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# Euclid meets Heisenberg

**Coordinatore del Corso:** Chiar.mo Prof. Giovanni Colombo

**Supervisore:** Chiar.mo Prof. Davide Vittone

**Cosupervisore:** Chiar.mo Prof. Antoine Julia

**Dottorando:** Marco Di Marco

Ai miei genitori

Une géométrie ne peut pas être plus vraie qu'une autre; elle peut seulement être plus commode.

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HENRI POINCARÉ, *La Science et l'Hypothèse*

Una geometria non può esser più vera di un'altra; può solo essere più comoda.

---

HENRI POINCARÉ, *La Scienza e l'Ipotesi*

One geometry cannot be more true than another; it can only be more convenient.

---

HENRI POINCARÉ, *Science and Hypothesis*

## Sunto

In questa tesi presentiamo quattro risultati originali che estendono analoghi classici del caso euclideo al contesto sub-Riemanniano. La tesi è organizzata in sei capitoli e un'appendice. Nei Capitoli 1 e 2 introduciamo le definizioni principali e alcuni risultati preliminari che verranno utilizzati negli ultimi quattro capitoli, nei quali sono presentati i contributi originali. Nel Capitolo 3 dimostriamo che il diametro di piccole palle in varietà sub-Riemanniane di classe  $C^{1,1}$  è uguale al doppio del raggio. Dimostriamo inoltre che, quando la regolarità della struttura viene ulteriormente ridotta a  $C^0$ , il diametro è arbitrariamente vicino al doppio del raggio. Entrambi i risultati valgono indipendentemente dalla validità della condizione di Hörmander. Nel Capitolo 4 introduciamo lo spazio  $SBV_X$  delle funzioni speciali a variazione  $X$ -limitata negli spazi di Carnot-Carathéodory e ne studiamo le principali proprietà. Dimostriamo anche un teorema di approssimazione, rispetto alla topologia  $BV_X$ , per le funzioni  $SBV_X$ . Nel Capitolo 5 dimostriamo un teorema di differenziabilità alla Stepanov per grafici intrinseci nei gruppi di Heisenberg sub-Riemanniani. Nel Capitolo 6 introduciamo e studiamo la nozione di sottovarietà regolare con bordo di classe  $C_{\mathbb{H}}^1$  nei gruppi di Heisenberg sub-Riemanniani. Come applicazione, dimostriamo una versione del Teorema di Stokes per sottovarietà  $C_{\mathbb{H}}^1$ -regolari con bordo e per il complesso di Rumin di forme differenziali nei gruppi di Heisenberg. Infine, nell' Appendice A, discutiamo di un lavoro in corso riguardante la relazione tra geometria di contatto e geometria sub-Riemanniana.

## Abstract

In this thesis, we present four original results that extend classical Euclidean counterparts to the sub-Riemannian setting. The thesis is organized in six chapters and one appendix. In Chapter 1 and 2 we introduce the main definitions and some preliminary results that are going to be used in the last four chapters, where the original contributions are presented. In Chapter 3 we prove that the diameter of small balls in  $C^{1,1}$  sub-Riemannian manifolds equals twice the radius. We also prove that, when the regularity of the structure is further lowered to  $C^0$ , the diameter is arbitrarily close to twice the radius. Both results hold independently of Hörmander condition. In Chapter 4 we introduce the space  $SBV_X$  of special functions with bounded  $X$ -variation in Carnot-Carathéodory spaces and study its main properties. We also prove an approximation result, with respect to the  $BV_X$  topology, for  $SBV_X$  functions. In Chapter 5 we prove a Stepanov differentiability type theorem for intrinsic graphs in sub-Riemannian Heisenberg groups. In Chapter 6 we introduce and study the notion of  $C^1_{\mathbb{H}}$ -regular submanifold with boundary in sub-Riemannian Heisenberg groups. As an application, we prove a version of Stokes' Theorem for  $C^1_{\mathbb{H}}$ -regular submanifolds with boundary and for the Rumin complex of differential forms on Heisenberg groups. Finally, in Appendix A, we discuss some work in progress on the relationship between contact geometry and sub-Riemannian geometry.

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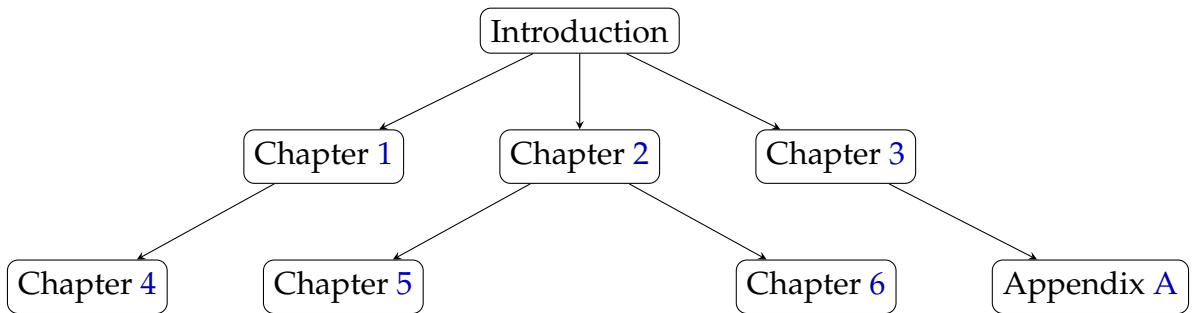
# Introduction

Since the birth of sub-Riemannian geometry [36, 42, 143, 153, 154] one of the most natural questions about the topic is the following.

*To what extent is analysis in sub-Riemannian manifolds similar to or different from analysis in Riemannian manifolds?*

Over the last century there have been results both on the *similar side* (see for instance [66, 141, 157, 158]) and on the *different side* (see for instance [39, 104, 137, 150]). The aim of this thesis is to add four new results [57, 58, 59, 60] on the *similar side*.

The thesis is organized in six chapters and one appendix; we suggest to the reader the following reading scheme.



In the first part of the thesis (Chapters 1 and 2) we recall some definitions and preliminary results that will be used in the second part of the thesis (Chapters 3, 4, 5, 6 and Appendix A), where the original contributions are presented. As one can see from the reading scheme diagram one can choose three different paths: Path A (Chapter 3 and Appendix A), Path B (Chapters 1 and 4) and Path C (Chapters 2, 5 and 6).

Before discussing Paths A, B and C in more detail, let us briefly recall the notions of *sub-Riemannian manifolds*, *Carnot-Carathéodory spaces* and *Heisenberg groups*, for which the inclusions below hold.

$$\left\{ \begin{array}{c} \text{Sub-Riemannian} \\ \text{manifolds} \end{array} \right\} \supset \left\{ \begin{array}{c} \text{Carnot-} \\ \text{Carathéodory} \\ \text{spaces} \end{array} \right\} \supset \left\{ \text{Heisenberg groups} \right\}.$$

A *sub-Riemannian manifold* is a manifold with a fixed admissible subspace (called also horizontal subspace), where in every tangent space the admissible subspaces are equipped with an inner product. The admissible paths are those curves whose velocities are admissible. The distance between two points is the infimum of the lengths of the admissible paths connecting the points. A *Carnot-Carathéodory space* is a special case of sub-Riemannian manifold, that is, the space  $\mathbb{R}^n$  endowed with a distance arising from a collection  $X = (X_1, \dots, X_m)$  of smooth and linearly independent vector fields, called horizontal vector fields, satisfying the Hörmander condition. *Heisenberg groups* are a special class of  $(2n + 1)$ -dimensional Carnot-Carathéodory spaces where the family of horizontal vector fields is fixed. In Path A we will focus on sub-Riemannian manifolds, in Path B on Carnot-Carathéodory spaces and in Path C on Heisenberg groups.

## Path A

The first path is the one given by Chapter 3 and Appendix A. In Chapter 3, where we present [60], we recall the definition of sub-Riemannian manifold and we present the following pair of observations (Theorems i and ii) about the diameters of small balls in sub-Riemannian manifolds.<sup>1</sup> The first result of Chapter 3 is the following theorem.

**Theorem i.** *Let  $M$  be a  $C^{1,1}$  sub-Riemannian manifold. Then, for every  $p \in M$  there exist a neighbourhood (with respect to the manifold topology)  $V$  of  $p$  and  $r_p > 0$  such that*

$$\text{diam}(B(q, r)) = 2r \quad \text{for every } 0 < r < r_p \text{ and } q \in V.$$

The inequality  $\text{diam}(B(q, r)) \leq 2r$  is trivial in every metric space; in general the equality does not hold. For example, the diameter of every open ball in an ultrametric space<sup>2</sup> has diameter strictly less than twice the radius. On the other hand, the equality  $\text{diam}(B(q, r)) = 2r$  is well-known for instance in  $\mathbb{R}^n$  and Banach spaces (for balls of arbitrary radii) and in Riemannian manifolds (for small radii). However, in the natural context of sub-Riemannian geometry the question apparently went under the radar: in fact, to our knowledge Theorem i is known only in Carnot groups (see e.g. [80, Proposition 2.4] or [115, Proposition 10.1.14]), while for more general sub-Riemannian manifolds one has only the partial result [63, Theorem 1.3], which we discuss below.

The proof of Theorem i is quite simple and is based on a classical *calibration* argument, see e.g. [120, 134]. Calibrations are usually exploited to prove length-minimality of a given curve; Theorem i stems from the fact that, actually, calibrations ensure minimality for a whole family of curves spanning a neighbourhood of a given point.

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<sup>1</sup>Actually our definition of sub-Riemannian manifolds does not require the horizontal distribution to be bracket-generating, so our results (Theorems i and ii) holds for a wider class of manifolds. See Definitions 3.3 and 3.12.

<sup>2</sup>We recall that a metric space  $(X, d)$  is an *ultrametric space* if for every  $a, b, c \in X$  we have  $d(a, c) \leq \max(d(a, b), d(b, c))$ . An easy example of ultrametric space is every set equipped with the discrete distance.

The second result presented in Chapter 3 is the following theorem, where we prove an estimate on the diameter of small balls for some more general control problems; namely, when the regularity assumptions on the sub-Riemannian structure are further relaxed and the horizontal distribution is only assumed to be continuous.

**Theorem ii.** *Let  $M$  be a  $C^0$  sub-Riemannian manifold. Then, for every  $p \in M$  and  $\varepsilon > 0$  there exist a neighbourhood (with respect to the manifold topology)  $V$  of  $p$  and  $r_{p,\varepsilon} > 0$  such that*

$$2r(1 - \varepsilon) \leq \text{diam}(B(q, r)) \leq 2r \quad \text{for every } 0 < r < r_{p,\varepsilon} \text{ and } q \in V.$$

Theorem ii was proved by S. Don and V. Magnani [63, Theorem 1.3] for smooth equiregular sub-Riemannian manifolds<sup>3</sup>: this provided a key result in the refined study of the measure of hypersurfaces performed in [63]. The proof of [63, Theorem 1.3] is based on the fact that the blow-up of equiregular sub-Riemannian manifolds at a fixed point is a “tangent” Carnot group and it relies on delicate, “locally uniform” estimates on the rate of convergence to the tangent group under blow-up. Besides working in a more general setting, our proof avoids this machinery and is based on a soft argument that provides a simple “quasi-calibration” for certain “quasi-optimal” curves. Finding a  $C^0$  sub-Riemannian manifold for which Theorem i fails to hold is still an open question.

The study of some properties of geodesics, which, as said before, was pivotal in the proofs of Theorems i and ii will be crucial<sup>4</sup> also in Appendix A, where we present some work in progress on the relationship between sub-Riemannian geometry and contact geometry. Here we only recall that a contact manifold is a smooth manifold equipped with a non-integrable hyperplane field. Every contact manifold can be equipped with a sub-Riemannian structure and the question we aim to answer is the following: how much do biLipschitz<sup>5</sup> maps between contact manifolds “remember” of the contact structures?

## Path B

Let us look at the path given by Chapters 1 and 4. In Chapter 1 we introduce Carnot-Carathéodory (CC) spaces and functions with bounded variation ( $BV_X$  functions) in CC spaces, while in Chapter 4 we present [57].

Functions with Bounded Variation (BV), and in particular their subclass of *special* functions with Bounded Variation (SBV), provide a natural framework for studying problems involving discontinuities, such as image processing, signal analysis, and variational problems. Over recent years, a considerable effort was put into the study of BV functions in metric spaces (see for instance [68, 70, 93, 113, 114] and [7, 8, 72, 103, 109, 127] for more general topics of Geometric Measure Theory in general metric spaces) and, particularly, in

<sup>3</sup>Notice that Theorem i holds under these assumptions.

<sup>4</sup>Actually, we will be more interested in *quasi-geodesics*.

<sup>5</sup>With respect to the Carnot-Carathéodory metrics.

Carnot-Carathéodory (CC) spaces. In Chapter 4 we introduce the space  $SBV_X$  of special functions with bounded  $X$ -variation and we study its properties. In particular, we extend to the setting of CC spaces the following approximation result for classical SBV functions proved by G. De Philippis, N. Fusco and A. Pratelli in [53].

**Theorem iii** ([53, Theorem A]). *Let  $\Omega \subset \mathbb{R}^n$  be an open set and let  $u \in SBV(\Omega)$ . Then, there exists a sequence of functions  $u_k \in SBV(\Omega)$  and of compact  $C^1$ -manifolds  $M_k \subset\subset \Omega$  such that  $\mathcal{J}_{u_k} \subseteq M_k \cap \mathcal{J}_u$ ,  $\mathcal{H}^{n-1}(\overline{\mathcal{J}_{u_k}} \setminus \mathcal{J}_{u_k}) = 0$  and*

$$\|u_k - u\|_{BV(\Omega)} \xrightarrow{k \rightarrow +\infty} 0, \quad u_k \in C^\infty(\Omega \setminus \overline{\mathcal{J}_{u_k}}).$$

Recall that smooth functions are not dense in BV with respect to the BV topology<sup>6</sup>, as their closure in BV is the Sobolev space  $W^{1,1}$ , i.e., BV functions whose derivatives admit no singular part (not even “nice” jumps) with respect to the Lebesgue measure. In this sense, Theorem iii provides a class of “nice” (although, clearly, not smooth) BV functions that are dense in SBV with respect to the BV topology. As explained in [53], this result is sharp and, besides being interesting *per se*, it led to the proof of a conjecture by L. Ambrosio, J. Bourgain, H. Brezis and A. Figalli [4] (see also [89]) about a formula for a BMO-seminorm (defined as an isotropic version of the BMO-norm introduced in [27]) for SBV functions.

Before stating the main result of Chapter 4 (Theorem vi), we need to briefly introduce the notion of special function of bounded variation in CC spaces. A *Carnot-Carathéodory space* (see Definition 1.1) is the space  $\mathbb{R}^n$  endowed with a distance arising from a collection  $X = (X_1, \dots, X_m)$  of smooth and linearly independent vector fields, called horizontal vector fields, satisfying the Hörmander condition. In Path B (and C), we will deal with *equiregular* CC spaces, where a homogeneous dimension  $Q$ , usually larger than the topological dimension  $n$ , can be defined. The space  $BV_X(\Omega)$  [32, 78] of functions with bounded  $X$ -variation (see Definition 1.3) consists of those functions  $u$  on an open set  $\Omega \subset \mathbb{R}^n$  whose derivatives  $X_1u, \dots, X_mu$  in the sense of distributions are represented by a vector-valued measure  $D_Xu$  with finite total variation  $|D_Xu|$ . The space  $BV_X$  has been the subject of intensive studies: see [25, 34, 49, 50, 76, 79, 80, 90, 91, 146] and the more recent [6, 9, 11, 28, 45, 62, 64, 65, 66, 121, 129, 151, 157].

In the classical Euclidean setting the space of SBV functions, first introduced in [52], naturally arises in the study of free discontinuity problems. The first original contribution in Chapter 4 is the introduction of special functions with Bounded Variation in CC spaces ( $SBV_X$  functions). Recall ([66]) that, if  $u \in BV_X(\Omega)$ , then  $D_Xu$  can be decomposed as

$$D_Xu = D_X^{\text{ap}}u \mathcal{L}^n + D_X^s u = D_X^{\text{ap}}u \mathcal{L}^n + D_X^j u + D_X^c u,$$

where  $D_X^{\text{ap}}u$  is the approximate  $X$ -gradient of  $u$ ,  $\mathcal{L}^n$  is the usual Lebesgue measure and

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<sup>6</sup>Smooth functions are dense in BV only with respect to the so-called strict topology in BV, see e.g. [5, Theorem 3.9].

$D_X^s u, D_X^j u, D_X^c u$  are, respectively, the singular part, the jump part and the Cantor part of  $D_X u$ . See Chapter 1 for precise definitions.

**Definition iv.** Let  $\Omega \subset \mathbb{R}^n$  be an open subset of an equiregular Carnot-Carathéodory space  $(\mathbb{R}^n, X)$  and let  $u \in BV_X(\Omega)$ . We say that  $u$  is a *special function of bounded  $X$ -variation*, and we write  $u \in SBV_X(\Omega)$ , if

- (1)  $D_X^c u = 0$ , and
- (2) the jump set  $\mathcal{J}_u$  of  $u$  is a countably  $X$ -rectifiable set.

A set is said to be countably  $X$ -rectifiable (see Definition 1.14) if it can be covered, up to a set which is negligible with respect to the Hausdorff measure  $\mathcal{H}^{Q-1}$ , by a countable family of  $C_X^1$ -hypersurfaces (Definition 1.13), that provide the intrinsic counterpart in CC spaces of classical  $C^1$ -hypersurfaces. Recall that, for classical BV functions, the jump set is always countably rectifiable; on the contrary, in CC spaces this – i.e., the validity of condition (2) above for every  $BV_X$  function  $u$  – is an important open problem. Let us however recall that, if the CC space satisfies the so-called *property  $\mathcal{R}$*  (“rectifiability”, see Definition 1.19), then condition (2) in Definition iv is automatically satisfied for every  $u \in BV_X$ ; see [66, Theorem 1.5]. There is a multitude of examples of CC spaces which satisfy property  $\mathcal{R}$ , such as Heisenberg groups, step 2 Carnot groups and Carnot groups of type  $\star$ ,<sup>7</sup> see [66, Theorem 4.3]. In Chapter 4 we tried to work in the widest possible generality, hence the extra requirement (2) in Definition iv. For this reason, let us also stress that our definition might be a priori different from the definition of SBV function in metric measure spaces (see [10]). We refer to [10] for a general overview of SBV functions in metric measure spaces and to [112] for an approximation result for BV functions via SBV functions in this context.

In Chapter 4 we study several properties of  $SBV_X$  (or locally  $SBV_X$ ) functions: we collect the main ones in the following theorem, which summarizes (some of) the results stated in Proposition 4.5, Theorem 4.6, Lemma 4.7, Theorem 4.9 and Theorem 4.11.

**Theorem v.** Let  $\Omega \subset \mathbb{R}^n$  be an open subset of an equiregular Carnot-Carathéodory space  $(\mathbb{R}^n, X)$ ; then, the following statements hold:

- (1)  $u \in SBV_{X,\text{loc}}(\Omega)$  if and only if  $D_X^s u = f \nu_R \mathcal{H}^{Q-1} \llcorner R$  for some countably  $X$ -rectifiable set  $R \subset \Omega$  with horizontal normal  $\nu_R$  and some  $f \in L_{\text{loc}}^1(R, \mathcal{H}^{Q-1})$ ;
- (2)  $SBV_X(\Omega)$  is a closed subspace of  $BV_X(\Omega)$ ;
- (3) the space  $SBV_{\text{loc}}(\Omega)$  of special function of (Euclidean) locally bounded variation is contained in  $SBV_{X,\text{loc}}(\Omega)$ ;

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<sup>7</sup>See Chapter 2 for the definition of Carnot groups.

(4) for every  $w \in L^1_{\text{loc}}(\Omega; \mathbb{R}^m)$  there exists  $u \in \text{SBV}_{X,\text{loc}}(\Omega)$  such that  $D_X^{\text{ap}} u = w$  a.e. in  $\Omega$ ;

(5) for every countably  $X$ -rectifiable set  $R \subseteq \Omega$  oriented by  $\nu_R$ , every  $\theta \in L^1(\mathcal{H}^{Q-1} \llcorner R)$  and every  $\delta > 0$  there exists  $u \in \text{SBV}_X(\Omega)$  such that

$$D_X^j u \equiv \theta \nu_R \mathcal{H}^{Q-1} \llcorner R, \quad \|u\|_{L^1(\Omega)} < \delta, \quad \text{and} \quad |D_X u|(\Omega) \leq (2 + \delta) \|\theta\|_{L^1(\mathcal{H}^{Q-1} \llcorner R)}.$$

Statements (3), (4) and (5), in particular, provide meaningful subclasses or examples of special functions of bounded  $X$ -variation which, in particular, turn out to form a quite large and interesting space.

We can now state the main result of Chapter 4.

**Theorem vi.** *Let  $\Omega$  be an open subset of an equiregular Carnot-Carathéodory space  $(\mathbb{R}^n, X)$  and let  $u \in \text{SBV}_X(\Omega)$ . Then, there exists a sequence of functions  $(u_k)_{k \in \mathbb{N}} \subset \text{SBV}_X(\Omega)$  and of  $C^1_X$ -hypersurfaces  $(M_k)_{k \in \mathbb{N}} \subset \Omega$  such that, for every  $k \in \mathbb{N}$ ,  $\mathcal{J}_{u_k} \subseteq M_k \cap \mathcal{J}_u$ ,  $\mathcal{J}_{u_k}$  is compact, and*

$$\|u - u_k\|_{\text{BV}_X(\Omega)} \xrightarrow{k \rightarrow +\infty} 0, \quad u_k \in C^\infty(\Omega \setminus \mathcal{J}_{u_k}).$$

Our proof of Theorem vi differs from the one of Theorem iii in that, rather than using mollifications with variable kernel as in [53], we exploit a partition-of-the-unity argument (reminiscent of [14, 58, 78, 157, 158]) that allows to approximate  $u$  out of a fixed compact set  $C_k$ . *A posteriori*, the set  $C_k$  coincides with the jump set  $\mathcal{J}_{u_k}$ , which in particular turns out to be compact itself, thus providing a slight improvement in Theorem iii.

We believe that Theorem vi will play a role in a possible, future BMO-type characterization of  $\text{BV}_X$  functions *à la* Ambrosio-Bourgain-Brezis-Figalli, [4].

## Path C

The last path is the one given by Chapters 2, 5 and 6. In Chapter 2 we introduce the Heisenberg groups  $\mathbb{H}^n$ , which are a special class of  $(2n + 1)$ -dimensional Carnot-Carathéodory spaces where the family of horizontal vector fields is given, for  $1 \leq j \leq n$ , by the vector fields

$$X_j := \partial_{x_j} - \frac{y_j}{2} \partial_t, \quad Y_j := \partial_{y_j} + \frac{x_j}{2} \partial_t,$$

where for  $p \in \mathbb{H}^n \equiv \mathbb{R}^{2n+1}$  we denoted its coordinates by  $p = (x_1, \dots, x_n, y_1, \dots, y_n, t)$ . Precise definitions for Heisenberg groups are given later in Section 2.1; here we only recall that the  $n$ -th Heisenberg group  $\mathbb{H}^n$  is a step 2 nilpotent Lie group diffeomorphic to  $\mathbb{R}^{2n+1}$  that is endowed with a left-invariant, bracket-generating horizontal sub-bundle  $H\mathbb{H}^n$  of dimension  $2n$ , with a one-parameter family  $(\delta_\lambda)_{\lambda>0}$  of group isomorphisms (called dilations), and with a left-invariant and homogeneous distance  $d$ . The Hausdorff dimension of the metric space  $(\mathbb{H}^n, d)$  is  $Q := 2n + 2$ . In Chapter 2 we introduce the geometrical objects and their main properties (*intrinsic* (Lipschitz) graphs in Section 2.2, *intrinsic*  $C^1$ -submanifolds in Section 2.3 and *intrinsic* differential forms in Section 2.4) that

we are going to need in Chapters 5 and 6. Finally in Section 2.5 we spend a few words on Carnot groups which can be considered a generalization of Heisenberg groups.

In Chapter 5 we present the results contained in [59]. The main outcome is the Stepanov theorem for intrinsic graphs in Heisenberg groups (Theorem viii) which, as in the Euclidean setting, is obtained as a consequence of Rademacher theorem.

The classical Rademacher theorem states that: if  $\Omega \subseteq \mathbb{R}^n$  is an open set and  $f : \Omega \rightarrow \mathbb{R}^m$  is Lipschitz continuous, then  $f$  is differentiable almost everywhere in  $\Omega$ . Stepanov theorem instead says the following (see e.g. [73, Theorem 3.1.9] and [152]).

**Theorem vii.** *Let  $\Omega \subseteq \mathbb{R}^n$  be an open set and let  $f : \Omega \rightarrow \mathbb{R}^m$ . Consider the set*

$$S_f := \left\{ a \in \Omega : \limsup_{x \rightarrow a} \frac{|f(x) - f(a)|}{|x - a|} < +\infty \right\}. \quad (i)$$

*Then  $f$  is differentiable almost everywhere on  $S_f$ .*

The classical proof of the Stepanov differentiability theorem can be found in many textbooks (see e.g. [73]) and it is essentially based on density theorems and the application of Rademacher's theorem to a Lipschitz extension of  $f|_{G_n}$ , where  $G_n$  are suitable measurable sets on which  $f$  is Lipschitz. In 1999, J. Malý [126] proposed, for *real-valued* functions defined on separable Banach spaces, an alternative and elegant proof without using any density theorem. In [26, 67], using differentiability points of the distance functions instead of density theorems, the authors were able to prove a Stepanov type theorem (in the Gateaux sense) for functions between infinite dimensional Banach spaces. Later, in [128], Malý and Zajíček, presented a new approach which shows how a Stepanov type theorem (in the Frechet sense) can be inferred from the corresponding theorem of Rademacher type.

In recent years, there has been significant and ongoing research aimed at extending classical analysis results from Euclidean spaces to more general metric-measure spaces (see e.g. [95, 98] and references therein). A major result is due to Cheeger [38], who found a deep generalization of Rademacher's theorem in the context of doubling metric measure spaces that satisfy a Poincaré inequality. More recently, the approach in [126] has been used in [19] to prove a Stepanov type theorem for real-valued maps defined on metric spaces endowed with a doubling Borel measure and later in [161] the result has been further generalized to maps between metric measure spaces and Banach spaces. Building on the foundation established in [38], a substantial body of literature has emerged, see e.g. [96, 98] for a complete overview. Let us also mention the recent [51].

The notion of Lipschitz submanifolds in sub-Riemannian geometry was introduced, at least in the setting of Carnot groups, by B. Franchi, R. Serapioni and F. Serra Cassano in [79, 81, 84] through the theory of intrinsic Lipschitz graphs. We postpone precise definitions to Section 2.2, but, roughly speaking, a subset  $S \subset \mathbb{G}$  of a Carnot group  $\mathbb{G}$  is intrinsic Lipschitz if at each point  $P \in S$  there is an intrinsic cone with vertex  $P$  and

fixed opening intersecting  $S$  only at  $P$ . Remarkably, this notion turned out to be the right one in the setting of the intrinsic rectifiability in the simplest Carnot group, namely the Heisenberg group  $\mathbb{H}^n$ . Indeed, it was proved in [79, 158] that the notion of rectifiable set in terms of an intrinsic regular hypersurfaces is equivalent to the one in terms of intrinsic Lipschitz graphs. Recently, the theory of intrinsic Lipschitz functions has played a crucial role in the study of quantitative rectifiability [40, 41] and has even been applied to problems in information theory [139, 140]. See also [44, 147] for further applications.

The main open question in this area of research is whether a Rademacher-type theorem holds for intrinsically Lipschitz functions between homogeneous subgroups of a Carnot group. Specifically, consider a splitting  $\mathbb{G} = \mathbb{W}\mathbb{V}$  of a Carnot group  $\mathbb{G}$  and let  $\phi : \mathbb{W} \rightarrow \mathbb{V}$  be an intrinsically Lipschitz function. The question is whether such a function is intrinsically differentiable almost everywhere (see Definition 2.28). This question has been affirmatively answered when  $\mathbb{V} \equiv \mathbb{R}$  and  $\mathbb{G}$  is either the Heisenberg group ([83], see also [37]) or a step 2 Carnot group ([77]) or a Carnot group of type  $\star$  ([84]) or a Carnot group of type  $\diamond$  ([116]). More recently, D. Vittone [158] has proved that the answer is also affirmative in the case of the Heisenberg group, even with no hypothesis on the splitting. When  $\mathbb{W}$  is a Carnot subgroup and  $\mathbb{V}$  is a normal subgroup, the Rademacher theorem is proved in [13]. Remarkably, in [99], the authors constructed intrinsic Lipschitz graphs of codimension 2 in Carnot groups which are nowhere intrinsically differentiable thus discovering a deep connection between the notion of intrinsic differentiability and the geometry of the underlying Carnot group.

The main result of Chapter 5 is the following.

**Theorem viii.** *Let  $\mathbb{W}$  and  $\mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$ . Let  $A \subseteq \mathbb{V}$  be an open set and  $\phi : A \rightarrow \mathbb{W}$ . Then  $\phi$  is intrinsically differentiable almost everywhere on the set where the intrinsic graph given by  $\phi$  is pointwise intrinsic Lipschitz.*

The proof of Theorem viii, although inspired by that of [73], differs in some fundamental points. First of all, the notion of intrinsic Lipschitz continuity differs from the classical metric one (see Definition 2.19). This means that an intrinsic Lipschitz function  $\phi : A \subseteq \mathbb{W} \rightarrow \mathbb{V}$  need not to be metric Lipschitz (with respect to the distances induced on  $\mathbb{W}$  and  $\mathbb{V}$ ), see [83]. Therefore Theorem viii does not fit into the classical framework of Lipschitz maps between metric measure spaces. Secondly, the notions of differentiability and Lipschitz continuity are indeed intrinsic geometric properties of  $\phi$ , which take into account not only the structure of  $\mathbb{W}, \mathbb{V}$  but also how they interact inside  $\mathbb{H}^n$ . To overcome these problems, we revisited the proof provided in [73] from a new geometric perspective; specifically, instead of working with the function  $\phi$ , we considered its intrinsic graph. In Subsection 5.2.2 we also provide an alternative proof of the theorem in the case of codimension one, using the approach developed in [126]. We point out that the two different proofs of Theorem viii are not dependent on the particular structure of  $\mathbb{H}^n$ . They can be extended

to intrinsic graphs in general Carnot groups  $\mathbb{G}$  endowed with a splitting  $\mathbb{W}\mathbb{V}$  for which a Rademacher theorem holds.

In Chapter 6, where we present the results contained in [58], the main outcome is a sub-Riemannian analogue of Stokes' Theorem (Theorem x) in Heisenberg groups. The classical Stokes Theorem states that, if  $M$  is a smooth  $m$ -dimensional manifold with boundary, then the equality

$$\int_M d\omega = \int_{\partial M} \omega$$

holds for every smooth  $(m - 1)$ -form on  $M$ .

Before stating our sub-Riemannian version of Stokes' Theorem we need to discuss the two keywords, *submanifolds* and *forms* in Heisenberg groups, that are needed for the very statement of Stokes' Theorem.

The first ingredient is provided by  $C_{\mathbb{H}}^1$ -regular (or  $\mathbb{H}$ -regular) submanifolds, i.e., submanifolds that are  $C^1$  regular from the viewpoint of intrinsic geometry. These submanifolds are of a completely different nature depending on whether they have *low dimension*  $1 \leq k \leq n$ , in which case they are  $k$ -dimensional submanifolds of class  $C^1$  tangent to  $H\mathbb{H}^n$ , or *low codimension*  $1 \leq k \leq n$ , then they are (locally) defined as non-critical level sets of maps from  $\mathbb{H}^n$  to  $\mathbb{R}^k$  that are continuously differentiable along horizontal directions. In particular,  $C_{\mathbb{H}}^1$ -regular submanifolds of low dimension  $k$  are (contained in) Legendrian<sup>8</sup> submanifolds and they have classical  $C^1$  regularity; their Hausdorff dimension coincides with the topological dimension,  $k$ . On the contrary,  $C_{\mathbb{H}}^1$ -regular submanifolds of low codimension  $k$  can have fractal Euclidean dimension, while their Hausdorff dimension is  $Q - k = 2n + 2 - k$ , thus larger than the topological dimension  $2n + 1 - k$ . Extensive studies of  $C_{\mathbb{H}}^1$ -regular submanifolds have been carried out, see e.g. [12, 16, 22, 23, 24, 43, 46, 56, 82, 110, 111]. We collect in Section 2.3 some basic facts about  $C_{\mathbb{H}}^1$ -regular submanifolds, but we suggest to the reader the beautiful paper [82], where the genesis of and motivations behind the theory are masterfully illustrated.

The second object we need is the complex of *Heisenberg-Rumin differential forms*

$$0 \rightarrow \mathbb{R} \rightarrow \Omega_{\mathbb{H}}^0 \xrightarrow{d} \Omega_{\mathbb{H}}^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_{\mathbb{H}}^n \xrightarrow{D} \Omega_{\mathbb{H}}^{n+1} \xrightarrow{d} \dots \xrightarrow{d} \Omega_{\mathbb{H}}^{2n+1} \rightarrow 0 \quad (\text{ii})$$

introduced by M. Rumin [144, 145] (see also [74, 119, 155]). For clarity of exposition, in this introduction we will regard the space  $\Omega_{\mathbb{H}}^k$  of Heisenberg  $k$ -forms as a subspace of classical differential  $k$ -forms: this is however a simplification and we postpone precise definitions to Section 2.4. The operators  $d$  appearing in (ii) are restriction of the standard exterior derivative, while the operator  $D : \Omega_{\mathbb{H}}^n \rightarrow \Omega_{\mathbb{H}}^{n+1}$  is a non-trivial second-order operator that constitutes M. Rumin's key discovery, as it ensures that the complex (ii) has the same cohomology as the de Rham complex. This complex arises naturally in Heisenberg groups

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<sup>8</sup>For the standard contact structure on the Heisenberg group.

and Theorem x below can be considered as one of the evidence that Rumin's complex is natural in this setting.

A Heisenberg form  $\omega$  can be integrated on an *oriented* (see Definition 6.11)  $C_{\mathbb{H}}^1$ -regular submanifold  $S$ . If  $S$  is of low dimension then  $\int_S \omega$  is the standard integral (see, however, Proposition 6.9); if, on the other hand,  $S$  has low codimension we set

$$\int_S \omega := \frac{1}{C_{n,k}} \int_S \langle t_S^{\mathbb{H}} | \omega \rangle dS^{Q-k}, \quad (\text{iii})$$

where  $t_S^{\mathbb{H}}$  denotes the Heisenberg unit tangent vector orienting  $S$ ,  $S^{Q-k}$  is the spherical Hausdorff measure of dimension  $Q - k$ , and  $C_{n,k}$  is a geometric constant with the property that  $\int_S \omega$  coincides with the standard integral when  $S$  is Euclidean smooth, see Lemma 6.12. The operator  $\omega \mapsto \int_S \omega$  is a *Heisenberg current*, which we denote by  $[[S]]$ . The theory of currents in Heisenberg groups (introduced in Section 6.2) is only at a very early stage [31, 82, 101, 158], but the formalism is quite convenient for our purposes and will be extensively used later in Chapter 6. It is worth noticing that the value of  $\int_S \omega$  does not depend on the particular choice of the distance  $d$ , see Lemma 6.16: in particular, the definition in (iii) seems to us natural and substantial – in a word, *geometric*.

The first original contribution in Chapter 6 is the study of  $C_{\mathbb{H}}^1$ -regular submanifolds with boundary in Heisenberg groups, see also [18, Section 5]. One could be tempted to consider as such those  $C_{\mathbb{H}}^1$ -regular submanifolds  $S$  whose boundary, defined as  $\partial S := \bar{S} \setminus S$ , is also a  $C_{\mathbb{H}}^1$ -regular submanifold; however, this definition would not be appropriate. To explain why, we use the following example in Euclidean geometry (similar examples could be produced in Heisenberg groups): the surface  $S = \{(x, y, x \sin \frac{1}{x}) : x > 0, y \in \mathbb{R}\}$  is  $C^\infty$  smooth in  $\mathbb{R}^3$  and its boundary  $\partial S = \{(0, y, 0) : y \in \mathbb{R}\}$  is also smooth; however,  $S$  is not of class  $C^1$  as a submanifold *with boundary*. The classical definition of  $C^1$  manifold with boundary  $S$  requires that, for every  $p \in \partial S$ , there is a  $C^1$  diffeomorphism  $\Phi$  between a neighbourhood of  $p$  and a closed half-ball such that  $\Phi(p)$  is the center of the ball; using a little bit of Whitney Extension Theorem, one realizes that this is equivalent to requiring additively that, for every  $p \in \partial S$ , the manifold  $S$  can be (locally) extended to a  $C^1$  submanifold  $S'$  whose interior contains  $p$ .

This discussion motivates the following definition.

**Definition ix.** Let  $1 \leq m \leq 2n + 1$ ; we say that  $S \subseteq \mathbb{H}^n$  is an  *$m$ -dimensional  $C_{\mathbb{H}}^1$ -regular submanifold with boundary* if the following conditions hold:

- (1)  $S$  is a non empty  $m$ -dimensional  $C_{\mathbb{H}}^1$ -regular submanifold;
- (2)  $\partial S$  is a non empty  $(m - 1)$ -dimensional  $C_{\mathbb{H}}^1$ -regular submanifold;
- (3) for every  $p \in \partial S$  there exist a neighbourhood  $U$  of  $p$  and an  $m$ -dimensional  $C_{\mathbb{H}}^1$ -regular submanifold  $S' \subseteq U$  such that  $U \cap \bar{S} \subseteq S'$  and, for every  $r > 0$ ,  $U(p, r) \cap (S' \setminus \bar{S}) \neq \emptyset$ .

In subsection 6.3.2 we provide some examples, coming from [61], of  $C_{\mathbb{H}}^1$ -regular submanifolds with boundary. In Definition 6.18 we also introduce the (local) notion of submanifolds with boundary that are  $C_{\mathbb{H}}^1$ -regular in an open set. Definition ix is stated to work for any dimension and codimension; however, it is quite clear that in the low-dimensional case  $C_{\mathbb{H}}^1$  regular submanifolds with boundary are exactly Euclidean  $C^1$  submanifolds with boundary that are tangent to the horizontal distribution.<sup>9</sup> In the case of low codimension  $k = 2n + 1 - m$ , one needs to distinguish between the non-critical case  $1 \leq k \leq n - 1$  and the critical case  $k = n$ : the first is somewhat easier and one can provide a useful equivalent definition in terms of level/superlevel sets, see Theorem 6.19. On the contrary, the case of critical codimension  $k = n$  is much more delicate, as it corresponds to the case in which the submanifold is of low codimension but its boundary is low dimensional. This is why in the critical case it is much more difficult to prove the main result of Chapter 6, which we now state.

**Theorem x.** *If  $1 \leq m \leq 2n + 1$ ,  $S \subset \mathbb{H}^n$  is an  $m$ -dimensional orientable  $C_{\mathbb{H}}^1$  regular submanifold with boundary and  $S, \partial S$  have locally finite measures in  $\mathbb{H}^n$ , then*

$$\int_S d_c \omega = \int_{\partial S} \omega \quad \text{for every } \omega \in \mathcal{D}_{\mathbb{H}}^{m-1}, \quad (\text{iv})$$

where  $\mathcal{D}_{\mathbb{H}}^{m-1}$  denotes the space of Heisenberg-Rumin differential  $(m - 1)$ -forms with compact support.

As customary, the symbol  $d_c$  stands for the exterior Rumin differential:  $d_c = d$  if  $m \neq n + 1$ , while in the critical case  $m = n + 1$  one writes  $d_c = D$ . The measures on  $S, \partial S$  referred to in Theorem x are the Hausdorff spherical measures on  $S, \partial S$  of the appropriate dimensions that are (respectively)  $m, m - 1$  (in the low dimensional case  $m \leq n$ ),  $m + 1, m$  (in the low-codimensional and non-critical case  $m \geq n + 2$ ) or  $n + 2, n$  (in the critical case  $m = n + 1$ ). A local version of Theorem x is also proved in Theorem 6.41, from which Theorem x follows. Observe that Stokes' Theorem can be written in the language of currents as  $\partial_c \llbracket S \rrbracket = \llbracket \partial S \rrbracket$ , and Theorem 6.41 is indeed stated in these terms.

In the low-dimensional case, Theorems x and 6.41 follows directly from the classical Stokes Theorem. For  $C_{\mathbb{H}}^1$ -regular submanifolds of low codimension  $1 \leq k \leq n$ , the theorems can be proved via an approximation scheme, performed in Section 6.4, in which the submanifold  $S$  and its boundary  $\partial S$  are  $C_{\mathbb{H}}^1$ -approximated by, respectively, smooth submanifolds  $S_h$  and their boundaries  $\partial S_h$ : the classical Stokes Theorem holds for each  $S_h$  and the equality  $\int_{S_h} d_c \omega = \int_{\partial S_h} \omega$  passes to the limit, as  $h \rightarrow \infty$ , to get (iv). The approximation scheme is particularly delicate in the case of critical codimension  $k = n$ : the major difficulty one encounters is that, while it is fairly easy to produce smooth

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<sup>9</sup>This implies that the boundary, whose tangent space is contained in the tangent space to the submanifold, is also tangent to the horizontal distribution.

approximations  $S_h$  of  $S$ , Frobenius Theorem makes it quite challenging to “cut” into  $S_h$  an  $n$ -dimensional boundary  $\partial S_h$  that is tangent to the horizontal distribution, as the latter is totally non-integrable. Note that this difficulty does not arise if  $k = n = 1$  in which case it is enough to find some approximation  $S_h$  and then “cut” it (thus creating the boundary  $\partial S_h$ ) along some horizontal curve. This idea was exploited in [86, Theorem 5.3], where Stokes’ Theorem in  $\mathbb{H}^1$  was proved for the critical dimension, although with a different formulation. However, following a technically quite elaborate argument, in Lemma 6.39, we were able to produce an approximating sequence  $S_h$  with the extra property that  $\partial S_h = \partial S$  for every  $h$ , which is even more than requested. We believe that the approximation results of Lemmata 6.38 and 6.39 might be of use for future applications.

## Future paths

In this final section of the Introduction, we outline several directions for further answering the central question of this thesis, which we restate here for the reader’s convenience.

*To what extent is analysis in sub-Riemannian manifolds  
similar to or different from analysis in Riemannian manifolds?*

Proceeding along Path A outside of the boundaries of this thesis the first step, of course, would be to conclude the work presented in Appendix A and continue to work to fully understand the relationship between sub-Riemannian geometry and contact geometry.

If, instead, we continue to walk on Path B, we wish to continue to build the theory of  $BV_X$  and  $SBV_X$  functions. As we said before, one of the main reason behind [57] was the possibility to use Theorem vi to obtain a BMO-type characterization of  $BV_X$  functions. Another possible direction could be to prove a closure and compactness theorem for  $SBV_X$  functions, in the same fashion of [5, Theorems 4.7 and 4.8]. After proving the latter, one could work on functionals in Carnot-Carathéodory spaces inspired, for example, by the classical Mumford-Shah functional. Another big open question regarding  $BV_X$  functions is the validity of the rank-one theorem for general Carnot-Carathéodory spaces. The rank-one theorem was already proven in [64] for a wide class of Carnot groups, but its validity in the first Heisenberg group  $\mathbb{H}^1$  remains an important open question that we would like to answer.

If, finally, we continue to walk on Path C outside of the boundaries of this thesis we have two possibilities. One is to try to extend the results of Chapters 5 and 6 to a generic Carnot group. Extending the results contained in Chapter 5 requires understanding in which Carnot groups, and for which type of splittings, Rademacher Theorem holds. Extending the result of Chapter 6 requires building a robust theory of intrinsic  $C^1$  submanifolds (with and without boundary) and understand the explicit appearance of the Rumin complex in generic Carnot groups. The other possibility coming from proceeding on Path C is to continue to import classical Euclidean results in the *easier* setting of Heisenberg groups. A

very interesting possibility is to help building the theory of Heisenberg currents, which, as said before, is still at a very early stage. A major result would be to prove the sub-Riemannian version of the polyhedral deformation theorem.

# Basic notations

$\oplus$	direct sum of vector spaces
$\circ$	composition of functions
$\mathbb{R}^n$	$n$ -dimensional Euclidean space
$\mathcal{L}^n$	Lebesgue measure in $\mathbb{R}^n$
$\mathbb{H}^n$	$n$ -th Heisenberg group
$\mathfrak{h}$	Lie algebra of $\mathbb{H}^n$
$H\mathbb{H}^n$	horizontal bundle of $\mathbb{H}^n$
$H_p\mathbb{H}^n$	horizontal subspace at $p \in \mathbb{H}^n$
$\theta$	contact form in $\mathbb{H}^n$
$\pi_p(q)$	projection of $q \in \mathbb{H}^n$ on $H_p\mathbb{H}^n$
$X$	$m$ -tuple of smooth and linearly independent vector fields on $\mathbb{R}^n$
$(\mathbb{R}^n, X)$	equiregular Carnot-Carathéodory space
$\langle x, y \rangle_{\mathbb{R}^n}$	standard Euclidean inner product of $x, y \in \mathbb{R}^n$
$ x _{\mathbb{R}^n}$	standard Euclidean norm of $x \in \mathbb{R}^n$
$\dot{\gamma}$	time derivative of a curve $\gamma$
$p \cdot q$	group product between $p, q \in \mathbb{H}^n$
$\delta_\lambda$	homogeneous dilations in $\mathbb{H}^n$ of parameter $\lambda$
$\tau_p$	left translation by an element $p \in \mathbb{H}^n$
$\star$	group convolution of $\mathbb{H}^n$
$*$	Euclidean convolution
$d_E$	Euclidean distance
$d_C$	Carnot-Carathéodory distance
$d$	left-invariant, homogeneous and rotationally invariant distance on $\mathbb{H}^n$
$\ \cdot\ $	norm induced by a distance $d$
$B_E(p, r)$	open Euclidean ball centered at $p$ with radius $r$
$B_C(p, r)$	open Carnot-Carathéodory ball centered at $p$ with radius $r$
$B_d(p, r)$	open ball induced by a distance $d$ centered at $p$ with radius $r$
$\mathcal{H}_E^k$	Euclidean $k$ -dimensional Hausdorff measure
$\mathcal{S}_E^k$	Euclidean $k$ -dimensional spherical Hausdorff measure
$\mathcal{H}_C^k$	$k$ -dimensional Hausdorff measure induced by $d_c$

$\mathcal{S}_C^k$	$k$ -dimensional spherical Hausdorff measure induced by $d_c$
$\mathcal{H}_d^k$	$k$ -dimensional Hausdorff measure induced by a distance $d$
$\mathcal{S}_d^k$	$k$ -dimensional spherical Hausdorff measure induced by a distance $d$
$f_{\#}\mu$	push-forward of the measure $\mu$ via $f$
$\mu \llcorner A$	restriction of a measure $\mu$ to a set $A$
$BV_X$	functions with bounded $X$ -variation
$SBV_X$	special function with bounded $X$ -variation
$\mathcal{J}_u$	Carnot-Carathéodory jump set of a $BV_X$ function $u$
$\mathcal{J}_u^E$	Euclidean jump set of a $BV$ function $u$
$Xf$	horizontal gradient of $f$ with respect to $X$
$\nabla_{\mathbb{H}}f$	horizontal gradient of $f$ in $\mathbb{H}^n$
$C^k(O)$	continuously $k$ -differentiable real functions in $O$
$C_c^k(O)$	functions in $C^k(O)$ with compact support in $O$
$C_X^1(O)$	continuously $X$ -differentiable functions in $O$
$C_{\mathbb{H}}^1(O)$	continuously $\nabla_{\mathbb{H}}$ -differentiable functions in $O$
$d_{\mathbb{H}}f_q$	Pansu differential of $f$ at $q \in \mathbb{H}^n$
$TS$	tangent bundle to a manifold $S$
$T_pS$	tangent space to $S$ at $p$
$T_p^{\mathbb{H}}S$	Heisenberg tangent cone to $S$ at $p$
$\mathfrak{gr}_{\varphi}$	intrinsic graph of a map $\varphi$
$\pi_{\mathbb{W}}(p), p_{\mathbb{W}}$	projection of $p \in \mathbb{H}^n$ on the subgroup $\mathbb{W}$
$\Omega_{\mathbb{H}}^k$	Heisenberg-Rumin differential $k$ -forms
$\mathcal{D}_{\mathbb{H}}^k(O)$	Heisenberg-Rumin differential $k$ -forms with compact support in $O$
$\mathbb{1}_E$	characteristic function of the set $E$
$u^*(p)$	approximate limit of $u$ at $p$
$\mathcal{S}_u$	set of points where $u$ does not admit an approximate limit
$D_X^{\text{ap}}u$	approximate $X$ -gradient of $u$
$D_E^{\text{ap}}u$	approximate Euclidean gradient of $u$
$\mathcal{D}_u$	set of point where $u$ is approximately $X$ -differentiable
$C_{\mathbb{W},\mathbb{V}}(p, \beta), C_{\beta}(p)$	intrinsic cone of vertex $p$ , base $\mathbb{W}$ , axis $\mathbb{V}$ and opening $\beta$
$d_c$	Rumin differential operator
$\mathcal{R}$	“rectifiability” property for Carnot-Carathéodory spaces
$\mathcal{D}$	“density” property for Carnot-Carathéodory spaces
$\langle v w \rangle$	standard pairing between a $k$ -vector $v$ and a $k$ -covector $w$
$\wedge$	exterior product

# Part I

## Preliminaries

In the first part of the thesis we recall the main definitions and the known results that we are going to need in the second part, where the original contributions are presented. We will present Carnot-Carathéodory spaces and functions of bounded variations in Chapter 1 and it will serve as an introduction to Chapter 4. We will present Heisenberg groups, intrinsic submanifolds, intrinsic graphs and intrinsic differential forms in Chapter 2 and it will serve as an introduction to Chapter 5 and Chapter 6.

