LEAST GRADIENT PROBLEM WITH NONHOMOGENEOUS TERM

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Abstract. Given a domain $\Omega \subset \mathbb{R}^N$, we consider the following generalization of the least gradient problem

$$\inf \left\{ \int_\Omega |Du| + \int_\Omega \psi\, u \;:\; u \in BV(\Omega), \quad u = f \text{ on } \partial\Omega \right\},$$

where $\psi \in L^N(\Omega)$ and $f \in L^1(\partial\Omega)$ are given functions. If the domain Ω satisfies a $\psi-$barrier condition and the boundary datum f is continuous, we show existence of a solution u to this problem provided that $\psi \in L^p(\Omega)$ with $p > N$ and ψ is sufficiently small in an appropriate sense. By a comparison theorem, we will also prove uniqueness of the solution provided that ψ is a constant λ and $|\lambda|$ is smaller than the Cheeger constant of Ω. When $N \leq 7$, we show that the solution u is continuous on $\overline{\Omega}$. These generalize previous results obtained for the classical BV least gradient problem (without the nonhomogeneous term ψ) in [18, 10, 9].

1. Introduction

In this paper, we study a variant of the least gradient problem. The classical version of the least gradient problem consists of minimizing the total variation of the vector measure Du among all BV functions u defined on an open bounded Lipschitz and simply connected domain $\Omega \subset \mathbb{R}^N$ with given boundary datum $f \in L^1(\partial\Omega)$:

$$\inf \left\{ \int_\Omega |Du| \;:\; u \in BV(\Omega),\, u|_{\partial\Omega} = f \right\}, \tag{1.1}$$

where $u|_{\partial\Omega}$ denotes the trace of u on the boundary $\partial\Omega$. It was first considered in this form in [18], where the authors prove existence and uniqueness of a solution to (1.1) in the case where Ω satisfies a "barrier condition" (in 2D, this assumption is equivalent to strictly convexity) and the boundary datum f is continuous. In fact, we can see clearly that a solution may not exist if Ω is just convex; assume that $\Omega = [0,1]^2$ with $f(x_1, x_2) = x_1$ on $[0, \frac{1}{2}] \times \{0\}$, $f(x_1, x_2) = 1 - x_1$ on $[\frac{1}{2}, 1] \times \{0\}$ and $f(x_1, x_2) = 0$ otherwise, then we see that the level sets (which are line segments; see [8, Chapter 10]) of a solution u to Problem (1.1) are contained in the segment $[0,1] \times \{0\}$, which means that u does not satisfy $u_{|\partial\Omega} = f$ and so, the problem (1.1) does not attain a minimum. However, in the case when Ω is not strictly convex, one needs to introduce admissibility conditions on the boundary data to obtain existence and uniqueness of solutions, see [4, 5, 16]. On the other hand, the continuity assumption on the boundary data can be relaxed. In [6], the authors proved that in two dimensions Problem (1.1) admits a solution as soon as $f \in BV(\partial\Omega)$; the idea there is to show that (1.1) is equivalent to an optimal transport problem between the positive and negative parts of the tangential derivative of f. However, we lose uniqueness of the solution to (1.1) if $f \notin C(\partial\Omega)$.

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The least gradient problem with Neumann boundary condition has been also considered in [7, 15, 14]. More precisely, the authors studied the following minimization problem:

$$\inf\left\{\int_\Omega |Du| - \int_{\partial\Omega} \phi\, u\, \mathrm{d}\mathcal{H}^1 \,:\, u \in BV(\Omega)\right\}, \tag{1.2}$$

where $\phi \in L^\infty(\partial\Omega)$ with $\int_{\partial\Omega}\phi = 0$. They show that Problem (1.2) reaches a minimum (which has to be clearly equal zero) as soon as the datum ϕ is small enough, that is $||\phi||_\star \leq 1$ where the norm $||\cdot||_\star$ is equivalent to $||\cdot||_{L^\infty(\partial\Omega)}$ and it is defined as follows:

$$||\phi||_\star := \sup\left\{\frac{\int_{\partial\Omega}\phi\, u}{\int_\Omega |Du|} \,:\, u \in BV(\Omega)\right\}.$$

If $||\phi||_\star < 1$ then $u = 0$ is the unique solution for Problem (1.2) while if $||\phi||_\star = 1$, then there are infinitely many minimizers. If $||\phi||_\star > 1$, the minimal value will be $-\infty$ and so, a solution u does not exist.

In this paper, we will consider the following BV least gradient problem with nonhomogeneous term:

$$\inf\left\{\int_\Omega |Du| + \int_\Omega \psi\, u \,:\, u \in BV(\Omega),\ \ u = f \ \text{ on } \partial\Omega\right\}, \tag{1.3}$$

where ψ is a given function in $L^N(\Omega)$. To the best of our knowledge, this problem with nonhomogeneous term has not been studied before in the literature. We note that Problem (1.1) is a particular case of (1.3) when $\psi = 0$. The main goal will be to prove existence and uniqueness of a solution to this problem. In fact, it is not difficult to check that the functional in Problem (1.3) is lower semicontinuous with respect to the L^1 convergence. Let $(u_n)_n$ be a minimizing sequence in (1.3). Since ψ has no sign, then it is not clear whether this sequence is uniformly bounded in $BV(\Omega)$ or not. Even if we assume that up to a subsequence, $u_n \rightharpoonup u$ weakly* in $BV(\Omega)$, then it is also not clear whether the limit function u satisfies the boundary condition $u = f$ on $\partial\Omega$ or not. Hence, the existence of a minimizer to Problem (1.3) seems to be a difficult task. Moreover, the existence of a solution to this problem should somehow be related to the nonhomogeneous term ψ, i.e. we cannot expect existence of a solution without appropriate conditions on ψ. On the other hand, we will also need some geometric assumption on Ω since we recall that in the case where $\psi = 0$, the domain Ω has to be strictly convex (or more generally, satisfies a barrier condition).

Inspired by [13], we will consider instead of Problem (1.3) the following relaxation problem:

$$\min\left\{\int_\Omega |Du| + \int_{\partial\Omega} |u - f|\, \mathrm{d}\mathcal{H}^{N-1} + \int_\Omega \psi\, u \,:\, u \in BV(\Omega)\right\}. \tag{1.4}$$

This can also be expressed in the following way: extend f into a BV function $\tilde{f}$ defined on a larger domain $\tilde{\Omega}$, and then consider

$$\min\left\{\int_{\tilde{\Omega}} |Du| + \int_\Omega \psi\, u \,:\, u \in BV(\tilde{\Omega}),\ \ u = \tilde{f} \ \text{ on } \tilde{\Omega}\backslash\overline{\Omega}\right\}. \tag{1.5}$$

Assume Problem (1.4) (or equivalently (1.5)) has a minimizer u. So, the fact that the solution u satisfies or not $u_{|\partial\Omega} = f$ could depend on f (and on the domain Ω as well as the nonhomogeneous term ψ). In case we have $u_{|\partial\Omega} = f$, then u will be also a solution to Problem (1.3). We note that when $\psi \neq 0$, an existence result cannot be obtained via the optimal transport approach used in [6], since it is not clear whether this more general formulation admits an equivalent optimal transport representation or not. Notice that if u is a solution to Problem (1.3), then u solves formally the following 1−Laplacian PDE with nonhomogeneous term:

$$\begin{cases}\nabla\cdot[\frac{\nabla u}{|\nabla u|}] = \psi & \text{in} \quad \Omega,\\ u = f & \text{on} \quad \partial\Omega.\end{cases} \tag{1.6}$$

In [14], the authors show through approximation by $p-$Laplacian problems as $p \to 1^+$ existence of a BV solution u to Problem (1.6) with Dirichlet boundary condition (i.e. $u = 0$ on $\partial\Omega$) but in a very weak sense (see [14, Definition 4.1]), assuming that the datum ψ is sufficiently small.

In Section 2, we will prove existence of a solution u to the relaxed problem (1.4) under the assumption that (we note that the existence of a minimizer to (1.4) in the case when $\psi = 0$ is trivial)

$$\inf\left\{\frac{\int_\Omega \psi\, u}{\int_\Omega |Du|}\ :\ u \in BV(\Omega),\ \ u = 0\ \text{ on }\ \partial\Omega\right\} > -1. \tag{1.7}$$

Then, we will show that any superlevel set $E_s := \{u \geq s\}$ of a solution u to Problem (1.4) is a $\psi-$minimal set. Thanks to this result, we will be able to show that u has the correct trace (i.e. $u = f$ on $\partial\Omega$) as soon as $f \in C(\partial\Omega)$ and $\partial\Omega$ satisfies a positivity condition on a sort of generalized mean curvature related to the nonhomogeneous term ψ (we refer to this as the $\psi-$barrier condition and this would be a generalization of the notion of barrier condition to the case when $\psi \neq 0$).

In Section 3, we will study the uniqueness of the solution to Problem (1.3). First, we show that the boundary of any superlevel set E_s of a minimizer u to (1.3) is smooth, up to a set of Hausdorff dimension at most $N-8$. Then, we will prove that the essential boundary of any superlevel set E_s must intersect $\partial\Omega$; this will play an important role in the proof of our comparison theorem. However, we need to restrict ourselves to the uniform case when ψ is constant (i.e. $\psi = \lambda$) to prove our comparison and uniqueness results. The key point here is that under this restriction, two different level sets cannot intersect at a point inside Ω.

