ADDENDUM TO: *EQUIVALENCE BETWEEN STRICT VISCOSITY SOLUTION AND VISCOSITY SOLUTION IN THE WASSERSTEIN SPACE AND REGULAR EXTENSION OF THE HAMILTONIAN IN $L^2_{\mathbb{P}}$.*

CHLOÉ JIMENEZ

This document contains additional proofs to the section 4.3. of the article [3]. Indeed, in an interesting discussion with Giulia Cavagnari, she made me realize that the second part of Proposition 4.27. (that is V_L is a supersolution), proved with the only help of the dynamic programming principle, (given in remark 4.26.) is not obvious at all. Indeed, as she mentioned, for some X, $V_L(t_0, X)$ might have no optimal trajectories. She is quite right and a lemma will be added in the present document to avoid this difficulty. I will also provide the proof of remark 4.26. which is not included in the orignal document

This addendum is organized as follows: in a first part we will recall briefly the setting of section 4.3; then I will give the proof the dynamic programming principle of remark 4.26.; in a third part I will prove that V_L is a supersolution of the suitable Hamilton-Jacobi-Bellman equation.

0.1. **Setting of the problem.** Given a bounded, uniformly continuous mapping $\mathcal{G} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, we study the following value function defined for all $t_0 \in [0, T]$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$:

$$\mathcal{V}(t_0, \mu) := \inf_{(v_t, \mu_t)} \{\mathcal{G}(\mu_T) : t \in [t_0, T] \mapsto (v_t, \mu_t) \text{ admissible and } \mu_{t_0} = \mu\}.$$

A couple (v_t, μ_t) is said to be admissible in $[t_0, T]$ if:

- $t \in [t_0, T] \mapsto \mu_t \in \mathcal{P}_2(\mathbb{R}^d)$ associated with v_t is in $AC^2([t_0, T], \mathcal{P}_2(\mathbb{R}^d))$.
- it exists $\mathbf{u} : [0, T] \times \mathbb{R}^d \to \mathbf{U}$ Borel such that:
$$v_t(x) = f(x, \mathbf{u}(t, x), \mu_t) \text{ a.e. } t,\ \mu_t\text{-a.e.}$$

Here $\mathbf{U}$ is a compact, convex and separable Banach space and $f : \mathbb{R}^d \times \mathbf{U} \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ satisfies the following assumptions:

- f is affine in $\mathbf{u}$, that is for all $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, $\mathbf{u}, \mathbf{v} \in \mathbf{U}$ and $t \in [0, 1]$:
$$f(x, (1-t)\mathbf{u} + t\mathbf{v}, \mu) = (1-t)f(x, \mathbf{u}, \mu) + tf(x, \mathbf{v}, \nu),$$
- f is continuous and it exists $L > 0$ such that for all $(x, \mathbf{u}, \mu) \in \mathbb{R}^d \times \mathbf{U} \times \mathcal{P}_2(\mathbb{R}^d)$ and $(y, \nu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$:
$$|f(x, \mathbf{u}, \mu) - f(y, \mathbf{u}, \nu)| \leq L\left(W_2(\mu, \nu) + |x - y|\right).$$

Given a fixed complete probability space $(\Omega, B(\Omega), \mathbb{P})$, we consider the lift of $\mathcal{V}$ which is given for all $t_0 \in [t_0, T]$ and $X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$ by (see [2] and [3]):

$$V_L(t_0, X) = \inf\{\mathcal{G}(X_T \sharp \mathbb{P}) : t \mapsto X_t \text{ is admissible and } X_{t_0} = X\},$$

Date: December 2, 2025.

a map $t \mapsto X_t$ is said to be admissible for $V_L(t_0, X)$ if for some $\mathbf{u} : [t_0, T] \times \Omega \to \mathbf{U}$ Borel:

- $t \mapsto X_t$ is in $W^{1,2}([t_0, T], L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d))$,
- $\dot{X}_t(\omega) = f(X_t(\omega), \mathbf{u}(t, \omega), X_t \sharp \mathbb{P})$ a.e. t, $\mathbb{P}$-a.e. $\omega \in \Omega$.