# 1 Carnot-Carathéodory spaces and functions of bounded variation

In this chapter we present some basic definitions and known facts about functions with bounded variation in Carnot-Carathéodory spaces ( $BV_X$  functions). Over the recent years, a considerable effort was put into the study of  $BV_X$  functions, see for instance [62] and the references therein. Before doing so, we spend a few words on Carnot-Carathéodory spaces (CC spaces) and their main properties. For a comprehensive introduction to CC spaces we refer the reader to [135] and the references therein.

For the rest of this chapter  $1 \leq m \leq n$  will denote fixed integers and  $X = (X_1, \dots, X_m)$  will denote a  $m$ -tuple of smooth and linearly independent vector fields on  $\mathbb{R}^n$ .

**Definition 1.1.** We say that an absolutely continuous curve  $\gamma: [0, T] \rightarrow \mathbb{R}^n$  is an  $X$ -subunit path joining  $p$  and  $q$  if  $\gamma(0) = p$ ,  $\gamma(T) = q$  and there exist  $h_1, \dots, h_m \in L^\infty([0, T])$  such that  $\sum_{j=1}^m h_j^2 \leq 1$  and

$$\dot{\gamma}(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t)), \quad \text{for a.e. } t \in [0, T].$$

For every  $p, q \in \mathbb{R}^n$  we define

$$d_C(p, q) := \inf\{T > 0 : \text{there exists an } X\text{-subunit path } \gamma \text{ joining } p \text{ and } q\},$$

where we agree that  $\inf \emptyset := +\infty$ .

By the Chow–Rashevskii Theorem (first proved in [42, 143], see for instance [1, Subsection 3.2.1] for a more modern proof), if for every  $p \in \mathbb{R}^n$  the linear span of all iterated commutators of the vector fields  $X_1, \dots, X_m$  computed at  $p$  has dimension  $n$  (i.e.  $X_1, \dots, X_m$  satisfy the Hörmander condition), then  $d_C$  is a distance: the latter means that for every couple of points of  $\mathbb{R}^n$  there always exists a  $X$ -subunit path joining them. In this case we say that  $(\mathbb{R}^n, X)$  is a Carnot-Carathéodory space of rank  $m$  and  $d_C$  is the associated Carnot-Carathéodory distance.

For every  $p \in \mathbb{R}^n$  and for every  $i \in \mathbb{N}$  we denote by  $\mathcal{L}^i(p)$  the linear span of all the commutators of  $X_1, \dots, X_m$  up to order  $i$  computed at  $p$ . We say that a Carnot-Carathéodory space  $(\mathbb{R}^n, X)$  is *equiregular* if there exist natural numbers  $n_0, n_1, \dots, n_s$  such that

$$0 = n_0 < n_1 < \dots < n_s = n \text{ and } \dim \mathcal{L}^i(p) = n_i, \quad \forall p \in \mathbb{R}^n, \forall i \in \{1, \dots, s\}.$$

The natural number  $s$  is called *step* of the Carnot-Carathéodory space. If  $(\mathbb{R}^n, X)$  is equiregular, then the *homogeneous dimension* is  $Q := \sum_{i=1}^s i(n_i - n_{i-1})$ .

We now collect some well-known facts about the geometry of an equiregular Carnot-Carathéodory space.

**Theorem 1.2.** *Let  $(\mathbb{R}^n, X)$  be an equiregular Carnot-Carathéodory space of step  $s$ . Then the following facts hold.*

(i) *For every compact set  $K \subseteq \mathbb{R}^n$ , there exists  $M \geq 1$  such that for any  $p, q \in K$*

$$\frac{1}{M}|p - q| \leq d_C(p, q) \leq M|p - q|^{\frac{1}{s}}.$$

(ii) *The Hausdorff dimension of the metric space  $(\mathbb{R}^n, d_C)$  coincides with the homogeneous dimension of  $(\mathbb{R}^n, X)$ .*

(iii) *The metric measure space  $(\mathbb{R}^n, d_C, \mathcal{L}^n)$  is locally Ahlfors  $Q$ -regular (where  $Q$  is the homogeneous dimension of  $(\mathbb{R}^n, X)$ ), i.e., for every compact set  $K \subseteq \mathbb{R}^n$  there exist  $R > 0$  and  $C > 1$  such that*

$$\frac{1}{C}r^Q \leq \mathcal{L}^n(B_C(p, r)) \leq Cr^Q,$$

*for every  $p \in K$  and for every  $r \in (0, R)$ . In particular, the metric measure space  $(\mathbb{R}^n, d_C, \mathcal{L}^n)$  is locally doubling.*

For a proof of (i) and (iii) see [138], for a proof of (ii) see [132].

In the rest of this thesis  $(\mathbb{R}^n, X)$  denotes an equiregular Carnot-Carathéodory space associated with the family  $X = (X_1, \dots, X_m)$ . We use  $d_C$  to denote the Carnot-Carathéodory distance associated with  $X$ ,  $B_C(\cdot, \cdot)$  to denote the associated open balls,  $\mathcal{H}_C^k$  to denote the associated Hausdorff  $k$ -measure and  $S_C^k$  to denote the associated spherical Hausdorff  $k$ -measure; on the other hand we will denote by  $d_E$  the usual Euclidean distance, by  $B_E(\cdot, \cdot)$  the associated open balls, by  $\mathcal{H}_E^k$  the associated Hausdorff  $k$ -measure and by  $S_E^k$  the associated spherical Hausdorff  $k$ -measure. By  $\Omega \subseteq (\mathbb{R}^n, X)$  we denote a fixed open set and by  $Q$  we denote the homogeneous dimension of  $(\mathbb{R}^n, X)$ . Later we will also use the following notation:

- for every  $1 \leq i \leq m$  and  $x \in \mathbb{R}^n$  we write

$$X_i(x) = (a_{i,1}(x), \dots, a_{i,n}(x))$$

where  $a_{i,t} \in C^\infty(\mathbb{R}^n)$  for  $1 \leq t \leq n$ ;

- for every  $1 \leq i \leq m$ ,  $\varphi \in C^1(\Omega)$  and  $x \in \Omega$  we write

$$(X_i\varphi)(x) := \sum_{t=1}^n a_{i,t}(x) \frac{\partial \varphi}{\partial x_t}(x);$$

- for every  $1 \leq i \leq m$  and  $x \in \mathbb{R}^n$  we write

$$(\operatorname{div} X_i)(x) := \sum_{t=1}^n \frac{\partial a_{i,t}}{\partial x_t}(x)$$

- for every  $1 \leq i \leq m$  we denote by  $X_i^*$  the formal adjoint of  $X_i$ , i.e., for every  $\varphi \in C^1(\Omega)$ ,  $x \in \Omega$  we write

$$(X_i^* \varphi)(x) := \sum_{t=1}^n \frac{\partial (a_{i,t} \varphi)}{\partial x_t}(x);$$

- given a Radon measure  $\mu$ , we use the notation

$$\int_A u d\mu := \frac{1}{\mu(A)} \int_A u d\mu,$$

to denote the average integral of a measurable function  $u$  on a  $\mu$ -measurable set  $A$  with  $\mu(A) > 0$ ;

- we will consider a Riemannian metric, namely a smoothly varying family of scalar products  $\langle \cdot, \cdot \rangle_p = \langle \cdot, \cdot \rangle: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  which makes the horizontal vectors  $X_1, \dots, X_m$  orthonormal at every point  $p \in \mathbb{R}^n$ .

**Definition 1.3.** We say that  $u \in L^1_{\text{loc}}(\Omega)$  is a *function of locally bounded X-variation*, and we write  $u \in \text{BV}_{X,\text{loc}}(\Omega)$ , if there exists a  $\mathbb{R}^m$ -valued Radon measure  $D_X u = (D_{X_1} u, \dots, D_{X_m} u)$  on  $\Omega$  such that, for every open set  $A \subset\subset \Omega$ , for every  $1 \leq i \leq m$  and for every  $\varphi \in C_c^1(A)$  one has

$$\int_A \varphi d(D_{X_i} u) = - \int_A u X_i^* \varphi d\mathcal{L}^n.$$

Moreover, if  $u \in L^1(\Omega)$  and  $D_X u$  has bounded total variation  $|D_X u|$ , then we say that  $u$  has *bounded X-variation* and we write  $u \in \text{BV}_X(\Omega)$ .

**Definition 1.4.** For every  $u \in \text{BV}_X(\Omega)$  we define the norm

$$\|u\|_{\text{BV}_X(\Omega)} := \|u\|_{L^1(\Omega)} + |D_X u|(\Omega).$$

The space  $\text{BV}_X(\Omega)$  equipped with the above norm is a Banach space.

**Proposition 1.5.** Let  $u \in \text{BV}_X(\Omega)$ .

- (i) Let  $\zeta \in C_c^\infty(\Omega)$ . Then  $u\zeta \in \text{BV}_X(\Omega)$  and

$$D_{X_i}(u\zeta) = \zeta D_{X_i} u + u X_i \zeta \mathcal{L}^n.$$

(ii) Let  $K \in C_c^\infty(B_E(0, r))$  be spherically symmetric. Then  $u * K \in C^\infty(\Omega)$  and for any  $y \in \Omega$  such that  $d_E(y, \partial\Omega) > r$  one has

$$X_i(u * K)(y) = (D_{X_i}u * K)(y) + R_i(u, K; y)$$

where

$$R_i(u, K; y) := \int_{\Omega} u(x) ((\operatorname{div} X_i)(x)K(x - y) - \langle X_i(y) - X_i(x), \nabla K(x - y) \rangle) dx. \quad (1.1)$$

*Proof.* Let us first prove (i). Let  $\varphi \in C_c^1(\Omega)$ . We have

$$\begin{aligned} - \int_{\Omega} \varphi \tilde{\zeta} d(D_{X_i}u) &= \int_{\Omega} u X_i^*(\varphi \tilde{\zeta}) d\mathcal{L}^n = \sum_{t=1}^n \int_{\Omega} u \frac{\partial(a_{i,t} \varphi \tilde{\zeta})}{\partial x_t} d\mathcal{L}^n = \\ &= \sum_{t=1}^n \int_{\Omega} u \tilde{\zeta} \frac{\partial(a_{i,t} \varphi)}{\partial x_j} d\mathcal{L}^n + \sum_{t=1}^n \int_{\Omega} u \varphi a_{i,t} \frac{\partial \tilde{\zeta}}{\partial x_t} d\mathcal{L}^n = \\ &= \int_{\Omega} u \tilde{\zeta} X_i^* \varphi d\mathcal{L}^n + \int_{\Omega} u \varphi X_i \tilde{\zeta} d\mathcal{L}^n. \end{aligned}$$

Rearranging the equation we get

$$\int_{\Omega} u \tilde{\zeta} X_i^* \varphi d\mathcal{L}^n = - \int_{\Omega} \varphi d(u X_i \tilde{\zeta} \mathcal{L}^n) - \int_{\Omega} \varphi d(\tilde{\zeta} D_{X_i}u)$$

which implies

$$D_{X_i}(u \tilde{\zeta}) = u X_i \tilde{\zeta} \mathcal{L}^n + \tilde{\zeta} D_{X_i}u.$$

For a proof of (ii) see [91, Lemma 2.6] and [88].  $\square$

**Definition 1.6.** For every  $u \in \operatorname{BV}_X(\Omega)$  we decompose

$$D_X u = D_X^a u + D_X^s u$$

where  $D_X^a u$  denotes the *absolutely continuous part* of  $D_X u$  (with respect to the usual Lebesgue measure  $\mathcal{L}^n$ ) and  $D_X^s u$  denotes the *singular part* of  $D_X u$ .

**Definition 1.7.** We say that a measurable set  $E \subseteq \mathbb{R}^n$  has *locally finite X-perimeter* (respectively, *finite X-perimeter*) in  $\Omega$  if its characteristic function  $\mathbb{1}_E$  belongs to  $\operatorname{BV}_{X,\operatorname{loc}}(\Omega)$  (respectively,  $\mathbb{1}_E \in \operatorname{BV}_X(\Omega)$ ). In such a case we define the *X-perimeter measure*  $P_E^X$  of  $E$  as  $P_E^X := |D_X \mathbb{1}_E|$ .

**Definition 1.8.** Let  $\Omega \subseteq (\mathbb{R}^n, X)$  be an open set and  $f: \Omega \rightarrow \mathbb{R}$ . We say that  $f \in C_X^1(\Omega)$  if  $f$  is continuous and its *horizontal gradient*  $Xf := (X_1 f, \dots, X_m f)$ , in the sense of distributions, is represented by a continuous function.

**Definition 1.9.** Let  $u \in L^1_{\text{loc}}(\Omega)$ ,  $z \in \mathbb{R}$  and  $p \in \Omega$ . We say that  $z$  is the *approximate limit* of  $u$  at  $p$  if

$$\lim_{r \rightarrow 0} \int_{B_C(p,r)} |u - z| d\mathcal{L}^n = 0.$$

If the approximate limit of  $u$  at  $p$  exists, it is also unique (see [66, Definition 2.19]). We hence denote by  $u^*(p)$  the approximate limit of  $u$  at  $p$  and by  $\mathcal{S}_u$  the subset of points in  $\Omega$  where  $u$  does not admit an approximate limit.

**Definition 1.10.** Let  $u \in L^1_{\text{loc}}(\Omega)$  and  $p \in \Omega \setminus \mathcal{S}_u$ . We say that  $u$  is *approximately X-differentiable* at  $p$  if there exist a neighbourhood  $U \subset \Omega$  of  $p$  and  $f \in C^1_X(U)$  such that  $f(p) = 0$  and

$$\lim_{r \rightarrow 0} \int_{B_C(p,r)} \frac{|u - u^*(p) - f|}{r} d\mathcal{L}^n = 0.$$

The set of points in  $\Omega$  where  $u$  is approximately  $X$ -differentiable is denoted by  $\mathcal{D}_u$ . The vector  $Xf(p) \in \mathbb{R}^m$  is uniquely determined (see [66, Proposition 2.30]): we call it *approximate X-gradient* of  $u$  at  $p$  and we denote it by  $D_X^{\text{ap}}u(p)$ . Similarly, we write  $D_{X_i}^{\text{ap}}u(p) := X_i f(p)$  for every  $i \in \{1, \dots, m\}$ .

The next two results collect some of the “fine” properties of  $BV_X$  functions proved in [66].

**Theorem 1.11** ([66, Theorem 1.1]). *Let  $u \in BV_X(\Omega)$ . Then  $u$  is approximately  $X$ -differentiable at  $\mathcal{L}^n$ -almost every point of  $\Omega$ . Moreover, the approximate  $X$ -gradient coincides  $\mathcal{L}^n$ -almost everywhere with the density of  $D_X^{\text{a}}u$  with respect to  $\mathcal{L}^n$ .*

**Theorem 1.12** ([66, Theorem 1.3]). *There exists  $\lambda: \mathbb{R}^n \rightarrow (0, +\infty)$  locally bounded away from 0 such that, for every open set  $\Omega \subset \mathbb{R}^n$  and every  $u \in BV_X(\Omega)$*

$$|D_X u| \geq \lambda |u^+ - u^-| \mathcal{H}_C^{\mathbb{Q}-1} \llcorner \mathcal{J}_u.$$

Moreover, for every Borel set  $B \subseteq \Omega$  the following implications hold:

$$\mathcal{H}_C^{\mathbb{Q}-1}(B) = 0 \quad \Rightarrow \quad |D_X u|(B) = 0; \tag{1.2}$$

$$\mathcal{H}_C^{\mathbb{Q}-1}(B) < +\infty \text{ and } B \cap \mathcal{S}_u = \emptyset \quad \Rightarrow \quad |D_X u|(B) = 0. \tag{1.3}$$

We now devote a few words to intrinsically  $C^1$  (or  $C^1_X$ ) hypersurfaces and the notion of  $X$ -rectifiability.

**Definition 1.13.** We say that  $S \subseteq (\mathbb{R}^n, X)$  is a  $C^1_X$ -hypersurface if for every  $p \in S$  there exist  $r > 0$  and  $f \in C^1_X(B_C(p, r))$  such that the following facts hold:

- (i)  $S \cap B_C(p, r) = \{q \in B_C(p, r) : f(q) = 0\}$ ,
- (ii)  $Xf \neq 0$  on  $B_C(p, r)$ .

We define the *horizontal normal* to  $S$  at  $p \in S$  as

$$\nu_S(p) := \frac{Xf(p)}{|Xf(p)|}.$$

Notice that  $\nu_S(p)$  is well defined up to a sign and, in particular, it does not depend on the choice of  $f$ , see [66, Corollary 2.14].

**Definition 1.14.** Let  $S \subseteq (\mathbb{R}^n, X)$ . We say that  $S$  is *countably  $X$ -rectifiable* if there exists a family  $\{S_h : h \in \mathbb{N}\}$  of  $C_X^1$ -hypersurfaces such that

$$\mathcal{H}_C^{Q-1} \left( S \setminus \bigcup_{h \in \mathbb{N}} S_h \right) = 0.$$

Moreover, if  $\mathcal{H}_C^{Q-1}(S) < +\infty$ , we say that  $S$  is  *$X$ -rectifiable*. We define the *horizontal normal* of a countably  $X$ -rectifiable set  $S$  at  $p \in S$  as

$$\nu_S(p) := \nu_{S_h}(p) \text{ if } p \in S_h \setminus \bigcup_{k < h} S_k.$$

Notice that  $\nu_S$  is well defined, up to a sign,  $\mathcal{H}_C^{Q-1}$ -a.e., see [66, Proposition 2.18].

The following Lemma provides an equivalent definition of  $X$ -rectifiability; although not difficult and probably well known (see [101, Lemma 2.4] for a proof in Heisenberg groups), we include a proof for the sake of completeness.

**Lemma 1.15.** *A set  $R \subseteq (\mathbb{R}^n, X)$  is  $X$ -rectifiable if and only if, for every  $\varepsilon > 0$ , there exists a  $C_X^1$ -hypersurface  $S_\varepsilon \subseteq (\mathbb{R}^n, X)$  such that  $\mathcal{H}_C^{Q-1}(S_\varepsilon) < \infty$  and*

$$\mathcal{H}_C^{Q-1}(R \setminus S_\varepsilon) < \varepsilon.$$

*Proof.* Since  $R$  is  $X$ -rectifiable we can write

$$R \subseteq S_0 \cup \bigcup_{i \in \mathbb{N}} S_i$$

where  $S_0$  is a  $\mathcal{H}_C^{Q-1}$ -negligible set and, for every  $i \in \mathbb{N}$ ,  $S_i$  is a  $C_X^1$ -hypersurface. It is not restrictive to assume  $\mathcal{H}_C^{Q-1}(S_i) < \infty$  for every  $i \in \mathbb{N}$ . For every  $\varepsilon > 0$  there exists a positive integer  $M$  such that

$$\mathcal{H}_C^{Q-1} \left( R \setminus \bigcup_{i \leq M} S_i \right) < \frac{\varepsilon}{2}.$$

We define the  $C_X^1$ -hypersurface  $S'_1 := \{p \in S_1 : d_C(p, \partial S_1) > r_1\}$ , where  $\partial S'_1 := \overline{S'_1} \setminus S'_1$  and  $r_1$  is chosen so that

$$\mathcal{H}_C^{Q-1}(R \cap \partial S'_1) = 0 \quad \text{and} \quad \mathcal{H}_C^{Q-1}((R \cap S_1) \setminus S'_1) < \frac{\varepsilon}{4}.$$

Let us prove that such  $r_1$  exists: for  $r > 0$  we define the set

$$S'_1(r) := \{p \in S_1 : d_C(p, \partial S_1) > r\}.$$

Since  $\{R \cap \partial S'_1(r) : r > 0\}$  is a family of uncountably many pairwise disjoint subsets of  $R$  and  $\mathcal{H}_C^{Q-1}(R) < \infty$ , then  $\mathcal{H}_C^{Q-1}(R \cap \partial S'_1(r)) = 0$  for arbitrarily small  $r > 0$ . Moreover, since  $R \cap S_1$  is the union of the nested sets  $(R \cap S_1) \setminus S'_1(r)$ , we have, by regularity of the Hausdorff measure,

$$\mathcal{H}_C^{Q-1}((R \cap S_1) \setminus S'_1(r)) \xrightarrow{r \rightarrow 0} 0.$$

Reasoning by induction, for every  $i = 2, \dots, M$  we can define the  $C_X^1$ -hypersurfaces

$$S'_i = \left\{ p \in S_i \setminus \bigcup_{j < i} \overline{S'_j} : d_C \left( p, \partial \left( S_i \setminus \bigcup_{j < i} \overline{S'_j} \right) \right) > r_j \right\},$$

where we used the fact that  $S_i \setminus \bigcup_{j < i} \overline{S'_j}$  is a  $C_X^1$ -hypersurface and  $r_i > 0$  is chosen so that

$$\mathcal{H}_C^{Q-1}(R \cap \partial S'_i) = 0 \quad \text{and} \quad \mathcal{H}_C^{Q-1} \left( R \cap \left( S_i \setminus \bigcup_{j < i} \overline{S'_j} \right) \setminus S'_i \right) < \frac{\varepsilon}{2^{i+2}}.$$

Now consider  $S_\varepsilon := \bigcup_{i=1}^M S'_i$ , which is a  $C_X^1$ -hypersurface because it is union of finitely many  $C_X^1$ -hypersurface at positive distance from each other. Then

$$\begin{aligned} \mathcal{H}_C^{Q-1}(R \setminus S_\varepsilon) &\leq \mathcal{H}_C^{Q-1} \left( R \setminus \bigcup_{i \leq M} S_i \right) + \mathcal{H}_C^{Q-1} \left( R \cap \left( \bigcup_{i \leq M} S_i \right) \setminus \left( \bigcup_{j \leq M} S'_j \right) \right) \\ &< \frac{\varepsilon}{2} + \mathcal{H}_C^{Q-1} \left( \bigcup_{i \leq M} \left( (R \cap S_i) \setminus \bigcup_{j \leq M} S'_j \right) \right) \\ &\leq \frac{\varepsilon}{2} + \mathcal{H}_C^{Q-1} \left( \bigcup_{i \leq M} \left( R \cap \left( S_i \setminus \bigcup_{j \leq i} S'_j \right) \right) \right) \\ &= \frac{\varepsilon}{2} + \mathcal{H}_C^{Q-1} \left( \bigcup_{i \leq M} \left( R \cap \left( S_i \setminus \bigcup_{j < i} \overline{S'_j} \right) \setminus S'_i \right) \right) \\ &< \varepsilon, \end{aligned}$$

where we used that  $\mathcal{H}_C^{Q-1}(R \cap \partial S'_j) = 0$ . Finally we observe that

$$\mathcal{H}_C^{Q-1}(S_\varepsilon) = \sum_{i \leq M} \mathcal{H}_C^{Q-1}(S'_i) \leq \sum_{i \leq M} \mathcal{H}_C^{Q-1}(S_i) < \infty.$$

This proves one implication, the other one is straightforward.  $\square$

**Definition 1.16.** Fix  $p \in (\mathbb{R}^n, X)$ ,  $R > 0$  and  $\nu \in \mathbb{S}^{m-1}$ . Let  $f \in C_X^1(B_C(p, R))$  be such that  $f(p) = 0$  and  $\frac{Xf(p)}{|Xf(p)|} = \nu$ . For every  $r \in (0, R)$  we set

$$\begin{aligned} B_\nu^+(p, r) &:= B_C(p, r) \cap \{f > 0\}, \\ B_\nu^-(p, r) &:= B_C(p, r) \cap \{f < 0\}. \end{aligned}$$

By using the *half-balls* from Definition 1.16, one can define intrinsic approximate jumps. Notice that condition (1.4) below does not depend on the choice of the function  $f$  used to construct the sets  $B_\nu^+(p, r)$  and  $B_\nu^-(p, r)$ , see [66, Proposition 2.26 and Remark 2.27].

**Definition 1.17.** Let  $u \in L_{loc}^1(\Omega)$  and  $p \in \Omega$ . We say that  $u$  has an *approximate X-jump* at  $p$  if there exist  $u^+, u^- \in \mathbb{R}$  with  $u^+ \neq u^-$  and  $\nu \in \mathbb{S}^{m-1}$  such that

$$\lim_{r \rightarrow 0} \int_{B_\nu^+(p, r)} |u - u^+| d\mathcal{L}^n = \lim_{r \rightarrow 0} \int_{B_\nu^-(p, r)} |u - u^-| d\mathcal{L}^n = 0. \quad (1.4)$$

The *jump set*  $\mathcal{J}_u$  is defined as the set of points where  $u$  has an approximate X-jump.

It is worth remarking that, by [66, Theorem 1.2 and Remark 2.25], the Hausdorff measure  $\mathcal{H}_C^{Q-1}$  on the jump set is  $\sigma$ -finite.

**Remark 1.18.** It was proved in [66, Theorem 1.5] that, if  $(\mathbb{R}^n, X)$  satisfies the additional property  $\mathcal{R}$  (see Definition 1.19 below), then the jump set  $\mathcal{J}_u$  is countably X-rectifiable. It is worth recalling that Heisenberg groups<sup>1</sup>, Carnot groups of step 2 and Carnot groups of type  $\star$  all satisfy property  $\mathcal{R}$ , see [66, Theorem 4.3].

**Definition 1.19.** We say that  $(\mathbb{R}^n, X)$  satisfies the *property  $\mathcal{R}$*  if for every open set  $\Omega \subseteq \mathbb{R}^n$  and every  $E \subseteq \mathbb{R}^n$  with locally finite X-perimeter in  $\Omega$ , the essential boundary  $\partial^* E \cap \Omega$  is countably X-rectifiable. Let us recall for completeness that the *essential boundary* of a measurable set  $E$  is  $\partial^* E := \mathbb{R}^n \setminus (E_0 \cup E_1)$ , where for  $\lambda \in [0, 1]$  we denote by  $E_\lambda$  the set of points  $p \in \mathbb{R}^n$  where  $E$  has density  $\lambda$ , i.e.,

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}^n(E \cap B(p, r))}{\mathcal{L}^n(B(p, r))} = \lambda.$$

We are ready to introduce the jump part of the derivative  $D_X u$ .

**Definition 1.20.** Let  $u \in BV_X(\Omega)$ . We define the *jump part* of  $D_X u$  as

$$D_X^j u := D_X^s u \llcorner \mathcal{J}_u$$

and the *Cantor part* of  $D_X u$  as

$$D_X^c u := D_X^s u \llcorner (\Omega \setminus \mathcal{J}_u).$$

<sup>1</sup>For the definitions of Heisenberg and Carnot groups see Chapter 2.

**Remark 1.21.** If  $(\mathbb{R}^n, X)$  satisfies both properties  $\mathcal{R}$  and  $\mathcal{D}$  (see Definition 1.22 below), then the jump part has the representation

$$D_X^j u = \sigma(\cdot, \nu_{\mathcal{J}_u})(u^+ - u^-) \nu_{\mathcal{J}_u} \mathcal{S}_C^{\mathbb{Q}-1} \llcorner \mathcal{J}_u.$$

for a suitable function  $\sigma: \mathbb{R}^n \times \mathbb{S}^{m-1} \rightarrow (0, +\infty)$ ; see [66, Theorem 1.7]. Again, Heisenberg groups, Carnot groups of step 2 and Carnot groups of type  $\star$  all satisfy both properties, see [66, Theorem 4.3].

**Definition 1.22.** We say that  $(\mathbb{R}^n, X)$  satisfies the *property  $\mathcal{D}$*  if there exists a function  $\sigma: \mathbb{R}^n \times \mathbb{S}^{m-1} \rightarrow (0, +\infty)$  such that, for every  $C_X^1$ -hypersurface  $S \subseteq \mathbb{R}^n$  and every  $p \in S$ , one has

$$\lim_{r \rightarrow 0} \frac{\mathcal{S}_C^{\mathbb{Q}-1}(S \cap B(p, r))}{r^{\mathbb{Q}-1}} = \sigma(p, \nu_S(p)).$$

We mention that the validity of property  $\mathcal{D}$  is related to the broader problem of computing the Federer density for the perimeter measure of surfaces; see, e.g. [85, 117, 122, 123]. If property  $\mathcal{D}$  holds, then the function  $\sigma$  of Definition 1.22, is actually explicit in many cases, see e.g. [63, 125].

## 2 A special case of Carnot-Carathéodory spaces: Heisenberg groups

Among the Carnot-Carathéodory spaces, probably the ones most studied are Heisenberg groups. We can define Heisenberg groups, using the same notation as in Definition 1.1, as  $\mathbb{H}^n := (\mathbb{R}^{2n+1}, X)$  where we denote each  $p \in \mathbb{R}^{2n+1}$  as  $p = (x, y, t)$  with  $x, y \in \mathbb{R}^n, t \in \mathbb{R}$  and  $X = (X_1, \dots, X_n, Y_1, \dots, Y_n)$  is a  $2n$ -tuple composed, for  $1 \leq j \leq n$ , of the vector fields

$$X_j := \partial_{x_j} - \frac{y_j}{2} \partial_t, \quad Y_j := \partial_{y_j} + \frac{x_j}{2} \partial_t.$$

Heisenberg groups are equiregular Carnot-Carathéodory spaces which, as stated before, satisfy both properties  $\mathcal{R}$  and  $\mathcal{D}$ . In particular all results present in Chapter 1 (and the ones that we are going to state in Chapter 4) hold in Heisenberg groups. In this chapter we will collect only the known results that we are going to use in Chapter 5 and Chapter 6. For a comprehensive introduction to Heisenberg groups we refer the reader to [33].

### 2.1 Heisenberg groups and Heisenberg algebras

**Definition 2.1.** For  $n \geq 1$  we denote by  $\mathbb{H}^n$  the  $n$ -th Heisenberg group, identified with  $\mathbb{R}^{2n+1}$  through exponential coordinates. We denote a point  $\mathbb{H}^n \ni p = (x, y, t)$  by  $x, y \in \mathbb{R}^n$  and  $t \in \mathbb{R}$ . If  $p = (x, y, t), q = (x', y', t') \in \mathbb{H}^n$ , the group operation is defined as

$$p \cdot q := (x + x', y + y', t + t' + \frac{1}{2} \langle x, y' \rangle_{\mathbb{R}^n} - \frac{1}{2} \langle x', y \rangle_{\mathbb{R}^n}).$$

If  $\mathbb{H}^n \ni p = (x, y, t)$ , its inverse is  $p^{-1} = (-x, -y, -t)$  and  $0 = (0, 0, 0) \in \mathbb{H}^n$  is the identity of  $\mathbb{H}^n$ .

Let  $p, q \in \mathbb{H}^n$ , we denote by  $\tau_p : \mathbb{H}^n \rightarrow \mathbb{H}^n$  the left translation, i.e.,  $\tau_p(q) := p \cdot q$ . For  $\lambda > 0$ , we denote by  $\delta_\lambda : \mathbb{H}^n \rightarrow \mathbb{H}^n$  the dilations of the Heisenberg groups defined, for  $p = (x, y, t) \in \mathbb{H}^n$ , by  $\delta_\lambda(x, y, t) := (\lambda x, \lambda y, \lambda^2 t)$ . Observe that dilations form a one-parameter family of group isomorphisms.

We denote by  $Q := 2n + 2$  the homogeneous dimension of  $\mathbb{H}^n$ . The Lebesgue measure  $\mathcal{L}^{2n+1}$  is the Haar measure on  $\mathbb{H}^n \cong \mathbb{R}^{2n+1}$  and it is  $Q$ -homogeneous with respect to dilations.

**Definition 2.2.** We denote by  $\mathfrak{h}^n$  (or by  $\mathfrak{h}$  when the dimension  $n$  is clear) the  $(2n + 1)$ -dimensional Lie algebra of left invariant vector fields in  $\mathbb{H}^n$ . The algebra  $\mathfrak{h}$  is generated by the vector fields  $X_1, \dots, X_n, Y_1, \dots, Y_n, T$  where (for  $1 \leq j \leq n$ )

$$X_j := \partial_{x_j} - \frac{y_j}{2} \partial_t, \quad Y_j := \partial_{y_j} + \frac{x_j}{2} \partial_t, \quad T := \partial_t.$$

We denote by  $\mathfrak{h}_1$  the horizontal subspace of  $\mathfrak{h}$ , i.e.,

$$\mathfrak{h}_1 := \text{span}(X_1, \dots, X_n, Y_1, \dots, Y_n),$$

and by  $\mathfrak{h}_2$  the linear span of  $T$ ; the Lie algebra  $\mathfrak{h}$  admits the 2-step stratification  $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ . Note also that, since  $\mathbb{H}^n$  is simply connected and nilpotent, the exponential map  $\exp: \mathfrak{h} \rightarrow \mathbb{H}^n$  is a global diffeomorphism.

If  $p \in \mathbb{H}^n$  we denote by  $H_p \mathbb{H}^n := \text{span}(X_1(p), \dots, X_n(p), Y_1(p), \dots, Y_n(p))$  and by  $H\mathbb{H}^n := \bigcup_{p \in \mathbb{H}^n} H_p \mathbb{H}^n$  the horizontal fiber bundle. We denote by  $\langle \cdot, \cdot \rangle$  the inner product on  $\mathfrak{h}$  that makes the basis  $X_1, \dots, X_n, Y_1, \dots, Y_n, T$  orthonormal and if  $p \in \mathbb{H}^n$  we denote by  $\langle \cdot, \cdot \rangle_p$  the corresponding inner product on  $H_p \mathbb{H}^n$  that makes the basis  $X_1(p), \dots, X_n(p), Y_1(p), \dots, Y_n(p)$  orthonormal. We will use  $|\cdot|$  and  $|\cdot|_p$  to denote the corresponding norms.

**Definition 2.3.** We define the  $k$ -th exterior algebra of  $k$ -vectors and  $k$ -covectors as, respectively,

$$\bigwedge_k \mathfrak{h} := \text{span}\{W_{i_1} \wedge \dots \wedge W_{i_k} \text{ with } 1 \leq i_1 \leq \dots \leq i_k \leq 2n + 1\}$$

$$\bigwedge^k \mathfrak{h} := \text{span}\{\theta_{i_1} \wedge \dots \wedge \theta_{i_k} \text{ with } 1 \leq i_1 \leq \dots \leq i_k \leq 2n + 1\}$$

where we used the notation  $W_i := X_i$  if  $1 \leq i \leq n$ ,  $W_i := Y_{i-n}$  if  $n + 1 \leq i \leq 2n$  and  $W_{2n+1} := T$ . Similarly we define  $\theta_i := dx_i$  if  $1 \leq i \leq n$ ,  $\theta_i := dy_{i-n}$  if  $n + 1 \leq i \leq 2n$  and  $\theta_{2n+1} := \theta := dt + \frac{1}{2} \sum_{j=1}^n (y_j dx_j - x_j dy_j)$ ; observe that  $\{\theta_1, \dots, \theta_{2n}, \theta\}$  is the dual basis to  $\{X_1, \dots, Y_n, T\}$ . In the same fashion we define the  $k$ -th exterior algebra of horizontal  $k$ -vectors and horizontal  $k$ -covectors as

$$\bigwedge_k \mathfrak{h}_1 := \text{span}\{W_{i_1} \wedge \dots \wedge W_{i_k} \text{ with } 1 \leq i_1 \leq \dots \leq i_k \leq 2n\}$$

$$\bigwedge^k \mathfrak{h}_1 := \text{span}\{\theta_{i_1} \wedge \dots \wedge \theta_{i_k} \text{ with } 1 \leq i_1 \leq \dots \leq i_k \leq 2n\}.$$

The (canonical) action of a  $k$ -covector  $\omega$  on a  $k$ -vector  $v$  is denoted by  $\langle v | \omega \rangle$ . The fiber of  $\bigwedge_k \mathfrak{h}$  over  $p \in \mathbb{H}^n$  is denoted by  $\bigwedge_{k,p} \mathfrak{h}$  and analogously for  $\bigwedge_{k,p} \mathfrak{h}_1$ . The inner product  $\langle \cdot, \cdot \rangle$  defined on  $\mathfrak{h}$  extends canonically to  $\bigwedge_k \mathfrak{h}$  and  $\bigwedge^k \mathfrak{h}$  (and then, by restriction, to  $\bigwedge_k \mathfrak{h}_1$  and  $\bigwedge^k \mathfrak{h}_1$ ). In the following we drop the subscripts in  $\langle \cdot, \cdot \rangle_{\bigwedge_k \mathfrak{h}}$  and  $|\cdot|_{\bigwedge_k \mathfrak{h}}$ .

**Definition 2.4.** We say that a function  $d : \mathbb{H}^n \times \mathbb{H}^n \rightarrow [0, +\infty)$  is a *left invariant and homogeneous distance* if

- (i)  $d(p, q) = d(r \cdot p, r \cdot q)$  for all  $p, q, r \in \mathbb{H}^n$ ,
- (ii)  $d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p, q)$  for all  $p, q \in \mathbb{H}^n$  and  $\lambda > 0$ .

We define the associated norm  $\|\cdot\|$  to  $d$  as  $\|p\| := d(0, p)$  for every  $p \in \mathbb{H}^n$ . Moreover, if for every  $(x, y, t), (x', y', t) \in \mathbb{H}^n$  such that  $|(x, y)|_{\mathbb{R}^{2n}} = |(x', y')|_{\mathbb{R}^{2n}}$  we have  $\|(x, y, t)\| = \|(x', y', t)\|$  we say that  $d$  is a *rotationally invariant* distance.

**Example 2.5.** There are many examples of left invariant, homogeneous and rotationally invariant distances on  $\mathbb{H}^n$ , the most notable being the following.

- (i) The *Carnot-Carathéodory* distance  $d_C$  defined as in Chapter 1 with the choice of horizontal vector fields given by  $X = (X_1, \dots, X_n, Y_1, \dots, Y_n)$ .
- (ii) The *infinity* distance  $d_\infty$  defined for  $(x, y, t) \in \mathbb{H}^n$  as

$$d_\infty(0, (x, y, t)) := \max\{|(x, y)|_{\mathbb{R}^{2n}}, 2|t|_{\mathbb{R}}^{\frac{1}{2}}\}.$$

- (iii) The *Korányi (or Cygan-Korányi)* distance  $d_K$  defined for  $(x, y, t) \in \mathbb{H}^n$  as

$$d_K(0, (x, y, t)) := \left( (|x|_{\mathbb{R}^n}^2 + |y|_{\mathbb{R}^n}^2) + 16t^2 \right)^{\frac{1}{4}}.$$

In view of the following proposition, for the rest of this thesis, we will denote by  $d$  a fixed left invariant, homogeneous and rotationally invariant distance on  $\mathbb{H}^n$ , by  $\|\cdot\|$  its associated norm and for any  $p \in \mathbb{H}^n, r > 0$  we will denote by  $B_d(p, r)$  the corresponding open balls.

**Proposition 2.6** ([54, Proposition 1.3.15]). *Let  $d_1$  and  $d_2$  be left invariant and homogeneous distances on  $\mathbb{H}^n$ . Then they are bi-Lipschitz equivalent, i.e., there exists  $C > 0$  such that for all  $p, q \in \mathbb{H}^n$*

$$\frac{1}{C}d_2(p, q) \leq d_1(p, q) \leq Cd_2(p, q).$$

*In particular every left invariant and homogeneous distance induces the Euclidean topology on  $\mathbb{H}^n$ .*

**Definition 2.7.** Let  $m \geq 0$ . Given any left invariant, homogeneous and rotationally invariant distance on  $\mathbb{H}^n$  we denote by  $\mathcal{S}^m$  the spherical Hausdorff measure on  $\mathbb{H}^n$  defined for  $E \subset \mathbb{H}^n$  by

$$\mathcal{S}_d^m(E) := \liminf_{r \rightarrow 0^+} \left\{ \sum_{i \in \mathbb{N}} (2r_i)^m : \exists (p_i)_i \subset \mathbb{H}^n, \exists (r_i)_i : 0 < r_i < r, E \subset \bigcup_{i \in \mathbb{N}} B_d(p_i, r_i) \right\}.$$

It is well known that  $\mathcal{S}_d^Q$  coincides with  $\mathcal{L}^{2n+1}$  up to a positive multiplicative constant; in particular,  $Q$  is the Hausdorff dimension of  $\mathbb{H}^n$ .

**Definition 2.8.** Let  $p \in \mathbb{H}^n$ ,  $q = (x_1, \dots, x_n, y_1, \dots, y_n, t) \in \mathbb{H}^n$  be given. We set

$$\pi_p(q) := \sum_{j=1}^n x_j X_j(p) + \sum_{j=1}^n y_j Y_j(p).$$

For fixed  $p$ , the map  $\pi_p$  is a projection, while for fixed  $q$ , the map  $p \rightarrow \pi_p(q)$  is a smooth section of the horizontal bundle  $H\mathbb{H}^n$  (it is in fact a left-invariant vector field).

**Proposition 2.9.** For any  $a, b, p \in \mathbb{H}^n$  the following relations hold:

- (1)  $\pi_p(a \cdot b) = \pi_p(a) + \pi_p(b)$ ;
- (2)  $d(a, b) \geq C |\pi_p(a) - \pi_p(b)|_p$ .

where  $C$  is a positive constant only depending on the distance  $d$ .

We omit the boring proof.

## 2.2 Intrinsic graphs in Heisenberg groups

**Definition 2.10.** We say that a homogeneous subgroup  $\mathbb{V} \subset \mathbb{H}^n$  is *horizontal* if it is contained in the *horizontal fiber*, i.e.,  $\mathbb{V} \subseteq \exp(\mathfrak{h}_1)$ . Horizontal subgroups can be identified with  $(\mathbb{R}^k, |\cdot|)$ : more precisely, if  $V_1, \dots, V_k \in \mathfrak{h}_1$  are such that  $\mathbb{V} = \exp(\text{span}(V_1, \dots, V_k))$ , then the map  $\mathbb{R}^k \ni x \mapsto \exp(x_1 V_1 + \dots + x_k V_k)$  is a biLipschitz diffeomorphism between  $(\mathbb{R}^k, |\cdot|)$  and  $(\mathbb{V}, d)$ .<sup>1</sup> On the other hand, we will say that a subgroup  $\mathbb{W}$  is *vertical* if it contains the center of the group, i.e.,  $\exp(\mathfrak{h}_2) \subseteq \mathbb{W}$ .<sup>2</sup>

**Remark 2.11.** All homogeneous subgroups of the Heisenberg group are either horizontal (and in this case they are abelian) or vertical (and in this case they are normal).

**Definition 2.12.** Let  $\mathbb{W}, \mathbb{V}$  be homogeneous subgroups of  $\mathbb{H}^n$ . We say that  $\mathbb{W}, \mathbb{V}$  are *complementary subgroups* in  $\mathbb{H}^n$  if  $\mathbb{W} \cap \mathbb{V} = \{0\}$  and  $\mathbb{W} \cdot \mathbb{V} = \mathbb{H}^n$ .

**Remark 2.13.** All possible couples of complementary subgroups in  $\mathbb{H}^n$  are formed by an (abelian) horizontal subgroup  $\mathbb{V}$  of dimension  $k$ , for  $1 \leq k \leq n$ , and by a (normal) vertical subgroup  $\mathbb{W}$  of topological dimension  $2n + 1 - k$ .

**Remark 2.14.** If  $\mathbb{W}$  and  $\mathbb{V}$  are complementary subgroups in  $\mathbb{H}^n$ , then each element  $p \in \mathbb{H}^n$  can be written in a unique way as  $p = w \cdot v$ , for  $w \in \mathbb{W}, v \in \mathbb{V}$ . The elements  $w, v$  are called the *projections* of  $p$  with respect to the decomposition  $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$  and we will use the notation  $w = p_{\mathbb{W}}, v = p_{\mathbb{V}}$ . Let us stress that the projections of a point  $p \in \mathbb{H}^n$  depend on both the complementary subgroups and also on the order in which they are taken.

<sup>1</sup>In particular the Hausdorff dimension of  $\mathbb{V}$  equals the topological dimension  $k$ .

<sup>2</sup>In this case the Hausdorff dimension of  $\mathbb{W}$  is greater than the topological one: for instance the metric dimension of  $\mathbb{H}^n$  coincides with the homogeneous dimension  $Q = 2n + 2$ .

Some properties of the projections are described in [84, Subsection 2.2], we collect below few of them.

