LIPSCHITZ REGULARITY OF ALMOST-MINIMIZERS IN TWO-PHASE FREE BOUNDARY PROBLEMS WITH GENERALIZED ORLICZ GROWTH

CHIARA LEONE, GIOVANNI SCILLA, FRANCESCO SOLOMBRINO, AND ANNA VERDE

Abstract. Optimal local Lipschitz regularity for scalar almost minimizers of two-phase free-boundary functionals

$$\mathcal{F}(v;\Omega) := \int_\Omega \varphi(x,|\nabla v|) + \lambda_1\chi_{\{v<0\}} + \lambda_2\chi_{\{v>0\}} + \min\{\lambda_1,\lambda_2\}\chi_{\{v=0\}}\,\mathrm{d}x\,,$$

with growth function φ a generalized Orlicz function and λ_1, λ_2 nonnegative bounded functions, is established, assuming a "small" density for either the positivity or the negativity set.

Contents

1. Introduction and the main results

For a bounded open set $\Omega \subset \mathbb{R}^d$ ($d \geq 2$), we will deal with the regularity of local almost-minimizers of the functional

$$\mathcal{F}(v;\Omega) := \int_\Omega \varphi(x,|\nabla v|) + \lambda_1(x)\chi_{\{v<0\}} + \lambda_2(x)\chi_{\{v>0\}} + \min\{\lambda_1(x),\lambda_2(x)\}\chi_{\{v=0\}}\,\mathrm{d}x\,, \tag{1.1}$$

for $v \in W^{1,\varphi}(\Omega)$, where φ is a generalized Orlicz function.

As relevant examples of energies undergoing non-standard growth we report here the perturbed *Orlicz*, the so-called *variable exponent*, and the *double-phase* case

$$a(x)\varphi(|\xi|), \qquad |\xi|^{p(x)}, \qquad \text{and} \qquad |\xi|^p + a(x)|\xi|^q \qquad \text{for } (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

while an exhaustive list of examples can be found in [10]. We point out that the unified approach we develop embraces all the relevant examples in literature, provided the bulk energy has a growth from below with exponent $p > 1$, and needs not distinguish between sub- and superquadratic energies.

Variational problems involving functionals like (1.1) occur in many contexts, especially in modeling the flow of two liquids in jets and cavities (see, *e.g.*, [2]).

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Lipschitz regularity for minimizers of two-phase free boundary functionals has been obtained in some cases. For $\varphi(x,t) = t^2$ it was proven in the seminal paper by Alt-Caffarelli-Friedman [1], whose proof heavily relies on a monotonicity formula. To the best of our knowledge, Lipschitz regularity remains wide-open in two phase scenarios involving more general functionals, except in those cases where a variant of the ACF monotonicity formula exists (see [5], for instance, and [4] for a discussion about this issue). Trying to overcome the challenges posed by the lack of a monotonicity formula, a first result dealing with the p-Laplacian is due to Karakhanyan [11], where he showed the local Lipschitz continuity of minimizers under the assumption of a suitably small uniform upper bound on the Lebesgue density of the negative phase[1] along the free boundary. Later on, Dipierro and Karakhanyan [8] circumvented the lack of a monotonicity formula by a completely different geometric type argument (see also [3]). Since we are dealing with *almost* minimizers in a generalized Orlicz framework, we will still resort to the smallness assumption on the density of one of the phases, deferring to forthcoming contributions the compliance of the method in [8] with the situation at hand.

In the autonomous Orlicz setting, contributions concerning the regularity of minimizers of two-phase problems are, for instance, [4, 14], where again the smallness assumption on the density of the negative phase is exploited. In particular, in [4] the class of the admissible Orlicz functions is restricted by some additional structural assumption which ensures suitable compactness properties useful when the blow-up arguments are addressed. Our approach, which already works in the autonomous and variational setting, does not require additional assumptions on the Orlicz functions, as it relies on a lower semicontinuity argument when applying blow-up techniques. When dealing with almost minimizers, one has to face some technical difficulties, due to the fact that almost minimizers, unlike minimizers, do not satisfy a PDE or a monotonicity formula. As far as we know, the first result about regularity of almost minimizers of two-phase functionals in the prototype case $\varphi(t) = t^2$ is due to David and Toro in [7]. Their argument is mainly based on an *almost*-monotonicity formula, which overcomes the lack of the classical one used in [1]. Subsequently, their result has been extended to the variable-coefficients setting in [6], still in the quadratic growth case.

Assuming a suitable upper bound on the density of one of the phases, with this work we address the case of almost minimizers, which is not covered, to the best of our knowledge, by the previous literature even in the p-case. In particular, the generalized Orlicz case represents a novel contribution, as it encompasses a wide range of significant examples, both for minimizers and almost minimizers.

Description of our result. In order to introduce the main result of our paper, we formulate our problem and specify the definition of almost-minimizer. Given two bounded and nonnegative functions $\lambda_1, \lambda_2 : \Omega \to \mathbb{R}$, we define

$$\lambda := \max\{\|\lambda_1\|_\infty, \|\lambda_2\|_\infty\}\,. \tag{1.2}$$

For a bounded open set $\Omega \subset \mathbb{R}^d$ ($d \geq 2$), we will deal with local almost-minimizers of the functional

$$\mathcal{F}(v;\Omega) := \int_\Omega \varphi(x, |\nabla v(x)|) + \lambda_1(x)\chi_{\{v<0\}}(x) + \lambda_2(x)\chi_{\{v>0\}}(x) + \min\{\lambda_1(x), \lambda_2(x)\}\chi_{\{v=0\}}(x)\,\mathrm{d}x\,, \tag{1.3}$$

for $v \in W^{1,\varphi}(\Omega)$. The precise notion of almost minimizers that we use is the following.

[1] Indeed, as we are going to state, the same argument works if the minimum density between the two phases is suitably small.

Definition 1.1. *We say that $u : \Omega \to \mathbb{R}$ is a (local)* almost-minimizer *for $\mathcal{F}$ in Ω, with constant κ and exponent β, if*

$$\mathcal{F}(u; B_r(x_0)) \leq \left(1 + \kappa r^{\beta}\right) \mathcal{F}(w; B_r(x_0)),$$

for every ball $B_r(x_0)$ such that $\overline{B_r(x_0)} \subset \Omega$ and every $w \in W^{1,\varphi}(B_r(x_0))$ such that $u = w$ on $\partial B_r(x_0)$.

We denote by

$$F^{\pm}(u) := F^{+}(u) \cup F^{-}(u)\,,$$

where $F^{+}(u) := \partial\{u > 0\} \cap \Omega$ and $F^{-}(u) := \partial\{u < 0\} \cap \Omega$. Moreover, for $x_0 \in F^{\pm}(u)$ we consider the density function of the negativity set along the free boundaries for u; that is,

$$\Theta_u^{-}(x_0, r) := \frac{\mathcal{L}^d(\{u < 0\} \cap B_r(x_0))}{\mathcal{L}^d(B_r(x_0))}\,. \tag{1.4}$$

In a similar fashion, we define

$$\Theta_u^{+}(x_0, r) := \frac{\mathcal{L}^d(\{u > 0\} \cap B_r(x_0))}{\mathcal{L}^d(B_r(x_0))}\,.$$

The main result of the paper is the following.

Theorem 1.1 (Lipschitz regularity). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, and $\Omega' \Subset \Omega$. Let $\varphi \in \Phi_c(\Omega)$, $\varphi(x, \cdot) \in C^1([0, \infty))$ be satisfying* (VA1)*, and such that φ_t comply with* (A0), $(\mathrm{inc})_{p-1}$ *and* $(\mathrm{dec})_{q-1}$ *for some $1 < p \leq q$. Let $u : \Omega \to \mathbb{R}$ be an almost-minimizer of $\mathcal{F}$ in Ω, where $\mathcal{F}$ is defined as in* (1.3)*. Then, there exist $r_0 = r_0(d, p, q, L, \lambda, \kappa, \beta, \Omega', u) > 0$ and a constant $C = C(d, p, q, L, \lambda, \kappa, \beta) > 0$ with the following property: for every $0 < r \leq r_0$ and $B_r(x_0) \Subset \Omega$, u is Lipschitz continuous in $B_r(x_0)$ provided that*

$$\min\{\Theta_u^{+}(z, r), \Theta_u^{-}(z, r)\} \leq C \ \textit{ for every } \ z \in F^{\pm}(u) \cap B_{\frac{3r}{4}}(x_0)\,. \tag{1.5}$$

For proving this result, we may take advantage of the analysis of the one-phase case carried out in [13]. Some results therein can be directly transferred to the case at hand (see, for instance, Theorem 3.4 and Theorem 3.5). Hence, in our discussion we mainly focus on the parts where the two-phase scenario needs additional work, which are mainly contained in Section 3.3.

