

ON THE LOWER SEMICONTINUITY OF THE LAGRANGIAN H -MASS

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ABSTRACT. In this paper we introduce a notion of H -mass for the Lagrangian formulation of branched optimal transport, and we prove its lower semicontinuity. This is a functional whose expression mirrors that of the Eulerian H -mass for normal currents and the Gilbert energy for discrete networks.

This result provides a simple variational framework for branched optimal transport in the Lagrangian setting, previously unavailable. The proof is inspired by the Euclidean dimension-reduction argument for currents, but requires new tools since convergence of traffic plans does not imply convergence of slices.

KEYWORDS: branched optimal transport, optimal networks, H -mass.

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1. INTRODUCTION

Branched optimal transport models describe variational problems in which the transportation cost favors the concentration of particles, and minimizers therefore exhibit a ramified structure. The discrete problem was introduced by Gilbert [17]. In that model, a network is a finite, oriented, weighted graph connecting sources and sinks, and its cost is obtained by summing, over all edges, a concave power of the weight times the length of the edge so that transporting mass together is strictly cheaper than splitting it into several disjoint paths. This type of discrete ramified transport model has been used in a variety of contexts (irrigation systems, river basins, urban traffic, biological networks, communication networks); see for instance the monograph by Bernot, Caselles and Morel [2] and the references therein.

A substantial amount of work has been devoted to developing continuous analogues of Gilbert's model. On the *Eulerian* side, Xia introduced *transport paths*, a formulation in terms of vector-valued measures (or normal 1-currents), see [28]. Simultaneously, on the *Lagrangian* side, Maddalena, Solimini and Morel introduced *irrigation patterns*, that is, parametrized families of curves, see [20]. The *traffic plan* formalism of Bernot, Caselles and Morel, see [3, 2], provides a unifying Lagrangian framework: a traffic plan is a probability measure on the space of 1-Lipschitz curves.

On the Eulerian side, if T is a rectifiable 1-current in \mathbb{R}^d one can write

$$T = \llbracket E, \tau, \theta \rrbracket,$$

where E is a 1-rectifiable set, τ its orientation and θ the multiplicity. The typical branched-transport functional is then the α -mass

$$\mathbb{M}^\alpha(T) = \int_E |\theta(x)|^\alpha d\mathcal{H}^1(x), \quad \alpha \in (0, 1),$$

and more generally one considers an H -mass

$$\mathbb{M}_H(T) = \int_E H(|\theta(x)|) d\mathcal{H}^1(x),$$

for suitable functions $H : [0, +\infty) \rightarrow [0, +\infty)$, see [4]. Lower semicontinuity properties of \mathbb{M}_H with respect to flat convergence of currents are by now well understood, starting from the relaxation results of [11, 4, 23, 19] and the dimension-reduction arguments developed there. These Eulerian lower semicontinuity results are a key ingredient in many recent results, including existence, stability, and generic uniqueness of optimal networks; we refer for instance to [7, 8, 5, 30, 29].

In the Lagrangian setting of traffic plans, Bernot–Caselles–Morel introduce the (Gilbert) α -energy of a traffic plan \mathbf{P} , which we recall here. A traffic plan is typically represented by a measurable parametrisation

$$\chi : \Omega \times \mathbb{R}_+ \rightarrow X,$$

where Ω is a probability space indexing the fibres, and each $\chi(\omega, \cdot)$ is 1-Lipschitz and eventually constant. The multiplicity of the traffic plan at a point (ω, t) is denoted by $|\chi(\omega, t)|_\chi$ and represents the amount of fibers passing through the point $\chi(\omega, t)$. For $\alpha \in [0, 1]$ the α -energy of \mathbf{P} , also called Gilbert energy, is defined by [2, Definition 3.33]

$$\mathcal{E}^\alpha(\mathbf{P}) := \int_{\Omega} \int_{\mathbb{R}_+} |\chi(\omega, t)|_\chi^{\alpha-1} |\dot{\chi}(\omega, t)| dt d\omega, \quad (1.1)$$

with the convention $0^{\alpha-1} = +\infty$. One of the main results presented in [2] is the proof of the lower semicontinuity of \mathcal{E}^α and the existence of minimizers for the irrigation and *who-goes-where* problems.

The quantity that (1.1) is meant to extend is the “Gilbert energy”. For a *loop-free* traffic plan \mathbf{P} , Bernot–Caselles–Morel prove the representation formula

$$\mathcal{E}^\alpha(\mathbf{P}) = \int_{\mathbb{R}^d} \theta_{\mathbf{P}}(x)^\alpha d\mathcal{H}^1(x), \quad (1.2)$$

where $\theta_{\mathbf{P}}(x)$ denotes the multiplicity of \mathbf{P} at the point x , see [2, Proposition 4.8]. Thus, (1.2) is the exact Lagrangian counterpart of Gilbert’s discrete cost and of the Eulerian α -mass for currents. The equivalence between these two quantities plays a fundamental role in many recent results on the well posedness of the problem, including [9, 10].

However, as emphasized in [2, Chapter 4], it is not possible to simply *define* the energy of a traffic plan by the right-hand side of (1.2). On the one hand, this would require a priori rectifiability assumptions on the network, which are not easy to handle. On the other hand, and more importantly for the present work, the authors explain that a direct variational treatment of the Gilbert energy appears out of reach, since the multiplicity map $x \mapsto \theta_{\mathbf{P}}(x)$ is not lower semicontinuous with respect to convergence of traffic plans. In particular, they explicitly write that “*a direct proof of the lower semicontinuity of the Gilbert energy does not seem to be easily attainable*” [2, p. 39]. This difficulty is precisely the reason for introducing the more involved definition (1.1) of α -energy in place of the more geometric α -mass suggested by (1.2).

The main purpose of this paper is to close this gap at the level of Lagrangian functionals. We introduce a notion of H -mass $\mathbb{M}_H(\mathbf{P})$ for traffic plans \mathbf{P} , defined directly in terms of the multiplicity $\theta_{\mathbf{P}}$ and of the network

$$\Sigma_{\mathbf{P}} := \{x \in X : \theta_{\mathbf{P}}(x) > 0\},$$

together with a measure describing the diffuse part of the plan. More precisely, for a Borel function $H : [0, +\infty) \rightarrow [0, +\infty)$ satisfying natural assumptions (monotonicity,

subadditivity, and lower semicontinuity), we set

$$\mathbb{M}_H(\mathbf{P}) := \int_{\Sigma_{\mathbf{P}}} H(\theta_{\mathbf{P}}(x)) d\mathcal{H}^1(x) + H'(0) \mu_{\mathbf{P}}^d(X),$$

where $\mu_{\mathbf{P}}^d$ is the diffuse part of a measure canonically associated with \mathbf{P} , compare also with [4, Propositions 2.32 and 3.19]. In particular, when $H(\theta) = \theta$ we recover the usual mass of a traffic plan, and when $H(\theta) = \theta^\alpha$ with $\alpha \in (0, 1)$ we obtain what we shall call the *Lagrangian α -mass*

$$\mathbb{M}^\alpha(\mathbf{P}) := \int_{\Sigma_{\mathbf{P}}} \theta_{\mathbf{P}}(x)^\alpha d\mathcal{H}^1(x),$$

which is the direct analogue of the Eulerian α -mass for currents and of the Gilbert energy for discrete networks. Compared with the α -energy \mathcal{E}^α in (1.1), the functional \mathbb{M}^α has a considerably simpler geometric expression, but its lower semicontinuity had remained open, see [2, Chapter 4].

Our main result is the lower semicontinuity of this Lagrangian H -mass with respect to weak* convergence of traffic plans, see Theorem 3.1. Roughly speaking, we prove that if $(\mathbf{P}_n)_n$ is a sequence of traffic plans converging to a traffic plan \mathbf{P} , then

$$\mathbb{M}_H(\mathbf{P}) \leq \liminf_{n \rightarrow \infty} \mathbb{M}_H(\mathbf{P}_n).$$

In particular, for $H(\theta) = \theta^\alpha$ we obtain the lower semicontinuity of the Lagrangian α -mass \mathbb{M}^α .

One can thus view our result as giving a posteriori justification for the “simple” formula suggested by Gilbert’s discrete energy in the Lagrangian framework, in the same spirit in which the results of [11] justify the analogous formula for currents.

1.1. Idea of the proof. The proof is inspired by the dimension-reduction arguments used in the Eulerian setting, but it also presents substantial new difficulties. In the case of currents, the lower semicontinuity of \mathbb{M}_H can be reduced to the 0-dimensional case by slicing: if $T_n \rightarrow T$ in the flat norm, then for almost every slice (with respect to a Lipschitz map) one has flat convergence of the slices $\langle T_n, f, y \rangle \rightarrow \langle T, f, y \rangle$, and one can exploit the lower semicontinuity of the corresponding “0-dimensional H -mass” together with an integral-geometric formula.

In the Lagrangian framework, we also work with slices, but the situation is more delicate: we introduce a notion of simple Lagrangian slices $\langle \mathbf{P}, f, y \rangle$, which are measures counting intersections between trajectories and level sets $\{f = y\}$ of Lipschitz functions, see §2.5. While it is still possible to establish lower semicontinuity for the H -mass of such measures under suitable assumptions, **the convergence of traffic plans does not imply any convergence of the slices**. In particular, one cannot immediately exploit the lower semicontinuity of the 0-dimensional H -mass as in the Eulerian case.

The key observation is that, even in the absence of convergence of the slices, we retain a one-sided control on the multiplicities: roughly speaking, for almost every slice and for almost every point in the intersection with the limit network, the amount of sliced mass that the sequence $(\mathbf{P}_n)_n$ places in small neighbourhoods of that point is *at least* as large as the sliced mass of the limit traffic plan, see Proposition 3.2. A careful decomposition of each slice into its atomic part and its diffuse part then permits us to exploit this inequality: using the monotonicity and subadditivity assumptions on H , we obtain the lower semicontinuity of the H -mass of the slices, separately on the atomic and on the diffuse component. In particular, for almost every choice of the slicing level y we achieve

$$\mathbb{M}_H(\langle \mathbf{P}, f, y \rangle) \leq \liminf_{n \rightarrow \infty} \mathbb{M}_H(\langle \mathbf{P}_n, f, y \rangle).$$

To pass from slice-wise to global lower semicontinuity, we first develop an integral-geometric representation of the Lagrangian H -mass via simple slices: we show that $\mathbb{M}_H(\mathbf{P})$ can be expressed as an average of the H -masses of the slices $\langle \mathbf{P}, f, y \rangle$, taken over appropriate families of Lipschitz functions f and levels y , see Corollary 2.26. Combining this representation with the slice-wise liminf inequality above, and using Fubini-type arguments, we finally deduce the global lower semicontinuity of \mathbb{M}_H for traffic plans, see Propositions 3.3 and 3.4.

