VERY WEAK NOTIONS OF DIFFERENTIABILITY

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1. Introduction

In a recent paper [7] devoted among other things to the differentiability properties of the flow associated to Sobolev vector fields in $\mathbb{R}^n$ (deeply studied in [3]), C. Le Bris and P. L. Lions proved that the flow map satisfies for any given time (even with some uniformity w.r.t. time) the property

$$(1)\qquad \lim_{r\to 0}\frac{f(x+ry)-f(x)-rW(x,y)}{r}=0 \quad \text{locally in measure in } \mathbb{R}^n_x\times\mathbb{R}^n_y$$

for some Borel map $W(x,y)$. Later on, assuming enough integrability (precisely a $L\log L$ one) of the spatial derivative of the vector field, the first author, M. Lecumberry and S. Maniglia proved that indeed the flow is approximately differentiable (see 3.1.2 of [6]). As a consequence we have also that the map W in (1) is linear w.r.t y and the flow satisfies Lipschitz properties, in the Luzin sense, typical of approximately differentiable maps (see for instance 3.1.8 of [6]). It is still not clear whether these results hold even for the flow associated to $W^{1,1}$ or even BV [1] vector fields.

These results stimulated us to compare the weak differentiability property (1), that we call directional differentiability in measure, to the more classical approximate differentiability, and to study the nature of W. The first main result is a simpler characterization of directional differentiability in measure that does not involve the averaging property w.r.t. y (see Theorem 3.2 and Theorem 3.4). Motivated by various (and really different) notions of pointwise derivatives, we compare their counterparts "in measure".

Theorem 1.1. *Let $f:\Omega\subset\mathbb{R}^n\to\mathbb{R}$ be a measurable function. Then the following assertions are equivalent:*

(i) *f in differentiable in measure in Ω;*
(ii) *f is Gâteaux differentiable in measure in Ω;*
(iii) *f is directionally differentiable in measure in Ω.*

Moreover, the derivative in measure g is linked to the directional derivative in measure W by

$$g(x)\cdot y = W(x,y) \qquad \text{for a.e. } (x,y)\in\Omega\times\mathbb{R}^n.$$

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Here a measurable function $g : \Omega \to \mathbb{R}^n$ is called a derivative in measure if

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - g(x)\cdot h}{|h|} = 0 \quad \text{locally in measure in } \Omega.$$

Analogously, g is called a Gâteaux derivative in measure if the difference quotients above tend to 0 along all lines passing through the origin.

The second main result is that all these differentiability properties are strictly weaker than approximate differentiability: in Theorem 4.2 we show the implication and in Section 5 we provide an example of a map differentiable in measure that is nowhere approximately differentiable.

2. Preliminaries

In this paper we denote by Ω an open subset of $\mathbb{R}^n$ and denote by $|A|$ the Lebesgue measure of a measurable subset A of an Euclidean space.

We recall that a sequence of measurable maps f_k defined on a Euclidean open domain A is said to converge locally in measure to a measurable map f if

$$\lim_{k\to\infty} |\{x \in K : |f_k(x) - f(x)| > \varepsilon\}| = 0$$

for any compact set $K \subset A$ and any $\varepsilon > 0$. Similarly we define local convergence in measure for other types of variable.

Lemma 2.1. *Suppose that $f : \Omega \to \mathbb{R}$ is a measurable function. Then $f(x + h) \to f(x)$ locally in Ω as $h \to 0$.*

Proof. This is an easy consequence of the Luzin theorem and of the fact that, for any compact set $K \subset \Omega$, the measure of the symmetric difference

$$(K + h \setminus K) \cup (K \setminus K + h)$$

tends to 0 as $h \to 0$. □

Lemma 2.2. *Let $f_k, f : \ \Omega \times \mathbb{R}^n \to \mathbb{R}$ be measurable functions. Suppose that, for each $y \in \mathbb{R}^n$, $f_k(\cdot, y) \to f(\cdot, y)$ locally in measure in Ω as $k \to \infty$. Then $f_k \to f$ locally in measure in $\Omega \times \mathbb{R}^n$ as $k \to \infty$.*

