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MINIMALITY OF FREE-BOUNDARY AXIAL HYPERPLANES IN HIGH DIMENSIONAL CIRCULAR CONES VIA CALIBRATION

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Abstract. Consider an $(n+1)$-dimensional circular cone. Using a calibration argument, we prove that if $n \geq 4$ and the aperture of the cone is sufficiently large, the intersection of the cone with an axial hyperplane is area-minimizing with respect to free-boundary variations inside the cone.

1. Introduction

Consider a volume $v > 0$ of liquid inside a container, here identified with a Lipschitz open set $\Omega \subset \mathbb{R}^3$. According to Gauss' formulation, in the absence of gravity, the configuration E assumed by the liquid minimizes the free-energy functional

$$P(F;\Omega) + \gamma\, P(F;\partial\Omega) \tag{1.1}$$

among all sets $F \subset \Omega$ such that $P(F;\Omega) < +\infty$ and satisfying the volume constraint $|F| = v$. Here, $\gamma \in [-1,1]$ is the so-called relative adhesion coefficient between the liquid and $\partial\Omega$, and $P(F;U)$ stands for the relative perimeter of F in a Borel set U (see [11,26] for more details).

From a mathematical point of view, given an open set $\Omega \subset \mathbb{R}^{n+1}$, a well-established internal regularity theory for minimizers of (1.1) is available in the literature. In fact, let E be a minimizer of (1.1) and indicate by $\partial^* E$ its reduced boundary. Letting $M := \overline{\partial^* E \cap \Omega}$, one can show that $M \cap \Omega$, that is, the internal portion of M, is formed by the union of a smooth hypersurface and a singular set Σ whose Hausdorff dimension $\dim_{\mathcal{H}} \Sigma$ does not exceed the critical value $n-7$, being in particular nonempty only when $n \geq 7$; see [6] and the references therein.

Much less is known regarding the boundary regularity for minimizers of (1.1), that is, the regularity of the free-boundary interface $M \cap \partial\Omega$. One of the first general regularity results, which holds for smooth containers $\Omega \subset \mathbb{R}^3$, is due to Taylor [35], who proved that $M \cap \partial\Omega$ does not exhibit singularities. This result has been generalized much later by De Philippis and Maggi in [6] for minimizers of anisotropic capillarity functionals. In the isotropic case, which includes the energy functional (1.1), their result ensures that, if $\partial\Omega$ is $C^{1,1}$, then M is the union of a $C^{1,1/2}$ hypersurface with boundary and a singular set Σ with the property that $\dim_{\mathcal{H}}(\Sigma \cap \partial\Omega) \leq n-3$. Further improvements on the Hausdorff dimension of $\Sigma \cap \partial\Omega$ have been provided in [4] when γ is close to either 0, 1 or -1. See also [32] for stability and rigidity properties of singular capillary cones arising in the capillarity setting.

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Focusing now on the special case $\gamma = 0$, corresponding to the relative perimeter in Ω, something stronger regarding the regularity of $M \cap \partial\Omega$ was proved by Grüter-Jost [15] and Grüter [13,14] in the setting of varifolds, under the assumption that $\partial\Omega$ is C^2. In particular, in [15], the authors demonstrate that if a varifold M is close to a free-boundary hyperplane in Ω, then M is $C^{1,\beta}$-graphical with respect to this hyperplane. Subsequently, in [13], the author exploits the latter result to show that, as in the interior case, the Hausdorff dimension of the singular set of the free-boundary satisfies $\dim_{\mathcal{H}}(\Sigma \cap \partial\Omega) \leq n - 7$.

The common denominator of the boundary regularity results mentioned above, both for $\gamma \neq 0$ and $\gamma = 0$, is that the container Ω must be smooth. The literature concerning boundary regularity becomes sparser when one looks for results involving nonsmooth containers Ω. In this situation, the main available studies typically consider specific nonsmooth domains or classes of domains. This is the case, for example, in the series of works [16–19] by Hildebrandt and Sauvigny, where the authors perform a thorough study of minimal surfaces in wedges (i.e., three-dimensional domains obtained as the intersection of two half-spaces); in particular, in [19, p. 73], they prove a regularity result for free-boundary minimal graphs on a plane orthogonal to the wedge.

The latter result has been recently generalized by Edelen and Li in [8]. In that work, they prove a boundary regularity theorem in the same spirit as Grüter-Jost [15], holding for a varifold M under the assumption that Ω is locally polyhedral and that M is close to a horizontal plane (for instance, if $\Omega = W^2 \times \mathbb{R}^{n-1}$ is a wedge, horizontal planes are those containing $W^2 \times \{0\}$). In the same work, Edelen and Li also prove that $\dim_{\mathcal{H}}(\Sigma \cap \partial\Omega) \leq n - 2$, with a sharper estimate holding when the dihedral angles of their cone models are nonobtuse; see [8, Theorem 1.2]. In the setting of their polyhedral domains, the singular set Σ is defined as the complement of the points around which the interface M is the graph of a $C^{1,\beta}$ function over a horizontal plane.

In this paper, we prove that in dimensions $n \geq 4$ and for circular cones with sufficiently large aperture, axial hyperplanes are area-minimizing with respect to free-boundary variations. This result fits naturally into some very recent contributions by the author in collaboration with Leonardi [23, 25]. In the latter, we establish a boundary monotonicity formula that holds for almost minimizers of the relative perimeter under minimal regularity assumptions on $\partial\Omega$, and we then employ this formula to prove a minimizing-cone property for a blow-up sequence at a boundary point of Ω. In [25], the authors prove a Vertex-skipping theorem, which states that the boundary of any almost minimizer of the relative perimeter E in a convex domain $\Omega \subset \mathbb{R}^3$ cannot contain isolated singularities (say, vertices) of $\partial\Omega$. We highlight that, after a dimension-reduction argument, this result combined with [8, Theorem 1.1] implies the bound $\dim_{\mathcal{H}}(\Sigma \cap \partial\Omega) \leq n - 3$, improving the estimate given in [8, Theorem 1.2 (1)]. The study of the behavior of E near a singularity of the container is also relevant from the perspective of free-boundary minimal surfaces, since the interface $M = \overline{\partial^* E \cap \Omega}$ is a free-boundary minimal hypersurface in Ω. We refer to [3] and the references therein for background on free-boundary minimal surfaces.