1.2. Applications. The lower semicontinuity of \mathbb{M}_H is a crucial property for many questions where the Lagrangian formulation is particularly natural, such as stability and regularity of optimal traffic plans and generalized network problems involving additional structure such as multi-material transport, orientation constraints, robustness with respect to damages, and so on, see [12, 22, 27, 6, 21, 15, 30, 13, 29, 14]. The lower semicontinuity of the Lagrangian H -mass established here provides a flexible tool that can be combined with these results and with the known equivalences between Eulerian and Lagrangian formulations (see e.g. [25, 26]) to obtain new existence and stability statements in settings where the Lagrangian viewpoint is essential.

As a concrete example in this direction, in a subsequent work we plan to adapt the methods developed in the present paper to prove stability results for the mailing problem in a very general framework, extending the analysis of [9] and removing technical restrictions on the cost. The possibility of working directly with the Lagrangian α -mass, rather than with the more involved α -energy \mathcal{E}^α , is particularly important in that context.

2. PRELIMINARIES

2.1. Notation. Throughout this paper, $X \subset \mathbb{R}^d$ will be a fixed compact and convex subset. Sets and functions on \mathbb{R}^d are assumed to be Borel measurable, and, unless we specify otherwise, measures on \mathbb{R}^d are positive, finite measures on the Borel σ -algebra (the obvious exceptions being the Hausdorff measures). We will also use the following notation:

- $|x|$: Euclidean norm of $x \in \mathbb{R}^d$;
- $B_r(x)$: open Euclidean ball centred at $x \in \mathbb{R}^d$ with radius $r > 0$;
- $\text{dist}(E, F)$: Euclidean distance between $E \subset \mathbb{R}^d$ and $F \subset \mathbb{R}^d$, namely $\text{dist}(E, F) := \inf_{x \in E, y \in F} |x - y|$;
- $\text{dist}(x, E)$: Euclidean distance between $x \in \mathbb{R}^d$ and $E \subset \mathbb{R}^d$, namely $\text{dist}(x, E) := \text{dist}(\{x\}, E)$;
- $U(E, \rho)$: open tubular neighbourhood of $E \subset \mathbb{R}^d$ of radius $\rho > 0$, namely $U(E, \rho) := \{x \in \mathbb{R}^d \mid \text{dist}(x, E) < \rho\}$;
- \bar{E} : closure of a subset $E \subset \mathbb{X}$, if \mathbb{X} is a topological space;
- χ_E : characteristic function of a set E , taking values 0 and 1;
- \mathcal{H}^k : k -dimensional Hausdorff measure;
- $\mathcal{M}_+(\mathbb{X})$: the space of (non-negative) Radon measures on \mathbb{X} , when \mathbb{X} is a metric space;
- $\mu_n \xrightarrow{*} \mu$: weak* convergence of measures, namely $\int \phi d\mu_n \rightarrow \int \phi d\mu$ for every $\phi \in \mathcal{C}^0(\mathbb{X})$, provided that \mathbb{X} is a compact metric space;
- $\mu \llcorner E$: restriction of a measure μ to $E \subset \mathbb{X}$, that is, $\mu \llcorner E(A) := \mu(E \cap A)$ for every A Borel;
- $f_{\#}\mu$: push-forward of a measure with respect to a map $f: \mathbb{X} \rightarrow \mathbb{Y}$, namely $f_{\#}\mu(E) := \mu(f^{-1}(E))$ if $E \subset \mathbb{Y}$ Borel;
- $\text{Gr}(k, d)$: the Grassmannian of k -dimensional vector subspaces of \mathbb{R}^d .

A 1-*rectifiable* set is a Borel subset $E \subset \mathbb{R}^d$ such that $E \subset E_0 \cup \bigcup_{i=1}^{+\infty} E_i$, where $\mathcal{H}^1(E_0) = 0$ and $E_i = f_i(\mathbb{R})$, with $f_i: \mathbb{R} \rightarrow \mathbb{R}^d$ a Lipschitz curve. Recall that for 1-rectifiable subsets the weak tangent line at $x \in E$, that we denote by $\text{Tan}(x, E)$, is defined for \mathcal{H}^1 -a.e. $x \in E$ (see [1, Chapter 2] for more details).

2.2. Traffic plans. This section is devoted to some preliminary notation and facts concerning traffic plans. The main references for the topic are [2, 26].

We denote

$$\text{Lip}_1 := \{\gamma: \mathbb{R}^+ \rightarrow X \mid \gamma \text{ is Lipschitz with } \text{Lip}(\gamma) \leq 1\}.$$

We consider on Lip_1 the topology induced by the distance

$$d_L(\gamma_1, \gamma_2) := \sup_{k \in \mathbb{N}^+} \frac{1}{k} \|\gamma_1 - \gamma_2\|_{\infty, [0, k]},$$

which metrizes the topology of uniform convergence on compact subsets of \mathbb{R}^+ . It is well-known (see [2, Lemma 3.4]) that (Lip_1, d_L) is a compact metric space.

Definition 2.1 (Stopping time of a curve). Given $\gamma \in \text{Lip}_1$, we define its *stopping time* as

$$T_\infty(\gamma) := \inf\{t \in \mathbb{R}^+ \mid \gamma \text{ is constant on } [t, +\infty)\}$$

Definition 2.2 (Full length of a curve). Given $\gamma \in \text{Lip}_1$, we define its *full length* as

$$L(\gamma) := \int_0^{+\infty} |\gamma'(t)| dt,$$

where the integral is well-defined thanks to Rademacher's theorem.

Lemma 2.3 (Lemma 3.20 of [2]). *The maps $\gamma \mapsto T_\infty(\gamma)$ and $\gamma \mapsto L(\gamma)$ are lower semicontinuous with respect to the topology induced by d_L .*

We also denote

$$\text{Lip} := \{\gamma \in \text{Lip}_1 \mid T_\infty(\gamma) < +\infty\}.$$

In what follows, we will use the following maps:

$$\begin{aligned} e_0: \text{Lip}_1 &\rightarrow X, & e_\infty: \text{Lip} &\rightarrow X. \\ \gamma &\mapsto \gamma(0) & \gamma &\mapsto \gamma(T_\infty(\gamma)) \end{aligned}$$

The map e_0 is continuous with respect to the topology induced by d_L , while e_∞ is Borel-measurable. In particular, we can extend e_∞ to a Borel-measurable function defined on the whole Lip_1 . We tacitly use this fact.

We are now ready to define traffic plans.

Definition 2.4 (Traffic plan). A *traffic plan* on X is a Radon measure $\mathbf{P} \in \mathcal{M}_+(\text{Lip}_1)$ such that

$$\int_{\text{Lip}_1} T_\infty(\gamma) d\mathbf{P}(\gamma) < +\infty.$$

We denote the space of traffic plans on X by $\mathbf{TP}(X)$.

Observe that, by definition, a traffic plan is concentrated on Lip .

Definition 2.5 (Marginals of a traffic plan). Let $\mathbf{P} \in \mathbf{TP}(X)$. The *starting marginal* and the *ending marginal* of \mathbf{P} are defined, respectively, as

$$\mu_{\mathbf{P}}^- := (e_0)_\# \mathbf{P}, \quad \mu_{\mathbf{P}}^+ := (e_\infty)_\# \mathbf{P}.$$

Definition 2.6 (Coupling of a traffic plan). Let $\mathbf{P} \in \mathbf{TP}(X)$. The *coupling* of \mathbf{P} is defined as

$$\pi_{\mathbf{P}} := (e_0, e_{\infty})_{\#} \mathbf{P}.$$

The notions of marginals and coupling can be interpreted as “boundary conditions”: the former are prescribed in the Lagrangian branched transport problem, while the latter is prescribed in the Mailing Problem (see [2, 9]). While the marginals only provide a “global” information about the displacement of the measures, the coupling provides a more specific information, that allows one to track “where every particle goes”, once it is transported via \mathbf{P} .

Let us also define the *multiplicities* of a traffic plan. These quantities will play a key role later.

Definition 2.7 (Multiplicities of a traffic plan at a point). Let $\mathbf{P} \in \mathbf{TP}(X)$ and let $x \in \mathbb{R}^d$. The *multiplicity* of \mathbf{P} at x is

$$\theta_{\mathbf{P}}(x) := \mathbf{P}(\{\gamma \in \text{Lip}_1 \mid \text{Im } \gamma \ni x\}).$$

The *full multiplicity* of \mathbf{P} at x is

$$\Theta_{\mathbf{P}}(x) := \int_{\text{Lip}_1} \mathcal{H}^0(\gamma^{-1}(x)) d\mathbf{P}(\gamma).$$

It can be readily checked that $\theta_{\mathbf{P}}(x) \leq \Theta_{\mathbf{P}}(x)$ for every $x \in \mathbb{R}^d$.

2.2.1. Rectifiable traffic plans. We now briefly recall the notion of rectifiability for traffic plans.

Let us begin by associating to every traffic plan \mathbf{P} a 1-rectifiable set.

Definition 2.8 (Network of a traffic plan). Let $\mathbf{P} \in \mathbf{TP}(X)$. Its *network* is

$$\Sigma_{\mathbf{P}} := \{x \in \mathbb{R}^d \mid \theta_{\mathbf{P}}(x) > 0\}.$$

Lemma 2.9 (Proposition 2.5 of [26]). *Let $\mathbf{P} \in \mathbf{TP}(X)$. Then $\Sigma_{\mathbf{P}}$ is a 1-rectifiable set.*

The definition of network allows one to define the notion of *rectifiability* for traffic plans.

Definition 2.10 (Rectifiable traffic plan). A traffic plan $\mathbf{P} \in \mathbf{TP}(X)$ is said to be *rectifiable* if $\mathcal{H}^1(\text{Im } \gamma \setminus \Sigma_{\mathbf{P}}) = 0$ for \mathbf{P} -a.e. $\gamma \in \text{Lip}_1$.

2.3. The Lagrangian H -mass. In what follows, we will consider a Borel-measurable function $H: [0, +\infty) \rightarrow [0, +\infty)$ such that

- (H1) $H(0) = 0$;
- (H2) H is subadditive and nondecreasing;
- (H3) H is lower semicontinuous, that is, for every $\theta \in [0, +\infty)$ and for every $(\theta_n)_{n \in \mathbb{N}}$ with $\theta_n \rightarrow \theta$, it holds that

$$H(\theta) \leq \liminf_{n \rightarrow +\infty} H(\theta_n).$$

Remark 2.11. By combining (H2) and (H3), it can be easily checked that H is countably subadditive (see [11, Remark 2.2]).

Remark 2.12. It follows from [18, Theorem 16.3.3], that there exists

$$H'(0) = \lim_{\theta \rightarrow 0^+} \frac{H(\theta)}{\theta} = \sup_{\theta > 0} \frac{H(\theta)}{\theta} \in [0, +\infty].$$

To define the H -mass of a traffic plan, we need to study the decomposition of a measure naturally associated with it. Let us be more precise.