**Proposition 2.15.** *Let  $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$  where  $\mathbb{W}$  and  $\mathbb{V}$  are complementary subgroups of  $\mathbb{H}^n$ . Let us denote by  $\pi_{\mathbb{W}}$  and  $\pi_{\mathbb{V}}$  the projection maps onto  $\mathbb{W}$  and  $\mathbb{V}$  respectively, namely*

$$\begin{aligned} \pi_{\mathbb{W}} : \mathbb{H}^n &\rightarrow \mathbb{W} & \pi_{\mathbb{V}} : \mathbb{H}^n &\rightarrow \mathbb{V} \\ \pi_{\mathbb{W}}(p) &:= p_{\mathbb{W}} & \pi_{\mathbb{V}}(p) &:= p_{\mathbb{V}}. \end{aligned}$$

Then:

1. *if  $\mathbb{W}$  is a normal subgroup, then  $\pi_{\mathbb{V}}$  is a Lipschitz homomorphism of groups; Similarly, if  $\mathbb{V}$  is normal, then  $\pi_{\mathbb{W}}$  is a Lipschitz homomorphism of groups.*
2. *If  $\mathbb{W}$  is a normal subgroup, then the following identities hold:*

$$\begin{aligned} \pi_{\mathbb{W}}(p \cdot q) &= \pi_{\mathbb{W}}(p) \cdot \pi_{\mathbb{V}}(p) \cdot \pi_{\mathbb{W}}(q) \cdot \pi_{\mathbb{V}}(p)^{-1} & \pi_{\mathbb{V}}(p \cdot q) &= \pi_{\mathbb{V}}(p) \cdot \pi_{\mathbb{V}}(q) \\ \pi_{\mathbb{W}}(p^{-1}) &= \pi_{\mathbb{V}}(p)^{-1} \cdot \pi_{\mathbb{W}}(p)^{-1} \cdot \pi_{\mathbb{V}}(p) & \pi_{\mathbb{V}}(p^{-1}) &= \pi_{\mathbb{V}}(p)^{-1}. \end{aligned}$$

3. *There exists a constant  $\tilde{C} = \tilde{C}(\mathbb{W}, \mathbb{V}) > 0$  such that*

$$\tilde{C}(\|\pi_{\mathbb{W}}(p)\| + \|\pi_{\mathbb{V}}(p)\|) \leq \|p\| \leq \|\pi_{\mathbb{W}}(p)\| + \|\pi_{\mathbb{V}}(p)\| \quad \text{for all } p \in \mathbb{H}^n.$$

**Definition 2.16.** Let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  such that  $\mathbb{V} = \exp(\mathfrak{v})$  and  $\mathbb{W} = \exp(\mathfrak{w})$  for some subalgebras  $\mathfrak{v}, \mathfrak{w}$  of  $\mathfrak{h}$  such that  $\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{w}$ . Given a function  $\phi : E \subset \mathbb{W} \rightarrow \mathbb{V}$ , we define its *intrinsic graph* as the set

$$\text{gr}_{\phi} := \{w \cdot \phi(w) : w \in E\}.$$

With a slight abuse of notation, we also say that a set  $S \subset \mathbb{H}^n$  is an (continuous, respectively smooth) *intrinsic graph* over  $\mathbb{W}$  along  $\mathbb{V}$  if there is a (continuous, resp. smooth) function  $\phi : E \subset \mathbb{W} \rightarrow \mathbb{V}$  such that  $S = \text{gr}_{\phi}$ . We define the *graph map*  $\Phi : E \rightarrow \mathbb{H}^n$  associated to  $\phi$  as  $\Phi(w) := w \cdot \phi(w)$  for every  $\zeta \in E$ . If, moreover, the subalgebras  $\mathfrak{w}$  and  $\mathfrak{v}$  are orthogonal, we say that  $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$  is an *orthogonal splitting* of  $\mathbb{H}^n$  and  $S$  is an *orthogonal graph*.

**Definition 2.17.** Let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$ . If  $\beta \geq 0$ , we define the *intrinsic cone*  $C_{\mathbb{W}, \mathbb{V}}(0, \beta)$  of vertex 0, base  $\mathbb{W}$ , axis  $\mathbb{V}$  and opening  $\beta$  as

$$C_{\mathbb{W}, \mathbb{V}}(0, \beta) := \{p \in \mathbb{H}^n : \|\pi_{\mathbb{W}}(p)\| \leq \beta \|\pi_{\mathbb{V}}(p)\|\}.$$

Moreover, for every  $p \in \mathbb{H}^n \setminus \{0\}$  we define the intrinsic cone of vertex  $p$ , base  $\mathbb{W}$ , axis  $\mathbb{V}$  and opening  $\beta$  as

$$C_{\mathbb{W}, \mathbb{V}}(p, \beta) := p \cdot C_{\mathbb{W}, \mathbb{V}}(0, \beta). \tag{2.1}$$

For the sake of brevity, when there is no risk of confusion, in the following we will denote such cone as  $C_{\beta}(p)$ .

**Remark 2.18.** Let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$ . For every  $p \in \mathbb{H}^n$ ,  $0 < \alpha < \beta$  we have  $C_\alpha(p) \subset C_\beta(p)$ . Moreover  $\delta_\lambda(C_\beta(0)) = C_\beta(0)$  for every  $\lambda > 0$ . Finally  $C_0(0) = \mathbb{V}$ , while  $\bigcup_{\beta > 0} C_\beta(0) = \mathbb{H}^n$ .

**Definition 2.19.** Let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  and let  $\phi : A \subseteq \mathbb{W} \rightarrow \mathbb{V}$ . We say that  $\phi$  is an *intrinsic Lipschitz map* if there exists  $M > 0$  such that, for all  $p \in \text{gr}_\phi$ ,

$$C_{\frac{1}{M}}(p) \cap \text{gr}_\phi = \{p\}.$$

If this is the case, we will say that  $\phi$  is *intrinsic  $M$ -Lipschitz*. Moreover, for  $E \subseteq A$ , we define the *Lipschitz constant* of  $\phi$  on  $E$  as

$$\text{Lip}(\phi, E) := \inf \left\{ M > 0 : C_{\frac{1}{M}}(p) \cap \text{gr}_{\phi|_E} = \{p\} \text{ for all } p \in \text{gr}_{\phi|_E} \right\}.$$

**Definition 2.20.** Let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  and let  $\phi : A \subseteq \mathbb{W} \rightarrow \mathbb{V}$ . We define the set

$$S_\phi := \{w \in A : \exists \beta > 0 \text{ and } U \text{ open in } A \text{ with } w \in U, C_\beta(w \cdot \phi(w)) \cap \text{gr}_{\phi|_U} = \{w \cdot \phi(w)\}\}.$$

If  $w \in S_\phi$  we say that  $\phi$  is *locally intrinsic Lipschitz* at  $w$  and we define the *pointwise intrinsic Lipschitz constant* of  $\phi$  at  $w$  as

$$\text{lip}(\phi, w) := \inf \left\{ M > 0 : \exists r > 0 \text{ such that } C_{\frac{1}{M}}(w \cdot \phi(w)) \cap \text{gr}_{\phi|_{B_{\mathbb{W}}(w,r)}} = \{w \cdot \phi(w)\} \right\}.$$

where with  $B_{\mathbb{W}}(w, r)$  we intend, here and in the following, the ball depending on the distance induced on  $\mathbb{W}$  by  $d$ .

**Remark 2.21.** By definition, intrinsic cones depend on the chosen distance  $d$ . However, the intrinsic Lipschitz continuity property of a given map  $\phi$  does not depend on the choice of  $d$ ; also the set  $S_\phi$  does not depend on  $d$ .

**Theorem 2.22** ([84, Theorem 3.9]). *Let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  and let  $\phi : \mathbb{W} \rightarrow \mathbb{V}$  be an intrinsic Lipschitz map. Denote by  $k$  the (metric) dimension of  $\mathbb{V}$ . Then  $\mathcal{S}_d^{Q-k} \llcorner \text{gr}_\phi$  is  $(Q - k)$ -Ahlfors regular on  $\text{gr}_\phi$ .*

**Definition 2.23.** Let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  and  $\phi : A \subseteq \mathbb{W} \rightarrow \mathbb{V}$ . We define the *graph distance*  $\rho_\phi : A \times A \rightarrow [0, +\infty)$  as

$$\rho_\phi(w_1, w_2) := \frac{1}{2} \left( \|(p_1^{-1} \cdot p_2)_{\mathbb{W}}\| + \|(p_2^{-1} \cdot p_1)_{\mathbb{W}}\| \right) \quad \forall w_1, w_2 \in A$$

where  $p_i = w_i \cdot \phi(w_i)$  for  $i = 1, 2$ .

**Proposition 2.24** ([147, Proposition 4.59] and [84, Remark 3.6]). *Let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  and  $\phi : A \subseteq \mathbb{W} \rightarrow \mathbb{V}$  be intrinsic  $M$ -Lipschitz. Then the graph distance  $\rho_\phi$  is a quasi-distance, i.e.,*

- (i)  $\rho_\phi(w_1, w_2) = \rho_\phi(w_2, w_1)$  for every  $w_1, w_2 \in A$ ,
- (ii)  $\rho_\phi(w_1, w_2) = 0$  if and only if  $w_1 = w_2$ ,
- (iii) there exists a constant  $C > 1$  such that

$$\rho_\phi(w_1, w_2) \leq C(\rho_\phi(w_1, w_3) + \rho_\phi(w_3, w_2)) \text{ for every } w_1, w_2, w_3 \in A.$$

Moreover, there exists a positive constant  $C = C(M) > 1$  such that

$$\frac{1}{C}\rho_\phi(w_1, w_2) \leq d(w_1 \cdot \phi(w_1), w_2 \cdot \phi(w_2)) \leq C\rho_\phi(w_1, w_2)$$

for every  $w_1, w_2 \in A$ . This means that  $\rho_\phi$  is equivalent to the distance on the graph of  $\phi$ .

**Remark 2.25.** If  $\mathbb{W}$  is a normal subgroup of  $\mathbb{H}^n$ , then Definition 2.20 is equivalent to the following one, which generalizes (i):

$$S_\phi = \left\{ w \in A : \limsup_{\mathbb{W} \ni y \rightarrow w} \frac{d(\phi(y), \phi(w))}{\rho_\phi(y, w)} < +\infty \right\}.$$

In fact, on one side, assume that there exists  $\beta > 0$  and  $U$  open in  $A$  with  $w \in U$  and

$$C_\beta(w \cdot \phi(w)) \cap \text{gr}_{\phi|U} = \{w \cdot \phi(w)\}.$$

Then, for every  $y \in U$  with  $y \neq w$ , it holds  $y \cdot \phi(y) \notin C_\beta(w \cdot \phi(w))$ . Denoting by  $p := w \cdot \phi(w)$  and  $q := y \cdot \phi(y)$ , by definition of cone, we get that  $p^{-1} \cdot q \notin C_\beta(0)$  and so

$$\|(p^{-1} \cdot q)_{\mathbb{W}}\| > \beta \|(p^{-1} \cdot q)_{\mathbb{V}}\| = \beta \|\phi(w)^{-1} \cdot \phi(y)\|,$$

where the last identity follows by the fact that  $\pi_{\mathbb{V}}$  is a group homomorphism (see Proposition 2.15). Hence

$$\begin{aligned} d(\phi(y), \phi(w)) &= \|\phi(w)^{-1} \cdot \phi(y)\| < \frac{1}{\beta} \|(p^{-1} \cdot q)_{\mathbb{W}}\| \\ &\leq \frac{1}{\beta} \left[ \|(p^{-1} \cdot q)_{\mathbb{W}}\| + \|(q^{-1} \cdot p)_{\mathbb{W}}\| \right] = \frac{2}{\beta} \rho_\phi(y, w) \end{aligned}$$

for every  $y \in U$  with  $y \neq w$ . It follows that

$$\limsup_{\mathbb{W} \ni y \rightarrow w} \frac{d(\phi(y), \phi(w))}{\rho_\phi(y, w)} \leq \frac{2}{\beta} < +\infty.$$

On the other hand, assume that

$$\limsup_{\mathbb{W} \ni y \rightarrow w} \frac{d(\phi(y), \phi(w))}{\rho_\phi(y, w)} < +\infty,$$

which implies that there exists  $L > 0$  and  $U \ni w$  open in  $A$  such that, for every  $y \in U$ ,

$$\|\phi^{-1}(w) \cdot \phi(y)\| = d(\phi(y), \phi(w)) \leq L\rho_\phi(y, w). \quad (2.2)$$

For every  $y \in U$ , let us denote as before  $q := y \cdot \phi(y)$  and  $p := w \cdot \phi(w)$ .

**Claim:** There exists  $C > 0$  (not depending on  $y$ ) such that  $\rho_\phi(w, y) \leq C\|(p^{-1} \cdot q)_\mathbb{W}\|$  for every  $y \in U$ .

Assuming the Claim, thanks to (2.2), we get that, for a suitable constant  $\tilde{L}$  and for every  $y \in U$ ,

$$\|(p^{-1} \cdot q)_\mathbb{V}\| = \|\phi(w)^{-1} \cdot \phi(y)\| \leq \tilde{L}\|(p^{-1} \cdot q)_\mathbb{W}\|,$$

which means that the point  $p^{-1}q$  does not belong to the cone  $C_{\frac{1}{\tilde{L}}}(0)$ . Therefore  $q \notin C_{\frac{1}{\tilde{L}}}(p)$  for every  $y \in U$  and

$$C_{\frac{1}{\tilde{L}}}(p) \cap \text{gr}_{\phi|U} = \{p\}$$

concluding the proof of the equivalence.

It remains to show the validity of the Claim: such a conclusion can be obtained proceeding as in the second part of the proof of Theorem 4.60 in [147]. In particular, by formula (100) in [147], one gets that, for every  $\epsilon > 0$  there exists a constant  $\bar{C} = \bar{C}(\epsilon)$  such that for every  $y \in A$

$$\rho_\phi(w, y) \leq \bar{C}(\epsilon)\|(p^{-1} \cdot q)_\mathbb{W}\| + \epsilon\|(p^{-1} \cdot q)_\mathbb{V}\|.$$

From (2.2), if  $y \in U$  we get

$$\rho_\phi(w, y) \leq \bar{C}(\epsilon)\|(p^{-1} \cdot q)_\mathbb{W}\| + \epsilon L\rho_\phi(w, y).$$

Fixing  $\epsilon < 1/L$ , we finally get

$$\rho_\phi(w, y) \leq \frac{\bar{C}}{1 - \epsilon L}\|(p^{-1} \cdot q)_\mathbb{W}\|$$

for every  $y \in U$ , proving the Claim.

**Definition 2.26.** Let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  and  $\phi : A \subseteq \mathbb{W} \rightarrow \mathbb{V}$ . We say that  $\phi$  is an *intrinsic linear map* if its graph  $\text{gr}_\phi$  is a homogeneous subgroup of  $\mathbb{H}^n$ .

**Remark 2.27.** Another characterization of the notion of intrinsic linear map can be given as follows, see [158]. Let  $k = \dim(\mathbb{V})$ . For every  $w \in \mathbb{W}$  define  $w_H \in \mathbb{R}^{2n+1-k}$  as

$$\begin{aligned} w_H &:= (x_{k+1}, \dots, y_n) & \text{if } k < n & \quad \text{and} \quad w = (x_{k+1}, \dots, y_n, t), \\ w_H &:= (y_1, \dots, y_n) & \text{if } k = n & \quad \text{and} \quad w = (y_1, \dots, y_n, t). \end{aligned}$$

Then,  $\phi$  is intrinsic linear if and only if there exists a  $k \times (2n - k)$  matrix  $M$  such that, for every  $w \in \mathbb{W}$ ,  $\phi(w) = Mw_H$  (identifying  $M$  with a linear map  $M : \mathbb{R}^{2n-k} \rightarrow \mathbb{R}^k \equiv \mathbb{V}$ ).

**Definition 2.28.** Let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  and  $\phi : A \subseteq \mathbb{W} \rightarrow \mathbb{V}$  where  $A$  is a relatively open set. We say that  $\phi$  is *intrinsically differentiable* at  $\bar{w} \in A$  if there exists an intrinsic linear map  $d\phi_{\bar{w}} : \mathbb{W} \rightarrow \mathbb{V}$  such that

$$\lim_{\mathbb{W} \ni w \rightarrow 0} \frac{d(\phi_{\bar{w}}(w), d\phi_{\bar{w}}(w))}{\|w\|} = 0,$$

where  $\phi_{\bar{w}}$  is the map whose intrinsic graph is given by  $(\bar{w} \cdot \phi(\bar{w}))^{-1} \cdot \text{gr}_{\phi}$ . The map  $d\phi_{\bar{w}}$  is called the *intrinsic differential* of  $\phi$  at  $\bar{w}$ .

**Remark 2.29.** If  $\mathbb{W}$  is a normal subgroup, then the explicit expression of  $\phi_{\bar{w}}$  is given by

$$\phi_{\bar{w}}(w) := \phi(\bar{w})^{-1} \cdot \phi(\bar{w} \cdot \phi(\bar{w}) \cdot w \cdot \phi(\bar{w})^{-1}). \quad (2.3)$$

If instead  $\mathbb{V}$  is normal, then  $\phi_{\bar{w}}$  can be written as follows

$$\phi_{\bar{w}}(w) := w^{-1} \cdot \phi(\bar{w})^{-1} \cdot w \cdot \phi(\bar{w} \cdot w). \quad (2.4)$$

A general expression for the map  $\phi_{\bar{w}}$ , which has as special cases formulas (2.3) and (2.4), can be found for instance in [84, Proposition 2.21].

Later we will need the two following results: an equivalent characterization of intrinsic differentiability and the Rademacher Theorem for intrinsic graphs. We now state both.

**Theorem 2.30** ([83, Theorem 4.15], see also [77, Theorem 3.2.8]). *Fix  $1 \leq k \leq n$  and let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  where  $\mathbb{V}$  is a horizontal subgroup of dimension  $k$ . Let  $A \subseteq \mathbb{W}$  be an open set and  $\phi : A \rightarrow \mathbb{V}$ . Fix  $\bar{w} \in A$  and define  $\bar{p} := \bar{w} \cdot \phi(\bar{w})$ . Then the following statements are equivalent:*

- (i)  $\phi$  is intrinsically differentiable at  $\bar{w}$ ;
- (ii) there exists a vertical subgroup  $\mathbb{T}_{\phi, \bar{p}}$ , complementary to  $\mathbb{V}$ , such that for every  $\alpha > 0$  there exists  $\bar{r} = \bar{r}(\phi, \bar{w}, \alpha) > 0$  such that

$$C_{\mathbb{T}_{\phi, \bar{p}}, \mathbb{V}}(\bar{p}, \alpha) \cap \text{gr}_{\phi|_{B_{\mathbb{W}}(\bar{w}, \bar{r})}} = \{\bar{p}\}.$$

**Theorem 2.31** ([158, Theorem 1.1 and Theorem 1.5]). *Fix  $1 \leq k \leq n$  and let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  where  $\mathbb{V}$  is a horizontal subgroup of dimension  $k$ . Let  $A \subseteq \mathbb{W}$  be a set and  $\phi : A \rightarrow \mathbb{V}$  be intrinsic Lipschitz. Then there exists an intrinsic Lipschitz map  $\tilde{\phi} : \mathbb{W} \rightarrow \mathbb{V}$  such that  $\tilde{\phi}|_A \equiv \phi$  and  $\tilde{\phi}$  is intrinsically differentiable almost everywhere on  $\mathbb{W}$ .*

## 2.3 $C_{\mathbb{H}}^1$ -regular functions and $C_{\mathbb{H}}^1$ -regular submanifolds

**Definition 2.32.** Let  $U \subset \mathbb{H}^n$  be an open set and  $f : U \rightarrow \mathbb{R}^k$ . We say that  $f$  is *Pansu differentiable* at  $p \in U$  if there is a (necessarily unique) group homomorphism  $d_{\mathbb{H}}f_p : \mathbb{H}^n \rightarrow \mathbb{R}^k$ , called *Pansu differential*, such that

$$d_{\mathbb{H}}f_p(q) := \lim_{\lambda \rightarrow 0} \frac{f(p \cdot \delta_{\lambda}(q)) - f(p)}{\lambda} \quad \forall q \in \mathbb{H}^n$$

locally uniformly in  $q$ .

Let  $U \subset \mathbb{R}^k$  be an open set and  $f : U \rightarrow \mathbb{H}^n$ . We say that  $f$  is *Pansu differentiable* at  $a \in U$  if there is a (necessarily unique) group homomorphism  $d_{\mathbb{H}}f_a : \mathbb{R}^k \rightarrow \mathbb{H}^n$ , called *Pansu differential*, such that

$$d_{\mathbb{H}}f_a(b) := \lim_{\lambda \rightarrow 0} \delta_{1/\lambda} \left( f(a)^{-1} \cdot f(a + \lambda b) \right) \quad \forall b \in \mathbb{R}^k$$

locally uniformly in  $b$ .

**Definition 2.33.** Let  $U \subset \mathbb{H}^n$  be an open set and  $f : U \rightarrow \mathbb{R}$ ; we say that  $f$  is of class  $C_{\mathbb{H}}^1$  ( $f \in C_{\mathbb{H}}^1(U)$ ) if  $f$  is continuous and its *horizontal gradient*  $\nabla_{\mathbb{H}}f := (X_1f, \dots, X_nf, Y_1f, \dots, Y_nf)$  (to be understood in the sense of distributions) is represented by a  $2n$ -ple of continuous functions on  $U$ .

We agree that, for every  $p \in U$ , the horizontal gradient  $\nabla_{\mathbb{H}}f(p)$  is identified with the horizontal vector

$$\nabla_{\mathbb{H}}f(p) = X_1f(p)X_1(p) + \dots + Y_nf(p)Y_n(p) \in H_p\mathbb{H}^n.$$

**Remark 2.34.** It is well-known (see e.g. [100, Proposition 2.4]) that, if  $f \in C_{\mathbb{H}}^1(U)$ , then  $f$  is Pansu differentiable at every  $p \in U$  and

$$d_{\mathbb{H}}f_p(q) = \sum_{i=1}^n x_i X_i f(p) + y_i Y_i f(p) \quad \forall q = (x, y, t) \in \mathbb{H}^n.$$

**Definition 2.35.** Let  $1 \leq k \leq n$ . A subset  $S \subset \mathbb{H}^n$  is a  $k$ -dimensional (or  $(2n + 1 - k)$ -codimensional)  $C_{\mathbb{H}}^1$ -regular (or  $\mathbb{H}$ -regular) submanifold if for any  $p \in S$  there are open sets  $U \subset \mathbb{H}^n$ ,  $V \subset \mathbb{R}^k$  and a function  $f : V \rightarrow U$  such that  $p \in U$ ,  $f$  is injective,  $f$  is continuously Pansu differentiable with  $d_{\mathbb{H}}f$  injective and

$$S \cap U = f(V).$$

**Remark 2.36.** By [82, Theorem 3.5], a  $C_{\mathbb{H}}^1$  regular submanifold  $S$  of dimension  $k \leq n$  is also a Euclidean  $C^1$  submanifold and  $T_p S \subset H_p \mathbb{H}^n$  for every  $p \in S$ . Moreover, the Hausdorff dimension of  $S$  equals the topological dimension  $k$  and the spherical Hausdorff measure  $\mathcal{S}^k \llcorner S$  is comparable with the Euclidean  $k$ -dimensional Hausdorff measure on  $S$ .

**Definition 2.37.** Let  $1 \leq k \leq n$ . A subset  $S \subset \mathbb{H}^n$  is a  $k$ -codimensional (or  $(2n + 1 - k)$ -dimensional)  $C_{\mathbb{H}}^1$  regular (or  $\mathbb{H}$ -regular) submanifold if for any  $p \in S$  there are an open set  $U \subset \mathbb{H}^n$  and a function  $f : U \rightarrow \mathbb{R}^k$  such that  $p \in U$ ,  $f = (f_1, \dots, f_k) \in C_{\mathbb{H}}^1(U, \mathbb{R}^k)$  and

$$\begin{aligned} S \cap U &= \{q \in U : f(q) = 0\} \\ \nabla_{\mathbb{H}}f_1 \wedge \dots \wedge \nabla_{\mathbb{H}}f_k &\neq 0 \text{ on } U \text{ (equivalently, } d_{\mathbb{H}}f \text{ is onto).} \end{aligned}$$

The *tangent group*  $T_{\mathbb{H}}^{\mathcal{S}}S(p)$  to  $S$  at  $p$  is the subgroup of  $\mathbb{H}^n$  defined by

$$T_{\mathbb{H}}^{\mathcal{S}}S(p) := \ker d_{\mathbb{H}}f_p.$$

The *horizontal normal* at  $p \in S$   $n_S^{\mathbb{H}}(p) \in \wedge_{k,p}\mathfrak{h}_1$  is defined by

$$n_S^{\mathbb{H}}(p) := \frac{\nabla_{\mathbb{H}}f_1(p) \wedge \cdots \wedge \nabla_{\mathbb{H}}f_k(p)}{|\nabla_{\mathbb{H}}f_1(p) \wedge \cdots \wedge \nabla_{\mathbb{H}}f_k(p)|}.$$

The *tangent vector*  $t_S^{\mathbb{H}}(p) \in \wedge_{2n+1-k,p}\mathfrak{h}$  is defined by

$$t_S^{\mathbb{H}}(p) := *n_S^{\mathbb{H}}(p)$$

where  $*$  is the Hodge operator. By [82, Proposition 3.29],  $t_S^{\mathbb{H}}(p)$  and  $n_S^{\mathbb{H}}(p)$  are well defined (up to a sign) and they do not depend on the defining function  $f$ . The tangent vector is never horizontal and it can always be written in the form

$$t_S^{\mathbb{H}}(p) = \tau_S^{\mathbb{H}}(p) \wedge T$$

for a unique (up to a sign)  $\tau_S^{\mathbb{H}}(p) \in \wedge_{2n-k,p}\mathfrak{h}_1$ .

Observe that  $n_S^{\mathbb{H}}(p)$ ,  $t_S^{\mathbb{H}}(p)$  and  $\tau_S^{\mathbb{H}}(p)$  can be (locally) chosen to be continuous in  $p$ . Moreover,  $T_{\mathbb{H}}^{\mathcal{S}}S(p) = \text{span } t_S^{\mathbb{H}}(p)$ .

**Definition 2.38.** Let  $S \subset \mathbb{H}^n$ . The *tangent cone* to  $S$  at 0 is the set

$$T_0^{\mathbb{H}}S := \{x \in \mathbb{H}^n : \text{there exist } (x_h)_{h \in \mathbb{N}} \subset S, r_h \rightarrow +\infty \text{ and } \delta_{r_h}x_h \rightarrow x\},$$

while the tangent cone to  $S$  at  $p$  is defined as

$$T_p^{\mathbb{H}}S := T_0^{\mathbb{H}}\tau_{p^{-1}}(S).$$

We have the following relationship between tangent cone and tangent group.

**Proposition 2.39** ([82, Proposition 3.29]). *Let  $S \subset \mathbb{H}^n$  be a  $C_{\mathbb{H}}^1$  submanifold of codimension  $1 \leq k \leq n$ . Then*

$$T_p^{\mathbb{H}}S = T_{\mathbb{H}}^{\mathcal{S}}S(p) \quad \text{for every } p \in S.$$

The following version of the Implicit Function Theorem says that every high dimensional  $C_{\mathbb{H}}^1$ -regular submanifold can be locally written as an orthogonal continuous graph.

**Theorem 2.40** ([82, Theorem 4.1], see also [100, Lemma 2.10]). *Let  $S \subset \mathbb{H}^n$  be a  $C_{\mathbb{H}}^1$  submanifold of codimension  $1 \leq k \leq n$ . Then there exist*

- an orthogonal splitting  $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$  where  $\mathbb{V}$  is a horizontal  $k$ -dimensional vector space,
- relatively open sets  $A \subset \mathbb{W}$  and  $B \subset \mathbb{V}$  such that  $p \in U := A \cdot B$ ,

- a continuous function  $\phi : A \rightarrow B$
- a function  $f = (f_1, \dots, f_k) \in C_{\mathbb{H}}^1(U, \mathbb{R}^k)$  with  $\nabla_{\mathbb{H}} f_1 \wedge \dots \wedge \nabla_{\mathbb{H}} f_k \neq 0$

such that

$$S \cap U = \{q \in U : f(q) = 0\} = \{\xi \cdot \phi(\xi) : \xi \in A\}.$$

In the same notations and assumption of Theorem 2.40 we now state the following area formula, which is a particular case of [48, Theorem 1.3 and Theorem 8.7].

**Theorem 2.41.** *Let  $\Phi$  be the graph map associated to  $\phi$ ,  $\Phi_{\#}$  be the push-forward of measures. Then*

$$\mathcal{S}_d^{Q-k} \llcorner (S \cap U) = C_{n,k} \Phi_{\#} \left( \left( \frac{|\nabla_{\mathbb{H}} f_1 \wedge \dots \wedge \nabla_{\mathbb{H}} f_k|}{\Delta} \circ \Phi \right) \mathcal{H}_E^{2n+1-k} \llcorner \mathbb{W} \right)$$

where

$$\Delta(q) := |\det[v_i f_j(q)]_{1 \leq i, j \leq k}| \neq 0 \text{ for } q \in U,$$

$v_1, \dots, v_k \in \mathfrak{v}$  form an orthonormal basis of the Lie algebra  $\mathfrak{v} \subset \mathfrak{h}$  of  $\mathbb{V}$  and  $C_{n,k}$  is a positive constant only depending on  $n, k$  and the distance  $d$ .

**Remark 2.42.** An explicit expression of the constant  $C_{n,k}$  in Theorem 2.41 is given by

$$C_{n,k} = \left( \sup \{ \mathcal{L}^{2n+1-k}(\mathbb{W} \cap B_d(p, 1)) : p \in B_d(0, 1) \} \right)^{-1},$$

see [124] and [47].

## 2.4 The Rumin complex

In this section we present the Rumin complex. Introduced in [144, 145] for contact manifolds, it has become clear over the years that Rumin complex is natural when dealing with differential forms in the Heisenberg groups; see for instance [17, 21, 58, 82, 102, 106, 107, 108, 119, 158].

**Definition 2.43.** For a given integer  $0 \leq k \leq 2n + 1$  we define the following sets of covectors:

$$\mathcal{I}^k := \{\lambda \wedge \theta + \mu \wedge d\theta : \lambda \in \wedge^{k-1} \mathfrak{h}, \mu \in \wedge^{k-2} \mathfrak{h}\},$$

$$\mathcal{J}^k := \{\lambda \in \wedge^k \mathfrak{h} : \lambda \wedge \theta = \lambda \wedge d\theta = 0\},$$

where we adopted the convention that  $\wedge^i \mathfrak{h} := \{0\}$  if  $i < 0$ . Then we define the *Heisenberg-Rumin differential  $k$ -forms* as

$$\Omega_{\mathbb{H}}^k := C^\infty \left( \mathbb{H}^n, \frac{\wedge^k \mathfrak{h}}{\mathcal{I}^k} \right) \quad \text{if } 0 \leq k \leq n,$$

$$\Omega_{\mathbb{H}}^k := C^\infty(\mathbb{H}^n, \mathcal{J}^k) \quad \text{if } n + 1 \leq k \leq 2n + 1.$$

It is worth noticing that, for  $k \geq n + 1$ , the exterior differentiation<sup>3</sup>  $d$  satisfies  $d(\Omega_{\mathbb{H}}^k) \subset (\Omega_{\mathbb{H}}^{k+1})$ . For every  $k$  we have  $d(C^\infty(\mathbb{H}^n; \mathcal{I}^k)) \subset C^\infty(\mathbb{H}^n; \mathcal{I}^{k+1})$ ; in particular, for  $k \leq n - 1$ ,  $d$  passes to the quotient defining an operator  $d : \Omega_{\mathbb{H}}^k \rightarrow \Omega_{\mathbb{H}}^{k+1}$ .

**Theorem 2.44** (See [144, 145]). *There exists a second-order differential operator  $D : \Omega_{\mathbb{H}}^n \rightarrow \Omega_{\mathbb{H}}^{n+1}$  such that the sequence*

$$0 \rightarrow \mathbb{R} \rightarrow \Omega_{\mathbb{H}}^0 \xrightarrow{d} \Omega_{\mathbb{H}}^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_{\mathbb{H}}^n \xrightarrow{D} \Omega_{\mathbb{H}}^{n+1} \xrightarrow{d} \dots \xrightarrow{d} \Omega_{\mathbb{H}}^{2n+1} \rightarrow 0$$

is locally exact.

*Proof.* Let us define the non-trivial second order operator  $D$ . First we define the Lefschetz operator  $L : \wedge^k \mathbb{R}^{2n} \rightarrow \wedge^{k+2} \mathbb{R}^{2n}$  as  $L(\lambda) := \lambda \wedge d\theta$ . When  $k \leq n - 1$  then  $L$  is injective; for  $k \geq n - 1$  then  $L$  is surjective; in particular for  $k = n - 1$  we have that  $L$  is bijective (see for instance [29, Proposition 1.1]). Now we observe that

$$\frac{\wedge^n \mathfrak{h}}{\mathcal{I}^n} = \frac{\wedge^n \mathfrak{h}_1}{\{\mu \wedge d\theta : \mu \in \wedge^{n-2} \mathfrak{h}_1\}}.$$

Then we define  $D$  on smooth sections of  $\wedge^n \mathfrak{h}_1$  as

$$D(\alpha) := d(\alpha - \theta \wedge L^{-1}((d\alpha)_{\mathfrak{h}_1})),$$

where  $\alpha$  is a smooth section of  $\wedge^n \mathfrak{h}_1$  and  $(\cdot)_{\mathfrak{h}_1}$  denotes the horizontal part (i.e. given  $\lambda \in \wedge^k \mathfrak{h}$  for  $1 \leq k \leq 2n$  then  $\lambda_{\mathfrak{h}_1} \in \wedge^k \mathfrak{h}_1$  is the unique covector such that  $\lambda = \lambda_{\mathfrak{h}_1} + \mu \wedge \theta$  for a (unique)  $\mu \in \wedge^{k-1} \mathfrak{h}_1$ ). If we prove that  $D$  passes to the quotient modulo smooth sections of  $\{\mu \wedge d\theta : \mu \in \wedge^{n-2} \mathfrak{h}_1\}$  then we will obtain that  $D$  is well defined as a linear operator  $\Omega_{\mathbb{H}}^n \rightarrow \Omega_{\mathbb{H}}^{n+1}$ . First we prove that  $D(\beta \wedge d\theta) = 0$  for every smooth section  $\beta \in \wedge^{n-2} \mathfrak{h}_1$ . Inasmuch,

$$\begin{aligned} L^{-1}((d(\beta \wedge d\theta))_{\mathfrak{h}_1}) &= L^{-1}((d\beta \wedge d\theta)_{\mathfrak{h}_1}) = L^{-1}((d\beta)_{\mathfrak{h}_1} \wedge d\theta) = (d\beta)_{\mathfrak{h}_1} \\ d(\beta \wedge d\theta) &= (-1)^{n-1} d(d\beta \wedge \theta) \end{aligned}$$

we deduce that

$$\begin{aligned} D(\beta \wedge d\theta) &= d(\beta \wedge d\theta + (-1)^n (d\beta)_{\mathfrak{h}_1} \wedge \theta) \\ &= (-1)^{n-1} d(d\beta \wedge \theta - (d\beta)_{\mathfrak{h}_1} \wedge \theta) \\ &= (-1)^{n-1} d((d\beta)_{\mathfrak{h}_1} \wedge \theta - (d\beta)_{\mathfrak{h}_1} \wedge \theta) = 0 \end{aligned}$$

as wished; the latter implying that  $D$  is well defined as a linear operator  $\Omega_{\mathbb{H}}^n \rightarrow \Omega_{\mathbb{H}}^{n+1}$ . We have to check that  $D(\Omega_{\mathbb{H}}^n) \subseteq \Omega_{\mathbb{H}}^{n+1}$ , i.e. that  $D\alpha \wedge \theta = D\alpha \wedge d\theta = 0$  for every  $\alpha \in \Omega_{\mathbb{H}}^n$ . Let

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<sup>3</sup>We use the same symbol,  $d$ , for both the distance in  $\mathbb{H}^n$  and for exterior differentiation; no confusion will ever arise.

us write  $(d\alpha)_v := d\alpha - (d\alpha)_{\mathfrak{h}_1}$  to denote the vertical part of  $d\alpha$ , which can be written as  $(d\alpha)_v = \theta \wedge \beta_\alpha$  for a suitable smooth section  $\beta_\alpha$  of  $\bigwedge^n \mathfrak{h}_1$ . Then

$$\begin{aligned} D\alpha &= d(\alpha + (-1)^n L^{-1}((d\alpha)_{\mathfrak{h}_1}) \wedge \theta) \\ &= \cancel{(d\alpha)_{\mathfrak{h}_1}} + (d\alpha)_v + (-1)^n d(L^{-1}((d\alpha)_{\mathfrak{h}_1})) \wedge \theta - \cancel{L^{-1}((d\alpha)_{\mathfrak{h}_1}) \wedge d\theta} \\ &= \theta \wedge (\beta_\alpha + d(L^{-1}((d\alpha)_{\mathfrak{h}_1}))). \end{aligned}$$

This implies that  $D\alpha \wedge \theta = 0$  and, as a consequence,  $0 = d(D\alpha \wedge \theta) = (-1)^{n+1} D\alpha \wedge d\theta$ , the latter implying that  $D$  is well defined as a linear operator  $\Omega_{\mathbb{H}}^n \rightarrow \Omega_{\mathbb{H}}^{n+1}$ . Now we are left to prove that the sequence is locally exact: we have four cases.

- (i) CASE  $k \leq n - 1$ . Let  $\alpha \in \Omega_{\mathbb{H}}^k$  such that  $d\alpha = 0$ . This means there exists a smooth section  $\tilde{\alpha}$  of  $\bigwedge^k \mathfrak{h}$  such that its projection on  $\Omega_{\mathbb{H}}^k$  is given by  $\alpha$  and

$$d\tilde{\alpha} = \theta \wedge \beta - d\theta \wedge \gamma$$

for suitable forms  $\beta \in \Omega^{k-1}, \gamma \in \Omega^{k-2}$ . We observe that the form  $\tilde{\alpha} - \theta \wedge \gamma$  also projects to  $\alpha$  on  $\Omega_{\mathbb{H}}^k$ . We compute

$$d(\tilde{\alpha} - \theta \wedge \gamma) = \theta \wedge \beta + d\theta \wedge \gamma - d\theta \wedge \gamma + \theta \wedge d\gamma = \theta \wedge (\beta + d\gamma),$$

the latter implying that  $d\tilde{\alpha} = \theta \wedge \beta$ ; differentiating we obtain

$$0 = d\theta \wedge \beta + \theta \wedge d\beta.$$

We observe  $d\theta \wedge \beta|_{H\mathbb{H}^n} = L(\beta|_{H\mathbb{H}^n}) = 0$  and by the injectivity of  $L$  we obtain  $\beta|_{H\mathbb{H}^n} = 0$ . This means we can write  $\beta = \theta \wedge \delta$  for some  $\delta \in \Omega_{\mathbb{H}}^{k-1}$ , finally obtaining  $d\tilde{\alpha} = \theta \wedge \beta = 0$ . Since  $d\tilde{\alpha} = 0$  there exists locally  $\mu \in \Omega^{k-1}$  such that  $\tilde{\alpha} = d\mu$ . The projection  $\tilde{\mu}$  of  $\mu$  on  $\Omega_{\mathbb{H}}^{k-1}$  is such that  $d\tilde{\mu} = \alpha$ .

- (ii) CASE  $k = n$ . Let  $\alpha \in \Omega_{\mathbb{H}}^n$  such that  $D\alpha = 0$ . Then by definition we have that the smooth section  $\omega := \alpha - \theta \wedge L^{-1}((d\alpha)_{\mathfrak{h}_1})$  of  $\bigwedge^n \mathfrak{h}$  is such that  $d\omega = 0$ . Then there exists locally a smooth section  $\beta$  of  $\bigwedge^{n-1} \mathfrak{h}$  such that  $\omega = d\beta$ . The projection  $\tilde{\beta}$  of  $\beta$  on  $\Omega_{\mathbb{H}}^{n-1}$  is such that  $\alpha = d\tilde{\beta}$ .
- (iii) CASE  $k = n + 1$ . Let  $\alpha \in \Omega_{\mathbb{H}}^{n+1}$  such that  $d\alpha = 0$ . Then there exists locally a smooth section  $\beta$  of  $\bigwedge^n \mathfrak{h}$  such that  $d\beta = \alpha$  which is a smooth section of  $\mathcal{J}^{n+1}$ . The projection  $\tilde{\beta}$  of  $\beta$  on  $\Omega_{\mathbb{H}}^n$  is such that  $D\tilde{\beta} = \alpha$ .
- (iv) CASE  $k \geq n + 2$ . Let  $\alpha \in \Omega_{\mathbb{H}}^k$  such that  $d\alpha = 0$ . This means there exists locally  $\beta \in \Omega^{k-1}$  such that  $\alpha = d\beta$ . By surjectivity of  $L$  we can write

$$\beta = \lambda \wedge \theta + \mu \wedge d\theta,$$



The sum in the formula above is finite because  $\mathfrak{g}$  is nilpotent. The resulting Lie group, which we denote by  $\mathbb{G}$ , is nilpotent and simply connected.

The identification  $\mathbb{G} = V = \mathfrak{g}$  corresponds to the identification between Lie algebra and Lie group via the exponential map  $\exp : \mathfrak{g} \rightarrow \mathbb{G}$ . Notice that  $p^{-1} = -p$  for every  $p \in \mathbb{G}$  and that 0 is the neutral element of  $\mathbb{G}$ .

For  $\lambda > 0$ , define the *dilations* as the maps  $\delta_\lambda : V \rightarrow V$  such that  $\delta_\lambda v = \lambda^j v$  for  $v \in V_j$ . We say that a subset  $M$  of  $V$  is *homogeneous* if  $\delta_\lambda(M) = M$  for all  $\lambda > 0$ .

Let  $\mathbb{P}$  be a homogeneous subgroup of  $\mathbb{G}$  and  $\theta$  a Haar measure on  $\mathbb{P}$ . Since  $\delta_\lambda|_{\mathbb{P}}$  is an automorphism of  $\mathbb{P}$ , there is  $c_\lambda > 0$  such that  $(\delta_\lambda)_\# \theta = c_\lambda \theta$ . Since the map  $\lambda \mapsto \delta_\lambda|_{\mathbb{P}}$  is a multiplicative one-parameter group of automorphisms, the map  $\lambda \mapsto c_\lambda$  is a continuous automorphism of the multiplicative group  $(0, +\infty)$ , hence  $c_\lambda = \lambda^{-d}$  for some  $d \in \mathbb{R}$ . As  $\delta_\lambda$  is contractive for  $\lambda < 1$ , we actually have  $d > 0$ . Since any other Haar measure of  $\mathbb{P}$  is a positive multiple of  $\theta$ , the constant  $d$  does not depend on the choice of the Haar measure. We call such exponent  $d$  the *homogeneous dimension* of  $\mathbb{P}$ . The homogeneous dimension of the ambient space  $\mathbb{G}$  is denoted by  $Q$  and it is easy to see that  $Q := \sum_{i=1}^s i \dim V_i$ .

A *homogeneous distance* on  $\mathbb{G}$  is a distance function  $d$  that is left-invariant and 1-homogeneous with respect to dilations, i.e.,

- (i)  $d(gx, gy) = d(x, y)$  for all  $g, x, y \in \mathbb{G}$ ;
- (ii)  $d(\delta_\lambda x, \delta_\lambda y) = \lambda d(x, y)$  for all  $x, y \in \mathbb{G}$  and all  $\lambda > 0$ .

When a stratified group  $\mathbb{G}$  is endowed with a homogeneous distance  $d$ , we call the metric Lie group  $(\mathbb{G}, d)$  a *Carnot group*.

We now introduce Carnot groups of type  $\star$  (see also [129]) and  $\diamond$  (see also [116]).

**Definition 2.47.** We say that a Carnot group  $(\mathbb{G}, d)$  is of *type  $\star$*  if its stratified Lie algebra  $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$  has the following property: there exists a basis  $(X_1, \dots, X_m)$  of  $V_1$  such that

$$[X_j, [X_j, X_i]] = 0 \quad \text{for } i, j = 1, \dots, m.$$

**Definition 2.48.** We say that a Carnot groups  $(\mathbb{G}, d)$  is of *type  $\diamond$*  if each subalgebra  $\mathfrak{p} \subseteq \mathfrak{g} = V_1 \oplus \cdots \oplus V_s$  for which  $\mathfrak{p} \cap V_1$  has codimension 1 in  $V_1$  satisfies one of the following equivalent conditions:

- (i) there exists a basis  $(X_1, \dots, X_m)$  of  $V_1$  such that

$$\text{ad}_{X_i}^2 X_j \in \mathfrak{p} \quad \text{and} \quad \text{ad}_{\text{ad}_{X_i}^k X_j}^2 (X_i) \in \mathfrak{p}$$

for all  $i, j = 1, \dots, m$  and  $k \geq 2$ ;

(ii) there exists a basis  $(Y_1, \dots, Y_{m-1})$  of  $V_1 \cap \mathfrak{g}$  and  $X \in V_1 \setminus \mathfrak{p}$  such that

$$\mathrm{ad}_X^2 Y_i \in \mathfrak{p}, \quad \mathrm{ad}_{Y_i}^2 X \in \mathfrak{p} \quad \text{and} \quad \mathrm{ad}_{\mathrm{ad}_X^k Y_i}^2(X) \in \mathfrak{p},$$

for all  $i = 1, \dots, m - 1$  and  $k \geq 2$ .

**Remark 2.49.** The sequence of inclusions below holds; see [129] for a proof of the second inclusion and [116] for a proof of the third inclusion.

$$\left\{ \begin{array}{c} \text{Heisenberg} \\ \text{groups} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{Carnot} \\ \text{groups of} \\ \text{step 2} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{Carnot} \\ \text{groups of} \\ \text{type } \star \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{Carnot} \\ \text{groups of} \\ \text{type } \diamond \end{array} \right\}.$$

# Part II

## Original contributions

In the second part of the thesis we present our original contributions.

In Chapter 3 we present the results from [60], i.e.,

- M. Di Marco, G. Somma, D. Vittone,  
*A note on the diameter of small sub-Riemannian balls.*  
Preprint (2025) available online at <https://arxiv.org/abs/2505.02790>.

In Chapter 4 we present the results from [57], i.e.,

- M. Di Marco, S. Don, D. Vittone,  
*SBV functions in Carnot-Carathéodory spaces.*  
Nonlinear Anal. 262 (2026), Paper No. 113944,  
<https://doi.org/10.1016/j.na.2025.113944>.

In Chapter 5 we present the results from [59], i.e.,

- M. Di Marco, A. Pinamonti, D. Vittone, K. Zambanini,  
*Stepanov differentiability theorem for intrinsic graphs in Heisenberg groups.*  
Adv. Calc. Var. 18 (2025), no.3, 963-978,  
<https://doi.org/10.1515/acv-2024-0118>.

In Chapter 6 we present the results from [58], i.e.,

- M. Di Marco, A. Julia, S. Nicolussi Golo, D. Vittone,  
*Submanifolds with boundary and Stokes' Theorem in Heisenberg groups.*  
Trans. Amer. Math. Soc. 378 (2025), no.7, 4955–4990,  
<https://doi.org/10.1090/tran/9410>.

and we also present some examples of  $C_{\mathbb{H}}^1$ -regular submanifold with boundary described in [61].

Finally, in Appendix A, we present some joint work, in progress with A. Julia, P. Massot, and A. Vaugon, on the relationship between contact geometry and sub-Riemannian geometry.

# 3 Diameter of small sub-Riemannian balls

The aim of this chapter is to present the results contained in [60]. Carnot-Carathéodory spaces, as defined in Definition 1.1, are the prototypical example of more general structures called sub-Riemannian manifolds. In this chapter we only give the definitions and some basic results about sub-Riemannian hamiltonians that we are going to need in order to prove Theorem 3.1 and Theorem 3.2 below. For an introduction to sub-Riemannian manifolds we refer the reader to [1].

**Theorem 3.1.** *Let  $M$  be a  $C^{1,1}$  sub-Riemannian manifold. Then, for every  $p \in M$  there exist a neighbourhood (with respect to the manifold topology)  $V$  of  $p$  and  $r_p > 0$  such that*

$$\text{diam}(B_C(q, r)) = 2r \quad \text{for every } 0 < r < r_p \text{ and } q \in V.$$

**Theorem 3.2.** *Let  $M$  be a  $C^0$  sub-Riemannian manifold. Then, for every  $p \in M$  and  $\varepsilon > 0$  there exist a neighbourhood (with respect to the manifold topology)  $V$  of  $p$  and  $r_{p,\varepsilon} > 0$  such that*

$$2r(1 - \varepsilon) \leq \text{diam}(B_C(q, r)) \leq 2r \quad \text{for every } 0 < r < r_{p,\varepsilon} \text{ and } q \in V.$$

In Section 3.1 we will give our definition of  $C^{1,1}$  sub-Riemannian manifolds and prove Theorem 3.1 using some properties of the sub-Riemannian Hamiltonian. In Section 3.2 we will give our definition of  $C^0$  sub-Riemannian manifolds and prove Theorem 3.2. As stated in the introduction, both results will be proved using a calibration argument and are independent of the Hörmander condition.