Recalling (1.7), we note that the existence of a solution to Problem (1.5) in the case when $\psi = \lambda$ is then related to the Cheeger constant:

$$\lambda_\Omega = \inf\left\{\frac{Per(F)}{|F|}\ :\ F \subset \Omega\right\}.$$

Indeed, we have

$$\begin{aligned}
\inf\left\{\frac{\lambda\int_\Omega u}{\int_\Omega |Du|}\ :\ u \in BV(\Omega),\ \ u = 0\ \text{ on }\ \partial\Omega\right\} &= \inf\left\{\frac{-|\lambda|\,|\int_\Omega u|}{\int_\Omega |Du|}\ :\ u \in BV(\Omega),\ \ u = 0\ \text{ on }\ \partial\Omega\right\} \\
&= -|\lambda|\ \sup\left\{\frac{|\int_\Omega u|}{\int_\Omega |Du|}\ :\ u \in BV(\Omega),\ \ u = 0\ \text{ on }\ \partial\Omega\right\} \\
&= \frac{-|\lambda|}{\inf\left\{\frac{\int_\Omega |Du|}{|\int_\Omega u|}\ :\ u \in BV(\Omega),\ \ u = 0\ \text{ on }\ \partial\Omega\right\}} = -\frac{|\lambda|}{\lambda_\Omega}.
\end{aligned}$$

Consequently, we will show that if $\psi = \lambda$ and $|\lambda| < \lambda_\Omega$ then Problem (1.3) has a unique solution u. However, it is not clear whether we lose uniqueness or not in the general case when ψ is not constant. In addition, we will prove only in low dimensions (when $N \leq 7$) that this solution u is continuous on $\overline{\Omega}$. But, we do not also know whether it is possible for a minimizer to be discontinuous when $N \geq 8$.

2. Existence

In this section, we prove existence of a solution to Problem (1.3). Let $\tilde{\Omega}$ be an open bounded Lipschitz domain such that $\Omega \subset \tilde{\Omega}$. Let $\tilde{f}$ be a function in $BV(\tilde{\Omega}\backslash\overline{\Omega})$ such that $\tilde{f} = f$ on $\partial\Omega$ (we extend $\tilde{f}$ by 0 on Ω). Let us also extend ψ by 0 on $\tilde{\Omega}\backslash\overline{\Omega}$. Then, we consider the relaxation problem (1.5):

$$\min\left\{\int_{\tilde{\Omega}} |Du| + \int_\Omega \psi\, u\ :\ u \in BV(\tilde{\Omega}),\ \ u = \tilde{f}\ \text{ on }\ \tilde{\Omega}\backslash\overline{\Omega}\right\}. \tag{2.1}$$

We start by proving existence of a minimizer u to Problem (2.1). Before that, we need to introduce the constant:

$$\Lambda := \inf\left\{\frac{\int_\Omega \psi\, u}{\int_{\overline{\Omega}} |Du|}\ :\ u \in BV(\tilde{\Omega}),\ u = 0 \text{ on } \tilde{\Omega}\backslash\overline{\Omega}\right\} \leq 0. \tag{2.2}$$

Then, we have the following:

Proposition 2.1. *Assume $\Lambda > -1$ and $\psi \in L^p(\Omega)$ with $p > N$. Then, Problem (2.1) reaches a minimum.*

Proof. Let $(u_n)_n$ be a minimizing sequence in Problem (2.1). Then, there is a constant $C < \infty$ such that

$$\int_{\tilde{\Omega}} |Du_n| + \int_\Omega \psi\, u_n \leq C, \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

Yet,

$$\int_\Omega \psi u_n = \int_\Omega \psi(u_n - \tilde{f}) \geq \Lambda \int_{\overline{\Omega}} |Du_n - D\tilde{f}|.$$

Hence,

$$(1+\Lambda)\int_{\overline{\Omega}} |Du_n| + \Lambda \int_{\partial\Omega} |f| \leq \int_{\overline{\Omega}} |Du_n| + \Lambda \int_{\overline{\Omega}} |Du_n - D\tilde{f}| \leq \int_{\overline{\Omega}} |Du_n| + \int_\Omega \psi\, u_n \leq C, \quad \text{for all } n \in \mathbb{N}.$$

Since $\Lambda > -1$, this implies that

$$\int_{\overline{\Omega}} |Du_n| \leq C, \quad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

But, $u_n - \tilde{f} \in BV(\tilde{\Omega})$ with $u_n - \tilde{f} = 0$ on $\tilde{\Omega}\backslash\overline{\Omega}$. Then, there is a uniform Poincaré constant $M < \infty$ such that

$$\int_{\tilde{\Omega}} |u_n - \tilde{f}| \leq M \int_{\tilde{\Omega}} |Du_n - D\tilde{f}|. \tag{2.5}$$

Thanks to (2.4) and (2.5), we infer that the sequence $(u_n)_n$ is bounded in $BV(\tilde{\Omega})$. Hence, up to a subsequence, $u_n \rightharpoonup \tilde{u}$ weakly* in $BV(\tilde{\Omega})$. In particular, we have $u_n \to \tilde{u}$ in $L^q(\tilde{\Omega})$ for all $1 \leq q < \frac{N}{N-1}$ and $\tilde{u} = \tilde{f}$ on $\tilde{\Omega}\backslash\overline{\Omega}$. So, $\tilde{u}$ is admissible in Problem (2.1). From the lower semicontinuity of the total variation, we get also that

$$\int_{\tilde{\Omega}} |D\tilde{u}| \leq \liminf_n \int_{\tilde{\Omega}} |Du_n|. \tag{2.6}$$

Moreover, one has

$$\int_\Omega \psi\, u_n \to \int_\Omega \psi\, \tilde{u}. \tag{2.7}$$

Thus, (2.6) and (2.7) imply together that

$$\int_{\tilde{\Omega}} |D\tilde{u}| + \int_\Omega \psi\, u \leq \liminf \left[\int_{\tilde{\Omega}} |Du_n| + \int_\Omega \psi\, u_n\right].$$

Consequently, $\tilde{u}$ is a minimizer for Problem (2.1). □

After shifting the minimisation problem (2.1) by the constant term $\int_{\tilde{\Omega}\backslash\overline{\Omega}} |D\tilde{f}|$, we also see that $u := \tilde{u}_{|\,\Omega}$ solves

$$\min\left\{\int_\Omega |Du| + \int_{\partial\Omega} |u - f|\, d\mathcal{H}^{N-1} + \int_\Omega \psi\, u\ :\ u \in BV(\Omega)\right\}. \tag{2.8}$$

Now, we come back to the original problem (1.3) and our goal is to show that a minimizer u in Problem (2.8) has no jump on $\partial\Omega$, i.e. the inner trace of u equals f on $\partial\Omega$. Hence, we consider again the minimization problem:

$$\min\left\{\int_\Omega |Du| + \int_\Omega \psi\, u \;:\; u \in BV(\Omega),\; u = f \text{ on } \partial\Omega\right\}. \tag{2.9}$$

Definition 2.1. *Assume Ω is a bounded open set and $\tilde{\Omega}$ is a large domain that contains Ω. We say that a function $u \in BV(\tilde{\Omega})$ is $\psi-$least gradient in Ω if for all functions $v \in BV(\tilde{\Omega})$ with $v = u$ a.e. in $\tilde{\Omega}\backslash\Omega$, we have*

$$\int_{\tilde{\Omega}} |Du| + \int_\Omega \psi u \le \int_{\tilde{\Omega}} |Dv| + \int_\Omega \psi v.$$

We say that $E \subset \tilde{\Omega}$ of finite perimeter is a $\psi-$minimal set in Ω if for all subsets $F \subset \tilde{\Omega}$ such that $F\backslash\Omega = E\backslash\Omega$ a.e., we have

$$Per(E,\tilde{\Omega}) + \int_E \psi \le Per(F,\tilde{\Omega}) + \int_F \psi,$$

where $Per(E,\tilde{\Omega})$ denotes the perimeter of E in $\tilde{\Omega}$ defined by $Per(E,\tilde{\Omega}) = \int_{\tilde{\Omega}} |D\chi_E|$, where χ_E is the characteristic function of E. In the sequel, we will write $Per(E)$ instead of $Per(E,\tilde{\Omega})$.

Let u be a minimizer in Problem (2.1). For every $s \in \mathbb{R}$, we define the superlevel set

$$E_s := \{x \in \tilde{\Omega} : u(x) \ge s\}.$$

Then, we show the following:

Proposition 2.2. *For almost every s such that $E_s \neq \emptyset$, the superlevel set E_s minimizes the following problem:*

$$\left\{Per(E) + \int_E \psi \;:\; E \subset \tilde{\Omega},\;\; E\Delta E_s \subset \Omega\right\}. \tag{2.10}$$

Proof. Let $v \in BV(\tilde{\Omega})$ be a nonnegative function such that $v = 0$ on $\tilde{\Omega}\backslash\overline{\Omega}$. From the minimality of u in Problem (2.1), we clearly have

$$\int_{\tilde{\Omega}} |Du| \le \int_{\tilde{\Omega}} |D(u+v)| + \int_\Omega \psi\, v.$$

Fix $s \in \mathbb{R}$. We define $u_1 := \max\{u-s, 0\}$ and $u_2 := \min\{u, s\}$. Then, we claim that u_1 and u_2 are $\psi-$least gradient functions in Ω. In fact, it is easy to see that $u = u_1 + u_2$. Moreover, we will show that

$$\int_{\tilde{\Omega}} |Du| = \int_{\tilde{\Omega}} |Du_1| + \int_{\tilde{\Omega}} |Du_2|.$$

Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ be a Lipschitz function then by the chain rule formula for BV functions (see [1, Theorem 3.96]), one has the following:

$$D[\phi(u)] = \phi'(u)\nabla u + [\phi(u_j^+) - \phi(u_j^-)]\cdot \nu_u\, \mathcal{H}^{N-1} \llcorner \mathcal{J}_u + \phi'(\tilde{u}) D^c u,$$

where ∇u is the absolutely continuous part and $D^c u$ is the Cantor part of Du, $\tilde{u}$ is the approximate limit of u at x defined as follows:

$$\lim_{r\to 0} \frac{1}{r^N} \int_{B(x,r)} |u(y) - \tilde{u}(x)|\, \mathrm{d}y = 0,$$

$\mathcal{J}_u$ is the jump set (which is countably $N-1$ rectifiable and so, at a.e. $x \in \mathcal{J}_u$ there exists a normal vector ν_u to $\mathcal{J}_u$), u_j^+ and u_j^- are the two "one-sided" approximate limits of u on $\mathcal{J}_u$ in the sense that

$$\lim_{r\to 0} \frac{1}{r^N} \int_{B(x,r)\cap\{y: <y-x, \nu_u> > 0\}} |u(y) - u_j^+(x)|\, \mathrm{d}y = 0,$$

$$\lim_{r\to 0}\frac{1}{r^N}\int_{B(x,r)\cap\{y:<y-x,\nu_u><0\}}|u(y)-u_j^-(x)|\,\mathrm{d}y=0.$$

Thanks to [1, Proposition 3.92], we note that $\phi'(u)\nabla u$ is well defined since $\nabla u=0$ $\mathcal{L}^N-$a.e. on the set where $\phi'(u)$ is not defined and $\phi'(u)D^c u$ is also a well defined measure since $\phi'(u)$ is undefined in a $|D^c u|-$negligible set.