In this framework, the main result proved here provides a counterexample to the validity of Vertex-skipping results in higher dimensions. The question of the validity of the Vertex-skipping for $n \geq 3$ was already tackled in [24]. In that work, using a suitable Lipschitz flow of deformations for a free-boundary surface in a Lipschitz container Ω, which in particular allows $\partial\Omega$ (including its singular set) to move, we establish a stability result for axial hyperplanes in circular cones, stated as follows. Given an integer $n \geq 2$, and for $\lambda > 0$, consider the circular cone

$$\Omega_\lambda := \left\{ (x,t) \in \mathbb{R}^{n+1} \,:\, t > \lambda\sqrt{x_1^2 + \cdots + x_n^2} \right\}.$$

Note that the opening angle $\alpha_\lambda \in (0, \pi)$ of Ω_λ is related with λ by $\lambda = \cot(\alpha_\lambda/2)$. Then, if $n = 2$, a hyperplane of $\mathbb{R}^3$ containing the axis of the cone (up to rotations, $x_1 = 0$) is unstable in Ω_λ for every $\lambda > 0$. This is in accordance with the aforementioned Vertex-skipping theorem. In partial contrast, when $n \geq 3$, we show that there exists a threshold $\lambda^*(n) > 0$ such that if $0 < \lambda \leq \lambda^*(n)$, i.e., the circular cone has a sufficiently large opening angle $\alpha \geq \alpha_{\lambda^*(n)}$, the axial hyperplane is stable in Ω_λ, whereas for $\lambda > \lambda^*(n)$ it is unstable. In this paper, we intend to analyze whether, for $n \geq 3$ and $\lambda \leq \lambda^*(n)$, an axial hyperplane is merely stable or even area-minimizing in Ω_λ. We will provide a basically positive answer for $n \geq 4$.

We say that a measurable set $E \subset \mathbb{R}^{n+1}$ has locally finite perimeter provided $P(E; U) < +\infty$ for every $U \subset\subset \mathbb{R}^{n+1}$, see [12,26]. For $(x,t) \in \mathbb{R}^{n+1}$ and $r > 0$, we denote by $B_r(x,t)$ the ball of radius r centered at (x,t) (with $B_r := B_r(0,0)$). Our main result is the following:

Theorem 1.1. *For $n \geq 4$, consider the threshold parameter*

$$\bar{\lambda}(n) := \frac{1}{2}\frac{n-3}{\sqrt{n-2}}, \tag{1.2}$$

and assume that $0 < \lambda \leq \bar{\lambda}(n)$. Let also

$$E := \{(x,t) \in \mathbb{R}^{n+1} \,:\, x_1 > 0\}.$$

Then E is a minimizer of the relative perimeter in Ω_λ, that is, given a locally-finite perimeter set $F \subset \mathbb{R}^{n+1}$ satisfying $E \Delta F \subset\subset B_R$, for some $R > 0$, one has

$$P(E; \Omega_\lambda \cap B_R) \leq P(F; \Omega_\lambda \cap B_R).$$

We emphasize that the above theorem provides a definitive counterexample to the validity of a Vertex-skipping theorem such as [25, Theorem 1.1] in higher dimensions, but only when $n \geq 4$. The case $n = 3$ is a limiting situation where, in view of [24], the portion of hyperplane $H_\lambda := \partial E \cap \Omega_\lambda$ is stable with respect to free-boundary variations in Ω_λ, $\lambda \geq \lambda^*$, but our minimality result, Theorem 1.1, does not apply. We also remark that, despite Ω_λ not being a polyhedral cone, Theorem 1.1 appears to suggest that the dimensional bound for the singular set of the free-boundary proved in [8, Theorem 1.2 (1)] cannot be improved beyond the threshold $n - 4$. In fact, when $n = 4$ and $\lambda \leq 1/2\sqrt{2}$, the result above implies that $\partial E \cap \partial\Omega_\lambda = \{0\}$, and so $0 \in \Sigma \cap \partial\Omega_\lambda$.

The proof of Theorem 1.1 is based on a calibration argument. The calibration method was originally introduced by Bombieri, De Giorgi, and Giusti in their celebrated work [2] to prove the minimality of Simon's cone in $\mathbb{R}^8$. Since then, several authors have contributed by proposing new versions of the original method [5,7,10,21,29] or by developing various generalizations to different contexts, such as the Steiner tree problem [27,28,33] and the Mumford-Shah functional [1,30].

Here, the calibration method is adapted to the free-boundary setting in the sense that we require the calibrating vector field Z to be tangent to $\partial\Omega_\lambda$. This turns out to be crucial for carrying out the calibration argument, as it allows us to exclude the contribution coming from the area of $(E\Delta F) \cap \partial\Omega_\lambda$. We point out that the vector field Z will not be defined in the entire closed cone $\overline{\Omega}_\lambda$, but only outside a 2-dimensional subset of $\overline{\Omega}_\lambda$. This will force us to perform an approximation argument to apply the Divergence Theorem on $(E\Delta F)\cap\Omega_\lambda$, see Section 5.

The existence of the vector field Z will be shown in Section 4, and its construction will be carried out in two steps. Let $S_\lambda := \partial\Omega_\lambda \setminus \{0\}$, and denote by $e_1,\dots,e_n,e_t$ the canonical basis of $\mathbb{R}^{n+1}$. We begin by calibrating the $(n-1)$-dimensional surface $S^0_\lambda := H_\lambda \cap S_\lambda$ in S_λ; that is, we build a divergence-free vector field Y, defined on a suitable subset S'_λ of S_λ, such that $Y|_{S^0_\lambda} = e_1$ and $|Y| \le 1$. In a slightly different formulation via differential forms, this was essentially done by Morgan in [31], using ideas from Lawlor's PhD thesis [20]. Second, we extend the calibration obtained in the previous step to (a suitable portion of) Ω_λ by vertical projection onto S_λ: we will show that, due to the structure of the metric on S_λ, the vector field Z obtained in this way is, in particular, divergence-free in its domain.

1.1. Organization of the paper.

In Section 2 we provide some preliminary notation and facts concerning differentiable manifolds (especially Riemannian ones) and sets of finite perimeter. In Section 3 we introduce further notation and perform some basic computations needed in the subsequent parts of the paper. Then, Section 4 is devoted to the construction of the calibration Z, whose main properties are collected in Theorem 4.1. Finally, in Section 5 we present the calibration argument; namely, we show how the existence of the vector field Z from Theorem 4.1 can be used to demonstrate Theorem 1.1.

2. Preliminary notions and basic notation

2.1. Manifolds and k-forms.

Given a n-dimensional manifold M, we denote by TM, T^*M the tangent and cotangent bundles of M respectively. For any $k \ge 1$, we define the bundle $\bigwedge^k T^*M$ as the disjoint union of the vector spaces $\bigwedge^k (T_pM)^*$ (and we observe that $\bigwedge^1 T^*M = T^*M$). When $k=0$, we identify $\bigwedge^0 T^*M$ with $C^\infty(M)$. A function $\omega : M \to \bigwedge^k T^*M$ with the property that $\omega(p) \in \bigwedge^k (T_pM)^*$, for all $p \in M$, is called k-form. Given a local chart $\varphi = (x_1,\dots,x_n)$ for M, we denote by ∂_i and dx_i, for $1 \le i \le n$, the local frames induced by φ on TM and T^*M respectively. We introduce the following multilinear algebra notation: given a basis $u_1,\dots,u_n$ of a n-dimensional vector space V and a multi-index $\alpha = (\alpha_1,\dots,\alpha_k) \in \mathbb{N}^k$ with $1 \le \alpha_1 < \cdots < \alpha_k \le n$, we set

$$u_\alpha = u_{\alpha_1} \wedge \cdots \wedge u_{\alpha_k}.$$

Moreover, fixing $1 \le i < j \le k$, we define

$$\begin{aligned} u^i_\alpha &= u_{\alpha_1} \wedge \cdots \wedge u_{\alpha_{i-1}} \wedge u_{\alpha_{i+1}} \wedge \cdots \wedge u_{\alpha_k} \\ u^{i,j}_\alpha &= u_{\alpha_1} \wedge \cdots \wedge u_{\alpha_{i-1}} \wedge u_{\alpha_{i+1}} \wedge \cdots \wedge u_{\alpha_{j-1}} \wedge u_{\alpha_{j+1}} \wedge \cdots \wedge u_{\alpha_k}. \end{aligned} \tag{2.1}$$

Any k-form can be then represented by

$$\omega = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_k)\in\mathbb{N}^k \\ 1\le\alpha_1<\cdots<\alpha_k\le n}} \omega_\alpha \, dx_\alpha,$$

for suitable scalar functions $\omega_\alpha : M \to \mathbb{R}$. We indicate by $\mathfrak{X}(M)$, $\mathcal{A}^k(M)$ the space of the vector fields on M and the space of the k-forms on M, respectively.