Given a traffic plan $\mathbf{P} \in \mathbf{TP}(X)$, define the measure

$$\mu_{\mathbf{P}} := \int_{\text{Lip}_1} \mathcal{H}^1 \llcorner (\text{Im } \gamma) d\mathbf{P}(\gamma).$$

Decomposing $\mu_{\mathbf{P}}$ as

$$\mu_{\mathbf{P}} = \mu_{\mathbf{P}} \llcorner \Sigma_{\mathbf{P}} + \mu_{\mathbf{P}} \llcorner (\Sigma_{\mathbf{P}})^c,$$

we would like to characterize $\mu_{\mathbf{P}} \llcorner \Sigma_{\mathbf{P}}$: for every $\phi \in \mathcal{C}^0(X)$, using Fubini's theorem, one obtains that

$$\begin{aligned} \langle \mu_{\mathbf{P}} \llcorner \Sigma_{\mathbf{P}}, \phi \rangle &= \int_{\text{Lip}_1} \int_{\text{Im } \gamma \cap \Sigma_{\mathbf{P}}} \phi(x) d\mathcal{H}^1(x) d\mathbf{P}(\gamma) \\ &= \int_{\text{Lip}_1} \int_{\Sigma_{\mathbf{P}}} \phi(x) \chi_{\text{Im } \gamma}(x) d\mathcal{H}^1(x) d\mathbf{P}(\gamma) \\ &= \int_{\Sigma_{\mathbf{P}}} \int_{\text{Lip}_1} \phi(x) \chi_{\{\text{Im } \gamma \ni x\}}(\gamma) d\mathbf{P}(\gamma) d\mathcal{H}^1(x) \\ &= \int_{\Sigma_{\mathbf{P}}} \phi(x) \theta_{\mathbf{P}}(x) d\mathcal{H}^1(x) \\ &= \langle \theta_{\mathbf{P}} \mathcal{H}^1 \llcorner \Sigma_{\mathbf{P}}, \phi \rangle. \end{aligned}$$

In particular, we have the following decomposition of $\mu_{\mathbf{P}}$:

$$\mu_{\mathbf{P}} = \theta_{\mathbf{P}} \mathcal{H}^1 \llcorner \Sigma_{\mathbf{P}} + \mu_{\mathbf{P}}^d, \text{ where } \mu_{\mathbf{P}}^d := \mu_{\mathbf{P}} \llcorner (\Sigma_{\mathbf{P}})^c. \quad (2.1)$$

Remark 2.13. Observe that $\mathbf{P} \in \mathbf{TP}(X)$ is rectifiable if and only if $\mu_{\mathbf{P}}^d = 0$.

Taking into account the decomposition in (2.1), the measure $\mu_{\mathbf{P}}^d$ represents the ‘‘diffuse’’ part of the traffic plan. By Remark 2.13, the rectifiability of a traffic plan intrinsically requires that no diffusion phenomena occur. However, for general traffic plans, one must also keep track of the diffused part. This leads to the following definition of H -mass.

Definition 2.14 (H -mass of a traffic plan). Let $\mathbf{P} \in \mathbf{TP}(X)$ and let $H: [0, +\infty) \rightarrow [0, +\infty)$ be a function satisfying (H1), (H2) and (H3). The H -mass of \mathbf{P} is defined as

$$\mathbb{M}_H(\mathbf{P}) := \int_{\Sigma_{\mathbf{P}}} H(\theta_{\mathbf{P}}(x)) d\mathcal{H}^1(x) + H'(0) \mu_{\mathbf{P}}^d(X),$$

where $\mu_{\mathbf{P}}^d$ is defined in (2.1).

Remark 2.15. Depending on the choice of H , one obtains functionals that are relevant in the literature:

- if $H(\theta) = \theta$, one refers to \mathbb{M}_H as the *mass*¹ of the traffic plan, usually denoted by \mathbb{M} ;
- if $H(\theta) = \theta^\alpha$ with $\alpha \in (0, 1)$, one refers to \mathbb{M}_H as the α -*mass* of the traffic plan, usually denoted by \mathbb{M}^α . This functional is relevant in Branched Optimal Transport, since it is the natural generalisation of Gilbert's energy to the continuous setting.

For notational convenience, we also define a localized version of the H -mass.

¹Observe that this notion of mass is different from the notion of total variation.

Definition 2.16 (Localized H -mass of a traffic plan). Let $\mathbf{P} \in \mathbf{TP}(X)$ be a traffic plan, let $H: [0, +\infty) \rightarrow [0, +\infty)$ be a function satisfying (H1), (H2) and (H3), and let $E \subset \mathbb{R}^d$ be a Borel subset. The H -mass of \mathbf{P} localized at E is

$$\mathbb{M}_H(\mathbf{P}; E) := \int_{\Sigma_{\mathbf{P}} \cap E} H(\theta_{\mathbf{P}}(x)) d\mathcal{H}^1(x) + H'(0) \mu_{\mathbf{P}}^d(E),$$

where $\mu_{\mathbf{P}}^d$ is defined in (2.1).

Remark 2.17. Given a measure $\mu \in \mathcal{M}_+(X)$, denoting by

$$A_\mu := \{x \in X \mid \mu(\{x\}) > 0\}$$

the set of atoms of μ , we have that A_μ is at most countable.

In view of Remark 2.17, we can also define the notion of H -mass for a measure.

Definition 2.18 (H -mass of a measure). Let $\mu \in \mathcal{M}_+(X)$ and let $H: [0, +\infty) \rightarrow [0, +\infty)$ be a function satisfying (H1), (H2) and (H3). The H -mass of μ is

$$\mathbb{M}_H(\mu) := \int_{A_\mu} H(\mu(\{x\})) d\mathcal{H}^0(x) + H'(0) \mu^d(X),$$

where $\mu^d := \mu \llcorner (A_\mu)^c$.

2.4. Regularity of intersection points between curves and level sets of Lipschitz functions. In this section, we collect some preliminary facts that will turn out to be useful later. Most of them are based on results concerning 1-rectifiability and the coarea formula, see [1, Chapter 2], for a comprehensive reference.

We start with the following definition of a *regular point*, see [10, Section 2.3].

Definition 2.19 (Regular point of a curve). Let $\gamma \in \text{Lip}$. We say that $x \in X$ is a *regular point of γ* if

- $x \in \text{Im } \gamma \setminus \{\gamma(0), \gamma(T_\infty(\gamma))\}$;
- $\text{Tan}(x, \text{Im } \gamma)$ exists;
- $\mathcal{H}^0(\gamma^{-1}(x)) < +\infty$;
- for every $t \in \gamma^{-1}(x)$, $\gamma'(t)$ exists and $\text{Tan}(x, \text{Im } \gamma) = \text{span}(\gamma'(t))$.

We denote the set of regular points of a curve γ by $\text{Reg}(\gamma)$.

Remark 2.20. It follows from [10, Section 2.3, Remark 1] that, given $\gamma \in \text{Lip}$, \mathcal{H}^1 -a.e. $x \in \text{Im } \gamma$ is a regular point of γ .

Lemma 2.21. Let $\mathbf{P} \in \mathbf{TP}(X)$ be a traffic plan and let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ be a Lipschitz function. Then, for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$, the quantity

$$\int_{\text{Lip}_1} \mathcal{H}^0(\text{Im } \gamma \cap f^{-1}(y)) d\mathbf{P}(\gamma)$$

is finite.

Proof. Using Fubini's theorem and the coarea formula, see [1, Theorem 2.93],

$$\begin{aligned} \int_{\mathbb{R}} \int_{\text{Lip}_1} \mathcal{H}^0(\text{Im } \gamma \cap f^{-1}(y)) d\mathbf{P}(\gamma) dy &= \int_{\text{Lip}_1} \int_{\mathbb{R}} \mathcal{H}^0(\text{Im } \gamma \cap f^{-1}(y)) dy d\mathbf{P}(\gamma) \\ &\leq \text{Lip}(f) \int_{\text{Lip}_1} \mathcal{H}^1(\text{Im } \gamma) d\mathbf{P}(\gamma) \\ &\leq \text{Lip}(f) \int_{\text{Lip}_1} T_\infty(\gamma) d\mathbf{P}(\gamma) < +\infty, \end{aligned}$$

and this concludes the proof. \square

In particular, by Lemma 2.21, setting

$$Y(f) := \{y \in \mathbb{R} \mid \mathcal{H}^0(\text{Im } \gamma \cap f^{-1}(y)) < +\infty \text{ for } \mathbf{P}\text{-a.e. } \gamma \in \text{Lip}\}, \quad (2.2)$$

one has $\mathcal{H}^1(\mathbb{R} \setminus Y(f)) = 0$.

Let us now consider a Lipschitz function $f: \mathbb{R}^d \rightarrow \mathbb{R}$. It is well-known that, if $E \subset \mathbb{R}^d$ is a 1-rectifiable subset, then for \mathcal{H}^1 -a.e. $x \in E$, the *tangential differential* at x , denoted by $d_\tau^E f(x)$, is well-defined (see [1, Chapter 2]).

We are now ready to prove the following lemma, which will allow us to choose level sets intersecting almost every curve (with respect to a traffic plan) at points with good differentiability properties.

Lemma 2.22. *Let $\mathbf{P} \in \mathbf{TP}(X)$ be a traffic plan and let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ be a Lipschitz function. Then, for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$, for \mathbf{P} -a.e. $\gamma \in \text{Lip}_1$, the following facts hold*

- (R1) *for every $x \in \text{Im } \gamma \cap f^{-1}(y)$, $\gamma^{-1}(x) \cap (0, T_\infty(\gamma))$ is finite and for every $t \in \gamma^{-1}(x) \cap (0, T_\infty(\gamma))$, γ is differentiable at t and $\text{span } \gamma'(t) = \text{Tan}(x, \text{Im } \gamma)$;*
- (R2) *for every $x \in \text{Im } \gamma \cap f^{-1}(y)$, $d_\tau^{\text{Im } \gamma} f(x)$ exists and $d_\tau^{\text{Im } \gamma} f(x)[v] \neq 0$ for every $v \in \text{Tan}(x, \text{Im } \gamma)$.*

Proof. Considering $Y(f)$ as in (2.2), we have $\mathcal{H}^1(\mathbb{R} \setminus Y(f)) = 0$. For $y \in Y(f)$, define

$$N_1(y) := \{\gamma \in \text{Lip} \mid \text{(R1) does not hold}\},$$

$$N_2(y) := \{\gamma \in \text{Lip} \mid \text{(R1) holds but (R2) does not hold}\}.$$

Proving that, for \mathcal{H}^1 -a.e. $y \in Y(f)$, $\mathbf{P}(N_1(y)) = \mathbf{P}(N_2(y)) = 0$, we would conclude.