## 3.1 Diameter of small sub-Riemannian balls: case $C^{1,1}$

**Definition 3.3.** We say that  $(M, \Delta, g)$  is an  $n$ -dimensional  $C^{1,1}$  sub-Riemannian manifold of rank  $m$  if

- $M$  is a connected smooth manifold of dimension  $n$ ;
- $\Delta = \sqcup_{p \in M} \Delta_p$  is a  $C^{1,1}$  distribution on  $M$ , i.e., a  $C^{1,1}$  map  $p \mapsto \Delta_p$  which assigns to each  $p \in M$  an  $m$ -dimensional vector subspace of  $T_p M$ ;
- $g$  is a  $C^{1,1}$  metric on  $\Delta$ .

For every  $v \in \Delta_p$ , we also set  $|v|_p := \sqrt{g_p(v, v)}$ . Every vector field  $X$  such that  $X(p) \in \Delta_p$  for every  $p \in M$  is said to be *horizontal*.

**Remark 3.4.** Usually the definition of sub-Riemannian manifold is given in terms of *smooth* distribution and metric and, moreover, the family of horizontal vector fields is assumed to be bracket-generating (i.e., the Hörmander condition holds). However, in order to prove Theorems 3.1 and 3.2 we will not need these assumptions.

**Definition 3.5.** Let  $M = (M, \Delta, g)$  be an  $n$ -dimensional  $C^{1,1}$  sub-Riemannian manifold of rank  $m$ . We say that an absolutely continuous curve  $\gamma : [a, b] \rightarrow M$  is an *admissible curve* joining  $p$  and  $q$  if  $\gamma(a) = p$ ,  $\gamma(b) = q$  and  $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$  for a.e.  $t \in [a, b]$ . The *length* of  $\gamma$  is

$$L(\gamma) := \int_a^b |\dot{\gamma}(t)|_{\gamma(t)} dt.$$

For every  $p, q \in M$ , the *Carnot-Carathéodory (CC) distance* is

$$d_C(p, q) := \inf\{L(\gamma) : \gamma \text{ is an admissible curve joining } p \text{ and } q\},$$

where we agree that  $\inf \emptyset := +\infty$ .

**Remark 3.6.** The Carnot-Carathéodory distances defined in Definition 1.1 and Definition 3.5 coincide when computed on equiregular Carnot-Carathéodory spaces, see for instance [62, Remark 1.2.3]. As a matter of fact, under suitable assumptions of regularity, Definition 1.1 and Definition 3.5 provide two different way to compute *the* Carnot-Carathéodory distance.

**Remark 3.7.** All the spaces mentioned in Chapter 1 and Chapter 2 such as Carnot-Carathéodory spaces and Carnot groups fall within our definition of  $C^{1,1}$  sub-Riemannian manifolds. In particular Theorem 3.1 holds for those spaces.

We now present a lemma<sup>1</sup> that will be pivotal in the proof of Theorem 3.1, about the existence of a *calibration*. Recall that the the existence of a calibration is a sufficient condition for the length-minimality of a given curve, see e.g. [120, 134, 142].

**Lemma 3.8.** *Let  $M = (M, \Delta, g)$  be an  $n$ -dimensional  $C^{1,1}$  sub-Riemannian manifold of rank  $m$ . For every  $p \in M$  there exists a neighbourhood  $W$  (with respect to the manifold topology) of  $p$ , a horizontal vector field  $Y$  on  $W$  and an exact 1-form  $\Lambda$  on  $W$  such that*

$$\begin{aligned} \langle \Lambda(q), v \rangle &\leq |v|_q && \text{for every } q \in W \text{ and } v \in \Delta_q, \\ \langle \Lambda(q), Y(q) \rangle &= |Y(q)|_q = 1 && \text{for every } q \in W. \end{aligned}$$

---

<sup>1</sup>Lemma 3.8 is well-known; however, for the sake of completeness, we will provide a proof after the proof of Theorem 3.1.

The 1-form  $\Lambda$  in Lemma 3.8 calibrates the integral curves of  $Y$ , which are therefore *all* length-minimizing: this remark is the key ingredient for proving Theorem 3.1, that we restate for the reader's convenience. Let us recall that the Riemannian analogue of Theorem 3.1 comes as a byproduct of the existence of a geodesic for a given point and an initial velocity (contained in the tangent space at that point), see for instance [118, Corollary 4.28].

**Theorem 3.9.** *Let  $M$  be a  $C^{1,1}$  sub-Riemannian manifold. Then, for every  $p \in M$  there exist a neighbourhood (with respect to the manifold topology)  $V$  of  $p$  and  $r_p > 0$  such that*

$$\text{diam}(B_C(q, r)) = 2r \quad \text{for every } 0 < r < r_p \text{ and } q \in V.$$

*Proof.* Let  $W, Y$  and  $\Lambda$  be as in Lemma 3.8 and fix an open subset  $V \subset\subset W$ . Since the Euclidean distance is locally controlled by above, up to a positive multiplicative constant, by the CC one, there exists  $r_p > 0$  such that  $B_C(q, 2r_p) \subseteq W$  for every  $q \in V$ . Consider the curve  $\gamma_0 : (-r, r) \rightarrow B_C(q, r) \subseteq W$  defined by  $\gamma_0(0) = q$  and  $\dot{\gamma}_0(t) = Y$  for every  $t \in (-r, r)$ . Fix also  $\delta \in (0, r)$  and let  $q_1 := \gamma_0(-r + \delta)$ ,  $q_2 := \gamma_0(r - \delta)$ ; then,  $\gamma_0$  is an admissible curve joining  $q_1$  and  $q_2$  and  $q_1, q_2 \in B_C(q, r)$ . Let  $\kappa : [a, b] \rightarrow M$  be another admissible curve joining  $q_1$  and  $q_2$ . If the support of  $\kappa$  is not contained in  $W$ , then  $L(\kappa) \geq 2r_p$ ; otherwise, the support of  $\kappa$  is contained in  $W$  and

$$\begin{aligned} L(\kappa) &= \int_a^b |\dot{\kappa}(t)|_{\kappa(t)} dt \geq \int_a^b \langle \Lambda(\kappa(t)), \dot{\kappa}(t) \rangle dt = \int_{\kappa} \Lambda \\ &= \int_{\gamma_0} \Lambda = \int_{-r+\delta}^{r-\delta} \langle \Lambda(\gamma_0(t)), Y(\gamma_0(t)) \rangle dt = 2(r - \delta). \end{aligned}$$

In any case, we obtain

$$\text{diam}(B_C(q, r)) \geq d_C(q_1, q_2) \geq 2(r - \delta)$$

and we conclude by letting  $\delta \searrow 0$ . □

We will now prove Lemma 3.8; before doing so we need to recall the definition of sub-Riemannian Hamiltonian and some preliminary results.

**Definition 3.10.** Let  $M = (M, \Delta, g)$  be an  $n$ -dimensional  $C^{1,1}$  sub-Riemannian manifold of rank  $m$  and assume that there exists a family of horizontal vector fields  $X_1, \dots, X_m$  of class  $C^{1,1}$  that form a global orthonormal frame of  $\Delta$ . The *sub-Riemannian Hamiltonian* is the function  $H : T^*M \rightarrow \mathbb{R}$  defined by  $H(q, \lambda) := \frac{1}{2} \sum_{i=1}^m \langle \lambda, X_i(q) \rangle^2$ . We can consider (in the canonical coordinates on  $T^*M$ ) the associated *Hamiltonian system*:

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial \lambda}(q, \lambda) \\ \dot{\lambda} = -\frac{\partial H}{\partial q}(q, \lambda). \end{cases} \quad (3.1)$$

If  $(q(t), \lambda(t))$  is a solution to (3.1), it is called *normal extremal* and  $q(t)$  *normal extremal trajectory*.

Observe that the  $C^{1,1}$  assumption on  $X_1, \dots, X_m$  provides the minimal regularity that guarantees existence and uniqueness of solutions to (3.1). We now state an important result about normal extremals; see [1, Theorem 4.25 and Corollary 4.27] for the proof.

**Theorem 3.11.** *Let  $M = (M, \Delta, g)$  be an  $n$ -dimensional  $C^{1,1}$  sub-Riemannian manifold of rank  $m$  and assume that there exists a family of horizontal vector fields  $X_1, \dots, X_m$  of class  $C^{1,1}$  that form a global orthonormal frame of  $\Delta$ . A curve  $(q, \lambda) : [a, b] \rightarrow T^*M$  is a normal extremal if and only if, for every  $i = 1, \dots, m$ ,  $h_i(t) = \langle \lambda(t), X_i(q(t)) \rangle$  for a.e.  $t \in [a, b]$ , where  $h = (h_1, \dots, h_m) \in L^\infty([a, b]; \mathbb{R}^m)$  is such that  $\dot{\gamma}(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t))$  for a.e.  $t \in [a, b]$ . In this case,  $|\dot{q}(t)|_{q(t)}$  is constant and it satisfies*

$$\frac{1}{2} |\dot{q}(t)|_{q(t)} = H(q(t), \lambda(t)) \quad \text{for every } t \in [a, b].$$

In particular,  $q(t)$  is arclength parametrized if and only if  $H(q(t), \lambda(t)) = \frac{1}{2}$ .

We are now ready to prove Lemma 3.8.

*Proof of Lemma 3.8.* For the sake of brevity, we will assume that there exists a family of horizontal vector fields  $X_1, \dots, X_m$  of class  $C^{1,1}$  that form a global orthonormal frame of  $\Delta$ ; this is not restrictive since all the arguments will be local. Up to fixing a chart  $U$  around  $p$ , we can assume that  $M = \mathbb{R}^n$  and  $p = 0$ . We can also suppose  $X_1 \equiv \partial_1$  on  $U$ ,  $X_2(0) = \partial_2, \dots, X_m(0) = \partial_m$ . Let  $(q(t), \lambda(t))$  be the solution of (3.1) with initial condition  $(q(0), \lambda(0)) = (0, e_1^*)$  (it is unique by assumption). In particular,  $q(t)$  is a normal extremal trajectory and it is not the constant curve  $(0, 0)$ . Moreover,  $H(q(0), \lambda(0)) = \frac{1}{2}$ , hence by Theorem 3.11  $q(t)$  is arclength parametrized, so that  $H(q(t), \lambda(t)) = \frac{1}{2}$ , that is,

$$\sum_{i=1}^m \langle \lambda(t), X_i(q(t)) \rangle^2 = 1.$$

Then, by Theorem 3.11 we have  $\dot{q}(0) = \partial_1$ , so that  $\langle e_1^*, \dot{q}(0) \rangle = 1$  and, if we set  $H' := \{0\} \times \mathbb{R}^{n-1}$ , we get  $\dot{q}(0) \notin T_0 H'$ . Observe that for a sufficiently small neighbourhood  $U' \subseteq H'$  of 0 we can find a (unique) non-vanishing  $C^{1,1}$ -function  $\tilde{\xi} : U' \rightarrow \text{span}\{e_1^*\} \subseteq T^*\mathbb{R}^n$  such that  $\tilde{\xi}(0) = e_1^*$  and  $H((0, x'), \tilde{\xi}(x')) = \frac{1}{2}$  for every  $(0, x') \in H'$ . Up to shrinking  $U'$ , we can denote by  $(Q(t, x'), \Lambda(t, x'))$  the solution at time  $t$  of (3.1) with initial condition  $(Q(0, x'), \Lambda(0, x')) = ((0, x'), \tilde{\xi}(x'))$ . Since

$$\dot{q}(0) = dQ_{(0,0)}[(1, 0)] \notin T_0 H' = dQ_{(0,0)}[\{0\} \times H'],$$

$dQ_{(0,0)}$  is invertible and, up to shrinking  $U'$ , there exists  $\varepsilon > 0$  such that  $Q|_{(-\varepsilon, \varepsilon) \times U'}$  is a diffeomorphism onto its image  $W \subseteq \mathbb{R}^n$ . Furthermore, for every  $x = Q(t, x') \in W$ , by Theorem 3.11 we have

$$H(x, \Lambda(t, x')) = H(Q(0, x'), \Lambda(0, x')) = H((0, x'), \tilde{\xi}(x')) = \frac{1}{2},$$

that is,

$$\sum_{i=1}^m \langle \Lambda(t, x'), X_i(x) \rangle^2 = 1. \quad (3.2)$$

Now, we want to show that  $\Lambda$  is a calibration that calibrates  $(-\varepsilon, \varepsilon) \ni t \mapsto Q(t, x')$  for every  $x' \in U'$ . Indeed, for every  $x = Q(t, x')$  and  $v = \sum_{i=1}^m h_i X_i(x) \in \Delta_x$ , we have

$$\langle \Lambda(t, x'), v \rangle = \sum_{i=1}^m h_i \langle \Lambda(t, x'), X_i(x) \rangle \leq \left( \sum_{i=1}^m h_i^2 \right)^{\frac{1}{2}} = |v|_x \quad (3.3)$$

thanks to (3.2) and the Cauchy-Schwarz inequality. In particular, if  $|v|_x = 1$ , the equality holds exactly when  $h_i = \langle \Lambda(t, x'), X_i(x) \rangle$  for every  $1 \leq i \leq m$ , i.e.,

$$v = \sum_{i=1}^m \langle \Lambda(t, x'), X_i(x) \rangle X_i(x) = \frac{\partial H}{\partial \lambda}(x, \Lambda(t, x')) = \frac{\partial H}{\partial \lambda}(Q(t, x'), \Lambda(t, x')) = \frac{\partial Q}{\partial t}(t, x').$$

It is well-known (see e.g. [120, Appendix C] or [142, Theorem 2.58]) that  $\Lambda$  is exact, and in fact that  $\Lambda = Q_*(dt)$  where  $Q_*$  denotes the pushforward by  $Q$ ; however, for the sake of completeness we include a proof of this fact. Define the vector field  $Y(x) := \frac{\partial Q}{\partial t}(Q^{-1}(x))$  for every  $x \in W$ ; notice that  $Y$  is unitary. Since  $(Q, \Lambda)$  solves (3.1), observe that

$$Y(x) = \sum_{i=1}^m \langle \Lambda(t, x'), X_i(x) \rangle X_i(x), \quad (3.4)$$

$$\begin{aligned} \frac{\partial \Lambda}{\partial t}(t, x') &= -\frac{\partial H}{\partial q}(x, \Lambda(t, x')) = -\sum_{i=1}^m \langle \Lambda(t, x'), X_i(x) \rangle \langle \Lambda(t, x'), dX_i(x) \rangle \\ &= -\langle \Lambda(t, x'), dY(x) \rangle + \sum_{i=1}^m d(\langle \Lambda(t, x'), X_i(x) \rangle) \langle \Lambda(t, x'), X_i(x) \rangle \\ &= -\langle \Lambda(t, x'), dY(x) \rangle + d(H(x, \Lambda(t, x'))) = -\langle \Lambda(t, x'), dY(x) \rangle. \end{aligned} \quad (3.5)$$

By (3.4) and (3.2), we obtain that  $\Lambda$  and  $Q_*(dt)$  coincide on  $dQ_{(t, x')}[(1, 0)] = Y(x)$ :

$$\langle \Lambda(t, x'), Y(x) \rangle = \sum_{i=1}^m \langle \Lambda(t, x'), X_i(x) \rangle^2 = 1 = \langle dt, (1, 0) \rangle = \langle Q_*(dt)(x), dQ_{(t, x')}[(1, 0)] \rangle.$$

Then, it suffices to show that, for every  $w \in \mathbb{R}^{n-1}$ ,  $\Lambda$  and  $Q_*(dt)$  agree on  $dQ_{(t, x')}[(0, w)]$ . Indeed,  $Q_*(dt)$  always vanishes on this vector, whereas  $\Lambda$  vanishes on it if  $t = 0$  (recall that  $\Lambda(0, x') = \xi(x')$  is a multiple of  $e_1^*$ ). But we have

$$\begin{aligned} \frac{d}{dt} \langle \Lambda(t, x'), dQ_{(t, x')}[(0, w)] \rangle &= \left\langle \frac{d}{dt} \Lambda(t, x'), dQ_{(t, x')}[(0, w)] \right\rangle + \left\langle \Lambda(t, x'), \frac{d}{dt} dQ_{(t, x')}[(0, w)] \right\rangle \\ &= -\langle \Lambda(t, x'), dY(x) \circ dQ_{(t, x')}[(0, w)] \rangle + \langle \Lambda(t, x'), dY(x) \circ dQ_{(t, x')}[(0, w)] \rangle = 0 \end{aligned}$$

thanks to (3.5) and the fact that  $Q$  is of class  $C^{1,1}$ . Hence,  $\Lambda$  has to identically vanish on  $dQ_{(t, x')}[(0, w)]$  too.  $\square$

## 3.2 Diameter of small sub-Riemannian balls: case $C^0$

**Definition 3.12.** An  $n$ -dimensional  $C^0$  sub-Riemannian manifold is a connected smooth manifold  $M$  of dimension  $n$  endowed with a family of continuous vector fields  $X_1, \dots, X_m$  such that, for every  $p \in M$ , there exists  $1 \leq i \leq m$  such that  $X_i(p) \neq 0$ . For  $p \in M$  we denote by  $\Delta_p := \text{span}\{X_1(p), \dots, X_m(p)\} \neq \{0\}$  the space of horizontal vectors at  $p$ .

**Definition 3.13.** Let  $M$  be an  $n$ -dimensional  $C^0$  sub-Riemannian manifold. and  $X_1, \dots, X_m$  its family of continuous vector fields. An absolutely continuous curve  $\gamma : [a, b] \rightarrow M$  is an *admissible curve* joining  $p$  and  $q$  if  $\gamma(a) = p$ ,  $\gamma(b) = q$  and there exists a measurable function  $h : [a, b] \rightarrow \mathbb{R}^m$  such that  $\dot{\gamma}(t) = \sum_{j=1}^m h_j(t)X_j(\gamma(t))$  for a.e.  $t \in [a, b]$ .

**Remark 3.14.** Since the vector fields  $X_1, \dots, X_m$  are not assumed to be linearly independent, the function  $h$  in Definition 3.13 is in general not unique. However, one can choose  $h$  so that it is measurable and, for a.e.  $t$ , where  $h(t)$  is the element of minimal norm in the affine space  $\{u \in \mathbb{R}^m : \dot{\gamma}(t) = \sum_{j=1}^m u_j(t)X_j(\gamma(t))\}$ ; see e.g. [1, Lemma 3.68]. We will write  $h_\gamma$  to denote the function  $h$  constructed in this way.

**Definition 3.15.** Let  $M$  be an  $n$ -dimensional  $C^0$  sub-Riemannian manifold. and  $X_1, \dots, X_m$  its family of continuous vector fields. The *length* of an admissible curve  $\gamma : [a, b] \rightarrow M$  is

$$L(\gamma) := \int_a^b |h_\gamma(t)| dt.$$

For every  $p, q \in M$ , the *Carnot-Carathéodory (CC) distance* is

$$d_C(p, q) := \inf\{L(\gamma) : \gamma \text{ is an admissible curve joining } p \text{ and } q\},$$

where we agree that  $\inf \emptyset := +\infty$ .

We are now ready to prove Theorem 3.2, that we restate for the reader's convenience.

**Theorem 3.16.** Let  $M$  be a  $C^0$  sub-Riemannian manifold. Then, for every  $p \in M$  and  $\varepsilon > 0$  there exist a neighbourhood (with respect to the manifold topology)  $V$  of  $p$  and  $r_{p,\varepsilon} > 0$  such that

$$2r(1 - \varepsilon) \leq \text{diam}(B_C(q, r)) \leq 2r \quad \text{for every } 0 < r < r_{p,\varepsilon} \text{ and } q \in V.$$

*Proof.* Clearly, by the triangle inequality we always have  $\text{diam}(B_C(q, r)) \leq 2r$ . For the other inequality, up to rearranging the vector fields we can assume that  $X_1(p) \neq 0$ . By continuity, there exists a neighbourhood (with respect to the manifold topology)  $U \subseteq M$  of  $p$  such that  $X_1 \neq 0$  on  $U$ .

Consider the surjective linear map  $A : \mathbb{R}^m \rightarrow \Delta_p$  defined by  $A(h) := \sum_{j=1}^m h_j X_j(p)$ . Let us write  $X_1(p) = A(\bar{h})$ , where  $\bar{h} \in \mathbb{R}^m$  is the element of minimal norm in the affine space

$A^{-1}(X_1(p))$ ; observe, in particular, that  $\bar{h}$  is orthogonal to  $\ker A$ . Let  $\lambda \in (\mathbb{R}^m)^*$  be defined by  $\langle \lambda, \bar{h} \rangle := |\bar{h}|$  and  $\lambda = 0$  on  $\bar{h}^\perp$ ; we define  $\lambda_p \in (\Delta_p)^*$  by

$$\langle \lambda_p, v \rangle := \langle \lambda, h \rangle \quad \text{whenever } v = A(h).$$

Observe that  $\lambda_p$  is well defined because  $\lambda = 0$  on  $\bar{h}^\perp \supseteq \ker A$ . We also observe that

$$\begin{aligned} |\langle \lambda_p, \sum_{j=1}^m h_j X_j(p) \rangle| &= |\langle \lambda, h \rangle| \leq |h| \quad \text{for every } h \in \mathbb{R}^m, \\ \langle \lambda_p, \sum_{j=1}^m \bar{h}_j X_j(p) \rangle &= \langle \lambda, \bar{h} \rangle = |\bar{h}|. \end{aligned}$$

Up to shrinking  $U$ , we can fix a smooth exact 1-form  $\omega$  on  $U$  such that  $\omega_p = \lambda_p$ ; by continuity (and up to shrinking  $U$  again) we find that

$$|\langle \omega_q, \sum_{j=1}^m h_j X_j(q) \rangle| \leq (1 + \varepsilon)|h| \quad \text{for every } h \in \mathbb{R}^m \text{ and } q \in U, \quad (3.6)$$

$$\left\langle \omega_q, \sum_{j=1}^m \frac{\bar{h}_j}{|\bar{h}|} X_j(q) \right\rangle \geq 1 - \varepsilon^2 \quad \text{for every } q \in U. \quad (3.7)$$

Now, consider an open neighbourhood  $V \subset\subset U$  of  $p$ ; since the Euclidean distance is locally controlled by above, up to a positive multiplicative constant, by the CC one, there exists  $r_{p,\varepsilon} > 0$  such that  $B_C(q, 2r_{p,\varepsilon}) \subseteq U$  for every  $q \in V$ . We claim that

$$\text{diam}(B_C(q, r)) \geq 2r(1 - \varepsilon) \quad \text{for every } r \in (0, r_{p,\varepsilon}) \text{ and } q \in V.$$

Indeed, for  $q \in V$  and  $r \in (0, r_{p,\varepsilon})$  consider a curve  $\gamma_0 : (-r, r) \rightarrow B_C(q, r) \subseteq U$  defined by  $\gamma_0(0) = q$  and  $\dot{\gamma}_0(t) = \sum_{j=1}^m \frac{\bar{h}_j}{|\bar{h}|} X_j(\gamma_0(t))$  for every  $t \in (-r, r)$ . Fix also  $\delta \in (0, r)$  and let  $q_1 := \gamma_0(-r + \delta)$ ,  $q_2 := \gamma_0(r - \delta)$ ; then,  $\gamma_0$  is an admissible curve joining  $q_1$  and  $q_2$  and  $q_1, q_2 \in B_C(q, r) \subseteq B_C(q, r_{p,\varepsilon})$ . Let  $\gamma : [a, b] \rightarrow M$  be another admissible curve joining  $q_1$  and  $q_2$ . If the support of  $\gamma$  is not contained in  $B_C(q, 2r_{p,\varepsilon})$ , then  $L(\gamma) \geq 2r_{p,\varepsilon}$ ; otherwise, the support of  $\gamma$  is contained in  $B_C(q, 2r_{p,\varepsilon}) \subseteq U$  and, since  $\omega$  is exact on  $U$ ,

$$\begin{aligned} L(\gamma) &= \int_a^b |h_\gamma(t)| dt \stackrel{(3.6)}{\geq} \frac{1}{1 + \varepsilon} \int_a^b \langle \omega_{\gamma(t)}, \dot{\gamma}(t) \rangle dt = \frac{1}{1 + \varepsilon} \int_\gamma \omega \\ &= \frac{1}{1 + \varepsilon} \int_{\gamma_0} \omega = \frac{1}{1 + \varepsilon} \int_{-r+\delta}^{r-\delta} \left\langle \omega_{\gamma_0(t)}, \sum_{j=1}^m \frac{\bar{h}_j}{|\bar{h}|} X_j(\gamma_0(t)) \right\rangle dt \stackrel{(3.7)}{\geq} 2(r - \delta)(1 - \varepsilon). \end{aligned}$$

In any case, we obtain

$$\text{diam}(B_C(q, r)) \geq d_C(q_1, q_2) \geq 2(r - \delta)(1 - \varepsilon)$$

and we conclude by letting  $\delta \searrow 0$ . □

# 4 SBV functions in Carnot-Carathéodory spaces

The aim of this chapter is to present the result contained in [57]. In Section 4.1 we introduce the space of special functions of bounded variation in Carnot-Carathéodory spaces ( $SBV_X$  functions) and we study some of its properties, which are summed up in the following theorem.

**Theorem 4.1.** *Let  $\Omega \subset \mathbb{R}^n$  be an open subset of an equiregular Carnot-Carathéodory space  $(\mathbb{R}^n, X)$ ; then, the following statements hold:*

- (i)  $u \in SBV_{X,\text{loc}}(\Omega)$  if and only if  $D_X^s u = f \nu_R \mathcal{H}_C^{Q-1} \llcorner R$  for some countably  $X$ -rectifiable set  $R \subset \Omega$  with horizontal normal  $\nu_R$  and some  $f \in L^1_{\text{loc}}(R, \mathcal{H}_C^{Q-1})$ ;
- (ii)  $SBV_X(\Omega)$  is a closed subspace of  $BV_X(\Omega)$ ;
- (iii) the space  $SBV_{\text{loc}}(\Omega)$  of special function of (Euclidean) locally bounded variation is contained in  $SBV_{X,\text{loc}}(\Omega)$ ;
- (iv) for every  $w \in L^1_{\text{loc}}(\Omega; \mathbb{R}^m)$  there exists  $u \in SBV_{X,\text{loc}}(\Omega)$  such that  $D_X^{\text{ap}} u = w$  a.e. in  $\Omega$ ;
- (v) for every countably  $X$ -rectifiable set  $R \subseteq \Omega$  oriented by  $\nu_R$ , every  $\theta \in L^1(\mathcal{H}_C^{Q-1} \llcorner R)$  and every  $\delta > 0$  there exists  $u \in SBV_X(\Omega)$  such that

$$D_X^j u \equiv \theta \nu_R \mathcal{H}_C^{Q-1} \llcorner R, \quad \|u\|_{L^1(\Omega)} < \delta, \quad \text{and} \quad |D_X u|(\Omega) \leq (2 + \delta) \|\theta\|_{L^1(\mathcal{H}_C^{Q-1} \llcorner R)}.$$

In Section 4.2 we prove the following approximation result that, as stated in the introduction, draws inspiration from its Euclidean analogous [53, Theorem A].

**Theorem 4.2.** *Let  $\Omega$  be an open subset of an equiregular Carnot-Carathéodory space  $(\mathbb{R}^n, X)$  and let  $u \in SBV_X(\Omega)$ . Then, there exists a sequence of functions  $(u_k)_{k \in \mathbb{N}} \subset SBV_X(\Omega)$  and of  $C^1_X$ -hypersurfaces  $(M_k)_{k \in \mathbb{N}} \subset \Omega$  such that, for every  $k \in \mathbb{N}$ ,  $\mathcal{J}_{u_k} \subseteq M_k \cap \mathcal{J}_u$ ,  $\mathcal{J}_{u_k}$  is compact, and*

$$\|u - u_k\|_{BV_X(\Omega)} \xrightarrow{k \rightarrow +\infty} 0, \quad u_k \in C^\infty(\Omega \setminus \mathcal{J}_{u_k}).$$

## 4.1 Special functions of bounded $X$ -variation

In the rest of this chapter we will use the same notations of Chapter 1; in particular  $\Omega \subset \mathbb{R}^n$  is a fixed open set of an equiregular Carnot-Carathéodory space  $(\mathbb{R}^n, X)$ .

**Definition 4.3.** Let  $u \in \text{BV}_X(\Omega)$ . We say that  $u$  is a *special function of bounded  $X$ -variation*, and we write  $u \in \text{SBV}_X(\Omega)$ , if

- (i)  $D_X^c u = 0$ ,
- (ii)  $\mathcal{J}_u$  is a countably  $X$ -rectifiable set.

If  $u$  is in  $\text{BV}_{X,\text{loc}}(\Omega)$  only, we say that  $u$  is a *special function of locally bounded  $X$ -variation*, and we write  $u \in \text{SBV}_{X,\text{loc}}(\Omega)$ .

As explained in Remark 1.18, when  $(\mathbb{R}^n, X)$  satisfies property  $\mathcal{R}$  then condition (ii) in Definition 4.3 is always automatically satisfied by any  $u \in \text{BV}_{X,\text{loc}}(\Omega)$ .

In the current section the results [157, Theorems 1.3, 1.4, 1.5, 1.6] will be crucial: we make the following key observation that will allow us to use the aforementioned theorems.

**Remark 4.4.** The metric balls in CC spaces, in general, are not  $X$ -Lipschitz domains (see [157, Definition 1.1]) since their boundaries contain characteristic points. However, a  $C_X^1$ -hypersurface  $S$  locally separates an open set into two  $X$ -regular domains (see [157, Section 2.4]) in the following sense: for each  $p \in S$ , there exists  $r > 0$  and  $f \in C_X^1(B_C(p, r))$  such that  $S \cap B_C(p, r) = \{f = 0\}$  with  $Xf \neq 0$  on  $B_C(p, r)$ , thus  $B^+ := B_C(p, r) \cap \{f > 0\}$  and  $B^- := B_C(p, r) \cap \{f < 0\}$  satisfy all the properties of  $X$ -regular domains if we consider the boundary in the relative topology with respect to  $B_C(p, r)$ .

We now provide an equivalent definition for special functions of bounded  $X$ -variation that will be useful in the sequel.

**Proposition 4.5.** *The following statements are equivalent:*

- (i)  $u \in \text{SBV}_{X,\text{loc}}(\Omega)$ ;
- (ii)  $u \in \text{BV}_{X,\text{loc}}(\Omega)$  and there exist a countably  $X$ -rectifiable set  $R \subset \Omega$  and a function<sup>1</sup>  $f: R \rightarrow \mathbb{R}$  such that

$$D_X^s u = f \nu_R \mathcal{H}_C^{Q-1} \llcorner R,$$

where  $\nu_R$  denotes the horizontal normal to  $R$ .

Moreover, the jump set  $\mathcal{J}_u$  coincides with  $R_0 := \{p \in R : f(p) \neq 0\}$  up to  $\mathcal{H}_C^{Q-1}$ -negligible sets.

<sup>1</sup>Observe that necessarily  $f \in L_{\text{loc}}^1(R, \mathcal{H}_C^{Q-1})$ , for otherwise  $u \notin \text{BV}_{X,\text{loc}}(\Omega)$ .

*Proof.* Assume (i); then, the jump set  $\mathcal{J}_u$  can be covered, up to a  $\mathcal{H}_C^{Q-1}$ -negligible set, by countably many  $C_X^1$ -hypersurfaces  $(S_j)_{j \in \mathbb{N}}$ , that we may assume to be pairwise disjoint. Since  $|D_X u|(S_j \setminus \mathcal{J}_u) = 0$  for every  $j$ , we obtain

$$D_X^s u = D_X^j u = D_X u \llcorner \mathcal{J}_u = \sum_{j \in \mathbb{N}} D_X u \llcorner (\mathcal{J}_u \cap S_j).$$

Observe that, locally, each hypersurface  $S_j$  separates the space  $\mathbb{R}^n$  into two  $X$ -regular open sets (see [157]): denoting by  $P_j^X$  the measure on  $S_j$  defined locally as the  $X$ -perimeter measure of (each of) these two components, using [157, Theorems 1.4 and 1.6] (see also [66, Proposition 3.7]) one finds

$$D_X^s u = \sum_{j \in \mathbb{N}} (u^+ - u^-) \nu_{S_j} P_j^X \llcorner \mathcal{J}_u,$$

where  $u^\pm$  are the *traces* (see [157]) of  $u$  on  $S_j$ . By [3, Theorem 4.2] we further obtain

$$D_X^s u = \sum_{j \in \mathbb{N}} (u^+ - u^-) \lambda \nu_{S_j} \mathcal{H}_C^{Q-1} \llcorner (\mathcal{J}_u \cap S_j)$$

for a suitable  $\lambda: \cup_j S_j \rightarrow (0, +\infty)$ . Up to changing the sign of  $\nu_{\mathcal{J}_u}$ , we can write  $\nu_{\mathcal{J}_u} = \nu_{S_j}$   $\mathcal{H}_C^{Q-1}$ -a.e. on  $\mathcal{J}_u \cap S_j$ , hence concluding that

$$D_X^s u = \sum_{j \in \mathbb{N}} (u^+ - u^-) \lambda \nu_{\mathcal{J}_u} \mathcal{H}_C^{Q-1} \llcorner (\mathcal{J}_u \cap S_j),$$

which proves (ii) with  $R := \mathcal{J}_u$  and  $f := (u^+ - u^-) \lambda$ .

Concerning the opposite implication, assume (ii); clearly, it is not restrictive to assume that  $R = R_0$ . Cover  $R$ , up to a  $\mathcal{H}_C^{Q-1}$ -negligible set, by countably many  $C_X^1$ -hypersurfaces  $(S_j)_{j \in \mathbb{N}}$ , that we may assume to be pairwise disjoint. Using again [157, Theorems 1.4 and 1.6] (see also [66, Proposition 3.7]) and [3, Theorem 4.2], for every fixed  $j$  we have

$$D_X^s u \llcorner S_j = (u^+ - u^-) \lambda \nu_{S_j} \mathcal{H}_C^{Q-1} \llcorner S_j,$$

where again  $u^\pm$  are the traces of  $u$  on  $S_j$  and  $\lambda: S_j \rightarrow (0, +\infty)$ . On the other hand, by assumption, we have

$$D_X^s u \llcorner S_j = f \nu_R \mathcal{H}_C^{Q-1} \llcorner (R \cap S_j),$$

which implies that (up to a change of sign for  $\nu_R$  and  $f$ )

$$\begin{aligned} u^+ - u^- &= 0 \text{ } \mathcal{H}_C^{Q-1}\text{-a.e. on } S_j \setminus R, \\ (u^+ - u^-) \lambda &= f \neq 0 \text{ and } \nu_{S_j} = \nu_R \text{ } \mathcal{H}_C^{Q-1}\text{-a.e. on } S_j \setminus R. \end{aligned}$$

Therefore,  $\mathcal{H}_C^{Q-1}$ -a.e. point of  $R \cap S_j$  is an approximate  $X$ -jump point and, in particular,  $\mathcal{H}_C^{Q-1}(R \setminus \mathcal{J}_u) = 0$ . The proof will be accomplished if we show that  $\mathcal{H}_C^{Q-1}(\mathcal{J}_u \setminus R) = 0$ ; if, instead,  $\mathcal{H}_C^{Q-1}(\mathcal{J}_u \setminus R) > 0$ , then by [66, Theorem 1.3 (i)] we would obtain

$$|D_X^s u|(\mathcal{J}_u \setminus R) = |D_X u|(\mathcal{J}_u \setminus R) > 0,$$

which clearly contradicts assumption (ii). This concludes the proof.  $\square$

We observe in passing that Proposition 4.5, together with the fact that the horizontal normals of two  $X$ -rectifiable sets  $R_1, R_2$  coincide (up to a sign)  $\mathcal{H}_C^{Q-1}$ -a.e. on  $R_1 \cap R_2$  (see [66, Proposition 2.18]), implies that the space  $SBV_X$  is closed with respect to the usual sum.

The following theorem provides our first main result about  $SBV_X$  functions.

**Theorem 4.6.** *The subspace  $SBV_X(\Omega)$  is closed in  $BV_X(\Omega)$ .*

*Proof.* If  $I$  is finite or countable,  $u_i \in SBV_X(\Omega)$  for any  $i \in I$  and  $\sum_{i \in I} u_i$  converges to  $u \in BV_X(\Omega)$  in the  $BV_X$  norm, then  $D_X u = \sum_{i \in I} D_X u_i$ . Since  $\sum_i D_X^a u_i$  is absolutely continuous with respect to  $\mathcal{L}^n$  and  $\sum_i D_X^s u_i$  is singular, we have

$$D_X^a u = \sum_{i \in I} D_X^a u_i, \quad D_X^s u = \sum_{i \in I} D_X^s u_i.$$

Proposition 4.5 implies that  $u \in SBV_X(\Omega)$ , hence  $SBV_X(\Omega)$  is closed in  $BV_X(\Omega)$ .  $\square$

In the following lemma we denote by  $Du = (D_1 u, \dots, D_n u)$  the derivatives of  $u$  in the sense of distribution; moreover, when  $\mu = (\mu_1, \dots, \mu_n)$  is a vector-valued Radon measure and  $X(x) = (a_1(x), \dots, a_n(x))$  is a smooth vector field, we denote by  $\langle \mu, X \rangle$  the (scalar) Radon measure  $\sum_{t=1}^n a_t \mu_t$ .

**Lemma 4.7.** *The following statements hold:*

- (i)  $BV_{\text{loc}}(\Omega) \subseteq BV_{X,\text{loc}}(\Omega)$  and  $D_{X_i} u = \langle Du, X_i \rangle$  for every  $i = 1, \dots, m$ .
- (ii)  $SBV_{\text{loc}}(\Omega) \subseteq SBV_{X,\text{loc}}(\Omega)$ .

*Proof.* (i) Let  $u \in BV_{\text{loc}}(\Omega)$ . For every open set  $A \subset\subset \Omega$ , for every  $1 \leq i \leq m$  and for every  $\varphi \in C_c^1(A)$  we have

$$\int_A u X_i^* \varphi d\mathcal{L}^n = \sum_{t=1}^n \int_A u \frac{\partial(a_{i,t} \varphi)}{\partial x_t} d\mathcal{L}^n = - \int_A \varphi d \left( \sum_{t=1}^n (a_{i,t} D_t u) \right),$$

as claimed.

(ii) For every  $u \in SBV_{\text{loc}}(\Omega)$  we write  $Du = D_E^{\text{ap}} u \mathcal{L}^n + (u_E^+ - u_E^-) \nu^E \mathcal{H}_E^{n-1} \llcorner \mathcal{J}_u^E$ , where  $D_E^{\text{ap}} u$  denotes the (Euclidean) approximate gradient of  $u$ ,  $\mathcal{J}_u^E$  is the (Euclidean) jump set of  $u$  oriented by its (Euclidean) unit normal  $\nu^E = (\nu_1^E, \dots, \nu_n^E)$  and  $u_E^+$  and  $u_E^-$  are the (Euclidean) traces of  $u$  on  $\mathcal{J}_u^E$ . By statement (i) we know that  $u \in BV_{X,\text{loc}}(\Omega)$  and

$$D_{X_i} u = \langle D_E^{\text{ap}} u, X_i \rangle \mathcal{L}^n + (u_E^+ - u_E^-) \langle \nu^E, X_i \rangle \mathcal{H}_E^{n-1} \llcorner \mathcal{J}_u^E.$$

We know that  $\mathcal{J}_u^E$  is countably rectifiable in the Euclidean sense, hence, up to modifying  $\mathcal{J}_u^E$  on a  $\mathcal{H}_E^{n-1}$ -negligible set, there exists a countable collection of  $C^1$ -hypersurfaces  $(S_j)_{j \in \mathbb{N}}$  such that

$$\mathcal{J}_u^E \subset \bigcup_{j \in \mathbb{N}} S_j.$$

Without loss of generality we can assume that the  $C^1$ -hypersurfaces  $(S_j)_{j \in \mathbb{N}}$  are pairwise disjoint; in this way  $\nu^E$  coincides with the Euclidean unit normal  $\nu_{S_j}^E$  to  $S_j$  on  $\mathcal{J}_u^E \cap S_j$ . For every  $j \in \mathbb{N}$  we introduce the characteristic set  $S_j^{\text{ch}} \subset S_j$  as

$$S_j^{\text{ch}} := \{p \in S_j : \text{span}(X_1(p), \dots, X_m(p)) \subseteq T_p S_j\}.$$

For  $p \in \mathcal{J}_u^E$  let  $k \in \mathbb{N}$  be such that  $p \in S_k$ ; if  $p \in S_k^{\text{ch}}$ , then  $\langle X_i(p), \nu^E(p) \rangle = \langle X_i(p), \nu_{S_k}^E(p) \rangle = 0$  for every  $i \in \{1, \dots, m\}$ , hence we can rewrite  $D_{X_i} u$  as

$$D_{X_i} u = \langle X_i, D_E^{\text{ap}} u \rangle \mathcal{L}^n + (u_E^+ - u_E^-) \langle X_i, \nu^E \rangle \mathcal{H}_E^{n-1} \llcorner R, \quad (4.1)$$

where

$$R := \mathcal{J}_u^E \setminus \bigcup_{j \in \mathbb{N}} S_j^{\text{ch}}.$$

Upon possibly replacing  $\mathcal{J}_u^E$  with  $\mathcal{J}_u^E \cap (\bigcup_{j \in \mathbb{N}} S_j)$ , we observe that the set  $R$  is countably  $X$ -rectifiable because  $R \subset \bigcup_{j \in \mathbb{N}} (S_j \setminus S_j^{\text{ch}})$  and each  $S_j \setminus S_j^{\text{ch}}$  is a  $C_X^1$ -hypersurface. For  $p \in R$ , let  $k \in \mathbb{N}$  be the unique integer such that  $p \in S_k \setminus S_k^{\text{ch}}$  and let  $f$  be a  $C^1$  defining function for  $S_k$  in a neighborhood of  $p$ ; then

$$(\nu_R)_i(p) = \frac{X_i f(p)}{|Xf(p)|} = \frac{\langle X_i(p), \nabla f(p) \rangle}{|Xf(p)|} = \frac{\langle X_i(p), \nu^E(p) \rangle}{|Xf(p)|} |\nabla f(p)|, \quad \forall i = 1, \dots, m,$$

i.e.,

$$\langle X_i, \nu^E \rangle = \sigma_1 (\nu_R)_i, \quad \forall i = 1, \dots, m \quad (4.2)$$

for a suitable function  $\sigma_1: R \rightarrow (0, +\infty)$ . As in the proof of Proposition 4.5 we observe that, locally, each  $C^1$ - and  $C_X^1$ -hypersurface  $S_j \setminus S_j^{\text{ch}}$  separates the space into two connected open components of locally finite  $X$ -perimeter; combining [157, Propositions 4.1 and 4.5, Remark 4.7 and Corollary 4.14] (see also [63]) it can be shown that these  $X$ -perimeter measures have an integral representation with respect to both  $\mathcal{H}_C^{\text{Q-1}} \llcorner (S_j \setminus S_j^{\text{ch}})$  and  $\mathcal{H}_E^{n-1} \llcorner (S_j \setminus S_j^{\text{ch}})$ . Ultimately, this gives

$$\mathcal{H}_E^{n-1} \llcorner R = \sigma_2 \mathcal{H}_C^{\text{Q-1}} \llcorner R \quad (4.3)$$

for a suitable  $\sigma_2: R \rightarrow (0, +\infty)$ . Combining (4.1), (4.2) and (4.3) we obtain

$$D_{X_i} u = \langle X_i, D_E^{\text{ap}} u \rangle \mathcal{L}^n + (u_E^+ - u_E^-) \sigma_1 \sigma_2 (\nu_R)_i \mathcal{H}_C^{\text{Q-1}} \llcorner R,$$

i.e.,

$$D_X^s u = (u_E^+ - u_E^-) \sigma_1 \sigma_2 \nu_R \mathcal{H}_C^{\text{Q-1}} \llcorner R.$$

Proposition 4.5 implies that  $u \in \text{SBV}_{X, \text{loc}}(\Omega)$ , as claimed.  $\square$

**Remark 4.8.** A deeper inspection of the proof of Lemma 4.7 (ii) and, in particular, of the results from [157] that were used reveals that, if  $u \in \text{SBV}_{\text{loc}}(\Omega)$  and  $R$  is as in the proof, then the traces  $u^\pm$  equal the Euclidean ones  $u_E^\pm$ .

The following result is an easy consequence of the celebrated Lusin-type theorem for gradients by G. Alberti [2].

**Theorem 4.9.** *For every  $w \in L^1_{\text{loc}}(\Omega; \mathbb{R}^m)$  there exists  $u \in \text{SBV}_{X, \text{loc}}(\Omega)$  such that  $D_X^{\text{ap}} u = w$   $\mathcal{L}^n$ -a.e. on  $\Omega$ .*

*Moreover, if  $\Omega$  is bounded, then there exists  $C = C(\Omega) > 0$  such that, for every  $w \in L^1(\Omega; \mathbb{R}^m)$ , the function  $u$  can be chosen in such a way that  $|D_X u|(\Omega) \leq C \|w\|_{L^1(\Omega)}$ .*

*Proof.* Consider the horizontal vector field  $X_w := w_1 X_1 + \dots + w_m X_m$ . Consider a sequence of open sets  $(\Omega_i)_{i \in \mathbb{N}}$  such that, for every  $i \in \mathbb{N}$ ,  $\Omega_i \subset\subset \Omega_{i+1}$ ,  $\Omega_i \subseteq \Omega$  and  $(\Omega_i)_{i \in \mathbb{N}}$  invades  $\Omega$  when  $i \rightarrow +\infty$ . Clearly,  $X_w \in L^1(\Omega_i; \mathbb{R}^n)$  for every  $i \in \mathbb{N}$ . By [2, Theorem 3] there exists  $u_i \in \text{SBV}(\Omega_i)$  whose (Euclidean) approximate gradient is  $X_w|_{\Omega_i}$ . For every  $i \in \mathbb{N}$  define the sets

$$U_i := \begin{cases} \Omega_1 & \text{if } i = 1 \\ \Omega_{i+1} \setminus \Omega_i & \text{if } i > 1. \end{cases}$$

The function  $u$  defined as  $u_i$  on  $U_i$  belongs to  $\text{SBV}_{\text{loc}}(\Omega)$  and its (Euclidean) approximate gradient is  $X_w$ . The latter, together with Lemma 4.7, proves the first part of the statement. The second part is a consequence of the estimate stated in [2, Theorem 3].  $\square$

We conclude with a result, Theorem 4.11, where we provide a recipe to produce lots of  $\text{SBV}_X$  functions: in fact, *any*  $L^1$  function on *any* countably  $X$ -rectifiable set can appear as the jump part of an  $\text{SBV}_X$  function.