2.2. **Riemannian manifolds.** Let (M, g) be a Riemannian manifold. Given two vectors $v, w \in T_pM$, we set $\langle v, w\rangle_g := g_p(v, w)$, $|v|_g := \sqrt{g_p(v, v)}$, with the convention that, if the subscript g is omitted, we are referring to the Euclidean scalar product. The metric g can be expressed in terms of the local frame for TM, by introducing the $\mathbb{R}$-valued functions

$$g_{i,j}(p) := \langle \partial_i(p), \partial_j(p)\rangle_g\,, \qquad \text{for } 1 \le i, j \le n.$$

With a little abuse, we denote by g the symmetric, positive-definite matrix $g = (g_{i,j})_{1\le i,j\le n}$, while we indicate by $\tilde g = (\tilde g_{i,j})_{1\le i,j\le n}$ the inverse of g. The metric g on M can be extended to $\bigwedge^k T^*M$, for all $k \ge 1$. When $k = 1$, we set

$$\langle dx_i, dx_j\rangle_g := \tilde g_{i,j}\,, \qquad \text{for } 1 \le i, j \le n,$$

and, when $k \ge 2$, given two multi-indices $\alpha = (\alpha_1, \dots, \alpha_k), \beta = (\beta_1, \dots, \beta_k) \in \mathbb{N}^k$, we define

$$\begin{aligned}\langle dx_\alpha, dx_\beta\rangle_g &:= \det(\langle dx_{\alpha_i}, dx_{\beta_j}\rangle_g)_{1\le i,j\le k}\\ &= \det(\tilde g_{\alpha_i,\beta_j})_{1\le i,j\le k}.\end{aligned} \tag{2.2}$$

These definitions can be then extended to $\bigwedge^k T^*M$ by multi-linearity. By employing the metric structure of a Riemannian manifold, one can establish an identification between the tangent and the cotangent bundles. In local coordinates, this identification acts as follows: given a vector field $X = \sum_{i=1}^n X_i\partial_i \in TM$ one can consider the 1-form $X^\flat$ defined by

$$X^\flat = \sum_{i,j=1}^n g_{i,j} X_i\, dx_j\,;$$

conversely, given a 1-form $\omega = \sum_{i=1}^n \omega_i\, dx_i \in T^*M$, one defines the vector field

$$\omega^\# = \sum_{i,j=1}^n \tilde g_{i,j}\, \omega_i\, \partial_j\,.$$

Note that $|X^\flat|_g = |X|_g$, $|\omega^\#|_g = |\omega|_g$, and also $X^{\flat\#} = X$, $\omega^{\#\flat} = \omega$.

Let now M and N be Riemannian manifolds and consider a diffeomorphism $F : M \to N$. Given a vector field $X \in \mathfrak{X}(M)$, the push-forward of X with respect to F is a vector field on N defined by

$$F_*X(q) = dF_{F^{-1}(q)}(X(F^{-1}(q)))\,, \qquad \text{for all } q \in N.$$

We denote by $\operatorname{div}^M X$ the divergence of a vector field X on a Riemannian manifold M, and we omit the superscript M when referring to the Euclidean divergence. One can easily verify that, if F is an isometry,

$$\operatorname{div}^N F_*X\,(q) = \operatorname{div}^M X\,(F^{-1}(q)), \qquad \text{for every } q \in N. \tag{2.3}$$

We conclude by stating the following formula for the divergence on M: if X is expressed in local coordinates by $X = \sum_{i=1}^n X_i\partial_i$, we have

$$\operatorname{div}^M X = \frac{1}{\sqrt{\det g}} \sum_{i=1}^n \partial_i \left(\sqrt{\det g}\, X_i\right), \tag{2.4}$$

where $\det g$ is the determinant of the matrix $g = (g_{i,j})_{1\le i,j\le n}$ introduced above.

2.3. **The Hodge-$\star$ operator.** Consider an oriented Riemannian manifold (M, g) and denote by ν its volume form (in local coordinates, $\nu = \sqrt{\det g}\, dx_1 \wedge \cdots \wedge dx_n$). For any $k \in \mathbb{N}$, the Hodge-$\star$ operator is a linear isometry $\star : \bigwedge^k T^*M \to \bigwedge^{n-k} T^*M$ which is uniquely defined by the condition

$$\omega \wedge (\star\psi) = \langle \omega, \psi \rangle_g \, \nu, \qquad \text{for all } k\text{-forms } \omega \text{ and } \psi. \tag{2.5}$$

It can be shown that $\star \circ \star = \star^2$ coincides with the multiplication by $(-1)^{k(n-k)}$, and that $\star\nu = 1$. In particular, when $k = n-1$, the operator $\star$ allows to associate to every $(n-1)$-form $\omega \in \mathcal{A}^{n-1}(M)$ a 1-form $\star\omega \in \mathcal{A}^1(M)$. Given a vector field $X \in \mathfrak{X}(M)$, denote by $i_X\nu \in \mathcal{A}^{n-1}(M)$ the contraction between X and the volume form ν. We have the following technical Lemma (see [22, p. 52]).

Lemma 2.1. *Let $X \in \mathfrak{X}(M)$. The following identities hold:*

(i) $\star X^\flat = i_X\nu$;

(ii) $\star d(i_X\nu) = \operatorname{div}^M X$.