Proof. Let $K \subset \Omega$ be a a compact set. We fix $\gamma > 0$ and set

$$g_k(y) = \Big|\{x \in K : \ |f_k(x, y) - f(x, y)| \ge \gamma\}\Big|.$$

Then from the Lebesgue Dominated Convergence Theorem we infer that

$$g_k \to 0 \quad \text{ in } L^1_{\text{loc}}(\mathbb{R}^n).$$

Now a standard use of the Fubini theorem yields the conclusion. □

3. Derivatives in measure

Definition 3.1. Let $f : \Omega \to \mathbb{R}$ be a measurable function. A measurable function $g : \Omega \to \mathbb{R}^n$ is called a (Fréchet) derivative of f in measure if

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - g(x) \cdot h}{|h|} = 0 \quad \text{locally in measure w.r.t. } x \in \Omega.$$

A measurable function $g : \Omega \to \mathbb{R}^n$ is called a Gâteaux derivative in measure of f if

$$\lim_{\delta \to 0} \frac{f(x+\delta y) - f(x) - \delta g(x) \cdot y}{\delta} = 0 \quad \begin{array}{l} \text{for each } y \in \mathbb{R}^n, \\ \text{locally in measure w.r.t. } x \in \Omega. \end{array}$$

A measurable function $W : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is called a directional derivative of f in measure if

$$\lim_{\delta \to 0} \frac{f(x+\delta y) - f(x) - \delta W(x,y)}{\delta} = 0 \quad \begin{array}{l} \text{locally in measure} \\ \text{w.r.t. } (x,y) \in \Omega \times \mathbb{R}^n. \end{array}$$

It is obvious that if g is the derivative in measure of f then g is also the Gâteaux derivative in measure of f, and that all derivatives in measure are unique, up to Lebesgue negligible sets.

Theorem 3.2. *Let $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}^n$ be measurable functions. Suppose that g is the Gâteaux derivative of f in measure. Then*

$$(x, y) \mapsto g(x) \cdot y$$

is the directional derivative of f in measure.

Proof. This follows immediately from Lemma 2.2. $\square$

The next result belongs to W. G. Jurkat [4] and N. G. de Bruijn [5], it appeared as a solution of a problem posed by P. Erdös.

Lemma 3.3. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a measurable function such that*

$$\varphi(y+z) = \varphi(y) + \varphi(z) \text{ for a.e. } (y,z) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{2}$$

Then there exists $a \in \mathbb{R}^n$ such that $\varphi(y) = a \cdot y$ for a.e. $y \in \mathbb{R}^n$.

Theorem 3.4. *Let $f : \Omega \to \mathbb{R}$ be a measurable function. Suppose that there exists a directional derivative $W : \Omega \times \mathbb{R} \to \mathbb{R}$ of f in measure. Then*

(a) *there exists a measurable function $g : \Omega \to \mathbb{R}^n$ such that $W(x,h) = g(x) \cdot h$ for a.e. $(x,h) \in \Omega \times \mathbb{R}^n$,*
(b) *g is the derivative of f in measure.*

Proof. (a) We will show "a.e. linearity" of $W(x,\cdot)$. We define

$$\begin{aligned}
F_\delta(x,y,z) &= \frac{f(x+\delta y)-f(x)-\delta W(x,y)}{\delta},\\
G_\delta(x,y,z) &= \frac{f(x+\delta y+\delta z)-f(x+\delta y)-\delta W(x+\delta y,z)}{\delta},\\
H_\delta(x,y,z) &= \frac{-f(x+\delta y+\delta z)+f(x)+\delta W(x,y+z)}{\delta}.
\end{aligned}$$

Then

$$F_\delta \to 0 \quad \text{as } \delta\to 0, \tag{3}$$

$$G_\delta \to 0 \quad \text{as } \delta\to 0, \tag{4}$$

$$H_\delta \to 0 \quad \text{as } \delta\to 0 \tag{5}$$

locally in measure with respect to x,y,z. Indeed, (3) and (5) are easy applications of Lemma 2.2. To verify (4) we need also the following shift argument. Let $\Omega''\subset\subset\Omega'\subset\subset\Omega$. For small δ we obtain