Outline of the paper. The rest of the paper is organized as follows. In Section 2 we fix the basic notation and recall some basic facts about Orlicz and generalized Orlicz functions. In Sections 3.1 and 3.2, we recall some ingredients in order to get the main result. Namely, we discuss Caccioppoli type estimates and higher integrability results for almost minimizers, as well the local Hölder continuity of almost minimizers, Theorem 3.4, and that of their gradients away from the free boundary, Theorem 3.5. Section 3.3 is entirely devoted to the proof of the main result: the main steps are Lemma 3.6, where an interior uniform bound for the gradient of an almost minimizer is provided; Proposition 3.8, showing that a suitable blow-up sequence of almost minimizers converges to the solution of a limit autonomous problem, and its consequence Proposition 3.9. This is the point where the role of the assumption (1.5) is apparent. Together with the blow-up argument, it allows us for proving the sublinearity of a bounded almost minimizer in a neighborhood of a free-boundary point.

2. Basic notation and preliminaries

We begin with some basic notation. Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set. For every $x \in \mathbb{R}^d$ and $r > 0$, we denote by $B_r(x) \subset \mathbb{R}^d$ the open ball centered at x with radius r. We will often use the shorthand B_r when either $x = 0$ or the center x is not relevant.

For $x \in \mathbb{R}^d$, $|x|$ represents the Euclidean norm of x. The m-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^m$ is denoted by γ_m for each $m \in \mathbb{N}$. The d-dimensional Lebesgue measure is denoted by $\mathcal{L}^d$. The closure of a set A is denoted by $\overline{A}$. The characteristic function of any set $A \subset \mathbb{R}^d$ is denoted by χ_A, which takes the value 1 on A and 0 elsewhere.

Given two functions $f, g : [0, +\infty) \to \mathbb{R}$, we write $f \sim g$ to indicate that f and g are equivalent, meaning there exist constants $c_1, c_2 > 0$ such that for all $t \geq 0$,

$$c_1 g(t) \leq f(t) \leq c_2 g(t).$$

The symbol $\lesssim$ is used to denote $\leq$ up to a constant. The space of measurable functions on Ω is denoted by $L^0(\Omega)$.

2.1. **Generalized Φ-functions and Orlicz spaces.** We provide some basic definitions and results concerning generalized Φ-functions and Orlicz spaces, focusing solely on the concepts that will be relevant to our discussion. For a more detailed and comprehensive treatment of the subject, we refer the reader to [9].

Definition 2.1. *Let $\varphi : [0, +\infty) \to [0, +\infty]$ be increasing with $\varphi(0) = 0$, $\lim_{t\to 0^+} \varphi(t) = 0$ and $\lim_{t\to+\infty} \varphi(t) = +\infty$. Such φ is called a*

(i) weak Φ-function *if $\frac{\varphi(t)}{t}$ is* almost increasing, *meaning that there exists $L \geq 1$ such that $\frac{\varphi(t)}{t} \leq L\frac{\varphi(s)}{s}$ for $0 < t \leq s$.*
(ii) convex Φ-function *if φ is left-continuous and convex.*

By virtue of Remark 2.1 below, each convex Φ-function is a weak Φ-function. If φ is a convex Φ-function, then there exists φ' the *right derivative* of φ, which is non-decreasing and right-continuous, and such that

$$\varphi(t) = \int_0^t \varphi'(s)\,\mathrm{d}s\,.$$

A special subclass of convex Φ-functions is represented by the N-functions (see, e.g., [12, Ch.I]).

Definition 2.2. *A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be an* N-function *if it admits the representation*

$$\varphi(t) = \int_0^t a(\tau)\,\mathrm{d}\tau$$

where $a(s)$ is right-continuous, non-decreasing for $s > 0$, $a(s) > 0$ for $s > 0$ and satisfies the conditions

$$a(0) = 0\,, \quad \lim_{s\to+\infty} a(s) = +\infty\,. \tag{2.1}$$

The function $a(t)$ is simply the right-hand derivative of $\varphi(t)$. As an immediate consequence of the definition, we observe that an N-function φ is continuous, satisfies $\varphi(0) = 0$, and is increasing. Furthermore, φ is a convex function and, as noted in Remark 2.1 below, it satisfies condition $(\mathrm{inc})_1$. Conditions (2.1) imply that

$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} = 0, \quad \lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty.$$

For our purposes, we need functions φ to depend also on the spatial variable x.

Definition 2.3. *Let $\varphi : \Omega \times [0, \infty) \to [0, \infty]$. We call φ a generalized weak Φ-function (resp., convex Φ-function, N-function) if*

(1) $x \mapsto \varphi(x, |f(x)|)$ is measurable for every $f \in L^0(\Omega)$;
(2) $t \mapsto \varphi(x, t)$ is a weak Φ-function (resp., a convex Φ-function, an N-function) for every $x \in \Omega$.

We write $\varphi \in \Phi_w(\Omega)$, $\varphi \in \Phi_c(\Omega)$ *and* $\varphi \in N(\Omega)$, *respectively. If* φ *does not depend on* x, *we will adopt the shorthands* $\varphi \in \Phi_w$, $\varphi \in \Phi_c$ *and* $\varphi \in N$, *respectively. For the right-derivative of a generalized convex* Φ*-function, we will use the notation* φ_t *in place of* φ'.

For a bounded function $\varphi : \Omega \times [0, +\infty) \to [0, +\infty)$ and a ball $B_r(x_0) \subset \Omega$ we define, for every $t \geq 0$,

$$\varphi^-_{r,x_0}(t) := \inf_{x \in B_r(x_0)} \varphi(x,t) \quad \text{and} \quad \varphi^+_{r,x_0}(t) := \sup_{x \in B_r(x_0)} \varphi(x,t). \tag{2.2}$$

Following the terminology of [9, 10], we give the following definitions. The first three ones concern with the regularity of φ with respect to the t- variable, (A1) imposes a bound on how much φ can change between nearby points, while the last one, (VA1), is a continuity assumption with respect to the spatial variable x. Note that (VA1) implies (A1), see [10, Remark 4.2].

Definition 2.4. *Let* $p, q > 0$. *A function* $\varphi : \Omega \times [0, +\infty) \to [0, +\infty)$ *satisfies*

- $(\mathrm{inc})_p$ *if* $t \in (0, +\infty) \mapsto \frac{\varphi(x,t)}{t^p}$ *is increasing for every* $x \in \Omega$
- $(\mathrm{dec})_q$ *if* $t \in (0, +\infty) \mapsto \frac{\varphi(x,t)}{t^q}$ *is decreasing for every* $x \in \Omega$
- (A0) *if there exists* $L \geq 1$ *such that* $\frac{1}{L} \leq \varphi(x, 1) \leq L$ *for every* $x \in \Omega$
- (A1) *if there exists* $L \geq 1$ *such that, for any ball* $B_r(x_0) \subset \Omega$,

$$\varphi^+_{r,x_0}(t) \leq L\, \varphi^-_{r,x_0}(t), \quad \forall t > 0 \ \text{ such that } \ \varphi^-_{r,x_0}(t) \in \left[1, \frac{1}{\mathcal{L}^d(B_r(x_0))}\right].$$

- (VA1) *if there exists an increasing continuous function* $\omega : [0, +\infty) \to [0, 1]$ *with* $\omega(0) = 0$ *such that, for any ball* $B_r(x_0) \subset \Omega$,

$$\varphi^+_{r,x_0}(t) \leq (1 + \omega(r))\varphi^-_{r,x_0}(t), \quad \forall t > 0 \ \text{ such that } \ \varphi^-_{r,x_0}(t) \in \left[\omega(r), \frac{1}{\mathcal{L}^d(B_r(x_0))}\right].$$

Remark 2.1. *If* $\varphi : \Omega \times [0, +\infty) \to [0, +\infty)$ *is convex and* $\varphi(x, 0) = 0$ *for every* $x \in \Omega$, *then* φ *satisfies* $(\mathrm{inc})_1$. *If* φ *satisfies* $(\mathrm{inc})_{p_1}$, *then it satisfies* $(\mathrm{inc})_{p_2}$ *for every* $0 < p_2 \leq p_1$. *If* φ *satisfies* $(\mathrm{dec})_{q_1}$, *then it satisfies* $(\mathrm{dec})_{q_2}$ *for every* $q_2 \geq q_1$.

