphi\,dy.$$

Combining this inequality with the weak $L^2(B_1;\mathbb{R}^2)$ convergence of $\{(\eta_{\varepsilon_j}+\hat v_j^2)^{1/2}\nabla\hat u_j\}_{j\in\mathbb{N}}$ to $\nabla\hat u=\nabla u(x_0)$, we infer that $(\eta_{\varepsilon_j}+\hat v_j^2)^{1/2}\nabla\hat u_j\to\nabla u(x_0)$ strongly in $L^2_{\mathrm{loc}}(B_1)$. Recalling the last convergence in (4.5), we deduce that $\frac{d\lambda_1}{d\mathcal{L}^2}(x_0)=|\nabla u(x_0)|^2$, hence we obtain (4.2). □

Remark 4.1. If $\nabla u\in L^\infty(\Omega;\mathbb{R}^2)$, we have an alternative proof of the previous result based on Anzellotti's duality pairing (see [3]). Indeed, recalling that $\operatorname{div}(\nabla u)=0$ in $\mathcal{D}'(\Omega)$, we can define as in

[3, Definition 1.4] the distribution $[\nabla u \cdot Du] \in \mathcal{D}'(\Omega)$ by

$$\langle [\nabla u \cdot Du], \varphi \rangle := -\int_\Omega u \nabla u \cdot \nabla \varphi \, dx \quad \text{ for all } \varphi \in \mathcal{C}^\infty_c(\Omega). \tag{4.6}$$

By [3, Theorem 1.5 and Theorem 2.4], $[\nabla u \cdot Du]$ extends to a bounded Radon measure in Ω whose absolutely continuous part is given by

$$[\nabla u \cdot Du]^a = |\nabla u|^2 \mathcal{L}^2 \llcorner \Omega. \tag{4.7}$$

We now make the connexion between the measures λ_1 and $[\nabla u \cdot Du]$. Indeed, let $\varphi \in \mathcal{C}^\infty_c(\Omega)$, taking $u_\varepsilon \varphi$ as test function in the first equation of (2.1) yields

$$\int_\Omega (\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2 \varphi \, dx = -\int_\Omega u_\varepsilon (\eta_\varepsilon + v_\varepsilon^2) \nabla u_\varepsilon \cdot \nabla \varphi \, dx.$$

Since $\{u_\varepsilon\}_{\varepsilon>0}$ and $\{v_\varepsilon\}_{\varepsilon>0}$ are bounded in $L^\infty(\Omega)$ and $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in strongly in $[L^2(\Omega)]^2$, we have $u_\varepsilon \to u$ and $(\eta_\varepsilon + v_\varepsilon^2)^{1/2} \to 1$ strongly in $L^4(\Omega)$. Recalling that $(\eta_\varepsilon + v_\varepsilon^2)^{1/2} \nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $L^2(\Omega; \mathbb{R}^2)$ and $(\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2 \mathcal{L}^2 \llcorner \Omega \rightharpoonup \lambda_1$ weakly* in $\mathcal{M}(\Omega)$, we obtain

$$\int_\Omega \varphi \, d\lambda_1 = -\int_\Omega u \nabla u \cdot \nabla \varphi \, dx. \tag{4.8}$$

In orther words, $\lambda_1 = [\nabla u \cdot Du]$, and thus (4.7) implies that $\lambda_1 - |\nabla u|^2 \mathcal{L}^2 \llcorner \Omega$ is a singular measure with respect to $\mathcal{L}^2$.

Remark 4.2. Anzellotti's pairing (4.6) is still well-defined if we assume only $\nabla u \in L^2(\Omega; \mathbb{R}^2)$ and $\mathrm{div}(\nabla u) = 0$. In this remark we show that this pairing can be used to give a weak formulation to the criticality condition $\partial_{\nu_u} u = 0$ on J_u, valid for critical points of the MS energy when J_u is smooth enough. We recall that this criticality condition comes from (2.3). Now, formally we have that $[\nabla u \cdot Du] = \nabla u \cdot Du = |\nabla u|^2 \mathcal{L}^2 \llcorner \Omega + (u^+ - u^-) \nabla u \cdot \nu_u \mathcal{H}^1 \llcorner J_u$. Hence a possible weak formulation of (2.3) could be

$$\begin{cases} \mathrm{div}(\nabla u) = 0 \text{ in } \mathcal{D}'(\Omega) \\ [\nabla u \cdot Du] = |\nabla u|^2 \mathcal{L}^2 \llcorner \Omega. \end{cases}$$

Notice that, in light of (4.8), the second condition is fulfilled when $(\eta_\varepsilon + v_\varepsilon^2)^{1/2} \nabla u_\varepsilon \to \nabla u$ strongly in $L^2(\Omega)$ which is the case in the context of [5].

Proposition 4.1 allows us to compute the density of μ, the weak* limit of $(\eta_\varepsilon + v_\varepsilon^2) \nabla u_\varepsilon \otimes \nabla u_\varepsilon \mathcal{L}^2 \llcorner \Omega$, with respect to $\mathcal{L}^2$.

Proposition 4.2. *There exists a measure $\mu^s \in \mathcal{M}(\Omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ which is singular with respect to the Lebesgue measure such that*

$$(\eta_\varepsilon + v_\varepsilon^2) \nabla u_\varepsilon \otimes \nabla u_\varepsilon \mathcal{L}^2 \llcorner \Omega \xrightharpoonup{\star} \mu = \nabla u \otimes \nabla u \mathcal{L}^2 \llcorner \Omega + \mu^s \quad \textit{weakly}^\star \textit{ in } \mathcal{M}(\Omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}}). \tag{4.9}$$

Proof. Let σ be defined as in Proposition 4.1. Since the family $\{(v_\varepsilon^2 + \eta_\varepsilon) \nabla u_\varepsilon \otimes \nabla u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^1(\Omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ thanks to the energy bound (2.5), we infer that, up to a subsequence, $(\eta_\varepsilon + v_\varepsilon^2) \nabla u_\varepsilon \otimes \nabla u_\varepsilon \mathcal{L}^2 \llcorner \Omega \xrightharpoonup{\star} \mu$ weakly* in $\mathcal{M}(\Omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ for some $\mu \in \mathcal{M}(\Omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$. Writing the Lebesgue decomposition of μ as $\mu = \mu^a + \mu^s$, we claim that $\mu^a = \nabla u \otimes \nabla u \mathcal{L}^2 \llcorner \Omega$.