Now, define $\phi_1(t):=\max\{t-s,0\}$ and $\phi_2(t)=\min\{t,s\}$. So, we have $u_1=\phi_1(u)$ and $u_2=\phi_2(u)$. Hence,

$$Du_1=\phi_1'(u)\nabla u+[\phi_1(u_j^+)-\phi_1(u_j^-)]\cdot\nu_u\,\mathcal{H}^{N-1}\llcorner\mathcal{J}_u+\phi_1'(\tilde u)D^c u$$

and

$$Du_2=\phi_2'(u)\nabla u+[\phi_2(u_j^+)-\phi_2(u_j^-)]\cdot\nu_u\,\mathcal{H}^{N-1}\llcorner\mathcal{J}_u+\phi_2'(\tilde u)D^c u.$$

In particular, we have

$$\int_{\tilde\Omega}|Du_1|=\int_{\{u>s\}}|\nabla u|+\int_{\mathcal{J}_u}|\phi_1(u_j^+)-\phi_1(u_j^-)|\,\mathrm{d}\mathcal{H}^{N-1}+|D^c u|(\{u>s\})$$

and

$$\int_{\tilde\Omega}|Du_2|=\int_{\{u<s\}}|\nabla u|+\int_{\mathcal{J}_u}|\phi_2(u_j^+)-\phi_2(u_j^-)|\,\mathrm{d}\mathcal{H}^{N-1}+|D^c u|(\{u<s\}).$$

Thus,

$$\int_{\tilde\Omega}|Du_1|+\int_{\tilde\Omega}|Du_2|$$

$$=\int_{\tilde\Omega}|\nabla u|+\int_{\mathcal{J}_u}[|\phi_1(u_j^+)-\phi_1(u_j^-)|+|\phi_2(u_j^+)-\phi_2(u_j^-)|]\,\mathrm{d}\mathcal{H}^{N-1}+|D^c u|(\tilde\Omega),$$

where we have used that $|\nabla u|(\{u=s\})=|D^c u|(\{u=s\})=0$. On the other hand, it is easy to check that

$$|\phi_1(u_j^+)-\phi_1(u_j^-)|+|\phi_2(u_j^+)-\phi_2(u_j^-)|=|u_j^+-u_j^-|.$$

Hence, we get

$$\int_{\tilde\Omega}|Du_1|+\int_{\tilde\Omega}|Du_2|+\int_\Omega\psi\,u_1+\int_\Omega\psi\,u_2=\int_{\tilde\Omega}|Du|+\int_\Omega\psi\,u\le\int_{\tilde\Omega}|D(u+v)|+\int_\Omega\psi\,(u+v)$$

$$\le\int_{\tilde\Omega}|D(u_1+v)|+\int_{\tilde\Omega}|Du_2|+\int_\Omega\psi\,(u_1+v)+\int_\Omega\psi\,u_2$$

and so,

$$\int_{\tilde\Omega}|Du_1|+\int_\Omega\psi\,u_1\le\int_{\tilde\Omega}|D(u_1+v)|+\int_\Omega\psi\,(u_1+v),$$

for any function $v\in BV(\tilde\Omega)$ with $v=0$ on $\tilde\Omega\backslash\overline{\Omega}$. In the same way, one can show that u_2 is also a $\psi-$least gradient function.

For every $\varepsilon>0$ small enough, we define $u_\varepsilon:=\frac{1}{\varepsilon}\min\{\max\{u-s,0\},\varepsilon\}$. Thanks to the above argument, we infer that the function u_ε also satisfies

$$\int_{\tilde\Omega}|Du_\varepsilon|\le\int_{\tilde\Omega}|D(u_\varepsilon+v)|+\int_\Omega\psi\,v,$$

for any function $v\in BV(\tilde\Omega)$ such that $v=0$ on $\tilde\Omega\backslash\overline{\Omega}$. Hence, u_ε is a $\psi-$least gradient function for all $\varepsilon>0$.

For every $v\in BV(\tilde\Omega)$, we will denote in the sequel by v^+ and v^- the inner and outer traces of v on $\partial\Omega$, respectively. We recall that v^+ and v^- are characterized by the fact that for $\mathcal{H}^{N-1}$ almost every $x\in\partial\Omega$,

$$\lim_{r\to 0}\frac{1}{r^N}\int_{B(x,r)\cap\Omega}|v(y)-v^+(x)|\,\mathrm{d}y=0\qquad\text{and}\qquad\lim_{r\to 0}\frac{1}{r^N}\int_{B(x,r)\backslash\Omega}|v(y)-v^-(x)|\,\mathrm{d}y=0.$$

Assume that

$$\mathcal{L}^N(\{x \in \tilde{\Omega} : u(x) = s\}) = \mathcal{H}^{N-1}(\{x \in \partial\Omega : u^{\pm}(x) = s\}) = 0.$$

We note that this is the case for a.e. s. Then, it is clear that $u_\varepsilon \to \chi_{E_s}$ in $L^q(\tilde{\Omega})$ for any $1 \le q < \infty$, when $\varepsilon \to 0$. Moreover, we claim that $u_\varepsilon^{\pm} \to \chi_{E_s}^{\pm}$ in $L^1(\partial\Omega)$. Let us prove that $u_\varepsilon^{+} \to \chi_{E_s}^{+}$ in $L^1(\partial\Omega)$ (the proof that $u_\varepsilon^{-} \to \chi_{E_s}^{-}$ in $L^1(\partial\Omega)$ is completely similar and so, it will be omitted). First of all, we see that

$$u_\varepsilon^+ = \frac{1}{\varepsilon}\min\{\max\{u^+ - s, 0\}, \varepsilon\}.$$

This follows from the fact that

$$\frac{1}{r^N}\int_{B(x,r)\cap\Omega}\left|\frac{1}{\varepsilon}\min\{\max\{u(y)-s,0\},\varepsilon\} - \frac{1}{\varepsilon}\min\{\max\{u^+(x)-s,0\},\varepsilon\}\right| \mathrm{d}y$$
$$\le \frac{1}{\varepsilon\, r^N}\int_{B(x,r)\cap\Omega}|u(y)-u^+(x)|\,\mathrm{d}y \to 0.$$

But again, we see that $u_\varepsilon^+ \to \chi_{\{u^+ \ge s\}}$ in $L^1(\partial\Omega)$. Yet, we claim that $\chi_{\{u^+\ge s\}} = \chi_{E_s}^+$. Fix $x \in \partial\Omega$ and assume that $u^+(x) > s$. Set $A_r := \{u < s\} \cap B(x,r) \cap \Omega$. Then, we must have

$$|u^+(x) - s|\,\frac{|A_r|}{r^N} \le \frac{1}{r^N}\int_{A_r}|u(y)-u^+(x)|\,\mathrm{d}y \le \frac{1}{r^N}\int_{B(x,r)\cap\Omega}|u(y)-u^+(x)|\,\mathrm{d}y \to 0.$$

Hence, $|A_r|/r^N \to 0$. Yet, one has

$$\frac{1}{r^N}\int_{B(x,r)\cap\Omega}|\chi_{E_s}(y)-1|\,\mathrm{d}y = \frac{1}{r^N}\int_{(B(x,r)\cap\Omega)\setminus A_r}|\chi_{E_s}(y)-1|\,\mathrm{d}y + \frac{1}{r^N}\int_{A_r}|\chi_{E_s}(y)-1|\,\mathrm{d}y$$
$$= \frac{1}{r^N}\int_{A_r}|\chi_{E_s}(y)-1|\,\mathrm{d}y \to 0.$$

Then, $\chi_{E_s}^+(x) = 1$. If $u^+(x) < s$, then one can show in the same way that $\chi_{E_s}^+(x) = 0$. Hence, $\chi_{E_s}^+ = \chi_{\{u^+\ge s\}}$.