2.4. **Locally finite perimeter sets under set operations.** Given a measurable set $E \subset \mathbb{R}^{n+1}$ and $s \in [0,1]$, we define

$$E^{(s)} := \left\{ (x,t) \in \mathbb{R}^{n+1} \, : \, \lim_{\rho \to 0^+} \frac{|E \cap B_\rho(x,t)|}{|B_\rho(x,t)|} = s \right\}.$$

If E is a locally finite perimeter set, we indicate by $\partial^* E$ its reduced boundary and by ν_E the inner unit normal to its reduced boundary [26]. Assume now that E and F have locally finite perimeter. We denote by $Q(E,F)^\pm$ the set

$$Q(E,F)^\pm := \{(x,t) \in \partial^* E \cap \partial^* F \, : \, \nu_E(x,t) = \pm \nu_F(x,t)\}.$$

It is well known that also $E \cap F$ and $E \setminus F$ have locally finite perimeter and, up to $\mathcal{H}^n$-negligible sets, one has (see [26, Theorem 16.3])

$$\partial^*(E \cap F) = \left(F^{(1)} \cap \partial^* E\right) \cup \left(E^{(1)} \cap \partial^* F\right) \cup Q(E,F)^+ \tag{2.6}$$

$$\partial^*(E \setminus F) = \left(F^{(0)} \cap \partial^* E\right) \cup \left(E^{(1)} \cap \partial^* F\right) \cup Q(E,F)^-, \tag{2.7}$$

and

$$\nu_{E \cap F} = \nu_E \llcorner F^{(1)} + \nu_F \llcorner E^{(1)} + \nu_E \llcorner Q(E,F)^+ \tag{2.8}$$

$$\nu_{E \setminus F} = \nu_E \llcorner F^{(0)} - \nu_F \llcorner E^{(1)} + \nu_E \llcorner Q(E,F)^-. \tag{2.9}$$

3. Further notation and preliminary computations

Let $n \geq 2$ be an integer number. For $x' \in \mathbb{R}^{n-1}$, we set $x = (x_1, x')$, and

$$r := |x'| = \sqrt{x_2^2 + \cdots + x_n^2} \qquad \text{and} \qquad \rho := |x| = \sqrt{x_1^2 + r^2}\,.$$

With this notation, the definition of Ω_λ given in the Introduction becomes

$$\Omega_\lambda = \{(x,t) \in \mathbb{R}^{n+1} \, : \, t > \lambda\, \rho\}.$$

We recall that $S_\lambda = \partial\Omega_\lambda \setminus \{0\}$ and we consider the following subsets of Ω_λ and S_λ, respectively:

$$\Omega'_\lambda := \{(x,t) \in \Omega_\lambda \, : \, x' \neq 0\} \qquad \text{and} \qquad S'_\lambda := \{(x,t) \in S_\lambda \, : \, x' \neq 0\}.$$

In other words, Ω'_λ (resp. S'_λ) coincides with Ω_λ (resp. S_λ) minus the 2-plane $x' = 0$. We observe that the hypersurface S_λ endowed with the Euclidean metric is a smooth Riemannian manifold. We consider the Riemannian manifold $M := \mathbb{R}^n \setminus \{0\}$ equipped with the metric

$$g_{i,j}(x) := \delta_{i,j} + \lambda^2 \frac{x_i\, x_j}{\rho^2}, \qquad \text{for } x \in M \text{ and } 1 \le i, j \le n. \tag{3.1}$$

A direct computation yields the expression of the metric on the cotangent bundle

$$\tilde{g}_{i,j}(x) = \delta_{i,j} - \frac{\lambda^2}{1+\lambda^2} \frac{x_i\, x_j}{\rho^2}, \qquad \text{for } x \in M \text{ and } 1 \le i, j \le n.$$

One can immediately check that $\mathcal{I} : M \to S_\lambda$, $\mathcal{I}(x) := (x, \lambda|x|)$, establishes an isometry between S_λ and M. Moreover, $\mathcal{I}$ restricts to an isometry between $M^0 := \{x \in M \,:\, x_1 = 0\}$ and $S^0_\lambda := S_\lambda \cap H_\lambda$. Consider the multi-index $\alpha = (2, \dots, n)$. Recalling notation (2.1), we introduce the $(n-2)$-form $\psi_0 \in \mathcal{A}^{n-2}(M)$ defined by

$$\psi_0(x) := \frac{\sqrt{1+\lambda^2}}{n-1} \sum_{i=2}^{n} (-1)^i\, x_i\, dx^i_\alpha, \qquad \text{for } x \in M. \tag{3.2}$$

Additionally, denoting by $M' := \{x \in M \,:\, x' \neq 0\}$, we let

$$\theta := \arctan(x_1/r) \in (-\pi/2, \pi/2), \qquad \text{for } x \in M'. \tag{3.3}$$

In the following technical result we gather some useful identities.

Lemma 3.1. *The following identities hold:*

$$\det g = 1 + \lambda^2 \tag{3.4}$$

$$|d\rho|_g = \frac{1}{\sqrt{1+\lambda^2}} \qquad |d\theta|_g = \frac{1}{\rho} \qquad \langle d\rho, d\theta\rangle_g = 0. \tag{3.5}$$

$$|\psi_0|_g = \frac{r}{n-1}\sqrt{1+\lambda^2}. \tag{3.6}$$

Proof. Fix $x \in M$, and, for $1 \le i \le n$, let

$$\partial_i = e_i + A_i\, e_t \qquad \text{where} \qquad A_i := \lambda\, \frac{x_i}{|x|}\,.$$

Standard properties of the determinant guarantee that $\det g = |\partial_1 \wedge \cdots \wedge \partial_n|^2$. Define the multi-index $\beta = (1, \dots, n-1)$. We have

$$\partial_1 \wedge \cdots \wedge \partial_n = e_\beta \wedge e_n + A_n\, e_\beta \wedge e_t + \sum_{i=1}^{n-1} (-1)^{n-1-i} A_i\, e^i_\beta \wedge e_t \wedge e_n. \tag{3.7}$$

Now the wedge products appearing in (3.7) are orthonormal each other, hence

$$|\partial_1 \wedge \cdots \wedge \partial_n|^2 = 1 + \sum_{i=1}^{n} A_i^2 = 1 + \lambda^2,$$

and this proves (3.4).

Let us now show the validity of (3.5). We start observing that, by the definitions of ρ and θ,

$$d\rho = \sum_{i=1}^{n} \frac{x_i}{\rho}\, dx_i \qquad \text{and} \qquad d\theta = \frac{1}{\rho^2}\left(r\, dx_1 - \frac{x_1}{r}\sum_{i=2}^{n} x_i\, dx_i \right). \tag{3.8}$$

Hence we have

$$\begin{aligned}
|d\rho|_g^2 &= \sum_{i,j=1}^{n} \frac{x_i x_j}{\rho^2} \langle dx_i\,,\, dx_j\rangle_g \\
&= \sum_{i,j=1}^{n} \frac{x_i x_j}{\rho^2}\left(\delta_{i,j} - \frac{\lambda^2}{1+\lambda^2}\frac{x_i x_j}{\rho^2}\right) \\
&= \frac{1}{\rho^2}\sum_{i=1}^{n} x_i^2 - \frac{\lambda^2}{\rho^4(1+\lambda^2)}\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{j=1}^{n} x_j^2\right) \\
&= \frac{1}{1+\lambda^2};
\end{aligned}$$

$$\begin{aligned}
|d\theta|_g^2 &= \frac{1}{\rho^4}\left(r^2 |dx_1|_g^2 - 2x_1 \sum_{i=2}^{n} x_i \langle dx_i\,,\, dx_1\rangle_g + \frac{x_1^2}{r^2}\sum_{i,j=2}^{n} x_i x_j \langle dx_i\,,\, dx_j\rangle_g \right) \\
&= \frac{1}{\rho^4}\left(r^2\left(1 - \frac{\lambda^2}{1+\lambda^2}\frac{x_1^2}{\rho^2}\right) + 2x_1 \sum_{i=2}^{n} \frac{\lambda^2}{1+\lambda^2}\frac{x_1 x_i^2}{\rho^2} + x_1^2\left(1 - \frac{\lambda^2}{1+\lambda^2}\frac{r^2}{\rho^2}\right)\right) \\
&= \frac{1}{\rho^4}\left(r^2 + x_1^2\right) \\
&= \frac{1}{\rho^2};
\end{aligned}$$