The following simple results can be found in [13, Proposition 2.4].

Proposition 2.2. *Let* $1 < p \leq q < +\infty$ *and* $\varphi \in \Phi_c(\Omega)$ *with right derivative* φ_t. *Assume that* φ_t *satisfies* $(\mathrm{inc})_{p-1}$ *and* $(\mathrm{dec})_{q-1}$. *Then*

(*i*) φ *satisfies* $(\mathrm{inc})_p$ *and* $(\mathrm{dec})_q$, *and the following estimate hold:*

$$\varphi(x,s) \min\{t^p, t^q\} \leq \varphi(x, ts) \leq \max\{t^p, t^q\}\varphi(x,s), \quad \forall x \in \Omega,\ \forall s, t \in [0, +\infty).$$

(ii) $\varphi(x,t)$ *and* $t\varphi_t(x,t)$ *are equivalent, in the sense that*

$$p\,\varphi(x,t) \leq t\,\varphi_t(x,t) \leq q\,\varphi(x,t), \quad \forall (x,t) \in \Omega \times [0, +\infty);$$

(iii) if, in addition, φ_t *complies with* (A0), *then also* φ *does with constants depending on* L, p, q. *More precisely,*

$$\frac{1}{Lq} \leq \varphi(x, 1) \leq \frac{L}{p}, \quad \forall x \in \Omega. \tag{2.3}$$

If, in addition, $\varphi(x, \cdot) \in C^1([0, +\infty))$ *for every* $x \in \Omega$, *then* $\varphi \in N(\Omega)$.

For $\varphi \in \Phi_w(\Omega)$, the *generalized Orlicz space* is defined by

$$L^\varphi(\Omega) := \left\{ f \in L^0(\Omega) : \|f\|_{L^\varphi(\Omega)} < \infty \right\}$$

with the (Luxemburg) norm

$$\|f\|_{L^{\varphi}(\Omega)} := \inf\left\{\lambda > 0 : \varrho_{\varphi}\Big(\frac{f}{\lambda}\Big) \le 1\right\}, \quad \text{where } \varrho_{\varphi}(f) := \int_{\Omega} \varphi(x, |f(x)|)\,\mathrm{d}x.$$

We denote by $W^{1,\varphi}(\Omega)$ the set of $f \in L^{\varphi}(\Omega)$ satisfying that $\partial_1 f, \dots, \partial_d f \in L^{\varphi}(\Omega)$, where $\partial_i f$ is the weak derivative of f in the x_i-direction, with the norm $\|f\|_{W^{1,\varphi}(\Omega)} := \|f\|_{L^{\varphi}(\Omega)} + \sum_i \|\partial_i f\|_{L^{\varphi}(\Omega)}$. Note that if φ satisfies $(\mathrm{dec})_q$ for some $q \ge 1$, then $f \in L^{\varphi}(\Omega)$ if and only if $\varrho_{\varphi}(f) < \infty$, and if φ satisfies (A0), $(\mathrm{inc})_p$ and $(\mathrm{dec})_q$ for some $1 < p \le q$, then $L^{\varphi}(\Omega)$ and $W^{1,\varphi}(\Omega)$ are reflexive Banach spaces. In addition we denote by $W^{1,\varphi}_0(\Omega)$ the closure of $C^{\infty}_0(\Omega)$ in $W^{1,\varphi}(\Omega)$.

3. Local Lipschitz regularity of almost minimizers

3.1. **Preliminary regularity results for almost-minimizers.** We start by stating a Caccioppoli-type inequality for almost minimizers of $\mathcal{F}$. The proof follows along the lines of [13, Lemma 3.2], so we omit further details.

Lemma 3.1. *Let $\varphi \in \Phi_c(\Omega)$ be such that $(\mathrm{dec})_q$ holds for some $q > 0$. Let u be an almost-minimizer of $\mathcal{F}$ in Ω, with constant $\kappa \le \kappa_0$ and exponent β, and let $x_0 \in \Omega$ and $B_{2r}(x_0) \Subset \Omega$, with $2r \le 1$. Then there exists a constant $c = c(q, \kappa_0)$ such that*

$$\fint_{B_r(x_0)} \varphi(x, |\nabla u|)\,\mathrm{d}x \le c\left(\fint_{B_{2r}(x_0)} \varphi\left(x, \frac{|u - (u)_{x_0,2r}|}{2r}\right)\mathrm{d}x + \lambda\right). \tag{3.1}$$

Also the following lemma, which contains a higher integrability result and reverse Hölder type estimates for the gradient of an almost minimizer of $\mathcal{F}$, can be proved exactly as in [13, Lemma 3.3].

Lemma 3.2. *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0), (A1), $(\mathrm{inc})_p$, $(\mathrm{dec})_q$ with constant $L \ge 1$ and $1 < p \le q$. Let $u \in W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ be an almost-minimizer of $\mathcal{F}$ in Ω with constant $\kappa \le \kappa_0$ and exponent β, and let $x_0 \in \Omega$ and $B_{2r}(x_0) \Subset \Omega$, with $\|\nabla u\|_{L^{\varphi}(B_{2r}(x_0))} \le 1$, and $2r \le 1$. Then*

(i) (Higher integrability) there exist $s_0 = s_0(d, p, q, L) > 0$ and $c = c(d, p, q, L, \kappa_0) \ge 1$ such that

$$\left(\fint_{B_r(x_0)} \varphi(x, |\nabla u|)^{1+s_0}\,\mathrm{d}x\right)^{\frac{1}{1+s_0}} \le c2^{\frac{ds_0}{1+s_0}}\delta^{-\frac{ds_0}{1+s_0}}\left(\fint_{B_{(1+\delta)r}(x_0)} \varphi(x, |\nabla u|)\,\mathrm{d}x + \Lambda\right), \tag{3.2}$$

for any $\delta \in (0, 1]$, where $\Lambda := \lambda + 1$. In particular, this implies $\varphi(\cdot, |\nabla u|) \in L^{1+s_0}_{\mathrm{loc}}(\Omega)$.

(ii) (Reverse Hölder type estimates) for every $t \in (0, 1]$, there exist $c_t = c_t(d, p, q, L, \kappa_0, t) > 0$ such that

$$\left(\fint_{B_r(x_0)} \varphi(x, |\nabla u|)^{1+s_0}\,\mathrm{d}x\right)^{\frac{1}{1+s_0}} \le c_t\left(\left(\fint_{B_{2r}(x_0)} \varphi(x, |\nabla u|)^t\,\mathrm{d}x\right)^{\frac{1}{t}} + \Lambda\right). \tag{3.3}$$

Note that, under our assumption on φ, it holds that

$$\|\nabla u\|_{L^{\varphi}(B_{2r}(x_0))} \le 1 \iff \int_{B_{2r}(x_0)} \varphi(x, |\nabla u|)\,\mathrm{d}x \le 1$$

(it is sufficient that $\varphi \in \Phi_w(\Omega)$ and $\varphi(x, \cdot)$ be left-continuous; see, e.g., [9, Lemma 3.2.3]).