Fix $v \in BV(\tilde{\Omega})$ with $v = 0$ on $\tilde{\Omega}\backslash\overline{\Omega}$. Set $v_\varepsilon = [\chi_{E_s} - u_\varepsilon]\cdot\chi_\Omega + v$. For every $\varepsilon > 0$, since u_ε is $\psi-$least gradient and $v_\varepsilon = 0$ on $\tilde{\Omega}\backslash\overline{\Omega}$, then we have

$$\int_{\tilde{\Omega}}|Du_\varepsilon| \le \int_{\tilde{\Omega}}|D(u_\varepsilon + v_\varepsilon)| + \int_\Omega \psi\, v_\varepsilon.$$

Hence,

$$\int_\Omega |Du_\varepsilon| + \int_{\partial\Omega}|u_\varepsilon^+ - u_\varepsilon^-| \le \int_\Omega |D(u_\varepsilon + v_\varepsilon)| + \int_{\partial\Omega}|u_\varepsilon^+ + v_\varepsilon^+ - u_\varepsilon^-| + \int_\Omega \psi\, v_\varepsilon.$$

Therefore,

$$\int_\Omega |Du_\varepsilon| + \int_{\partial\Omega}|u_\varepsilon^+ - u_\varepsilon^-| \le \int_\Omega |D(\chi_{E_s} + v)| + \int_{\partial\Omega}|\chi_{E_s}^+ + v^+ - u_\varepsilon^-| + \int_\Omega \psi\,(\chi_{E_s} - u_\varepsilon) + \int_\Omega \psi\, v.$$

Since $\psi \in L^p(\Omega)$ with $p > 1$, $u_\varepsilon \to \chi_{E_s}$ in $L^q(\Omega)$ (for any $q < \infty$) and $u_\varepsilon^{\pm} \to \chi_{E_s}^{\pm}$ in $L^1(\partial\Omega)$, then by the lower semicontinuity of the total variation, we get that

$$\int_\Omega |D\chi_{E_s}| + \int_{\partial\Omega}|\chi_{E_s}^+ - \chi_{E_s}^-| \le \int_\Omega |D(\chi_{E_s} + v)| + \int_{\partial\Omega}|\chi_{E_s}^+ + v^+ - \chi_{E_s}^-| + \int_\Omega \psi\, v.$$

This yields that

$$\int_{\tilde{\Omega}}|D\chi_{E_s}| \le \int_{\tilde{\Omega}}|D(\chi_{E_s} + v)| + \int_\Omega \psi\, v. \tag{2.11}$$

To conclude the proof, assume E is a subset of $\tilde{\Omega}$ such that $E\Delta E_s \subset \Omega$. Hence, by (2.11) with $v = \chi_E - \chi_{E_s}$, we infer the following inequality:

$$Per(E_s, \tilde{\Omega}) + \int_{E_s}\psi \;\le\; Per(E, \tilde{\Omega}) + \int_E \psi. \quad \square$$

Moreover, we have the following result which extends somehow Proposition 2.2 to the case of all s. In [2, Theorem 1], the authors show that if u is a classical (i.e. $\psi = 0$) least gradient function in Ω, then any superlevel set E_s of u is of least area in Ω.

Proposition 2.3. *For all s, the superlevel set E_s minimizes the following: for all $v \in BV(\Omega)$ such that $v = 0$ on $\partial\Omega$, we have*

$$\int_\Omega |D\chi_{E_s}| \leq \int_\Omega |D(\chi_{E_s} + v)| + \int_\Omega \psi v.$$

Proof. Let $(s_n)_n$ be a sequence such that $s_n \to s$, $s_n < s$, and E_{s_n} is a $\psi-$minimal set. It is easy to see that $\chi_{E_{s_n}} \to \chi_{E_s}$ in $L^1(\tilde{\Omega})$. Let K be a compact set and A be an open set such that

$$K \subset A \subset\subset \Omega,$$
$$\mathcal{H}^{N-1}(\partial A) < \infty,$$

and

$$\int_{\partial A} |D\chi_{E_{s_n}}| = 0, \qquad \text{for all } n.$$

For every n, let us denote by g_n the function that is zero on A and $\chi_{E_{s_n}}$ on $\tilde{\Omega}\backslash A$. Then, we clearly have

$$\int_{\tilde{\Omega}} |Dg_n| = \int_{\partial A} |\chi_{E_{s_n}}| \, \mathrm{d}\mathcal{H}^{N-1} + \int_{\tilde{\Omega}\backslash A} |D\chi_{E_{s_n}}|.$$

In particular, one has

$$\int_{\tilde{\Omega}} |D\chi_{E_{s_n}}| + \int_\Omega \psi\, \chi_{E_{s_n}} \leq \int_{\tilde{\Omega}} |Dg_n| + \int_\Omega \psi\, g_n = \int_{\partial A} |\chi_{E_{s_n}}| \, \mathrm{d}\mathcal{H}^{N-1} + \int_{\tilde{\Omega}\backslash A} |D\chi_{E_{s_n}}| + \int_{\Omega\backslash A} \psi\, \chi_{E_{s_n}}.$$

Hence,

$$\int_A |D\chi_{E_{s_n}}| \leq \int_{\partial A} |\chi_{E_{s_n}}| \, \mathrm{d}\mathcal{H}^{N-1} - \int_A \psi\, \chi_{E_{s_n}} \leq \mathcal{H}^{N-1}(\partial A) + \int_A |\psi|.$$

Then, this implies that $\chi_{E_s} \in BV(A)$. In addition to the assumptions above, one can always assume the following:

$$\int_{\partial A} |D\chi_{E_s}| = 0$$

and

$$\lim_n \int_{\partial A} |\chi_{E_{s_n}} - \chi_{E_s}| = 0.$$

Let $h \in BV(\tilde{\Omega})$ such that $h = \chi_{E_s}$ on $\tilde{\Omega}\backslash K$. For every n, let us denote by h_n the function that is equal to h on A and $\chi_{E_{s_n}}$ on $\tilde{\Omega}\backslash A$. Then, we have

$$\int_{\tilde{\Omega}} |D\chi_{E_{s_n}}| + \int_\Omega \psi\, \chi_{E_{s_n}} \leq \int_{\tilde{\Omega}} |Dh_n| + \int_\Omega \psi\, h_n.$$

Yet,

$$\int_{\tilde{\Omega}} |Dh_n| = \int_A |Dh| + \int_{\partial A} |\chi_{E_{s_n}} - \chi_{E_s}| \, \mathrm{d}\mathcal{H}^{N-1} + \int_{\tilde{\Omega}\backslash A} |D\chi_{E_{s_n}}|.$$

Thus, we get that

$$\int_A |D\chi_{E_{s_n}}| + \int_A \psi\, \chi_{E_{s_n}} \leq \int_A |Dh| + \int_{\partial A} |\chi_{E_{s_n}} - \chi_{E_s}| \, \mathrm{d}\mathcal{H}^{N-1} + \int_A \psi h.$$

Letting $n \to \infty$, we infer that

$$\int_A |D\chi_{E_s}| + \int_A \psi\, \chi_{E_s} \leq \liminf_n \left[\int_A |D\chi_{E_{s_n}}| + \int_A \psi\, \chi_{E_{s_n}} \right] \leq \int_A |Dh| + \int_A \psi h.$$

Consequently,

$$\int_{\tilde{\Omega}} |D\chi_{E_s}| \leq \int_{\tilde{\Omega}} |D(\chi_{E_s} + v)| + \int_\Omega \psi v,$$

for all $v \in BV(\tilde{\Omega})$ such that $\text{spt}(v) \subset K$. Finally, we can extend this inequality to any $v \in BV(\tilde{\Omega})$ with $v^+ = 0$ on $\partial\Omega$ by approximation. $\square$

Notice that a set with finite perimeter $E \subset \mathbb{R}^N$ is defined up to a set of measure zero; thus, each such set defines an equivalence class of sets F with $|E\Delta F| = 0$. To avoid this ambiguity, we will focus on a particular element $E^{(1)}$ (the set of points with density 1) of this class, which remains unchanged even if a set of measure zero is added to E. For this aim, we define the $N-$dimensional density of E at a point $x \in \mathbb{R}^N$ as follows:

$$\theta_N[E](x) := \lim_{r\to 0} \frac{|E \cap B(x,r)|}{|B_r|}.$$

Thanks to the Lebesgue points theorem, $\theta_N[E](x)$ exists at *a.e.* point $x \in \mathbb{R}^n$ with $\theta_N[E](x) = 1$ for a.e. $x \in E$ and $\theta_N[E](x) = 0$ for a.e. $x \in \mathbb{R}^N \backslash E$. Given $\lambda \in [0,1]$, the set of points of density λ of E is defined as

$$E^{(\lambda)} = \{x \in \mathbb{R}^N : \theta_N[E](x) = \lambda\}.$$

So, we clearly have

$$|E\Delta E^{(1)}| = 0 \qquad \text{and} \qquad |[\mathbb{R}^N \backslash E]\Delta E^{(0)}| = 0.$$

In particular, it is easy to see that $E^{(1)}$ is insensitive to modifications of E on sets of measure 0. On the other hand, we define the essential boundary of E as the set of points where the density is neither 0 nor 1, i.e.

$$\partial_e E = \mathbb{R}^N \backslash [E^{(0)} \cup E^{(1)}].$$

In addition, we will also use in the sequel the notion of reduced boundary $\partial^\star E$ of E which is defined as the set of points $x \in \text{spt}[D\chi_E]$ where the limit

$$\nu_E(x) := \lim_{r\to 0} \frac{D\chi_E(B(x,r))}{|D\chi_E|(B(x,r))}$$

exists and belongs to S^{N-1}. We note that replacing E with $E^{(1)}$ does not change the reduced boundary $\partial^\star E$. On the other hand, we clearly have

$$\partial^\star E \subset \partial_e E \subset \partial E.$$

Thanks to [1, Theorem 3.61], one has

$$\mathcal{H}^{N-1}(\partial_e E \backslash \partial^\star E) = 0.$$

By [1, Theorem 3.59], one can show that the reduced boundary $\partial^\star E$ is countably $(N-1)$ rectifiable. Moreover, we have

$$|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^\star E.$$

In order to prove existence of solutions to Problem (2.9), we need to introduce the following condition on Ω that generalizes the notion of barrier condition in [18] to the case when $\psi \neq 0$.