For $i = 1, 2$, we denote by $\sigma_i \in \mathcal{M}(\Omega)$ the nonnegative measure such that

$$(v_\varepsilon^2 + \eta_\varepsilon)|\partial_i u_\varepsilon|^2 \mathcal{L}^2 \llcorner \Omega \xrightharpoonup{\star} |\partial_i u|^2 \mathcal{L}^2 \llcorner \Omega + \sigma_i \text{ weakly}^\star \text{ in } \mathcal{M}(\Omega).$$

Let σ be the singular measure provided by Proposition 4.1. As $\sigma = \sigma_1 + \sigma_2$ with σ_1 and σ_2 non negative, we infer that σ_1 and σ_2 are absolutely continuous with respect to σ. Since σ is singular with respect to $\mathcal{L}^2$, we infer that both σ_1 and σ_2 are singular with respect to $\mathcal{L}^2$ as well. In addition, since $(\eta_\varepsilon+v_\varepsilon^2)|\partial_i u_\varepsilon|^2\mathcal{L}^2\llcorner\Omega \overset{\star}{\rightharpoonup} \mu^a_{ii}+\mu^s_{ii}$, we obtain, by uniqueness of the Lebesgue decomposition, that $\mu^s_{ii}=\sigma_i$ and $\mu^a_{ii}=|\partial_i u|^2\mathcal{L}^2\llcorner\Omega$.

Let us write $(v_\varepsilon^2+\eta_\varepsilon)\partial_1 u_\varepsilon\,\partial_2 u_\varepsilon\mathcal{L}^2\llcorner\Omega \overset{\star}{\rightharpoonup} \partial_1 u\,\partial_2 u\mathcal{L}^2\llcorner\Omega+\sigma_{12}$ weakly* in $\mathcal{M}(\Omega)$ for some $\sigma_{12}\in\mathcal{M}(\Omega)$. For every test function $\varphi\in\mathcal{C}_c(\Omega)$, Proposition 4.1 and Lemma 3.1 yield

$$\begin{aligned}\int_\Omega \left|\sqrt{v_\varepsilon^2+\eta_\varepsilon}\nabla u_\varepsilon-\nabla u\right|^2\varphi\,dx &= \int_\Omega (v_\varepsilon^2+\eta_\varepsilon)|\nabla u_\varepsilon|^2\varphi\,dx+\int_\Omega|\nabla u|^2\varphi\,dx\\ &\quad -2\int_\Omega\sqrt{v_\varepsilon^2+\eta_\varepsilon}\nabla u_\varepsilon\cdot\nabla u\varphi\,dx\\ &\underset{\varepsilon\to0}{\longrightarrow}\int_\Omega|\nabla u|^2\varphi\,dx+\int_\Omega\varphi\,d\sigma+\int_\Omega|\nabla u|^2\varphi\,dx-2\int_\Omega|\nabla u|^2\varphi\,dx\\ &=\int_\Omega\varphi\,d\sigma.\end{aligned}$$

Thus, we have $|\sqrt{v_\varepsilon^2+\eta_\varepsilon}\nabla u_\varepsilon-\nabla u|^2\mathcal{L}^2\llcorner \overset{\star}{\rightharpoonup}\sigma$ weakly* in $\mathcal{M}(\Omega)$. Hence

$$\begin{aligned}2&\left|(\sqrt{v_\varepsilon^2+\eta_\varepsilon}\partial_1 u_\varepsilon-\partial_1 u)(\sqrt{v_\varepsilon^2+\eta_\varepsilon}\partial_2 u_\varepsilon-\partial_2 u)\right|\\ &\leqslant\left(\sqrt{v_\varepsilon^2+\eta_\varepsilon}\partial_1 u_\varepsilon-\partial_1 u\right)^2+\left(\sqrt{v_\varepsilon^2+\eta_\varepsilon}\partial_2 u_\varepsilon-\partial_2 u\right)^2\\ &=\left|\sqrt{v_\varepsilon^2+\eta_\varepsilon}\nabla u_\varepsilon-\nabla u\right|^2\overset{\star}{\rightharpoonup}\sigma.\end{aligned}\tag{4.10}$$

Next, expanding the left-hand side of (4.10) and using Lemma 3.1, we obtain for all $\varphi\in\mathcal{C}_c(\Omega)$,

$$\begin{aligned}\int_\Omega&\left(\sqrt{v_\varepsilon^2+\eta_\varepsilon}\partial_1 u_\varepsilon-\partial_1 u\right)\left(\sqrt{v_\varepsilon^2+\eta_\varepsilon}\partial_2 u_\varepsilon-\partial_2 u\right)\,\varphi\,dx\\ &=\int_\Omega(v_\varepsilon^2+\eta_\varepsilon)\partial_1 u_\varepsilon\,\partial_2 u_\varepsilon\,\varphi\,dx-\int_\Omega\sqrt{v_\varepsilon^2+\eta_\varepsilon}\,\partial_1 u_\varepsilon\,\partial_2 u\,\varphi\,dx-\int_\Omega\sqrt{v_\varepsilon^2+\eta_\varepsilon}\,\partial_2 u_\varepsilon\,\partial_1 u\varphi\,dx\\ &\quad+\int_\Omega\partial_1 u\,\partial_2 u\varphi\,dx\\ &\underset{\varepsilon\to0}{\longrightarrow}\int_\Omega\partial_1 u\,\partial_2 u\,\varphi+\int_\Omega\varphi\,d\sigma_{12}-\int_\Omega\partial_1 u\,\partial_2 u\,\varphi\,dx=\int_\Omega\varphi\,d\sigma_{12}.\end{aligned}\tag{4.11}$$