Remark 3.3. *As noted in* [13, Remark 3.4], *the choice of radii can be done in order to exploit the estimates of Lemma 3.2. More precisely, if $L \geq 1$ is that of condition* (A0) *and s_0 is the exponent of Lemma 3.2,*
(i) with fixed $\Omega' \Subset \Omega$, there exists $r_0 \in (0,1)$, $r_0 = r_0(d, L, \omega(\cdot), \|\varphi(\cdot, |\nabla u|)^{1+s_0}\|_{L^1(\Omega')})$ satisfying

$$r_0 \leq \frac{1}{2}, \quad \omega(2r_0) \leq \frac{1}{L}, \quad \mathcal{L}^d(B_{2r_0}) \leq \min\left\{\frac{1}{2L}, 2^{-\frac{2(1+s_0)}{s_0}}\left(\int_{\Omega'} \varphi(x, |\nabla u|)^{1+s_0}\,\mathrm{d}x\right)^{-\frac{2+s_0}{s_0}}\right\}, \tag{3.4}$$

and such that for any $B_{2r}(x_0) \subset \Omega'$ with $r \in (0, r_0]$, it holds that

$$\int_{B_{2r}(x_0)} \varphi(x, |\nabla u|)\,\mathrm{d}x \leq 1\,.$$

(ii) If, in addition, $u \in L^\infty_{\mathrm{loc}}(\Omega)$, the choice of s_0 and, accordingly, of r_0 can be done in such a way that the dependence of r_0 on u is through $\|u\|_{L^\infty(\Omega')}$.

3.2. **Local Hölder continuity.** In this section, we recall two main regularity results for almost minimizers, which will be instrumental in the proof of the local Lipschitz continuity result. The first concerns the $C^{0,\alpha}$-regularity of any almost-minimizer u for $\mathcal{F}$, locally within Ω, for *any* exponent $\alpha \in (0,1)$. It has been proved for the one-phase scenario in [13, Theorem 3.7] and the proof is *verbatim* the same in our case, as it only relies on the aforementioned Caccioppoli and higher integrability estimates, together with the properties of φ. Later, we recall that ∇u is locally $C^{0,\alpha}$ for *some* exponent $\alpha \in (0,1)$ away from the free-boundary $F^\pm(u)$. Since the open sets considered in Theorem 3.5 are assumed not to meet the free-boundary, it holds $u \neq 0$ in the whole set, and up to discussing along the same lines the additional case $u < 0$ with respect to [13, Theorem 3.8], no change in the proof strategy is needed

This is the precise statement of the $C^{0,\alpha}$-regularity result.

Theorem 3.4 ($C^{0,\alpha}$-regularity). *Let $\varphi \in \Phi_c(\Omega)$, $\varphi(x,\cdot) \in C^1([0,\infty))$ be satisfying* (VA1), *and such that φ_t comply with* (A0), $(\mathrm{inc})_{p-1}$ *and* $(\mathrm{dec})_{q-1}$ *for some $1 < p \leq q$. If $u \in W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ is an almost minimizer of $\mathcal{F}$ with constant $\kappa \leq \kappa_0$ and exponent β, then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for any $\alpha \in (0,1)$. More precisely, if $\Omega' \Subset \Omega$ is fixed, there exists $0 < R_0 < \frac{1}{2}\mathrm{dist}(\Omega', \partial\Omega)$ complying with* (3.4) *(thus depending on u) and a constant $c = c(d, p, q, \alpha, \kappa_0, R_0, \Omega')$ such that*

$$[u]_{C^\alpha(\Omega')} \leq c\left(\int_\Omega |\nabla u|\,\mathrm{d}x + \Lambda\right). \tag{3.5}$$

The following result establishes the local Hölder continuity of the gradient of almost-minimizers, away from the free boundary.

Theorem 3.5 ($C^{1,\alpha}$-regularity in $\{u \neq 0\}$). *Let $\varphi \in \Phi_c(\Omega)$, $\varphi(x,\cdot) \in C^1([0,\infty))$ be such that φ_t comply with* (A0), $(\mathrm{inc})_{p-1}$ *and* $(\mathrm{dec})_{q-1}$ *for some $1 < p \leq q$. If $u \in W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ is an almost minimizer of $\mathcal{F}$ with constant $\kappa \leq \kappa_0$ and exponent β and φ satisfies* (VA1) *with*

$$\omega(r) \leq cr^\theta, \quad \text{for all } r \in (0,1] \text{ and for some } \theta \in (0,1), \tag{3.6}$$

then $u \in C^{1,\bar{\alpha}}_{\mathrm{loc}}(\{u \neq 0\})$ for some $\bar{\alpha} \in (0,1)$ depending on $d, p, q, L, \beta, \theta$. More precisely, for any $\Omega' \Subset \{u \neq 0\} \cap \Omega$, there exists an exponent $\bar{\alpha} = \bar{\alpha}(d, p, q, L, \beta, \theta)$ and a constant $C = C(\Omega', d, p, q, L, \kappa_0, \beta, \theta)$ such that

$$[\nabla u]_{C^{\bar{\alpha}}(\Omega')} \leq C\left(\int_\Omega |\nabla u|\,\mathrm{d}x + \Lambda\right). \tag{3.7}$$

3.3. **Proof of the local Lipschitz continuity.**

Lemma 3.6. *Under the assumptions of Theorem 3.5, let u be a bounded almost minimizer of $\mathcal{F}$ in $B_1(0)$ with constant $\kappa \le \kappa_0$ and exponent β. Assume that $u(x) \neq 0$ for all $x \in B_1(0)$. Then*

$$|\nabla u(0)| \le C\,, \tag{3.8}$$

where the constant C depends on $p, q, d, L, \kappa_0, \beta, \theta, \Lambda, \|u\|_{L^\infty(B_1(0))}$.

Proof. We have

$$|\nabla u(0)| \le |\nabla u(0) - (\nabla u)_{B_{\frac14}(0)}| + |(\nabla u)_{B_{\frac14}(0)}|\,. \tag{3.9}$$

Now, the first term on the right hand side above can be estimated by Theorem 3.5: observe that, since we are assuming u to be bounded, by Remark 3.3, (ii) the constant appearing there depends on $p, q, d, L, \kappa_0, \beta, \theta, \Lambda, \|u\|_{L^\infty(B_1(0))}$. We then have

$$\begin{aligned} |\nabla u(0) - (\nabla u)_{B_{\frac14}(0)}| &\le \fint_{B_{\frac14}(0)} |\nabla u(0) - \nabla u(x)|\,\mathrm{d}x \\ &\le 4^{-\alpha}[\nabla u]_{C^\alpha(B_{\frac14}(0))} \\ &\le C\left(\int_{B_{\frac12}(0)} |\nabla u|\,\mathrm{d}x + \Lambda\right), \end{aligned} \tag{3.10}$$

while

$$|(\nabla u)_{B_{\frac14}(0)}| \le \mathcal{L}^d(B_{\frac14})\left(\int_{B_{\frac12}(0)} |\nabla u|\,\mathrm{d}x + \Lambda\right). \tag{3.11}$$

Further, from the Caccioppoli inequality Lemma 3.1, (2.3), $(\mathrm{inc})_1$ and the boundedness of u on $B_1(0)$, we get

$$\begin{aligned} \int_{B_{\frac12}(0)} |\nabla u|\,\mathrm{d}x \le Lq \int_{B_{\frac12}(0)} (\varphi(x, |\nabla u|) + 1)\,\mathrm{d}x &\le c\int_{B_1(0)} \left(\varphi(x, 2\|u\|_{L^\infty(B_1(0))}) + \Lambda\right)\mathrm{d}x \\ &\le c\left(\max\left\{\|u\|^p_{L^\infty(B_1(0))}, \|u\|^q_{L^\infty(B_1(0))}\right\} + \Lambda\right). \end{aligned} \tag{3.12}$$

Combining (3.9)–(3.12) we obtain (3.8), and this concludes the proof. □

The next lemma concerns the asymptotic properties for suitable blow-ups of the sequence $\varphi(x,t)$. It is proved in [13, Lemma 3.10 and Appendix A].