Regarding $N_1(y)$, using Fubini's theorem,

$$\int_{Y(f)} \mathbf{P}(N_1(y)) dy = \int_{\text{Lip}_1} \mathcal{H}^1(\{y \in Y(f) \mid N_1(y) \ni \gamma\}) d\mathbf{P}(\gamma). \quad (2.3)$$

If $\gamma \in \text{Lip}$,

$$\{y \in Y(f) \mid N_1(y) \ni \gamma\} \subset f(\text{Im } \gamma \setminus \text{Reg}(\gamma)). \quad (2.4)$$

Indeed, if y is such that $N_1(y) \ni \gamma$, then there exists $x \in \text{Im } \gamma \cap f^{-1}(y)$ such that $x \notin \text{Reg}(\gamma)$. Furthermore, $f(x) = y$, so $y \in f(\text{Im } \gamma \setminus \text{Reg}(\gamma))$. In particular, using (2.4),

$$\begin{aligned} \mathcal{H}^1(\{y \in Y(f) \mid N_1(y) \ni \gamma\}) &\leq \mathcal{H}^1(f(\text{Im } \gamma \setminus \text{Reg}(\gamma))) \\ &\leq \text{Lip}(f) \mathcal{H}^1(\text{Im } \gamma \setminus \text{Reg}(\gamma)) = 0, \end{aligned} \quad (2.5)$$

where the last equality is due to Remark 2.20. Since \mathbf{P} is concentrated on Lip , by combining (2.3) and (2.5), we conclude that $\mathbf{P}(N_1(y)) = 0$ for \mathcal{H}^1 -a.e. $y \in Y(f)$.

Concerning $N_2(y)$, arguing as above,

$$\int_{Y(f)} \mathbf{P}(N_2(y)) dy = \int_{\text{Lip}_1} \mathcal{H}^1(\{y \in Y(f) \mid N_2(y) \ni \gamma\}) d\mathbf{P}(\gamma). \quad (2.6)$$

For a fixed $\gamma \in \text{Lip}$, for \mathcal{H}^1 -a.e. $x \in \text{Im } \gamma$, $d_\tau^{\text{Im } \gamma} f(x)$ exists and, by the coarea formula (see [1, Theorem 2.93]), if $\tau_\gamma(x)$ is a vector with $|\tau_\gamma(x)| = 1$ spanning $\text{Tan}(x, \text{Im } \gamma)$, then

$$\int_{\mathbb{R}} \int_{\text{Im } \gamma \cap f^{-1}(y)} \frac{1}{|d_\tau^{\text{Im } \gamma} f(x)[\tau_\gamma(x)]|} d\mathcal{H}^0(x) dy = \mathcal{H}^1(\text{Im } \gamma) < +\infty,$$

In particular, for \mathcal{H}^1 -a.e. $y \in Y(f)$, $d_\tau^{\text{Im } \gamma} f(x)[v] \neq 0$ for every $v \in \text{Tan}(x, \text{Im } \gamma)$, for every $x \in \text{Im } \gamma \cap f^{-1}(y)$. In other words, $\gamma \notin N_2(y)$ for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$, so, for $\gamma \in \text{Lip}$,

$$\mathcal{H}^1(\{y \in Y(f) \mid N_2(y) \ni \gamma\}) = 0. \quad (2.7)$$

Combining (2.6) and (2.7), we conclude that $\mathbf{P}(N_2(y)) = 0$ for \mathcal{H}^1 -a.e. $y \in Y(f)$. \square

2.5. Simple Lagrangian slicing. In [10], a Lagrangian analog of the usual notion of slicing for currents was introduced. In this section, we present a different notion of slicing from the one presented therein, called *simple slicing*. This turns out to be useful in representing the H -mass by means of an integral-geometric formula.

Let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ be a Lipschitz function. Given a curve $\gamma \in \text{Lip}_1$, for \mathcal{L}^1 -a.e. $y \in \mathbb{R}$, the measure

$$s_\gamma(f, y) := \mathcal{H}^0 \llcorner (\text{Im } \gamma \cap f^{-1}(y))$$

is well-defined and 0-rectifiable.

To define the simple slice of a traffic plan, one would like to integrate all the measures $s_\gamma(f, y)$ with respect to the said traffic plan. This is a well-defined operation, provided that the quantity

$$\int_{\text{Lip}_1} \mathcal{H}^0(\text{Im } \gamma \cap f^{-1}(y)) d\mathbf{P}(\gamma)$$

is finite. By Lemma 2.21, this holds for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$.

Definition 2.23 (Simple slice of a traffic plan). Let $\mathbf{P} \in \mathbf{TP}(X)$ be a traffic plan, let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ be Lipschitz and let $y \in \mathbb{R}$ be such that

$$\int_{\text{Lip}_1} \mathcal{H}^0(\text{Im } \gamma \cap f^{-1}(y)) d\mathbf{P}(\gamma) < +\infty.$$

The *simple slice* of \mathbf{P} with respect to f at y is the measure

$$\langle \mathbf{P}, f, y \rangle := \int_{\text{Lip}_1} s_\gamma(f, y) d\mathbf{P}(\gamma).$$

It is immediate to prove that, when defined, $\langle \mathbf{P}, f, y \rangle$ is concentrated on $f^{-1}(y)$. In order to write the H -mass of a slice, one needs to characterize its atomic part.

Lemma 2.24. *Let $\mathbf{P} \in \mathbf{TP}(X)$ be a traffic plan, let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ and let $y \in \mathbb{R}$ be such that $\langle \mathbf{P}, f, y \rangle$ is well-defined. If $A(\mathbf{P}, f, y)$ denotes the set of atoms of $\langle \mathbf{P}, f, y \rangle$, then*

$$A(\mathbf{P}, f, y) = \Sigma_{\mathbf{P}} \cap f^{-1}(y).$$

Proof. Since $\langle \mathbf{P}, f, y \rangle$ is concentrated on $f^{-1}(y)$, it is enough to prove that, given $x \in f^{-1}(y)$, then

$$x \in \Sigma_{\mathbf{P}} \iff x \in A(\mathbf{P}, f, y).$$

This is true, since

$$\begin{aligned} \langle \mathbf{P}, f, y \rangle(\{x\}) &= \int_{\text{Lip}_1} \mathcal{H}^0(\text{Im } \gamma \cap f^{-1}(y) \cap \{x\}) d\mathbf{P}(\gamma) \\ &= \int_{\text{Lip}_1} \chi_{\{\text{Im } \gamma \ni x\}}(\gamma) d\mathbf{P}(\gamma) \\ &= \theta_{\mathbf{P}}(x), \end{aligned}$$

where we have used the fact that

$$\mathcal{H}^0(\text{Im } \gamma \cap f^{-1}(y) \cap \{x\}) = \begin{cases} 1 & \text{if } x \in \text{Im } \gamma \\ 0 & \text{otherwise.} \end{cases}$$

This concludes the proof. \square

Using Lemma 2.24, we can write the atomic part of $\langle \mathbf{P}, f, y \rangle$: given $\phi \in \mathcal{C}^0(X)$,

$$\begin{aligned}
\langle \langle \mathbf{P}, f, y \rangle \llcorner A(\mathbf{P}, f, y), \phi \rangle &= \langle \langle \mathbf{P}, f, y \rangle \llcorner (\Sigma_{\mathbf{P}} \cap f^{-1}(y)), \phi \rangle \\
&= \int_{\text{Lip}_1} \int_{\text{Im } \gamma \cap \Sigma_{\mathbf{P}} \cap f^{-1}(y)} \phi(x) d\mathcal{H}^0(x) d\mathbf{P}(\gamma) \\
&= \int_{\text{Lip}_1} \int_{\Sigma_{\mathbf{P}} \cap f^{-1}(y)} \phi(x) \chi_{\text{Im } \gamma}(x) d\mathcal{H}^0(x) d\mathbf{P}(\gamma) \\
&= \int_{\Sigma_{\mathbf{P}} \cap f^{-1}(y)} \phi(x) \int_{\text{Lip}_1} \chi_{\{\text{Im } \gamma \ni x\}}(\gamma) d\mathbf{P}(\gamma) d\mathcal{H}^0(x) \\
&= \int_{\Sigma_{\mathbf{P}} \cap f^{-1}(y)} \phi(x) \theta_{\mathbf{P}}(x) d\mathcal{H}^0(x) \\
&= \langle \theta_{\mathbf{P}} \mathcal{H}^0 \llcorner (\Sigma_{\mathbf{P}} \cap f^{-1}(y)), \phi \rangle.
\end{aligned}$$

Therefore, we have characterized the atomic part of $\langle \mathbf{P}, f, y \rangle$, and we can decompose the slice as

$$\langle \mathbf{P}, f, y \rangle = \theta_{\mathbf{P}} \mathcal{H}^0 \llcorner (\Sigma_{\mathbf{P}} \cap f^{-1}(y)) + \langle \mathbf{P}, f, y \rangle^d.$$

We can use this decomposition to compute the H -mass of the slice (see Definition 2.18).

2.6. An integral-geometric formula for the Lagrangian H -mass. We will now focus on the representation of the H -mass of a traffic plan via an integral-geometric formula. Recall that on $\text{Gr}(1, d)$ we can consider a probability measure $\gamma_{1,d}$ which is invariant under the action of $O(d)$ on $\text{Gr}(1, d)$, $O(d)$ denoting the d -dimensional orthogonal group (for more details concerning such a measure, consult [24, Chapter 3]). Given a line $\ell \in \text{Gr}(1, d)$, let $\pi_{\ell}: \mathbb{R}^d \rightarrow \mathbb{R}$ be the orthogonal projection onto ℓ (here we tacitly identify $\ell \equiv \mathbb{R}$ for $\gamma_{1,d}$ -a.e. $\ell \in \text{Gr}(1, d)$). We have the following integral-geometric formula for 1-rectifiable sets (see [16, Sections 2.10.15, 3.2.26]).