$$\begin{aligned}
&\Big|\big\{(x,y,z)\in\Omega'':\ G_\delta(x,y,z))\ge\gamma\big\}\Big|\\
&\quad=\Big|\big\{(x,y,z)\in\Omega'':\ |f(x+\delta y+\delta z)-f(x+\delta y)-\delta W(x+\delta y,z)|\ge\gamma\delta\big\}\Big|\\
&\quad\le\Big|\big\{(x',y,z)\in\Omega':|f(x'+\delta z)-f(x')-\delta W(x',z)|\ge\gamma\delta\big\}\Big|\to 0
\end{aligned}$$

where we used change of variables $x'=x+\delta y$. This proves (4). From Lemma 2.1 and Lemma 2.2 we infer that

$$W(x+\delta y,z)-W(x,z)\to 0 \quad \text{as } \delta\to 0. \tag{6}$$

Adding (3)–(6) we obtain that

$$W(x,y+z)-W(x,y)-W(x,z)=0 \quad \text{for a.e. } (x,y,z)\in\Omega\times\mathbb{R}^n\times\mathbb{R}^n.$$

This verifies the assumptions of Lemma 3.3 and thus proves (a).

(b) Let $\boldsymbol{\alpha}_n$ be the measure of the unit ball in $\mathbb{R}^n$. We fix a set $\Omega'\subset\subset\Omega$ and $\varepsilon,\gamma>0$. We find $\delta_1>0$ such that

$$B(x,2\delta_1)\subset\Omega,\qquad x\in\Omega'.$$

Consider the sets

$$\begin{aligned}
A_\delta(x)=\{y\in B(0,2)\colon |f(x+\delta y)-f(x)-\delta g(x)\cdot y|\ge\gamma\delta\},\\
0<\delta<\delta_1,\quad x\in\Omega'.
\end{aligned}$$

Since $(x,y)\mapsto g(x)\cdot y$ is a directional derivative of f in measure, by the Fubini theorem,

$$\int_{\Omega'}|A_\delta(x)|\,dx\to 0 \quad \text{as } \delta\to 0.$$

Therefore there exists $\delta_2 > 0$ such that for $0 < \delta < \delta_2$ we have

$$|\{x \in \Omega' \colon |A_\delta(x)| \geq \frac{\boldsymbol{\alpha}_n}{4}\}| \leq \varepsilon.$$

By the Luzin theorem, there exists a compact set $K \subset \Omega$ such that $|\Omega \setminus K| < \varepsilon$ and $g|_K$ is continuous. By the uniform continuity, there exists $\delta_3 > 0$ such that for $0 < \delta < \delta_3$ we have

$$x, x' \in K,\ |x - x'| < \delta \implies |g(x) - g(x')| < \gamma.$$

Let $h \in \mathbb{R}^n$, $|h| = r < \frac{1}{2}\min\{\delta_1, \delta_2, \delta_3\}$. Consider the sets

$$\begin{aligned}
E &:= \{x \in \Omega' :\ |A_r(x)| \geq \tfrac{1}{4}\boldsymbol{\alpha}_n\},\\
E' &:= \{x \in \Omega' :\ |A_r(x+h)| \geq \tfrac{1}{4}\boldsymbol{\alpha}_n\},\\
F &:= \Omega' \setminus K,\\
F' &:= \{x \in \Omega' :\ x + h \notin K\}.
\end{aligned}$$

Then

$$|E \cup E' \cup F \cup F'| \leq 4\varepsilon.$$

Let $x \in \Omega \setminus (E \cup E' \cup F \cup F')$. Then both x and $x+h$ belong to K and

$$\begin{aligned}
|B(0,1) \cap A_r(x)| &\leq \tfrac{1}{4}\boldsymbol{\alpha}_n,\\
|\{y \in B(0,1) : y - \tfrac{h}{r} \in A_r(x+h)\}| &\leq \tfrac{1}{4}\boldsymbol{\alpha}_n.
\end{aligned}$$

Thus there exists $y \in B(0,1) \setminus A_r(x)$ such that $y' := y - \frac{h}{r} \notin A_r(x+h)$. Since $|y'| < 2$, the fact that $y' \notin A_r(x+h)$ means that $x + h + ry' \in \Omega$ and

$$|f(x+h+ry') - f(x+h) - rg(x+h) \cdot y'| \leq \gamma r.$$

Since $h + ry' = ry$, we have

$$\begin{aligned}
|f(x+h) - f(x) - g(x) \cdot h| &\leq |f(x+h) - f(x+h+ry') + rg(x+h) \cdot y'|\\
&\quad + |f(x+ry) - f(x) - rg(x) \cdot y| + |r(g(x) - g(x+h)) \cdot y'|\\
&\leq \gamma r + \gamma r + \gamma r|y'| \leq 4|h|\gamma.
\end{aligned}$$