**Lemma 4.10.** *Let  $S \subseteq \Omega$  be a  $C^1_X$ -hypersurface oriented by a normal  $\nu$ , let  $\theta \in L^1(\mathcal{H}_C^{Q-1} \llcorner S)$  and  $\delta > 0$ . Then there exists  $u \in \text{SBV}_X(\Omega)$  such that*

$$D_X^j u \equiv \theta \nu \mathcal{H}_C^{Q-1} \llcorner S, \quad \|u\|_{L^1(\Omega)} < \delta, \quad \text{and} \quad |D_X u|(\Omega) \leq (2 + \delta) \|\theta\|_{L^1(\mathcal{H}_C^{Q-1} \llcorner S)}.$$

*Proof.* Fix a countable family  $(B_j)_{j \in \mathbb{N}}$  of balls, contained in  $\Omega$  and with centers on  $S$ , and functions  $f_j \in C^1_X(B_j)$  such that, for every  $j \in \mathbb{N}$ ,

$$S \cap B_j = \{q \in B_j : f_j(q) = 0\}, \quad X f_j \neq 0 \text{ on } B_j, \quad S \subset \bigcup_{j \in \mathbb{N}} B_j.$$

We can also assume that  $\langle X f_j, \nu \rangle > 0$  on  $S \cap B_j$ . Without loss of generality, we can assume that each ball  $B_j$  intersects only a finite number of other balls of the collection. Now consider a partition of the unity associated with  $(B_j)_{j \in \mathbb{N}}$ , i.e., a collection of functions  $(\zeta_j)_{j \in \mathbb{N}}$  such that, for every  $j \in \mathbb{N}$ ,

$$\zeta_j \in C_c^\infty(B_j), \quad 0 \leq \zeta_j \leq 1, \quad \text{and} \quad \sum_{j \in \mathbb{N}} \zeta_j \equiv 1 \text{ on } S.$$

Fix  $j \in \mathbb{N}$ , we define

$$B_j^+ := \{q \in B_j : f_j(q) > 0\}, \quad B_j^- := \{q \in B_j : f_j(q) < 0\}. \quad (4.4)$$

For every  $j$ , let  $\theta_j := \theta/\sigma_j \in L^1(S \cap B_j)$ , where  $\sigma_j$  is a function on  $S \cap B_j$  with  $\inf \sigma_j > 0$  that will be introduced later. Using [157, Theorem 1.5] we can find  $\tilde{u}_j \in C^\infty(B_j^+) \cap W_X^{1,1}(B_j^+)^2$  such that  $\text{spt } \tilde{u}_j \subset\subset B_j$  and

$$\|\tilde{u}_j\|_{L^1(B_j^+)} \leq \frac{\delta}{2j}, \quad \|X\tilde{u}_j\|_{L^1(B_j^+)} \leq \left(1 + \frac{\delta}{2j}\right) \|\zeta_j\theta_j\|_{L^1(P_X^{B_j^+} \llcorner S)},$$

and, for  $\mathcal{H}_C^{\mathcal{Q}-1}$ -almost every  $q \in S \cap B_j$ , we have

$$\lim_{r \rightarrow 0} \int_{B_v^+(q,r)} |\tilde{u}_j - \zeta_j\theta_j| d\mathcal{L}^n = 0.$$

where  $B_v^+(q,r)$  is defined as in Definition 1.16. We define  $u_j$  on  $\Omega$  as

$$u_j := \begin{cases} \tilde{u}_j & \text{on } B_j^+ \\ 0 & \text{on } \mathbb{R}^n \setminus B_j^+. \end{cases}$$

By [157, Theorem 5.3 and Theorem 1.3],  $u_j \in \text{BV}_X(\Omega)$  and, by using the representation of the  $X$ -perimeter measure and the coarea formula for  $\text{BV}_X$  functions (see [3, 78]) we can find a constant  $C_j > 1$  and a  $\mathcal{H}_C^{\mathcal{Q}-1}$ -measurable function  $\sigma_j : S \cap B_j \rightarrow [1/C_j, C_j]$  such that

$$\begin{aligned} D_X u_j &= (X\tilde{u}_j)\mathcal{L}^n \llcorner B_j^+ + \zeta_j\theta_j\sigma_j\nu\mathcal{H}_C^{\mathcal{Q}-1} \llcorner (B_j \cap S) \\ &= (X\tilde{u}_j)\mathcal{L}^n \llcorner B_j^+ + \zeta_j\theta_j\nu\mathcal{H}_C^{\mathcal{Q}-1} \llcorner (B_j \cap S), \end{aligned}$$

the latter implying that  $u_j \in \text{SBV}_X(\Omega)$ . The function  $u := \sum_{j \in \mathbb{N}} u_j$  satisfies the statement of the Lemma: clearly  $u \in \text{SBV}_X(\Omega)$ ,  $D_X^j u \equiv \theta\nu\mathcal{H}_C^{\mathcal{Q}-1} \llcorner S$  and  $\|u\|_{L^1(\Omega)} < \delta$ . Now let us prove that the estimate on  $|D_X u|(\Omega)$ . We observe

$$\begin{aligned} |D_X u|(\Omega) &\leq \sum_{j \in \mathbb{N}} |D_X u_j|(\Omega) \leq \sum_{j \in \mathbb{N}} \left( \|X\tilde{u}_j\|_{L^1(B_j^+)} + \|\zeta_j\theta\|_{L^1(\mathcal{H}_C^{\mathcal{Q}-1} \llcorner S \cap B_j)} \right) \\ &\leq \sum_{j \in \mathbb{N}} \left( \left(1 + \frac{\delta}{2j}\right) \|\zeta_j\theta_j\|_{L^1(P_X^{B_j^+} \llcorner S)} + \|\zeta_j\theta\|_{L^1(\mathcal{H}_C^{\mathcal{Q}-1} \llcorner S \cap B_j)} \right). \end{aligned}$$

Again, by the representation of the  $X$ -perimeter measure, we have

$$\|\zeta_j\theta_j\|_{L^1(P_X^{B_j^+} \llcorner S)} = \|\zeta_j\theta\|_{L^1(\mathcal{H}_C^{\mathcal{Q}-1} \llcorner S \cap B_j)},$$

the latter implying that

$$|D_X u|(\Omega) \leq \sum_{j \in \mathbb{N}} \left(2 + \frac{\delta}{2j}\right) \|\zeta_j\theta\|_{L^1(\mathcal{H}_C^{\mathcal{Q}-1} \llcorner S \cap B_j)} \leq (2 + \delta) \|\theta\|_{L^1(\mathcal{H}_C^{\mathcal{Q}-1} \llcorner S)},$$

concluding the proof. □

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<sup>2</sup>Remember that  $W_X^{1,1}$  is the space of functions  $u$  such that both  $u$  and  $Xu$  belong to  $L^1$ .

**Theorem 4.11.** *Let  $S \subseteq \Omega$  be a countably  $X$ -rectifiable set oriented by  $\nu$ ; let  $\theta \in L^1(\mathcal{H}_C^{Q-1} \llcorner S)$  and  $\delta > 0$  be fixed. Then there exists  $u \in \text{SBV}_X(\Omega)$  such that*

$$D_X^j u \equiv \theta \nu \mathcal{H}_C^{Q-1} \llcorner S, \quad \|u\|_{L^1(\Omega)} < \delta, \quad \text{and} \quad |D_X u|(\Omega) \leq (2 + \delta) \|\theta\|_{L^1(\mathcal{H}_C^{Q-1} \llcorner S)}.$$

*Proof.* Since  $S$  is countably  $X$ -rectifiable there exists a countable collection  $(S_i)_{i \in \mathbb{N}}$  of  $C_X^1$ -hypersurfaces such that

$$\mathcal{H}_C^{Q-1} \left( S \setminus \bigcup_{i \in \mathbb{N}} S_i \right) = 0.$$

Without loss of generality we can assume that the  $C_X^1$ -hypersurfaces  $(S_i)_{i \in \mathbb{N}}$  are pairwise disjoint. We extend  $\theta$  to 0 outside  $S$  and, for every  $i \in \mathbb{N}$ , we define

$$\theta_i := \theta|_{S_i}.$$

For every  $i \in \mathbb{N}$  we use Lemma 4.10 to obtain a function  $u_i \in \text{SBV}_{X,\text{loc}}(\Omega)$  such that

$$D_X^j u_i \equiv \theta_i \nu \mathcal{H}_C^{Q-1} \llcorner S_i, \quad \|u_i\|_{L^1(\Omega)} < \frac{\delta}{2^i}, \quad \text{and} \quad |D_X u_i|(\Omega) \leq \left(2 + \frac{\delta}{2^i}\right) \|\theta_i\|_{L^1(\mathcal{H}_C^{Q-1} \llcorner S_i)}.$$

The function  $u := \sum_{i \in \mathbb{N}} u_i$  satisfies the statement of the Theorem.  $\square$

## 4.2 Approximation of $\text{SBV}_X$ functions

### 4.2.1 Construction of the approximating sequence

This subsection is devoted to the construction of the approximating sequence  $(u_k)_{k \in \mathbb{N}}$  of  $\text{SBV}_X$  functions that will be used to prove Theorem 4.2. In the following Construction 4.12 we start by proving that, if  $u \in \text{BV}_X(\Omega)$  and  $\mathcal{J}_u$  is countably  $X$ -rectifiable, then it is possible to approximate  $\mathcal{J}_u$  with a  $X$ -rectifiable set that can be in turn approximated with a  $C_X^1$ -hypersurface. We underline that the following construction is valid, as a particular case, for functions  $u \in \text{SBV}_X(\Omega)$ .

**Construction 4.12.** Fix  $u \in \text{BV}_X(\Omega)$  with a countably  $X$ -rectifiable  $\mathcal{J}_u$ . For every  $\eta > 0$  we define the set

$$\mathcal{J}_{u,\eta} := \left\{ x \in \mathcal{J}_u : |u^+(x) - u^-(x)| \geq \frac{1}{\eta} \right\} \cap B(0, \eta). \quad (4.5)$$

Since the jump set  $\mathcal{J}_u$  is countably  $X$ -rectifiable, also  $\mathcal{J}_{u,\eta}$  is countably  $X$ -rectifiable for every  $\eta > 0$ . Let us moreover observe that the set  $\mathcal{J}_{u,\eta}$  is  $X$ -rectifiable, i.e., that  $\mathcal{H}_C^{Q-1}(\mathcal{J}_{u,\eta}) < \infty$ . In fact, thanks to Theorem 1.12, there exists a positive constant  $C > 0$ , only depending on  $\eta$  such that

$$\mathcal{H}_C^{Q-1}(\mathcal{J}_{u,\eta}) \leq C\eta |D_X u|(\Omega).$$

Since the family  $\mathcal{J}_{u,\eta}$  is increasing and invades  $\mathcal{J}_u$  when  $\eta \rightarrow +\infty$  we also have

$$|D_X u|(\mathcal{J}_u \setminus \mathcal{J}_{u,\eta}) \xrightarrow{\eta \rightarrow +\infty} 0$$

so that, for every  $k \in \mathbb{N}$ , we can choose an  $\eta_k > 0$  such that  $(\eta_k)_{k \in \mathbb{N}}$  is increasing and

$$|D_X u|(\mathcal{J}_u \setminus \mathcal{J}_{u, \eta_k}) < \frac{1}{k}. \quad (4.6)$$

For the sake of brevity let us write  $\mathcal{J}_u^k := \mathcal{J}_{u, \eta_k}$ . Now, for every  $\delta > 0$ , using Lemma 1.15, we can find a  $C_X^1$ -hypersurface  $M_\delta$  such that

$$\mathcal{H}_C^{Q-1}(\mathcal{J}_u^k \setminus M_\delta) < \delta.$$

By Theorem 1.12 one has  $|D_X u| \llcorner \mathcal{J}_u \ll \mathcal{H}_C^{Q-1} \llcorner \mathcal{J}_u$ , so for every  $k \in \mathbb{N}$  we can find a  $C_X^1$ -hypersurface  $M_k$  such that

$$|D_X u|(\mathcal{J}_u^k \setminus M_k) < \frac{1}{k}. \quad (4.7)$$

Before starting the construction of the approximating sequence we need the following Lemma.

**Lemma 4.13.** *Let  $u \in \text{BV}_X(\Omega)$  be such that  $\mathcal{J}_u$  is countably  $X$ -rectifiable and consider the function*

$$\begin{aligned} j_u : \mathcal{J}_u &\rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{m-1} \\ x &\rightarrow (u^+(x), u^-(x), \nu_{\mathcal{J}_u}(x)). \end{aligned}$$

Then for every  $k \in \mathbb{N}$  there exist a compact set  $C_k \subseteq \mathcal{J}_u^k \cap M_k$  (where  $\mathcal{J}_u^k$  and  $M_k$  are defined as in Construction 4.12) and a representative of  $j_u$  such that  $j_u|_{C_k}$  is continuous and

$$|D_X u|((\mathcal{J}_u^k \cap M_k) \setminus C_k) < \frac{1}{k}. \quad (4.8)$$

*Proof.* By [66, Proposition 2.28] we can choose a Borel representative of  $j_u$  so it suffices to use Lusin's Theorem [73, Theorem 2.3.5].  $\square$

Using the previous Lemma we can construct the required approximating sequence. Recall that we want to obtain a sequence of functions  $(u_k)_{k \in \mathbb{N}}$  such that  $u_k \in \text{SBV}_X(\Omega)$ ,  $u_k \in C^\infty(\Omega \setminus \mathcal{J}_{u_k})$  and  $u_k$  converges to  $u$  in the  $\text{BV}_X$  norm. Fix a representative of  $u \in \text{SBV}_X(\Omega)$  and  $k \in \mathbb{N}$  and consider the compact set  $C_k$  given by Lemma 4.13. For  $\ell \in \mathbb{N}$  we define the sets

$$\begin{aligned} A_k^1 &:= \left\{ x \in \Omega : d_E(x, C_k) > \frac{1}{2} \right\}, \\ A_k^\ell &:= \left\{ x \in \Omega : \frac{1}{\ell+1} < d_E(x, C_k) < \frac{1}{\ell-1} \right\} \text{ if } \ell > 1. \end{aligned}$$

We observe that

$$\bigcup_{\ell \in \mathbb{N}} A_k^\ell = \Omega \setminus C_k$$

and that, for every  $\bar{\ell} \in \mathbb{N}$ ,  $A_k^{\bar{\ell}}$  intersects at most two of the sets of the family  $(A_k^i)_i$ , namely  $A_k^{\bar{\ell}+1}$  and<sup>3</sup>  $A_k^{\bar{\ell}-1}$ . Now for  $s \in \mathbb{N}$  we define the bounded open sets

$$\begin{aligned} A_k^{\ell,1} &:= A_k^\ell \cap \{|x|_{\mathbb{R}^n} < 2\}, \\ A_k^{\ell,s} &:= A_k^\ell \cap \{s-1 < |x|_{\mathbb{R}^n} < s+1\} \text{ if } s > 1. \end{aligned}$$

We observe that

$$\bigcup_{s \in \mathbb{N}} A_k^{\ell,s} = A_k^\ell$$

and that, for every  $\bar{s} \in \mathbb{N}$ ,  $A_k^{\ell,\bar{s}}$  intersects at most two of the sets of the family  $(A_k^{\ell,i})_i$ , namely  $A_k^{\ell,\bar{s}+1}$  and<sup>4</sup>  $A_k^{\ell,\bar{s}-1}$ . Consider a partition of unity on  $\Omega \setminus C_k$  associated with  $(A_k^{\ell,s})_{\ell,s \in \mathbb{N}}$ , that is, functions  $\zeta_k^{\ell,s} \in C_c^\infty(A_k^{\ell,s})$  such that  $0 \leq \zeta_k^{\ell,s} \leq 1$  and  $\sum_{\ell,s \in \mathbb{N}} \zeta_k^{\ell,s} \equiv 1$  on  $\Omega \setminus C_k$ . Let us also define

$$Z_k^{\ell,s} := \left\{ x \in \mathbb{R}^n : d_E(x, \text{spt}(\zeta_k^{\ell,s})) \leq \frac{d_E(\partial A_k^{\ell,s}, \text{spt}(\zeta_k^{\ell,s}))}{5} \right\}. \quad (4.9)$$

Notice that  $Z_k^{\ell,s}$  is compact and  $Z_k^{\ell,s} \subset A_k^{\ell,s}$ . Fix a mollification kernel, i.e., a spherically symmetric non-negative function  $K \in C_c^\infty(B_E(0,1))$  such that  $\int_{\mathbb{R}^n} K d\mathcal{L}^n = 1$ . For  $\varepsilon > 0$  we define  $K_\varepsilon(x) = \varepsilon^{-n} K(x/\varepsilon)$ . For  $k \in \mathbb{N}$  we finally define

$$u_k := \sum_{\ell,s \in \mathbb{N}} (\zeta_k^{\ell,s} u) * K_{\varepsilon_k^{\ell,s}} \text{ on } \Omega \setminus C_k \quad (4.10)$$

where  $*$  denotes the classical Euclidean convolution and the  $\varepsilon_k^{\ell,s}$  are chosen so small that, for every  $1 \leq i \leq m$ ,  $1 \leq h \leq n$ ,  $1 \leq t \leq n$ ,  $h \neq t$ , we have<sup>5</sup>

$$\varepsilon_k^{\ell,s} < \frac{1}{2^{\ell+s} k'} \quad (4.11)$$

$$\varepsilon_k^{\ell,s} < \frac{d_E(\partial A_k^{\ell,s}, \text{spt}(\zeta_k^{\ell,s}))}{10}, \quad (4.12)$$

$$\left\| K_{\varepsilon_k^{\ell,s}} * (u X_i \zeta_k^{\ell,s}) - u X_i \zeta_k^{\ell,s} \right\|_{L^1(\Omega)} < \frac{1}{2^{\ell+s} k'} \quad (4.13)$$

$$\left\| \left( u \zeta_k^{\ell,s} \frac{\partial a_{i,t}}{\partial x_t} \right) * W_{\varepsilon_k^{\ell,s}}^t \right\|_{L^1(\Omega)} < \frac{1}{n 2^{\ell+s} k'} \quad (4.14)$$

$$\left\| \left( u \zeta_k^{\ell,s} \frac{\partial a_{i,t}}{\partial x_h} \right) * W_{\varepsilon_k^{\ell,s}}^{t,h} \right\|_{L^1(\Omega)} < \frac{1}{n^2 2^{\ell+s} k'} \quad (4.15)$$

$$\left\| (\zeta_k^{\ell,s} u) * K_{\varepsilon_k^{\ell,s}} - \zeta_k^{\ell,s} u \right\|_{L^1(\Omega)} < \frac{1}{2^{\ell+s} k'} \quad (4.16)$$

$$\left\| (\zeta_k^{\ell,s} D_{X_i}^{ap} u) * K_{\varepsilon_k^{\ell,s}} - \zeta_k^{\ell,s} D_{X_i}^{ap} u \right\|_{L^1(\Omega)} < \frac{1}{2^{\ell+s} k'} \quad (4.17)$$

<sup>3</sup>For convenience we also define  $A_k^\ell := \emptyset$  if  $\ell < 1$ .

<sup>4</sup>For convenience we also define  $A_k^{\ell,s} := \emptyset$  if either  $\ell < 1$  or  $s < 1$ .

<sup>5</sup>Notice the existence of such  $\varepsilon_k^{\ell,s}$  for (4.14) and (4.15) is guaranteed, for instance, by [78, Lemma 1.2.1 (i)].

$$\varepsilon_k^{\ell,s} < \frac{1}{100(\ell+1)(\ell+2)}, \quad (4.18)$$

$$\varepsilon_k^{\ell,s} < \frac{1}{C \|\nabla K\|_{L^\infty} \|u\|_{L^1(Z_k^{\ell,s})} 2^{\ell+s} k}. \quad (4.19)$$

The number  $C > 0$  appearing in (4.19) is a constant that will be chosen in Proposition 4.18 below, and  $W_{\varepsilon_k^{\ell,s}}^t$  and  $W_{\varepsilon_k^{\ell,s}}^{t,h}$  appearing in (4.14) and (4.15) are defined as

$$W_{\varepsilon_k^{\ell,s}}^t(x) := \left( K_{\varepsilon_k^{\ell,s}}(x) + x_t \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x) \right), \quad (4.20)$$

$$W_{\varepsilon_k^{\ell,s}}^{t,h}(x) := x_h \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x), \quad (4.21)$$

being  $x_v$  the  $v$ -th component of  $x$ ,  $v = 1, \dots, n$ .

**Remark 4.14.** Fix  $\ell, s, k \in \mathbb{N}$ . Then for any  $x \in A_k^{\ell,s}$  we have, using condition (4.18), that

$$B_E(x, \varepsilon_k^{\ell,s}) \subseteq \bigcup_{\substack{\ell-1 \leq \alpha \leq \ell+1 \\ s-1 \leq \beta \leq s+1}} A_k^{\alpha,\beta}$$

Let us also observe that, thanks to condition (4.12), we have that  $\text{spt}[(\tilde{\zeta}_k^{\ell,s} u) * K_{\varepsilon_k^{\ell,s}}] \subseteq A_k^{\ell,s}$ , the latter implying that the sum in (4.10) is locally finite, hence  $u_k \in C^\infty(\Omega \setminus C_k)$ . Moreover,  $u_k$  is defined out of a  $\mathcal{L}^n$ -negligible set  $C_k$  and, from (4.16),  $u_k \in L^1(\Omega)$  and  $\|u_k - u\|_{L^1(\Omega)} \xrightarrow{k \rightarrow +\infty} 0$ . Later, using Lemma 4.17 and Proposition 4.18, we will prove in Proposition 4.19 that  $u_k \in \text{SBV}_X(\Omega)$ .

**Lemma 4.15.** Let  $u, u_k$  and  $C_k$  be defined as above. Then for every  $M > 0$  and every  $y \in C_k$  one has

$$\lim_{r \rightarrow 0} r^{-M} \int_{B_E(y,r) \cap \Omega} |u_k - u| d\mathcal{L}^n = 0$$

*Proof.* Fix  $r > 0$  and  $y \in C_k$ . From the fact that  $\tilde{\zeta}_k^{\ell,s} \in C_c^\infty(A_k^{\ell,s})$  and (4.12) we have that

$$u_k(x) - u(x) = \sum_{s \in \mathbb{N}} \sum_{\ell=\ell_0}^{\infty} \left( (\tilde{\zeta}_k^{\ell,s} u) * K_{\varepsilon_k^{\ell,s}}(x) - \tilde{\zeta}_k^{\ell,s} u(x) \right), \quad \forall x \in B_E(y,r) \cap \Omega$$

where  $\ell_0 \in \mathbb{N}$  is defined as  $\ell_0 := [1/r]$  and  $[\cdot]$  denotes the floor function. From (4.16) we obtain

$$\|u_k - u\|_{L^1(B_E(y,r) \cap \Omega)} \leq \sum_{s \in \mathbb{N}} \sum_{\ell=\ell_0}^{\infty} \frac{1}{2^{\ell+s} k}.$$

From the definition of  $\ell_0$  and the fact that for every  $M > 0$  one has  $\lim_{r \rightarrow 0^+} r^{-M} 2^{-1/r} = 0$  we obtain the thesis.  $\square$

**Lemma 4.16.** Let  $u, u_k$  and  $C_k$  be defined as above. Then for every  $M > 0$  and for every  $y \in C_k$  one has

$$\lim_{r \rightarrow 0} r^{-M} \int_{B_C(y,r) \cap \Omega} |u_k - u| d\mathcal{L}^n = 0$$

*Proof.* For any  $p, q \in \overline{(B_E(y, 1))}$  one has  $d_E(p, q) \leq C d_C(p, q)$  where  $C \geq 1$  is a constant only depending on the vector fields  $X_i$ 's and on a compact set  $K \supset \supset (B_E(y, 1))$  (see [138]). Hence for any sufficiently small  $r > 0$ , we have  $B_C(y, r) \subseteq B_E(y, Cr)$ . The latter implies that for any such  $r > 0$  we have

$$r^{-M} \int_{B_C(y, r) \cap \Omega} |u_k - u| d\mathcal{L}^n \leq r^{-M} \int_{B_E(y, Cr) \cap \Omega} |u_k - u| d\mathcal{L}^n.$$

The result then follows upon letting  $r \rightarrow 0$  and using Lemma 4.15.  $\square$

**Lemma 4.17.** *Let  $u, u_k$  and  $C_k$  be defined as above. Then for every  $y \in C_k$*

$$\lim_{r \rightarrow 0} \int_{B_{v_{\mathcal{J}_u}(y)}^+(y, r)} |u_k(x) - u^+(y)| dx = 0, \quad \lim_{r \rightarrow 0} \int_{B_{v_{\mathcal{J}_u}(y)}^-(y, r)} |u_k(x) - u^-(y)| dx = 0.$$

*Proof.* We will prove only that  $\lim_{r \rightarrow 0} \int_{B_{v_{\mathcal{J}_u}(y)}^+(y, r)} |u_k(x) - u^+(y)| dx = 0$ , the other limit is analogous. For the sake of brevity we write  $B_r^+ := B_{v_{\mathcal{J}_u}(y)}^+(y, r)$  and  $u^+ := u^+(y)$ . Since

$$\int_{B_r^+} |u_k(x) - u^+| dx \leq \int_{B_r^+} |u_k(x) - u(x)| dx + \int_{B_r^+} |u(x) - u^+| dx,$$

and  $\int_{B_r^+} |u(x) - u^+| dx \xrightarrow{r \rightarrow 0} 0$ , it suffices to prove that

$$\lim_{r \rightarrow 0} \int_{B_r^+} |u_k(x) - u(x)| dx = 0.$$

We observe that

$$\begin{aligned} \int_{B_r^+} |u_k(x) - u(x)| dx &= \frac{1}{\mathcal{L}^n(B_r^+)} \int_{B_r^+} |u_k(x) - u(x)| dx \leq \frac{1}{\mathcal{L}^n(B_r^+)} \int_{B_C(p, r) \cap \Omega} |u_k(x) - u(x)| dx \\ &= \frac{\mathcal{L}^n(B_C(y, r))}{\mathcal{L}^n(B_r^+)} \frac{1}{\mathcal{L}^n(B_C(y, r))} \int_{B_C(p, r) \cap \Omega} |u_k(x) - u(x)| dx. \end{aligned}$$

From [62, Proposition 2.1.5] we have  $\frac{\mathcal{L}^n(B_C(p, r))}{\mathcal{L}^n(B_r^+)} \xrightarrow{r \rightarrow 0} 2$  so the result follows upon letting  $r \rightarrow 0$ , using Lemma 4.16 and the fact that there exists a positive constant  $C > 0$  such that  $\mathcal{L}^n(B_C(y, r)) \geq Cr^Q$  (see for instance [138, Theorem 1]).  $\square$

## 4.2.2 Estimates on the total variation

Fix  $k \in \mathbb{N}$ ,  $i \in \{1, \dots, m\}$  and  $y \in \Omega \setminus C_k$ . By Proposition 1.5 we have

$$\begin{aligned} (X_i u_k)(y) &= X_i \left( \sum_{\ell, s \in \mathbb{N}} (\tilde{\zeta}_k^{\ell, s} u) * K_{\varepsilon_k^{\ell, s}} \right) (y) = \sum_{\ell, s \in \mathbb{N}} X_i [(\tilde{\zeta}_k^{\ell, s} u) * K_{\varepsilon_k^{\ell, s}}] (y) = \\ &= \sum_{\ell, s \in \mathbb{N}} \left[ (D_{X_i} (\tilde{\zeta}_k^{\ell, s} u) * K_{\varepsilon_k^{\ell, s}})(y) + R_i(\tilde{\zeta}_k^{\ell, s} u, K_{\varepsilon_k^{\ell, s}}; y) \right] = \\ &= \sum_{\ell, s \in \mathbb{N}} \left\{ \left[ \left( u X_i \tilde{\zeta}_k^{\ell, s} \mathcal{L}^n + \tilde{\zeta}_k^{\ell, s} D_{X_i} u \right) * K_{\varepsilon_k^{\ell, s}} \right] (y) + R_i(\tilde{\zeta}_k^{\ell, s} u, K_{\varepsilon_k^{\ell, s}}; y) \right\} = \\ &= \sum_{\ell, s \in \mathbb{N}} \left[ (\tilde{\zeta}_k^{\ell, s} D_{X_i} u) * K_{\varepsilon_k^{\ell, s}}(y) + \int_{\mathbb{R}^n} K_{\varepsilon_k^{\ell, s}}(y - x) u(x) (X_i \tilde{\zeta}_k^{\ell, s})(x) dx + R_i(\tilde{\zeta}_k^{\ell, s} u, K_{\varepsilon_k^{\ell, s}}; y) \right]. \end{aligned}$$

For the sake of brevity let us define

$$S_i^{k,\ell,s}(y) := \int_{\mathbb{R}^n} K_{\varepsilon_k^{\ell,s}}(y-x)u(x)(X_i\tilde{\zeta}_k^{\ell,s})(x)dx, \quad R_i^{k,\ell,s}(y) := R_i(\tilde{\zeta}_k^{\ell,s}u, K_{\varepsilon_k^{\ell,s}}; y) \quad (4.22)$$

so that

$$(X_i u_k)(y) = \sum_{\ell,s \in \mathbb{N}} \left( (\tilde{\zeta}_k^{\ell,s} D_{X_i} u) * K_{\varepsilon_k^{\ell,s}}(y) + S_i^{k,\ell,s}(y) + R_i^{k,\ell,s}(y) \right). \quad (4.23)$$

Now we want to estimate the  $L^1$ -norm of the two remainders  $R_i^{k,\ell,s}$  and  $S_i^{k,\ell,s}$ : part of the following Proposition is a rewriting of [78, Lemma 2.1.1] in a language more useful for our purposes.

**Proposition 4.18.** *If  $S_i^{k,\ell,s}$  and  $R_i^{k,\ell,s}$  are defined as above (see (1.1) and (4.22)), then for any  $i = 1, \dots, m$*

$$\left\| \sum_{\ell,s \in \mathbb{N}} S_i^{k,\ell,s} \right\|_{L^1(\Omega)} \leq \frac{1}{k} \quad \text{and} \quad \left\| \sum_{\ell,s \in \mathbb{N}} R_i^{k,\ell,s} \right\|_{L^1(\Omega)} \leq \frac{3}{k}.$$

*Proof.* We start by estimating  $S_i^{k,\ell,s}$ . Fix  $y \in \Omega \setminus C_k$ : since  $\sum_{\ell,s \in \mathbb{N}} X_i \tilde{\zeta}_k^{\ell,s} \equiv 0$ , then

$$\begin{aligned} \sum_{\ell,s \in \mathbb{N}} S_i^{k,\ell,s}(y) &= \sum_{\ell,s \in \mathbb{N}} K_{\varepsilon_k^{\ell,s}} * (u X_i \tilde{\zeta}_k^{\ell,s})(y) \\ &= \sum_{\ell,s \in \mathbb{N}} K_{\varepsilon_k^{\ell,s}} * (u X_i \tilde{\zeta}_k^{\ell,s})(y) - \sum_{\ell,s \in \mathbb{N}} (u X_i \tilde{\zeta}_k^{\ell,s})(y) \\ &= \sum_{\ell,s \in \mathbb{N}} \left( K_{\varepsilon_k^{\ell,s}} * (u X_i \tilde{\zeta}_k^{\ell,s}) - u X_i \tilde{\zeta}_k^{\ell,s} \right)(y). \end{aligned}$$

Using (4.13) and the fact that  $\mathcal{L}^n(C_k) = 0$ , we immediately obtain  $\left\| \sum_{\ell,s \in \mathbb{N}} S_i^{k,\ell,s} \right\|_{L^1(\Omega)} \leq 1/k$ . To estimate  $R_i^{k,\ell,s}$  we start by observing that for every  $y \in \Omega \setminus C_k$  one has

$$R_i^{k,\ell,s}(y) = \int_{\Omega} \tilde{\zeta}_k^{\ell,s}(x)u(x) \left( (\operatorname{div} X_i)(x)K_{\varepsilon_k^{\ell,s}}(x-y) - \langle X_i(y) - X_i(x), \nabla K_{\varepsilon_k^{\ell,s}}(x-y) \rangle \right) dx \quad (4.24)$$

$$= \sum_{t=1}^n \int_{\Omega} \tilde{\zeta}_k^{\ell,s}(x)u(x) \left( \frac{\partial a_{i,t}}{\partial x_t}(x)K_{\varepsilon_k^{\ell,s}}(x-y) - (a_{i,t}(y) - a_{i,t}(x)) \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x-y) \right) dx \quad (4.25)$$

Now, for every  $y, x \in Z_k^{\ell,s}$  (see (4.9)) we can expand each  $a_{i,t}$  with a Taylor's expansion

$$a_{i,t}(y) - a_{i,t}(x) = \sum_{h=1}^n \frac{\partial a_{i,t}}{\partial x_h}(x)(y-x)_h + T_{i,t}(y, x) \quad (4.26)$$

where  $(y-x)_h$  denotes the  $h$ -component of the vector  $y-x \in \mathbb{R}^n$  and

$$|T_{i,t}(y, x)| \leq C_{i,t}^{k,\ell,s} |y-x|^2 \quad (4.27)$$

where  $C_{i,t}^{k,\ell,s}$  is a positive constant only depending on the  $L^\infty$ -norm of the second derivatives of  $a_{i,t}$  on  $Z_k^{\ell,s}$ . Replacing (4.26) into (4.24) we obtain:

$$\begin{aligned}
R_i^{k,\ell,s}(y) &= \sum_{t=1}^n \int_{\Omega} \zeta_k^{\ell,s}(x) u(x) \frac{\partial a_{i,t}}{\partial x_t}(x) K_{\varepsilon_k^{\ell,s}}(x-y) dx \\
&\quad - \sum_{t,h=1}^n \int_{\Omega} \zeta_k^{\ell,s}(x) u(x) \frac{\partial a_{i,t}}{\partial x_h}(x) (y-x)_h \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x-y) dx \\
&\quad - \sum_{t=1}^n \int_{\Omega} \zeta_k^{\ell,s}(x) u(x) T_{i,t}(y,x) \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x-y) dx \\
&= \sum_{t=1}^n \int_{\Omega} \zeta_k^{\ell,s}(x) u(x) \frac{\partial a_{i,t}}{\partial x_t}(x) \left( K_{\varepsilon_k^{\ell,s}}(x-y) - (y-x)_t \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x-y) \right) dx \\
&\quad - \sum_{t,h=1,t \neq h}^n \int_{\Omega} \zeta_k^{\ell,s}(x) u(x) \frac{\partial a_{i,t}}{\partial x_h}(x) (y-x)_h \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x-y) dx \\
&\quad - \sum_{t=1}^n \int_{\Omega} \zeta_k^{\ell,s}(x) u(x) T_{i,t}(y,x) \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x-y) dx.
\end{aligned}$$

Recall that  $K$  (and therefore  $K_{\varepsilon_k^{\ell,s}}$ ) is spherically symmetric so that we can write

$$\begin{aligned}
R_i^{k,\ell,s}(y) &= \sum_{t=1}^n \left( u \zeta_k^{\ell,s} \frac{\partial a_{i,t}}{\partial x_t} \right) * W_{\varepsilon_k^{\ell,s}}^t(y) + \sum_{t,h=1,t \neq h}^n \left( u \zeta_k^{\ell,s} \frac{\partial a_{i,t}}{\partial x_h} \right) * W_{\varepsilon_k^{\ell,s}}^{t,h}(y) \\
&\quad - \sum_{t=1}^n \int_{\Omega} \zeta_k^{\ell,s}(x) u(x) T_{i,t}(y,x) \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x-y) dx
\end{aligned}$$

where  $W_{\varepsilon_k^{\ell,s}}^t$  and  $W_{\varepsilon_k^{\ell,s}}^{t,h}$  are defined as in (4.20). Then we have

$$\begin{aligned}
\left\| \sum_{\ell,s \in \mathbb{N}} R_i^{k,\ell,s} \right\|_{L^1(\Omega)} &\leq \underbrace{\sum_{\ell,s \in \mathbb{N}} \sum_{t=1}^n \left\| \left( u \zeta_k^{\ell,s} \frac{\partial a_{i,t}}{\partial x_t} \right) * W_{\varepsilon_k^{\ell,s}}^t \right\|_{L^1(\Omega)}}_{(A)} + \underbrace{\sum_{\ell,s \in \mathbb{N}} \sum_{\substack{t,h=1 \\ t \neq h}}^n \left\| \left( u \zeta_k^{\ell,s} \frac{\partial a_{i,t}}{\partial x_h} \right) * W_{\varepsilon_k^{\ell,s}}^{t,h} \right\|_{L^1(\Omega)}}_{(B)} \\
&\quad + \underbrace{\sum_{\ell,s \in \mathbb{N}} \sum_{t=1}^n \int_{Z_k^{\ell,s}} \int_{B_E(x, \varepsilon_k^{\ell,s})} \left| \zeta_k^{\ell,s}(x) u(x) T_{i,t}(y,x) \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t}(x-y) \right| dy dx}_{(C)}
\end{aligned}$$

From (4.14) and (4.15) we obtain that (A)  $\leq 1/k$  and (B)  $\leq 1/k$ . Concerning (C) we have,

using (4.27) and the fact that  $\left| \frac{\partial K_{\varepsilon_k^{\ell,s}}}{\partial x_t} \right| < \|\nabla K\|_{L^\infty} (\varepsilon_k^{\ell,s})^{-n-1}$  and  $|x - y| < \varepsilon_k^{\ell,s}$ , that

$$\begin{aligned} (C) &\leq \|\nabla K\|_{L^\infty} \sum_{\ell,s \in \mathbb{N}} \sum_{t=1}^n \int_{Z_k^{\ell,s}} \int_{B_E(x, \varepsilon_k^{\ell,s})} (\varepsilon_k^{\ell,s})^{-n-1} |\zeta_k^{\ell,s}(x)u(x)| |T_{i,t}(y, x)| dy dx \\ &\leq \|\nabla K\|_{L^\infty} \sum_{\ell,s \in \mathbb{N}} \sum_{t=1}^n \int_{Z_k^{\ell,s}} (\varepsilon_k^{\ell,s})^{-n-1} |\zeta_k^{\ell,s}(x)u(x)| C_{i,t}^{k,\ell,s} \int_{B_E(x, \varepsilon_k^{\ell,s})} |y - x|^2 dy dx \\ &\leq \|\nabla K\|_{L^\infty} \sum_{\ell,s \in \mathbb{N}} \sum_{t=1}^n C_{i,t}^{k,\ell,s} \varepsilon_k^{\ell,s} \int_{Z_k^{\ell,s}} |\zeta_k^{\ell,s}(x)u(x)| dx \end{aligned}$$

and using (4.19) with the specific choice of

$$C := \sum_{t=1}^n C_{i,t}^{k,\ell,s},$$

we obtain  $(C) \leq 1/k$ , concluding the proof.  $\square$

**Proposition 4.19.** *For every  $k \in \mathbb{N}$  the function  $u_k$  defined in (4.10) satisfies the following properties:*

- (i)  $u_k \in \text{BV}_X(\Omega \setminus C_k)$ ,
- (ii)  $u_k \in \text{BV}_X(\Omega)$ ,
- (iii)  $u_k \in \text{SBV}_X(\Omega)$ .

*Proof.*

(i) We know from Proposition 1.5 that for every  $\ell, s \in \mathbb{N}$  one has  $u_k^{\ell,s} \in \text{BV}_X(\Omega \setminus C_k)$  where, for the sake of brevity, we defined  $u_k^{\ell,s} := (\zeta_k^{\ell,s} u) * K_{\varepsilon_k^{\ell,s}}$ . To prove  $u_k \in \text{BV}_X(\Omega \setminus C_k)$ , it is enough to show that

$$\left\| \sum_{\ell,s \in \mathbb{N}} u_k^{\ell,s} \right\|_{\text{BV}_X(\Omega \setminus C_k)} = \left\| \sum_{\ell,s \in \mathbb{N}} u_k^{\ell,s} \right\|_{L^1(\Omega)} + \left| D_X \left( \sum_{\ell,s \in \mathbb{N}} u_k^{\ell,s} \right) \right| (\Omega \setminus C_k) < \infty. \quad (4.28)$$

As we mentioned before, thanks to (4.16),  $u_k \in L^1(\Omega)$  so that the first addend on the right hand side of (4.28) is finite. We are left to estimate the second term. Since  $u_k \in C^\infty(\Omega \setminus C_k)$  for every  $i \in \{1, \dots, m\}$ , we have that

$$D_{X_i} u_k = X_i u_k \mathcal{L}^n \text{ on } \Omega \setminus C_k,$$

meaning that

$$\left| D_{X_i} \left( \sum_{\ell,s \in \mathbb{N}} u_k^{\ell,s} \right) \right| (\Omega \setminus C_k) = \left\| \sum_{\ell,s \in \mathbb{N}} X_i u_k^{\ell,s} \right\|_{L^1(\Omega)}.$$

Hence, by the decomposition (4.23), we write

$$\|X_i u_k\|_{L^1(\Omega)} \leq \left\| \sum_{\ell, s \in \mathbb{N}} (\zeta_k^{\ell, s} D_{X_i} u) * K_{\varepsilon_k^{\ell, s}} \right\|_{L^1(\Omega)} + \left\| \sum_{\ell, s \in \mathbb{N}} S_i^{k, \ell, s} \right\|_{L^1(\Omega)} + \left\| \sum_{\ell, s \in \mathbb{N}} R_i^{k, \ell, s} \right\|_{L^1(\Omega)}.$$

Thanks to Proposition 4.18, we are only left to prove the boundedness of the first term of the right hand side of the inequality above. Using [5, Theorem 2.2 (b)] we have

$$\left\| \sum_{\ell, s \in \mathbb{N}} (\zeta_k^{\ell, s} D_{X_i} u) * K_{\varepsilon_k^{\ell, s}} \right\|_{L^1(\Omega)} \leq \sum_{\ell, s \in \mathbb{N}} \left\| (\zeta_k^{\ell, s} D_{X_i} u) * K_{\varepsilon_k^{\ell, s}} \right\|_{L^1(A_k^{\ell, s})} \leq \sum_{\ell, s \in \mathbb{N}} |D_{X_i} u|(A_k^{\ell, s} + \varepsilon_k^{\ell, s}).$$

where we have written  $A_k^{\ell, s} + \varepsilon_k^{\ell, s}$  to denote

$$A_k^{\ell, s} + \varepsilon_k^{\ell, s} := \bigcup_{x \in A_k^{\ell, s}} B_E(x, \varepsilon_k^{\ell, s}).$$

By Remark 4.14 we have

$$A_k^{\ell, s} + \varepsilon_k^{\ell, s} \subseteq \bigcup_{\substack{\ell-1 \leq \alpha \leq \ell+1 \\ s-1 \leq \beta \leq s+1}} A_k^{\alpha, \beta}$$

so that

$$|D_{X_i} u|(A_k^{\ell, s} + \varepsilon_k^{\ell, s}) \leq |D_{X_i} u| \left( \bigcup_{\substack{\ell-1 \leq \alpha \leq \ell+1 \\ s-1 \leq \beta \leq s+1}} A_k^{\alpha, \beta} \right).$$

and, finally,

$$\sum_{\ell, s \in \mathbb{N}} |D_{X_i} u|(A_k^{\ell, s} + \varepsilon_k^{\ell, s}) \leq 9 \sum_{\ell, s \in \mathbb{N}} |D_{X_i} u|(A_k^{\ell, s}) \leq 27 \sum_{\ell \in \mathbb{N}} |D_{X_i} u|(A_k^\ell) \leq 81 |D_{X_i} u|(\Omega)$$

which, together with the fact that  $u \in \text{BV}_X(\Omega)$ , allows us conclude the boundedness of  $\|u_k\|_{\text{BV}_X(\Omega \setminus C_k)}$ .

(ii) We aim to prove that  $u_k \in \text{BV}_X(\Omega)$ . Suppose first that, in addition to our previous assumptions, we have  $u \in C^\infty(\Omega)$ . Then  $C_k = \emptyset$  and clearly  $u_k \in C^\infty(\Omega) \subset C_X^1(\Omega)$  for any  $k \in \mathbb{N}$ , hence

$$\begin{aligned} |D_{X_i} u_k|(\Omega) &= \|X_i u_k\|_{L^1(\Omega)} \\ &\leq \left\| \sum_{\ell, s \in \mathbb{N}} (\zeta_k^{\ell, s} D_{X_i} u) * K_{\varepsilon_k^{\ell, s}} \right\|_{L^1(\Omega)} + \left\| \sum_{\ell, s \in \mathbb{N}} S_i^{k, \ell, s} \right\|_{L^1(\Omega)} + \left\| \sum_{\ell, s \in \mathbb{N}} R_i^{k, \ell, s} \right\|_{L^1(\Omega)} \\ &\leq 81 |D_{X_i} u|(\Omega) + \frac{4}{k}, \end{aligned}$$

the latter implying that  $u_k \in \text{BV}_X(\Omega)$ .

Now we drop the smoothness assumption on  $u$  and we just assume that  $u \in \text{BV}_X(\Omega)$ . We know, by [78, Theorem 2.2.2], that there exists a sequence  $(u^t)_{t \in \mathbb{N}}$  such that, for every  $t \in \mathbb{N}$  we have  $u^t \in C^\infty(\Omega) \cap \text{BV}_X(\Omega)$  and

$$\|u^t - u\|_{L^1(\Omega)} \xrightarrow{t \rightarrow +\infty} 0, \quad |D_{X_i} u^t|(\Omega) \xrightarrow{t \rightarrow +\infty} |D_{X_i} u|(\Omega), \quad |D_{X_i} u^t|(\Omega) \leq |D_{X_i} u|(\Omega) + \frac{1}{t}$$

for every  $i \in \{1, \dots, m\}$ . Now for every  $t \in \mathbb{N}$  consider the approximation sequence  $(u_k^t)_{k \in \mathbb{N}}$  constructed as in (4.10). For the observations we just made on the approximation of smooth functions we know that  $u_k^t \in C_X^1(\Omega) \cap \text{BV}_X(\Omega)$ . Let us prove that  $\|u_k^t - u_k\|_{L^1(\Omega)} \xrightarrow{t \rightarrow +\infty} 0$ . First we observe that, thanks to Remark 4.14, we have  $\|u_k\|_{L^1(\Omega)} \leq 81 \|u\|_{L^1(\Omega)}$  so that

$$\|u_k^t - u_k\|_{L^1(\Omega)} = \|(u^t - u)_k\|_{L^1(\Omega)} \leq 81 \|u^t - u\|_{L^1(\Omega)}.$$

The inequality above shows that  $\|u_k^t - u_k\|_{L^1(\Omega)} \xrightarrow{t \rightarrow +\infty} 0$ . Then we observe

$$|D_{X_i} u_k^t|(\Omega) \leq 81 |D_{X_i} u^t|(\Omega) + \frac{4}{k} \leq 81 |D_{X_i} u|(\Omega) + \frac{81}{t} + \frac{4}{k} \leq 81 |D_{X_i} u|(\Omega) + 85.$$

Passing to the  $\liminf$  for  $t \rightarrow +\infty$  in the above inequality and using the lower semicontinuity of the total variation is enough to obtain  $u_k \in \text{BV}_X(\Omega)$ .

(iii) From Lemma 4.17, the construction of  $C_k$  in Lemma 4.13 and the fact that  $u_k \in \text{BV}_X(\Omega) \cap C^\infty(\Omega \setminus C_k)$  we obtain that  $u_k \in \text{SBV}_X(\Omega)$ .  $\square$

The following Lemma will be the last step needed to prove our main result.