By using (4.10) and (4.11) along with a measure theoreric argument (see e.g. in [30, Lemma 5.1]), we conclude that $|\sigma_{12}|\leqslant\frac{1}{2}\sigma$, from which we deduce that σ_{12} is absolutely continuous with respect to σ. Hence, σ_1,σ_2 and σ_{12} are singular with respect to $\mathcal{L}^2$ thanks to Proposition 4.1. We have thus established that

$$\mu=\mu^a+\mu^s=\nabla u\otimes\nabla u\mathcal{L}^2\llcorner\Omega+\begin{pmatrix}\sigma_1&\sigma_{12}\\ \sigma_{12}&\sigma_2\end{pmatrix},$$

with σ_1, σ_2 and σ_{12} singular with respect to $\mathcal{L}^2$. By uniqueness of the Lebesgue decomposition, we infer that

$$\mu^a=\nabla u\otimes\nabla u\mathcal{L}^2\llcorner\Omega,\quad \mu^s=\begin{pmatrix}\sigma_1&\sigma_{12}\\ \sigma_{12}&\sigma_2\end{pmatrix}$$

which completes the proof of the proposition. □

Remark 4.3. Propositions 4.1 and 4.2 show that one should not expect the strong L^2-convergence of $\{\sqrt{\eta_\varepsilon+v_\varepsilon^2}\nabla u_\varepsilon\}_{\varepsilon>0}$ to ∇u. This is an obstacle to be able to pass to the limit in the second inner variation as done in [5, Theorem 1.3].

5. Proof of Theorem 2.1

In this subsection, we prove Theorem 2.1 by passing to the limit in (2.4). Recall the definition of μ (4.9). Observe that $\mu_{12} = \mu_{21}$ and that, passing to the trace, $(\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2 \mathcal{L}^2 \llcorner \Omega \rightharpoonup \mu_{11} + \mu_{22}$. As in [5, Lemma 5.1], we have the following limit conservation law involving μ, the first moment $\overline{A}$ of V, and the boundary measure $m \in \mathcal{M}(\partial\Omega)$ defined in Lemma 3.2.

Lemma 5.1. *For all vector field $X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$, we have*

$$\langle (\mu_{11}+\mu_{22})\mathrm{Id} - 2\mu, DX\rangle + \int_{\widehat{J_u}} \overline{A} : DX \, d\mathcal{H}^1$$
$$= -\int_{\partial\Omega} (X\cdot\nu)\, dm + \int_{\partial\Omega} |\partial_\tau g|^2 (X\cdot\nu)\, d\mathcal{H}^1 - 2\int_{\partial\Omega} (\nabla u\cdot\nu)(X\cdot\tau)\partial_\tau g \, d\mathcal{H}^1, \quad (5.1)$$

where τ is a unit tangent vector to $\partial\Omega$ and $\partial_\tau g = \nabla g\cdot\tau$ is the tangential derivative of g on $\partial\Omega$.

Proof. Let $X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$ be an arbitrary vector field. By [5, Proposition 4.2],

$$\int_\Omega (2\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon \otimes \sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon - (\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2 \mathrm{Id}) : DX \, dx$$
$$+ \int_\Omega \left[2\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon - \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|\nabla v_\varepsilon|^2\right)\mathrm{Id}\right] : DX \, dx$$
$$= \int_{\partial\Omega} \left[(\eta_\varepsilon+1)|\partial_\nu u_\varepsilon|^2 + \varepsilon|\partial_\nu v_\varepsilon|^2 - (\eta_\varepsilon+1)|\partial_\tau g|^2\right](X\cdot\nu) d\mathcal{H}^1$$
$$+ 2(\eta_\varepsilon+1)\int_{\partial\Omega} \partial_\nu u_\varepsilon (X\cdot\tau)(\partial_\tau g)\, d\mathcal{H}^1, \quad (5.2)$$

We now study separately each term of this expression.

Using the definition (2.9) of the measure μ, we have

$$\int_\Omega (2\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon \otimes \sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon - (\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2 \mathrm{Id}) : DX \, dx$$
$$\xrightarrow[\varepsilon\to 0]{} \langle 2\mu - (\mu_{11}+\mu_{22})\mathrm{Id}, DX\rangle. \quad (5.3)$$

Next, thanks to the equi-partition of the energy (3.1) and the varifold convergence (3.10), we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_\Omega \left[2\varepsilon\nabla v_\varepsilon \otimes \nabla v_\varepsilon - \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|\nabla v_\varepsilon|^2\right)\right] : DX\, dx \\
=\ & \lim_{\varepsilon\to 0} \int_{\{|\nabla v_\varepsilon|\neq 0\}\cap\Omega} 2\varepsilon|\nabla v_\varepsilon|^2 \left(\frac{\nabla v_\varepsilon}{|\nabla v_\varepsilon|}\otimes\frac{\nabla v_\varepsilon}{|\nabla v_\varepsilon|} - \mathrm{Id}\right) : DX\, dx \\
=\ & \lim_{\varepsilon\to 0} \int_{\{|\nabla w_\varepsilon|\neq 0\}\cap\Omega} |\nabla w_\varepsilon| \left(\frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|}\otimes\frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} - \mathrm{Id}\right) : DX\, dx \qquad (5.4)\\
=\ & -\lim_{\varepsilon\to 0} \int_{\overline{\Omega}\times\mathbf{G}_1} A : DX(x)\, dV_\varepsilon(x,A) \\
=\ & -\int_{\overline{\Omega}\times\mathbf{G}_1} A : DX(x)\, dV(x,A) = -\int_{\widehat{J_u}} \overline{A} : DX d\mathcal{H}^1.
\end{aligned}$$