This shows that

$$\frac{f(x+h) - f(x) - g(x) \cdot h}{|h|} \to 0$$

in measure w.r.t. x as $h \to 0$. $\square$

4. Approximate differentiability implies differentiability in measure

Definition 4.1. Let $f : \Omega \to \mathbb{R}$ be a measurable function and $x \in \Omega$. We say that $a \in \mathbb{R}^n$ is an approximate derivative of f at x if

$$\left\{h\colon \frac{|f(x+h)-f(x)-a\cdot h|}{|h|} > \varepsilon\right\} \quad \begin{array}{l}\text{has null Lebesgue density at } 0\\ \text{for any } \varepsilon > 0.\end{array} \tag{7}$$

As we are concerned here with convergence in measure, it is worth mentioning that (7) is equivalent to the convergence

$$\lim_{\delta\to 0} \frac{f(x+\delta y)-f(x)-\delta a\cdot y}{\delta} = 0 \quad \text{locally in measure w.r.t. } y \in \mathbb{R}^n.$$

Theorem 4.2. *Suppose that $f : \Omega \to \mathbb{R}$ is approximately differentiable a.e. Then f is differentiable in measure.*

Proof. Let g be the approximate derivative of f. By 3.1.16 in [6] there exist $f_j \in C^1_c(\Omega)$ and pairwise disjoint measurable sets A_j such that

$$\begin{aligned} f_j &= f \text{ on } A_j,\\ \nabla f_j &= g \text{ on } A_j,\\ |\Omega \setminus \bigcup_j A_j| &= 0. \end{aligned}$$

We fix $\Omega' \subset\subset \Omega$ and $\gamma > 0$ and find $\delta > 0$ such that $B(x,\delta) \subset \Omega$ for each $x \in \Omega'$. We set

$$E(h) := \{x \in \Omega'\colon |f(x+h)-f(x)-g(x)h| \ge \gamma|h|\}, \quad h \in B(0,\delta).$$

Then

$$E(h) \subset N \cup \bigcup_j (E_j(h) \cup F_j(h)),$$

where $|N| = 0$ and

$$\begin{aligned} E_j(h) &= \big\{x \in \Omega' \cap A_j :\ x+h \notin A_j\big\},\\ F_j(h) &= \big\{x \in \Omega' \cap A_j :\ |f_j(x+h)-f_j(x)-\nabla f_j(x)h| \ge \gamma|h|\big\}. \end{aligned}$$

Then

$$|E_j(h)| \to 0 \quad \text{as } h \to 0$$

by Lemma 2.1. Using the continuous differentiability of f_j, we obtain that also

$$|F_j(h)| \to 0 \quad \text{as } h \to 0.$$

Choose $\varepsilon > 0$. Since $E_j(h) \cup F_j(h) \subset A_j$ and $\sum_j |A_j| \le |\Omega| < \infty$, we can find an index m independent of h such that

$$\sum_{j=m+1}^{\infty} |E_j(h) \cup F_j(h)| \le \varepsilon.$$

Then

$$\limsup_{h\to 0} |E(h)| \le \sum_{j=1}^{m} \limsup_{h\to 0} |E_j(h) \cup F_j(h)| + \varepsilon = \varepsilon.$$

It follows that $|E(h)| \to 0$ as required. □

5. An example

5.1. **Triadic interval.** An interval $[a, b] \subset \mathbb{R}$ is called *triadic* if there exists $q \in \mathbb{N}$ such that $3^q(b-a) = 1$ and $3^q a \in \mathbb{Z}$.