Definition 2.2. *We say that Ω satisfies the $\psi-$barrier condition if for every point $x_0 \in \partial\Omega$ and $\varepsilon > 0$ sufficiently small, if E minimizes*

$$\left\{ Per(W) \pm \int_W \psi \,:\, W \subset \Omega,\ W \backslash B(x_0,\varepsilon) = \Omega \backslash B(x_0,\varepsilon) \right\}, \tag{2.12}$$

then

$$\partial E^{(1)} \cap \partial\Omega \cap B(x_0,\varepsilon) = \emptyset.$$

Remark 2.1. *We note that the $\psi-$barrier condition is related to both the shape of Ω as well as the function ψ. Assume $\Omega \subset \mathbb{R}^2$ has a smooth boundary and $\psi = \lambda$, then it is not difficult to check that Ω satisfies the $\lambda-$barrier condition as soon as it is uniformly convex and the curvature of $\partial\Omega$ is greater than $|\lambda|$.*

Now, we are ready to prove one of the main results in this paper.

Theorem 2.4. *Assume Ω satisfies the $\psi-$barrier condition, $\psi \in L^p(\Omega)$ with $p > N$, $\Lambda > -1$ and, $f \in C(\partial\Omega)$. Then, every minimizer of Problem (2.1) is also a minimizer of Problem (2.9). In particular, Problem (2.9) has a solution.*

Proof. Let u be a solution for Problem (2.1) (thanks to Proposition 2.1, u exists). Assume that the inner trace of u on $\partial\Omega$ is not exactly f. So, there will be a point $x_0 \in \partial\Omega$ and a constant $\delta > 0$ small such that

$$\operatorname{ess\,sup}_{y\in\Omega,\,|x_0-y|<r}[u(y) - f(x_0)] \geq \delta \quad \text{or} \quad \operatorname{ess\,sup}_{y\in\Omega,\,|x_0-y|<r}[f(x_0) - u(y)] \geq \delta, \tag{2.13}$$

for every $r > 0$. Let us assume that the first statement in (2.13) holds (the proof will be similar if the second statement in (2.13) holds). Since $f \in C(\partial\Omega)$, then we may assume that the extension $\tilde{f}$ is also continuous on $\tilde{\Omega}\backslash\Omega$. Fix $f(x_0) < s < f(x_0) + \delta$. Thanks to the continuity of $\tilde{f}$, then we also have $u(y) < s$, for all $y \in B(x_0, \varepsilon)\backslash\Omega$; where $\varepsilon > 0$ is sufficiently small. Now, consider the superlevel set

$$E_s := \Big\{x \in \tilde{\Omega} \,:\, u(x) \geq s\Big\}.$$

Hence, we clearly get

$$x_0 \in \partial E_s^{(1)} \cap \partial\Omega \quad \text{and} \quad E_s^{(1)} \cap B(x_0, \varepsilon) \subset \overline{\Omega}. \tag{2.14}$$

On the other hand, thanks to Proposition 2.2 and up to choosing a suitable $s \in (f(x_0), f(x_0) + \delta)$, we have that the superlevel set E_s minimizes the following problem:

$$\Big\{Per(E, \tilde{\Omega}) + \int_E \psi \,:\, E \subset \tilde{\Omega},\;\; E\Delta E_s \subset \Omega \;\text{ a.e.}\Big\}.$$

The aim is to arrive to a contradiction with the assumption that Ω satisfies the $\psi-$barrier condition. Let V be a minimal set in Problem (2.12), i.e. V minimizes

$$\Big\{Per(W, \tilde{\Omega}) + \int_W \psi \,:\, W \subset \Omega,\; W\backslash B(x_0, \varepsilon) = \Omega\backslash B(x_0, \varepsilon)\Big\}. \tag{2.15}$$

Since Ω satisfies the $\psi-$barrier condition, then we have $\partial V^{(1)} \cap \partial\Omega \cap B(x_0, \varepsilon) = \emptyset$. Now, we define $V' := V \cup (E_s \cap \Omega)$. Then, we claim that V' also minimizes Problem (2.15). But, this yields obviously to a contradiction with the assumption that Ω satisfies the $\psi-$barrier condition since we clearly have $x_0 \in \partial V'^{(1)} \cap \partial\Omega \cap B(x_0, \varepsilon)$. Let us prove our claim; assume V' is not a minimizer in (2.15). First, we see that $V'\backslash B(x_0, \varepsilon) = \Omega\backslash B(x_0, \varepsilon)$. So, by the minimality of V in Problem (2.15), we have

$$Per(V, \tilde{\Omega}) + \int_V \psi < Per(V', \tilde{\Omega}) + \int_{V'} \psi.$$

Set $F = [E_s\backslash V] \cap B(x_0, \varepsilon)$. Then, it is clear that $F \neq \emptyset$ since $x_0 \in \partial E_s^{(1)} \cap \partial\Omega$ while $\partial V^{(1)} \cap \partial\Omega \cap B(x_0, \varepsilon) = \emptyset$ and, $F \subset \Omega$ as $E_s^{(1)} \cap B(x_0, \varepsilon) \subset \overline{\Omega}$. Moreover, we have clearly $V' = V \cup F$. Thanks to [11, Theorem 16.3], one has

$$Per(V', \tilde{\Omega})$$
$$= \mathcal{H}^{N-1}(\partial^\star(V\cup F)) = \mathcal{H}^{N-1}(\partial^\star V\cap F^{(0)})+\mathcal{H}^{N-1}(\partial^\star F\cap V^{(0)})+\mathcal{H}^{N-1}(\{x \in \partial^\star V\cap\partial^\star F : \nu_V(x) = \nu_F(x)\}).$$

Yet,

$$Per(V, \tilde{\Omega}) = \mathcal{H}^{N-1}(\partial^\star V \cap F^{(0)}) + \mathcal{H}^{N-1}(\partial^\star V \cap F^{(1)}) + \mathcal{H}^{N-1}(\partial^\star V \cap \partial^\star F).$$

But, we have $\partial^\star V \cap F^{(1)} = \emptyset$. Since otherwise it means that there is a point $x \in \partial^\star V \cap F^{(1)}$ and so, $|V \cap B(x, r)|/|B_r| \to \theta \in (0, 1)$ and $|F \cap B(x, r)|/|B_r| \to 1$. Since $V \cap F = \emptyset$, this yields that

$|V' \cap B(x,r)|/|B_r| \to 1+\theta > 1$, which is a contradiction. On the other hand, it is easy to see that $\{x \in \partial^\star V \cap \partial^\star F : \nu_V(x) = \nu_F(x)\} = \emptyset$. Hence, we get

$$\mathcal{H}^{N-1}(\partial^\star V \cap \partial^\star F) < \mathcal{H}^{N-1}(\partial^\star F \cap V^{(0)}) + \int_F \psi. \tag{2.16}$$

Set $E' = E_s \backslash F$. Then, we clearly have $E' \Delta E_s \subset \Omega$. Then, by the minimality of E_s in Problem (2.10), one has

$$Per(E_s, \tilde{\Omega}) + \int_{E_s} \psi \le Per(E', \tilde{\Omega}) + \int_{E'} \psi.$$

But, we have

$$Per(E_s, \tilde{\Omega}) = \mathcal{H}^{N-1}(\partial^\star E_s \cap F^{(0)}) + \mathcal{H}^{N-1}(\partial^\star E_s \cap F^{(1)}) + \mathcal{H}^{N-1}(\partial^\star E_s \cap \partial^\star F)$$

and

$$Per(E', \tilde{\Omega}) = \mathcal{H}^{N-1}(\partial^\star E_s \cap F^{(0)}) + \mathcal{H}^{N-1}(\partial^\star F \cap E_s^{(1)}) + \mathcal{H}^{N-1}(\{x \in \partial^\star E_s \cap \partial^\star F : \nu_{E_s}(x) = -\nu_F(x)\}).$$

Since $F \subset E_s$, then we clearly have $\{x \in \partial^\star E_s \cap \partial^\star F : \nu_{E_s}(x) = -\nu_F(x)\} = \emptyset$. This yields that

$$Per(E', \tilde{\Omega}) = \mathcal{H}^{N-1}(\partial^\star E_s \cap F^{(0)}) + \mathcal{H}^{N-1}(\partial^\star F \cap E_s^{(1)}).$$

Moreover, it is easy to check that $\partial^\star E_s \cap F^{(1)} = \emptyset$ (otherwise, there will be a point x such that $|E_s \cap B(x,r)|/|B_r| \to \theta \in (0,1)$ and $|F \cap B(x,r)|/|B_r| \to 1$, which is not possible since $F \subset E_s$). Hence, we get

$$\mathcal{H}^{N-1}(\partial^\star E_s \cap \partial^\star F) + \int_F \psi \le \mathcal{H}^{N-1}(\partial^\star F \cap E_s^{(1)}). \tag{2.17}$$

However, we claim that

$$\mathcal{H}^{N-1}(\partial^\star F \cap E_s^{(1)}) \le \mathcal{H}^{N-1}(\partial^\star V \cap \partial^\star F). \tag{2.18}$$

Indeed, assume $x \in \partial^\star F \cap E_s^{(1)}$. Hence, we have $|E_s \cap B(x,r)|/|B_r| \to 1$ and $|F \cap B(x,r)|/|B_r| \to \theta \in (0,1)$. Assume $x \in V^{(0)}$. Then, we have

$$\frac{|V \cap E_s \cap B(x,r)|}{|B_r|} \le \frac{|V \cap B(x,r)|}{|B_r|} \to 0.$$

Hence,

$$\frac{|E_s \cap B(x,r)|}{|B_r|} = \frac{|F \cap B(x,r)|}{|B_r|} + \frac{|V \cap E_s \cap B(x,r)|}{|B_r|} \to \theta, \tag{2.19}$$

which is a contradiction since $|E_s \cap B(x,r)|/|B_r| \to 1$. Now, assume that $x \in V^{(1)}$. Then, one has

$$\frac{|V \cap B(x,r)|}{|B_r|} \to 1.$$

Clearly, this is not possible since otherwise we get

$$\frac{|V' \cap B(x,r)|}{|B_r|} = \frac{|F \cap B(x,r)|}{|B_r|} + \frac{|V \cap B(x,r)|}{|B_r|} \to 1+\theta > 1.$$

Consequently, $x \in \partial_e V$. Yet, $\mathcal{H}^{N-1}(\partial_e V \backslash \partial^\star V) = 0$. Thus, up to a $\mathcal{H}^{N-1}-$negligible set, $\partial^\star F \cap E_s^{(1)} \subset \partial^\star V \cap \partial^\star F$.