Finally, according to the regularity properties of $\partial\Omega$ and g, Lemma 3.2 yields

$$\int_{\partial\Omega} \left[(\eta_\varepsilon+1)|\partial_\nu u_\varepsilon|^2+\varepsilon|\partial_\nu v_\varepsilon|^2-(\eta_\varepsilon+1)|\partial_\tau g|^2\right](X\cdot\nu)\,d\mathcal{H}^1 \underset{\varepsilon\to 0}{\longrightarrow} \int_{\partial\Omega}(X\cdot\nu)\,dm-\int_{\partial\Omega}(X\cdot\nu)|\partial_\nu g|^2\,d\mathcal{H}^1 \quad (5.5)$$

and

$$2(\eta_\varepsilon+1)\int_{\partial\Omega}(\partial_\nu u_\varepsilon)(X\cdot\tau)\partial_\tau g\,d\mathcal{H}^1 \underset{\varepsilon\to 0}{\longrightarrow} 2\int_{\partial\Omega}(\nabla u\cdot\nu)(X\cdot\tau)\partial_\tau g\,d\mathcal{H}^1. \quad (5.6)$$

Combining (5.3), (5.4),(5.5), (5.6) yields (5.1). □

Our strategy is then to analyse each term of the left-hand-side of (5.1) to recover (2.8). We now set

$$T:=(\mu_{11}+\mu_{22})\,\mathrm{Id}-2\mu+\overline{A}\mathcal{H}^1\llcorner\widehat{J_u}\in\mathcal{M}(\overline{\Omega};\mathbb{M}^{2\times 2}). \quad (5.7)$$

We observe that (5.1) rewrites as

$$\langle T,DX\rangle=-\int_{\partial\Omega}(X\cdot\nu)\,dm+\int_{\partial\Omega}|\partial_\tau g|^2(X\cdot\nu)\,d\mathcal{H}^1 -2\int_{\partial\Omega}(\nabla u\cdot\nu)(X\cdot\tau)\partial_\tau g\,d\mathcal{H}^1 \quad \text{for all } X\in\mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2), \quad (5.8)$$

or still

$$-\operatorname{div}(T)=-\nu m\llcorner\partial\Omega+|\partial_\tau g|^2\nu\mathcal{H}^1\llcorner\partial\Omega-2(\nabla u\cdot\nu)(\partial_\tau g)\tau\mathcal{H}^1\llcorner\partial\Omega \quad \text{in } \mathcal{D}'(\mathbb{R}^2;\mathbb{R}^2). \quad (5.9)$$

Let us write the Lebesgue-Besicovitch decomposition for μ and T,

$$\begin{cases}\mu=\mu^a+\mu^j+\mu^c,\\ T=T^a+T^j+T^c,\end{cases} \quad (5.10)$$

where μ^a and T^a are absolutely continuous with respect to $\mathcal{L}^2$, μ^j and T^j are absolutely continuous with respect to $\mathcal{H}^1\llcorner\widehat{J_u}$, and μ^c and T^c are singular with respect to both $\mathcal{L}^2$ and $\mathcal{H}^1\llcorner\widehat{J_u}$. We observe that, according to Proposition 4.2,

$$\begin{cases}T^a=(\mu^a_{11}+\mu^a_{22})\mathrm{Id}-2\mu^a=(|\nabla u|^2\mathrm{Id}-2\nabla u\otimes\nabla u)\mathcal{L}^2,\\ T^j=(\mu^j_{11}+\mu^j_{22})\mathrm{Id}-2\mu^j+\overline{A}\mathcal{H}^1\llcorner\widehat{J_u},\\ T^c=(\mu^c_{11}+\mu^c_{22})\mathrm{Id}-2\mu^c.\end{cases} \quad (5.11)$$

Let Θ be the density of T^j with respect to $\mathcal{H}^1\llcorner\widehat{J_u}$, so that

$$T^j=\Theta\mathcal{H}^1\llcorner\widehat{J_u}=(\mu^j_{11}+\mu^j_{22})\mathrm{Id}-2\mu^j+\overline{A}\mathcal{H}^1\llcorner\widehat{J_u}. \quad (5.12)$$

We now establish algebraic properties of the measures T^j and T^c. The theory developed in [17] aims precisely at describing singular measures satisfying a linear PDE. It states that the polar of the singular part of the measure belongs to the wave-cone of the linear operator.

Lemma 5.2. *For $\mathcal{H}^1$-a.e. $x\in\widehat{J_u}$, the matrix $\Theta(x)\in\mathbb{M}^{2\times 2}_{\mathrm{sym}}$ is an orthogonal projector. Moreover, $T^c=0$ and there exists a scalar measure $\lambda^c\in\mathcal{M}(\overline{\Omega})$ such that $\mu^c=\lambda^c\,\mathrm{Id}$.*

Proof. According to (5.9), the limiting stress-energy tensor is a measure T (extended by zero in $\mathbb{R}^2\setminus\overline{\Omega}$) which satisfies the linear PDE $\operatorname{div}(T)\in\mathcal{M}(\mathbb{R}^2;\mathbb{R}^2)$. The wave cone of the divergence operator (acting on symmetric matrices) is defined by

$$\Lambda:=\bigcup_{\xi\in\mathbb{R}^2,\,|\xi|=1}\{A\in\mathbb{M}^{2\times 2}_{\mathrm{sym}}:\ A\xi=0\}$$

and it is immediate to check that it corresponds to the set of singular matrices. Hence according to [17, Corollary 1.13] and using that T^j and T^c are singular to each other and both singular with respect to $\mathcal{L}^2$, one finds that

$$\mathrm{rank}\,(\Theta) \le 1 \quad \mathcal{H}^1\text{-a.e. on } \widehat{J_u} \tag{5.13}$$

and

$$\mathrm{rank}\left(\frac{dT^c}{d|T^c|}\right) \le 1 \quad |T^c|\text{-a.e. in } \overline{\Omega}. \tag{5.14}$$

Since the matrix-valued measure $(\mu_{11}+\mu_{22})\mathrm{Id} - 2\mu$ has zero trace, it follows from (5.11) together with Proposition 3.4 that

$$\mathrm{Tr}\,(\Theta) = \mathrm{Tr}(\overline{A}) = 1 \quad \mathcal{H}^1\text{-a.e. on } \widehat{J_u}.$$

Hence, recalling (5.13), for $\mathcal{H}^1$-a.e. $x \in \widehat{J_u}$, the eigenvalues of $\Theta(x) \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ are exactly 0 and 1 so that $\Theta(x)$ is indeed an orthogonal projector.