In the next section, we show that V_L satisfies a dynamic programming principle inherited from the corresponding one for $\mathcal{V}$.

0.2. **Proof of the Remark 4.26.**

The remark 4.26. states that for all $s < t$ and $X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d)$:

$$V_L(s, X) = \inf \{V(t, X_t) : \tau \mapsto X_\tau \text{ is admissible for } V_L(s, X) \text{ and } X_s = X\}.$$

We denote by $W(s, X)$ the right hand side of the equality.
Step 1: It is easy to see that:

$$W(s, X) \leq V_L(s, X).$$

Indeed take $\tau \in [s, T] \mapsto X_\tau$ an admissible for $V_L(s, X)$. On the one hand its restriction to $[s, t]$ is admissible for $W(s, X)$ and, on the other hand, its restriction to $[t, T]$ is admissible for $V_L(t, X_t)$ so that:

$$W(s, X) \leq V_L(t, X_t) \leq \mathcal{G}(X_T \sharp \mathbb{P}).$$

Taking the infimum of the right hand side on all admissible trajectories for $V_L(s, X)$ gives the result.
Step 2: We use the following dynamic programming principle proved in [4]:

$$\mathcal{V}(s, \mu) = \inf_{(v_\tau, \mu_\tau)} \{\mathcal{V}(t, \mu_t) : (v_\tau, \mu_\tau) \text{ is admissible for } \mathcal{V}(s, \mu) \text{ and } \mu_s = \mu\}$$

with μ the law of X. Denote by $\mathcal{W}(s, \mu)$ the right side of the equality. Then for any admissible trajectory $\tau \in [s, t] \mapsto X_\tau$ for $W(s, X)$, by Proposition 4.22 of [3] (see also [2] and [5]), $\tau \in [s, t] \mapsto X_\tau \sharp \mathbb{P}$ is admissible for $\mathcal{W}(s, \mu)$ and as $\mathcal{V}(t, X_t \sharp \mathbb{P}) = V_L(t, X_t)$, we get:

$$\mathcal{W}(s, \mu) \leq W(s, X).$$

Finally, recalling step 1:

$$V_L(s, X) = \mathcal{V}(s, \mu) = \mathcal{W}(s, \mu) \leq W(s, X) \leq V_L(s, X).$$

□

0.3. V_L **is a viscosity supersolution.**

In this section we prove the following proposition using the dynamic programming principle above:

Proposition 0.1. *The lift V_L of $\mathcal{V}$ is the unique viscosity supersolution of the following equation:*

$$\begin{cases} \partial_t U(t, X) + \overline{\mathrm{H}}_2(X, D_X U(t, X)) = 0 \ \forall (t, X) \in [0, T[\times L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \\ U(T, X) = \mathcal{G}(X \sharp \mathbb{P}) \ \forall X \in L^2_{\mathbb{P}}(\Omega, \mathbb{R}^d) \end{cases}$$

with: $\overline{\mathrm{H}}_2(X, Z) = \inf \left\{ \int_\Omega f(X, \mathbf{v}(X, Z), X \sharp \mathbb{P}) \cdot Z \, d\mathbb{P} : \ \mathbf{v} : \mathbb{R}^{2d} \to \mathbf{U} \text{ Borel} \right\}.$

We will need the following lemma:

Lemma 0.2. *Let* $t \in [0,T]$, $X \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$ *and* $(p_t, Z) \in D^- V_L(t,X)$ *(or* $D^+V_L(t,X)$*). It exists* (X', Z') *such that:*

$$(X,Z)\sharp \mathbb{P} = (X',Z')\sharp\mathbb{P}, \quad Z' \in D^-V_L(t,X') \textit{ (resp. } D^+V_L(t,X') \textit{)}$$

and an optimal trajectory exists for $V_L(t,X')$*.*

Proof of the Lemma: We do the proof for D^-V_L, the proof for D^+V_L being the same. Let $\gamma = (X,Z)\sharp\mathbb{P}$ and μ the law of X. Recall that:

$$(p_t, Z) \in D^-V_L(t,X) \Leftrightarrow (p_t,\gamma) \in D^-_{AGS}\mathcal{V}(t,\mu)$$