Now, we claim that $\partial^\star F \cap V^{(0)} \subset \partial^\star E_s \cap \partial^\star F$, up to a $\mathcal{H}^{N-1}$ negligible set. Fix $x \in \partial^\star F \cap V^{(0)}$ and assume $x \notin \partial_e E_s$. We have $|V \cap B(x,r)|/|B_r| \to 0$ and $|F \cap B(x,r)|/|B_r| \to \theta \in (0,1)$. Hence, it is clear that $x \notin E_s^{(0)}$ since $F \subset E_s$. Therefore, $x \in E_s^{(1)}$ and so, we have $|E_s \cap B(x,r)|/|B_r| \to 1$, which is again a contradiction (see (2.19)). Hence,

$$\mathcal{H}^{N-1}(\partial^\star F \cap V^{(0)}) \le \mathcal{H}^{N-1}(\partial^\star E_s \cap \partial^\star F). \tag{2.20}$$

Combining (2.16), (2.17), (2.18) & (2.20), we get

$$\begin{aligned}\mathcal{H}^{N-1}(\partial^\star E_s \cap \partial^\star F) + \int_F \psi \leq \mathcal{H}^{N-1}(\partial^\star F \cap E_s^{(1)}) &\leq \mathcal{H}^{N-1}(\partial^\star V \cap \partial^\star F)\\ &< \mathcal{H}^{N-1}(\partial^\star F \cap V^{(0)}) + \int_F \psi\\ &\leq \mathcal{H}^{N-1}(\partial^\star E_s \cap \partial^\star F) + \int_F \psi.\end{aligned}$$

But, this is clearly a contradiction because of the strict inequality. Hence, $u = f$ on $\partial\Omega$ and so, u minimizes Problem (2.9). □

3. Uniqueness

In this section, we prove uniqueness of the solution to Problem (2.9). In the case when $\psi = 0$, existence and uniqueness are known to hold provided that Ω satisfies a barrier condition and the boundary datum f is continuous; see [18]. Thus, in the general case when $\psi \neq 0$, one might expect uniqueness to remain valid under suitable assumptions on ψ (and possibly on Ω). However, we will be able to establish uniqueness only in certain particular cases, when ψ is constant and $|\psi|$ is strictly smaller than the Cheeger constant of Ω. The key idea here is to prove a comparison principle on the solutions of (2.9) or more precisely, on the superlevel sets of a solution u. From Proposition 2.10, we recall that each superlevel set E_s is $\psi-$minimal. Since the proof of our comparison principle requires some regularity on the boundaries of the superlevel sets, so we begin with the following result that shows that the singular set of the boundary of any superlevel set E_s is of dimension at most $N-8$ (so, in dimensions up to seven it is empty).

Proposition 3.1. *Assume that $\psi \in L^p(\Omega)$ with $p > N$. Then, there will be an open set $W \subset \Omega$ such that $\partial E_s^{(1)} \cap W$ is of class $C^{1,\gamma}$ (for some γ that depends on p and N) with $\mathcal{H}^s(\Omega\backslash W) = 0$ for all $s > N-8$. In addition, $\partial E_s^{(1)} \cap W$ is of class $C^{2,\alpha}$ (resp. $C^{k+2,\alpha}$) provided that $\psi \in C^{0,\alpha}(\overline{\Omega})$ (resp. $\psi \in C^{k,\alpha}(\overline{\Omega})$). Moreover, the mean curvature H of $\partial E_s^{(1)} \cap W$ is given by the following:*

$$H = \frac{-\psi}{N-1}.$$

Proof. Fix $x_0 \in \partial E_s^{(1)} \cap \Omega$ and $\varepsilon > 0$ sufficiently small so that $B(x_0,\varepsilon) \subset \Omega$. From the minimality of E_s in Problem (2.10), it is clear that for any set $F \subset \tilde{\Omega}$ such that $E_s \Delta F \subset \overline{B(x_0,\varepsilon)}$, we have

$$Per(E_s, \overline{B(x_0,\varepsilon)}) + \int_{E_s \cap B(x_0,\varepsilon)} \psi \leq Per(F, \overline{B(x_0,\varepsilon)}) + \int_{F \cap B(x_0,\varepsilon)} \psi,$$

where

$$Per(F, \overline{B(x_0,\varepsilon)}) = \int_{\overline{B(x_0,\varepsilon)}} |D\chi_F|.$$

Thus, the $C^{1,\gamma}$ regularity of $\partial E_s^{(1)} \cap \Omega$ follows directly from [12, Theorem 3.1]. Assume x_0 is a regular point. After rotation and translation of axes, we may assume that $x_0 = 0$ and $\partial E_s^{(1)} \cap B(x_0,\varepsilon)$ is the graph of a function $v^\star$ and that the inward unit normal vector to $\partial E_s^{(1)}$ at x_0 is $e_N := < 0, ..., 0, 1 >$. Then, it is easy to see that $v^\star$ minimizes the following problem:

$$\min\left\{\mathcal{J}(v)\,:\, v \in \mathrm{Lip}(B'(0,\varepsilon)),\ v = v^\star \text{ on } \partial B'(0,\varepsilon)\right\},$$

where

$$\mathcal{J}(v) := \int_{B'(0,\varepsilon)} \sqrt{1+|\nabla v|^2} - \int_{B'(0,\varepsilon)} \int_0^{v(x')} \psi(x',x'')\,\mathrm{d}x''\,\mathrm{d}x'$$

and, $B'(0,\varepsilon)$ is the ball centered at 0 with radius ε in $\mathbb{R}^{N-1}$. Now, fix $\phi \in C_0^\infty(B'(0,\varepsilon))$. For any δ small, we have $\mathcal{J}(v^\star) \le \mathcal{J}(v^\star + \delta\phi)$. Then, the optimality condition at $\delta = 0$ yields that

$$\int_{B'(0,\varepsilon)} \frac{\nabla v^\star(x')}{\sqrt{1+|\nabla v^\star(x')|^2}} \cdot \nabla\phi(x') = \int_{B'(0,\varepsilon)} \psi(x', v^\star(x'))\, \phi(x').$$

Hence, one has

$$-\nabla \cdot \left[\frac{\nabla v^\star(x')}{\sqrt{1+|\nabla v^\star(x')|^2}} \right] = \psi(x', v^\star(x')). \tag{3.1}$$

or equivalently,

$$\sum_{i.j} a_{ij} v^\star_{ij} = -\psi(x', v^\star(x')) \qquad \text{where} \qquad a_{ij} = \frac{(1+|\nabla v^\star|^2)\delta_{ij} - v^\star_i v^\star_j}{(1+|\nabla v^\star|^2)^{3/2}}.$$

Yet, the right hand side in (3.1) is clearly in $C^{0,\alpha}(\overline{\Omega})$. Moreover, it is easy to check that there are two positive constants $0 < \lambda < \Lambda < \infty$ such that $\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2$. In addition, $a_{ij} \in C^{0,\gamma}(B'(0,\varepsilon))$, for all i, j. Then, thanks to Schauder estimates (see [3]), this implies that $v^\star$ is $C^{2,\alpha}$ in $B'(0, \frac{\varepsilon}{2})$. In addition, the mean curvature H of $\partial E_s^{(1)}$ at a point $x = (x', v^\star(x'))$ is given by

$$(N-1)\, H(x) = \nabla \cdot \left[\frac{\nabla v^\star(x')}{\sqrt{1+|\nabla v^\star(x')|^2}} \right] = -\psi(x', v^\star(x')).$$ □

Before proving our comparison principle, we need to introduce several lemmas. We start by the following:

Lemma 3.2. *Assume E_1 and E_2 are two $\psi-$minimal sets with $E_1\backslash\Omega \subset E_2\backslash\Omega$. Then, $E_1 \cap E_2$ and $E_1 \cup E_2$ are also $\psi-$minimal sets.*

Proof. We note that the proof is similar to the one in Theorem 2.4; we include it here for the sake of completeness. Let us prove that $E_1 \cap E_2$ is $\psi-$minimal. Set $F = E_1\backslash E_2$. Thanks to the assumption that $E_1\backslash\Omega \subset E_2\backslash\Omega$, then we see easily that $F \subset \Omega$. Now, we define $E_1' = E_1\backslash F = E_1 \cap E_2$. Then, we have $E_1 \Delta E_1' \subset \Omega$. Thanks to the minimality of E_1, we get

$$Per(E_1, \tilde{\Omega}) + \int_{E_1} \psi \le Per(E_1', \tilde{\Omega}) + \int_{E_1'} \psi. \tag{3.2}$$