Using (5.11) and (5.14), it follows that for $|T^c|$-a.e. $x \in \overline{\Omega}$, $\frac{dT^c}{d|T^c|}(x) \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ is a trace free matrix with zero determinant, which implies that $\frac{dT^c}{d|T^c|}(x) = 0$. It thus follows that $T^c = 0$ and using again (5.11),

$$\mu^c = \frac{\mu^c_{11} + \mu^c_{22}}{2}\mathrm{Id},$$

which completes the proof of the result by setting $\lambda^c := \frac{\mu^c_{11}+\mu^c_{22}}{2}$. □

Note that, in the previous argument, we crucially used the fact that the dimension is set to be equal to two.

Reporting the information obtained in (5.7), (5.11) and Lemma 5.2 inside (5.8) yields for all $X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$,

$$\int_\Omega (|\nabla u|^2 \mathrm{Id} - 2\nabla u \otimes \nabla u) : DX\,dx + \int_{\widehat{J_u}} \Theta : DX\,d\mathcal{H}^1$$
$$= -\int_{\partial\Omega} (X\cdot\nu)\,dm + \int_{\partial\Omega} |\partial_\tau g|^2 (X\cdot\nu)\,d\mathcal{H}^1 - 2\int_{\partial\Omega} (\nabla u\cdot\nu)(X\cdot\tau)\partial_\tau g\,d\mathcal{H}^1. \tag{5.15}$$

We already know that for $\mathcal{H}^1$-a.e. $x \in \widehat{J_u}$, the matrix $\Theta(x)$ is an orthogonal projector. We now make this information more precise by showing that it actually consists in the orthogonal projection onto the (approximate) tangent space to the rectifiable set $\widehat{J_u}$ at x. We proceed in two steps, by distinguishing interior points in Ω to boundary points $\partial\Omega$.

We first consider interior points. The proof of the following result is similar to that of [5, Lemma 5.3] with several simplications due to the fact that we are here working in dimension two.

Lemma 5.3. *For $\mathcal{H}^1$-a.e $x \in \widehat{J_u} \cap \Omega = J_u$, one has $\Theta(x) = \mathrm{Id} - \nu_u(x)\otimes\nu_u(x)$.*

Proof. We perform a blow-up argument on (5.15). Let $x_0 \in J_u = \widehat{J_u}\cap\Omega$ be such that

(i) J_u admits an approximate tangent space at x_0, which is given by $T_{x_0}J_u = \nu_u(x_0)^\perp$.

(ii) $\lim_{\rho\to 0} \frac{\mathcal{H}^1(J_u\cap B_\rho(x_0))}{2\pi\rho} = 1$;

(iii) $\lim_{\rho\to 0} \frac{1}{\rho}\int_{B_\rho(x_0)} |\nabla u|^2\,dx = 0$;

(iv) x_0 is a Lebesgue point of Θ with respect to $\mathcal{H}^1 \llcorner J_u$.

It turns out that $\mathcal{H}^1$-a.e points x_0 in $J_u\cap\Omega$ fulfill those four items as a consequence of the Besicovitch Differentiation Theorem (see [1, Theorem 2.22]), the $\mathcal{H}^1$-rectifiability of J_u (see [1, Theorem 2.63]), and the fact the measures $|\nabla u|^2\mathcal{L}^2 \llcorner \Omega$ and $\mathcal{H}^1 \llcorner J_u$ are singular to each other.

Let $\zeta \in \mathcal{C}_c(\mathbb{R}^2;\mathbb{R}^2)$ be a test vector-field such that $\mathrm{Supp}(\zeta) \subset B_1$ and define $\varphi_\delta(x) = \zeta\left(\frac{x-x_0}{\delta}\right)$ for every $\delta > 0$ small enough such that $B_\delta(x_0) \subset\subset \Omega$. Note that $\mathrm{Supp}(\varphi_\delta) \subset \Omega$ so that φ_δ can be taken as a test function in (5.15). It leads to

$$\int_{J_u \cap B_\delta(x_0)} \Theta : D\varphi_\delta \, d\mathcal{H}^1 = -\int_{B_\delta(x_0)} (|\nabla u|^2 \mathrm{Id} - 2\nabla u \otimes \nabla u) : D\varphi_\delta \, dx, \tag{5.16}$$

which rewrites as

$$\frac{1}{\delta}\int_{J_u \cap B_\delta(x_0)} \Theta(x) : D\zeta\left(\frac{x-x_0}{\delta}\right) d\mathcal{H}^1(x) = -\frac{1}{\delta}\int_{B_\delta(x_0)} (|\nabla u|^2 \mathrm{Id} - 2\nabla u \otimes \nabla u) : D\zeta\left(\frac{x-x_0}{\delta}\right) dx. \tag{5.17}$$

Since x_0 satisfies item (iii), one has

$$\left|\frac{1}{\delta}\int_{B_\delta(x_0)} (|\nabla u|^2 \mathrm{Id} - 2\nabla u \otimes \nabla u) : D\zeta\left(\frac{x-x_0}{\delta}\right) dx\right| \leqslant C\|D\zeta\|_{L^\infty(B_1)} \frac{1}{\delta}\int_{B_\delta(x_0)} |\nabla u|^2 \, dx \longrightarrow 0. \tag{5.18}$$

On the other hand, by item (iv) we have

$$\left|\frac{1}{\delta}\int_{J_u \cap B_\delta(x_0)} (\Theta(x) - \Theta(x_0)) : D\zeta\left(\frac{x-x_0}{\delta}\right) d\mathcal{H}^1(x)\right| \leqslant C\|D\zeta\|_{L^\infty(B_1)} \frac{1}{\delta}\int_{J_u \cap B_\delta(x_0)} |\Theta(x) - \Theta(x_0)| \, d\mathcal{H}^1(x) \longrightarrow 0. \tag{5.19}$$