Theorem 2.25. *There exists $c = c(d) > 0$ such that, if $E \subset \mathbb{R}^d$ is 1-rectifiable, then*

$$\mathcal{H}^1(E) = c \int_{\text{Gr}(1,d) \times \mathbb{R}} \mathcal{H}^0(E \cap \pi_{\ell}^{-1}(y)) d(\gamma_{1,d} \otimes \mathcal{H}^1)(\ell, y).$$

Corollary 2.26 (Integral-geometric formula for the H -mass). *There exists $c = c(d) > 0$ such that, if $\mathbf{P} \in \mathbf{TP}(X)$ is a traffic plan and $E \subset \mathbb{R}^d$ is a Borel subset, then*

$$\mathbb{M}_H(\mathbf{P}; E) = \int_{\text{Gr}(1,d) \times \mathbb{R}} \mathbb{M}_H(\langle \mathbf{P}, \pi_{\ell}, y \rangle \llcorner E) d(\gamma_{1,d} \otimes \mathcal{H}^1)(\ell, y). \quad (2.8)$$

Proof. To prove (2.8), it is enough to check that

$$\begin{aligned}
&\int_{\Sigma_{\mathbf{P}} \cap E} H(\theta_{\mathbf{P}}(x)) d\mathcal{H}^1(x) \\
&= c \int_{\text{Gr}(1,d) \times \mathbb{R}} \left(\int_{\Sigma_{\mathbf{P}} \cap \pi_{\ell}^{-1}(y) \cap E} H(\theta_{\mathbf{P}}(x)) d\mathcal{H}^0(x) \right) d(\gamma_{1,d} \otimes \mathcal{H}^1)(\ell, y)
\end{aligned} \quad (2.9)$$

and that

$$\mu_{\mathbf{P}}^d(E) = c \int_{\text{Gr}(1,d) \times \mathbb{R}} \langle \mathbf{P}, \pi_{\ell}, y \rangle^d(E) d(\gamma_{1,d} \otimes \mathcal{H}^1)(\ell, y). \quad (2.10)$$

The proof of (2.9) can be found in [11, Lemma 3.1]. Hence, we focus only on (2.10). Observe that, by definition,

$$\langle \mathbf{P}, \pi_\ell, y \rangle^d(E) = \int_{\text{Lip}_1} \mathcal{H}^0(\text{Im } \gamma \cap \pi_\ell^{-1}(y) \cap (\Sigma_{\mathbf{P}})^c \cap E) d\mathbf{P}(\gamma),$$

so, using Fubini's theorem,

$$\begin{aligned} & c \int_{\text{Gr}(1,d) \times \mathbb{R}} \langle \mathbf{P}, \pi_\ell, y \rangle^d(E) d(\gamma_{1,d} \otimes \mathcal{H}^1)(\ell, y) \\ &= \int_{\text{Lip}_1} \left(c \int_{\text{Gr}(1,d) \times \mathbb{R}} \mathcal{H}^0(\text{Im } \gamma \cap E \cap (\Sigma_{\mathbf{P}})^c \cap \pi_\ell^{-1}(y)) d(\gamma_{1,d} \otimes \mathcal{H}^1)(\ell, y) \right) d\mathbf{P}(\gamma). \end{aligned}$$

Applying Theorem 2.25 to the set $\text{Im } \gamma \cap E \cap (\Sigma_{\mathbf{P}})^c$, we obtain that

$$\begin{aligned} c \int_{\text{Gr}(1,d) \times \mathbb{R}} \langle \mathbf{P}, \pi_\ell, y \rangle^d(E) d(\gamma_{1,d} \otimes \mathcal{H}^1)(\ell, y) &= \int_{\text{Lip}_1} \mathcal{H}^1(\text{Im } \gamma \cap E \cap (\Sigma_{\mathbf{P}})^c) d\mathbf{P}(\gamma) \\ &= \mu_{\mathbf{P}}^d(E), \end{aligned}$$

and this concludes the proof. \square

3. MAIN RESULT

Our goal is to prove the following result, concerning the lower semicontinuity of the H -mass.

Theorem 3.1. *Let $A \subset X$ be an open subset, let $(\mathbf{P}_n)_{n \in \mathbb{N}} \subset \mathbf{TP}(X)$, let $\mathbf{P} \in \mathbf{TP}(X)$ be such that $\mathbf{P}_n \xrightarrow{*} \mathbf{P}$ and let $H: [0, +\infty) \rightarrow [0, +\infty)$ be a function satisfying (H1), (H2) and (H3). Then*

$$\mathbb{M}_H(\mathbf{P}; A) \leq \liminf_{n \rightarrow +\infty} \mathbb{M}_H(\mathbf{P}_n; A). \quad (3.1)$$

The proof of Theorem 3.1 relies on proving the lower semicontinuity for the H -mass for the simple slices, assuming that we have the convergence of traffic plans. Observe that this would follow trivially by [11, Proposition 2.6], if we knew that the convergence of traffic plans implies the convergence of the simple slices. However, this is not true in general. Hence, our strategy relies on a ‘‘counting’’ argument that exploits the connectedness of the curves and their regularity. All preliminary facts regarding this approach will be presented in Section 3.1.

After proving the lower semicontinuity of the sliced H -mass, we will be able to prove the main result with a simple computation, that exploits the integral-geometric equality (see Corollary 2.26).

3.1. Counting intersection points between curves and hyperplanes. As mentioned above, the convergence of the traffic plans does not imply the convergence of the corresponding simple slices. Nevertheless, we are able to retrieve the lower semicontinuity of the slices evaluated on a fixed open subset. The goal of this section is to prove this fact.

For notational convenience, for $\ell \in \text{Gr}(1, d)$ and $y \in \mathbb{R}$ we denote

$$H_{\ell, y}^- := \pi_\ell^{-1}(-\infty, y), \quad H_{\ell, y}^+ := \pi_\ell^{-1}(y, +\infty).$$

Proposition 3.2. *Let $O \subset X$ be an open subset, let $(\mathbf{P}_n)_{n \in \mathbb{N}} \subset \mathbf{TP}(X)$ and let $\mathbf{P} \in \mathbf{TP}(X)$ be such that $\mathbf{P}_n \xrightarrow{*} \mathbf{P}$. Then, for $\gamma_{1,d}$ -a.e. $\ell \in \text{Gr}(1, d)$, for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$,*

$$\langle \mathbf{P}, \pi_\ell, y \rangle(O) \leq \liminf_{n \rightarrow +\infty} \langle \mathbf{P}_n, \pi_\ell, y \rangle(O). \quad (3.2)$$

Proof. Using Lemma 2.22 with the traffic plan \mathbf{P} and the maps $f := \pi_\ell$, we infer that for $\gamma_{1,d}$ -a.e. $\ell \in \text{Gr}(1, d)$, for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$, for \mathbf{P} -a.e. $\gamma \in \text{Lip}_1$, $\text{Im } \gamma \cap \pi_\ell^{-1}(y) \cap O$ is finite. Then, for every $x \in \text{Im } \gamma \cap \pi_\ell^{-1}(y) \cap O$, setting

$$O_x := B_{\frac{\rho_0}{2}}(x),$$

with

$$\rho_0 := \min \left\{ \min_{\substack{(w,z) \in (\text{Im } \gamma \cap \pi_\ell^{-1}(y) \cap O)^2 \\ w \neq z}} |w - z|, \min_{z \in \text{Im } \gamma \cap \pi_\ell^{-1}(y) \cap O} \text{dist}(z, \partial O) \right\},$$

there exist $t^\pm(x) \in (0, T_\infty(\gamma))$ such that²

- (1) $\gamma([t^-(x), t^+(x)]) \subset O_x$;
- (2) $\gamma(t^\pm(x)) \in H_{\ell,y}^\pm$.

For the rest of the proof, we will fix such $\ell \in \text{Gr}(1, d)$ and $y \in \mathbb{R}$.

Recall that the simple slice is defined as

$$\langle \mathbf{P}, \pi_\ell, y \rangle (O) := \int_{\text{Lip}_1} \mathcal{H}^0(\text{Im } \gamma \cap \pi_\ell^{-1}(y) \cap O) d\mathbf{P}(\gamma).$$

By proving that the function

$$\gamma \mapsto \mathcal{H}^0(\text{Im } \gamma \cap \pi_\ell^{-1}(y) \cap O)$$

is lower semicontinuous for \mathbf{P} -a.e. $\gamma \in \text{Lip}_1$, then the thesis follows by [1, Lemma 1.61], and by arguing exactly as in [1, Proposition 1.62].

For fixed ℓ and y , the above properties hold for \mathbf{P} -a.e. $\gamma \in \text{Lip}_1$. Fix any of such γ , and consider any sequence $(\gamma_n)_{n \in \mathbb{N}} \subset \text{Lip}_1$ such that $\gamma_n \xrightarrow{d_K} \gamma$. Recall that $\gamma_n \xrightarrow{d_K} \gamma$ if and only if $\gamma_n \rightarrow \gamma$ uniformly on every compact subset $K \subset \mathbb{R}^+$. This means that if $K \subset \mathbb{R}^+$ is compact, for every $\rho > 0$ there exists $n_\rho \in \mathbb{N}$ such that

$$\gamma_n(K) \subset U(\gamma(K), \rho) \quad \forall n \geq n_\rho, \quad (3.3)$$

where $U(\gamma(K), \rho)$ denotes the open tubular neighbourhood of $\gamma(K)$ with radius ρ . Fix

$$x \in \text{Im } \gamma \cap \pi_\ell^{-1}(y) \cap O.$$

By (1) and by the openness of O_x , there exists $\rho_x > 0$ such that

$$U(\gamma([t^-(x), t^+(x)]), \rho_x) \subset O_x.$$

Then, using (3.3) with $\rho = \rho_x$ and, there exists $n_{1,x} \in \mathbb{N}$ such that, for every $n \geq n_{1,x}$,

$$\gamma_n([t^-(x), t^+(x)]) \subset U(\gamma([t^-(x), t^+(x)]), \rho_x) \subset O_x.$$

Moreover, by (2), it can be easily proved that there exists $n_x \geq n_{1,x}$ such that, for every $n \geq n_x$,

$$\gamma_n(t^\pm(x)) \in H_{\ell,y}^\pm.$$

This shows, by the continuity of the map π_ℓ and by connectedness, that for every $x \in \text{Im } \gamma \cap \pi_\ell^{-1}(y) \cap O$, there exists $n_x \in \mathbb{N}$ such that, for every $n \geq n_x$,

$$\text{Im } \gamma_n \cap \pi_\ell^{-1}(y) \cap O_x \neq \emptyset.$$

Choosing

$$\bar{n} := \max_{x \in \text{Im } \gamma \cap \pi_\ell^{-1}(y) \cap O} n_x,$$

²In what follows, if $s, t \in \mathbb{R}^+$ and $s \neq t$, we denote $[s, t] := [\min\{s, t\}, \max\{s, t\}]$.

by the disjointness of the neighbourhoods O_x it holds that

$$\mathcal{H}^0(\text{Im } \gamma_n \cap \pi_\ell^{-1}(y) \cap O) \geq \mathcal{H}^0(\text{Im } \gamma \cap \pi_\ell^{-1}(y) \cap O) \quad \forall n \geq \bar{n}.$$

In particular,

$$\mathcal{H}^0(\text{Im } \gamma \cap \pi_\ell^{-1}(y) \cap O) \leq \liminf_{n \rightarrow +\infty} \mathcal{H}^0(\text{Im } \gamma_n \cap \pi_\ell^{-1}(y) \cap O),$$

and this concludes the proof. \square

3.2. Lower semicontinuity of the sliced H -mass. In this section, we prove the following results. The first one proves the lower semicontinuity of the sliced H -masses when we have $H'(0) = +\infty$. The second result, instead, assumes the finiteness of $H'(0)$. The arguments used for both proofs are similar, but we prefer to split the result for more clarity.

