STRATIFICATION OF THE SINGLE BLOW-UP SET FOR RADON MEASURES

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Abstract. We show that the set of points where the blow-up, in the sense of Preiss, of a signed Radon measure on $\mathbb{R}^n$ is unique and its invariant subspace has dimension k is k-rectifiable. As simple applications, we obtain a rectifiability criterion for signed Radon measures and the extension of a result, due to Mattila, on measures having unique blow-up almost everywhere.

1. Introduction

The notion of blow-up of a geometric or analytic object, i.e. the limit of its enlargements around a point as the zooming factor goes to infinity, is of fundamental importance in many areas of Mathematics. The blow-up of a function or a manifold provides a simplified, but somewhat essential, description of the local behavior of the object "at infinitesimal scale". For this and other reasons, the classification of the blow-ups of various mathematical items has become a central theme in geometric and functional analysis. For a Radon measure μ in $\mathbb{R}^n$, a natural definition of blow-up at a point $x \in \mathbb{R}^n$ is the one in the sense of Preiss, [Pre87]: the limit in the weak*-topology of Radon measures (in duality with $C_c(\mathbb{R}^n)$) of the rescalings

$$c_j (\tau_{x,r_j})_\# \mu,$$

for some sequences $c_j > 0$, $r_j \to 0$, where $\tau_{x,r}(y) := \frac{y-x}{r}$ and $_\#$ is the push-forward of Radon measures. We say that μ has unique blow-up at $x \in \operatorname{supp}\mu$ if there exists a Radon measure μ_x with either $|\mu_x|(B_1(0)) = 1$ or $\mu_x = 0$ and such that, for every sequence $r_j \to 0^+$, there are a subsequence, not relabeled, and a constant $c \neq 0$ for which

$$\mu_{x,r_j} := \frac{1}{|\mu|(B_{r_j}(x))} (\tau_{x,r_j})_\# \mu \overset{*}{\rightharpoonup} c\mu_x, \tag{1.1}$$

where $|\mu|$ is the total variation measure of μ and $\overset{*}{\rightharpoonup}$ denotes the weak*-convergence of Radon measures. We will denote by $\mathcal{S}_\mu$ the set of points where μ has unique blow-up.

Mattila studied positive measures having a unique blow-up at almost every point: a way to rephrase his main result [Mat05, Theorem 3.2] is that, for μ-a.e. $x \in \mathcal{S}_\mu$, the blow-up of μ at x is a multiple of the k-dimensional Hausdorff measure $\mathcal{H}^k$ on a k-plane. Although this describes the blow-ups of μ at μ-a.e. point in $\mathcal{S}_\mu$, it can be interesting to study the behavior of μ in the remaining part of $\mathcal{S}_\mu$.

This problem has already received some attention: in [Del21] it is proved that, for a given $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the set Σ_u of points x where $\frac{1}{r^n} u\left(\frac{\cdot - x}{r}\right)$ converges, in the L^1_{loc}-topology, to a non-constant function can be covered by a countable union of $(n-1)$-dimensional Lipschitz graphs.

- The results in [Del21] rely on the fact that the measure μ is induced by a L^1_{loc} function u and that the rescalings converge in the L^1_{loc}-topology, while in $\mathcal{S}_\mu$ the convergence is intended in the weak*-topology of Radon measures. Thus the subset of $\mathcal{S}_\mu$ where the unique blow-up of u is not a constant function can be larger than Σ_u. Is this set rectifiable as well? What can be said if μ is a general Radon measure?
- It is easily seen (Lemma 5) that the unique blow-up μ_x of μ at a point in $x \in \mathcal{S}_\mu$ has an *invariant linear subspace* D_x of $\mathbb{R}^n$, namely $(\tau_{y,1})_\# \mu_x = \mu_x$ for $y \in D_x$. Therefore

$$\mathcal{S}_\mu = \bigcup_{k=1}^{n} \mathcal{S}^k_\mu, \qquad \text{where} \qquad \mathcal{S}^k_\mu := \{x \in \mathcal{S}_\mu \colon \dim D_x = k\}. \tag{1.2}$$

  Do these $\mathcal{S}^k_\mu$ enjoy finer rectifiability properties?

The main result of this note provides an answer to these questions.

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Theorem 1. *Let μ be a signed Radon measure on $\mathbb{R}^n$ and $k \in \{0,1,\dots,n\}$. Then $\mathcal{S}^k_\mu$ is contained in a countable union of k-dimensional Lipschitz graphs. Moreover, the approximate tangent plane to $\mathcal{S}^k_\mu$ at x coincides with D_x for $\mathcal{H}^k$-a.e. $x \in \mathcal{S}^k_\mu$.*

The idea of the proof is that, for every $x \in \mathcal{S}^k_\mu$, there is a radius $r > 0$ for which the points in $\mathcal{S}^k_\mu \cap B_r(x)$ with similar properties are contained in a cone $x + C$ around $x + D_x$. Otherwise, by contradiction, one can find a sequence of such points $y_j \notin x + C$ with $r_j = |x - y_j| \to 0$ and such that the measures $c_j \mu_{y_j}$ and μ_{y_j,r_j} are as close as we want. Now, on one hand $c_j \mu_{y_j}$ converges to a measure ν whose invariant subspace is close to D_x; on the other hand μ_{y_j,r_j}, being translations of μ_{x,r_j} with respect to points outside C, converge to a translation of μ_x whose invariant subspace is not contained in C, contradicting the closeness of $c_j \mu_{y_j}$ and μ_{y_j,r_j}.