Combining (5.17), (5.18), (5.19) and item (ii) yields

$$0 = \Theta(x_0) : \lim_{\delta \to 0} \frac{1}{\delta}\int_{J_u \cap B_\delta(x_0)} D\zeta\left(\frac{x-x_0}{\delta}\right) d\mathcal{H}^1(x) = \int_{T_{x_0} J_u \cap B_1} \Theta(x_0) : D\zeta(x) d\mathcal{H}^1(x).$$

Recalling that $\Theta(x_0)$ is an orthonormal projector, there exists a unit vector $e(x_0) \in \mathbb{R}^2$ such that $\Theta(x_0) = \mathrm{Id} - e(x_0) \otimes e(x_0)$. In particular, the measure $W = (\mathcal{H}^1 \llcorner T_{x_0} J_u) \otimes \delta_{\Theta(x_0)} \in \mathcal{M}(B_1 \times \mathbf{G}_1)$ is a 1-varifold and the previous computation shows that W is stationary in B_1, i.e., $\delta W = 0$ in B_1. Hence, according to the monotonicity formula (1.7), for all $0 < r < \rho < 1$,

$$\frac{\mathcal{H}^1(T_{x_0} J_u \cap B_\rho)}{\rho} - \frac{\mathcal{H}^1(T_{x_0} J_u \cap B_r)}{r} = \int_{T_{x_0} J_u \cap (B_\rho \setminus B_r)} \frac{|e(x_0) \cdot y|^2}{|y|^3} \, d\mathcal{H}^1(y).$$

Since the left-hand side of the previous equality vanishes, we obtain

$$\int_{T_{x_0} J_u \cap (B_\rho \setminus B_r)} \frac{|e(x_0) \cdot y|^2}{|y|^3} d\mathcal{H}^1(y) = 0.$$

We deduce that that $y \cdot e(x_0) = 0$ for $\mathcal{H}^1$-a.e. $y \in T_{x_0} J_u \cap B_1$, which implies that $T_{x_0} J_u = e(x_0)^\perp$ and $e(x_0) = \pm \nu_u(x_0)$. ☐

We next address the case of boundary points following the ideas of [5, Lemma 5.4].

Lemma 5.4. *For $\mathcal{H}^1$-a.e x in $\widehat{J_u} \cap \partial\Omega$, we have $\Theta(x) = \mathrm{Id} - \nu(x) \otimes \nu(x) = \mathrm{Id} - \nu_u(x) \otimes \nu_u(x)$, where ν is the outward unit normal to $\partial\Omega$.*

Proof. Let $x_0 \in \widehat{J_u} \cap \partial\Omega$ be such that :

(i) x_0 is a Lebesgue point of Θ with respect to $\mathcal{H}^1 \llcorner \widehat{J_u} \cap \partial\Omega$;

(ii) $\widehat{J_u}$ admits an approximate tangent space at x_0 which is given by $T_{x_0}\widehat{J_u} = \nu_u(x_0)^\perp$;

(iii) $\nu_u(x_0) = \nu(x_0)$;

(iv) $\lim\limits_{\rho\to 0} \frac{\mathcal{H}^1(\widehat{J_u}\cap B_\rho(x_0))}{2\pi\rho} = 1$;

(v) $\lim\limits_{\rho\to 0} \frac{1}{\rho}\int_{B_\rho(x_0)\cap\Omega} |\nabla u|^2\,dx = 0$;

(vi) $m(\{x_0\}) = 0$.

Let us first prove that $\mathcal{H}^1$-a.e points x in $\widehat{J_u}\cap\partial\Omega$ fullfil those conditions. The validity of item (i) follows from the Besicovitch Differentiation Theorem (see [1, Theorem 2.22]). Then, the rectifiability of $\widehat{J_u}$, the locality of the approximate tangent space (see [1, Theorem 2.85 and Remark 2.87]) and the Besicovitch-Mastrand-Mattila Theorem (see [1, Theorem 2.63]) ensure that item (ii), item (iii) and item (iv) are also satisfied. The validity of item (v) is a direct consequence of the fact that the measures $|\nabla u|^2\mathcal{L}^2 \llcorner \Omega$ and $\mathcal{H}^1 \llcorner \widehat{J_u}$ are singular to each other. Finally, since m is a finite measure, it follows that its set of atoms is at most countable, hence $\mathcal{H}^1$-negligible which ensure that item (vi) holds.

We now prove that $\nu(x_0)$ belongs to the kernel of $\Theta(x_0)$, which, combined with Lemma 5.2 together with item (iii), implies that $\Theta(x_0) = \mathrm{Id} - \nu_u(x_0)\otimes\nu_u(x_0) = \mathrm{Id} - \nu(x_0)\otimes\nu(x_0)$.

Let $\psi \in \mathcal{C}^1_c(B_1)$ be a scalar test function, and $\zeta \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$ a test vector field. For $\delta > 0$, set

$$\varphi_\delta = \psi\left(\frac{\cdot - x_0}{\delta}\right)\zeta \in \mathcal{C}^1_c(B_\delta(x_0);\mathbb{R}^2),$$

and notice that

$$D\varphi_\delta = \psi\left(\frac{\cdot - x_0}{\delta}\right)D\zeta \ + \zeta\otimes\frac{1}{\delta}\nabla\psi\left(\frac{\cdot - x_0}{\delta}\right).$$

Using φ_δ as a test function in (5.15) yields

$$\begin{aligned}\int_{B_\delta(x_0)\cap\Omega} (|\nabla u|^2 - 2\nabla u\otimes\nabla u) : D\varphi_\delta \ dx + \int_{\widehat{J_u}\cap B_\delta(x_0)} \Theta : D\varphi_\delta \ d\mathcal{H}^1 \\ = -\int_{\partial\Omega}(\varphi_\delta\cdot\nu)\,dm + \int_{\partial\Omega}|\partial_\tau g|^2(\varphi_\delta\cdot\nu)\,d\mathcal{H}^1 \\ - 2\int_{\partial\Omega\cap B_\delta(x_0)}(\nabla u\cdot\nu)\partial_\tau g(\tau\cdot\varphi_\delta)\,d\mathcal{H}^1.\end{aligned} \tag{5.20}$$