Proposition 3.3. *Let $A \subset X$ be an open subset, let $(\mathbf{P}_n)_{n \in \mathbb{N}} \subset \mathbf{TP}(X)$ and let $\mathbf{P} \in \mathbf{TP}(X)$ be such that $\mathbf{P}_n \xrightarrow{*} \mathbf{P}$. Assume that $H: [0, +\infty) \rightarrow [0, +\infty)$ is a function satisfying (H1), (H2), (H3) and $H'(0) = +\infty$. Then, for $\gamma_{1,d}$ -a.e. $\ell \in \text{Gr}(1, d)$ and for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$,*

$$\mathbb{M}_H(\langle \mathbf{P}, \pi_\ell, y \rangle \llcorner A) \leq \liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A). \quad (3.4)$$

Proof. For $\gamma_{1,d}$ -a.e. $\ell \in \text{Gr}(1, d)$ and for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$,

$$\Sigma_{\mathbf{P}} \cap \pi_\ell^{-1}(y) \cap A$$

is at most countable, and so are

$$\Sigma_{\mathbf{P}_n} \cap \pi_\ell^{-1}(y) \cap A,$$

since they are sets of atoms of $\langle \mathbf{P}, \pi_\ell, y \rangle \llcorner A$ and $\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A$, for every $n \in \mathbb{N}$, respectively. If

$$\liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A) = +\infty,$$

we trivially conclude, as there is nothing to prove. Otherwise,

$$\liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A) < +\infty.$$

In this case, choose a (non-relabelled) subsequence $(\mathbf{P})_{n \in \mathbb{N}}$ that realizes the inferior limit, namely

$$\liminf_{k \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_k, \pi_\ell, y \rangle \llcorner A) = \lim_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A).$$

In particular, there exists $M > 0$ such that

$$\mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A) \leq M \quad \forall n \in \mathbb{N}.$$

Since $H'(0) = +\infty$, the boundedness of the H -mass of the slices implies that the slices are purely atomic. In other words,

$$\langle \mathbf{P}_n, \pi_\ell, y \rangle^d(A) = 0 \quad \forall n \in \mathbb{N}.$$

Step 1: the slices of \mathbf{P} must be atomic. Assume that $\langle \mathbf{P}, \pi_\ell, y \rangle^d(A) > 0$. In particular, there exists $\delta > 0$ such that $\langle \mathbf{P}, \pi_\ell, y \rangle^d(A) \geq \delta$. Then, for every $K \in \mathbb{N}^+$, there exist C_1, \dots, C_K such that

- $C_i \subset A$ is a closed ball;
- $C_i \cap C_j = \emptyset$ if $i \neq j$;
- $\langle \mathbf{P}, \pi_\ell, y \rangle(C_i) \geq \frac{\delta}{2K}$ for every $i \in \{1, \dots, K\}$.

For every fixed K , consider

$$\rho_K := \min \left\{ \min_{\substack{(i,j) \in \{1,\dots,K\}^2 \\ i \neq j}} \{\text{dist}(C_i, C_j)\}, \min_{i \in \{1,\dots,K\}} \text{dist}(C_i, \partial A) \right\} > 0.$$

In particular, setting, for $i \in \{1, \dots, K\}$,

$$B_i := U \left(C_i, \frac{\rho_K}{2} \right),$$

that is, a slightly enlarged open ball concentric to C_i , we still have the open balls B_1, \dots, B_K being disjoint.

Using Proposition 3.2 with $O = B_i$ for every i , we obtain that for $\gamma_{1,d}$ -a.e. $\ell \in \text{Gr}(1, d)$, for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$,

$$\langle \mathbf{P}, \pi_\ell, y \rangle (B_i) \leq \liminf_{n \rightarrow +\infty} \langle \mathbf{P}_n, \pi_\ell, y \rangle (B_i) \quad \forall i \in \{1, \dots, K\}.$$

In particular, since $\langle \mathbf{P}, \pi_\ell, y \rangle \geq \frac{\delta}{2K}$, there exists $\bar{n} \in \mathbb{N}$ such that, for every $n \geq \bar{n}$,

$$\frac{\delta}{4K} \leq \langle \mathbf{P}_n, \pi_\ell, y \rangle (B_i) \quad \forall i \in \{1, \dots, K\}. \quad (3.5)$$

In particular, for every $n \geq \bar{n}$, since H is non-decreasing and the slices of \mathbf{P}_n are atomic,

$$H \left(\frac{\delta}{4K} \right) \leq H(\langle \mathbf{P}_n, \pi_\ell, y \rangle (B_i \cap \Sigma_{\mathbf{P}_n})) \quad \forall i \in \{1, \dots, K\}.$$

Thus, summing over all the indices i , for every $n \geq \bar{n}$, by Remark 2.11,

$$\begin{aligned} KH \left(\frac{\delta}{4K} \right) &\leq \sum_{i=1}^K H(\langle \mathbf{P}_n, \pi_\ell, y \rangle (B_i \cap \Sigma_{\mathbf{P}_n})) = \sum_{i=1}^K H \left(\sum_{x \in \Sigma_{\mathbf{P}_n} \cap B_i} \theta_{\mathbf{P}_n}(x) \right) \\ &\leq \sum_{k=1}^N \sum_{x \in \Sigma_{\mathbf{P}_n} \cap B_i} H(\theta_{\mathbf{P}_n}(x)) \leq \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle). \end{aligned}$$

In particular, passing to the inferior limit,

$$KH \left(\frac{\delta}{4K} \right) \leq \liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A).$$

However, since $K \in \mathbb{N}^+$ is arbitrary, we can let $K \rightarrow +\infty$. Since $H'(0) = +\infty$, we obtain

$$\liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A) = +\infty,$$

thus contradicting the finiteness of the inferior limit. This proves that, for $\gamma_{1,d}$ -a.e. $\ell \in \text{Gr}(1, d)$ and for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$, $\langle \mathbf{P}, \pi_\ell, y \rangle^d(A) = 0$.

Step 2: upper bound on the H -mass of “localized” slices of \mathbf{P} . Fix $\varepsilon \in (0, 1)$. Fix any ℓ and y that satisfy all the above properties; in particular,

$$\langle \mathbf{P}, \pi_\ell, y \rangle \llcorner A = \langle \mathbf{P}, \pi_\ell, y \rangle \llcorner (A \cap \Sigma_{\mathbf{P}}).$$

There exist $N = N(\varepsilon) \in \mathbb{N}$ and $x_1, \dots, x_N \in \Sigma_{\mathbf{P}} \cap \pi_\ell^{-1}(y) \cap A$ such that

$$\mathbb{M}_H(\langle \mathbf{P}, \pi_\ell, y \rangle \llcorner A) - \sum_{j=1}^N H(\theta_{\mathbf{P}}(x_j)) \leq \varepsilon \quad \text{if } \mathbb{M}_H(\langle \mathbf{P}, \pi_\ell, y \rangle \llcorner A) < +\infty \quad (3.6)$$

or

$$\sum_{j=1}^N H(\theta_{\mathbf{P}}(x_j)) \geq \frac{1}{\varepsilon} \quad \text{otherwise.} \quad (3.7)$$

Since H is lower semicontinuous, for every $j \in \{1, \dots, N\}$ there exists $\eta_j = \eta_j(\varepsilon) > 0$ such that, whenever $|\Theta - \theta_{\mathbf{P}}(x_j)| < \eta_j$, then

$$H(\Theta) \geq (1 - \varepsilon)H(\theta_{\mathbf{P}}(x_j)). \quad (3.8)$$

We now fix

$$\bar{\eta} := \min \left\{ \min_{j \in \{1, \dots, N\}} \{\eta_j\}, \min_{j \in \{1, \dots, N\}} \{\theta_{\mathbf{P}}(x_j)\} \right\}, \quad (3.9)$$

and we also choose $r > 0$ such that the balls $B_r(x_j)$, with $j \in \{1, \dots, N\}$, are contained in A and disjoint. From now on, we fix³ $\ell \in \text{Gr}(1, d)$ and $y \in \mathbb{R}$ such that all the above properties hold, and also Proposition 3.2 holds when choosing $O = B_r(x_j)$, for every $j \in \{1, \dots, N\}$. By Proposition 3.2, for every $j \in \{1, \dots, N\}$,

$$\langle \mathbf{P}, \pi_\ell, y \rangle (B_r(x_j)) \leq \liminf_{n \rightarrow +\infty} \langle \mathbf{P}_n, \pi_\ell, y \rangle (B_r(x_j)),$$

so, for every $j \in \{1, \dots, N\}$, there exists $n_j \in \mathbb{N}$ such that for every $n \geq n_j$

$$\theta_{\mathbf{P}}(x_j) - \frac{\bar{\eta}}{2} \leq \langle \mathbf{P}, \pi_\ell, y \rangle (B_r(x_j)) - \frac{\bar{\eta}}{2} \leq \langle \mathbf{P}_n, \pi_\ell, y \rangle (B_r(x_j)). \quad (3.10)$$

Choosing

$$\Theta_j := \theta_{\mathbf{P}}(x_j) - \frac{\bar{\eta}}{2},$$

by (3.8) we obtain that, for every $j \in \{1, \dots, N\}$,

$$(1 - \varepsilon)H(\theta_{\mathbf{P}}(x_j)) \leq H(\Theta_j).$$

In particular, choosing

$$\bar{n} := \max_{j \in \{1, \dots, N\}} n_j,$$

using (3.10) and recalling that the slices $\langle \mathbf{P}_n, \pi_\ell, y \rangle$ are purely atomic, for every $n \geq \bar{n}$, we have

$$H(\theta_{\mathbf{P}}(x_j)) \leq \frac{1}{1 - \varepsilon} H(\langle \mathbf{P}_n, \pi_\ell, y \rangle (B_r(x_j) \cap \Sigma_{\mathbf{P}_n})). \quad (3.11)$$

Step 3: final estimates. We can now conclude: if (3.6) holds, for every $n \geq \bar{n}$,

$$\begin{aligned} \mathbb{M}_H(\langle \mathbf{P}, \pi_\ell, y \rangle \llcorner A) &\leq \sum_{j=1}^N H(\theta_{\mathbf{P}}(x_j)) + \varepsilon \stackrel{(3.11)}{\leq} \frac{1}{1 - \varepsilon} \sum_{j=1}^N H(\langle \mathbf{P}_n, \pi_\ell, y \rangle (B_r(x_j) \cap \Sigma_{\mathbf{P}_n})) + \varepsilon \\ &\leq \frac{1}{1 - \varepsilon} \sum_{j=1}^N H \left(\sum_{x \in B_r(x_j) \cap \Sigma_{\mathbf{P}_n}} \theta_{\mathbf{P}_n}(x) \right) + \varepsilon \\ &\leq \frac{1}{1 - \varepsilon} \sum_{j=1}^N \sum_{x \in B_r(x_j) \cap \Sigma_{\mathbf{P}_n}} H(\theta_{\mathbf{P}_n}(x)) + \varepsilon \\ &\leq \frac{1}{1 - \varepsilon} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A) + \varepsilon. \end{aligned}$$

Thus, passing to the inferior limit as $n \rightarrow +\infty$,

$$\mathbb{M}_H(\langle \mathbf{P}, \pi_\ell, y \rangle \llcorner A) \leq \frac{1}{1 - \varepsilon} \liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A) + \varepsilon.$$

³observe that $\gamma_{1,d}$ -a.e. $\ell \in \text{Gr}(1, d)$ and \mathcal{H}^1 -a.e. $y \in \mathbb{R}$ satisfy all the required properties.