A first outcome of Theorem 1 is a rectifiability criterion for signed Radon measures in terms of their blow-ups which, to the best of our knowledge, is new.

Corollary 2. *Let μ be a signed Radon measure on $\mathbb{R}^n$ and $k \in \{0,1,\dots,n\}$. Let us assume that, for $|\mu|$-a.e. $x \in \mathbb{R}^n$, there exists a Radon measure ν_x whose invariant linear subspace V_x is k-dimensional and with the property that, for every sequence $r_j \to 0^+$ there are a subsequence, not relabeled, and a constant $c \neq 0$ such that*

$$\frac{1}{r_j^k} (\tau_{x,r_j})_\# \mu \overset{*}{\rightharpoonup} c \nu_x.$$

Then μ is k-rectifiable: there exists a signed function $\theta \in L^1_{\mathrm{loc}}(\mathcal{S}^k_\mu, \mathcal{H}^k)$ such that $\mu = \theta \mathcal{H}^k \llcorner \mathcal{S}^k_\mu$, where $\mathcal{S}^k_\mu$ is defined in (1.2).

The above result extends to signed Radon measures the well-known criterion for positive ones, see for instance [Sim83, Theorem 11.8], where one usually sets $\nu_x = \theta_0 \mathcal{H}^k \llcorner V_x$ for some k-dimensional linear subspace V_x and assumes the existence of the limit $r^{-k}(\tau_{x,r_j})_\# \mu \overset{*}{\rightharpoonup} \theta_0 \mathcal{H}^k \llcorner V_x$ as $r \to 0^+$.
We point out that Corollary 2 cannot be trivially deduced from the corresponding result for positive measures, since the fact that μ has a certain blow-up at a point does not provide any suitable information on the blow-ups of $|\mu|$ at the same point, not even their uniqueness; nor does it appear that the classical proof of [Sim83, Theorem 11.8] can be readily adapted to the case of signed measures.

A further consequence of Theorem 1 is the following extension of [Mat05, Theorem 3.2] to signed Radon measures. It is based on the intuitive idea that, since each $\mathcal{S}^k_\mu$ is k-rectifiable, it is reasonable to expect that its subset of points where the blow-up of μ is more diffused than a k-dimensional measure is $|\mu|$-negligible.

Corollary 3. *Let μ be a signed Radon measure on $\mathbb{R}^n$ and $k \in \{0,1,\dots,n\}$. Then, for $|\mu|$-a.e. $x \in \mathcal{S}^k_\mu$, the unique blow-up of μ at x is a multiple of $\mathcal{H}^k \llcorner D_x$.*

We conclude this introduction with a comment on the definition of unique blow-up at a point. As already said above, we say that μ_x is the unique blow-up of μ at $x \in \operatorname{supp} \mu$ if, for every sequence $r_j \to 0$, *there exists* a subsequence for which (1.1) holds true. This differs from the definition used in [Mat05] for positive Radon measures, where one does not require the convergence of a subsequence for every choice of $\{r_j\}_j$: according to [Mat05], one says that μ has unique blow-up at $x \in \mathbb{R}$ if there exists ν such that any non-trivial weak*-limit (in case it exists) of $c_j(\tau_{x,r_j})_\# \mu$, where $c_j > 0$ and $r_j \to 0^+$, is of the form $c\nu$ for some $c > 0$.
[Mat05, Lemma 2.5] shows, provided μ is positive, that this condition actually implies $\mu_{x,r} \overset{*}{\rightharpoonup} c\mu_x$ as $r \to 0$ and, subsequently, our definition (they are actually equivalent). However, this does not hold in the case of signed Radon measures: let us consider two sequences of positive real numbers a_j, ρ_j such that

$$\lim_{j \to +\infty} \frac{1}{a_j} \sum_{i=j+1}^{+\infty} a_i = 0, \qquad \lim_{j \to +\infty} \frac{\rho_{j+1}}{\rho_j} = 0.$$

Then the Radon measure on $\mathbb{R}$ defined as

$$\mu := \sum_{j \in \mathbb{N}} a_j \left(\delta_{2\rho_j + \rho_j^2} - \delta_{2\rho_j - \rho_j^2} \right)$$

has the null measure as unique blow-up at $x = 0$ in the sense of [Mat05], but $\mu_{0,\rho_j} = \frac{1}{|\mu|(B_{\rho_j}(0))} (\tau_{0,\rho_j})_\# \mu$ has no converging subsequences because $|\mu_{x,\rho_j}|\big(B_2(0)\big) \to +\infty$ as $j \to +\infty$.

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2. Notation

Throughout the paper, μ denotes a fixed signed Radon measure on $\mathbb{R}^n$.