Observe that

$$\left|\int_{B_\delta(x_0)\cap\Omega}(|\nabla u|^2 - 2\nabla u\otimes\nabla u) : (D\zeta)\psi\left(\frac{\cdot - x_0}{\delta}\right)dx\right| \leqslant C\|\psi\|_{L^\infty(B_1)}\|D\zeta\|_{L^\infty(\mathbb{R}^2)}\int_{B_\delta(x_0)\cap\Omega}|\nabla u|^2\,dx \xrightarrow[\delta\to 0]{} 0$$

and, thanks to item (v)

$$\left|\int_{B_\delta(x_0)\cap\Omega}(|\nabla u|^2 - 2\nabla u\otimes\nabla u) : \zeta\otimes\frac{1}{\delta}\nabla\psi\left(\frac{\cdot - x_0}{\delta}\right)dx\right| \leqslant C\|\nabla\psi\|_{L^\infty(B_1)}\|D\zeta\|_{L^\infty(\mathbb{R}^2)}\frac{1}{\delta}\int_{B_\delta(x_0)\cap\Omega}|\nabla u|^2 dx \xrightarrow[\delta\to 0]{} 0.$$

We then deduce that

$$\int_{B_\delta(x_0)\cap\Omega}(|\nabla u|^2 - 2\nabla u\otimes\nabla u) : D\varphi_\delta \ dx \xrightarrow[\delta\to 0]{} 0. \tag{5.21}$$

Next, we have

$$\left|\int_{\widehat{J_u}\cap B_\delta(x_0)} \Theta : (D\zeta)\psi\left(\frac{\cdot - x_0}{\delta}\right) d\mathcal{H}^1\right| \leqslant C\|\psi\|_{L^\infty(B_1)}\|D\zeta\|_{L^\infty(\mathbb{R}^2)}\mathcal{H}^1(\widehat{J_u}\cap B_\delta(x_0)) \xrightarrow[\delta\to 0]{} 0. \quad (5.22)$$

and according to item (i)

$$\left|\int_{\widehat{J_u}\cap B_\delta(x_0)} (\Theta(x)-\Theta(x_0)) : \zeta(x)\otimes\frac{1}{\delta}\nabla\psi\left(\frac{x-x_0}{\delta}\right) d\mathcal{H}^1\right|$$
$$\leqslant C\|\nabla\psi\|_{L^\infty(B_1)}\|\zeta\|_{L^\infty(\mathbb{R}^2)}\frac{1}{\delta}\int_{\widehat{J_u}\cap B_\delta(x_0)} |\Theta(x)-\Theta(x_0)|\, d\mathcal{H}^1(x) \xrightarrow[\delta\to 0]{} 0. \quad (5.23)$$

Combining (5.22), (5.23), the continuity of ζ and item (ii) yield

$$\begin{aligned}\lim_{\delta\to 0}\int_{\widehat{J_u}\cap B_\delta(x_0)} \Theta : D\varphi_\delta\, d\mathcal{H}^1 &= \lim_{\delta\to 0}\frac{1}{\delta}\int_{\widehat{J_u}\cap B_\delta(x_0)} \Theta(x) : \zeta(x)\otimes\nabla\psi\left(\frac{x-x_0}{\delta}\right) d\mathcal{H}^1(x)\\
&= \lim_{\delta\to 0}\frac{1}{\delta}\int_{\widehat{J_u}\cap B_\delta(x_0)} \Theta(x_0) : \zeta(x)\otimes\nabla\psi\left(\frac{x-x_0}{\delta}\right) d\mathcal{H}^1(x)\\
&= \lim_{\delta\to 0}\frac{1}{\delta}\int_{\widehat{J_u}\cap B_\delta(x_0)} \Theta(x_0) : \zeta(x_0)\otimes\nabla\psi\left(\frac{x-x_0}{\delta}\right) d\mathcal{H}^1(x)\\
&= \int_{T_{x_0}\widehat{J_u}\cap B_1} \Theta(x_0) : \zeta(x_0)\otimes\nabla\psi(y)\, d\mathcal{H}^1(y). \quad (5.24)\end{aligned}$$

Finally item (vi) leads to

$$\left| -\int_{\partial\Omega}(\varphi_\delta\cdot\nu)\, dm + \int_{\partial\Omega}|\partial_\tau g|^2(\varphi_\delta\cdot\nu)\, d\mathcal{H}^1 - 2\int_{\partial\Omega\cap B_\delta(x_0)}(\nabla u\cdot\nu)\partial_\tau g(\tau\cdot\varphi_\delta)\, d\mathcal{H}^1\right|$$
$$\leqslant \|\zeta\|_{L^\infty(\mathbb{R}^2)}\|\psi\|_{L^\infty(B_1)}\left[m(B_\delta(x_0)) + \int_{B_\delta(x_0)\cap\partial\Omega}(|\nabla u\cdot\nu| + |\nabla g|^2)\, d\mathcal{H}^1\right] \xrightarrow[\delta\to 0]{} 0. \quad (5.25)$$

Gathering (5.20), (5.21), (5.24), (5.25) leads to

$$\int_{T_{x_0}\widehat{J_u}\cap B_1} \Theta(x_0) : \zeta(x_0)\otimes\nabla\psi\, d\mathcal{H}^1 = 0. \quad (5.26)$$

We now specify (5.26) for ζ such that $\zeta(x_0) = \nu(x_0)$. Denoting by $\tau(x_0)$ an orthonormal vector to $\nu(x_0)$ we decompose $\nabla\psi$ as

$$\nabla\psi = (\partial_{\tau(x_0)}\psi)\tau(x_0) + (\partial_{\nu(x_0)}\psi)\nu(x_0).$$