Lemma 3.7. *Let $R > 0$ be such that $B_{2R}(0) \Subset \Omega$, and $(r_j)_{j\in\mathbb{N}}$, $(\sigma_j)_{j\in\mathbb{N}}$ be sequences of nonnegative numbers, with $R < \frac{1}{2r_j}$ for every j, $r_j \to 0$ as $j \to +\infty$, and*

$$\sigma_j \to +\infty \quad \textit{and} \quad \varphi(0,\sigma_j)r_j \to 0 \quad \textit{as } j \to +\infty. \tag{3.13}$$

We define, for every j,

$$\varphi_j(x,t) := \frac{\varphi(r_j x, \sigma_j t)}{\varphi(0,\sigma_j)}\,,\ x \in B_{2R}(0)\,,\ t > 0\,. \tag{3.14}$$

Then,

(i) the functions

$$\varphi_j^-(t) := \inf_{y\in B_{2R}(0)} \varphi_j(y,t)\,, \quad \textit{and} \quad \varphi_j^+(t) := \sup_{y\in B_{2R}(0)} \varphi_j(y,t)\,, \tag{3.15}$$

are weak Φ *functions satisfying* $(\mathrm{inc})_p$ *and* $(\mathrm{dec})_q$*. Moreover, for* j *large enough,*

$$\min\{t^p,t^q\} \lesssim \varphi_j^-(t) \le \max\{t^p,t^q\}, \quad \min\{t^p,t^q\} \le \varphi_j^+(t) \lesssim \max\{t^p,t^q\}, \tag{3.16}$$

where the hidden constants are independent of j*;*

(ii) there exists $j_0 \in \mathbb{N}$ *such that* φ_j *complies with* (A0) *for* $j \ge j_0$ *with* $L = 2$*;*

(iii) there exists $j_0 \in \mathbb{N}$ *such that* φ_j *complies with* (VA1) *for* $j \ge j_0$ *with the same* ω*;*

(iv) there exists a convex function $\varphi_\infty \in C^1([0,+\infty))$*, whose derivative* φ_∞' *complies with* $(\mathrm{inc})_{p-1}$ *and* $(\mathrm{dec})_{q-1}$ *such that*

$$\varphi_j(x,t) \to \varphi_\infty(t) \ \ \textit{uniformly on } B_{2R}(0)\times K, \textit{ where } K \subset [0,+\infty) \textit{ is compact.} \tag{3.17}$$

(v) Let $(v_j)_{j\in\mathbb{N}} \subset W^{1,1}(B_R(0))$ *be such that*

$$\sup_{j\in\mathbb{N}} \int_{B_R(0)} \varphi_j(y,|\nabla v_j|)\,\mathrm{d}y \le C\,,$$

and $v_j \to v_0$ *a.e. in* $B_R(0)$*. Then*

$$\int_{B_R} \varphi_\infty(|\nabla v_0|)\,\mathrm{d}y \le \liminf_{j\to+\infty} \int_{B_R} \varphi_j(y,|\nabla v_j|)\,\mathrm{d}y. \tag{3.18}$$

Let $(\varphi_j)_{j\in\mathbb{N}}$ be the sequence defined in (3.14) and, correspondingly, consider the scaled functions $\lambda_i^j(x) := \frac{\lambda_i(r_jx)}{\varphi(0,\sigma_j)}$, $i = 1,2$, and the scaled functional

$$\begin{aligned}\hat{\mathfrak{F}}_j(v,\Omega) := &\int_\Omega \varphi_j(x,|\nabla v|)\,\mathrm{d}x \\ &+ \int_\Omega \lambda_1^j(x)\chi_{\{v<0\}}(x) + \lambda_2^j(x)\chi_{\{v>0\}}(x) + \max\{\lambda_1^j(x),\lambda_2^j(x)\}\chi_{\{v=0\}}(x)\,\mathrm{d}x\,.\end{aligned} \tag{3.19}$$

With given u, we also consider for every j the blow-up function

$$v_j(x) := \frac{u(r_jx)}{\sigma_j r_j}\,, \quad x \in B_{2R}(0)\,. \tag{3.20}$$

We then have the following result about the asymptotic behavior of a blow-up sequence defined by scaling an almost minimizer of $\mathcal{F}$.

Proposition 3.8. *Let* R*,* $(r_j)_{j\in\mathbb{N}}$*,* $(\sigma_j)_{j\in\mathbb{N}}$*,* $(\varphi_j)_{j\in\mathbb{N}}$ *and* φ_∞ *be as in Lemma 3.7. Let* u *be a bounded almost minimizer of* $\mathcal{F}$ *in* $B_2(0)$ *with constant* $\kappa \le \kappa_0$ *and exponent* β*. Then, for every* j*, the function* v_j *defined in* (3.20) *is an almost minimizer of the scaled functional* $\hat{\mathfrak{F}}_j$ (3.19) *in* $B_{2R}(0)$*, with constant* $\hat\kappa := \kappa r_j^\beta$ *and the same exponent* β*. Moreover, if* $\|v_j\|_{L^\infty(B_{2R}(0))} \le M$*, there exists* $v_\infty \in W^{1,1}(B_R(0))$ *such that, up to a subsequence,* $v_j \rightharpoonup v_\infty$ *weakly in* $W^{1,p}(B_R(0))$*, and uniformly in* $B_R(0)$*, and* v_∞ *is* φ_∞*-harmonic in* $B_R(0)$*.*

Proof. Let $B_\rho(x_0)$ be a ball such that $\overline{B_\rho(x_0)} \subset B_{\frac{1}{r_j}}(0)$, and $w \in W^{1,\varphi}(B_\rho(x_0))$ such that $w = v_j$ on $\partial B_\rho(x_0)$. Setting $y_0 := r_jx_0$, we then have

$$u(y) = \sigma_j r_j w(\tfrac{y}{r_j}) =: \widetilde{w}_j(y)\,, \quad \text{on } \partial B_{r_j\rho}(y_0)$$

and, by the almost minimality of u, we get

$$\mathcal{F}(u,B_{r_j\rho}(y_0)) \le (1+\kappa(r_j\rho)^\beta)\mathcal{F}(\widetilde{w}_j,B_{r_j\rho}(y_0))$$

Now, set $\hat{\kappa}_j := (\kappa r_j)^\beta$. Multiplying both the sides of the inequality by $\frac{1}{\varphi(0,\sigma_j)}$, with the change of variables $x = \frac{y}{r_j}$, and recalling the definition of φ_j we have

$$\hat{\mathfrak{F}}_j(v_j, B_\rho(x_0)) \leq (1 + \hat{\kappa}_j \rho^\beta)\hat{\mathfrak{F}}_j(w_j, B_\rho(x_0))$$

that is, v_j is an almost minimizer of the functional $\hat{\mathfrak{F}}_j$ defined in (3.19).

Now, we notice that applying Lemma 3.1 to φ_j, and observing that $\hat{\kappa}_j \leq \kappa$ for all j, we obtain for v_j the Caccioppoli-type estimate

$$\int_{B_\rho(y)} \varphi_j(x, |\nabla v_j|)\, \mathrm{d}x \leq c\left(\int_{B_{2\rho}(y)} \varphi_j\left(x, \frac{|v_j - (v_j)_{y,2\rho}|}{2\rho}\right) \mathrm{d}x + \frac{\lambda}{\varphi(0,\sigma_j)}\rho^d\right) \tag{3.21}$$

for any $B_{2\rho}(y) \Subset B_{2R}(0)$, where the constant c only depends on d, p, q, κ, β, and is a uniform constant with respect to j.

Recall that $\|v_j\|_{L^\infty(B_{2R}(0))} \leq M$. By (3.16), φ_j satisfy (inc)$_p$ and (dec)$_q$, with constants independent of j. They also satisfy (A0) and (VA1) with L and ω independent of j, by Lemma 3.7. Then, the radius r_0 of Remark 3.3(ii) can be chosen independently of j. It follows that, applying Proposition 3.4 to each v_j, each of them is locally α-Hölder continuous on $B_{2R}(0)$, and the $C^{0,\alpha}$-estimate (3.5) on $B_R(0)$, holds with a uniform bound not depending on j.