5.2. **Cantor-Vitali function.** The Cantor-Vitali function is the unique continuous function $\Phi : [0,1] \to [0,1]$ satisfying the functional equation

$$\begin{cases} \Phi(x) = \frac{1}{2}, & x \in [\frac{1}{3}, \frac{2}{3}], \\ \Phi(x) = \frac{1}{2}\Phi(3x), & x \in [0, \frac{1}{3}], \\ \Phi(x) = \frac{1}{2}\Big(1 + \Phi(3x-2)\Big), & x \in [\frac{2}{3}, 1]. \end{cases}$$

We say that a function f is *cantorian* on $[a, b]$ if f is constant or an affine copy of the Cantor-Vitali function, namely

$$f(x) - f(a) = \Phi\Big(\frac{x-a}{b-a}\Big)\,(f(b) - f(a)), \qquad x \in [a, b]. \tag{8}$$

It is easy to observe that any function f, which is cantorian on a triadic interval $[a, b]$, is also cantorian on all triadic subintervals of $[a, b]$. Also, we need to notice that the family of all cantorian functions on $[a, b]$ is a linear space.

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is piecewise cantorian on the scale σ if f is cantorian on each interval $[(p-1)\sigma,\ p\sigma]$, $p \in \mathbb{Z}$.

5.3. **A cantorian saw.** We construct the auxiliary function $\Psi : \mathbb{R} \to \mathbb{R}$ as the 3-periodic function which is defined by

$$\Psi(x) = \begin{cases} \Phi(x - 3p), & x \in [3p, 3p+1],\ p \in \mathbb{Z}, \\ 1, & x \in [3p+1, 3p+2],\ p \in \mathbb{Z}, \\ \Phi(3p - x), & x \in [3p-1, 3p],\ p \in \mathbb{Z}. \end{cases}$$

5.4. **A sequence of intervals.** The following sequence of intervals plays an important role in our construction:

$$
\begin{aligned}
&I_1 = [0,\ 1],\\
&I_2 = [0,\ \tfrac{1}{3}],\\
&I_3 = [\tfrac{1}{3},\ \tfrac{2}{3}],\\
&I_4 = [\tfrac{2}{3},\ 1],\\
&I_5 = [0,\ \tfrac{1}{9}],\\
&\quad \dots,\\
&I_k = [(p-1)3^{-q},\ p3^{-q}]; \quad q \in \{0,1,2,\dots\},\ p \in \{1,2,\dots,3^q\},\\
&\qquad\qquad\qquad\qquad\qquad\quad\; k = \tfrac{1}{2}(3^q - 1) + p.
\end{aligned}
$$

Lemma 5.1. *There exists a sequence $\{r_k\}$ of real numbers with the following properties:*

(a) $r_k \in \{3^{-q} :\ q \in \mathbb{N}\}$*;*
(b) $r_{k+1} + r_{k+2} + \cdots \le r_k$*;*
(c) $r_k \le 3^{-k}$*;*
(d) $\frac{1}{9} r_k \ge 2^{k+1} r_{k+1}$*;*
(e) *If s is any piecewise cantorian function on the scale r_{k-1}, then*

$$\Bigl|\Bigl\{x \in [0,1] :\ s \text{ is nonconstant on } [x - 2^k r_k, x + 2^k r_k]\Bigr\}\Bigr| < 2^{-k}.$$

Proof. Notice that conditions (a), (b), (c), (d) are trivially satisfied provided we choose in the k-th step $r_k = 3^{-q}$ with q large enough (to ensure (b) it suffices that $r_k \le r_{k-1}/9$). Concerning (e), it suffices to notice that the canonical Cantor-Vitali function Φ satisfies

$$\Bigl|\Bigl\{x \in [0,1] :\ \Phi \text{ is nonconstant on } [x - l, x + l]\Bigr\} \to 0$$

as $l \to 0$ and then, given l_k such that the quantity above is less than 2^{-k}, choose r_k such that $2^k r_k / r_{k-1} < l_k$. □

5.5. **Construction.** We consider the sequence $\{I_k\}$ of intervals from 5.4 and the sequence $\{r_k\}$ from Lemma 5.1 and define

$$
\begin{aligned}
f_k(x) &= r_k \Psi\Bigl(\frac{x}{r_k}\Bigr)\chi_{I_k}(x), \qquad x \in [0,1], \quad k = 1,2,\dots,\\
f &= \sum_{k=1}^{\infty} f_k. \qquad (9)
\end{aligned}
$$