Since $\mathrm{Supp}(\psi)\subset B_1$, then $\int_{T_{x_0}\widehat{J_u}\cap B_1}\partial_{\tau(x_0)}\psi\, d\mathcal{H}^1 = 0$ and (5.26) becomes

$$\int_{T_{x_0}\widehat{J_u}\cap B_1} \Theta(x_0) : (\nu(x_0)\otimes\nu(x_0))\partial_{\nu(x_0)}\psi\, d\mathcal{H}^1 = 0 \quad \text{for all } \psi\in\mathcal{C}^1_c(B_1).$$

We conclude that

$$\nu(x_0)\cdot(\Theta(x_0)\nu(x_0)) = \Theta(x_0) : (\nu(x_0)\otimes\nu(x_0)) = 0$$

and thus, since $\Theta(x_0)$ is a projector, we obtain $\Theta(x_0)\nu(x_0) = 0$, as claimed. □

We are now in position to complete the proof of our main result.

Proof of Theorem 2.1. According to (5.15), Lemma 5.3 and Lemma 5.4, we have that for all $X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$,

$$\int_\Omega (|\nabla u|^2 - 2\nabla u \otimes \nabla u) : DX\,dx + \int_{\widehat{J_u}} (\mathrm{Id} - \nu_u \otimes \nu_u) : DX\,d\mathcal{H}^1$$
$$= -\int_{\partial\Omega} (X\cdot\nu)\,dm + \int_{\partial\Omega} |\partial_\tau g|^2 (X\cdot\nu)\,d\mathcal{H}^1 - 2\int_{\partial\Omega} (\nabla u\cdot\nu)(X\cdot\tau)\partial_\tau g\,d\mathcal{H}^1.$$

Specifying to vector fields $X \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^2)$ such that $X\cdot\nu = 0$ on $\partial\Omega$ leads to

$$\int_\Omega (|\nabla u|^2 - 2\nabla u \otimes \nabla u) : DX\,dx + \int_{\widehat{J_u}} (\mathrm{Id} - \nu_u \otimes \nu_u) : DX\,d\mathcal{H}^1 = -2\int_{\partial\Omega} (\nabla u\cdot\nu) X\cdot\nabla g\,d\mathcal{H}^1,$$

which completes the proof of Theorem 2.1. □

Remark 5.1. The assumption of convergence of the energy can also be used to pass to the limit in the second inner variation in general, cf. [23, 24, 25, 5]. Here the sole convergence of the phase field energy is not sufficient a priori to pass to the limit in the second inner variation of the AT energy.

Thanks to the varifold convergence and the equi-partition of energy, we first observe as in [5, Corollary 5.1] that if $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ is a family of critical points of AT_ε satisfying the assumptions of Theorem 2.1 then, up to a subsequence

$$\nabla w_\varepsilon \otimes \nabla w_\varepsilon \mathcal{L}^2 \llcorner \Omega \rightharpoonup (\mathrm{Id} - \overline{A})\mathcal{H}^1 \llcorner J_u \quad \text{weakly* in } \mathcal{M}(\Omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$$

and

$$\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon \mathcal{L}^2 \llcorner \Omega \rightharpoonup \frac{1}{2}(\mathrm{Id} - \overline{A})\mathcal{H}^1 \llcorner J_u \quad \text{weakly* in } \mathcal{M}(\Omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}}).$$

Recalling that $(\mathrm{Id} - \overline{A})\mathcal{H}^1 \llcorner J_u = \nu_u \otimes \nu_u \mathcal{H}^1 \llcorner J_u - ((\mathrm{Tr}(\mu^j))\mathrm{Id} - 2\mu^j)$

Using Propositions 4.1 and 4.2, the expression of the second inner variation of the AT energy computed in [5, Lemma A.3], we find that if $X \in \mathcal{C}^\infty_c(\Omega;\mathbb{R}^2)$,

$$\begin{aligned}
\lim_{\varepsilon\to 0} \delta^2 AT_\varepsilon(u_\varepsilon, v_\varepsilon)[X] \quad = \quad & \int_\Omega |\nabla u|^2 ((\mathrm{div} X)^2 - \mathrm{Tr}((DX)^2))\,dx + \langle \mathrm{Tr}(\mu^s), (\mathrm{div} X)^2 - \mathrm{Tr}((DX)^2)\rangle \\
& -4\int_\Omega ((\nabla u \otimes \nabla u) : DX)\mathrm{div} X\,dx - 4\langle \mu^s, (\mathrm{div} X) DX\rangle \\
& +4\int_\Omega (\nabla u \otimes \nabla u) : (DX)^2\,dx + 4\langle \mu^s, (DX)^2\rangle \\
& +2\int_\Omega |DX^T \nabla u|^2\,dx + 2\langle \mu^s, (DX)(DX)^T\rangle \\
& +\int_{J_u} ((\mathrm{div} X)^2 - \mathrm{Tr}((DX)^2))\,d\mathcal{H}^1 \\
& -2\int_{J_u} (\nu_u \otimes \nu_u) : (DX)\mathrm{div} X\,d\mathcal{H}^1 + 2\langle (\mathrm{Tr}(\mu^j))\mathrm{Id} - 2\mu^j, (DX)\mathrm{div} X\rangle \\
& +2\int_{J_u} (\nu_u \otimes \nu_u) : (DX)^2\,d\mathcal{H}^1 - 2\langle (\mathrm{Tr}(\mu^j))\mathrm{Id} - 2\mu^j, (DX)^2\rangle \\
& +\int_{J_u} |DX^T \nu_u|^2\,d\mathcal{H}^1 - \langle (\mathrm{Tr}(\mu^j))\mathrm{Id} - 2\mu^j, (DX)(DX)^T\rangle.
\end{aligned}$$

Thus it does not seem to have any compensation phenomenon to get rid-off the terms involving the singular measure μ, and stability of the the second inner variation might not be satisfied.

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