Since φ_j^- is (inc)$_p$, combining the bounds $\|v_j\|_{L^\infty(B_{2R}(0))} \leq M$, (3.21) and (3.16) we obtain

$$\sup_{j\geq j_0} \int_{B_\rho(0)} |\nabla v_j|^p\, \mathrm{d}x \leq \sup_{j\geq j_0} \int_{B_\rho(0)} \left(\frac{\varphi_j^-(|\nabla v_j|)}{\varphi_j^-(1)} + 1\right) \mathrm{d}x \leq C\,, \tag{3.22}$$

for a constant C depending on $d, p, q, L, \kappa_0, M, R, \lambda$. Hence the above inequality also holds for $\rho = R$, whence we infer the existence of a function $v_\infty \in W^{1,p}(B_R(0))$ such that, up to a subsequence,

$$v_j \rightharpoonup v_\infty \quad \text{weakly in } W^{1,p}(B_R(0)). \tag{3.23}$$

By (3.5) on $B_R(0)$, 3.23 also gives

$$v_j \to v_\infty \quad \text{uniformly in } B_R(0)$$

since the sequence (v_j) is equibounded by M on $B_R(0)$.

So we are left to prove that v_∞ is φ_∞-harmonic in $B_R(0)$. We first notice that using (3.21) for $\frac{R}{2} < \rho < R$, exploiting the uniform bound $\|v_j\|_{L^\infty(B_{2R}(0))} \leq M$ and letting $\rho \to R$, we also get that the sequence of positive measures $\mu_j := \varphi_j(\cdot, |\nabla v_j|)\,\mathcal{L}^d$ is equibounded on $B_R(0)$. Thus, we can find a Radon measure μ on $B_R(0)$ such that

$$\mu_j \rightharpoonup^* \mu \quad \text{on } B_R(0)$$

up to a subsequence (not relabeled).

Let us fix $w \in W^{1,\varphi_\infty}(B_R(0))$ be such that $\{w \neq v_\infty\} \Subset B_R(0)$. Since φ_∞ satisfies (dec)$_q$, we can find a sequence $(w^\varepsilon)_{\varepsilon>0} \subset W^{1,\infty}(B_R(0))$ of regularizations of w, strongly converging to w in $W^{1,\varphi_\infty}(B_R(0))$ as $\varepsilon \to 0$ (see, e.g., [9, Lemma 6.4.5]).

Let $\rho < \rho' \in (0, R)$, with $\mu(\partial B_{\rho'}) = \mu(\partial B_\rho) = 0$ and $\{w \neq v_\infty\} \Subset B_\rho$. Let $\eta \in C_c^\infty(B_{\rho'})$ be such that $\eta = 1$ on B_ρ, $0 \leq \eta \leq 1$, $|\nabla\eta| \leq \frac{2}{\rho'-\rho}$, and define $\zeta_j = \eta w^\varepsilon + (1-\eta)v_j$. Since $\{\zeta_j \neq v_j\} \Subset B_{\rho'}$, using the almost minimality of v_j, straightforward computations lead to

$$\begin{aligned}
&\int_{B_{\rho'}} \varphi_j(x, |\nabla v_j|)\,\mathrm{d}x \le (1+\kappa(r_j\rho')^\beta)\hat{\mathfrak{F}}_j(\zeta_j, B_{\rho'}) \\
&\le \int_{B_\rho} \varphi_j(x, |\nabla w^\varepsilon|)\,\mathrm{d}x + c(1+\kappa(r_j\rho')^\beta)\int_{B_{\rho'}\setminus B_\rho}\left(\varphi_j(x,|\nabla v_j|)+\varphi_j(x,|\nabla w^\varepsilon|)+\varphi_j\left(x, \frac{|w^\varepsilon - v_j|}{\rho'-\rho}\right)\right)\mathrm{d}x \\
&\quad + \frac{\lambda}{\varphi(0,\sigma_j)}\mathcal{L}^d(B_\rho) + \kappa(r_j\rho')^\beta\hat{\mathfrak{F}}_j(w^\varepsilon, B_\rho) + (1+\kappa(r_j\rho')^\beta)\frac{\lambda}{\varphi(0,\sigma_j)}\,\mathcal{L}^d(B_{\rho'}\setminus B_\rho)
\end{aligned} \tag{3.24}$$

for a suitable constant $c \ge 1$ depending only on L and p, q.

First, we note that

$$\limsup_{j\to+\infty}\int_{B_{\rho'}\setminus B_\rho}\varphi_j(x,|\nabla v_j|)\,\mathrm{d}x \le \mu(B_{\rho'}\setminus B_\rho) \quad \text{and} \quad \hat{\mathfrak{F}}_j(w^\varepsilon, B_\rho) \le C\mathcal{L}^d(B_\rho)$$

for j sufficiently large, and

$$\frac{\lambda}{\varphi(0,\sigma_j)}\mathcal{L}^d(B_{\rho'}) + \kappa(r_j\rho')^\beta\hat{\mathfrak{F}}_j(w^\varepsilon, B_\rho) + (1+\kappa(r_j\rho')^\beta)\frac{\lambda}{\varphi(0,\sigma_j)}\,\mathcal{L}^d(B_{\rho'}\setminus B_\rho) \to 0 \tag{3.25}$$

as $j\to+\infty$, for fixed ρ, ρ', ε.

Now we deal with the convergence of the integral terms above. Using the uniform convergence (3.17) we have that

$$\lim_{j\to+\infty}\int_{B_{\rho'}\setminus B_\rho}\varphi_j(x,|\nabla w^\varepsilon|)\,\mathrm{d}x = \int_{B_{\rho'}\setminus B_\rho}\varphi_\infty(|\nabla w^\varepsilon|)\,\mathrm{d}x,$$

since $|\nabla w^\varepsilon|$ is bounded. Likewise, we have

$$\lim_{j\to+\infty}\int_{B_{\rho'}\setminus B_\rho}\varphi_j\left(x, \frac{|w^\varepsilon - v_j|}{\rho'-\rho}\right)\mathrm{d}x = \int_{B_{\rho'}\setminus B_\rho}\varphi_\infty\left(\frac{|w^\varepsilon - v_\infty|}{\rho'-\rho}\right)\mathrm{d}x\,.$$

Therefore, passing to the liminf as $j\to+\infty$ in (3.24), we have

$$\begin{aligned}
&\liminf_{j\to+\infty}\int_{B_{\rho'}}\varphi_j(x,|\nabla v_j|)\,\mathrm{d}x \\
&\qquad\le \int_{B_\rho}\varphi_\infty(|\nabla w^\varepsilon|)\,\mathrm{d}x + c\left[\int_{B_{\rho'}\setminus B_\rho}\left(\varphi_\infty(|\nabla w^\varepsilon|)+\varphi_\infty\left(\frac{|w^\varepsilon - v_\infty|}{\rho'-\rho}\right)\right)\mathrm{d}x\right] + \mu(B_{\rho'}\setminus B_\rho)\,.
\end{aligned}$$

Now we let $\varepsilon\to 0$ and, recalling that $w = v_\infty$ outside B_ρ, we easily obtain

$$\liminf_{j\to+\infty}\int_{B_\rho}\varphi_j(x,|\nabla v_j|)\,\mathrm{d}x \le \int_{B_\rho}\varphi_\infty(|\nabla w|)\,\mathrm{d}x + c\int_{B_{\rho'}\setminus B_\rho}\varphi_\infty(|\nabla w|)\mathrm{d}x + \mu(B_{\rho'}\setminus B_\rho)\,.$$

Therefore, with the lower semicontinuity result (3.18), letting ρ' tend to ρ we finally get that for every $\rho\in(0,R)$ and any $w\in W^{1,\varphi_\infty}(B_R)$ such that $\{w\neq v_\infty\}\Subset B_\rho$ we have

$$\int_{B_\rho}\varphi_\infty(|\nabla v_\infty|)\,\mathrm{d}x \le \int_{B_\rho}\varphi_\infty(|\nabla w|)\,\mathrm{d}x\,,$$

as desired. □

A crucial tool for proving the Lipschitz continuity of an almost minimizer is the following Proposition.