Yet,

$$Per(E_1', \tilde{\Omega})$$
$$= \mathcal{H}^{N-1}(\partial^\star E_1 \cap F^{(0)}) + \mathcal{H}^{N-1}(\partial^\star F \cap E_1^{(1)}) + \mathcal{H}^{N-1}(\{x \in \partial^\star E_1 \cap \partial^\star F : \nu_{E_1}(x) = -\nu_F(x)\}).$$

Hence,

$$\mathcal{H}^{N-1}(\partial^\star E_1 \cap \partial^\star F) + \int_F \psi \le \mathcal{H}^{N-1}(\partial^\star F \cap E_1^{(1)}). \tag{3.3}$$

On the other hand, set $E_2' = E_2 \cup F = E_1 \cup E_2$. Again, we clearly have $E_2 \Delta E_2' \subset \Omega$. Since E_2 is a $\psi-$minimal set, then we have

$$Per(E_2, \tilde{\Omega}) + \int_{E_2} \psi \le Per(E_2', \tilde{\Omega}) + \int_{E_2'} \psi. \tag{3.4}$$

Yet,

$$Per(E_2', \tilde{\Omega})$$
$$= \mathcal{H}^{N-1}(\partial^\star E_2 \cap F^{(0)}) + \mathcal{H}^{N-1}(\partial^\star F \cap E_2^{(0)}) + \mathcal{H}^{N-1}(\{x \in \partial^\star E_2 \cap \partial^\star F : \nu_{E_2}(x) = \nu_F(x)\}).$$

But, this yields to

$$\mathcal{H}^{N-1}(\partial^\star E_2 \cap \partial^\star F) \le \mathcal{H}^{N-1}(\partial^\star F \cap E_2^{(0)}) + \int_F \psi. \tag{3.5}$$

However, it is not difficult to check that $\mathcal{H}^{N-1}(\partial^\star F \cap E_1^{(1)}) = \mathcal{H}^{N-1}(\partial^\star E_2 \cap \partial^\star F)$ and $\mathcal{H}^{N-1}(\partial^\star E_1 \cap \partial^\star F) = \mathcal{H}^{N-1}(\partial^\star F \cap E_2^{(0)})$. Combining (3.3) & (3.5), we infer that

$$\mathcal{H}^{N-1}(\partial^\star E_2 \cap \partial^\star F) = \mathcal{H}^{N-1}(\partial^\star F \cap E_2^{(0)}) + \int_F \psi.$$

In particular, the inequalities in (3.2) & (3.4) must be equalities. But, this means that $E_1 \cap E_2$ and $E_1 \cup E_2$ are $\psi-$minimal sets. □

Lemma 3.3. *For almost every $s \in [\min_{\partial\Omega} f, \max_{\partial\Omega} f]$, we have $\partial E_s^{(1)} \cap \partial\Omega \subset f^{-1}(s)$. In addition, we have*

$$\{x \in \partial\Omega \cap \partial E_s^{(1)} : E_s^{(1)} \cap B(x,\varepsilon) \subset \overline{\Omega} \ \text{ for some } \varepsilon > 0\} = \emptyset.$$

Proof. Assume that there is a point $x_0 \in \partial E_s^{(1)} \cap \partial\Omega$ with $f(x_0) < s$ (resp. $f(x_0) > s$). We note that since $f \in C(\partial\Omega)$, then we may assume that the extension $\tilde{f}$ is continuous over $\tilde{\Omega}\backslash\Omega$. In particular, if $\varepsilon > 0$ is sufficiently small then $\tilde{f} < s$ (resp. $\tilde{f} > s$) on $B(x_0,\varepsilon)$. Hence, one has

$$E_s^{(1)} \cap B(x_0,\varepsilon) \cap \tilde{\Omega}\backslash\Omega = \emptyset.$$

Therefore, we have $E_s^{(1)} \cap B(x_0,\varepsilon) \subset \overline{\Omega}$. Recalling (2.14) and arguing as in Theorem 2.4, we arrive again to a contradiction. □

In the next lemmas, we will show that any connected component of the closure of the reduced boundary of a superlevel set intersects $\partial\Omega$. For the sequel, we set

$$\Lambda^{\pm}_{\psi,m} := \min\left\{ Per(F) \pm \int_F \psi \,:\, F \subset \Omega, \quad |F| = m > 0 \right\}.$$

Lemma 3.4. *Assume $\Lambda^{+}_{\psi,m} > 0$ for every $m > 0$. Let F be a connected component of $E_s^{(1)}$ such that $F \cap \Omega \neq \emptyset$. Then, the closure of the reduced boundary $\overline{\partial^\star F}$ must intersect the boundary $\partial\Omega$.*

Proof. Assume $\overline{\partial^\star F} \cap \partial\Omega = \emptyset$ (so, $\mathrm{dist}(\overline{\partial^\star F}, \partial\Omega) > 0$). Hence, it is not difficult to see that $\partial F \subset \Omega$ since we recall from Proposition 3.1 that $\partial F \backslash \partial^\star F$ is of dimension at most $N-8$. Now, set $\tilde{E} = E_s \backslash F$. We clearly have $\tilde{E} \Delta E_s \subset \overline{\Omega}$. Then, thanks to the minimality of E_s in Problem (2.10), one has

$$Per(E_s) + \int_{E_s} \psi \le Per(\tilde{E}) + \int_{\tilde{E}} \psi. \tag{3.6}$$

Yet, one has

$$Per(E_s) = \mathcal{H}^{N-1}(\partial^\star E_s \cap F^{(0)}) + \mathcal{H}^{N-1}(\partial^\star E_s \cap F) + \mathcal{H}^{N-1}(\partial^\star E_s \cap \partial^\star F)$$

and

$$Per(\tilde{E}) = \mathcal{H}^{N-1}(\partial^\star E_s \cap F^{(0)}) + \mathcal{H}^{N-1}(\partial^\star F \cap E_s^{(1)}) + \mathcal{H}^{N-1}(\{x \in \partial^\star E_s \cap \partial^\star F : \nu_{E_s}(x) = -\nu_F(x)\}).$$

But, we see easily that $\mathcal{H}^{N-1}(\partial^\star F \cap E_s^{(1)}) = 0$ and $\mathcal{H}^{N-1}(\partial^\star E_s \cap F) = 0$. Moreover, since F is a connected component of E_s, then $\mathcal{H}^{N-1}(\{x \in \partial^\star E_s \cap \partial^\star F : \nu_{E_s}(x) = -\nu_F(x)\}) = 0$. Therefore, by (3.6), we get that

$$\mathcal{H}^{N-1}(\partial^\star F) + \int_F \psi \le 0.$$

Yet, this inequality yields obviously to a contradiction thanks to the assumption that $\Lambda^{+}_{\psi,m} > 0$ with $m = |F| > 0$. □

Lemma 3.5. *Assume $\Lambda^{\pm}_{\psi,m} > 0$ for every $m > 0$. If $\mathcal{C}$ is a connected component of $\overline{\partial^\star E_s}$ and $\mathcal{C} \cap \Omega \neq \emptyset$, then $\mathcal{C} \cap \partial\Omega \neq \emptyset$.*

Proof. First of all, assume G is a connected component of E_s and $\mathcal{C}$ is a connected component of $\overline{\partial^\star G}$. Assume $\mathcal{C} \cap \partial\Omega = \emptyset$. Since Ω is simply connected and thanks to Lemma 3.4, then $\mathcal{C}$ must be contained in the inner boundary $\partial_i G$ of G. In particular, the region F inside $\partial_i G$ satisfies $F \subset \Omega \backslash E_s$, $\mathcal{C} \subset \partial F$ and, $|F| > 0$. Now, we define $\tilde{E} = E_s \cup F$. So, we clearly have $\tilde{E} \Delta E_s \subset \Omega$. Hence,

$$Per(E_s) + \int_{E_s} \psi \leq Per(\tilde{E}) + \int_{\tilde{E}} \psi.$$

But,

$$Per(E_s) = \mathcal{H}^{N-1}(\partial^\star E_s \cap F^{(0)}) + \mathcal{H}^{N-1}(\partial^\star E_s \cap F^{(1)}) + \mathcal{H}^{N-1}(\partial^\star E_s \cap \partial^\star F)$$

and

$$Per(\tilde{E}) = \mathcal{H}^{N-1}(\partial^\star E_s \cap F^{(0)}) + \mathcal{H}^{N-1}(\partial^\star F \cap E_s^{(0)}) + \mathcal{H}^{N-1}(\{x \in \partial^\star E_s \cap \partial^\star F : \nu_{E_s}(x) = \nu_F(x)\}).$$

However, we see easily that $\mathcal{H}^{N-1}(\partial^\star E_s \cap F^{(1)}) = 0$ while $\partial^\star E_s \cap \partial^\star F = \partial^\star F$. In addition, we have $\mathcal{H}^{N-1}(\partial^\star F \cap E_s^{(0)}) = 0$ and $\mathcal{H}^{N-1}(\{x \in \partial^\star E_s \cap \partial^\star F : \nu_{E_s}(x) = \nu_F(x)\}) = 0$. Hence, we get

$$\mathcal{H}^{N-1}(\partial^\star F) \leq \int_F \psi.$$

But, this contradicts now the assumption that $\Lambda^-_{\psi,m} > 0$, where $m = |F| > 0$. □

Now, we are ready to prove the following geometric comparison principle. However, we have to restrict ourselves to the case when ψ is constant. In the sequel, we will denote by λ_Ω the Cheeger constant of Ω, i.e.

$$\lambda_\Omega = \inf\left\{\frac{Per(F)}{|F|} \;:\; F \subset \Omega\right\}.$$

Proposition 3.6. *Assume that $\psi = \lambda$ with $|\lambda| < \lambda_\Omega$ and Ω satisfies the $\psi-$barrier condition. Let E_1, $E_2 \subset \tilde{\Omega}$ be two $\psi-$minimal sets such that*

$$\overline{E_1} \backslash \overline{\Omega} \subset \mathring{E}_2 \backslash \overline{\Omega}.$$

Then, the following holds

$$E_1 \subset E_2.$$

Proof. We assume without loss of generality that $E_i^{(1)} = E_i$ $(i = 1, 2)$. Assume by contradiction that E_1 is not included in E_2 and so, $\mathcal{L}^N(E_1 \backslash E_2) > 0$. First, we claim that

$$\partial E_1 \backslash \mathring{E}_2 \subset \Omega. \tag{3.7}$$

It is clear that $\partial E_1 \backslash \mathring{E}_2 \subset \overline{\Omega}$. Now, assume there is a point $x_0 \in [\partial E_1 \backslash \mathring{E}_2] \cap \partial\Omega$. So, there will be an $\varepsilon > 0$ small enough such that $E_1 \cap B(x_0, \varepsilon) \subset \overline{\Omega}$ since otherwise we get a contradiction with the assumption that $\overline{E_1} \backslash \overline{\Omega} \subset \mathring{E}_2 \backslash \overline{\Omega}$. But, this contradicts Lemma 3.3.