General notation

$B_r(x)$	Open ball of radius r and center x in $\mathbb{R}^n$;
ω_k	Lebesgue measure of the unit ball in $\mathbb{R}^k$;
$C(z,V,s)$	Open cone with center z, axis the k-dimensional linear subspace V and opening angle $\gamma \in \big(0, \frac{\pi}{2}\big]$: $\{y \in \mathbb{R}^n \colon \operatorname{dist}(y, z + D_x) < (\sin\gamma)\lvert y - z\rvert\}$;
$\tau_{x,r}$	Translation with base x and scaling factor $\frac{1}{r}$: $\tau_{x,r}(y) \coloneqq \frac{y-x}{r}$ for every $y \in \mathbb{R}^n$;

Measures

$\mathcal{H}^s$	s-dimensional Hausdorff measure in $\mathbb{R}^n$;
$\lvert\mu\rvert$	Total variation measure of the signed Radon measure μ;
$\#$	Push-forward of Radon measures;
$\overset{*}{\rightharpoonup}$	Weak*-convergence of Radon measures, in duality with $C_c(\mathbb{R}^n)$;
d	Distance compatible with the weak*-convergence of locally uniformly bounded Radon measures: if $\sup_{j\in\mathbb{N}} \lvert\nu_j\rvert(C) < +\infty$ for every bounded set $C \subset \mathbb{R}^n$, then $\nu_j \overset{*}{\rightharpoonup} \nu \iff d(\nu_j,\nu) \to 0$.
$\mu_{x,r}$	Normalized rescaling of μ at $x \in \operatorname{supp}\mu$ at scale r: $\mu_{x,r} \coloneqq \frac{1}{\lvert\mu\rvert(B_r(x))}(\tau_{x,r})_{\#}\mu$;
μ_x	Normalized unique blow-up of μ at $x \in \mathbb{R}^n$ as in (1.1);
D_x	Invariant subspace of μ_x, given by Lemma 5;
$\mathcal{S}_\mu$	Set of points in $\mathbb{R}^n$ where the blow-up of μ is unique;
$\mathcal{S}_\mu^k$	Set of points in $\mathbb{R}^n$ where the invariant subspace of the unique blow-up of μ has dimension k;
Hom_k^β	Set of measures ν such that for some $\alpha \in [0,\beta]$ it holds $(\tau_{0,\lambda})_{\#}\nu = \lambda^\alpha \nu$ for all $\lambda > 0$, whose invariant subspace has dimension at least k and $\lvert\nu\rvert\big(B_1(0)\big) \le 1$;

3. Proof of the results

We start with a simple remark on the definition of unique blow-up at a point: for every point $x \in \mathcal{S}_\mu$ such that $\mu_x \neq 0$, the constant c in (1.1) is bounded and away from 0.

Lemma 4. *Let μ be a signed Radon measure on $\mathbb{R}^n$ and let $\mu_x \neq 0$ be the unique blow-up of μ at $x \in \mathcal{S}_\mu$. Then there exists $\eta = \eta(x) > 0$ such that, for every sequence μ_{x,r_j} converging to $c\mu_x$ as $r_j \to 0^+$, it holds $\eta \le \lvert c\rvert \le 1$.*

Proof. The upper bound on $\lvert c\rvert$ trivially follows from the lower semi-continuity of the total variation measure of open sets under weak*-convergence and the fact that, by definition, it holds $\lvert\mu_{x,r}\rvert\big(B_1(0)\big) \le 1$ for every $r > 0$, while $\lvert\mu_x\rvert\big(B_1(0)\big) = 1$. It remains to show the lower bound.
By contradiction, let us assume that, for every $m \in \mathbb{N}$, there exists a sequence $r_j(m) \to 0$ such that $\mu_{x,r_j(m)} \overset{*}{\rightharpoonup} c_m\mu_x$ with $\lvert c_m\rvert \le \frac{1}{m}$. Since the distance d metrizes the weak-*convergence of Radon measures, there exists $M_m \in \mathbb{N}$ such that

$$d(\mu_{x,r_j(m)}, c_m\mu_x) < \frac{1}{m} \qquad \forall j \ge M_m.$$

We can thus define a sequence $\rho_m \downarrow 0$ such that $\rho_m \coloneqq r_{j_m}(m)$ for some $j_m \ge M_m$. Since the sequence of measures $c_m\mu_x$ converges strongly to 0, it holds

$$\lim_{m\to+\infty} d(\mu_{x,\rho_m}, 0) = 0. \tag{3.1}$$

On the other hand, by definition of unique blow-up at x, the sequence μ_{x,ρ_m} must have a weakly*-converging subsequence, whose limit is 0 by (3.1), hence contradicting the assumption $c \neq 0$ in (1.1). ☐

We now show that the unique blow-up of μ at x is homogeneous and that it has an invariant subspace.

Lemma 5. *Let μ be a signed Radon measure on $\mathbb{R}^n$ and let $\mu_x \neq 0$ be the unique blow-up of μ at $x \in \mathcal{S}_\mu$. Then there exists $\alpha = \alpha(x) \geq 0$ such that μ_x and $|\mu_x|$ are $(\alpha - n)$-homogeneous,*[1] *namely*

$$(\tau_{0,\lambda})_\# \mu_x = \lambda^\alpha \mu_x, \quad (\tau_{0,\lambda})_\# |\mu_x| = \lambda^\alpha |\mu_x| \qquad \forall \lambda > 0.$$

For μ_x, and more generally for every $(\alpha - n)$-homogeneous Radon measure, the set

$$C_x \coloneqq \{y \in \mathbb{R}^n \colon (\tau_{y,\lambda})_\# \mu_x = \lambda^\alpha (\tau_{y,1})_\# \mu_x \ \ \forall \lambda > 0\}$$

coincides with the invariant subspace D_x of μ_x, that is

$$D_x \coloneqq \{y \in \mathbb{R}^n \colon (\tau_{y,1})_\# \mu_x = \mu_x\}.$$

Moreover D_x is a linear subspace of $\mathbb{R}^n$ and, provided $\mu_x \neq 0$, it holds $\dim D_x \leq \alpha$.