(see Definition 4.12., Example 4.13 in [3] and the results in [5]).
It was proved in [4], that an optimal trajectory $\tau \in [t,T] \mapsto \mu_\tau$ exists for $\mathcal{V}(t,\mu)$. Denote by e_τ the evaluation map:

$$e_\tau : (x,\sigma) \in \mathbb{R}^d \times \mathcal{C}([t,T],\mathbb{R}^d) \mapsto \sigma(\tau) \in \mathbb{R}^d.$$

Let $\eta \in \mathbb{P}(\mathbb{R}^d \times \mathcal{C}([t,T],\mathbb{R}^d))$ associated to $\tau \mapsto \mu_\tau$ by the superposition principle ([1], Theorem 8.2.1.), in particular $e_\tau\sharp\eta = \mu_\tau$. Let us disintegrate γ and η and glue them together:

$$\eta = \eta^x \otimes \mu(x), \quad \gamma = \gamma^x \otimes \mu(x), \quad m(x,z,\sigma) = \gamma^x(z) \otimes \eta^x(\sigma) \otimes \mu(x).$$

It exists $(X',Z',S) \in L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)^2 \times L^2_{\mathbb{P}}(\Omega,\mathcal{C}([t,T],\mathbb{R}^d))$ such that $(X',Z',S)\sharp\mathbb{P} = m$. As a consequence $(X',Z')\sharp\mathbb{P} = \gamma$. Then, as $(p_t,\gamma) \in D^-_{AGS}\mathcal{V}(t,\mu)$:

$$(p_t, Z') \in D^-V_L(t,X').$$

Moreover denoting by $S_\tau = e_\tau \circ S$, we get $\tau \mapsto S_\tau$ an admissible trajectory for $V_L(t,X')$ (see for instance [5], proof of Theorem 2.1., i)). As we have:

$$V(t,X') = \mathcal{G}(\mu_T) = \mathcal{G}(X_T\sharp)\mathbb{P}$$

the trajectory $\tau \mapsto S_\tau$ is optimal for $V_L(t,X')$. □

Proof of the Proposition: Let $(t,X) \in [t_0,T[\times L^2_{\mathbb{P}}(\Omega,\mathbb{R}^d)$, $(p_t,Z) \in D^-V_L(t,X)$. Take X' , Z' given by the above Lemma and $\tau \mapsto X'_\tau$ optimal for $V_L(t_0,X')$, denote by $\mathbf{u} : [t_0,T] \times \Omega \to \mathbf{U}$ the associated control.
By Remark 4.26. for $h > 0$ small enough:

$$0 = V_L(t+h, X'_{t+h}) - V_L(t,X').$$

Moreover, as $(p_t,Z') \in D^-V_L(t,X')$:

$$V_L(t+h,X'_{t+h}) \geq V_L(t,X') + p_t h + \langle X'_{t+h} - X', Z'\rangle + o\left(\sqrt{h^2 + \|X_{t+h} - X_t\|^2}\right)$$

so that:

$$0 \geq p_t h + \langle \int_t^{t+h} f(X'_\tau, \mathbf{u}(\tau,\cdot), X'_t\sharp\mathbb{P})\, d\tau \, , \, Z'\rangle + o\left(\sqrt{h^2 + \|X_{t+h} - X_t\|^2}\right). \tag{1}$$

Note that, it exists a constant $M \geq 0$ such that (see [2], Proposition 4.9.):

$$\|X'_\tau - X\| \leq M|\tau - t|.$$

This implies, by the Lipschitz property of f:

$$\| \int_t^{t+h} f(X'_\tau, \mathbf{u}(\tau,\cdot), X'_t\sharp\mathbb{P})\, d\tau - \int_t^{t+h} f(X', \mathbf{u}(\tau,\cdot), X'\sharp\mathbb{P})\, d\tau \|$$

$$\leq 2L \int_t^{t+h} \|X'_\tau - X\| \, d\tau \leq 2LM \int_t^{t+h} |\tau - t| \, d\tau = LMh^2.$$