By Proposition 3.1, we know that $\partial E_i \cap \Omega$ $(i = 1, 2)$ is smooth up to a set of Hausdorff dimension at most $N - 8$. So, we have $\mathcal{L}^N(\partial(E_1 \backslash E_2)) = 0$ and so, $E_1 \backslash E_2$ coincides with its nonempty interior, up to a negligible set. In particular, there exists a connected component $\mathcal{C}$ of $\partial^\star E_1$ such that $\mathcal{C} \backslash \overline{E_2} \neq \emptyset$. By Lemma 3.5, we must have $\overline{\mathcal{C}} \cap \partial\Omega \neq \emptyset$. Then, one has

$$\overline{\mathcal{C}} \cap \partial\Omega \subset \partial E_1 \cap \partial\Omega \subset \mathring{E}_2.$$

Then, $\mathcal{C} \cap \mathring{E}_2 \neq \emptyset$. This implies that $\mathcal{C} \cap \partial E_2$ is of dimension at least $N - 2$ since it separates $\mathcal{C}$ to two nonempty components $\mathcal{C} \backslash \overline{E_2}$ and $\mathcal{C} \cap \mathring{E}_2$.

Thanks again to Proposition 3.1, we see that $\mathcal{C} \cap \mathring{E}_2$ is an open $(N-1)-$connected manifold and so, the topological dimension of its boundary is $N-2$. Let us denote by χ the intersection of the $(N-2)-$dimensional boundary of the manifold $\mathcal{C} \cap \mathring{E}_2$ with ∂E_2. Hence, we have

$$\chi \subset \partial(E_1 \cap E_2) \qquad \text{and} \qquad \dim(\chi) = N-2.$$

By Proposition 3.1 & Lemma 3.2, we know that $\partial(E_1 \cap E_2)$ is of class C^2 up to a set $\mathcal{N}$ of Hausdorff dimension $N-8$. In particular, $\mathcal{H}^{N-2}(\chi \setminus \mathcal{N}) > 0$. Fix $x \in \chi \setminus \mathcal{N}$. Then, one can assume that $\partial^\star(E_1 \cap E_2)$ and $\partial^\star E_i$ (with either $i=1$ or $i=2$) are the graphs of two C^2 functions (say v and v_i) around x. Moreover, the mean curvature of $\partial^\star(E_1 \cap E_2)$ and $\partial^\star E_i$ in the neighborhood of x should be equal to $H = \frac{-\lambda}{N-1}$. But, we may clearly assume that $v_i \leq v$ since $E_1 \cap E_2 \subset E_i$. Thus, thanks to the comparison principle in [17, Lemma 2.4], we infer that $v = v_i$ and so, $E_1 = E_2$ in a small neighborhood of x. But, this yields clearly to a contradiction. □

As an immediate consequence of the comparison principle above, we obtain the following elegant estimate which bounds the difference between two $\psi-$least gradient functions in terms of the difference of their corresponding boundary data.

Proposition 3.7. *Suppose that u_1 and u_2 are two solutions to Problem (1.3) with $u_1 = f_1$ and $u_2 = f_2$ on $\partial\Omega$, respectively. If $f_1 \leq f_2$ on $\partial\Omega$, then $u_1 \leq u_2$ a.e. in Ω. Moreover, we have*

$$|u_1 - u_2| \leq \max_{\partial\Omega} |f_1 - f_2|.$$

Proof. Assume $f_1 \leq f_2$ on $\partial\Omega$. Let us extend f_1 and f_2 by two continuous functions over $\tilde{\Omega}\backslash\Omega$ with $\tilde{f}_1 \leq \tilde{f}_2$ on $\tilde{\Omega}\backslash\Omega$ (this is possible since we can always replace $\tilde{f}_1$ and $\tilde{f}_2$ with $\min\{\tilde{f}_1, \tilde{f}_2\}$ and $\max\{\tilde{f}_1, \tilde{f}_2\}$). Now, assume that

$$|\{x \in \tilde{\Omega} : u_1(x) > u_2(x)\}| > 0.$$

It is clear that

$$\{x \in \tilde{\Omega} : u_1(x) > u_2(x)\} \subset \Omega.$$

Let s_1 and s_2 be two constants such that

$$|\{x \in \tilde{\Omega} : u_1(x) > s_1 > s_2 > u_2(x)\}| > 0. \tag{3.8}$$

Define

$$E_i := \{x \in \tilde{\Omega} : u_i(x) \geq s_i\}, \;\; i = 1, 2.$$

Then, we clearly have $|E_1 \backslash E_2| > 0$. Yet, $\tilde{f}_1 \leq \tilde{f}_2$ on $\tilde{\Omega}\backslash\Omega$. Hence, thanks to the continuity of $\tilde{f}_1$ and $\tilde{f}_2$, one has

$$\overline{E_1}\backslash\overline{\Omega} = \{x \in \tilde{\Omega}\backslash\overline{\Omega} \,:\, \tilde{f}_1 \geq s_1\} \subset \{x \in \tilde{\Omega}\backslash\overline{\Omega} \,:\, \tilde{f}_2 \geq s_1\} \subset \{x \in \tilde{\Omega}\backslash\overline{\Omega} \,:\, \tilde{f}_2 > s_2\} = \mathring{E}_2\backslash\overline{\Omega}.$$

Hence, thanks to the comparison principle in Proposition 3.6, we infer that $E_1 \subset E_2$. But, this is in contradiction with $|E_1\backslash E_2| > 0$.

Finally, set $c = \max_{\partial\Omega} |f_1 - f_2|$. Then, we clearly have $f_1 \leq f_2 + c$ on $\partial\Omega$. On the other, it is easy to see that $u_2 + c$ is $\psi-$least gradient. Hence, we should have

$$u_1 \leq u_2 + c.$$

Replacing u_1 and u_2, we get

$$|u_1 - u_2| \leq c. \quad \square$$

In particular, we get the following uniqueness result.

Theorem 3.8. *Assume that Ω satisfies the $\psi-$barrier condition, $\psi = \lambda$ with $|\lambda| < \lambda_\Omega$ and, $f \in C(\partial\Omega)$. Then, the solution of Problem (1.3) is unique.*

Finally, we conclude this paper by the following regularity result on the solution u, which holds only in low dimensions.

Theorem 3.9. *Under the assumptions of Theorem 3.8, the solution u of Problem (1.3) is continuous on $\overline{\Omega}$ as soon as $N \leq 7$.*

Proof. The continuity of u on $\partial\Omega$ follows from the fact that we already proved in Theorem 2.4 that $u = f$ on $\partial\Omega$ in the sense that:

$$\lim_{r\to 0} \operatorname{ess\,sup}_{y\in B(x,r)\cap\Omega}|u(y) - f(x)| = 0, \text{ for all } x \in \partial\Omega.$$

Now, assume that there is a point $x_0 \in \Omega$ such that u is discontinuous at x_0. Then, there exist two numbers s_1 and s_2 such that

$$\lim_{n\to\infty} \operatorname{ess\,inf}_{B(x_0,\frac{1}{n})} u < s_1 < s_2 < \lim_{n\to\infty} \operatorname{ess\,sup}_{B(x_0,\frac{1}{n})} u.$$

Consider the superlevel sets $E_{s_1} := \{x \in \tilde{\Omega} : u(x) \geq s_1\}$ and $E_{s_2} := \{x \in \tilde{\Omega} : u(x) \geq s_2\}$. Hence, we see that $x_0 \in \overline{E_{s_2}} \cap \overline{\tilde{\Omega} \setminus E_{s_1}}$. However, we have $E_{s_2} \subset E_{s_1}$. Then, we infer that

$$x_0 \in \partial E_{s_1} \cap \partial E_{s_2}.$$

Thanks to Proposition 3.1, since $N \leq 7$ then ∂E_{s_1} and ∂E_{s_2} are of class C^2 in the neighborhood of x_0 and the mean curvature of ∂E_{s_1} and ∂E_{s_2} is the same

$$H = \frac{-\lambda}{N-1}.$$

Thus, we infer that ∂E_{s_1} and ∂E_{s_2} coincide. But, by Lemma 3.3, $\partial E_s \cap \partial\Omega \subset f^{-1}(s)$. This contradicts the continuity of f on $\partial\Omega$. □

Remark 3.1. *Following the same lines as in the proof of Theorem 3.9, one can show that in higher dimensions, when $N > 8$, the solution u is continuous on $\overline{\Omega}$, except possibly on a set of Hausdorff dimension $N - 8$.*

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