$$\begin{aligned}
\langle d\rho\,,\, d\theta\rangle_g &= \frac{1}{\rho^2}\left\langle \sum_{i=1}^{n}\frac{x_i}{\rho}\, dx_i\,,\, r\, dx_1 - \frac{x_1}{r}\sum_{i=2}^{n} x_i\, dx_i \right\rangle_g \\
&= \frac{1}{\rho^2}\left(\frac{r}{\rho}\, x_1\, |dx_1|_g^2 + \left(\frac{r}{\rho} - \frac{x_1^2}{r\rho}\right)\sum_{i=2}^{n} x_i \langle dx_1\,,\, dx_i\rangle_g - \frac{x_1}{r\rho}\sum_{i,j=2}^{n} x_i x_j \langle dx_i\,,\, dx_j\rangle_g \right) \\
&= \frac{1}{\rho^2}\left(\frac{r}{\rho}\, x_1 \left(1 - \frac{\lambda^2}{1+\lambda^2}\frac{x_1^2}{\rho^2}\right) - \frac{\lambda^2}{1+\lambda^2}\left(\frac{r}{\rho} - \frac{x_1^2}{r\rho}\right)\sum_{i=2}^{n}\frac{x_1 x_i}{\rho^2} - \frac{x_1}{\rho}\, r\left(1 - \frac{\lambda^2}{1+\lambda^2}\frac{r^2}{\rho^2}\right)\right) \\
&= \frac{1}{\rho^2}\left(\frac{r}{\rho} x_1 \left(1 - \frac{\lambda^2}{1+\lambda^2}\frac{x_1^2}{\rho^2} - \right) - \frac{\lambda^2}{1+\lambda^2}\left(\frac{r}{\rho} - \frac{x_1^2}{r\rho}\right) x_1\, \frac{r^2}{\rho^2} - \frac{x_1}{\rho}\, r\left(1 - \frac{\lambda^2}{1+\lambda^2}\frac{r^2}{\rho^2}\right)\right) \\
&= 0.
\end{aligned}$$

Ultimately, we prove (3.6). We have

$$\begin{aligned}|\psi_0|_g^2 &= \langle \psi_0, \psi_0 \rangle_g \\ &= \frac{1+\lambda^2}{(n-1)^2} \left\langle \sum_{i=2}^{n} (-1)^i \, x_i \, dx_\alpha^i \, , \sum_{j=2}^{n} (-1)^j \, x_j \, dx_\alpha^j \right\rangle_g \\ &= \frac{1+\lambda^2}{(n-1)^2} \sum_{i,j=2}^{n} (-1)^{i+j} \, x_i \, x_j \, \langle dx_\alpha^i \, , dx_\alpha^j \rangle_g .\end{aligned} \tag{3.9}$$

Denote by $\tilde{g}_1$ the $(n-1)\times(n-1)$ submatrix of $\tilde{g}$ obtained by cancelling the first row and the first column of $\tilde{g}$. By the definition of the metric on $\bigwedge^k T^*M$ given in (2.2) (in this case $k = n-2$) and the properties of the adjugate matrix (see [34, p. 73]), we have

$$\langle dx_\alpha^i \, , dx_\alpha^j \rangle_g = (-1)^{i+j} \, (\tilde{g}_1^{-1})_{i,j} \, \det(\tilde{g}_1). \tag{3.10}$$

A mere computation yields

$$(\tilde{g}_1^{-1})_{i,j} = \delta_{i,j} + \frac{\lambda^2}{\rho^2 + \lambda^2 \, x_1^2} \, x_i \, x_j, \tag{3.11}$$

The same calculation performed above to compute $\det g$ guarantees also that

$$\det(\tilde{g}_1)^{-1} = \det(\tilde{g}_1^{-1}) = 1 + \frac{\lambda^2 \, r^2}{\rho^2 + \lambda^2 \, x_1^2}. \tag{3.12}$$

Consequently, inserting (3.12) and (3.11) inside (3.10), by (3.9), we infer that

$$\begin{aligned}|\psi_0|_g^2 &= \frac{1+\lambda^2}{n-1} \sum_{i,j=2}^{n} x_i \, x_j \left(\delta_{i,j} + \frac{\lambda^2}{\rho^2 + \lambda^2 \, x_1^2} \, x_i \, x_j \right) \left(1 + \frac{\lambda^2 \, r^2}{\rho^2 + \lambda^2 \, x_1^2} \right)^{-1} \\ &= \frac{1+\lambda^2}{n-1} \, r^2,\end{aligned}$$

that is precisely (3.6). □

4. Construction of the calibration

The main result of this section is the following Theorem.

Theorem 4.1. *For $n \geq 4$, consider the threshold $\bar{\lambda}(n)$ defined in (1.2). Then, if $0 < \lambda \leq \bar{\lambda}(n)$, there exists a vector field $Z : \Omega'_\lambda \cup S'_\lambda \to \mathbb{R}^{n+1}$ such that:*

(i) $Z\,(x,t) = Z\,(x, \lambda|x|)$, *for all* $(x,t) \in \Omega'_\lambda \cup S'_\lambda$;

(ii) $|Z(x,t)| \leq 1$, *for all* $(x,t) \in \Omega'_\lambda \cup S'_\lambda$;

(iii) $Z(x,t) = e_1$, *for all* $(x,t) \in \overline{H}_\lambda \cap (\Omega'_\lambda \cup S'_\lambda)$;

(iv) $Z(x,t) \in T_{(x,t)} S_\lambda$, *for all* $(x,t) \in S'_\lambda$;

(v) Z *is smooth and* $\operatorname{div} Z = 0$ *in* Ω'_λ.

A key passage for the proof of Theorem 4.1, is the construction of a calibration for the $(n-1)$-dimensional surface S_λ^0 inside S_λ. This is the subject of the next result.

Theorem 4.2. *For $n \geq 4$, let $0 < \lambda \leq \bar{\lambda}(n)$. Then there exists a vector field $Y \in \mathfrak{X}(S'_\lambda)$ satisfying the following conditions:*

(i) $|Y(p)| \leq 1$, *for all* $p \in S'_\lambda$;

(ii) $Y(p) = e_1$, *for all* $p \in S^0_\lambda$;

(iii) Y *is smooth and* $\mathrm{div}^{S_\lambda} Y = 0$ *on* S'_λ.

The proof of Theorem 4.2 can be in part found in [31]. For the sake of clarity, we provide a complete proof here.