Since $\varepsilon \in (0, 1)$ is arbitrary, we conclude. If (3.7) holds, with the exact same estimates we obtain that, for $n \geq \bar{n}$,

$$\frac{1}{\varepsilon} \leq \sum_{j=1}^N H(\theta_{\mathbf{P}}(x_j)) \leq \frac{1}{1-\varepsilon} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A).$$

Thus, passing to the inferior limit as $n \rightarrow +\infty$,

$$\frac{1}{\varepsilon} \leq \frac{1}{1-\varepsilon} \liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A).$$

However, since $\varepsilon \in (0, 1)$ is arbitrary, by letting $\varepsilon \rightarrow 0^+$ we would contradict the finiteness of the right-hand side. In other words, (3.7) cannot occur, and this concludes the proof. \square

Proposition 3.4. *Let $A \subset X$ be an open subset, let $(\mathbf{P}_n)_{n \in \mathbb{N}} \subset \mathbf{TP}(X)$ and let $\mathbf{P} \in \mathbf{TP}(X)$ be such that $\mathbf{P}_n \xrightarrow{*} \mathbf{P}$. Assume that $H: [0, +\infty) \rightarrow [0, +\infty)$ is a function satisfying (H1), (H2), (H3) and $H'(0) < +\infty$. Then, for $\gamma_{1,d}$ -a.e. $\ell \in \text{Gr}(1, d)$ and for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$,*

$$\mathbb{M}_H(\langle \mathbf{P}, \pi_\ell, y \rangle \llcorner A) \leq \liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A). \quad (3.12)$$

Proof. For $\gamma_{1,d}$ -a.e. $\ell \in \text{Gr}(1, d)$ and for \mathcal{H}^1 -a.e. $y \in \mathbb{R}$,

$$\Sigma_{\mathbf{P}} \cap \pi_\ell^{-1}(y) \cap A$$

is at most countable, and so are

$$\Sigma_{\mathbf{P}_n} \cap \pi_\ell^{-1}(y) \cap A,$$

since they are sets of atoms of

$$\langle \mathbf{P}, \pi_\ell, y \rangle \llcorner A \quad \text{and} \quad \langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A \quad \forall n \in \mathbb{N},$$

respectively.

Fix $\varepsilon \in (0, 1)$. For any ℓ and y as above, there exist $N = N(\varepsilon) \in \mathbb{N}$ and

$$x_1, \dots, x_N \in \Sigma_{\mathbf{P}} \cap \pi_\ell^{-1}(y) \cap A$$

such that

$$\int_{\Sigma_{\mathbf{P}} \cap \pi_\ell^{-1}(y) \cap A} H(\theta_{\mathbf{P}}(x)) d\mathcal{H}^0(x) - \sum_{j=1}^N H(\theta_{\mathbf{P}}(x_j)) \leq \varepsilon \quad \text{if } \mathbb{M}_H(\langle \mathbf{P}, \pi_\ell, y \rangle \llcorner \Sigma_{\mathbf{P}} \cap A) < +\infty \quad (3.13)$$

or

$$\sum_{j=1}^N H(\theta_{\mathbf{P}}(x_j)) \geq \frac{1}{\varepsilon} \quad \text{otherwise.} \quad (3.14)$$

Since H is lower semicontinuous, for every $j \in \{1, \dots, N\}$ there exists $\eta_j = \eta_j(\varepsilon) > 0$ such that, whenever $|\Theta - \theta_{\mathbf{P}}(x_j)| < \eta_j$, then

$$H(\Theta) \geq \left(1 - \frac{\varepsilon}{2}\right) H(\theta_{\mathbf{P}}(x_j)). \quad (3.15)$$

Furthermore, since $H'(0) < +\infty$, then there exists $\delta > 0$ such that, whenever $\theta \in (0, \delta)$,

$$(1 - \varepsilon)H'(0)\theta \leq H(\theta). \quad (3.16)$$

Choose

$$\bar{\eta} = \bar{\eta}(\varepsilon) := \min \left\{ \min_{j \in \{1, \dots, N\}} \{\eta_j\}, \min_{j \in \{1, \dots, N\}} \{\theta_{\mathbf{P}}(x_j)\}, \min_{j \in \{1, \dots, N\}} \left\{ \frac{\varepsilon H(\theta_{\mathbf{P}}(x_j))}{2 H'(0)} \right\}, \delta \right\}. \quad (3.17)$$

Then, there exists $K = K(\varepsilon) \in \mathbb{N}$, $K \geq N$, and there exist $B_1, \dots, B_K \subset A$ open balls such that

- the balls are pairwise disjoint;
- for every $j \in \{1, \dots, N\}$ there exists a unique $i(j) \in \{1, \dots, K\}$ such that $x_j \in B_{i(j)}$;
- $\sum_{i=1}^K \langle \mathbf{P}, \pi_\ell, y \rangle^d(B_i) \geq \langle \mathbf{P}, \pi_\ell, y \rangle^d(A) - \varepsilon$;
- $\langle \mathbf{P}, \pi_\ell, y \rangle^d(B_i \cap A) \leq \frac{\bar{\eta}}{2}$.

From now on, we fix ℓ and y such that Proposition 3.2 holds when choosing $O := B_i$, for every $i \in \{1, \dots, K\}$. For notational convenience, we split the set of indices i for the balls B_i as follows:

$$I^a := \{i(j) \in \{1, \dots, K\} \mid j \in \{1, \dots, N\}\},$$

$$I^d := \{1, \dots, K\} \setminus I^a.$$

Step 1: upper bound on the H -mass of slices of \mathbf{P} , localized at balls containing the chosen atoms. For every $i \in I^a$, by Proposition 3.2 we have

$$\langle \mathbf{P}, \pi_\ell, y \rangle(B_i) \leq \liminf_{n \rightarrow +\infty} \langle \mathbf{P}_n, \pi_\ell, y \rangle(B_i).$$

In particular, for every $i = i(j) \in I^a$ there exists $n_i \in \mathbb{N}$ such that, for every $n \geq n_i$,

$$\theta_{\mathbf{P}}(x_j) - \frac{\bar{\eta}}{4} \leq \langle \mathbf{P}, \pi_\ell, y \rangle(B_i) - \frac{\bar{\eta}}{4} \leq \langle \mathbf{P}_n, \pi_\ell, y \rangle(B_i). \quad (3.18)$$

In particular, setting, for $j \in \{1, \dots, N\}$,

$$\Theta_j := \theta_{\mathbf{P}}(x_j) - \frac{\bar{\eta}}{4},$$

we can use (3.8) to conclude that, for every $j \in \{1, \dots, N\}$,

$$\left(1 - \frac{\varepsilon}{2}\right) H(\theta_{\mathbf{P}}(x_j)) \leq H(\Theta_j),$$

and so, choosing $\bar{n} := \max_{i \in I^a} n_i$, using (3.18) and the monotonicity of H , for every $n \geq \bar{n}$, for every $j \in \{1, \dots, N\}$

$$\left(1 - \frac{\varepsilon}{2}\right) H(\theta_{\mathbf{P}}(x_j)) \leq H(\langle \mathbf{P}_n, \pi_\ell, y \rangle(B_{i(j)})). \quad (3.19)$$

We now estimate the left-hand side of (3.19) from below: by the choice of $\bar{\eta}$ and the construction of the open balls,

$$\begin{aligned} \left(1 - \frac{\varepsilon}{2}\right) H(\theta_{\mathbf{P}}(x_j)) &= (1 - \varepsilon)H(\theta_{\mathbf{P}}(x_j)) + \frac{\varepsilon}{2} \frac{H(\theta_{\mathbf{P}}(x_j))}{H'(0)} H'(0) \\ &\geq (1 - \varepsilon)H(\theta_{\mathbf{P}}(x_j)) + H'(0)\bar{\eta} \\ &\geq (1 - \varepsilon) \left[H(\theta_{\mathbf{P}}(x_j)) + H'(0) \langle \mathbf{P}, \pi_\ell, y \rangle^d(B_{i(j)}) \right]. \end{aligned}$$

Regarding the right-hand side of (3.19), using Remarks 2.11 and 2.12,

$$\begin{aligned} H(\langle \mathbf{P}_n, \pi_\ell, y \rangle(B_{i(j)})) &\leq H(\langle \mathbf{P}_n, \pi_\ell, y \rangle(\Sigma_{\mathbf{P}_n} \cap B_{i(j)})) + H(\langle \mathbf{P}_n, \pi_\ell, y \rangle^d(B_{i(j)})) \\ &\leq H(\langle \mathbf{P}_n, \pi_\ell, y \rangle(\Sigma_{\mathbf{P}_n} \cap B_{i(j)})) + H'(0) \langle \mathbf{P}_n, \pi_\ell, y \rangle^d(B_{i(j)}) \\ &\leq \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner B_{i(j)}). \end{aligned}$$

Combining these computations, we have that there exists $\bar{n} \in \mathbb{N}$ such that, for every $n \geq \bar{n}$, for every $j \in \{1, \dots, N\}$,

$$H(\theta_{\mathbf{P}}(x_j)) + H'(0) \langle \mathbf{P}, \pi_\ell, y \rangle^d(B_{i(j)}) \leq \frac{1}{1 - \varepsilon} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner B_{i(j)}). \quad (3.20)$$

Step 2: upper bound on the H -mass of slices of \mathbf{P} , localized at the remaining balls. Now, for $i \in I^d$, we use Proposition 3.2 to obtain that

$$\langle \mathbf{P}, \pi_\ell, y \rangle^d(B_i) \leq \liminf_{n \rightarrow +\infty} \langle \mathbf{P}_n, \pi_\ell, y \rangle(B_i).$$

Then,

$$H(\langle \mathbf{P}, \pi_\ell, y \rangle^d(B_i)) \leq H(\liminf_{n \rightarrow +\infty} \langle \mathbf{P}_n, \pi_\ell, y \rangle(B_i)) \leq \liminf_{n \rightarrow +\infty} H(\langle \mathbf{P}_n, \pi_\ell, y \rangle(B_i)). \quad (3.21)$$

Regarding the left-hand side of (3.21), by the choice of $\bar{\eta}$, by the construction of the balls B_i and by (3.16),

$$H(\langle \mathbf{P}, \pi_\ell, y \rangle^d(B_i)) \geq (1 - \varepsilon)H'(0) \langle \mathbf{P}, \pi_\ell, y \rangle^d(B_i).$$