So that:

$$\langle X'_{t+h} - X', Z' \rangle = \langle \int_t^{t+h} f(X'_\tau, \mathbf{u}(\tau,\cdot), X'_t \sharp \mathbb{P}) \, d\tau, Z' \rangle$$

$$\geq \langle \int_t^{t+h} f(X', \mathbf{u}(\tau,\cdot), X' \sharp \mathbb{P}) \, d\tau, Z' \rangle - LMh^2.$$

Now setting $m(\omega, x, z) = (Id, X', Z')\sharp\mathbb{P} = m^{x,z}(\omega) \otimes \gamma(x,z)$, as (X, Z) has the same law of (X', Z'), as f is affine in u:

$$\begin{aligned}
&\langle X'_{t+h} - X', Z' \rangle \geq \langle \int_t^{t+h} f(X', \mathbf{u}(\tau,\cdot), X'\sharp\mathbb{P}) \, d\tau, Z' \rangle - LMh^2 \\
&\geq -LMh^2 + \int_t^{t+h} \left[\int_\Omega f(X'(\omega), \mathbf{u}(\tau,\omega), X\sharp\mathbb{P}) \cdot Z'(\omega) \, d\mathbb{P}(\omega) \right] d\tau \\
&\geq -LMh^2 + \int_t^{t+h} \left[\int_{\Omega\times\mathbb{R}^{2d}} f(x, \mathbf{u}(\tau,\omega), X\sharp\mathbb{P}) \cdot z \, dm(\omega,x,z) \right] d\tau \\
&\geq -LMh^2 + \int_t^{t+h} \left[\int_{\mathbb{R}^{2d}} f(x, \int_\Omega \mathbf{u}(\tau,\omega) \; dm^{x,z}(\omega), x) \cdot z \, d\gamma(x,z) \right] d\tau \\
&\geq -LMh^2 + \int_t^{t+h} \left[\int_\Omega f(X, \mathbf{v}(\tau, X, Z), X\sharp\mathbb{P}) \cdot Z(\omega) \, d\mathbb{P}(\omega) \right] d\tau \\
&\geq -LMh^2 + h\overline{\mathrm{H}}_2(X,Z)
\end{aligned}$$

where, for almost every τ, we have set: $\mathbf{v}(\tau, x, z) = \displaystyle\int_\Omega \mathbf{u}(\tau,\omega) \; dm^{x,z}(\omega)$. Then, recalling (1), this leads to:

$$0 \geq p_t h - LMh^2 + h\overline{\mathrm{H}}_2(X,Z) + o\left(\sqrt{h^2 + \|X_{t+h} - X_t\|^2}\right).$$

Dividing by h and making h go to 0 gives the desired inequality. ☐

References

[1] L. Ambrosio, N. Gigli, G. Savaré, Gradient flows in metric spaces and in the space of probability measures, 2nd ed., Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.

[2] G. Cavagnari, S. Lisini, C. Orrieri and G. Savaré, *Lagragian, Eulerian and Kantorovitch formulations of multi-agent optimal control problems: equivalence and Gamma-convergence*, J. Differ. Equations 322, 268-364 (2022).

[3] C. Jimenez, *Equivalence between strict viscosity solution and viscosity solution in the Wasserstein space and regular extension of the Hamiltonian in* $L^2_{\mathbb{P}}$. J. Convex Anal. 31, No. 2, 619-670 (2024).

[4] C. Jimenez, A. Marigonda, M. Quincampoix *Optimal Control of Multiagent Systems in the Wasserstein Space*, (2020) Calculus of Variations and Partial Differential Equations.

[5] C. Jimenez, A. Marigonda, M. Quincampoix *Dynamical systems and Hamilton-Jacobi-Bellman equations on the Wasserstein space and their* L^2 *representations*, to be published in SIAM Journal of Mathematical Analysis (2023).

Chloé Jimenez: Univ Brest, CNRS UMR 6205,
Laboratoire de Mathématiques de Bretagne Atlantique,
6, avenue Victor Le Gorgeu, 29200 Brest, France.
E-mail address: `chloe.jimenez@univ-brest.fr`