Proof. Let α be the multi-index $(2, \dots, n)$, and consider the $(n-1)$-form $\omega_0 = \sqrt{1+\lambda^2}\, dx_\alpha$. A direct computation shows that, in M',

$$\omega_0 = d\psi_0 = \frac{n-1}{r}\, dr \wedge \psi_0, \tag{4.1}$$

where ψ_0 is as in (3.2). Additionally, set $u := x_1/r$. Given a differentiable function h, we define

$$\psi_h(x) := h(u)\,\psi_0(x) \qquad \text{and} \qquad \omega_h(x) := d\psi_h(x), \qquad \text{for } x \in M'.$$

By the properties of the exterior derivative, we have

$$\omega_h = dh \wedge \psi_0 + h\,\omega_0\,, \tag{4.2}$$

where

$$dh = \frac{h'(u)}{r}\, dx_1 - \frac{x_1}{r^2}\, h'(u)\, dr\,. \tag{4.3}$$

Putting together (4.2), (4.3) and using (4.1), we infer that

$$\begin{aligned} \omega_h(x) &= \left(\frac{h'(u)}{r}\, dx_1 - \frac{x_1}{r^2}\, h'(u)\, dr\right) \wedge \psi_0(x) + h(u)\,\omega_0(x) \\ &= \left[\frac{r}{n-1}\left(\frac{h'(u)}{r}\, dx_1 - \frac{x_1}{r^2}\, h'(u)\, dr\right) + h(u)\, dr\right] \wedge \left(\frac{n-1}{r}\,\psi_0(x)\right) \\ &= \left[\left(h(u) - \frac{x_1}{(n-1)\,r} h'(u)\right) dr + \frac{h'(u)}{n-1}\, dx_1\right] \wedge \left(\frac{n-1}{r}\,\psi_0(x)\right). \end{aligned} \tag{4.4}$$

Step 1. We look for a continuously differentiable function $h : \mathbb{R} \to \mathbb{R}$ such that:

$$h(0) = 1, \qquad h'(0) = 0 \qquad \text{and} \qquad |\omega_h(x)|_g \leq 1 \quad \text{for all } x \in M'. \tag{4.5}$$

Applying the Cauchy-Schwartz inequality for wedge products (see [9, pag. 32], and note that 1-forms are always simple), by (4.4), we obtain

$$|\omega_h(x)|_g \leq \left|\left(h(u) - \frac{x_1}{(n-1)r} h'(u)\right) dr + \frac{h'(u)}{n-1} dx_1\right|_g \cdot \left|\frac{n-1}{r}\psi_0(x)\right|_g.$$

Thanks to (3.6), to have $|\omega_h|_g \leq 1$, it suffices to impose

$$\left|\left(h(u) - \frac{x_1}{(n-1)\,r} h'(u)\right) dr + \frac{h'(u)}{n-1}\, dx_1\right|_g^2 \leq \frac{1}{1+\lambda^2}\,. \tag{4.6}$$

We now observe that, taking $\theta \in (-\pi/2, \pi/2)$ as in (3.3), one has

$$u = \tan\theta, \qquad x_1 = \rho\sin\theta \qquad \text{and} \qquad r = \rho\cos\theta.$$

The identities (3.5) permit us to rewrite (4.6) in the following way:

$$h(u)^2 + (1+\lambda^2)\left[\frac{(1+u^2)\,h'(u)}{n-1} - u\,h(u)\right]^2 \le 1+u^2\,. \tag{4.7}$$

Thus we are reduced to find a solution h to the Cauchy problem for the differential inequality (4.7) with initial conditions $h(0)=1$, $h'(0)=0$. In order to solve (4.7), we apply the ansatz

$$h(u) = \frac{\cos(\beta(\arctan u))}{\cos(\arctan u)} = \frac{\cos(\beta(\theta))}{\cos\theta}\,,$$

where $\beta : (-\pi/2,\pi/2) \to (-\pi/2,\pi/2)$ is a continuously differentiable function with $\beta(0)=0$. By this approach, $h(0)=1$. Moreover, we have

$$h'(u) = \frac{-\sin(\beta(\theta))\beta'(\theta)\cos\theta + \cos(\beta(\theta))\sin\theta}{(1+u^2)\cos^2(\theta)}\,, \tag{4.8}$$

and then, in particular, $h'(0)=0$. We can now rewrite the differential inequality (4.7) in terms of β and θ. Thanks to (4.8), we have

$$\begin{aligned}\frac{(1+u^2)\,h'(u)}{n-1} - u\,h(u) &= \frac{-\sin(\beta(\theta))\beta'(\theta)\cos\theta + \cos(\beta(\theta))\sin\theta}{(n-1)\cos^2(\theta)} - \tan(\theta)\frac{\cos(\beta(\theta))}{\cos\theta} \\ &= \frac{1}{n-1}\left(-\beta'(\theta)\frac{\sin(\beta(\theta))}{\cos\theta} + \frac{\tan\theta}{\tan(\beta(\theta))}\frac{\sin(\beta(\theta))}{\cos\theta}\right) - \frac{\tan\theta}{\tan(\beta(\theta))}\frac{\sin(\beta(\theta))}{\cos\theta} \\ &= \frac{\sin(\beta(\theta))}{\cos\theta}\left(-\frac{\beta'(\theta)}{n-1} + \frac{2-n}{n-1}\frac{\tan\theta}{\tan(\beta(\theta))}\right)\,.\end{aligned}$$

Thus, by the definition of h and the identity $\cos^{-2}(\theta) = 1+u^2$, (4.7) becomes

$$\left|\beta'(\theta) + (n-2)\frac{\tan\theta}{\tan(\beta(\theta))}\right| \le \frac{n-1}{\sqrt{1+\lambda^2}}, \qquad \text{for } \theta\in(-\pi/2,\pi/2). \tag{4.9}$$

We want to find a solution β to the above differential inequality with initial condition $\beta(0)=0$. We claim that such a solution can be found among the functions $\beta_\gamma : (-\pi/2,\pi/2) \to (-\pi/2,\pi/2)$ defined by

$$\beta_\gamma(\theta) := \operatorname{sgn}(\theta)\arccos\left(\cos^{1+\gamma}(\theta)\right) \qquad \text{for } \gamma > 0,$$

i.e., that there exists $\gamma > 0$ such that β_γ satisfies (4.9) and $\beta_\gamma(0)=0$. Note that, for any $\gamma>0$, $\beta_\gamma(0)=0$. Let us rewrite (4.9) for β_γ. We observe that β_γ is differentiable, and

$$\beta_\gamma'(\theta) = \operatorname{sgn}(\theta)\frac{(1+\gamma)\cos^\gamma\theta\,\sin\theta}{\sqrt{1-\cos^{2(1+\gamma)}(\theta)}}\,.$$

Moreover, we have

$$\frac{1}{\tan(\beta_\gamma(\theta))} = \frac{\cos(\beta_\gamma(\theta))}{\operatorname{sgn}(\beta_\gamma(\theta))\sqrt{1-\cos^2(\beta_\gamma(\theta))}} = \frac{\operatorname{sgn}(\theta)\cos^{1+\gamma}(\theta)}{\sqrt{1-\cos^{2(1+\gamma)}(\theta)}},$$

and consequently,

$$\left|\beta'(\theta) + (n-2)\frac{\tan\theta}{\tan(\beta(\theta))}\right| = (n-1+\gamma)\left|\frac{\tan\theta}{\tan(\beta_\gamma(\theta))}\right|\,.$$