Proposition 3.9. *Let u be an almost minimizer of $\mathcal{F}$ in $B_1(x_0)$, where $x_0 \in F^{\pm}(u)$, such that*

$$\sup_{x\in B_1(x_0)} |u(x)| \le M\,. \tag{3.26}$$

Then there exists a constant $C_0 = C_0(d,p,q,L,\kappa_0,\beta) \le 1$ such that

$$|u(x)| \le \frac{2M}{C_0}|x-x_0| \tag{3.27}$$

for all $x \in B_r(x_0)$, provided that $\min\{\Theta^+_u(x_0,r),\Theta^-_u(x_0,r)\} \le C_0$ for any $0<r<1$.

Proof. We may assume, without loss of generality, that $x_0 = 0$, and, throughout the proof, we will omit the center in the notation for a ball centered at x_0. We set

$$S(k,u) := \sup_{x\in B_{r_k}} |u(x)|\,, \quad r_k := 2^{-k}\,, \quad k \ge 0\,, \tag{3.28}$$

and our aim is to prove that there exists a constant $0 < C_0 \le 1$ such that

$$S(k+1,u) \le \max\left\{\frac{Mr_{k+1}}{C_0}, \frac{S(k,u)}{2}\right\}\,, \tag{3.29}$$

provided that $\min\{\Theta^+_u(x_0,r_k),\Theta^-_u(x_0,r_k)\} \le C_0$.

Indeed, once (3.29) has been established, arguing by induction we can prove that

$$S(k,u) \le \frac{Mr_k}{C_0}\,, \qquad \text{for every } k \ge 0\,. \tag{3.30}$$

From this, given $r \in (0,1]$ and chosen $k \ge 0$ such that $r_{k+1} < r \le r_k$, we obtain

$$\|u\|_{L^\infty(B_r)} \le \|u\|_{L^\infty(B_{r_k})} = S(k,u) \le \frac{Mr_k}{C_0} = 2\frac{Mr_{k+1}}{C_0} \le 2\frac{Mr}{C_0}\,, \tag{3.31}$$

and then (3.27).

In order to prove (3.29), we argue by contradiction, and, for every $j \ge 1$, we assume the existence of u_j almost minimizer of $\mathcal{F}$ in B_1, with constant κ and exponent β, and of an integer k_j such that

$$S(k_j+1,u_j) > \max\left\{jMr_{k_j+1}, \frac{S(k_j,u_j)}{2}\right\}\,, \tag{3.32}$$

and

$$\min\{\Theta^+_{u_j}(0,r_{k_j}),\Theta^-_{u_j}(0,r_{k_j})\} \le \frac{1}{j}\,. \tag{3.33}$$

Note that, by (3.32), $\|u_j\|_{L^\infty(B_1)} \le M$ implies $jr_{k_j+1} < 1$, hence $k_j > \log_2(j) - 1$ for every j, so that $k_j \to +\infty$. Furthermore, with the uniform bound $\|u_j\|_{L^\infty(B_1)} \le M$ and the same argument used in Proposition 3.8, we can show that for any $\eta \in (0,1)$ the u_j are uniformly locally η-Hölder continuous in B_1. Since $u_j(0) = 0$, we obtain

$$\sup_{x\in B_{r_{k_j+1}}} |u_j(x)| = \sup_{x\in B_{r_{k_j+1}}} |u_j(x)-u_j(0)| \le C_\eta r^\eta_{k_j+1} < C_\eta r^\eta_{k_j}\,, \tag{3.34}$$

where C_η is independent of j.

Now, we set

$$\sigma_j := \frac{S(k_j+1,u_j)}{r_{k_j}}\,, \tag{3.35}$$

and we consider the scaled function

$$v_j(x) := \frac{u_j(r_{k_j}x)}{\sigma_j r_{k_j}} = \frac{u_j(r_{k_j}x)}{S(k_j+1,u_j)}\,, \quad x \in B_{\frac{1}{r_{k_j}}}\,. \tag{3.36}$$

Note that, by (3.32),

$$\sigma_j \geq j^{\frac{M}{2}} \to +\infty \quad \text{as } j \to +\infty, \tag{3.37}$$

and since the density is scaling invariant, from (3.33) it follows that

$$\min\left\{ \frac{\mathcal{L}^d(\{v_j > 0\} \cap B_1)}{\mathcal{L}^d(B_1)}, \frac{\mathcal{L}^d(\{v_j < 0\} \cap B_1)}{\mathcal{L}^d(B_1)} \right\} \leq \frac{1}{j} \to 0\,. \tag{3.38}$$

Setting

$$\varphi_j(x,t) := \frac{\varphi(r_{k_j}x, \sigma_j t)}{\varphi(0,\sigma_j)}\,, \quad \lambda_i^j(x) := \frac{\lambda_i(r_{k_j}x)}{\varphi(0,\sigma_j)}\,,\; i = 1,2\,,$$

with this choice of σ_j, φ_j and λ_1^j, λ_2^j, we introduce the scaled functional $\hat{\mathfrak{F}}_j$ defined as in (3.19). Since, by (3.37), $\sigma_j > 1$ for j large enough, and $\varphi(0,t)$ is $(\text{dec})_q$, we have, in view of (3.34) for $\eta = 1 - \frac{1}{2q}$

$$\varphi(0,\sigma_j) r_{k_j} \leq \varphi(0,1)\sigma_j^q r_{k_j} = \varphi(0,1)\left(\frac{S(k_j+1,u_j)}{r_{k_j}^{1-\frac{1}{2q}}}\right)^q r_{k_j}^{\frac{1}{2}} \leq \varphi(0,1) C^q_{1-\frac{1}{2q}} r_{k_j}^{\frac{1}{2}}\,, \qquad \text{for } j \text{ large enough,}$$

whence (3.13) follows.

Taking into account (3.32), for $x \in B_1$, we have

$$|v_j(x)| \leq \frac{S(k_j,u_j)}{S(k_j+1,u_j)} \leq 2\frac{S(k_j,u_j)}{S(k_j,u_j)} = 2. \tag{3.39}$$

By Proposition 3.8 v_j is an almost minimizer of $\hat{\mathfrak{F}}_j$ in $B_1(0)$ with constant $\kappa r_{k_j}^{\beta} \leq \kappa_0$ and exponent β, there exist a $C^1([0,+\infty))$ convex function φ_∞ whose derivative φ'_∞ complies with $(\text{inc})_{p-1}$ and $(\text{dec})_{q-1}$, a function $v_\infty \in W^{1,1}(B_1(0))$ such that, up to a subsequence, $v_j \to v_\infty$ uniformly in $B_1(0)$, and v_∞ is φ_∞-harmonic in $B_{\frac{1}{2}}(0)$. Since, by (3.39) and (3.38), it holds that either $0 \leq v_\infty \leq 2$ or $-2 \leq v_\infty \leq 0$ in $B_{\frac{1}{2}}(0)$, and $v_\infty(0) = 0$ being $v_j(0) \equiv 0$, from the strong minimum/maximum principle we must have $v_\infty \equiv 0$ in $B_{\frac{1}{2}}(0)$. However, from (3.36) we deduce that $\sup_{x \in B_{\frac{1}{2}}(0)} |v_\infty(x)| = 1$ and this gives a contradiction. The proof is concluded.

□

We are now in position to prove the main result, Theorem 1.1.

Proof of Theorem 1.1. Let u be an almost minimizer of $\mathcal{F}$ in Ω, with constant $\kappa \leq \kappa_0$ and exponent β. Let $\Omega' \Subset \Omega$, define r_0 as in (3.4) and set

$$r_1 := \frac{1}{4}\min\{2r_0, \operatorname{dist}(\Omega', \partial\Omega)\} \quad \text{ and } \quad \Omega_{r_1} := \{x \in \Omega :\; \operatorname{dist}(x, \partial\Omega) \geq r_1\}\,.$$

We recall that, by virtue of Theorem 3.4, $u \in C^{0,\alpha}(\Omega_{r_1})$ for any fixed $\alpha \in (0,1)$. We then set $M := \|u\|_{L^\infty(\Omega_{r_1})}$, which depends on r_0, and hence on u and Ω' via the integral

$$\int_{\Omega'} \varphi(x, |\nabla u|)^{1+s_0}\, dx.$$

Now, let $x_0 \in \Omega' \cap \{u \neq 0\}$ be arbitrarily fixed. In order to estimate $|\nabla u(x_0)|$ we distinguish between two cases, according to $\tau := \operatorname{dist}(x_0, F^{\pm}(u))$.