Theorem 5.2. *The function f in (9) has null derivative in measure.*

Proof. We shall show that

$$g_h \to 0 \quad \text{in measure as } h \to 0 \text{, where} \quad g_h(x) = \frac{f(x+h)-f(x)}{h}.$$

We denote

$$\begin{aligned} s_k &= f_1 + \cdots + f_k, \qquad t_k = f_{k+1} + f_{k+2} + \cdots . \\ E_k &= \big\{x \in [0,1] :\ s_{k-1} \text{ is nonconstant on } [x - 2^k r_k, x + 2^k r_k]\big\}, \\ F_k &= \{x \in [0,1] :\ (x - 2^k r_k, x + 2^k r_k) \not\subset (0,1) \setminus I_k\}. \end{aligned}$$

The function s_{k-1} is piecewise cantorian on the scale r_{k-1}, so that by Lemma 5.1

$$|E_k| < 2^{-k} \to 0$$

Obviously

$$|F_k| \le 2^{k+2} r_k + |I_k| \to 0.$$

We have

$$\begin{gathered} |t_k(x+h) - t_k(x)| \le r_{k+1} + r_{k+2} + \cdots < 2r_{k+1} < 2^{-k} h, \\ 2^{k+1} r_{k+1} < |h|. \end{gathered} \tag{10}$$

Hence

$$\begin{gathered} x \notin E_k \cup F_k \implies |f(x+h) - f(x)| \le |t_k(x+h) - t_k(x)| < 2^{-k}|h|, \\ 2^{k+1} r_{k+1} < |h| \le 2^k r_k. \end{gathered}$$

This is enough to conclude that $g_h \to 0$ in measure. □

Theorem 5.3. *If f is the function in* (9)*, then f is nowhere approximately differentiable.*

Proof. Let us fix $x \in [0,1]$ and set $K = \{k \in \mathbb{N} :\ x \in I_k\}$ (in what follows, we will not mark the fact that the constructed objects depend on x). Fix $k \in K$ and notice that on any interval $[3pr_k, (3p+3)r_k]$, with $p \in \mathbb{N}$, the function s_{k-1} is monotone (being Cantorian on the scale $r_{k-1} \ge 3r_k$). On the other hand, if this interval is contained in I_k, the function f_k satisfies

$$f_k((3p+1)r_k) - f_k(3pr_k) = r_k, \qquad f_k((3p+3)r_k) - f_k((3p+2)r_k) = -r_k.$$

Therefore, taking into account that $x \in I_k$, we can find $p_k \in \mathbb{N}$ such that the interval $[a_k, b_k] := [(p_k - 1)r_k, p_k r_k]$ satisfies

(a) $[a_k, b_k] \subset I_k \cap [x - 6r_k, x + 6r_k]$,
(b) $x \notin [a_k, b_k]$,
(c) $|s_k(b_k) - s_k(a_k)| \ge r_k$.

Now, by a symmetry argument we may assume that $x < a_k$ and $s_k(b_k) > s_k(a_k)$. Denote

$$\alpha_k = s_k(a_k), \qquad \delta_k = s_k(b_k) - s_k(a_k) \geq r_k.$$

We still distinguish two cases:
Upper case: Suppose that

$$s_k(x) \geq \alpha_k + \tfrac{1}{4}\delta_k.$$

Set

$$U_k = [a_k + \tfrac{7}{9}r_k,\ a_k + \tfrac{8}{9}r_k], \qquad V_k = [a_k + \tfrac{1}{9}r_k,\ a_k + \tfrac{2}{9}r_k].$$

If $x + h \in U_k$ we use (8) with $a = a_k$ and $b = b_k$ (recall that s_k is cantorian on $[a_k, b_k]$) to estimate

$$\frac{s_k(x+h) - s_k(x)}{h} \geq \overline{d}_k := \frac{\alpha_k + \frac{3}{4}\delta_k - s_k(x)}{a_k + \frac{8}{9}r_k - x}.$$

If $x + h \in V_k$, by a similar argument we obtain

$$\frac{s_k(x+h) - s_k(x)}{h} \leq \underline{d}_k := \frac{\alpha_k + \frac{1}{4}\delta_k - s_k(x)}{a_k + \frac{1}{9}r_k - x}.$$