**Lemma 4.20.** *Let  $v, w \in \text{SBV}_X(\Omega)$  and  $R \subseteq \mathcal{J}_v \cap \mathcal{J}_w$ . Let  $j_v$  and  $j_w$  be defined as in Lemma 4.13 and such that  $j_v = j_w$   $\mathcal{H}_C^{Q-1}$ -a.e. on  $R$ . Then*

$$|D_X(v - w)|(R) = 0.$$

*Proof.* Let us observe that  $v - w \in \text{SBV}_X(\Omega)$  and  $\mathcal{H}_C^{Q-1}(R \cap \mathcal{S}_{v-w}) = 0$ . In fact, for  $\mathcal{H}^{Q-1}$ -a.e.  $p \in R$  one has  $j_v(p) = j_w(p)$  and, letting  $\nu := \nu_{\mathcal{J}_v}(p) = \nu_{\mathcal{J}_w}(p)$ , we notice that

$$\begin{aligned} \lim_{r \rightarrow 0} \int_{B_C(p,r)} |v - w| d\mathcal{L}^n &\leq \lim_{r \rightarrow 0} \int_{B_v^+(p,r)} |v - v^+| d\mathcal{L}^n + \lim_{r \rightarrow 0} \int_{B_v^+(p,r)} |w - w^+| d\mathcal{L}^n + \\ &+ \lim_{r \rightarrow 0} \int_{B_v^-(p,r)} |v - v^-| d\mathcal{L}^n + \lim_{r \rightarrow 0} \int_{B_v^-(p,r)} |w - w^-| d\mathcal{L}^n = 0. \end{aligned}$$

The latter implies that  $v - w$  has approximate limit 0 at  $p$ , i.e.,  $p \in R \setminus \mathcal{S}_{v-w}$ . This proves that  $\mathcal{H}_C^{Q-1}(R \cap \mathcal{S}_{v-w}) = 0$  and by Theorem 1.12 (1.2)

$$|D_X(v - w)|(R \cap \mathcal{S}_{v-w}) = 0. \quad (4.29)$$

Moreover, the measure  $\mathcal{H}^{Q-1} \llcorner (R \setminus \mathcal{S}_{v-w})$  is  $\sigma$ -finite and Theorem 1.12 (1.3) implies that

$$|D_X(v - w)|(R \setminus \mathcal{S}_{v-w}) = 0. \quad (4.30)$$

The desired equality  $|D_X(v - w)|(R) = 0$  follows from (4.29) and (4.30).  $\square$

### 4.2.3 Proof of the approximation Theorem

We are ready to prove our approximation result, Theorem 4.2, that we restate for the reader's convenience.

**Theorem 4.21.** *Let  $u \in \text{SBV}_X(\Omega)$ . Then there exists a sequence of functions  $(u_k)_{k \in \mathbb{N}} \subset \text{SBV}_X(\Omega)$  and of  $C_X^1$ -hypersurfaces  $(M_k)_{k \in \mathbb{N}} \subset \Omega$  such that, for every  $k \in \mathbb{N}$ ,  $\mathcal{J}_{u_k} \subseteq M_k \cap \mathcal{J}_u$ ,  $\mathcal{J}_{u_k}$  is compact, and*

$$\|u - u_k\|_{\text{BV}_X(\Omega)} \xrightarrow{k \rightarrow +\infty} 0, \quad u_k \in C^\infty(\Omega \setminus \mathcal{J}_{u_k}).$$

*Proof.* Let  $C_k$  and  $(u_k)_{k \in \mathbb{N}}$  be defined as in Lemma 4.13 and (4.10). By definition of  $\text{BV}_X$ -norm we have

$$\|u - u_k\|_{\text{BV}_X(\Omega)} = \|u - u_k\|_{L^1(\Omega)} + |D_X(u - u_k)|(\Omega). \quad (4.31)$$

Thanks to (4.16) we estimate

$$\|u - u_k\|_{L^1(\Omega)} < \frac{1}{k}. \quad (4.32)$$

Concerning the other summand, we estimate

$$\begin{aligned} & |D_X(u - u_k)|(\Omega) \\ & \leq |D_X(u - u_k)|(\Omega \setminus \mathcal{J}_u) + |D_X(u - u_k)|(\mathcal{J}_u \setminus \mathcal{J}_u^k) + |D_X(u - u_k)|(\mathcal{J}_u^k \setminus M_k) \\ & \quad + |D_X(u - u_k)|((\mathcal{J}_u \cap M_k) \setminus C_k) + |D_X(u - u_k)|(C_k). \end{aligned} \quad (4.33)$$

Because of Lemma 4.17, Proposition 4.19 and Lemma 4.20 we have

$$|D_X(u - u_k)|(C_k) = 0. \quad (4.34)$$

Then, since  $u_k \in C^\infty(\Omega \setminus C_k)$ , (4.6), (4.7) and (4.8) imply that

$$\begin{aligned} |D_X(u - u_k)|((\mathcal{J}_u \cap M_k) \setminus C_k) &= |D_X u|((\mathcal{J}_u \cap M_k) \setminus C_k) < \frac{1}{k}, \\ |D_X(u - u_k)|(\mathcal{J}_u \setminus \mathcal{J}_u^k) &= |D_X u|(\mathcal{J}_u \setminus \mathcal{J}_u^k) < \frac{1}{k}, \\ |D_X(u - u_k)|(\mathcal{J}_u^k \setminus M_k) &= |D_X u|(\mathcal{J}_u^k \setminus M_k) < \frac{1}{k}. \end{aligned} \quad (4.35)$$

By Theorem 1.11, one has  $D_X u = D_X^{\text{ap}} u \mathcal{L}^n + D_X^j u$  so that, for every  $i \in \{1, \dots, m\}$ ,

$$\begin{aligned} & |D_{X_i}(u - u_k)|(\Omega \setminus \mathcal{J}_u) = \left\| D_{X_i}^{\text{ap}} u - X_i u_k \right\|_{L^1(\Omega)} \\ & \leq \left\| D_{X_i}^{\text{ap}} u - \sum_{\ell, s \in \mathbb{N}} [(\bar{\alpha}_k^{\ell, s} D_{X_i} u) * K_{\varepsilon_k^{\ell, s}}] \right\|_{L^1(\Omega)} + \left\| \sum_{\ell, s \in \mathbb{N}} R_i^{k, \ell, s} \right\|_{L^1(\Omega)} + \left\| \sum_{\ell, s \in \mathbb{N}} S_i^{k, \ell, s} \right\|_{L^1(\Omega)}. \end{aligned}$$

By Proposition 4.18

$$\left\| \sum_{\ell, s \in \mathbb{N}} R_i^{k, \ell, s} \right\|_{L^1(\Omega)} + \left\| \sum_{\ell, s \in \mathbb{N}} S_i^{k, \ell, s} \right\|_{L^1(\Omega)} \leq \frac{4}{k}, \quad (4.36)$$

while

$$\begin{aligned}
\left\| \sum_{\ell,s \in \mathbb{N}} [(\tilde{\zeta}_k^{\ell,s} D_{X_i} u) * K_{\varepsilon_k^{\ell,s}}] - D_{X_i}^{\text{ap}} u \right\|_{L^1(\Omega)} &= \left\| \sum_{\ell,s \in \mathbb{N}} [(\tilde{\zeta}_k^{\ell,s} (D_{X_i}^{\text{ap}} u \mathcal{L}^n + D_{X_i}^j u)) * K_{\varepsilon_k^{\ell,s}}] - D_{X_i}^{\text{ap}} u \right\|_{L^1(\Omega)} \\
&\leq \left\| \sum_{\ell,s \in \mathbb{N}} [(\tilde{\zeta}_k^{\ell,s} D_{X_i}^{\text{ap}} u) * K_{\varepsilon_k^{\ell,s}}] - \sum_{\ell,s \in \mathbb{N}} \tilde{\zeta}_k^{\ell,s} D_{X_i}^{\text{ap}} u \right\|_{L^1(\Omega)} \\
&\quad + \left\| \sum_{\ell,s \in \mathbb{N}} (\tilde{\zeta}_k^{\ell,s} D_{X_i}^j u) * K_{\varepsilon_k^{\ell,s}} \right\|_{L^1(\Omega)}.
\end{aligned}$$

Thanks to (4.17) we have

$$\left\| \sum_{\ell,s \in \mathbb{N}} [(\tilde{\zeta}_k^{\ell,s} D_{X_i}^{\text{ap}} u) * K_{\varepsilon_k^{\ell,s}}] - \sum_{\ell,s \in \mathbb{N}} \tilde{\zeta}_k^{\ell,s} D_{X_i}^{\text{ap}} u \right\|_{L^1(\Omega)} \leq \frac{1}{k'}, \quad (4.37)$$

while

$$\left\| \sum_{\ell,s \in \mathbb{N}} (\tilde{\zeta}_k^{\ell,s} D_{X_i}^j u) * K_{\varepsilon_k^{\ell,s}} \right\|_{L^1(\Omega)} \leq \sum_{\ell,s \in \mathbb{N}} \left\| (\tilde{\zeta}_k^{\ell,s} D_{X_i}^j u) * K_{\varepsilon_k^{\ell,s}} \right\|_{L^1(A_k^{\ell,s})}.$$

Fix  $\ell, s \in \mathbb{N}$ . By [5, Theorem 2.2 (b)] we can write

$$\left\| (\tilde{\zeta}_k^{\ell,s} D_{X_i}^j u) * K_{\varepsilon_k^{\ell,s}} \right\|_{L^1(A_k^{\ell,s})} \leq |D_{X_i}^j u|(A_k^{\ell,s} + \varepsilon_k^{\ell,s}),$$

which in turn, by Remark 4.14, satisfies the following inclusion

$$A_k^{\ell,s} + \varepsilon_k^{\ell,s} \subseteq \bigcup_{\substack{\ell-1 \leq \alpha \leq \ell+1 \\ s-1 \leq \beta \leq s+1}} A_k^{\alpha,\beta}$$

Hence

$$|D_{X_i}^j u|(A_k^{\ell,s} + \varepsilon_k^{\ell,s}) \leq |D_{X_i}^j u| \left( \bigcup_{\substack{\ell-1 \leq \alpha \leq \ell+1 \\ s-1 \leq \beta \leq s+1}} A_k^{\alpha,\beta} \right).$$

Finally, we obtain

$$\sum_{\ell,s \in \mathbb{N}} \left\| (\tilde{\zeta}_k^{\ell,s} D_{X_i}^j u) * K_{\varepsilon_k^{\ell,s}} \right\|_{L^1(A_k^{\ell,s})} \leq \sum_{\ell,s \in \mathbb{N}} |D_{X_i}^j u| \left( \bigcup_{\substack{\ell-1 \leq \alpha \leq \ell+1 \\ s-1 \leq \beta \leq s+1}} A_k^{\alpha,\beta} \right) \leq 81 |D_{X_i}^j u|(\Omega \setminus C_k) \leq \frac{81}{k}. \quad (4.38)$$

Combining (4.32), (4.33), (4.34), (4.35), (4.36), (4.37), (4.38) with (4.31) and letting  $k \rightarrow +\infty$  one achieves the desired conclusion.  $\square$

# 5 Stepanov differentiability theorem for intrinsic graphs in Heisenberg groups

The aim of this chapter is to present the results contained in [59]. To be more specific we want to prove the following result. Recall that for an intrinsic graph we defined the set  $S_\phi$  in Definition 2.20.

**Theorem 5.1.** *Let  $\mathbb{W}$  and  $\mathbb{V}$  be complementary subgroup of  $\mathbb{H}^n$ . Let  $A \subseteq \mathbb{V}$  be an open set and  $\phi : A \rightarrow \mathbb{W}$ . Then  $\phi$  is intrinsically differentiable almost everywhere on  $S_\phi$ .*

This chapter is divided into two sections: in Section 5.1 we present the proof of Theorem 5.1 when  $\mathbb{V}$  is horizontal and  $\mathbb{W}$  is vertical and in Section 5.2 we present the proof of Theorem 5.1 when  $\mathbb{W}$  is horizontal and  $\mathbb{V}$  is vertical<sup>1</sup>. In this case we present two proofs; one that works for graphs of arbitrary codimension, inspired by the classical proof of Stepanov Theorem that may be found, for example, in [73] and an alternative proof for graphs of codimension 1, inspired by [126].

## 5.1 Proof of Stepanov Theorem: graphs with vertical target

The first case is a fairly easy consequence of the “classical” Stepanov theorem for maps between Carnot groups [159]. Indeed, by applying [16, Proposition 3.25], we can reduce the intrinsic differentiability of a map to the Pansu differentiability of its graph map.

**Theorem 5.2.** *Fix  $1 \leq k \leq n$  and let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  where  $\mathbb{V}$  is a horizontal subgroup of dimension  $k$ . Let  $A \subseteq \mathbb{V}$  be an open set and  $\phi : A \rightarrow \mathbb{W}$ . Then  $\phi$  is intrinsically differentiable almost everywhere on  $S_\phi$ .*

*Proof.* Let  $\phi : A \subseteq \mathbb{V} \rightarrow \mathbb{W}$  be as in the assumptions. Define  $\Phi_\phi : A \rightarrow \mathbb{H}^n$  to be the graph map:

$$\Phi_\phi(v) := v \cdot \phi(v).$$

We start by proving the following elementary fact:  $\bar{v} \in S_\phi$  if and only if  $\Phi_\phi$  is pointwise (metric) Lipschitz continuous at  $\bar{v}$ , that is

$$\limsup_{A \ni v \rightarrow \bar{v}} \frac{\|(\Phi_\phi(\bar{v}))^{-1} \cdot \Phi_\phi(v)\|}{\|\bar{v}^{-1} \cdot v\|} < +\infty. \quad (5.1)$$

---

<sup>1</sup>In this chapter, in order to maintain a coherent notation,  $\mathbb{V}$  (resp.  $\mathbb{W}$ ) will always denote an horizontal (resp. vertical) subgroup of  $\mathbb{H}^n$ , unless otherwise specified.

Suppose first  $\bar{v} \in S_\phi$ . This means that there exists  $\beta > 0$  and  $U \subset A$  open containing  $\bar{v}$  such that

$$C_{\mathbb{V}, \mathbb{W}}(\Phi_\phi(\bar{v}), \beta) \cap \text{gr}_{\phi|_U} = \{\Phi_\phi(\bar{v})\}.$$

Equivalently, after a left translation and (2.1),

$$C_{\mathbb{V}, \mathbb{W}}(0, \beta) \cap (\Phi_\phi(\bar{v}))^{-1} \cdot \text{gr}_{\phi|_U} = \{0\}.$$

For notational simplicity, let us write  $\bar{p} := \Phi_\phi(\bar{v})$ . By Definition 2.17, we get that, for every point  $p = v \cdot \phi(v) \in \text{gr}_{\phi|_U}$ ,

$$\|(\bar{p}^{-1} \cdot p)_{\mathbb{W}}\| \leq \frac{1}{\beta} \|(\bar{p}^{-1} \cdot p)_{\mathbb{V}}\|.$$

Since  $\mathbb{W}$  is normal, the projection on  $\mathbb{V}$  is a group homomorphism (Proposition 2.15): hence  $(\bar{p}^{-1} \cdot p)_{\mathbb{V}} = \bar{v}^{-1} \cdot v$ . Therefore we get

$$\|\bar{p}^{-1} \cdot p\| \leq \|(\bar{p}^{-1} \cdot p)_{\mathbb{W}}\| + \|(\bar{p}^{-1} \cdot p)_{\mathbb{V}}\| \leq \left(1 + \frac{1}{\beta}\right) \|\bar{v}^{-1} \cdot v\|,$$

for every  $v \in U$ , which implies (5.1).

On the other hand, assuming (5.1) and keeping the same notation as before, there exists  $U \subset A$  open with  $\bar{v} \in U$  and  $L > 0$  such that

$$\|\bar{p}^{-1} \cdot p\| \leq L \|\bar{v}^{-1} \cdot v\| = L \|(\bar{p}^{-1} \cdot p)_{\mathbb{V}}\|$$

for all  $v \in U$ . By Proposition 2.15, there exists a constant  $\tilde{C} > 0$  such that  $\|\bar{p}^{-1} \cdot p\| \geq \tilde{C} \|(\bar{p}^{-1} \cdot p)_{\mathbb{W}}\|$ . We deduce that

$$\|(\bar{p}^{-1} \cdot p)_{\mathbb{W}}\| \leq \frac{L}{\tilde{C}} \|(\bar{p}^{-1} \cdot p)_{\mathbb{V}}\|$$

which implies, arguing as in the first part of the proof, that  $\bar{v} \in S_\phi$ .

We move now to the proof of the Theorem: since  $\mathbb{V}$  is horizontal, we can identify  $\mathbb{V} \cong \mathbb{R}^k$  for some  $1 \leq k \leq n$ . Hence, by applying the ‘‘classical’’ Stepanov differentiability theorem for maps between Carnot groups (see [159, Theorem 3.1] and [160, Theorem 1]), we deduce that the graph map  $\Phi_\phi : A \subset \mathbb{V} \cong \mathbb{R}^k \rightarrow \mathbb{H}^n$  is Pansu-differentiable almost everywhere in the set of points where  $\Phi_\phi$  is pointwise Lipschitz continuous, which coincides by the previous argument with  $S_\phi$ . Then we conclude by applying [16, Proposition 3.25]: a map  $\phi : A \subset \mathbb{V} \rightarrow \mathbb{W}$  is intrinsically differentiable at  $\bar{v}$  if and only if the graph map  $\Phi_\phi : A \subset \mathbb{V} \rightarrow \mathbb{H}^n$  is Pansu-differentiable at  $\bar{v}$ .  $\square$

**Remark 5.3.** We point out that all the results used in the proof of Theorem 5.2 hold in general Carnot groups  $\mathbb{G}$  which can be written as  $\mathbb{G} = \mathbb{V}\mathbb{W}$  where  $\mathbb{V}$  is a horizontal subgroup and  $\mathbb{W}$  is normal; see, in particular, [13]. Therefore, Theorem 5.2 holds even in the generality described above for maps  $\phi : A \subseteq \mathbb{V} \rightarrow \mathbb{W}$ .

## 5.2 Proof of Stepanov Theorem: graphs with horizontal target

Let us now move to the proof of Stepanov differentiability theorem for maps from a normal subgroup to an abelian one. Theorem 5.4 below combined with Theorem 5.2 completes the proof of Theorem 5.1.

### 5.2.1 Federer-inspired proof for graphs of arbitrary codimension

**Theorem 5.4.** *Fix  $1 \leq k \leq n$  and let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  where  $\mathbb{V}$  is a horizontal subgroup of dimension  $k$ . Let  $A \subseteq \mathbb{W}$  be an open set and  $\phi : A \rightarrow \mathbb{V}$ . Then  $\phi$  is intrinsically differentiable almost everywhere on  $S_\phi$ .*

*Proof.* For convenience we split the proof into several steps.

*Step 1: Split  $S_\phi$  into countably many sets where  $\phi$  is intrinsic Lipschitz.*

For  $j \in \mathbb{N}$  we define

$$E_j := \left\{ w \in A : C_{1/j}(w \cdot \phi(w)) \cap \text{gr}_{\phi|_{B_{\mathbb{W}}(w, 1/j)}} = \{w \cdot \phi(w)\} \right\}. \quad (5.2)$$

Then each  $E_j$  is measurable and it is clear that  $S_\phi = \bigcup_{j \in \mathbb{N}} E_j$ . Then we express each  $E_j$  as the union of measurable sets  $E_{j,1}, E_{j,2}, \dots$  such that  $\text{diam}(E_{j,i}) < \frac{1}{j}$  for every  $i, j \in \mathbb{N}$ . We can do that for example intersecting  $E_j$  with countably many balls of diameter smaller than  $\frac{1}{j}$ . Then we have that  $S_\phi = \bigcup_{j,i \in \mathbb{N}} E_{j,i}$  and  $\phi|_{E_{j,i}}$  is intrinsic Lipschitz.

*Step 2: Use Theorem 2.31 to extend each  $\phi|_{E_{j,i}}$ .*

By Theorem 2.31 for every  $j, i \in \mathbb{N}$  there exists an intrinsic Lipschitz map  $\tilde{\phi} : \mathbb{W} \rightarrow \mathbb{V}$  such that  $\tilde{\phi}|_{E_{j,i}} \equiv \phi|_{E_{j,i}}$  and  $\tilde{\phi}$  is intrinsically differentiable almost everywhere on  $\mathbb{W}$ . Fix  $\bar{w} \in E_{j,i}$  such that  $\bar{w}$  is a point of intrinsic differentiability for  $\tilde{\phi}$  and  $\bar{w} \cdot \phi(\bar{w})$  is a density point of  $\text{gr}_{\phi|_{E_{j,i}}}$  with respect to  $\mathcal{S}_d^{Q-k} \llcorner \text{gr}_{\tilde{\phi}}$  (recall that from Theorem 2.22  $\mathcal{S}_d^{Q-k} \llcorner \text{gr}_{\tilde{\phi}}$  is a  $(Q-k)$ -Ahlfors regular measure on  $\text{gr}_{\tilde{\phi}}$ , so that Lebesgue density theorem holds). By [158, Remark 4.6], there exists a constant  $\bar{C} > 0$  such that, denoting by  $\Phi$  the graph map  $\Phi(w) := w \cdot \tilde{\phi}(w)$ ,

$$\bar{C}^{-1} \mathcal{S}_d^{Q-k} \llcorner \text{gr}_{\tilde{\phi}} \leq \Phi_{\#}(\mathcal{L}^{2n+1-k} \llcorner \mathbb{W}) \leq \bar{C} \mathcal{S}_d^{Q-k} \llcorner \text{gr}_{\tilde{\phi}}.$$

This implies that the set of points  $\bar{w}$  with the previous properties is a full-measure set in  $E_{j,i}$ .

If we prove that  $\bar{w}$  is also a point of intrinsic differentiability for  $\phi$  then we are done.

*Step 3: Without loss of generality, one can assume  $\bar{w} = 0$  and  $\phi(\bar{w}) = 0$ .*

Assuming that  $\bar{w} = 0$  and  $\phi(\bar{w}) = 0$  is equivalent to replacing the function  $\phi$  with

the translated function  $\phi_{\bar{w}}$  (see (2.3)) since  $\phi_{\bar{w}}(0) = 0$ . Notice that, by definition,  $\phi$  is intrinsically differentiable at  $\bar{w}$  if and only if  $\phi_{\bar{w}}$  is intrinsically differentiable at 0. Hence, it suffices to show that all the properties we will use of the map  $\phi$  are true also for  $\phi_{\bar{w}}$ . Again, the differentiability of  $\tilde{\phi}$  is preserved by translation of the graph. The same holds for the intrinsic Lipschitz property of  $\tilde{\phi}$ . Moreover  $\bar{w} \cdot \phi(\bar{w})$  is a density point of  $\text{gr}_{\phi|E_{j,i}}$  if and only if 0 is a density point of  $(\bar{w} \cdot \phi(\bar{w}))^{-1} \cdot \text{gr}_{\phi|E_{j,i}}$  (by invariance of the distance and the measure). The last condition to be verified is to show that there exists  $\delta > 0$  such that for all  $w \in E_{j,i}$  one has

$$B_{\mathbb{W}}(\phi(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot w \cdot \phi(\bar{w}), \delta) \subseteq \phi(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot B_{\mathbb{W}}(w, \frac{1}{j}) \cdot \phi(\bar{w}). \quad (5.3)$$

In fact, if (5.3) holds, then we obtain *Step 3* upon noticing that the set corresponding to  $B_{\mathbb{W}}(w, \frac{1}{j})$  after the translation of the graph of  $\phi$  is exactly  $\phi(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot B_{\mathbb{W}}(w, \frac{1}{j}) \cdot \phi(\bar{w})$ . Hence, the property defining  $E_j$  in (5.2) remains true for the function  $\phi_{\bar{w}}$  replacing  $\frac{1}{j}$  with  $\delta$ .

We are left to prove (5.3). Since

$$\lim_{\mathbb{W} \ni a \rightarrow 0} \|\phi(\bar{w}) \cdot a \cdot \phi(\bar{w})^{-1}\| = 0,$$

there exists  $\delta > 0$  such that, if  $\|a\| < \delta$ , then

$$\|\phi(\bar{w}) \cdot a \cdot \phi(\bar{w})^{-1}\| < \frac{1}{j}. \quad (5.4)$$

We define  $\tilde{w} := \phi(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot w \cdot \phi(\bar{w})$  and we claim that

$$\phi(\bar{w}) \cdot B_{\mathbb{W}}(\tilde{w}, \delta) \cdot \phi(\bar{w})^{-1} \subseteq B_{\mathbb{W}}(\phi(\bar{w}) \cdot \tilde{w} \cdot \phi(\bar{w})^{-1}, \frac{1}{j}). \quad (5.5)$$

If (5.5) holds, then

$$\begin{aligned} B_{\mathbb{W}}(\tilde{w}, \delta) &\subseteq \phi(\bar{w})^{-1} \cdot B_{\mathbb{W}}(\phi(\bar{w}) \cdot \tilde{w} \cdot \phi(\bar{w})^{-1}, \frac{1}{j}) \cdot \phi(\bar{w}) \\ &= \phi(\bar{w})^{-1} B_{\mathbb{W}}(\bar{w}^{-1} \cdot w, \frac{1}{j}) \cdot \phi(\bar{w}) \\ &= \phi(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot B_{\mathbb{W}}(w, \frac{1}{j}) \cdot \phi(\bar{w}). \end{aligned}$$

proving (5.3). We are left to prove (5.5). Let  $y \in \phi(\bar{w}) \cdot B_{\mathbb{W}}(\tilde{w}, \delta) \cdot \phi(\bar{w})^{-1}$ . Then  $y = \phi(\bar{w}) \cdot x \cdot \phi(\bar{w})^{-1}$  for some  $x \in B_{\mathbb{W}}(\tilde{w}, \delta)$ . Since  $d(x, \tilde{w}) < \delta$ , by (5.4) we have  $\|\phi(\bar{w}) \cdot \tilde{w}^{-1} \cdot x \cdot \phi(\bar{w})^{-1}\| < \frac{1}{j}$ . The latter implies that

$$d(y, \phi(\bar{w}) \cdot \tilde{w} \cdot \phi(\bar{w})^{-1}) = \|\phi(\bar{w}) \cdot \tilde{w}^{-1} \cdot \phi(\bar{w})^{-1} \cdot y\| = \|\phi(\bar{w}) \cdot \tilde{w}^{-1} \cdot x \cdot \phi(\bar{w})^{-1}\| < \frac{1}{j},$$

finally proving (5.5).

*Step 4: Use the equivalent characterization from Theorem 2.30.*

Since  $\tilde{\phi}$  is intrinsically differentiable at 0, by Theorem 2.30, there exists a vertical subgroup  $\mathbb{T}_{\tilde{\phi}, 0}$  such that for every  $\alpha > 0$  there exists  $\tilde{r} = \tilde{r}(\tilde{\phi}, 0, \alpha) > 0$  such that

$$C_{\mathbb{T}_{\tilde{\phi}, 0}, \mathbb{V}}(0, \alpha) \cap \text{gr}_{\tilde{\phi}|B_{\mathbb{W}}(0, \tilde{r})} = \{0\}. \quad (5.6)$$

Using again Theorem 2.30, if we show that for every  $\alpha > 0$  there exists  $\bar{r} = \bar{r}(\phi, 0, \alpha) > 0$  such that

$$C_{\mathbb{T}_{\tilde{\phi},0},\mathbb{V}}(0, \alpha) \cap \mathbf{gr}_{\phi|_{B_{\mathbb{W}}(0,\bar{r})}} = \{0\},$$

then we get that  $\phi$  is intrinsically differentiable at 0. Assume not: then there exists a certain  $\alpha > 0$  such that for every  $\bar{r} > 0$  one has

$$C_{\mathbb{T},\mathbb{V}}(0, \alpha) \cap \mathbf{gr}_{\phi|_{B_{\mathbb{W}}(0,\bar{r})}} \neq \{0\}. \quad (5.7)$$

where, for the sake of brevity, we write  $\mathbb{T} := \mathbb{T}_{\tilde{\phi},0}$ . From (5.7) we obtain that there is a sequence of points  $(x_h)_{h \in \mathbb{N}} \subseteq \mathbb{W}$  such that  $x_h \xrightarrow{h \rightarrow \infty} 0$  and

$$p_h := x_h \cdot \phi(x_h) \in C_{\mathbb{T},\mathbb{V}}(0, \alpha). \quad (5.8)$$

On the other hand  $\tilde{\phi}$  is intrinsically differentiable at 0 so, by (5.6), for  $h$  sufficiently large

$$\tilde{p}_h := x_h \cdot \tilde{\phi}(x_h) \notin C_{\mathbb{T},\mathbb{V}}(0, 2\alpha). \quad (5.9)$$

*Step 5: Prove that  $(p_h)_{\mathbb{T}} = (\tilde{p}_h)_{\mathbb{T}}$ .*

Here and in the following of the proof we will use the following notations: in order to indicate the projections of a point  $q \in \mathbb{H}^n$  with respect to the splitting  $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$  we will use  $q = q_{\mathbb{W}} \cdot q_{\mathbb{V}}$ ; in order to indicate the components of a point  $q \in \mathbb{H}^n$  with respect to the splitting  $\mathbb{H}^n = \mathbb{T} \cdot \mathbb{V}$  we will use  $q = q_{\mathbb{T}} \cdot q_{\mathbb{V}}^{\mathbb{T}}$ . Notice that, in general,  $q_{\mathbb{V}} \neq q_{\mathbb{V}}^{\mathbb{T}}$ . We observe that

$$(p_h)_{\mathbb{W}} \cdot (p_h)_{\mathbb{V}} = p_h = (p_h)_{\mathbb{T}} \cdot (p_h)_{\mathbb{V}}^{\mathbb{T}} = ((p_h)_{\mathbb{T}})_{\mathbb{W}} \cdot \underbrace{((p_h)_{\mathbb{T}})_{\mathbb{V}} \cdot (p_h)_{\mathbb{V}}^{\mathbb{T}}}_{\in \mathbb{V}}.$$

By the uniqueness of the components (see Remark 2.14) we conclude that  $(p_h)_{\mathbb{W}} = ((p_h)_{\mathbb{T}})_{\mathbb{W}}$  and so  $x_h = ((p_h)_{\mathbb{T}})_{\mathbb{W}}$ . In the same fashion we obtain that  $x_h = ((\tilde{p}_h)_{\mathbb{T}})_{\mathbb{W}}$  and so  $((p_h)_{\mathbb{T}})_{\mathbb{W}} = ((\tilde{p}_h)_{\mathbb{T}})_{\mathbb{W}}$ . Now we observe that

$$\mathbb{T} \ni ((p_h)_{\mathbb{T}})^{-1} \cdot (\tilde{p}_h)_{\mathbb{T}} = (((p_h)_{\mathbb{T}})_{\mathbb{V}})^{-1} \cdot \underbrace{(((p_h)_{\mathbb{T}})_{\mathbb{W}})^{-1} \cdot ((\tilde{p}_h)_{\mathbb{T}})_{\mathbb{W}}}_{=0} \cdot ((\tilde{p}_h)_{\mathbb{T}})_{\mathbb{V}} \in \mathbb{V}$$

but  $\mathbb{T}$  and  $\mathbb{V}$  are complementary so  $\mathbb{T} \cap \mathbb{V} = \{0\}$  implying that  $(p_h)_{\mathbb{T}} = (\tilde{p}_h)_{\mathbb{T}}$ .

*Step 6: Prove that  $d(\phi(x_h), \tilde{\phi}(x_h)) \geq K\|x_h\|$  for some  $K > 0$  and for  $h$  sufficiently large.*

We observe preliminarily that, by the fact that  $((p_h)_{\mathbb{T}})_{\mathbb{W}} = x_h$ ,

$$(p_h)_{\mathbb{V}}^{\mathbb{T}} = ((p_h)_{\mathbb{T}})^{-1} \cdot p_h = (((p_h)_{\mathbb{T}})_{\mathbb{V}})^{-1} \cdot x_h^{-1} \cdot x_h \cdot \phi(x_h) = (((p_h)_{\mathbb{T}})_{\mathbb{V}})^{-1} \cdot \phi(x_h) \quad (5.10)$$

and, in the same fashion, since  $(p_h)_{\mathbb{T}} = (\tilde{p}_h)_{\mathbb{T}}$ ,

$$(\tilde{p}_h)_{\mathbb{V}}^{\mathbb{T}} = (((\tilde{p}_h)_{\mathbb{T}})_{\mathbb{V}})^{-1} \cdot \tilde{\phi}(x_h) = (((p_h)_{\mathbb{T}})_{\mathbb{V}})^{-1} \cdot \tilde{\phi}(x_h). \quad (5.11)$$

From (5.8) and (5.9) we obtain, for  $h$  sufficiently large,

$$\begin{cases} \|(p_h)_\mathbb{T}\| \leq \alpha \|(p_h)_\mathbb{V}^\mathbb{T}\|, \\ \|(\tilde{p}_h)_\mathbb{T}\| > 2\alpha \|(\tilde{p}_h)_\mathbb{V}^\mathbb{T}\|. \end{cases} \quad (5.12)$$

By the left-invariance of the distance we have

$$\begin{aligned} d(\phi(x_h), \tilde{\phi}(x_h)) &= d\left(\left(\left((p_h)_\mathbb{T}\right)_\mathbb{V}\right)^{-1} \cdot \phi(x_h), \left(\left((p_h)_\mathbb{T}\right)_\mathbb{V}\right)^{-1} \cdot \tilde{\phi}(x_h)\right) \\ &= d\left(\left(p_h\right)_\mathbb{V}^\mathbb{T}, \left(\tilde{p}_h\right)_\mathbb{V}^\mathbb{T}\right) \\ &= \left\| \left(\left(\tilde{p}_h\right)_\mathbb{V}^\mathbb{T}\right)^{-1} \cdot \left(p_h\right)_\mathbb{V}^\mathbb{T} \right\| \\ &\geq \left\| \left(p_h\right)_\mathbb{V}^\mathbb{T} \right\| - \left\| \left(\tilde{p}_h\right)_\mathbb{V}^\mathbb{T} \right\| \\ &\geq \frac{1}{2\alpha} \left\| \left(p_h\right)_\mathbb{T} \right\|. \end{aligned}$$

where we used, in order from the first to the last line, the left-invariance of the distance, (5.10) and (5.11), the definition of norm associated to the distance, the reverse triangular inequality and, finally, (5.12) combined with Step 5. From Proposition 2.15, we obtain that

$$\left\| \left(p_h\right)_\mathbb{T} \right\| = \left\| x_h \cdot \left(\left(p_h\right)_\mathbb{T}\right)_\mathbb{V} \right\| \geq \tilde{C} \|x_h\| \quad (5.13)$$

where  $\tilde{C}$  is a constant only depending on  $\mathbb{W}$  and  $\mathbb{V}$ . The latter proves that  $d(\phi(x_h), \tilde{\phi}(x_h)) \geq K \|x_h\|$  for some  $K > 0$ .

*Step 7: Construction of an auxiliary sequence  $(q_h)_{h \in \mathbb{N}}$  with certain properties.*

Since we are assuming that 0 is a density point of  $\text{gr}_{\phi|_{E_{j,i}}}$  in  $\text{gr}_{\tilde{\phi}}$ , by [19, pag. 409] there exists a sequence  $(q_h)_{h \in \mathbb{N}} \subseteq \text{gr}_{\phi|_{E_{j,i}}}$  such that

$$d(q_h, \tilde{p}_h) = o(d(0, \tilde{p}_h)) = o(\|\tilde{p}_h\|).$$

In other words, for every  $\varepsilon > 0$  there exists  $\bar{h} \in \mathbb{N}$  such that for every  $h > \bar{h}$

$$d(q_h, \tilde{p}_h) \leq \varepsilon \|\tilde{p}_h\|. \quad (5.14)$$

We observe that

$$\|\tilde{p}_h\| = \|x_h \cdot \tilde{\phi}(x_h)\| \leq \|x_h\| + \|\tilde{\phi}(x_h)\| \quad (5.15)$$

and since  $\tilde{\phi}$  is intrinsically Lipschitz and  $\tilde{\phi}(0) = \phi(0) = 0$  we obtain  $\|\tilde{\phi}(x_h)\| \leq L \|x_h\|$  for some constant  $L > 0$ . The latter together with (5.14) and (5.15) implies that for every  $\varepsilon > 0$  there exists  $\bar{h}_1 \in \mathbb{N}$  such that for every  $h > \bar{h}_1$

$$d(q_h, \tilde{p}_h) \leq \varepsilon \|x_h\|. \quad (5.16)$$

Moreover, for every  $h \in \mathbb{N}$  there exists  $y_h \in E_{j,i}$  such that

$$q_h = y_h \cdot \phi(y_h) = y_h \cdot \tilde{\phi}(y_h),$$

so that we can rewrite (5.16) as

$$d(y_h \cdot \tilde{\phi}(y_h), x_h \cdot \tilde{\phi}(x_h)) \leq \varepsilon \|x_h\|. \quad (5.17)$$

Since  $\mathbb{W}$  is normal, the projection on  $\mathbb{V}$  is Lipschitz continuous (again Proposition 2.15) and there exists a constant  $D > 0$  such that

$$d(\tilde{\phi}(y_h), \tilde{\phi}(x_h)) \leq Dd(y_h \cdot \tilde{\phi}(y_h), x_h \cdot \tilde{\phi}(x_h)).$$

The latter together with (5.17) implies that for every  $\varepsilon > 0$  there exists  $\bar{h}_2 \in \mathbb{N}$  such that for every  $h > \bar{h}_2$

$$d(\tilde{\phi}(y_h), \tilde{\phi}(x_h)) \leq \varepsilon \|x_h\|. \quad (5.18)$$

Moreover, since  $y_h \in E_{j,i} \subseteq E_j$  it follows from (5.2) and (5.3) that

$$C_{1/j}(q_h) \cap \text{gr}_{\phi|_{B_{\mathbb{W}}(y_h, \delta)}} = \{q_h\}, \quad (5.19)$$

where  $\delta > 0$  is found as in (5.3).

*Step 8: Proof that  $d(x_h, y_h) \xrightarrow{h \rightarrow +\infty} 0$ .*

By (5.16) and the fact that  $x_h \rightarrow 0$ , we know that  $d(q_h, \tilde{p}_h) \xrightarrow{h \rightarrow +\infty} 0$ . If  $\tilde{C}$  is as in (5.13), by the explicit expression of the projections given in Proposition 2.15 (2), we have

$$\tilde{C} \|\tilde{\phi}(x_h)^{-1} \cdot x_h^{-1} \cdot y_h \cdot \tilde{\phi}(x_h)\| = \tilde{C} \|(\tilde{p}_h^{-1} \cdot q_h)_{\mathbb{W}}\| \leq \|\tilde{p}_h^{-1} \cdot q_h\| = d(q_h, \tilde{p}_h)$$

which implies

$$\|\tilde{\phi}(x_h)^{-1} \cdot x_h^{-1} \cdot y_h \cdot \tilde{\phi}(x_h)\| \xrightarrow{h \rightarrow +\infty} 0. \quad (5.20)$$

Recall that  $\tilde{\phi}$  is intrinsic Lipschitz, therefore continuous, so  $x_h \xrightarrow{h \rightarrow +\infty} 0$  implies  $\tilde{\phi}(x_h) \rightarrow \tilde{\phi}(0) = \phi(0) = 0$ . We conclude that  $d(x_h, y_h) \xrightarrow{h \rightarrow +\infty} 0$  upon observing that from (5.20) we get

$$d(x_h, y_h) = \|x_h^{-1} \cdot y_h\| = \|\tilde{\phi}(x_h) \cdot \tilde{\phi}(x_h)^{-1} \cdot x_h^{-1} \cdot y_h \cdot \tilde{\phi}(x_h) \cdot \tilde{\phi}(x_h)^{-1}\| \xrightarrow{h \rightarrow +\infty} 0.$$

The latter implies that, for  $h$  sufficiently large,  $x_h \in B_{\mathbb{W}}(y_h, \delta)$ .

*Step 9: Conclude the proof obtaining a contradiction.*

Because of Step 8 and (5.19), we have for  $h$  large enough

$$p_h = x_h \cdot \phi(x_h) \notin C_{1/j}(q_h) = q_h \cdot C_{1/j}(0). \quad (5.21)$$

The latter is true unless  $p_h = q_h$ , but in that case we would get  $x_h = y_h$  by uniqueness of the projections. Therefore  $x_h \in E_{j,i}$  and hence  $\phi(x_h) = \tilde{\phi}(x_h)$ : so we would obtain a contradiction with Step 6. From (5.21) we obtain

$$q_h^{-1} \cdot p_h \notin C_{1/j}(0) \Rightarrow \|(q_h^{-1} \cdot p_h)_{\mathbb{W}}\| > \frac{1}{j} \|(q_h^{-1} \cdot p_h)_{\mathbb{V}}\|. \quad (5.22)$$

From an explicit computation of the projections (see Proposition 2.15), we get

$$\begin{cases} (q_h^{-1} \cdot p_h)_{\mathbb{V}} = \tilde{\phi}(y_h)^{-1} \cdot \phi(x_h), \\ (q_h^{-1} \cdot p_h)_{\mathbb{W}} = \tilde{\phi}(y_h)^{-1} \cdot y_h^{-1} \cdot x_h \cdot \tilde{\phi}(y_h). \end{cases} \quad (5.23)$$

In particular  $(q_h^{-1} \cdot p_h)_{\mathbb{W}} = (q_h^{-1} \cdot \tilde{p}_h)_{\mathbb{W}}$ . From (5.23), (5.22) and Proposition 2.15 (3) we then obtain

$$\|\tilde{\phi}(y_h)^{-1} \cdot \phi(x_h)\| = \|(q_h^{-1} \cdot p_h)_{\mathbb{V}}\| < j \|(q_h^{-1} \cdot p_h)_{\mathbb{W}}\| = j \|(q_h^{-1} \cdot \tilde{p}_h)_{\mathbb{W}}\| \leq Cj \|q_h^{-1} \cdot \tilde{p}_h\|. \quad (5.24)$$

Finally

$$\begin{aligned} \|\tilde{\phi}(x_h)^{-1} \cdot \phi(x_h)\| &\leq \|\tilde{\phi}(x_h)^{-1} \cdot \tilde{\phi}(y_h)\| + \|\tilde{\phi}(y_h)^{-1} \cdot \phi(x_h)\| \\ &\leq \varepsilon \|x_h\| + jCd(\tilde{p}_h, q_h) \\ &\leq \varepsilon \|x_h\| + jC\varepsilon \|x_h\| \\ &= \underbrace{(1 + jC)}_M \varepsilon \|x_h\|, \end{aligned}$$

where in the first three lines we used the triangle inequality, (5.18) and (5.24), and, finally, (5.16). Combining the latter with Step 6 (that is  $d(\phi(x_h), \tilde{\phi}(x_h)) \geq K\|x_h\|$ ) we get

$$\|\tilde{\phi}(x_h)^{-1} \cdot \phi(x_h)\| \leq M\varepsilon \frac{d(\phi(x_h), \tilde{\phi}(x_h))}{K} = M\varepsilon \frac{\|\tilde{\phi}(x_h)^{-1} \cdot \phi(x_h)\|}{K} \quad (5.25)$$

where  $K$  is the same constant coming from Step 6. Simplifying  $\|\tilde{\phi}(x_h)^{-1} \cdot \phi(x_h)\|$  from both sides of (5.25) we obtain

$$1 \leq \frac{M\varepsilon}{K}$$

but then we get a contradiction from the arbitrariness of  $\varepsilon$ , concluding the proof.  $\square$

**Remark 5.5.** We emphasize that the proof of Theorem 5.4 is not dependent on the particular structure of  $\mathbb{H}^n$ . It can be extended to general Carnot groups  $\mathbb{G} = \mathbb{W}\mathbb{V}$ , where  $\mathbb{W}$  is a normal subgroup and  $\mathbb{V}$  is horizontal, provided that a Rademacher-type theorem holds. This is particularly relevant for graphs of codimension 1 either in step 2 Carnot groups or, more generally, in Carnot groups of type  $\star$ . However, it is important to note that the validity of a Rademacher-type theorem for intrinsic graphs of arbitrary codimension has only been established in the Heisenberg case.

## 5.2.2 Malý-inspired proof for graphs of codimension 1

In this subsection we study the case of *1-codimensional* graphs in  $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ , which means that we restrict to the case  $\dim(\mathbb{V}) = 1$ . Under this assumption,  $\mathbb{V} = \{\exp(tV) : t \in \mathbb{R}\}$  for a fixed  $V \in \mathfrak{h}_1$  (the first layer of the stratification of the Lie algebra of  $\mathbb{H}^n$ ) and we can naturally identify  $\mathbb{V} \equiv \mathbb{R}$  with its usual order relation. In particular we can define *infimum*

and supremum of functions: if  $\phi_\alpha : \mathbb{W} \rightarrow \mathbb{V}$  are such that  $\phi_\alpha(w) = \exp(g_\alpha(w)V)$ , for some  $g_\alpha : \mathbb{W} \rightarrow \mathbb{R}$ , we define

$$\inf_{\alpha \in I} \phi_\alpha(w) := \exp \left( \inf_{\alpha \in I} g_\alpha(w)V \right).$$

In the same way we can define the supremum of a family of maps from  $\mathbb{W}$  to  $\mathbb{V}$ .

Infimum and supremum of intrinsic Lipschitz maps are themselves intrinsic Lipschitz maps:

**Lemma 5.6** ([83, Proposition 4.24]). *Let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  where  $\mathbb{V}$  is a horizontal subgroup of dimension 1. Then, for all  $L > 0$  there exists  $\tilde{L} \geq L$  with the following property: if  $\{\phi_\alpha : \mathbb{W} \rightarrow \mathbb{V}\}$  is a family of intrinsic  $L$ -Lipschitz maps, then the function  $\phi := \inf_\alpha \phi_\alpha$  is either well defined and intrinsic  $\tilde{L}$ -Lipschitz, or  $\phi \equiv -\infty$ . The same property holds for the supremum.*

Before presenting the alternative proof, inspired by the proof of J. Malý [126], we need an auxiliary lemma.

**Lemma 5.7.** *Let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$ , where  $\mathbb{V}$  is a horizontal subgroup of dimension 1. Let  $A \subset \mathbb{W}$  be open and  $\bar{w} \in A$ . Suppose  $\psi, \phi, \eta : A \rightarrow \mathbb{V}$  are such that  $\psi \leq \phi \leq \eta$  on  $A$  (where we identify  $\mathbb{V} \equiv \mathbb{R}$ ),  $\psi(\bar{w}) = \phi(\bar{w}) = \eta(\bar{w})$  and assume also that the functions  $\psi$  and  $\eta$  are intrinsically differentiable at  $\bar{w}$ . Then  $\phi$  is intrinsically differentiable at  $\bar{w}$  and*

$$d\psi_{\bar{w}} \equiv d\phi_{\bar{w}} \equiv d\eta_{\bar{w}}.$$

*Proof.* For convenience we split the proof into 4 steps.

*Step 1:* We start by proving that, if a map  $\varphi : A \subset \mathbb{W} \rightarrow \mathbb{V}$  is intrinsically differentiable at  $\bar{w} = 0$ ,  $\varphi(0) = 0$  and  $\varphi(w) \geq 0$  for every  $w \in A$ , then  $d\varphi_0 \equiv 0$ . Notice that in this case  $\varphi_0(w) = \varphi(w)$ . By differentiability we know that

$$\frac{\|d\varphi_0(w)^{-1} \cdot \varphi(w)\|}{\|w\|} \rightarrow 0 \quad \text{if } w \rightarrow 0. \quad (5.26)$$

Since  $\mathbb{V} \equiv \mathbb{R}$  is horizontal, we can rewrite (5.26) as

$$\frac{\varphi(w) - d\varphi_0(w)}{\|w\|} \rightarrow 0 \quad \text{if } w \rightarrow 0.$$

By definition of limit, for every  $\epsilon > 0$  there exists  $\delta > 0$  such that, if  $\|w\| < \delta$ , then

$$\frac{\varphi(w)}{\|w\|} < \frac{d\varphi_0(w)}{\|w\|} + \epsilon.$$

Since  $\varphi(w) \geq 0$  and by homogeneity of  $d\varphi_0$  we get

$$d\varphi_0 \left( \delta \frac{1}{\|w\|} w \right) > -\epsilon.$$

Notice that  $\delta \frac{1}{\|w\|} w \in \partial B(0,1) \cap \mathbb{W}$ , independently on  $\delta$ . By arbitrariness of  $\epsilon$  we infer that  $d\varphi_0(u) \geq 0$  for every  $u \in \partial B(0,1) \cap \mathbb{W}$ , hence also for every  $u \in \mathbb{W}$ . This is possible only if  $d\varphi_0 \equiv 0$ , because an intrinsic linear map from  $\mathbb{W}$  to  $\mathbb{V}$  is actually  $H$ -linear and, if  $d\varphi_0(u) > 0$  for some  $u$ , then  $d\varphi_0(u^{-1}) = -d\varphi_0(u) < 0$ .