Hence, for this family of functions $\beta_\gamma(\theta)$, (4.9) transforms into

$$\left(1+\frac{\gamma}{n-1}\right)^2 \tan^2\theta \le \frac{\tan^2(\beta_\gamma(\theta))}{1+\lambda^2}, \qquad \text{for } \theta \in (-\pi/2, \pi/2). \tag{4.10}$$

Let us set $z := \cos^{-2}(\theta)$. In this way, $\cos^{-2}(\beta_\gamma(\theta)) = z^{1+\gamma}$, and condition (4.10) becomes

$$\left(1+\frac{\gamma}{n-1}\right)^2 \le \frac{1}{1+\lambda^2}\frac{z^{1+\gamma}-1}{z-1}, \qquad \text{for } z \ge 1. \tag{4.11}$$

Let $w(z) := (z^{1+\gamma}-1)/(z-1)$. For $z \ge 1$, w is monotonically increasing, hence it suffices to test condition (4.11) only as $z \to 1^+$. Since $\lim_{z\to 1^+} w(z) = 1+\gamma$, we get the condition

$$\left(1+\frac{\gamma}{n-1}\right)^2 - \frac{1+\gamma}{1+\lambda^2} \le 0. \tag{4.12}$$

In particular, (4.12) implies (4.11). We now observe that the function on the LHS of (4.12) has a global minimizer in

$$\bar\gamma := (n-1)\left(\frac{n-1}{2(1+\lambda^2)} - 1\right).$$

Let $\gamma = \bar\gamma$. If (4.12) is satisfied, by construction, the function $\beta = \beta_\gamma$ solves (4.9) and $\beta(0) = 0$. A mere computation gives that (4.12) holds for $\gamma = \bar\gamma$ if and only if

$$n \ge 4 \qquad \text{and} \qquad 0 < \lambda \le \bar\lambda(n) := \frac{1}{2}\frac{n-3}{\sqrt{n-2}}, \tag{4.13}$$

and this concludes the proof of Step 1. In particular, when (4.13) is fulfilled, the function h for which (4.5) holds true is given by

$$h(u) = \cos^\gamma(\arctan u), \qquad \text{where} \qquad \gamma = (n-1)\left(\frac{n-1}{2(1+\lambda^2)} - 1\right).$$

Notice that h is smooth, and so $\omega_h \in \mathcal{A}^{n-1}(M)$ is smooth.

Step 2. For $x \in M'$, define

$$X = (-1)^{n-1}(\star\omega_h)^\# \in \mathfrak{X}(M') \qquad \text{and} \qquad Y = \mathcal{I}_* X \in \mathfrak{X}(S'_\lambda).$$

We now show that Y satisfies (i), (ii) and (iii). By construction, $|\omega_h|_g \le 1$, and since $\star$ and $\mathcal{I}$ are isometries, we trivially infer that $|Y(p)| \le 1$, for all $p \in S'_\lambda$, that is (i). Moreover, by (4.5) and (4.4), since $u = 0$ as $x_1 = 0$, we have $\omega_h = \omega_0 = \sqrt{1+\lambda^2}\, dx_2 \wedge \cdots \wedge dx_n$ on M^0. It is easy to check that, by (2.5) (note that $\nu = \sqrt{\det g}\, dx_1 \wedge \cdots \wedge dx_n = \sqrt{1+\lambda^2}\, dx_1 \wedge \cdots \wedge dx_n$), the following equality holds

$$\star\left(\sqrt{1+\lambda^2}\, dx_2 \wedge \cdots \wedge dx_n\right) = (-1)^{n-1} e_1^\flat,$$

hence $X(x) = e_1$, for every $x \in M^0$. By the definition of $\mathcal{I}$, we have $Y(p) = e_1$, for all $p \in S^0_\lambda$, that is precisely (ii). Finally, let us prove (iii). The smoothness of Y immediately follows from that of ω_h. Now, by (i) of Lemma 2.1, $\star X^\flat = i_X\nu$, hence we have

$$i_X\nu = (-1)^{n-1} \star^2 \omega_h = \omega_h.$$

By (ii) of Lemma 2.1 and the definition of ω_h, we deduce that

$$\operatorname{div}^M X = \star d\omega_h = \star dd\psi_h = 0,$$

and then (iii) immediately follows from (2.3). $\square$

Let us now proceed with the proof of Theorem 4.1. The idea is to extend the vector field Y of Theorem 4.2 to Ω'_λ by vertically projecting onto S'_λ. This operation will produce a divergence-free vector field in Ω'_λ.

Proof (of Theorem 4.1). Consider the vector field

$$Z(x,t) := Y(x, \lambda|x|)\,.$$

Since Y is defined on S'_λ, it is clear that Z is defined in Ω'_λ. By construction, (i) of Theorem 4.1 holds, while the statements (ii), (iii), (iv) immediately follow from the properties of Y. It remains to demonstrate (v). For sure, Z is smooth because Y is. We need to show that $\operatorname{div} Z = 0$ in Ω'_λ. Consider the (non-orthonormal) frame $\partial_1, \dots, \partial_n, \partial_t$ for $T\Omega'_\lambda$ defined by

$$\partial_i(x,t) = \partial_i(x) = e_i + \lambda \frac{x_i}{|x|}\, e_t \qquad \text{for } 1 \le i \le n \qquad \text{and} \qquad \partial_t = e_t.$$

In addition, let $\hat g$ the matrix whose entries are the scalar products $\langle \partial_i, \partial_j \rangle$, $\langle \partial_i, \partial_t \rangle$, for $1 \le i, j \le n$. By (3.7), the determinant of $\hat g$ satisfies

$$\begin{aligned} \det \hat g &= |\partial_1 \wedge \cdots \wedge \partial_n \wedge e_t|^2 \\ &= |e_1 \wedge \cdots \wedge e_n \wedge e_t|^2 \\ &= 1, \end{aligned}$$

and, in particular, is constant. Represent now Y and Z in coordinates with respect to $\partial_1, \dots, \partial_n$ and $\partial_1, \dots, \partial_n, \partial_t$ respectively. Then, for suitable $Y_i \in C^\infty(S'_\lambda)$, $Z_i, Z_t \in C^\infty(\Omega'_\lambda)$,

$$Y(x, \lambda|x|) = \sum_{i=1}^n Y_i(x, \lambda|x|)\partial_i(x) \qquad \text{and} \qquad Z(x,t) = \sum_{i=1}^n Z_i(x,t)\partial_i(x) + Z_t(x,t)\partial_t,$$

for all $x \in \mathbb{R}^n$ and $t \ge \lambda|x|$. By construction, we have

$$Z_i(x,t) = Y_i(x, \lambda|x|) \quad \text{for any } 1 \le i \le n \qquad \text{and} \qquad Z_t \equiv 0.$$

Consequently, since by (3.4) also $\det g = 1 + \lambda^2$ is constant, (2.4) ensures that

$$\operatorname{div} Z(x,t) = \sum_{i=1}^n \partial_i Z_i(x,t) = \sum_{i=1}^n \partial_i Y_i(x, \lambda|x|) = \operatorname{div}^{S'_\lambda} Y(x, \lambda|x|) = 0.$$

This closes the proof. $\square$

5. Minimality of E

In this section, we demonstrate Theorem 1.1. The idea is to apply the Divergence Theorem to the vector field Z in the intersection between $E \Delta F$ and a sequence of sets invading $\Omega'_\lambda \subset \Omega_\lambda$. Subsequently, since Z is tangent to S'_λ, by an approximation argument, we will be able to infer the minimality condition of Theorem 1.1.