Proof. • *μ_x and $|\mu_x|$ are $(\alpha - n)$-homogeneous.* We have

$$(\tau_{0,\lambda})_\# \mu_{x,r} = \frac{1}{|\mu|\big(B_r(x)\big)} (\tau_{x,\lambda r})_\# \mu = \frac{|\mu|\big(B_{\lambda r}(x)\big)}{|\mu|\big(B_r(x)\big)} \mu_{x,\lambda r} \qquad \forall r, \lambda > 0.$$

Let us choose a sequence $r_j \to 0$ such that $\mu_{x,r_j} \overset{*}{\rightharpoonup} c\mu_x \neq 0$ and a subsequence such that $\mu_{x,\lambda r_{j_i}}$ converge as well to $c'\mu_x \neq 0$. Then the above equation implies

$$(\tau_{0,\lambda})_\# c\mu_x = (c'q)\mu_x, \qquad \text{where } \ q \coloneqq \lim_{i \to +\infty} \frac{|\mu|\big(B_{\lambda r_{j_i}}(x)\big)}{|\mu|\big(B_{r_{j_i}}(x)\big)}. \tag{3.2}$$

This in particular proves that the product $c'q$ does not depend on the subsequence r_{j_i} (nor on the sequence r_j) and depends only on λ, thus we can define a map $\lambda \mapsto \psi(\lambda) = \frac{c'q}{c}$ such that

$$(\tau_{0,\lambda})_\# \mu_x = \psi(\lambda)\mu_x \qquad \forall \lambda > 0.$$

The fact that $\psi(1) = 1$, that $\psi(\lambda) \neq 0$ for every $\lambda > 0$ and the continuity of the map $\lambda \mapsto (\tau_{0,\lambda})_\# \mu_x$ with respect to the weak*-topology imply $\psi(\lambda) > 0$ for every $\lambda > 0$. Since for each open set $U \subseteq \mathbb{R}^n$ it holds

$$\begin{aligned}(\tau_{0,\lambda})_\# |\mu_x|(U) &= \sup\left\{\int f(y)\, \mathrm{d}(\tau_{0,\lambda})_\# \mu_x(y) \colon f \in C_c(U), \|f\|_{C^0} \leq 1\right\} \\ &= \psi(\lambda) \sup\left\{\int f(y)\, \mathrm{d}\mu_x(y) \colon f \in C_c(U), \|f\|_{C^0} \leq 1\right\} \\ &= \psi(\lambda)|\mu_x|(U),\end{aligned}$$

we infer $(\tau_{0,\lambda})_\# |\mu_x| = \psi(\lambda)\mu_x$ for all $\lambda > 0$ as well. As in [Mat05, Lemma 2.5 (2)], it is easily seen that $\psi(\lambda\rho) = \psi(\lambda)\psi(\rho)$, thus there exists $\alpha \geq 0$ such that $\psi(\lambda) = \lambda^\alpha$ for every $\lambda > 0$.

- *$C_x = D_x$.* We consider an $(\alpha - n)$-homogeneous Radon measure ν which is fixed throughout the rest of the proof and we prove the statements for the sets C, D corresponding to ν. By definition of push-forward and of $\tau_{y,\lambda}$, it follows

$$C = \{y \in \mathbb{R}^n \colon \nu(y + \lambda A) = \lambda^\alpha \nu(y + A) \ \ \forall \lambda > 0, \ \forall A \subset \mathbb{R}^n \text{ bounded and Borel}\},$$

$$D = \{y \in \mathbb{R}^n \colon \nu(y + A) = \nu(A) \ \ \forall A \subset \mathbb{R}^n \text{ bounded and Borel}\},$$

where $y + \lambda A \coloneqq \{y + \lambda z \colon z \in A\} = \tau_{y,\lambda}^{-1}(A)$. Clearly $0 \in C, D$ and let $A \subset \mathbb{R}^n$ be any bounded Borel set. If $y \in C$, then

$$\nu(y + A) = \nu\left(2\left(y + \frac{-y + A}{2}\right)\right) \overset{(0 \in C)}{=} 2^\alpha \nu\left(y + \frac{-y + A}{2}\right) \overset{(y \in C)}{=} \nu(y - y + A) = \nu(A),$$

thus we infer $y \in D$. On the other hand, if $y \in D$, we have

$$\nu(y + \lambda A) \overset{(y \in D)}{=} \nu(\lambda A) \overset{(0 \in C)}{=} \lambda^\alpha \nu(A) \overset{(y \in D)}{=} \lambda^\alpha \nu(y + A) \qquad \forall \lambda > 0,$$

concluding $y \in C$.