We first treat the case $\tau \le r_1$, and choose $y_0 \in F^{\pm}(u)$ such that $|y_0 - x_0| = \tau$. Since $B_{2\tau}(y_0) \subset \Omega_{r_1}$, we have $|u| \le M$ in $B_{2\tau}(y_0)$. Then, by virtue of Proposition 3.9, for every $x \in B_\tau(x_0) \subset B_{2\tau}(y_0)$ we have

$$|u(x)| \le \frac{2M}{C_0}|x - y_0| \le \frac{4M}{C_0}\tau. \tag{3.40}$$

Now, let us consider the scaled function $u_\tau(x) := \frac{u(x_0+\tau x)}{\tau}$, $x \in B_1(0)$. Since u is an almost minimizer of $\mathcal{F}$ in $B_\tau(x_0)$ with constant κ and exponent β, a straightforward computation shows that u_τ is an almost minimizer in $B_1(0)$, with constant $\kappa\tau^\beta$ and exponent β, of the functional $\mathcal{F}_\tau$ defined as

$$\mathcal{F}_\tau(w,\Omega) := \int_\Omega \varphi_\tau(x, |\nabla w|)\,dx + \int_\Omega \lambda_1^\tau(x)\chi_{\{w<0\}}(x) + \lambda_2^\tau(x)\chi_{\{w>0\}}(x) + \min\{\lambda_1^\tau(x), \lambda_2^\tau(x)\}\chi_{\{w=0\}}(x)\,dx\,,$$

where $\varphi_\tau(x,t) := \varphi(x_0 + \tau x, t)$ and $\lambda_i^\tau(x) := \lambda_i(x_0 + \tau x)$, $i = 1, 2$. It is easy to check that $\varphi_\tau \in \Phi_c(B_1(0))$, $\varphi_\tau(x,\cdot) \in C^1([0,\infty))$ and that $(\varphi_\tau)_t$ complies with (A0), $(\mathrm{inc})_{p-1}$ and $(\mathrm{dec})_{q-1}$. We only have to remark that also (VA1) holds. For this, let $\rho \in (0,1)$, $B_\rho(y) \subset B_1(0)$ and recall that φ satisfies (VA1) on $B_{\tau\rho}(x_0 + \tau y)$, so that

$$\varphi^+_{B_{\tau\rho}(x_0+\tau y)}(t) \le (1 + \omega(\tau\rho))\varphi^-_{B_{\tau\rho}(x_0+\tau y)}(t)\,, \quad \text{if } \varphi^-_{B_{\tau\rho}(x_0+\tau y)}(t) \in \left[\omega(\tau\rho), 1/\mathcal{L}^d(B_{\tau\rho})\right].$$

Now, since $\varphi^\pm_\tau(t) = \varphi^\pm_{B_{\tau\rho}(x_0+\tau y)}(t)$, where $\varphi^\pm_\tau(t)$ are computed on $B_\rho(y)$, and $\tau \le 1$, from the previous estimate we infer

$$\varphi^+_\tau(t) \le (1 + \rho^\beta)\varphi^-_\tau(t)\,, \quad \text{if } \varphi^-_\tau(t) \in \left[\rho^\beta, 1/\mathcal{L}^d(B_\rho)\right].$$

Moreover, by (3.40), $|u_\tau| \le \frac{4M}{C_0}$ in $B_1(0)$. Therefore, by Lemma 3.6, we deduce that

$$|\nabla u(x_0)| = |\nabla u_\tau(0)| \le C_1\,,$$

where the constant C_1 depends on $p, q, d, L, \kappa_0, \beta, \Lambda, u, \Omega'$.

If, instead, $\tau \ge r_1$, we can perform an analogous argument as before with $u_{r_1}(x) := \frac{u(x_0+r_1x)}{r_1}$, $x \in B_1(0)$ in place of u_τ, which satisfies $\|u_{r_1}\|_{L^\infty(B_1(0))} \le \frac{M}{r_1}$, and $\mathcal{F}_{r_1}$ in place of $\mathcal{F}_\tau$. This concludes the proof.

□

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References

[1] H. W. Alt, L. A. Caffarelli and A. Friedmann, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. **282** (1984), 431–461.

[2] H. W. Alt, L. A. Caffarelli and A. Friedmann, *Jets with Two Fluids II: Two Free Boundaries*, Indiana U Math J., **33**(3) (1984), 367–391.

[3] M. Bayrami and M. Fotouhi, Regularity in the two-phase Bernoulli problem for the p-Laplace operator. *Calc. Var.* **63** (2024), nr. 183. https://doi.org/10.1007/s00526-024-02789-3

[4] J. E. M. Braga and D. R. Moreira, Uniform Lipschitz regularity for classes of minimizers in two-phase free boundary problems in Orlicz spaces with small density on the negative phase, *Ann. I. H. Poincaré* **31** (2014), 823–850.

[5] L. A. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. III. Existence theory, compactness, and dependence on X*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 15 (1988), 583–602.

[6] G. David, M. Engelstein, M. Smit Vega Garcia and T. Toro, *Regularity for almost-minimizers of variable coefficient Bernoulli-type functionals*, Math. Z., 299 (2021), 2131–2169.

[7] G. David and T. Toro, *Regularity of almost minimizers with free boundary*, Calc. Var. Partial Differential Equations, 54 (2015), 455–524.

[8] S. Dipierro and A. Kharakhanyan, *Stratification of free boundary points for a two-phase variational problem*, Adv. Math. **328** (2018), 40–81.

[9] P. Harjulehto, P. Hästö, *Orlicz Spaces and Generalized Orlicz Spaces*. Springer International Publishing, 2019

[10] P. Hästö and J. Ok, *Maximal regularity for local minimizers of non-autonomous functionals*, J. Eur. Math. Soc. (JEMS) **24** (2022), 1285–1334.

[11] A. Kharakhanyan, *On the Lipschitz regularity of solutions of a minimum problem with free boundary*, Interfaces Free Bound. **10** (2008), 79–86.

[12] M. A. Krasnosel'skiǐ and Ya. B. Rutickiǐ, *Convex Functions and Orlicz Spaces*. P. Noordhoff LTD. - Groningen - The Netherlands, (1961).

[13] C. Leone, G. Scilla, F. Solombrino and A. Verde, Lipschitz regularity of almost-minimizers in one-phase problems with generalized Orlicz growth, Preprint (2025), ArXiv:2508.01393.

[14] J. A. Santos and S. H. Monari Soares, Lipschitz regularity of solutions to two-phase free boundary problems governed by a non-uniformly elliptic operator, *Nonlinear Anal* **261** (2025), 113893.

(Chiara Leone) Department of Mathematics and Applications "R. Caccioppoli", University of Naples Federico II, Via Cintia, Monte S. Angelo, 80126 Naples, Italy
Email address: chiara.leone@unina.it

(Giovanni Scilla) Department of Mathematics and Applications "R. Caccioppoli", University of Naples Federico II, Via Cintia, Monte S. Angelo, 80126 Naples, Italy
Email address, Giovanni Scilla: giovanni.scilla@unina.it

(Francesco Solombrino) Department of Biological and Environmental Sciences and Technologies, University of Salento, Complesso Ecotekne, via Monteroni 165, 73100 Lecce, Italy
Email address: francesco.solombrino@unisalento.it

(Anna Verde) Department of Mathematics and Applications "R. Caccioppoli", University of Naples Federico II, Via Cintia, Monte S. Angelo, 80126 Naples, Italy
Email address: anna.verde@unina.it