Using the upper case assumption and writing

$$\overline{d}_k = \frac{\alpha_k + \frac{1}{4}\delta_k - s_k(x) + \frac{1}{2}\delta_k}{a_k + \frac{1}{9}r_k - x + \frac{7}{9}r_k}$$

we have

$$\begin{aligned}\overline{d}_k - \underline{d}_k &= \frac{\frac{1}{2}\delta_k(a_k - x + \frac{1}{9}r_k) - \frac{7}{9}r_k\big(\alpha_k + \frac{1}{4}\delta_k - s_k(x)\big)}{(a_k + \frac{1}{9}r_k - x)(a_k + \frac{8}{9}r_k - x)} \\ &\geq \frac{\frac{1}{2}\delta_k\,\frac{1}{9}r_k}{(b_k - x)^2} \geq \frac{\frac{1}{18}\delta_k r_k}{(6r_k)^2} \geq \frac{1}{648}.\end{aligned}$$

Lower case: Suppose that

$$s_k(x) \leq \alpha_k + \tfrac{1}{4}\delta_k.$$

Set

$$U_k = [a_k + \tfrac{3}{9}r_k,\ a_k + \tfrac{4}{9}r_k], \qquad V_k = [a_k + \tfrac{5}{9}r_k,\ a_k + \tfrac{6}{9}r_k].$$

If $x + h \in U_k$, arguing as in the upper case we estimate

$$\frac{s_k(x+h) - s_k(x)}{h} \geq \overline{d}_k := \frac{\alpha_k + \frac{1}{2}\delta_k - s_k(x)}{a_k + \frac{4}{9}r_k - x}.$$

If $x + h \in V_k$, we obtain

$$\frac{s_k(x+h) - s_k(x)}{h} \leq \underline{d}_k := \frac{\alpha_k + \frac{1}{2}\delta_k - s_k(x)}{a_k + \frac{5}{9}r_k - x}.$$

Using the lower case assumption we have

$$\begin{aligned}\overline{d}_k - \underline{d}_k &= \frac{\frac{1}{9}r_k\big(\alpha_k + \frac{1}{2}\delta_k - s_k(x)\big)}{(a_k + \frac{4}{9}r_k - x)(a_k + \frac{5}{9}r_k - x)}\\ &\ge \frac{\frac{1}{9}r_k\frac{1}{4}\delta_k}{(b_k - x)^2} \ge \frac{\frac{1}{36}\delta_k r_k}{(6r_k)^2} \ge \frac{1}{1296}.\end{aligned}$$

Both cases: By the preceding estimates and (10) (recall that $h \ge a_k - x + \frac{r_k}{9} \ge 2^{k+1}r_{k+1}$) we have

$$\begin{aligned}&\sup\Big\{\inf_{x+h\in E}\frac{f(x+h)-f(x)}{h}:\ E\subset[x-6r_k, x+6r_k],\ |E|\ge\tfrac{1}{9}r_k\Big\}\\ &-\inf\Big\{\sup_{x+h\in E}\frac{f(x+h)-f(x)}{h}:\ E\subset[x-6r_k, x+6r_k],\ |E|\ge\tfrac{1}{9}r_k\Big\}\\ &\ge\frac{1}{1296}-2^{1-k}.\end{aligned}$$

Since this occurs for all $k \in K$ and therefore for infinitely many k, the function f cannot be approximately differentiable at x. $\square$

References

[1] L. Ambrosio: *Transport equation and Cauchy problem for BV vector fields.* Invent. Math., **158** (2004), 227–260.

[2] L. Ambrosio; M. Lecumberry; S. Maniglia: *Lipschitz regularity and approximate differentiability of the DiPerna-Lions flow.* Preprint, 2005.

[3] R. J. Di Perna; P. L. Lions: *Ordinary differential equations, transport theory and Sobolev spaces.* Invent. Math., **98** (1989), 511–547.

[4] W. B. Jurkat: *On Cauchy's functional equation.* Proc. Amer. Math. Soc., **16** (1965), 683–686.

[5] N. G. de Bruijn: *On almost additive functions.* Colloq. Math. **15** (1966), 59–63.

[6] H. Federer: *Geometric Measure Theory.* Springer, 1969.

[7] C. Le Bris; P. L. Lions: *Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications.* Ann. Mat. Pura Appl. **183** (2004), no. 1, 97–130.

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