- *D_x is a linear space.* For every $y, z \in D$, we have

$$\nu(y - z + A) \overset{(y \in D)}{=} \nu(-z + A) \overset{(z \in D)}{=} \nu(z - z + A) = \nu(A),$$

thus $y - z \in D$. For every $y \in D$ and $\lambda > 0$, it holds

$$\nu(\lambda y + A) = \nu\left(\lambda\left(y + \frac{A}{\lambda}\right)\right) \overset{(0 \in C)}{=} \lambda^\alpha \nu\left(y + \frac{A}{\lambda}\right) \overset{(y \in D)}{=} \lambda^\alpha \nu\left(\frac{A}{\lambda}\right) \overset{(0 \in C)}{=} \nu(A),$$

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[1] According to this definition, the measure induced by a p-homogenous locally integrable function f, with respect to the Lebesgue measure, is p-homogeneous as well.

showing $\lambda y \in D$.

- *The dimension of D_x is at most α.* Let us assume $\nu \neq 0$ and $k := \dim D$. Up to multiplying by a constant, we can assume $|\nu|\big(B_1(0)\big) = 1$, so that the homogeneity provides

$$|\nu|\big(B_r(0)\big) = r^\alpha \qquad \forall r > 0. \tag{3.3}$$

Let $Q := [-1,1]^n$ be the unit cube. By a dyadic decomposition, for every $j \in \mathbb{N}$ the cube Q contains 2^{kj} disjoint open balls of radius 2^{-j} with centers on D. Thus (3.3) and the invariance of $|\nu|$ with respect translations on D yield

$$|\nu|(Q) \geq 2^{j(k-\alpha)} \qquad \forall j \in \mathbb{N},$$

which implies $k \leq \alpha$. ☐

Remark 6. In the first step of the above proof, (3.2), $|c|, |c'| \in [\eta(x), 1]$ and $\psi(\lambda) = \frac{c'q}{c} = \lambda^\alpha$ imply

$$\lambda^\alpha \eta(x) \leq \liminf_{r\to 0} \frac{|\mu|\big(B_{\lambda r}(x)\big)}{|\mu|\big(B_r(x)\big)} \leq \limsup_{r\to 0} \frac{|\mu|\big(B_{\lambda r}(x)\big)}{|\mu|\big(B_r(x)\big)} \leq \frac{\lambda^\alpha}{\eta(x)} \qquad \forall \lambda > 0. \tag{3.4}$$

We now prove that, if a Radon measure σ is homogeneous with respect to a point which is sufficiently distant from the invariant subspace of another homogeneous measure ν, then the two measures cannot be too close. See Section 2 for the relevant notations.

Lemma 7. *For every $\beta, M \geq 0$ and every $\eta, \gamma, \delta > 0$, there exists $\varepsilon > 0$ with the following property. Let ν, σ be signed Radon measures in $\mathbb{R}^n$ and let V be a k-dimensional linear subspace such that:*

- *$|\nu|\big(B_1(0)\big) \leq 1$ and $|\sigma|\big(B_1(0)\big) \leq M$;*
- *ν is $(\alpha - n)$-homogeneous for $\alpha \in [0, \beta]$ and V is its invariant subspace given by Lemma 5;*
- *$d(c\nu, \omega) \geq \delta$ for every $|c| \in [\eta, 1]$ and every $\omega \in \mathrm{Hom}^\beta_{k+1}$, that is the set of measures ω which are $(\alpha - n)$-homogeneous for some $\alpha \in [0, \beta]$, whose invariant subspace has dimension at least $k+1$ and such that $|\omega|\big(B_1(0)\big) \leq 1$;*
- *$(\tau_{y,\lambda})_\# \sigma = \lambda^{\alpha'} (\tau_{y,1})_\# \sigma$ for every $\lambda > 0$, where $y \in \big(\overline{B_{\frac12}(0)} \setminus B_{\frac14}(0)\big) \setminus C(0, V, \gamma)$ and $\alpha' \in [0, \beta]$.*

Then $d(c\nu, \sigma) > \varepsilon$ for every $c \in [\eta, 1]$.

Proof. Let us assume, by contradiction, that there exist $M, \beta \geq 0$ and $\eta, \gamma, \delta > 0$ such that, for every $j \in \mathbb{N}$, there are measures ν_j, σ_j, k-dimensional subspaces V_j and $y_j \in \big(\overline{B_{\frac12}(0)} \setminus B_{\frac14}(0)\big) \setminus C(0, V_j, \gamma)$ satisfying the hypotheses of the statement with $\alpha_j, \alpha'_j \leq \beta$ and $d(c_j \nu_j, \sigma_k) \leq \frac{1}{j}$ for $|c_j| \in [\eta, 1]$. Up to selecting a subsequence and without loss of generality, we can assume $c_j > 0$ for every $j \in \mathbb{N}$.
The assumptions on ν_j and σ_j imply that they are uniformly bounded on each bounded subset of $\mathbb{R}^n$; thus, up to subsequences, we can assume

$$\nu_j \overset{*}{\rightharpoonup} \nu, \qquad \sigma_j \overset{*}{\rightharpoonup} \sigma,$$
$$\alpha_j \to \alpha \leq \beta, \qquad \alpha'_j \to \alpha' \leq \beta, \qquad c_j \to \bar{c} \in [\eta, 1], \qquad V_j \to V,$$
$$y_j \to y \in \big(\overline{B_{\frac12}(0)} \setminus B_{\frac14}(0)\big) \setminus C(0, V, \gamma),$$

where the convergence of V_j to V is intended locally in the Hausdorff distance. Since d is continuous with respect to weak*-convergence, it holds $d(\bar{c}\nu, \sigma) = 0$, that is $\bar{c}\nu = \sigma$. Moreover:

- The lower semicontinuity of the total variation measure of open sets with respect to weak*-convergence provides $|\nu|\big(B_1(0)\big) \leq 1$ and $|\sigma|\big(B_1(0)\big) \leq M$.
- Again the continuity of d with respect to weak*-convergence yields
$$d(c\nu, \omega) \geq \delta \qquad \forall |c| \in [\eta, 1], \ \ \forall \omega \in \mathrm{Hom}^\beta_{k+1}.$$
In particular $\nu \neq 0$.
- Since for each $\lambda > 0$ the functions $\tau_{y,\lambda}$ converge uniformly to $\tau_{z,\lambda}$ as $y \to z$, from $V_j \to V$ and $\nu_j \overset{*}{\rightharpoonup} \nu$, we have that ν is $(\alpha - n)$-homogeneous and that its invariant subspace D contains V, thus $\dim D \geq k$. On the other hand, the above point implies $\dim D \leq k$, therefore $D = V$.
- Since for each $\lambda > 0$ the functions $\tau_{y_j,\lambda}$ converge uniformly to $\tau_{y,\lambda}$, we have $(\tau_{y,\lambda})_\# \nu = \lambda^{\alpha'} (\tau_{y,1})_\# \nu$. In order to infer $y \in V$, it remains to show that $\alpha' = \alpha$. To this aim, let us first observe that $|\nu|$ is $(\alpha - n)$-homogeneous with respect to the origin, as in the proof of Lemma 5, and $(\alpha' - n)$-homogeneous with respect to y. Thus $B_{\lambda - |y|}(0) \leq B_\lambda(y) \subset B_{\lambda + |y|}(0)$ yields
$$(\lambda - |y|)^\alpha |\nu|\big(B_1(0)\big) \leq \lambda^{\alpha'} |\nu|\big(B_1(y)\big) \leq (\lambda + |y|)^\alpha |\nu|\big(B_1(0)\big) \qquad \forall \lambda > 0,$$
which, for large λ and taking into account $\nu \neq 0$, implies $\alpha = \alpha'$.

The last point yields $y \in D = V$, which however contradicts $y \notin C(0,V,\gamma)$. $\square$

We now proceed with the proof of Theorem 1.

Proof of Theorem 1. The statement is trivial for $k=n$, thus we assume $k \in \{0,1,\dots,n-1\}$ be fixed throughout the proof. Let us moreover fix $\gamma \in \left(0,\frac{1}{16}\right)$. It is possible to choose a finite number m of k-dimensional linear subspaces $\{V_\ell\}_{\ell=1}^m$ such that every k-dimensional linear subspace V is contained in $\overline{C(0,V_\ell,\gamma)}$ for some $\ell \in \{1,\dots,m\}$. For every $\ell \in \{1,\dots,m\}$ and every $i \in \mathbb{N}$, we define the set

$$E_{\ell,i} \coloneqq \left\{ x \in \mathcal{S}^k \colon D_x \subset \overline{C(0,V_\ell,\gamma)},\ \ \alpha(x) \le i,\ \ \eta(x) \ge \frac{1}{i},\ \ d(c\mu_x,\omega) \ge \frac{1}{i}\ \ \forall |c| \in \left[\frac{1}{i},1\right],\ \forall \omega \in \mathrm{Hom}^i_{k+1} \right\},$$

where D_x and $\alpha(x)$ are respectively the invariant subspace and the homogeneity exponent of μ_x given by Lemma 5, $\eta = \eta(x)$ is given by Lemma 4, while d and Hom^i_{k+1} are respectively the distance and the set of $(\alpha - n)$-homogeneous measures described in Section 2. Since $\mathcal{S}^k = \bigcup_{i\in\mathbb{N}} \bigcup_{\ell=1}^m E_{i,\ell}$, it is enough to prove that each $E_{i,\ell}$ can be covered by a countable union of Lipschitz graphs.
Let us fix $i \in \mathbb{N}$ and $\ell \in \{1,\dots,m\}$. Then fix the value ε given by Lemma 7 for $\beta = i$, $\delta = \frac{1}{i}$, $M = i \cdot 2^\alpha$ and, for every $\rho > 0$, define the set

$$E_{i,\ell,\rho} \coloneqq \left\{ x \in E_{i,\ell} \colon \inf_{|c|\in[\eta,1]} d(\mu_{x,r}, c\mu_x) < \varepsilon\ \ \forall r \in (0,\rho) \right\}.$$