*Step 2:* We show now that, if  $\rho, \sigma : A \subset \mathbb{W} \rightarrow \mathbb{V}$  are intrinsically differentiable at 0 with  $\rho(0) = \sigma(0) = 0$ , then  $\rho^{-1}\sigma$  is also intrinsically differentiable at 0 and moreover  $d(\rho^{-1}\sigma)_0 = d\rho_0^{-1} \cdot d\sigma_0$ .

It is easy to check that  $d\rho_0^{-1} \cdot d\sigma_0$  is an intrinsic linear map, since in our setting intrinsic linear maps are exactly  $H$ -linear maps (see [16, Proposition 3.23]). So we are left to prove that

$$\frac{\|(d\rho_0^{-1}(w) \cdot d\sigma_0(w))^{-1} \cdot (\rho^{-1}\sigma)_0(w)\|}{\|w\|} \rightarrow 0 \quad \text{as } w \rightarrow 0.$$

Since  $\rho(0) = \sigma(0) = 0$ , then  $(\rho^{-1}\sigma)_0 = \rho^{-1}\sigma$ . Hence, by commutativity of  $\mathbb{V}$  and the triangular inequality,

$$\frac{\|(d\rho_0^{-1}(w) \cdot d\sigma_0(w))^{-1} \cdot (\rho^{-1}\sigma)_0(w)\|}{\|w\|} \leq \frac{\|d\rho_0(w)^{-1} \cdot \rho(w)\|}{\|w\|} + \frac{\|d\sigma_0(w)^{-1} \cdot \sigma(w)\|}{\|w\|}$$

The last two quantities tends to 0 as  $w \rightarrow 0$  since  $\rho, \sigma$  are intrinsically differentiable at 0 and we conclude.

*Step 3:* Let us now prove that if  $\psi$  and  $\eta$  are as in the statement of the Lemma then  $d\psi_{\bar{w}} \equiv d\eta_{\bar{w}}$ . Consider the translated functions

$$\begin{aligned} \psi_{\bar{w}}(w) &= \psi(\bar{w})^{-1} \psi(\bar{w} \psi(\bar{w}) w \psi(\bar{w})^{-1}), \\ \eta_{\bar{w}}(w) &= \eta(\bar{w})^{-1} \eta(\bar{w} \eta(\bar{w}) w \eta(\bar{w})^{-1}). \end{aligned}$$

Since  $\psi \leq \eta$  and  $\psi(\bar{w}) = \eta(\bar{w})$ , we get  $\psi_{\bar{w}} \leq \eta_{\bar{w}}$  and clearly  $\psi_{\bar{w}}(0) = \eta_{\bar{w}}(0) = 0$ . Notice also that since  $\psi$  and  $\eta$  are intrinsically differentiable at  $\bar{w}$  so are  $\psi_{\bar{w}}$  and  $\eta_{\bar{w}}$  at 0. Hence, by *Step 2*, the map  $\psi_{\bar{w}}^{-1} \eta_{\bar{w}}$  is intrinsically differentiable at 0 and  $d(\psi_{\bar{w}}^{-1} \eta_{\bar{w}})_0 = d(\psi_{\bar{w}})_0^{-1} \cdot d(\eta_{\bar{w}})_0$ . Moreover  $\psi_{\bar{w}}^{-1} \eta_{\bar{w}} \geq 0$ . Thus, by *Step 1*,  $d(\psi_{\bar{w}}^{-1} \eta_{\bar{w}})_0 \equiv 0$ , which implies  $d(\psi_{\bar{w}})_0 \equiv d(\eta_{\bar{w}})_0$ . Hence, by definition,  $d\psi_{\bar{w}} = d\eta_{\bar{w}}$ .

*Step 4:* Let us finally prove the main conclusion. Let  $\psi, \phi, \eta$  be as in the statement. As before we observe that  $\psi_{\bar{w}} \leq \phi_{\bar{w}} \leq \eta_{\bar{w}}$ . Now let  $\theta := d\psi_{\bar{w}} \equiv d\eta_{\bar{w}}$ . Then for every  $w \in \mathbb{W}$  near 0 we have

$$\frac{\psi_{\bar{w}}(w) - d\psi_{\bar{w}}(w)}{d(0, w)} \leq \frac{\phi_{\bar{w}}(w) - \theta(w)}{d(0, w)} \leq \frac{\eta_{\bar{w}}(w) - d\eta_{\bar{w}}(w)}{d(0, w)}.$$

Then the left hand side and the right hand side go to 0 when  $w \in B(0, s) \cap \mathbb{W}$  for  $s \rightarrow 0$ . This concludes the proof.  $\square$

Now we can present the alternative proof for Stepanov Theorem for 1-codimensional intrinsic graphs, that we restate for the reader's convenience.

**Theorem 5.8.** *Let  $\mathbb{W}, \mathbb{V}$  be complementary subgroups of  $\mathbb{H}^n$  where  $\mathbb{V}$  is a horizontal subgroup of dimension 1. Let  $A \subseteq \mathbb{W}$  be an open set and  $\phi : A \rightarrow \mathbb{V}$ . Then  $\phi$  is intrinsically differentiable almost everywhere on  $S_\phi$ .*

*Proof.* Let  $\{U_j\}_{j \in \mathbb{N}}$  be an enumeration of all rational balls (i.e. Euclidean balls with rational center and rational radius) contained in  $A$  such that  $\phi$  is bounded on  $U_j$  (here we are identifying  $\mathbb{W} \equiv \mathbb{R}^{2n}$ ). Is it clear that  $S_\phi \subseteq \bigcup_{i \in \mathbb{N}} U_i$ . For each  $j \in \mathbb{N}$  we define two intrinsic Lipschitz functions  $\eta_j$  and  $\psi_j$  on  $U_j$  by setting

$$\eta_j(w) := \inf\{\eta(w) : \eta \geq \phi \text{ on } B_j, \text{Lip}(\eta, U_j) \leq j\}, \quad (5.27)$$

$$\psi_j(w) := \sup\{\psi(w) : \psi \leq \phi \text{ on } B_j, \text{Lip}(\psi, U_j) \leq j\}. \quad (5.28)$$

By Lemma 5.6 (combined with extension Theorem in [157, Proposition 3.4] or [84, Theorem 4.1]), for every  $j \in \mathbb{N}$  there exists  $\tilde{j} \geq j$  such that  $\eta_j$  and  $\psi_j$  are intrinsic  $\tilde{j}$ -Lipschitz on  $U_j$ . Define now

$$N = \bigcup_{j \in \mathbb{N}} \{w \in U_j : \eta_j \text{ or } \psi_j \text{ is not intrinsically differentiable at } w\}.$$

By Theorem 2.31, we have that  $\mathcal{L}^{2n}(N) = 0$ . Let  $\bar{w} \in S_\phi \setminus N$ : we will prove that  $\phi$  is intrinsically differentiable at  $\bar{w}$ , concluding the proof. By definition of  $S_\phi$ , there exist  $\beta > 0$  and  $r > 0$  such that

$$C_\beta(\bar{w} \cdot \phi(\bar{w})) \cap \text{gr}_{\phi|_{B_{\mathbb{W}}(\bar{w}, r)}} = \{\bar{w} \cdot \phi(\bar{w})\}.$$

Since  $\mathbb{V}$  has dimension 1 we can write  $\mathbb{V} = \{\exp(tV) \mid t \in \mathbb{R}\}$ . Moreover, the ‘‘positive part’’ of the cone  $C_\beta^+(\bar{w} \cdot \phi(\bar{w})) := C_\beta(\bar{w} \cdot \phi(\bar{w})) \cap \exp(\{Z \mid \langle Z, V \rangle \geq 0\})$  is the graph of an intrinsic  $M$ -Lipschitz function  $\gamma : \mathbb{W} \rightarrow \mathbb{V}$  for some  $M > 0$  (see Lemma 4.20 in [83] and also [84]). Consider now  $i \geq M$  such that

$$B(\bar{w}, r/2) \subseteq U_i \subseteq B(\bar{w}, r).$$

Clearly  $\phi(\bar{w}) \leq \eta_i(\bar{w})$ . On the other hand  $\gamma$  is a suitable competitor in the family defined in (5.27): hence  $\eta_i(\bar{w}) \leq \gamma(\bar{w}) = \phi(\bar{w})$ . The same argument works for  $\psi_i$  and we deduce  $\psi_i(\bar{w}) = \phi(\bar{w}) = \eta_i(\bar{w})$ . Hence we conclude using Lemma 5.7.  $\square$

**Remark 5.9.** Similarly to Theorem 5.2 and Theorem 5.4, the proof of Theorem 5.8 can be extended to a more general context. Specifically, the same approach can be applied to one-codimensional graphs within general Carnot groups that satisfy a Rademacher-type theorem. The equivalence between intrinsic linear and  $H$ -linear maps (from normal to abelian subgroups), utilized in Lemma 5.7, can be derived as shown in [16], using, for example, [55, Proposition 3.4].

# 6 Submanifolds with boundary and Stokes' Theorem in Heisenberg groups

The aim of this chapter is to present the results contained in [58]. The main result of this chapter is the following version of Stokes' Theorem.

**Theorem 6.1.** *If  $1 \leq m \leq 2n + 1$ ,  $S \subset \mathbb{H}^n$  is an  $m$ -dimensional orientable  $C_{\mathbb{H}}^1$  regular submanifold with boundary and  $S, \partial S$  have locally finite measures in  $\mathbb{H}^n$ , then*

$$\int_S d_c \omega = \int_{\partial S} \omega \quad \text{for every } \omega \in \mathcal{D}_{\mathbb{H}}^{m-1}, \quad (6.1)$$

where  $\mathcal{D}_{\mathbb{H}}^{m-1}$  denotes the space of Heisenberg-Rumin differential  $(m - 1)$ -forms with compact support.

This chapter is divided into 5 sections. In Section 6.1 we present some preliminary results about properties of  $C_{\mathbb{H}}^1$ -regular submanifolds (without boundary and both low and high dimensional) that will be pivotal in the proof of the results contained in Section 6.3 and Section 6.4. In Section 6.2 we introduce currents in Heisenberg groups, and we define integration of Heisenberg-Rumin differential forms (which were defined in Section 2.4). In Section 6.3 we give our definition of  $C_{\mathbb{H}}^1$ -regular submanifold with boundary, we provide some equivalent characterizations, local properties and explicit examples and we describe how, given an orientation on the interior of the submanifold, a natural orientation can be induced on its boundary. In Section 6.4 we give the proof of two approximation results, Lemma 6.38 and Lemma 6.39, that allows us to approximate locally, in the sense of currents, a  $C_{\mathbb{H}}^1$ -regular submanifold with boundary with a sequence of  $C^1, C_{\mathbb{H}}^1$ -regular submanifolds with boundary. Finally in Section 6.5 we prove Theorem 6.1 by using the aforementioned approximating results and a partition of the unity type of argument.

## 6.1 Preliminary results

### 6.1.1 Properties of low dimensional $C_{\mathbb{H}}^1$ -regular submanifolds

In this subsection we present the following lemma which allows us to compare, locally, the distance between two points on a low dimensional  $C_{\mathbb{H}}^1$ -regular submanifold with the Euclidean distance between the horizontal projections of such points. Lemma 6.2 will be used later in the proof of Lemma 6.25.

**Lemma 6.2.** *Let  $1 \leq k \leq n$  and  $S \subset \mathbb{H}^n$  be a  $k$ -dimensional  $C_{\mathbb{H}}^1$  regular submanifold. Then for any  $p \in S$  there exist an open set  $U \subset \mathbb{H}^n$  with  $p \in U$  and a positive constant  $C$  such that for any  $q, q' \in U \cap S$  one has*

$$d(q, q') \leq C |\pi(q) - \pi(q')|_{\mathbb{R}^{2n}}$$

where  $\pi : \mathbb{H}^n \rightarrow \mathbb{R}^{2n}$  is the projection map defined by  $\pi(x, y, t) := (x, y)$ .

*Proof.* By Proposition 2.6 it is sufficient to prove the result when  $d = d_{\infty}$  (as defined in Example 2.5). Fix  $p \in S$ . Let  $V \subseteq \mathbb{R}^k$  and  $O \subseteq \mathbb{H}^n$  be open sets such that  $p \in O$  and  $f = (f_1, \dots, f_{2n+1}) : V \rightarrow \mathbb{H}^n$  be the defining function of  $S$  coming from Definition 2.35, i.e.,  $f(V) = O \cap S$ . Let  $U = B_d(p, r)$  where  $r > 0$  will be chosen later. Let  $q, q' \in U \cap S$  and  $a, b, b' \in V$  such that  $f(a) = p, f(b) = q$  and  $f(b') = q'$ . One has

$$\begin{aligned} d(q, q') &= d(f(b), f(b')) = d((f(b))^{-1} \cdot f(b'), 0) \\ &\leq d(d_{\mathbb{H}}f_a(b - b'), (f(b))^{-1} \cdot f(b')) + d(d_{\mathbb{H}}f_a(b - b'), 0). \end{aligned} \quad (6.2)$$

From [100, Proposition 2.4] we obtain

$$d(d_{\mathbb{H}}f_a(b - b'), (f(b))^{-1} \cdot f(b')) \leq |b - b'|_{\mathbb{R}^k}$$

and from [82, Proposition 2.7] one has (since the last component of  $d_{\mathbb{H}}f_a$  is always 0)

$$d(d_{\mathbb{H}}f_a(b - b'), 0) = |d_{\mathbb{H}}f_a(b - b')|_{\mathbb{R}^{2n+1}}.$$

Using this inequalities in (6.2) together with the fact that, by the injectivity of  $d_{\mathbb{H}}f_a$ , there exists a positive constant  $C$  such that  $|b - b'|_{\mathbb{R}^k} \leq C |d_{\mathbb{H}}f_a(b - b')|_{\mathbb{R}^{2n}}$  we get

$$d(q, q') \leq (1 + C) |d_{\mathbb{H}}f_a(b - b')|_{\mathbb{R}^{2n+1}}.$$

By [82, Proposition 2.7] we have

$$d_{\mathbb{H}}f_a(b - b') = \begin{pmatrix} \langle \nabla f_1(a), b - b' \rangle_{\mathbb{R}^k} \\ \vdots \\ \langle \nabla f_{2n}(a), b - b' \rangle_{\mathbb{R}^k} \\ 0 \end{pmatrix}.$$

We claim that there exists a positive constant  $C$  such that for every  $1 \leq j \leq 2n$

$$|\langle \nabla f_j(a), b - b' \rangle_{\mathbb{R}^k}| \leq C |f_j(b) - f_j(b')|;$$

this would prove the Lemma. We have (without loss of generality we can suppose  $b \neq b'$ )

$$|\langle \nabla f_j(a), b - b' \rangle_{\mathbb{R}^k}| = |b - b'|_{\mathbb{R}^k} \left| \left\langle \nabla f_j(a), \frac{b - b'}{|b - b'|_{\mathbb{R}^k}} \right\rangle_{\mathbb{R}^k} \right| = |b - b'|_{\mathbb{R}^k} \left| \frac{\partial f_j}{\partial v}(a) \right| \quad (6.3)$$

where we used  $\frac{\partial f_j}{\partial v}$  to denote the derivative of  $f_j$  in the direction  $v = \frac{b-b'}{|b-b'|_{\mathbb{R}^k}}$ . By definition we have

$$\frac{\partial f_j}{\partial v}(a) = \lim_{b, b' \rightarrow a} \frac{f_j(b) - f_j(b')}{b - b'};$$

therefore, possibly reducing  $V$ , we can assume that

$$\left| \frac{\partial f_j}{\partial v}(a) \right| \leq 1 + \left| \frac{f_j(b) - f_j(b')}{b - b'} \right| \quad \forall b, b' \in V.$$

If we use this information in (6.3) we obtain

$$|\langle \nabla f_j(a), b - b' \rangle_{\mathbb{R}^k}| \leq |b - b'|_{\mathbb{R}^k} + |f_j(b) - f_j(b')| \quad (6.4)$$

for every  $b, b' \in V$ . From [82, Theorem 3.5]  $f_j$  is  $C^1$  and injective with injective (classical) differential  $df_a$  so, possibly reducing  $V$ ,  $f_j$  is biLipschitz continuous, i.e., there exist  $K > 0$  such that for any  $b, b' \in V$  one has

$$\frac{1}{K}|b - b'|_{\mathbb{R}^k} \leq |f_j(b) - f_j(b')| \leq K|b - b'|_{\mathbb{R}^k}.$$

From the latter and (6.4) we eventually obtain

$$|\langle \nabla f_j(a), b - b' \rangle_{\mathbb{R}^k}| \leq |b - b'|_{\mathbb{R}^k} + |f_j(b) - f_j(b')| \leq (K + 1)|f_j(b) - f_j(b')|$$

for any  $b, b' \in V$ . The proof is concluded provided we choose  $U = B_d(p, r)$  with  $r > 0$  small enough such that  $B_d(p, r) \cap S \subseteq f(V)$ .  $\square$

### 6.1.2 Properties of high dimensional $C_{\mathbb{H}}^1$ -regular submanifolds

In this subsection we present three preliminary results about high dimensional  $C_{\mathbb{H}}^1$ -regular submanifolds: Lemmata 6.3, 6.5 and Theorem 6.6 which will be pivotal in the sequel.

The following Lemma 6.3 immediately follows from [100, Lemma 2.14 (iii)].

**Lemma 6.3.** *Let  $S \subset \mathbb{H}^n$  be a  $C_{\mathbb{H}}^1$  submanifold of codimension  $1 \leq k \leq n$ . Then, for every compact set  $K \subset S$  and every  $\varepsilon > 0$  there exists  $\delta > 0$  such that*

$$\forall p \in K, \forall q \in S \cap U(p, \delta) \quad d(q, p \cdot T_p^{\mathbb{H}} S) < \varepsilon d(p, q).$$

An equivalent definition of high dimensional  $C_{\mathbb{H}}^1$ -regular submanifold is provided in the subsequent Lemma 6.5, the proof of which requires group convolution, which we now introduce. See also [75, Chapter 1].

**Definition 6.4.** Let  $k$  be a positive integer,  $H \in C_c^\infty(\mathbb{H}^n, \mathbb{R})$  and  $G : \mathbb{H} \rightarrow \mathbb{R}^k$ . We use  $G \star H$  to denote the *group convolution* between  $G$  and  $H$ , which is defined, for every  $x \in \mathbb{H}^n$ , as

$$(G \star H)(x) := \int_{\mathbb{H}^n} G(y^{-1} \cdot x) H(y) d\mathcal{L}^{2n+1}(y) = \int_{\mathbb{H}^n} G(y) H(x \cdot y^{-1}) d\mathcal{L}^{2n+1}(y).$$

Notice that  $G \star H$  is a smooth map satisfying

$$W(G \star H) = (WG) \star H$$

for every  $W \in \mathfrak{h}$ .

Let  $\varepsilon > 0$ . In the following we will use  $K_\varepsilon$  to denote the *standard mollification kernel* that is,  $K_\varepsilon := \varepsilon^{-Q} K \circ \delta_{1/\varepsilon}$ , where  $K \in C_c^\infty(B_d(0, 1))$  is a fixed non-negative function (a kernel) such that  $\int_{\mathbb{H}^n} K d\mathcal{L}^{2n+1} = 1$ .

We use the compact notation  $\{f = 0\}$  to denote the level set  $\{p \in D : f(p) = 0\}$  of a function  $f : D \rightarrow \mathbb{R}^k$ ; the domain  $D$  of  $f$  will always be clear from the context. Moreover, for  $f \in C_{\mathbb{H}}^1(\mathbb{H}^n, \mathbb{R}^k)$  and  $q \in \mathbb{H}^n$  we denote with  $\widehat{\nabla}_{\mathbb{H}} f(q)$  the  $k \times k$  matrix  $\text{col}[X_1 f | \cdots | X_k f](q)$  and we denote by  $\text{Id}_{k \times k}$  the  $k \times k$  identity matrix. The following lemma will be useful later.

**Lemma 6.5.** *Let  $1 \leq k \leq n$  and  $S \subset \mathbb{H}^n$ . The following statements are equivalent:*

- (i)  $S$  is a  $C_{\mathbb{H}}^1$  submanifold of codimension  $k$ ;
- (ii) for every  $p \in S$  there exist an open set  $U$  with  $p \in U$ , a function  $f : \mathbb{H}^n \rightarrow \mathbb{R}^k$  and  $\delta_0 > 0$  such that, up to an isometry of  $\mathbb{H}^n$ ,
  - (1)  $f \in C_{\mathbb{H}}^1(\mathbb{H}^n, \mathbb{R}^k)$ ,
  - (2)  $|\nabla_{\mathbb{H}} f|$  is bounded on  $\mathbb{H}^n$ ,
  - (3)  $f \in C^\infty(\mathbb{H}^n \setminus \{f = 0\}, \mathbb{R}^k)$ ,
  - (4)  $\widehat{\nabla}_{\mathbb{H}} f(q) \geq \delta_0 \text{Id}_{k \times k}$  for every  $q \in \mathbb{H}^n$  in the sense of quadratic forms<sup>1</sup>,
  - (5)  $S \cap U = \{q \in U : f(q) = 0\}$ .

*Proof.* The implication (ii)  $\Rightarrow$  (i) is clear. Let us prove the implication (i)  $\Rightarrow$  (ii). Fix  $p \in S$  and an open ball  $B_d(p, 3r)$  where  $r > 0$  will be chosen later. Then there exists a function  $g \in C_{\mathbb{H}}^1(B_d(p, 3r), \mathbb{R}^k)$  with  $\nabla_{\mathbb{H}} g$  of maximal rank on  $B_d(p, 3r)$  and

$$S \cap B_d(p, 3r) = \{q \in B_d(p, 3r) : g(q) = 0\}.$$

Using the Whitney Extension Theorem (see [79, Theorem 6.8] and [54, Theorem 2.3.8]), it is not restrictive to assume that  $g \in C_{\mathbb{H}}^1(\mathbb{H}^n, \mathbb{R}^k) \cap C^\infty(\mathbb{H}^n \setminus [\{g = 0\} \cap B_d(p, 3r)], \mathbb{R}^k)$ . By [82, Proposition 3.25], up to an isometry of  $\mathbb{H}^n$  we can assume that  $X_1 g(p), \dots, X_k g(p) \in \mathbb{R}^k$  are linearly independent; in particular, there exists  $L \in GL(\mathbb{R}^k)$  such that  $L(X_i g(p)) = e_i$ , where  $e_1, \dots, e_k$  is the canonical basis of  $\mathbb{R}^k$ . Upon replacing  $g$  with  $L \circ g$  we have  $\widehat{\nabla}_{\mathbb{H}} g(p) = \text{Id}_{k \times k}$ . Possibly reducing  $r$ , we can assume that  $|\nabla_{\mathbb{H}} g|$  is bounded on  $B_d(p, 3r)$  and that  $\widehat{\nabla}_{\mathbb{H}} g \geq \frac{3}{4} \text{Id}_{k \times k}$  on  $B_d(p, 3r)$ .

<sup>1</sup>That is,  $\langle \widehat{\nabla}_{\mathbb{H}} f(q)v, v \rangle_{\mathbb{R}^k} \geq \delta_0 |v|_{\mathbb{R}^k}^2$  for every  $q \in \mathbb{H}^n$  and for every  $v \in \mathbb{R}^k$ .

Fix  $\chi_1 \in C_c^\infty(\mathbb{H}^n)$  such that  $0 \leq \chi_1 \leq 1$ ,  $\chi_1 \equiv 1$  on  $B_d(p, 1)$  and  $\chi_1 \equiv 0$  on  $\mathbb{H}^n \setminus B_d(p, 2)$ ; set  $C := \|\nabla_{\mathbb{H}} \chi_1\|_{C^0}$  and  $\chi_r(q) := \chi_1(\delta_{1/r}(q))$  for  $q \in \mathbb{H}^n$ . Then the functions  $\chi_r \in C_c^\infty(\mathbb{H}^n)$  satisfy

$$\begin{cases} 0 \leq \chi_r \leq 1 \\ \chi_r \equiv 1 \text{ on } B_d(p, r) \\ \chi_r \equiv 0 \text{ on } \mathbb{H}^n \setminus B_d(p, 2r) \\ |\nabla_{\mathbb{H}} \chi_r| \leq \frac{C}{r}. \end{cases} \quad (6.5)$$

We now consider the differential  $d_{\mathbb{H}}g_p : \mathbb{H}^n \rightarrow \mathbb{R}^k$  of  $g$  at  $p$  and define  $h : \mathbb{H}^n \rightarrow \mathbb{R}^k$  by

$$h := \chi_r g + (1 - \chi_r) d_{\mathbb{H}}g_p.$$

Clearly,  $h \in C_{\mathbb{H}}^1(\mathbb{H}^n, \mathbb{R}^k)$ ,  $h$  is Lipschitz continuous and  $S \cap B_d(p, r) = \{p \in B_d(p, r) : h(p) = 0\}$ .

We claim that

$$\widehat{\nabla}_{\mathbb{H}} h \geq \frac{1}{2} \text{Id}_{k \times k} \quad \text{on } \mathbb{H}^n; \quad (6.6)$$

we have

$$\begin{aligned} \widehat{\nabla}_{\mathbb{H}} h &= \widehat{\nabla}_{\mathbb{H}} \chi_r \otimes (g - d_{\mathbb{H}}g_p) + \chi_r \widehat{\nabla}_{\mathbb{H}} g + (1 - \chi_r) \widehat{\nabla}_{\mathbb{H}} g(p) \\ &\geq \widehat{\nabla}_{\mathbb{H}} \chi_r \otimes (g - d_{\mathbb{H}}g_p) + \chi_r \frac{3}{4} \text{Id}_{k \times k} + (1 - \chi_r) \text{Id}_{k \times k} \\ &\geq \widehat{\nabla}_{\mathbb{H}} \chi_r \otimes (g - d_{\mathbb{H}}g_p) + \frac{3}{4} \text{Id}_{k \times k}. \end{aligned}$$

We are left to estimate the first term  $\widehat{\nabla}_{\mathbb{H}} \chi_r \otimes (g - d_{\mathbb{H}}g_p)$ . We have that

$$\|\widehat{\nabla}_{\mathbb{H}} \chi_r \otimes (g - d_{\mathbb{H}}g_p)\|_{C^0(\mathbb{H}^n)} \leq \frac{C}{r} \|g - d_{\mathbb{H}}g_p\|_{C^0(U(p, 2r))} \leq \frac{C}{r} o(r) = o(1),$$

where we used the Taylor expansion of  $g$  at  $p$ . This proves the claim (6.6) provided  $r > 0$  is chosen small enough.

Now we use a standard mollification procedure (as for instance in [158, Proposition 2.10]) to construct a new function  $f$  which satisfies the properties (1)-(5) upon setting  $U = B_d(p, r)$ . First we define  $Z_h := \{p \in \mathbb{H}^n : h(p) = 0\}$ . For  $j \in \mathbb{N}$  we choose

- bounded open sets  $(U_j)_{j \in \mathbb{N}}$  such that  $\overline{U_j} \subset \mathbb{H}^n \setminus Z_h$  and  $\mathbb{H}^n \setminus Z_h = \bigcup_j U_j$ ,
- positive numbers  $\varepsilon_j$  such that  $\varepsilon_j < d(U_j, Z_h)$ ,
- nonnegative functions  $u_j \in C_c^\infty(U_j)$  forming a partition of the unity on  $\mathbb{H}^n \setminus Z_h$ .

We can also assume that  $\sum_j \mathbb{1}_{U_j} \leq M$  for some  $M > 0$ , where  $\mathbb{1}_{U_j}$  is the characteristic function of  $U_j$  so that the sum  $\sum_j u_j$  is locally finite. Possibly reducing  $\varepsilon_j > 0$ , as specified later, we define

$$f := \begin{cases} \sum_j u_j (h \star K_{\varepsilon_j}) & \text{on } \mathbb{H}^n \setminus Z_h \\ 0 & \text{on } Z_h, \end{cases}$$

where  $K_{\varepsilon_j}$  are standard mollification kernels. The function  $f$  is clearly smooth on  $\mathbb{H}^n \setminus Z_h$ , so in order to check that  $f$  is continuous on the whole  $\mathbb{H}^n$  we have to check its continuity at points of  $Z_h$ . Up to reducing  $\varepsilon_j$ , we can assume that

$$|(h \star K_{\varepsilon_j}) - h| \leq d(U_j, Z_h) \text{ on } U_j$$

so that

$$|f(x) - h(x)| \leq \sum_j u_j(x) |(h \star K_{\varepsilon_j})(x) - h(x)| \leq d(x, Z_h).$$

This implies the continuity of  $f$  since for every  $\bar{x} \in Z_h$  one has

$$\lim_{x \rightarrow \bar{x}} |f(x) - f(\bar{x})| = \lim_{x \rightarrow \bar{x}} |f(x)| \leq \lim_{x \rightarrow \bar{x}} |h(x)| + d(x, \bar{x}) = 0.$$

Now we want to prove that  $f \in C_{\mathbb{H}}^1(\mathbb{H}^n, \mathbb{R}^k)$ . Since  $f$  is by definition smooth on  $\mathbb{H}^n \setminus Z_h$  we just need to prove that for every  $W \in \mathfrak{h}_1$  and  $\bar{x} \in Z_h$

$$\lim_{x \rightarrow \bar{x}, x \notin Z_h} (Wf)(x) = (Wh)(\bar{x}).$$

For every  $x \in \mathbb{H}^n \setminus Z_h$  we get (using  $\sum_j Wu_j \equiv 0$ )

$$\begin{aligned} & |(Wf)(x) - (Wh)(x)| \\ &= \left| \sum_j (Wu_j)(x) (h \star K_{\varepsilon_j})(x) + \sum_j u_j(x) (Wh \star K_{\varepsilon_j})(x) - (Wh)(x) \right| \\ &\leq \left| \sum_j (Wu_j)(x) [(h \star K_{\varepsilon_j})(x) - h(x)] \right| + \left| \sum_j u_j(x) [(Wh \star K_{\varepsilon_j})(x) - (Wh)(x)] \right| \\ &\leq Cd(x, Z_h) \end{aligned}$$

and letting  $x \rightarrow \bar{x}$  we get the continuity of the horizontal derivatives, i.e., property (1).

Now we prove property (2), i.e., that  $|\nabla_{\mathbb{H}} f|$  is bounded. It is enough to prove that  $|\nabla_{\mathbb{H}} f|$  is bounded on  $\mathbb{H}^n \setminus Z_h$ . We observe that

$$\begin{aligned} \nabla_{\mathbb{H}} f &= \sum_j (h \star K_{\varepsilon_j}) \otimes (\nabla_{\mathbb{H}} u_j) + \sum_j u_j ((\nabla_{\mathbb{H}} h) \star K_{\varepsilon_j}) \\ &= \sum_j (h \star K_{\varepsilon_j} - h) \otimes (\nabla_{\mathbb{H}} u_j) + \sum_j u_j ((\nabla_{\mathbb{H}} h) \star K_{\varepsilon_j}) \end{aligned}$$

on  $\mathbb{H}^n \setminus Z_h$ . The last sum is bounded by  $\sup |\nabla_{\mathbb{H}} h|$ . Up to reducing  $\varepsilon_j$ , we can assume that

$$|h \star K_{\varepsilon_j} - h| \leq (\sup |\nabla_{\mathbb{H}} u_j|)^{-1} \text{ on } U_j$$

so that the second to last sum is bounded by  $M$ .

As for property (3), we observe that  $\{f = 0\} = Z_h$ : in fact, by [158, Theorem 1.4] both  $\{f = 0\}$  and  $Z_h$  are entire intrinsic graphs on  $\exp(\text{span}\{X_{k+1}, \dots, Y_n, T\})$ . Since  $Z_h \subset \{f = 0\}$ , the two sets coincide, hence  $f$  is smooth on  $\mathbb{H}^n \setminus Z_h = \mathbb{H}^n \setminus \{f = 0\}$ .

We prove property (4); using (6.6) we get

$$\begin{aligned}\widehat{\nabla}_{\mathbb{H}}f &= \sum_j (h \star K_{\varepsilon_j} - h) \otimes (\widehat{\nabla}_{\mathbb{H}}u_j) + \sum_j u_j ((\widehat{\nabla}_{\mathbb{H}}h) \star K_{\varepsilon_j}) \\ &\geq \sum_j (h \star K_{\varepsilon_j} - h) \otimes (\widehat{\nabla}_{\mathbb{H}}u_j) + \frac{1}{2} \text{Id}_{k \times k}.\end{aligned}$$

Given  $\eta > 0$ , possibly reducing  $\varepsilon_j$  we can assume that

$$|h \star K_{\varepsilon_j} - h| \leq \eta (\sup |\widehat{\nabla}_{\mathbb{H}}u_j|)^{-1} \text{ on } U_j$$

so that  $\sup_{\mathbb{H}^n} |\sum_j (h \star K_{\varepsilon_j} - h) \otimes (\widehat{\nabla}_{\mathbb{H}}u_j)| \leq M\eta$ . If  $\eta$  is small enough we get

$$\widehat{\nabla}_{\mathbb{H}}f \geq \frac{1}{4} \text{Id}_{k \times k},$$

which is property (4).

Eventually, property (5) trivially follows upon setting  $U = B_d(p, r)$  and noticing that

$$S \cap U = \{p \in U : g(p) = 0\} = \{p \in U : h(p) = 0\} = \{p \in U : f(p) = 0\}.$$

The proof is accomplished.  $\square$

Combining Lemma 6.5 with Theorem 2.40, Theorem 2.41 and [158, Theorem 1.6 and Proposition 2.10], we can state the following result which sums all the local properties of high dimensional  $C_{\mathbb{H}}^1$ -regular submanifolds that we are going to need to prove Stokes' Theorem.

**Theorem 6.6.** *Let  $S \subset \mathbb{H}^n$  be a  $C_{\mathbb{H}}^1$  submanifold of codimension  $1 \leq k \leq n$ . Then  $S$  is locally an orthogonal continuous graph, i.e., for each  $p \in S$  there exist*

- an orthogonal splitting  $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$  where  $\mathbb{V}$  is a horizontal  $k$ -dimensional vector space,
- relatively open sets  $A \subset \mathbb{W}$  and  $B \subset \mathbb{V}$  such that  $p \in U := A \cdot B$ ,
- a continuous function  $\phi : A \rightarrow B$
- a function  $f = (f_1, \dots, f_k) \in C_{\mathbb{H}}^1(U, \mathbb{R}^k)$  with  $\nabla_{\mathbb{H}}f_1 \wedge \dots \wedge \nabla_{\mathbb{H}}f_k \neq 0$

such that

$$S \cap U = \{q \in U : f(q) = 0\} = \{\xi \cdot \phi(\xi) : \xi \in A\}.$$

Further, if  $v_1, \dots, v_k \in \mathfrak{v}$  form an orthonormal basis of the Lie algebra  $\mathfrak{v} \subset \mathfrak{h}$  of  $\mathbb{V}$  and we put

$$\Delta(q) := |\det[v_i f_j(q)]_{1 \leq i, j \leq k}| \neq 0 \text{ for } q \in U$$

then

$$S_d^{Q-k} \llcorner (S \cap U) = C_{n,k} \Phi_{\#} \left( \left( \frac{|\nabla_{\mathbb{H}}f_1 \wedge \dots \wedge \nabla_{\mathbb{H}}f_k|}{\Delta} \circ \Phi \right) \mathcal{H}_E^{2n+1-k} \llcorner \mathbb{W} \right) \quad (6.7)$$

where  $\Phi$  is the graph map associated to  $\phi$ ,  $\Phi_{\#}$  denotes the push-forward of measures,  $\mathcal{H}_E^{2n+1-k}$  denotes the classical Euclidean  $(2n+1-k)$ -Hausdorff measure and  $C_{n,k}$  (see Remark 2.42) is a positive constant only depending on  $n, k$  and the distance  $d$ .

Moreover, the function  $f$  above can be chosen to be defined in the whole  $\mathbb{H}^n$ , and also in such a way that  $f \in C_{\mathbb{H}}^1(\mathbb{H}^n, \mathbb{R}^k) \cap C^\infty(\mathbb{H}^n \setminus \{x \in \mathbb{H}^n : f(x) = 0\}, \mathbb{R}^k)$  and there exists a sequence of smooth functions  $\phi_h : \mathbb{W} \rightarrow \mathbb{V}$  such that

$$\begin{aligned} \phi_h &\rightarrow \phi \text{ uniformly in } A \text{ as } h \rightarrow +\infty \\ \text{gr}_{\phi_h} &= \{q \in \mathbb{H}^n : f(q) = (\frac{1}{h}, 0, \dots, 0)\}. \end{aligned}$$

## 6.2 Currents and integration on $C_{\mathbb{H}}^1$ -regular submanifolds

In this section we introduce Heisenberg currents and we define integration on  $C_{\mathbb{H}}^1$ -regular submanifold. In the low dimensional case this is almost trivial (see Proposition 6.10), but in the high dimensional case we will need an auxiliary result (Lemma 6.12). In Lemma 6.14 we show that we can approximate, in the sense of currents, a  $C_{\mathbb{H}}^1$ -regular submanifold (without boundary) with a sequence of smooth and  $C_{\mathbb{H}}^1$ -regular submanifolds. Finally in Lemma 6.16 we show that our definition of integration is intrinsic in the sense that it does not depend on the specific choice of the distance (see Remark 6.17).

**Definition 6.7.** Given an open set  $O \subset \mathbb{H}^n$  we define the Heisenberg-Rumin differential  $k$ -forms with compact support in  $O$  as

$$\begin{aligned} \mathcal{D}_{\mathbb{H}}^k(O) &:= C_c^\infty \left( O, \frac{\wedge^k \mathfrak{h}}{\mathcal{I}^k} \right) && \text{if } 0 \leq k \leq n, \\ \mathcal{D}_{\mathbb{H}}^k(O) &:= C_c^\infty(O, \mathcal{J}^k) && \text{if } n+1 \leq k \leq 2n+1. \end{aligned}$$

We denote the dual of  $\mathcal{D}_{\mathbb{H}}^k(O)$  (endowed with the natural topology) by  $\mathcal{D}_{\mathbb{H},k}(O)$ . An element of  $\mathcal{D}_{\mathbb{H},k}(O)$  is called *Heisenberg current* of dimension  $k$  (or Heisenberg  $k$ -current).

**Definition 6.8.** Let  $O \subset \mathbb{H}^n$  be an open set and  $T \in \mathcal{D}_{\mathbb{H},k}(O)$  with  $1 \leq k \leq 2n+1$ . The boundary  $\partial_c T \in \mathcal{D}_{\mathbb{H},k-1}(O)$  of  $T$  is the Heisenberg  $(k-1)$ -current defined by duality with  $d_c$ , i.e.,

$$\begin{aligned} \partial_c T(\omega) &:= T(d\omega) \text{ if } k \neq n+1 \\ \partial_c T(\omega) &:= T(D\omega) \text{ if } k = n+1 \end{aligned}$$

for every  $\omega \in \mathcal{D}_{\mathbb{H}}^{k-1}(O)$ .

We want to define the current canonically associated to a  $C_{\mathbb{H}}^1$ -regular submanifold; this, of course, requires the notion of orientability.

Recalling Remark 2.36, for submanifolds of low dimension  $1 \leq k \leq n$  we define orientability as the classical Euclidean one.

Let then  $O \subset \mathbb{H}^n$  be an open set and  $S \subset O$  be an oriented  $C_{\mathbb{H}}^1$  submanifold of dimension  $1 \leq k \leq n$  and with locally finite measure; we claim that for every smooth  $(k-1)$ -form  $\lambda$  and every smooth  $(k-2)$ -form  $\mu$

$$\int_S \lambda \wedge \theta = 0 \quad \text{and} \quad \int_S \mu \wedge d\theta = 0. \quad (6.8)$$

The first equality in (6.8) is clear because  $TS \subset H\mathbb{H}^n = \ker \theta$ , see again Remark 2.36. Concerning the second equality, let us fix a sequence  $(S_j)_j$  of  $k$ -dimensional submanifolds  $S_j \subset S$  with  $C^1$  boundary and such that  $S_j \nearrow S$ ; then

$$\begin{aligned} \int_S \mu \wedge d\theta &= (-1)^{k-2} \int_S (d(\mu \wedge \theta) - d\mu \wedge \theta) = (-1)^k \int_S d(\mu \wedge \theta) \\ &= (-1)^k \lim_{j \rightarrow \infty} \int_{S_j} d(\mu \wedge \theta) = (-1)^k \lim_{j \rightarrow \infty} \int_{\partial S_j} \mu \wedge \theta = 0, \end{aligned}$$

where we used the fact that  $T\partial S_j \subset TS \subset \ker \theta$ .

We can now state an important consequence of (6.8).

**Proposition 6.9.** *If  $O \subset \mathbb{H}^n$  is an open set,  $S \subset O$  is an oriented  $C_{\mathbb{H}}^1$ -regular submanifold of dimension  $1 \leq k \leq n$  whose measure is locally finite in  $O$  and  $\omega \in \mathcal{D}_{\mathbb{H}}^k(O)$  is a Heisenberg  $k$ -form, then the integral*

$$\int_S \tilde{\omega}$$

*does not depend on the representative  $\tilde{\omega} \in C_c^\infty(O, \wedge^k \mathfrak{h})$  of the equivalence class  $\omega$  in the quotient  $\mathcal{D}_{\mathbb{H}}^k(O) = C_c^\infty(O, \wedge^k \mathfrak{h}/\mathcal{I}^k)$ .*

Proposition 6.9 implies that the following definition is well-posed.

**Definition 6.10.** Let  $O \subset \mathbb{H}^n$  be an open set and  $S$  be an oriented  $C_{\mathbb{H}}^1$  submanifold of dimension  $1 \leq k \leq n$  such that  $\mathcal{S}^k \llcorner S$  is locally finite in  $O^2$ . If  $\omega \in \mathcal{D}_{\mathbb{H}}^k(O)$  we define

$$\int_S \omega := \int_S \tilde{\omega}$$

for any representative  $\tilde{\omega} \in C_c^\infty(O, \wedge^k \mathfrak{h})$  of the equivalence class  $\omega$  in the quotient  $\mathcal{D}_{\mathbb{H}}^k(O) = C_c^\infty(O, \wedge^k \mathfrak{h}/\mathcal{I}^k)$ .

We denote by  $\llbracket S \rrbracket \in \mathcal{D}_{\mathbb{H},k}(O)$  the Heisenberg current associated with  $S$  and defined by

$$\llbracket S \rrbracket(\omega) := \int_S \omega \quad \forall \omega \in \mathcal{D}_{\mathbb{H}}^k(O). \quad (6.9)$$

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<sup>2</sup>By Remark 2.36, the measure  $\mathcal{S}_d^k \llcorner S$  is locally finite in  $O$  if and only if the  $k$ -dimensional Euclidean Hausdorff measure on  $S$  is locally finite in  $O$ .

Let us now consider the case of submanifolds of low codimension; we use the following definition of orientability, see also [30].

**Definition 6.11.** Let  $S$  be a  $C_{\mathbb{H}}^1$  submanifold of codimension  $1 \leq k \leq n$ . We say that  $S$  is orientable if we can define  $t_S^{\mathbb{H}}$  (or, equivalently,  $n_S^{\mathbb{H}}$ ) continuously on the whole  $S$ .

We will use the following lemma to define Heisenberg currents canonically associated with  $C_{\mathbb{H}}^1$ -regular oriented submanifold.

**Lemma 6.12** ([158, Lemma 3.31]). *Let  $n \geq 1$  and  $1 \leq k \leq n$  be integers and let  $O \subset \mathbb{H}^n$  be open. Then for every oriented  $C^1$  submanifold  $S \subset O$  of codimension  $k$  which is also  $C_{\mathbb{H}}^1$  regular and such that  $\mathcal{S}_d^{\mathbb{Q}-k} \llcorner S$  is locally finite in  $O$ , we have*

$$\int_S \langle t_S^{\mathbb{H}} | \omega \rangle d\mathcal{S}_d^{\mathbb{Q}-k} = C_{n,k} \int_S \omega \quad \forall \omega \in \mathcal{D}_{\mathbb{H}}^{2n+1-k}(O).$$

where  $C_{n,k}$  is as in Remark 2.42.

**Definition 6.13.** Let  $O \subset \mathbb{H}^n$  be an open set and  $S$  be an oriented  $C_{\mathbb{H}}^1$  submanifold of codimension  $1 \leq k \leq n$  such that the measure  $\mathcal{S}_d^{\mathbb{Q}-k} \llcorner S$  is locally finite in  $O$ . For every  $\omega \in \mathcal{D}_{\mathbb{H}}^{2n+1-k}(O)$  we define

$$\int_S \omega := \frac{1}{C_{n,k}} \int_S \langle t_S^{\mathbb{H}} | \omega \rangle d\mathcal{S}_d^{\mathbb{Q}-k},$$

where  $C_{n,k}$  is as in Remark 2.42. The Heisenberg current  $\llbracket S \rrbracket \in \mathcal{D}_{\mathbb{H},2n+1-k}(O)$  associated with  $S$  is

$$\llbracket S \rrbracket(\omega) := \int_S \omega \quad \forall \omega \in \mathcal{D}_{\mathbb{H}}^{2n+1-k}(O).$$

Lemma 6.12 states that, when  $S$  is both  $C^1$  smooth and  $C_{\mathbb{H}}^1$  regular of codimension  $k \leq n$ , then the Heisenberg current associated with  $S$  coincide with the usual Euclidean one on  $\mathcal{D}_{\mathbb{H}}^{2n+1-k}$ .