The right-hand side, instead, can be estimated as follows, using Remark 2.11:

$$\begin{aligned} \liminf_{n \rightarrow +\infty} H(\langle \mathbf{P}_n, \pi_\ell, y \rangle(B_i)) &\leq \liminf_{n \rightarrow +\infty} \left[H(\langle \mathbf{P}_n, \pi_\ell, y \rangle(B_i \cap \Sigma_{\mathbf{P}_n})) + H'(0) \langle \mathbf{P}_n, \pi_\ell, y \rangle^d(B_i) \right] \\ &\leq \liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner B_i). \end{aligned}$$

Combining such estimates with (3.21),

$$H'(0) \langle \mathbf{P}, \pi_\ell, y \rangle^d(B_i) \leq \frac{1}{1 - \varepsilon} \liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner B_i). \quad (3.22)$$

Step 3: final estimates. If (3.13) holds, using (3.20) and (3.22), for every $n \geq \bar{n}$,

$$\begin{aligned} \mathbb{M}_H(\langle \mathbf{P}, \pi_\ell, y \rangle \llcorner A) &= \int_{\Sigma_{\mathbf{P}} \cap \pi_\ell^{-1}(y) \cap A} H(\theta_{\mathbf{P}}(x)) d\mathcal{H}^0(x) + H'(0) \langle \mathbf{P}, \pi_\ell, y \rangle^d(A) \\ &\leq \sum_{j=1}^N H(\theta_{\mathbf{P}}(x_j)) + H'(0) \sum_{i=1}^K \langle \mathbf{P}, \pi_\ell, y \rangle^d(B_i) + (1 + H'(0))\varepsilon \\ &= \sum_{j=1}^N \left[H(\theta_{\mathbf{P}}(x_j)) + H'(0) \langle \mathbf{P}, \pi_\ell, y \rangle^d(B_{i(j)}) \right] \\ &\quad + \sum_{i \in I^d} H'(0) \langle \mathbf{P}, \pi_\ell, y \rangle^d(B_i) + (1 + H'(0))\varepsilon \\ &\leq \frac{1}{1 - \varepsilon} \sum_{j=1}^N \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner B_{i(j)}) \\ &\quad + \frac{1}{1 - \varepsilon} \sum_{i \in I^d} \liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner B_i) + (1 + H'(0))\varepsilon \\ &\leq \frac{1}{1 - \varepsilon} \mathbb{M}_H \left(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner \left(A \setminus \bigcup_{i \in I^d} B_i \right) \right) \\ &\quad + \frac{1}{1 - \varepsilon} \liminf_{n \rightarrow +\infty} \mathbb{M}_H \left(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner \bigcup_{i \in I^d} B_i \right) + (1 + H'(0))\varepsilon. \end{aligned}$$

In particular, passing to the inferior limit as $n \rightarrow +\infty$,

$$\begin{aligned} \mathbb{M}_H(\langle \mathbf{P}, \pi_\ell, y \rangle \llcorner A) &\leq \frac{1}{1-\varepsilon} \liminf_{n \rightarrow +\infty} \mathbb{M}_H \left(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner \left(A \setminus \bigcup_{i \in I^d} B_i \right) \right) \\ &\quad + \frac{1}{1-\varepsilon} \liminf_{n \rightarrow +\infty} \mathbb{M}_H \left(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner \bigcup_{i \in I^d} B_i \right) + (1 + H'(0))\varepsilon \\ &\leq \frac{1}{1-\varepsilon} \liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A) + (1 + H'(0))\varepsilon. \end{aligned}$$

In particular, since $\varepsilon \in (0, 1)$ is arbitrary, letting $\varepsilon \rightarrow 0^+$, we conclude that (3.12) holds.

If, instead, (3.14) holds, with the same argument one proves that

$$\frac{1}{\varepsilon} \leq \frac{1}{1-\varepsilon} \liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A) + (1 + H'(0))\varepsilon,$$

and since $\varepsilon \in (0, 1)$ is arbitrary, by letting $\varepsilon \rightarrow 0^+$, we conclude. \square

3.3. Conclusion. Using Corollary 2.26 and the results obtained in Section 3.2, we are able to conclude the proof of the main result.

Proof of Theorem 3.1. We use Corollary 2.26 and Propositions 3.3 or 3.4, combined with Fatou's lemma:

$$\begin{aligned} \mathbb{M}_H(\mathbf{P}; A) &= c \int_{\text{Gr}(1,d) \times \mathbb{R}} \mathbb{M}_H(\langle \mathbf{P}, \pi_\ell, y \rangle \llcorner A) d(\gamma_{1,d} \otimes \mathcal{H}^1)(\ell, y) \\ &\leq c \int_{\text{Gr}(1,d) \times \mathbb{R}} \liminf_{n \rightarrow +\infty} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A) d(\gamma_{1,d} \otimes \mathcal{H}^1)(\ell, y) \\ &\leq \liminf_{n \rightarrow +\infty} c \int_{\text{Gr}(1,d) \times \mathbb{R}} \mathbb{M}_H(\langle \mathbf{P}_n, \pi_\ell, y \rangle \llcorner A) d(\gamma_{1,d} \otimes \mathcal{H}^1)(\ell, y) \\ &= \liminf_{n \rightarrow +\infty} \mathbb{M}_H(\mathbf{P}_n; A). \end{aligned} \quad \square$$

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REFERENCES

- [1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000, pp. xviii+434.
- [2] M. Bernot, V. Caselles, and J.-M. Morel. *Optimal transportation networks*. Vol. 1955. Lecture Notes in Mathematics. Models and theory. Springer-Verlag, Berlin, 2009, pp. x+200.
- [3] M. Bernot, V. Caselles, and J.-M. Morel. "The structure of branched transportation networks". In: *Calc. Var. Partial Differential Equations* 32.3 (2008), pp. 279–317.
- [4] A. Brancolini and B. Wirth. "General transport problems with branched minimizers as functionals of 1-currents with prescribed boundary". In: *Calc. Var. Partial Differential Equations* 57 (2018).

- [5] G. Caldini, A. Marchese, and S. Steinbrüchel. “Generic uniqueness of optimal transportation networks”. In: *Calculus of Variations and Partial Differential Equations* 62:235 (2023).
- [6] M. Carioni, A. Marchese, A. Massaccesi, A. Pluda, and R. Tione. “The oriented mailing problem and its convex relaxation”. In: *Nonlinear Anal.* 199 (2020), pp. 112035, 12.
- [7] M. Colombo, A. De Rosa, and A. Marchese. “Improved stability of optimal traffic paths”. In: *Calc. Var. Partial Differential Equations* 57.1 (2018), Paper No. 28, 33.
- [8] M. Colombo, A. De Rosa, and A. Marchese. “On the well-posedness of branched transportation”. In: *Comm. Pure Appl. Math.* 74.4 (2021), pp. 833–864.
- [9] M. Colombo, A. De Rosa, and A. Marchese. “Stability for the mailing problem”. In: *J. Math. Pures Appl. (9)* 128 (2019), pp. 152–182.
- [10] M. Colombo, A. De Rosa, A. Marchese, P. Pegon, and A. Prouff. “Stability of optimal traffic plans in the irrigation problem”. In: *Discrete Contin. Dyn. Syst.* 42.4 (2022), pp. 1647–1667.
- [11] M. Colombo, A. De Rosa, A. Marchese, and S. Stuvard. “On the lower semicontinuous envelope of functionals defined on polyhedral chains”. In: *Nonlinear Anal.* 163 (2017), pp. 201–215.
- [12] S. Conti, M. Goldman, F. Otto, and S. Serfaty. “A branched transport limit of the Ginzburg-Landau functional”. In: *J. Éc. polytech. Math.* 5 (2018), pp. 317–375.
- [13] A. Cosenza, M. Goldman, and M. Koser. “New dimensional bounds for a branched transport problem”. In: *arXiv preprint arXiv:2411.14547* (2024).
- [14] L. De Masi, A. Marchese, and A. Massaccesi. “Variational models of robust optimal transport”. In: *arXiv preprint 2511.05146* (2025).
- [15] G. De Philippis, M. Goldman, and B. Ruffini. “From energy bounds to dimensional estimates in a branched transport model for type-I superconductors”. In: *arXiv preprint arXiv:2304.12715* (2023).
- [16] H. Federer. *Geometric measure theory*. Vol. 153. Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag New York, Inc., New York, 1969, pp. xiv+676.
- [17] E. N. Gilbert. “Minimum Cost Communication Networks”. In: *Bell System Technical Journal* 46.9 (1967), pp. 2209–2227.
- [18] M. Kuczma. *An introduction to the theory of functional equations and inequalities*. Birkhäuser Verlag, Basel, 2009, pp. xiv+595.
- [19] J. Lohmann, B. Schmitzer, and B. Wirth. “Formulas for the h-mass on 1-currents with coefficients in \mathbb{R}^m ”. In: *Calculus of Variations and Partial Differential Equations* 64 (2025), p. 288.
- [20] F. Maddalena, S. Solimini, and J.-M. Morel. “A variational model of irrigation patterns”. In: *Interfaces Free Bound.* 5.4 (2003), pp. 391–415.
- [21] A. Marchese, A. Massaccesi, S. Stuvard, and R. Tione. “A multi-material transport problem with arbitrary marginals”. In: *Calc. Var. Partial Differential Equations* 60 (2021), Art. 88.
- [22] A. Marchese, A. Massaccesi, and R. Tione. “A multimaterial transport problem and its convex relaxation via rectifiable G-currents”. In: *SIAM J. Math. Anal.* 51.3 (2019), pp. 1965–1998.
- [23] A. Marchese and B. Wirth. “Approximation of rectifiable 1-currents and weak-* relaxation of the h-mass”. In: *J. Math. Anal. Appl.* 479.2 (2019), pp. 2268–2283.
- [24] P. Mattila. *Geometry of sets and measures in Euclidean spaces*. Vol. 44. Cambridge Studies in Advanced Mathematics. Fractals and rectifiability. Cambridge University Press, Cambridge, 1995, pp. xii+343.

- [25] E. Paolini and E. Stepanov. “Optimal transportation networks as flat chains”. In: *Interfaces Free Bound.* 8.4 (2006), pp. 393–436.
- [26] P. Pegon. “On the Lagrangian branched transport model and the equivalence with its Eulerian formulation”. In: *Topological optimization and optimal transport*. Vol. 17. Radon Ser. Comput. Appl. Math. De Gruyter, Berlin, 2017, pp. 281–303.
- [27] P. Pegon, F. Santambrogio, and Q. Xia. “A fractal shape optimization problem in branched transport”. In: *J. Math. Pures Appl. (9)* 123 (2019), pp. 244–269.
- [28] Q. Xia. “Optimal paths related to transport problems”. In: *Commun. Contemp. Math.* 5.2 (2003), pp. 251–279.
- [29] Q. Xia and H. Sun. “Transport multi-paths with capacity constraints”. In: *Journal of Mathematical Analysis and Applications* 550.1:129499 (2025).
- [30] Q. Xia and S. Xu. “Ramified optimal transportation with payoff on the boundary”. In: *SIAM J. Math. Anal.* 55.1 (2023), pp. 186–209.

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