If $x \in E_{i,\ell}$, then it belongs to $E_{i,\ell,\rho}$ for some $\rho > 0$, because the existence of a sequence $r_j \to 0$ such that $d(\mu_{x,r_j}, c\mu_x) \ge \varepsilon$ for every $j \in \mathbb{N}$ and every $|c| \in [\eta,1]$ would contradict $x \in \mathcal{S}$. Therefore

$$E_{i,\ell} = \bigcup_{\rho \in \mathbb{Q}\cap(0,+\infty)} E_{i,\ell,\rho}$$

and it suffice to prove that, for a fixed $\rho > 0$, the set $E_{i,\ell,\rho}$ can be covered with a countable union of Lipschitz graphs. In order to do so, we claim that, for every $x \in E_{i,\ell,\rho}$, there exists $r > 0$ such that

$$E_{i,\ell,\rho} \cap B_r(x) \setminus C(x, D_x, 3\gamma) = \{x\}. \tag{3.5}$$

Let us fix $x \in E_{i,\ell,\rho}$, let μ_x be $(\alpha - n)$-homogeneous and let us assume, toward a contradiction, that for every $j \in \mathbb{N}$ there exists

$$y_j \in E_{i,\ell,\rho} \cap \overline{B_{\frac{r_j}{2}}(x)} \setminus \Big(B_{\frac{r_j}{4}}(x) \cup C(x, D_x, 3\gamma) \Big), \tag{3.6}$$

for some $r_j \le \rho$ such that $r_j \downarrow 0$. This in particular implies the existence of constants c_j with $|c_j| \in \left[\frac{1}{i},1\right]$ such that

$$d(c_j \mu_{y_j}, \mu_{y_j,r_j}) < \varepsilon \qquad \forall j \in \mathbb{N}. \tag{3.7}$$

Up to selecting a non relabeled subsequence and without loss of generality, we can assume $c_j > 0$ and that $\mu_{x,r_j} \overset{*}{\rightharpoonup} c\mu_x$ for some $c \in \left[\frac{1}{i},1\right]$. We now analyze the weak*-limit (up to a suitable subsequence) of the two sequences of measures in (3.7).

- The first component $c_j \mu_{y_j}$ of (3.7) satisfies the homogeneity of Lemma 5 with $c_j \in \left[\frac{1}{i},1\right]$, $\alpha(y_j) \le i$ and $|\mu_{y_j}|\big(B_1(0)\big) = 1$; therefore the sequence of measures μ_{y_j} is weakly relatively compact and converges, up to a subsequence, to a Radon measure ν, which, by the same arguments in the proof of Lemma 7, is $(\alpha' - n)$-homogeneous for some $\alpha' \le i$, its invariant subspace V has dimension k and

$$|\nu|\big(B_1(0)\big) \le 1, \qquad d(t\nu,\omega) > \frac{1}{i}\quad \forall |t| \in \left[\frac{1}{i},1\right],\ \ \forall \omega \in \mathrm{Hom}^i_{k+1}. \tag{3.8}$$

  Moreover

$$D_{\mu_{y_j}} \subset \overline{C(0,V_\ell,\gamma)} \subset \overline{C(0,D_x,2\gamma)} \quad \forall j \in \mathbb{N} \implies V \subset \overline{C(0,D_x,2\gamma)}.$$

- The second component μ_{y_j,r_j} of (3.7) satisfies

$$\mu_{y_j,r_j} = \frac{|\mu|\big(B_{r_j}(x)\big)}{|\mu|\big(B_{r_j}(y_j)\big)} \left(\tau_{\frac{y_j - x}{r_j},1} \right)_{\#} \mu_{x,r_j}. \tag{3.9}$$

  Since $B_{\frac{r_j}{2}}(x) \subset B_{r_j}(y_j)$, we have

$$\frac{|\mu|\big(B_{r_j}(x)\big)}{|\mu|\big(B_{r_j}(y_j)\big)} \le \frac{|\mu|\big(B_{r_j}(x)\big)}{|\mu|\big(B_{\frac{r_j}{2}}(x)\big)},$$

where, by (3.4), it holds

$$\limsup_{j\to+\infty} \frac{|\mu|\big(B_{r_j}(x)\big)}{|\mu|\big(B_{\frac{r_j}{2}}(x)\big)} \le i \cdot 2^\alpha .$$

Taking into account also (3.6), up to a subsequence we thus have

$$\frac{|\mu|\big(B_{r_j}(x)\big)}{|\mu|\big(B_{r_j}(y_j)\big)} \to q \le i \cdot 2^\alpha, \qquad \frac{y_j - x}{r_j} \to y \in \overline{B_{\frac{1}{2}}(0)} \setminus \big(B_{\frac{1}{4}}(0) \cup C(0, D_x, 3\gamma)\big).$$

Therefore, (3.9) and the convergence $\mu_{x,r_j} \overset{*}{\rightharpoonup} c\mu_x$ yield

$$\mu_{y_j,r_j} \overset{*}{\rightharpoonup} cq(\tau_{y,1})_{\#}\mu_x .$$

Passing to the limit in (3.7), we have

$$d\big(\nu, cq(\tau_{y,1})_{\#}\mu_x\big) \le \varepsilon. \tag{3.10}$$

We now observe that, by (3.8), ν satisfies the assumptions of Lemma 7 with $\beta = i$ and $\delta = \frac{1}{i}$, whereas the measure $\sigma := cq(\tau_{y,1})_{\#}\mu_x$ satisfies $|\sigma|\big(B_1(0)\big) \le i \cdot 2^\alpha$ and it is $(\alpha - n)$-homogeneous with respect to $-y$, because

$$(\tau_{-y,\lambda})_{\#}\sigma = c(\tau_{0,\lambda})_{\#}\mu_x = c\lambda^\alpha \mu_x = \lambda^\alpha (\tau_{-y,1})_{\#}\sigma .$$