Proof (of Theorem 1.1). Up to rescaling, we can assume $R = 1$. For $\varepsilon > 0$, let

$$A^\varepsilon := \{(x,t) \in \mathbb{R}^{n+1} \,:\, |x'| \le \varepsilon\} \qquad \text{and} \qquad \Omega^\varepsilon_\lambda := \Omega_\lambda \setminus A^\varepsilon.$$

We observe that, for every $\varepsilon > 0$, $\Omega^\varepsilon_\lambda \subset \Omega'_\lambda$. When $0 < \lambda \le \bar\lambda(n)$, we can consider the vector field $Z : \Omega'_\lambda \to \mathbb{R}^{n+1}$ of Theorem 4.1. By the Divergence Theorem (see [26]) and since Z is divergence-free, we have

$$0 = \int_{(F\setminus E)\cap\Omega^\varepsilon_\lambda} \operatorname{div} Z = \int_{\partial^*((F\setminus E)\cap\Omega^\varepsilon_\lambda)} \Big\langle Z\,,\,\nu_{(F\setminus E)\cap\Omega^\varepsilon_\lambda}\Big\rangle\, d\mathcal{H}^{n-1}. \tag{5.1}$$

Owing to (2.6), there holds

$$\partial^*((F\setminus E)\cap\Omega^\varepsilon_\lambda) = (\Omega^\varepsilon_\lambda \cap \partial^*(F\setminus E)) \cup \Big((F\setminus E)^{(1)} \cap \partial\Omega^\varepsilon_\lambda\Big) \cup Q\,(F\setminus E, \Omega^\varepsilon_\lambda)^+\,,$$

and this in combination with (5.1) and (2.8) ensures that

$$\begin{aligned} 0 = \int_{(F\setminus E)\cap\Omega^\varepsilon_\lambda} \operatorname{div} Z &= \int_{\Omega^\varepsilon_\lambda\cap\partial^*(F\setminus E)} \big\langle Z\,,\,\nu_{F\setminus E}\big\rangle\, d\mathcal{H}^{n-1} + \\ &\quad + \int_{\left((F\setminus E)^{(1)}\cap\partial\Omega^\varepsilon_\lambda\right)\cup Q\left(F\setminus E,\Omega^\varepsilon_\lambda\right)^+} \big\langle Z\,,\,\nu_{\Omega^\varepsilon_\lambda}\big\rangle\, d\mathcal{H}^{n-1} \\ &= (I) + (II). \end{aligned} \tag{5.2}$$

By (2.7), because $Z = e_1 = \nu_E$ on $\partial^* E \cap \Omega_\lambda = H_\lambda$, one has

$$\begin{aligned} (I) &= \int_{\Omega^\varepsilon_\lambda\cap E^{(0)}\cap\partial^* F} \langle Z\,,\,\nu_F\rangle\, d\mathcal{H}^{n-1} - \int_{\Omega^\varepsilon_\lambda\cap F^{(1)}\cap\partial^* E} \langle Z\,,\,\nu_E\rangle\, d\mathcal{H}^{n-1} - \int_{\Omega^\varepsilon_\lambda\cap Q(E,F)^-} \langle Z\,,\,\nu_E\rangle\, d\mathcal{H}^{n-1} \\ &\le P(F; E^{(0)}\cap\Omega^\varepsilon_\lambda) - P(E; F^{(1)}\cap\Omega^\varepsilon_\lambda) \\ &= P(F; \bar E^c\cap\Omega^\varepsilon_\lambda) - P(E; F^{(1)}\cap\Omega^\varepsilon_\lambda). \end{aligned} \tag{5.3}$$

Let us now estimate (II). Because $E\Delta F \subset\subset B_1$, clearly $\partial^*(F\setminus E) \subset B_1$ and $(F\setminus E)^{(1)} \subset B_1$. Thus, by applying (2.7) to $\Omega^\varepsilon_\lambda = \Omega_\lambda \setminus A^\varepsilon$ and (2.9) to its normal, we have

$$\begin{aligned} |(II)| &\le \int_{B_1\cap\partial\Omega^\varepsilon_\lambda} \big|\big\langle Z\,,\,\nu_{\Omega^\varepsilon_\lambda}\big\rangle\big|\, d\mathcal{H}^{n-1} \\ &\le \int_{B_1\cap A^\varepsilon\cap\partial\Omega_\lambda} |\langle Z\,,\,\nu_{\Omega_\lambda}\rangle|\, d\mathcal{H}^{n-1} + \int_{B_1\cap\Omega_\lambda\cap\partial A^\varepsilon} |\langle Z\,,\,\nu_{A^\varepsilon}\rangle|\, d\mathcal{H}^{n-1}. \end{aligned} \tag{5.4}$$

Now, since Z is tangent to S'_λ, the first integral on the right hand side of (5.4) vanishes. This implies that

$$|(II)| \le \mathcal{H}^{n-1}(B_1\cap\partial A^\varepsilon) = C(n)\,\varepsilon^{n-2}.$$

This in combination with (5.2) and (5.3) yields

$$P(E; F^{(1)}\cap\Omega^\varepsilon_\lambda) \le P(F; \bar E^c\cap\Omega^\varepsilon_\lambda) + C(n)\,\varepsilon^{n-2}. \tag{5.5}$$

By applying the Divergence Theorem in $(E\setminus F)\cap\Omega^\varepsilon_\lambda$ and arguing very similarly as above, we also obtain

$$\begin{aligned} P(E; F^{(0)}\cap\Omega^\varepsilon_\lambda) &\le P(F; E^{(1)}\cap\Omega^\varepsilon_\lambda) + C(n)\,\varepsilon^{n-2} \\ &= P(F; E\cap\Omega^\varepsilon_\lambda) + C(n)\,\varepsilon^{n-2} \end{aligned} \tag{5.6}$$

Summing up (5.5) and (5.6), noticing that

$$P(E; F^{(1)} \setminus B_1) = P(E; F^{(0)} \setminus B_1) = 0 \qquad \text{and} \qquad P(F; E \setminus B_1) = P(F; \bar{E}^c \setminus B_1) = 0,$$

we deduce that

$$P(E; (F^{(0)} \cup F^{(1)}) \cap \Omega_\lambda^\varepsilon \cap B_1) \leq P(F; (\bar{E}^c \cup E) \cap \Omega_\lambda^\varepsilon \cap B_1) + C(n)\, \varepsilon^{n-2}.$$

By applying Federer's Theorem [26, Theorem 16.2], we then get

$$P(E; \Omega_\lambda^\varepsilon \cap B_1) \leq P(F; \Omega_\lambda^\varepsilon \cap B_1) + C(n)\, \varepsilon^{n-2}.$$

Finally, passing to the limit as $\varepsilon \to 0^+$ in the relation above, we obtain that

$$P(E; \Omega_\lambda \cap B_1) \leq P(F; \Omega_\lambda \cap B_1),$$

and we conclude. □

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