We moreover point out that, since $y \notin C(0, D_x, 3\gamma)$ while $V \subset \overline{C(0, D_x, 2\gamma)}$, it holds $y \notin C(0, V, \gamma)$. Hence σ satisfies the hypothesis of Lemma 7 as well; however its conclusion is in contradiction with (3.10). This proves the existence of r such that (3.5) holds true and [Mat95, Lemma 15.13] (see also the observation below its proof) implies that $E_{i,\ell,\rho}$ can be covered by a countable union of Lipschitz graphs. From (3.5) we also infer that, for $\mathcal{H}^k$-a.e. $x \in E_{i,\ell,\rho}$, the approximate tangent plane $T_x E_{i,\ell,\rho}$ is contained in $\overline{C(0, D_x, 3\gamma)}$. Since γ can be chosen arbitrarily small, we have

$$T_x E_{i,\ell,\rho} = D_x \qquad \text{for } \mathcal{H}^k\text{-a.e. } x \in E_{i,\ell,\rho}.$$

By standard arguments it holds $T_x E_{i,\ell,\rho} = T_x \mathcal{S}^k$ for $\mathcal{H}^k$-a.e. $x \in E_{i,\ell,\rho}$, which proves the last assertion of the statement. □

We now prove Corollary 2.

Proof of Corollary 2. We first of all observe that, for every $x \in \mathbb{R}^n$ satisfying the hypotheses of the statement, it holds

$$\sup_{r\in(0,1]} \frac{|\mu|\big(B_r(x)\big)}{r^k} =: s(x) < +\infty,$$

otherwise there would be a sequence $r_j \to 0$ for which the above ratio is unbounded, contradicting the existence of a converging subsequence of $r_j^{-k}(\tau_{x,r_j})_{\#}\mu$. Since this happens $|\mu|$-a.e., we have

$$|\mu| = |\mu| \llcorner \mathcal{S}^k_\mu$$

and $\mathcal{S}^k_\mu$ is k-rectifiable by Theorem 1. We now show that $|\mu| \ll \mathcal{H}^k \llcorner E$. Indeed, let us consider $A \subset \mathcal{S}^k_\mu$ with $\mathcal{H}^k(A) = 0$. Then

$$A = \bigcup_{i\in\mathbb{N}} A_i, \qquad A_i := \{x \in A \colon s(x) \le i\}$$

and let us fix $i \in \mathbb{N}$. Since $\mathcal{H}^k(A_i) = 0$, for every $\varepsilon > 0$ there exists a countable covering $\{B_{r_\ell}(x_\ell)\}_{\ell\in\mathbb{N}}$ of A_i such that[2]

$$\sum_{\ell\in\mathbb{N}} r_\ell^k < \varepsilon \qquad \text{and} \qquad x_\ell \in A_i \quad \forall \ell \in \mathbb{N}.$$

Thus we have

$$|\mu|(A_i) \le \sum_{\ell\in\mathbb{N}} |\mu|\big(B_{r_\ell}(x_\ell)\big) \le i \sum_{\ell\in\mathbb{N}} r_\ell^k < i \cdot \varepsilon .$$

The arbitrariness of ε implies $|\mu|(A_i) = 0$. Since this is true for every $i \in \mathbb{N}$, we infer $|\mu|(A) = 0$, hence $|\mu| \ll \mathcal{H}^k \llcorner \mathcal{S}^k_\mu$. Recalling that $|\mu|$ is a Radon measure and $\mathcal{H}^k \llcorner \mathcal{S}^k_\mu$ is σ-finite, the Radon-Nikodym Theorem implies the existence of a function $\theta \in L^1_{\mathrm{loc}}(\mathcal{S}^k_\mu, \mathcal{H}^k)$ such that $\mu = \theta \mathcal{H}^k \llcorner \mathcal{S}^k_\mu$. □

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[2] According to the definition of $\mathcal{H}^k$, the balls covering A are not necessarily centered on A, but $x \in A_i \cap B_r(y)$ implies $B_r(y) \subset B_{2r}(x)$.

We conclude with the proof of Corollary 3.

Proof of Corollary 3. From Theorem 1 we know that $\mathcal{S}^k_\mu$ is contained in the union of a family $\{M_i\}_{i\in\mathbb{N}}$ of k-dimensional Lipschitz graphs. Since $|\mu|$-a.e. $x \in M_i$ is a point of density 1 for M_i with respect to $|\mu|$, the blow-ups of μ and $\mu \llcorner M_i$ at x are the same, therefore

$$\mu_x = (\mu \llcorner M_i)_x \qquad \text{for } |\mu|\text{-a.e. } x \in M_i.$$

For such points, this implies that $\operatorname{supp}\mu_x$ is contained in a closed cone C around a k-dimensional linear subspace V; we observe that C does not contain any half-space of dimension larger than k. On the other hand, from Lemma 5 we infer the implication

$$y \in \operatorname{supp}\mu_x \implies \{\lambda y + z \colon \lambda > 0, z \in D_x\} \subseteq \operatorname{supp}\mu_x.$$

Then it must hold either $\mu_x = 0$ or $\operatorname{supp}\mu_x = D_x$. The invariance of μ_x on D_x implies the conclusion. ☐

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