**Lemma 6.14.** *Let  $O \subset \mathbb{H}^n$  be an open set,  $S \subset O$  be a  $C_{\mathbb{H}}^1$  submanifold of codimension  $1 \leq k \leq n$  such that  $\mathcal{S}_d^{\mathbb{Q}-k} \llcorner S$  is locally finite in  $O$ . Let  $\llbracket S \rrbracket \in \mathcal{D}_{\mathbb{H},2n+1-k}(O)$  be the associated current. Then for each  $p \in S$  there exist an open set  $U$  with  $p \in U$  and a sequence  $(S_h)_{h \in \mathbb{N}}$  of smooth  $C_{\mathbb{H}}^1$  submanifolds of codimension  $k$  with locally finite measure on  $O$  such that, for every  $\omega \in \mathcal{D}_{\mathbb{H}}^{2n+1-k}(U)$ ,*

$$\llbracket S_h \rrbracket(\omega) \xrightarrow{h \rightarrow +\infty} \llbracket S \rrbracket(\omega).$$

*Proof.* By Theorem 6.6 for each point  $p \in S$  there exist an open set  $U$ , a function  $f = (f_1, \dots, f_k) \in C_{\mathbb{H}}^1(\mathbb{H}^n, \mathbb{R}^k) \cap C^\infty(\mathbb{H}^n \setminus \{f = 0\}, \mathbb{R}^k)$  such that  $\nabla_{\mathbb{H}} f_1 \wedge \dots \wedge \nabla_{\mathbb{H}} f_k \neq 0$  on  $\mathbb{H}^n$ , an orthogonal splitting  $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$  and a map  $\phi : E \subset \mathbb{W} \rightarrow \mathbb{V}$  such that

$$S \cap U = \{x \in U : f(x) = 0\} = \{\tilde{\xi} \cdot \phi(\xi) : \xi \in E\}$$

The smooth submanifolds  $S_h := \left\{ f = \left( \frac{1}{h}, 0, \dots, 0 \right) \right\}$  satisfy (possibly reducing the subsets  $U$  and  $E$ )

$$S_h \cap U = \left\{ x \in U : f(x) = \left( \frac{1}{h}, 0, \dots, 0 \right) \right\} = \{ \zeta \cdot \phi_h(\zeta) : \zeta \in E \}$$

for suitable smooth maps  $\phi_h : E \rightarrow \mathbb{V}$  that converge uniformly to  $\phi$  on  $E$ . For  $\omega \in \mathcal{D}_{\mathbb{H}}^{2n+1-k}(U)$  we have

$$\llbracket S_h \rrbracket(\omega) = \frac{1}{C_{n,k}} \int_{S_h} \langle t_{S_h}^{\mathbb{H}}(q) | \omega(q) \rangle d\mathcal{S}_d^{Q-k}(q) = \frac{1}{C_{n,k}} \int_{S_h \cap U} \langle t_{S_h}^{\mathbb{H}}(q) | \omega(q) \rangle d\mathcal{S}_d^{Q-k}(q).$$

We define  $\Phi_h(\zeta) := \zeta \cdot \phi_h(\zeta)$  and  $\Delta_h(p)$  as

$$\Delta_h(p) := \left| \det \left[ v_i \left( f_j - \frac{\delta_{1j}}{h} \right) (p) \right]_{1 \leq i, j \leq k} \right|$$

where  $v_i \in \mathfrak{h}_1$  are those determined in Theorem 6.6. By the same Theorem we get

$$\begin{aligned} \llbracket S_h \rrbracket(\omega) &= \frac{1}{C_{n,k}} \int_{S_h \cap U} \langle t_{S_h}^{\mathbb{H}}(q) | \omega(q) \rangle d\mathcal{S}_d^{Q-k}(q) = \\ &= \int_E \langle t_{S_h}^{\mathbb{H}}(\Phi_h(\zeta)) | \omega(\Phi_h(\zeta)) \rangle \frac{|\nabla_{\mathbb{H}} \left( f_1 - \frac{1}{h} \right) \wedge \nabla_{\mathbb{H}} f_2 \wedge \dots \wedge \nabla_{\mathbb{H}} f_k|}{\Delta_h}(\Phi_h(\zeta)) d\zeta \end{aligned}$$

Now we make the following observations:

(a) Since  $\phi_h \xrightarrow{h \rightarrow +\infty} \phi$  uniformly on  $E$ , then  $\Phi_h \xrightarrow{h \rightarrow +\infty} \Phi$  (where  $\Phi(\zeta) := \zeta \cdot \phi(\zeta)$ ) uniformly on  $E$ .

(b)  $\nabla_{\mathbb{H}} \left( f_1 - \frac{1}{h} \right) = \nabla_{\mathbb{H}} f_1$  and  $f \in C_{\mathbb{H}}^1(U, \mathbb{R}^k)$ ; in particular, possibly reducing  $E$  and  $U$ ,

$$\left( \nabla_{\mathbb{H}} \left( f_1 - \frac{1}{h} \right) \wedge \dots \wedge \nabla_{\mathbb{H}} f_k \right) \circ \Phi_h \xrightarrow{h \rightarrow +\infty} (\nabla_{\mathbb{H}} f_1 \wedge \dots \wedge \nabla_{\mathbb{H}} f_k) \circ \Phi$$

uniformly on  $E$ .

(c) Also  $t_{S_h}^{\mathbb{H}} \circ \Phi_h \xrightarrow{h \rightarrow +\infty} t_S^{\mathbb{H}} \circ \Phi$  uniformly on  $E$  because

$$\begin{aligned} \lim_{h \rightarrow +\infty} t_{S_h}^{\mathbb{H}}(\Phi_h(\zeta)) &= * \left( \lim_{h \rightarrow +\infty} n_{S_h}^{\mathbb{H}}(\Phi_h(\zeta)) \right) \\ &= * \left( \lim_{h \rightarrow +\infty} \frac{\nabla_{\mathbb{H}} \left( f_1 - \frac{1}{h} \right) \wedge \dots \wedge \nabla_{\mathbb{H}} f_k}{|\nabla_{\mathbb{H}} \left( f_1 - \frac{1}{h} \right) \wedge \dots \wedge \nabla_{\mathbb{H}} f_k|}(\Phi_h(\zeta)) \right) \\ &= * \left( \lim_{h \rightarrow +\infty} \frac{\nabla_{\mathbb{H}} f_1 \wedge \dots \wedge \nabla_{\mathbb{H}} f_k}{|\nabla_{\mathbb{H}} f_1 \wedge \dots \wedge \nabla_{\mathbb{H}} f_k|}(\Phi_h(\zeta)) \right) \\ &= * \left( \frac{\nabla_{\mathbb{H}} f_1 \wedge \dots \wedge \nabla_{\mathbb{H}} f_k}{|\nabla_{\mathbb{H}} f_1 \wedge \dots \wedge \nabla_{\mathbb{H}} f_k|}(\Phi(\zeta)) \right) \\ &= *(n_S^{\mathbb{H}}(\Phi(\zeta))) \\ &= t_S^{\mathbb{H}}(\Phi(\zeta)). \end{aligned}$$

(d) Since  $\omega$  is smooth, then  $\omega \circ \Phi_h \xrightarrow{h \rightarrow +\infty} \omega \circ \Phi$  uniformly on  $E$ .

(e) By definition of  $\Delta_h$  (given above) and  $\Delta$  (as in Theorem 6.6),  $\Delta_h \circ \Phi \xrightarrow{h \rightarrow +\infty} \Delta \circ \Phi$  uniformly on  $E$ .

(f) There exists  $C > 0$  such that  $\Delta_h \geq C$  and  $\Delta \geq C$ .

With these observations in mind we get

$$\begin{aligned} \lim_{h \rightarrow +\infty} \llbracket S_h \rrbracket(\omega) &= \lim_{h \rightarrow +\infty} \int_E \langle t_{S_h}^{\mathbb{H}}(\Phi_h(\xi)) | \omega(\Phi_h(\xi)) \rangle \frac{|\nabla_{\mathbb{H}}(f_1 - \frac{1}{h}) \wedge \cdots \wedge \nabla_{\mathbb{H}} f_k|}{\Delta_h}(\Phi_h(\xi)) d\xi \\ &= \int_E \langle t_S^{\mathbb{H}}(\Phi(\xi)) | \omega(\Phi(\xi)) \rangle \frac{|\nabla_{\mathbb{H}} f_1 \wedge \cdots \wedge \nabla_{\mathbb{H}} f_k|}{\Delta}(\Phi(\xi)) d\xi \\ &= \frac{1}{C_{n,k}} \int_{S \cap U} \langle t_S^{\mathbb{H}}(q) | \omega(q) \rangle d\mathcal{S}_d^{Q-k}(q) \\ &= \llbracket S \rrbracket(\omega), \end{aligned}$$

as desired.  $\square$

**Remark 6.15.** Lemma 6.14 shows that it is possible to (locally) approximate Heisenberg currents associated with  $C_{\mathbb{H}}^1$  regular submanifolds by (currents associated with) smooth submanifolds. In Lemmata 6.38 and 6.39 we will do the analogue for  $C_{\mathbb{H}}^1$  regular submanifolds with boundary.

**Lemma 6.16.** Let  $1 \leq k \leq n$  and  $m \geq 0$  be integers,  $O \subseteq \mathbb{H}^n$  be an open set. Let  $d_1, d_2$  be two left invariant, homogeneous and rotationally invariant distances on  $\mathbb{H}^n$  and let  $\mathcal{S}_1^m, \mathcal{S}_2^m$  be the associated spherical Hausdorff measures; denote by  $C_{n,k}^1$  and  $C_{n,k}^2$  the constants provided by Lemma 6.12. Let  $S \subseteq O$  be an oriented  $k$ -codimensional  $C_{\mathbb{H}}^1$ -regular submanifold such that  $\mathcal{S}_1^{Q-k} \llcorner S$  and  $\mathcal{S}_2^{Q-k} \llcorner S$  are locally finite on  $O$ ; then,

$$\frac{1}{C_{n,k}^1} \int_S \langle t_S^{\mathbb{H}} | \omega \rangle d\mathcal{S}_1^{Q-k} = \frac{1}{C_{n,k}^2} \int_S \langle t_S^{\mathbb{H}} | \omega \rangle d\mathcal{S}_2^{Q-k}.$$

for every  $\omega \in \mathcal{D}_{\mathbb{H}}^{2n+1-k}(O)$ .

*Proof.* By Lemma 6.14 for every  $p \in S$  there exist an open set  $U$  with  $p \in U$  and a sequence  $(S_h)_{h \in \mathbb{N}}$  of smooth and  $C_{\mathbb{H}}^1$   $k$ -codimensional submanifolds such that, for every  $\omega \in \mathcal{D}_{\mathbb{H}}^{2n+1-k}(U)$

$$\int_{S_h} \omega \xrightarrow{h \rightarrow +\infty} \frac{1}{C_{n,k}^j} \int_S \langle t_S^{\mathbb{H}} | \omega \rangle d\mathcal{S}_j^{Q-k}. \quad (6.10)$$

for  $j = 1, 2$ . By Lemma 6.12, for every  $h \in \mathbb{N}$  we have

$$\int_{S_h} \omega = \frac{1}{C_{n,k}^1} \int_{S_h} \langle t_{S_h}^{\mathbb{H}} | \omega \rangle d\mathcal{S}_1^{Q-k} = \frac{1}{C_{n,k}^2} \int_{S_h} \langle t_{S_h}^{\mathbb{H}} | \omega \rangle d\mathcal{S}_2^{Q-k}.$$

Letting  $h \rightarrow +\infty$  and using (6.10) we get

$$\frac{1}{C_{n,k}^1} \int_S \langle t_S^{\mathbb{H}} | \omega \rangle d\mathcal{S}_1^{Q-k} = \frac{1}{C_{n,k}^2} \int_S \langle t_S^{\mathbb{H}} | \omega \rangle d\mathcal{S}_2^{Q-k}. \quad (6.11)$$

We want to extend this result to every  $\omega \in \mathcal{D}_{\mathbb{H}}^{2n+1-k}(O)$ . For each  $p \in S$  we just showed that there exists an open set  $U_p \subset O$  such that (6.11) holds for every  $\omega \in \mathcal{D}_{\mathbb{H}}^{2n+1-k}(U_p)$ . We extract from the family  $(U_p)_{p \in S}$  a countable (or, possibly, finite) sub-family  $(U_i)_{i \in \mathbb{N}}$  such that  $S \subset \bigcup_{i \in \mathbb{N}} U_i$ . We fix a partition of the unity, i.e., functions  $\zeta_i \in C_c^\infty(U_i), i \in \mathbb{N}$ , such that

$$0 \leq \zeta_i \leq 1 \quad \text{and} \quad \sum_{i \in \mathbb{N}} \zeta_i = 1 \text{ on } S \quad (6.12)$$

It is not restrictive to assume that the covering  $(U_i)_i$  of  $S$  is locally finite, so that the sum in (6.12) is well defined. Let  $\omega \in \mathcal{D}_{\mathbb{H}}^{2n-k+1}(O)$  be fixed. Then

$$\begin{aligned} \frac{1}{C_{n,k}^1} \int_S \langle t_S^{\mathbb{H}} | \omega \rangle d\mathcal{S}_1^{Q-k} &= \frac{1}{C_{n,k}^1} \sum_{i \in \mathbb{N}} \int_{S \cap U_i} \langle t_S^{\mathbb{H}} | \zeta_i \omega \rangle d\mathcal{S}_1^{Q-k} \\ &= \frac{1}{C_{n,k}^2} \sum_{i \in \mathbb{N}} \int_{S \cap U_i} \langle t_S^{\mathbb{H}} | \zeta_i \omega \rangle d\mathcal{S}_2^{Q-k} = \frac{1}{C_{n,k}^2} \int_S \langle t_S^{\mathbb{H}} | \omega \rangle d\mathcal{S}_2^{Q-k}. \end{aligned}$$

□

**Remark 6.17.** Lemma 6.16 shows that Heisenberg currents associated with low codimensional submanifolds do not depend on the choice of the left invariant, homogeneous and rotationally invariant distance  $d$  (and, consequently, from the associated spherical Hausdorff measure).

### 6.3 $C_{\mathbb{H}}^1$ -regular submanifolds with boundary

After providing the definition of  $C_{\mathbb{H}}^1$ -regular submanifolds with boundary we will give equivalent characterizations first in the case of low and non-critical codimension  $k < n$ , then in the (easy) case of low dimension (i.e., codimension  $k \geq n + 1$ ), and eventually in the case of critical codimension  $k = n$ . Then we will introduce and discuss the natural way to induce an orientation on the boundary.

**Definition 6.18.** Let  $1 \leq k \leq 2n + 1$  and  $O \subseteq \mathbb{H}^n$  be an open set. Let  $S \subseteq \mathbb{H}^n$  and define the *boundary*  $\partial S$  of  $S$  as  $\partial S := \bar{S} \setminus S$ . We say that  $S \subseteq \mathbb{H}^n$  is a *k-codimensional (or  $(2n + 1 - k)$ -dimensional)  $C_{\mathbb{H}}^1$ -regular submanifold with boundary in  $O$*  if the following conditions hold:

- (1)  $O \cap S$  is a non empty  $k$ -codimensional  $C_{\mathbb{H}}^1$ -regular submanifold;
- (2)  $O \cap \partial S$  is a non empty  $k + 1$ -codimensional  $C_{\mathbb{H}}^1$ -regular submanifold;
- (3) for all  $p \in O \cap \partial S$  there exist a neighbourhood  $U$  of  $p$  and a  $k$ -codimensional  $C_{\mathbb{H}}^1$ -regular submanifold  $S' \subseteq U$  such that  $U \cap \bar{S} \subseteq S'$  and, for every  $r > 0$ ,  $U(p, r) \cap (S' \setminus \bar{S}) \neq \emptyset$ .

We will omit reference to the open set  $O$  in case the latter is clear from the context.

**Theorem 6.19.** *Let  $1 \leq k \leq n - 1$ ,  $O \subseteq \mathbb{H}^n$  be an open set and  $S \subseteq O$  such that  $O \cap \partial S \neq \emptyset$ . Then the following statements are equivalent:*

- (i)  $S$  is a  $k$ -codimensional  $C_{\mathbb{H}}^1$ -regular submanifold with boundary in  $O$ ;
- (ii) the statement (1) from Definition 6.18 holds and for all  $p \in O \cap \partial S$  there exist a neighbourhood  $U$  of  $p$  and functions  $f_1, \dots, f_{k+1} \in C_{\mathbb{H}}^1(U)$  such that

$$\begin{aligned} & \nabla_{\mathbb{H}} f_1, \dots, \nabla_{\mathbb{H}} f_{k+1} \text{ are linearly independent in } U, \\ & U \cap S = \{q \in U : f_1(q) = \dots = f_k(q) = 0, f_{k+1}(q) > 0\}, \\ & f_{k+1}(p) = 0. \end{aligned}$$

**Remark 6.20.** If (ii) of Theorem 6.19 holds, then

$$U \cap \bar{S} = \{q \in U : f_1(q) = \dots = f_k(q) = 0, f_{k+1}(q) \geq 0\}.$$

In fact, the left hand side in the previous equality is clearly contained in the right hand one; the opposite inclusion can be easily proved by taking into account that the map  $(f_1, \dots, f_{k+1}) : U \rightarrow \mathbb{R}^k$  is open (see for instance [100, Lemma 2.10]). In particular,

$$U \cap \partial S = \{q \in U : f_1(q) = \dots = f_k(q) = 0, f_{k+1}(q) = 0\}$$

and  $O \cap \partial S$  is a  $C_{\mathbb{H}}^1$ -regular submanifold of codimension  $k + 1$ .

**Remark 6.21.** By a standard mollification procedure (see for instance in [158, Proposition 2.10]), the functions  $f_1, \dots, f_{k+1}$  in (ii) of Theorem 6.19 can be chosen to be of class  $C^\infty$  on the open set  $U \setminus \{q \in U : f_1(q) = \dots = f_k(q) = 0\}$ .

*Proof of Theorem 6.19.* Let us prove (ii)  $\Rightarrow$  (i): since properties (1) and (2) in Definition 6.18 hold, respectively, by assumption and by Remark 6.20, we only have to prove property (3). Fix  $p \in O \cap \partial S$ . Let  $U, f_1, \dots, f_{k+1}$  be as in (ii). We define

$$S' := \{q \in U : f_1(q) = \dots = f_k(q) = 0\}.$$

It is clear that  $S'$  is a  $k$ -codimensional  $C_{\mathbb{H}}^1$ -regular submanifold and  $U \cap \bar{S} \subseteq S'$ . We must prove that, for every  $r > 0$ ,  $B_d(p, r) \cap (S' \setminus \bar{S}) \neq \emptyset$ . The map  $(f_1, \dots, f_{k+1}) : U \rightarrow \mathbb{R}^k$  is open, so for every  $r > 0$  there exists a  $\varepsilon > 0$  such that

$$B(0, \varepsilon) = \{a \in \mathbb{R}^{k+1} : |a|_{\mathbb{R}^{k+1}} < \varepsilon\} \subseteq (f_1, \dots, f_{k+1})(B_d(p, r)),$$

since, by Remark 6.20,  $f_1(p) = \dots = f_{k+1}(p) = 0$ . Therefore, there exists  $q \in B_d(p, r)$  such that  $(f_1, \dots, f_{k+1})(q) = (0, \dots, 0, -\frac{\varepsilon}{2})$ , i.e.,  $q \in B_d(p, r) \cap (S' \setminus \bar{S})$ , that is  $B_d(p, r) \cap (S' \setminus \bar{S}) \neq \emptyset$ .

We now prove (i)  $\Rightarrow$  (ii). Fix  $p \in O \cap \partial S$  and let  $S'$  be as in (3) of Definition 6.18; then, there exist a neighbourhood  $U$  of  $p$  and functions  $f_1, \dots, f_k \in C_{\mathbb{H}}^1(U)$  such that, setting

$$S' = \{q \in U : f_1(q) = \dots = f_k(q) = 0\},$$

we have

- $\nabla_{\mathbb{H}}f_1, \dots, \nabla_{\mathbb{H}}f_k$  are linearly independent on  $U$ ,
- $U \cap \bar{S} \subseteq S'$ ,
- for every  $r > 0$ ,  $B_d(p, r) \cap (S' \setminus \bar{S}) \neq \emptyset$ .

By property (2), possibly reducing  $U$ , there exist  $g_1, \dots, g_{k+1} \in C_{\mathbb{H}}^1(U)$  such that

- $\nabla_{\mathbb{H}}g_1, \dots, \nabla_{\mathbb{H}}g_{k+1}$  are linearly independent on  $U$ ,
- $U \cap \partial S = \{q \in U : g_1(q) = \dots = g_{k+1}(q) = 0\}$ .

Let  $\bar{j} \in \{1, \dots, k+1\}$  be such that  $\nabla_{\mathbb{H}}f_1(p), \dots, \nabla_{\mathbb{H}}f_k(p), \nabla_{\mathbb{H}}g_{\bar{j}}(p)$  are linearly independent. Setting  $f_{k+1} = g_{\bar{j}}$  and, possibly, reducing  $U$ ,  $\nabla_{\mathbb{H}}f_1, \dots, \nabla_{\mathbb{H}}f_{k+1}$  are linearly independent on  $U$ . We claim that

$$U \cap \partial S = \{q \in U : f_1(q) = \dots = f_{k+1}(q) = 0\}. \quad (6.13)$$

The inclusion

$$\begin{aligned} U \cap \partial S &\subseteq \{q \in U : f_1(q) = \dots = f_{k+1}(q) = 0\} \\ &= S' \cap \{q \in U : f_{k+1}(q) = 0\} \end{aligned}$$

holds because  $U \cap \partial S \subseteq U \cap \bar{S} \subseteq U \cap S'$  and, if  $q \in U \cap \partial S$ , then  $f_{k+1}(q) = g_{\bar{j}}(q) = 0$ . For the opposite inclusion

$$U \cap \partial S \supseteq \{q \in U : f_1(q) = \dots = f_{k+1}(q) = 0\}$$

The set  $\Delta := \{q \in U : f_1(q) = \dots = f_{k+1}(q) = 0\}$  is a  $C_{\mathbb{H}}^1$  submanifold of codimension  $k+1$ . Notice that  $p \in \Delta \cap \partial S$  and

$$T_p^{\mathbb{H}}\Delta = T_p^{\mathbb{H}}\partial S = T_p^{\mathbb{H}}S' \cap (\nabla_{\mathbb{H}}g_{\bar{j}}(p))^{\perp}.$$

Possibly restricting  $U$ , by Theorem 6.6 there exist a splitting  $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$  (with  $\mathbb{V}$  horizontal and of dimension  $k+1$ ), open subsets  $A \subset \mathbb{W}, B \subset \mathbb{V}$  and continuous maps  $\phi, \psi : A \rightarrow B$  such that  $U = A \cdot B$  and

$$\text{gr}_{\psi} = U \cap \partial S, \quad \text{gr}_{\phi} = U \cap \Delta.$$

The inclusion  $\text{gr}_{\psi} \subseteq \text{gr}_{\phi}$  proved above implies that  $\psi = \phi$  and the claim (6.13) follows. Moreover, possibly reducing  $A$  and  $U$ , by the Implicit Function Theorem [100, Lemma 2.10], there exist a real number  $t_0 > 0$  and a continuous map  $\varphi : A \times (-t_0, t_0) \rightarrow \mathbb{V}$  such that

- the graph map  $\Phi(w, t) := w \cdot \varphi(w, t)$  is an homeomorphism between  $A \times (-t_0, t_0)$  and  $\{q \in S' : f_{k+1}(q) \in (-t_0, t_0)\}$ ,
- $f_{k+1}(w \cdot \varphi(w, t)) = t$  for all  $w \in A$  and  $t \in (-t_0, t_0)$ .

Without loss of generality we can suppose that  $A$  is a connected open subset of  $\mathbb{W}$ . Moreover, possibly reducing  $U, A$  and  $t_0$ , we can assume that

- $U \cap \partial S = \Phi(A \times \{0\})$ ,
- $U \cap S' = \Phi(A \times (-t_0, t_0))$ .

We observe that  $U \cap (S' \setminus \partial S)$  has two connected components  $S'_+, S'_-$  defined by

$$S'_+ := \{q \in U : f_1(q) = \cdots = f_k(q) = 0, f_{k+1}(q) > 0\}$$

$$S'_- := \{q \in U : f_1(q) = \cdots = f_k(q) = 0, f_{k+1}(q) < 0\}.$$

By definition  $U \cap S$  is a  $C_{\mathbb{H}}^1$ -regular submanifold: it follows that it is locally connected, relatively open and relatively closed in  $U \cap (S' \setminus \partial S) = S'_- \sqcup S'_+$ . This implies that there are only three possibilities

- either  $U \cap S = S'_+$ ,
- or  $U \cap S = S'_-$ ,
- or  $U \cap S = S'_+ \cup S'_-$ .

If (a) holds, then the proof is accomplished. If (b) holds, it is enough to replace  $f_{k+1}$  with  $-f_{k+1}$ . Case (c) would lead to  $U \cap (S' \setminus \bar{S}) = \emptyset$ , a contradiction. This concludes the proof.  $\square$

The following property of low-dimensional  $C_{\mathbb{H}}^1$  submanifolds with boundary is an easy consequence of [82, Theorem 3.5], see Remark 2.36.

**Lemma 6.22.** *Let  $1 \leq k \leq n$ ,  $O \subseteq \mathbb{H}^n$  be an open set and let  $S$  be a  $k$ -dimensional  $C_{\mathbb{H}}^1$ -regular submanifold with boundary in  $O$ . Then,  $O \cap \bar{S}$  is a Euclidean  $k$ -dimensional  $C^1$ -regular submanifold with boundary such that  $T\bar{S} \subseteq H\mathbb{H}^n$  and  $T\partial S \subseteq H\mathbb{H}^n$ .*

In Lemma 6.22 and in the following, when we say that  $S$  is both a Euclidean  $C^1$  and  $C_{\mathbb{H}}^1$  submanifold with boundary in an open set  $O$ , we mean that  $O \cap \bar{S}$  is both a  $C^1$ -regular Euclidean submanifold with boundary (in the classical sense) and  $S$  is a  $C_{\mathbb{H}}^1$ -regular submanifold with boundary in  $O$  as in Definition 6.18.

The following Lemma shows a property of critical dimensional  $C_{\mathbb{H}}^1$  submanifolds with boundary, which will be crucial in the definition of the induced orientation on the boundary in Subsection 6.3.1.

**Lemma 6.23.** *Let  $O \subseteq \mathbb{H}^n$  be an open set and let  $S$  be an  $(n+1)$ -dimensional  $C_{\mathbb{H}}^1$  submanifold with boundary in  $O$ ; denote by  $S'$  a local extension of  $S$  as in Definition 6.18. Then for every  $p \in O \cap \partial S$  there exists a fundamental system of neighborhoods  $\mathcal{N}$  such that, for every neighbourhood  $N \in \mathcal{N}$ ,  $(N \cap S') \setminus \partial S$  has two connected components and one of the two is  $N \cap S$ .*

*Proof.* Fix  $p \in O \cap \partial S$ . By Definition 6.18 there exist a neighbourhood  $U$  of  $p$  and an  $(n+1)$ -dimensional  $C_{\mathbb{H}}^1$  submanifold  $S' \subseteq U$  such that  $U \cap \bar{S} \subseteq S'$ . By Theorem 6.6 we can locally write  $S'$  as the intrinsic graph of an continuous map  $\phi : \mathbb{W} \rightarrow \mathbb{V}$ , where  $\mathbb{W} := T_p^{\mathbb{H}} S'$  and  $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$  is an orthogonal splitting. We also fix orthonormal bases  $w_1, \dots, w_n, T$  of the Lie (sub)algebra of  $\mathbb{W}$  and  $v_1, \dots, v_n$  of the Lie (sub)algebra of  $\mathbb{V}$ , respectively; observe that, since  $\partial S$  has an horizontal  $n$ -dimensional tangent space,  $T_p^{\mathbb{H}} \partial S = \text{span}(w_1(p), \dots, w_n(p))$ . Let  $\pi_{\mathbb{W}} : \mathbb{H}^n \rightarrow \mathbb{W}$  be the projection defined in Remark 2.14; we observe that, possibly restricting  $U$ , the set  $\Gamma := \pi_{\mathbb{W}}(U \cap \partial S)$  is an  $n$ -dimensional  $C^1$  submanifold in  $\mathbb{W}$ . This is due to the fact that, up to restricting  $U$ ,  $U \cap \partial S$  and  $\pi_{\mathbb{W}}$  are transverse, i.e., for every  $q \in U \cap \partial S$

$$T_q \partial S \cap \ker d\pi_{\mathbb{W}} = T_q \partial S \cap \text{span}(v_1(q), \dots, v_n(q)) = \{0\};$$

in fact, the previous equality is true for  $q = p$  and, by continuity, it remains true for  $q$  close enough to  $p$ . Therefore, for every  $w \in \Gamma$  there exists a fundamental system of neighborhoods  $\mathcal{M}$  for  $\pi_{\mathbb{W}}(p)$  such that, for any  $M \in \mathcal{M}$ ,  $M \setminus \Gamma$  has two connected components. Now  $U \cap S$  is a  $C_{\mathbb{H}}^1$ -regular submanifold, hence it is relatively open in  $U \cap (S' \setminus \partial S)$ . This, together with the fact that, up to restricting  $U$ ,  $\pi_{\mathbb{W}}$  is an homeomorphism from  $U \cap S'$  onto its image, implies that  $\pi_{\mathbb{W}}(U \cap S)$  has to be either one of the two connected components of  $M \setminus \Gamma$  or the whole  $M \setminus \Gamma$ . Thanks to property (3) in Definition 6.18, one has  $\pi_{\mathbb{W}}(U \cap S) \neq M \setminus \Gamma$ , so that  $\pi_{\mathbb{W}}(U \cap S)$  is exactly one of the two connected components of  $M \setminus \Gamma$ . Then the fundamental system of neighbourhoods  $\mathcal{N}$  defined by  $\mathcal{N} := U \cap \pi_{\mathbb{W}}^{-1}(\mathcal{M})$  satisfies the requested property.  $\square$

**Remark 6.24.** Under the assumption and notation of Lemma 6.23, we observe that the  $n$ -dimensional  $C^1$  submanifold  $\Gamma = \pi_{\mathbb{W}}(U \cap \partial S)$  can be written, locally around  $\pi_{\mathbb{W}}(p)$ , as an  $n$ -dimensional graph in the  $T$  direction. More precisely, let us identify  $\mathbb{W} \cong \mathbb{R}^{n+1}$  by

$$\mathbb{R}^{n+1} \ni (s_1, \dots, s_n, t) \longleftrightarrow \exp(s_1 w_1 + \dots + s_n w_n + t T) \in \mathbb{W};$$

then there exists  $\gamma \in C^1(\mathbb{R}^n)$  and a relatively open set  $A \subset \mathbb{W}$  containing  $\pi_{\mathbb{W}}(p)$  such that

$$A \cap \Gamma = \{(s_1, \dots, s_n, t) \in A \subset \mathbb{W} \cong \mathbb{R}^{n+1} : t = \gamma(s_1, \dots, s_n)\}.$$

In particular, we have that either

$$A \cap \pi_{\mathbb{W}}(U \cap S) = \{(s_1, \dots, s_n, t) \in A \subset \mathbb{W} \cong \mathbb{R}^{n+1} : t > \gamma(s_1, \dots, s_n)\} \quad (6.14)$$

or

$$A \cap \pi_{\mathbb{W}}(U \cap S) = \{(s_1, \dots, s_n, t) \in A \subset \mathbb{W} \cong \mathbb{R}^{n+1} : t < \gamma(s_1, \dots, s_n)\}. \quad (6.15)$$

We will informally say that  $S$  lies above its boundary around  $p$  if (6.14) holds, while we say that  $S$  lies below its boundary around  $p$  if (6.15) holds. By continuity, it is easy to check that, for every  $p \in \partial S$ , there exists a neighbourhood  $V$  of  $p$  with the following property: if  $S$  lies above (resp., below) its boundary around  $p$ , then  $S$  lies above (resp., below) its boundary also around every  $q \in V \cap \partial S$ .

Finally, the following lemma contains a technical property of  $C_{\mathbb{H}}^1$ -regular submanifolds with boundary that holds in the case of critical codimension  $k = n$ . Such a property will be useful later in the proof of Lemma 6.39.

**Lemma 6.25.** *Let  $S$  be an  $(n + 1)$ -dimensional  $C_{\mathbb{H}}^1$  submanifold with boundary. Define  $\pi : \mathbb{H}^n \rightarrow \mathbb{R}^{2n}$  by  $\pi(x, y, t) := (x, y)$ . Then, for every  $p \in \partial S$  there exist an open set  $U$  with  $p \in U$ , a  $C_{\mathbb{H}}^1$ -regular submanifold  $S' \subset U$  as in (3) of Definition 6.18, a locally defining function  $f \in C_{\mathbb{H}}^1(U, \mathbb{R}^n)$  for  $S'$ , a function  $\hat{g} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$  and  $\delta_1 > 0$  such that, up to an isometry of  $\mathbb{H}^n$ ,*

- (1)  $\hat{g} \in C^1(\mathbb{R}^{2n}, \mathbb{R}^n)$ ,
- (2)  $|\nabla \hat{g}|$  is bounded on  $\mathbb{R}^{2n}$ ,
- (3)  $(\nabla \hat{g}) \circ \pi = \nabla_{\mathbb{H}} f$  on  $\Sigma = U \cap \partial S$ ,
- (4)  $\text{col}[\partial_{x_1} \hat{g} | \cdots | \partial_{x_n} \hat{g}](z) \geq \delta_1 \text{Id}_{n \times n}$  for every  $z \in \mathbb{R}^{2n}$  in the sense of quadratic forms,
- (5)  $\pi(\Sigma) \subseteq \{\hat{g} = 0\}$ ,

where in (3) we identify horizontal vectors with elements in  $\mathbb{R}^{2n}$ .

*Proof.* Fix  $p \in \partial S$  and let  $U, S'$  be an open neighbourhood of  $p$  and an  $(n + 1)$ -dimensional  $C_{\mathbb{H}}^1$  submanifold as in (3) of Definition 6.18. By Lemma 6.5, possibly restricting  $U$  there exist a function  $f : \mathbb{H}^n \rightarrow \mathbb{R}^n$  and  $\delta_0 > 0$  such that, up to an isometry of  $\mathbb{H}^n$ ,

- $f \in C_{\mathbb{H}}^1(\mathbb{H}^n, \mathbb{R}^n)$ ,
- $|\nabla_{\mathbb{H}} f|$  is bounded on  $\mathbb{H}^n$ ,
- $f \in C^\infty(\mathbb{H}^n \setminus \{f = 0\}, \mathbb{R}^n)$ ,
- $\widehat{\nabla}_{\mathbb{H}} f(q) \geq \delta_0 \text{Id}_{n \times n}$  for every  $q \in \mathbb{H}^n$  in the sense of quadratic forms,
- $S' \cap U = \{q \in U : f(q) = 0\}$ .

By Remark 2.36,  $\partial S$  is a Euclidean  $C^1$  submanifold whose tangent space is contained in the horizontal distribution. In particular,  $\partial S$  is transversal to the vector field  $T$  and, up to reducing  $U$ ,  $\pi|_{U \cap \partial S}$  is a  $C^1$  diffeomorphism between  $\Sigma = U \cap \partial S$  and its image. We consider, for  $1 \leq j \leq n$ , the map

$$(\nabla_{\mathbb{H}} f_j) \circ \pi^{-1} : \pi(\Sigma) \rightarrow \mathbb{R}^{2n},$$

where  $\nabla_{\mathbb{H}}f_j(p) \in H_p\mathbb{H}^n$  is identified with  $(X_1f_j(p), \dots, Y_nf_j(p)) \in \mathbb{R}^{2n}$ . Clearly,  $\nabla_{\mathbb{H}}f_j \circ \pi^{-1}$  is continuous and bounded on  $\pi(\Sigma)$ . Up to reducing  $U$ , we can assume that  $\nabla_{\mathbb{H}}f_j \circ \pi^{-1}$  is defined on  $\overline{\pi(\Sigma)}$ . We use the classical Whitney Extension Theorem to extend the null function on  $\overline{\pi(\Sigma)}$  to a  $C^1$  map with gradient  $\nabla_{\mathbb{H}}f_j \circ \pi^{-1}$  on  $\overline{\pi(\Sigma)}$ . In order to use Whitney Extension Theorem we have to check that for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$|\langle \nabla_{\mathbb{H}}f_j \circ \pi^{-1}(z), z - z' \rangle_{\mathbb{R}^{2n}}| \leq \varepsilon |z - z'|_{\mathbb{R}^{2n}} \quad \text{for every } z, z' \in \overline{\pi(\Sigma)} \text{ with } |z - z'|_{\mathbb{R}^{2n}} < \delta.$$

Let  $z, z' \in \overline{\pi(\Sigma)}$  be fixed. Since, up to restricting  $U$ ,  $\pi^{-1}$  is bijective on  $\overline{\pi(\Sigma)}$ , there exist unique  $q, q' \in \overline{\Sigma}$  such that  $\pi(q) = z$  and  $\pi(q') = z'$ . We have to prove that for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that, if  $q, q' \in \overline{\Sigma}$  and  $|\pi(q) - \pi(q')|_{\mathbb{R}^{2n}} < \delta$  then

$$|\langle \nabla_{\mathbb{H}}f_j(q), \pi(q) - \pi(q') \rangle_{\mathbb{R}^{2n}}| \leq \varepsilon |\pi(q) - \pi(q')|_{\mathbb{R}^{2n}}. \quad (6.16)$$

This is equivalent to

$$|\langle \nabla_{\mathbb{H}}f_j(q), \pi(q^{-1} \cdot q') \rangle_{\mathbb{R}^{2n}}| \leq \varepsilon |\pi(q^{-1} \cdot q')|_{\mathbb{R}^{2n}}.$$

Let  $r \in q \cdot T_q^{\mathbb{H}}S'$  such that  $d(q', q \cdot T_q^{\mathbb{H}}S') = d(q', r)$ . Then

$$\begin{aligned} |\langle \nabla_{\mathbb{H}}f_j(q), \pi(q^{-1} \cdot q') \rangle_{\mathbb{R}^{2n}}| &\leq \underbrace{|\langle \nabla_{\mathbb{H}}f_j(q), \pi(q^{-1} \cdot r) \rangle_{\mathbb{R}^{2n}}|}_{=0} + |\langle \nabla_{\mathbb{H}}f_j(q), \pi(r^{-1} \cdot q') \rangle_{\mathbb{R}^{2n}}| \\ &\leq \underbrace{|\nabla_{\mathbb{H}}f_j(q)|_{\mathbb{R}^{2n}}}_{\leq C} \underbrace{|\pi(r^{-1} \cdot q')|_{\mathbb{R}^{2n}}}_{\leq Cd(r, q')} \leq Cd(r, q') \end{aligned}$$

By Lemma 6.3 there exists  $\delta_2 > 0$  such that, if  $d(q, q') < \delta_2$ , then

$$d(r, q') \leq \varepsilon d(q, q').$$

By Lemma 6.2 there exist a  $\delta_1 > 0$  and a constant  $C' > 0$  such that, if  $d(p, q') < \delta_1$  and  $d(p, q) < \delta_1$ , then

$$d(q, q') \leq C' |\pi(q) - \pi(q')|_{\mathbb{R}^{2n}}.$$

The claim (6.16) follows and we can use the (classical Euclidean) Whitney Extension Theorem (see e.g [71, Theorem 6.10]) to obtain a function  $g_j \in C^1(\mathbb{R}^{2n}, \mathbb{R})$  such that  $g_j|_{\overline{\pi(\Sigma)}} = 0$  and  $(\nabla g_j) \circ \pi = \nabla_{\mathbb{H}}f_j$  on  $\Sigma$ . We define the function  $g : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$  as  $g = (g_1, \dots, g_n)$ . The function  $g$  satisfies statements (1), (3) and (5) (with  $g$  in place of  $\hat{g}$ ), while it satisfies statements (2) and (4) only in a neighbourhood of  $\overline{\pi(\Sigma)}$ . To conclude, one can use a strategy as in the proof of Lemma 6.5 to get a function  $\hat{g}$  with the desired properties.  $\square$

### 6.3.1 Boundary orientation

In this subsection we show that, given an orientation on a  $C_{\mathbb{H}}^1$ -regular submanifold with boundary, there exists a natural way to define an induced orientation on the boundary

which will be consistent with Theorem 6.1, in analogy with the classical Euclidean case. In the low dimensional case, because of Lemma 6.22, we can define the induced orientation on the boundary as usual, so we focus first on the low codimensional case and then on the critical dimensional case.

**Remark 6.26.** Let  $1 \leq k \leq n$ ,  $O \subseteq \mathbb{H}^n$  be an open set and  $S \subseteq \mathbb{H}^n$  a  $k$ -codimensional  $C_{\mathbb{H}}^1$  submanifold with boundary in  $O$ . Let  $p \in O \cap \partial S$  and  $S'$  be the local extension of  $S$  as in Definition 6.18. Then the tangent cone  $T_p^{\mathbb{H}}S'$  does not depend on the choice of the local extension  $S'$  and we denote it by  $T_p^{\mathbb{H}}S$ .

**Lemma 6.27.** Let  $1 \leq k \leq n - 1$ ,  $O \subseteq \mathbb{H}^n$  be an open set and  $S \subseteq \mathbb{H}^n$  a  $k$ -codimensional  $C_{\mathbb{H}}^1$  submanifold with boundary in  $O$ . Let  $p \in O \cap \partial S$  and assume there exist an open set  $U$  with  $p \in U$  and two families of locally defining functions  $f_1, \dots, f_{k+1} \in C_{\mathbb{H}}^1(U)$  and  $g_1, \dots, g_{k+1} \in C_{\mathbb{H}}^1(U)$  for  $S$ , as in Theorem 6.19. Then there exists a positive constant  $C > 0$  such that

$$(d_{\mathbb{H}}f_{k+1})_p = C(d_{\mathbb{H}}g_{k+1})_p \quad \text{on } T_p^{\mathbb{H}}S.$$

*Proof.* By Theorem 6.19 and Remark 6.20 we have

$$\begin{aligned} &\nabla_{\mathbb{H}}f_1, \dots, \nabla_{\mathbb{H}}f_{k+1} \text{ are linearly independent in } U, \\ &U \cap S = \{q \in U : f_1(q) = \dots = f_k(q) = 0, f_{k+1}(q) > 0\}, \\ &U \cap \partial S = \{q \in U : f_1(q) = \dots = f_{k+1}(q) = 0\}, \\ &\nabla_{\mathbb{H}}g_1, \dots, \nabla_{\mathbb{H}}g_{k+1} \text{ are linearly independent in } U, \\ &U \cap S = \{q \in U : g_1(q) = \dots = g_k(q) = 0, g_{k+1}(q) > 0\}, \\ &U \cap \partial S = \{q \in U : g_1(q) = \dots = g_{k+1}(q) = 0\}. \end{aligned}$$

We claim that

$$\lim_{\lambda \rightarrow 0} \delta_{1/\lambda}(\tau_{p^{-1}}S) = T_p^{\mathbb{H}}S \cap \{q \in \mathbb{H}^n : (d_{\mathbb{H}}f_{k+1})_p(q) > 0\}.$$

in the sense of local Hausdorff convergence of sets. For the sake of brevity, let us assume  $p = 0$ . For  $\lambda > 0$  we define for  $i = 1, \dots, k+1$  the maps  $(f_i)_{\lambda} := \frac{1}{\lambda}(f_i \circ \delta_{\lambda})$  and the  $C_{\mathbb{H}}^1$  submanifolds  $S_{\lambda} := \delta_{1/\lambda}(U \cap S)$ ,  $(\partial S)_{\lambda} := \delta_{1/\lambda}(U \cap \partial S)$ ,  $S'_{\lambda} := \delta_{1/\lambda}(S')$  where  $S' := \{q \in U : f_1(q) = \dots = f_k(q) = 0\}$ . We observe that

$$\begin{aligned} S_{\lambda} &= \{q \in \delta_{1/\lambda}(U) : (f_1)_{\lambda}(q) = \dots = (f_k)_{\lambda}(q) = 0, (f_{k+1})_{\lambda}(q) > 0\}, \\ (\partial S)_{\lambda} &= \{q \in \delta_{1/\lambda}(U) : (f_1)_{\lambda}(q) = \dots = (f_{k+1})_{\lambda}(q) = 0\}, \\ S'_{\lambda} &= \{q \in \delta_{1/\lambda}(U) : (f_1)_{\lambda}(q) = \dots = (f_k)_{\lambda}(q) = 0\} \end{aligned}$$

and in particular that

$$\begin{aligned} (\partial S)_{\lambda} &= S'_{\lambda} \cap \{q \in \delta_{1/\lambda}(U) : (f_{k+1})_{\lambda}(q) = 0\}, \\ S_{\lambda} &= S'_{\lambda} \cap \{q \in \delta_{1/\lambda}(U) : (f_{k+1})_{\lambda}(q) > 0\}. \end{aligned}$$

Letting  $\lambda \rightarrow 0$ , by [100, Lemma 2.14 (ii) and (iii)] we obtain

$$\begin{aligned} T_0^{\mathbb{H}}\partial S &= T_0^{\mathbb{H}}S \cap \{q \in \mathbb{H}^n : (d_{\mathbb{H}}f_{k+1})_0(q) = 0\}, \\ \lim_{\lambda \rightarrow 0} \delta_{1/\lambda}(S) &= T_0^{\mathbb{H}}S \cap \{q \in \mathbb{H}^n : (d_{\mathbb{H}}f_{k+1})_0(q) > 0\} \end{aligned}$$

in the sense of local Hausdorff convergence of sets, proving the claim. In the same way we obtain

$$\begin{aligned} T_0^{\mathbb{H}}\partial S &= T_0^{\mathbb{H}}S \cap \{q \in \mathbb{H}^n : (d_{\mathbb{H}}g_{k+1})_0(q) = 0\}, \\ \lim_{\lambda \rightarrow 0} \delta_{1/\lambda}(S) &= T_0^{\mathbb{H}}S \cap \{q \in \mathbb{H}^n : (d_{\mathbb{H}}g_{k+1})_0(q) > 0\} \end{aligned}$$

in the sense of local Hausdorff convergence of sets. By the uniqueness of the blow-up we obtain

$$\begin{aligned} T_0^{\mathbb{H}}S \cap \{q \in \mathbb{H}^n : (d_{\mathbb{H}}f_{k+1})_0(q) > 0\} &= T_0^{\mathbb{H}}S \cap \{q \in \mathbb{H}^n : (d_{\mathbb{H}}g_{k+1})_0(q) > 0\}, \\ T_0^{\mathbb{H}}S \cap \{q \in \mathbb{H}^n : (d_{\mathbb{H}}f_{k+1})_0(q) = 0\} &= T_0^{\mathbb{H}}S \cap \{q \in \mathbb{H}^n : (d_{\mathbb{H}}g_{k+1})_0(q) = 0\}. \end{aligned}$$

This, together with the fact that  $(d_{\mathbb{H}}f_{k+1})_0$  and  $(d_{\mathbb{H}}g_{k+1})_0$  are linear maps (on  $T_0^{\mathbb{H}}S$ ) with the same kernel, implies that there exists a positive constant  $C > 0$  such that

$$(d_{\mathbb{H}}f_{k+1})_0 = C(d_{\mathbb{H}}g_{k+1})_0 \quad \text{on } T_0^{\mathbb{H}}S,$$

as desired. □

**Definition 6.28.** Let  $1 \leq k \leq n - 1$ ,  $O \subseteq \mathbb{H}^n$  be an open set and  $S \subseteq \mathbb{H}^n$  a  $k$ -codimensional  $C_{\mathbb{H}}^1$  submanifold with boundary in  $O$ . Fix  $p \in O \cap \partial S$  and let  $U$  and  $f_1, \dots, f_{k+1} \in C_{\mathbb{H}}^1(U)$  be as in Theorem 6.19. We say that a vector  $v \in T_p^{\mathbb{H}}S$  is *outward pointing* if  $(d_{\mathbb{H}}f_{k+1})_p(v) < 0$ .

Definition 6.28 is well posed and does not depend on the choice of the family of locally defining functions  $f_1, \dots, f_{k+1}$  because of Lemma 6.27.

**Definition 6.29.** Let  $1 \leq k \leq n - 1$ ,  $O \subseteq \mathbb{H}^n$  be an open set and  $S \subseteq \mathbb{H}^n$  a  $k$ -codimensional  $C_{\mathbb{H}}^1$  submanifold with boundary in  $O$ . We say that  $V : O \cap \partial S \rightarrow T_p^{\mathbb{H}}S$  is an *outward pointing vector field* if, for any  $p \in O \cap \partial S$ ,  $V(p)$  is an outward pointing vector.

**Lemma 6.30.** Let  $1 \leq k \leq n - 1$ ,  $O \subseteq \mathbb{H}^n$  be an open set and  $S \subseteq \mathbb{H}^n$  a  $k$ -codimensional  $C_{\mathbb{H}}^1$  submanifold with boundary in  $O$ . Then there exists a unique continuous unit outward pointing vector field  $v_{\partial S}$  along  $O \cap \partial S$  such that  $v_{\partial S}(p) \perp T_p^{\mathbb{H}}\partial S$  for every  $p \in O \cap \partial S$ .

*Proof.* Fix  $p \in O \cap \partial S$ . By Theorem 6.19 and Remark 6.20 there exist a neighbourhood  $U$  of  $p$  and functions  $f_1, \dots, f_{k+1} \in C_{\mathbb{H}}^1(U)$  such that

$$\begin{aligned} \nabla_{\mathbb{H}}f_1, \dots, \nabla_{\mathbb{H}}f_{k+1} &\text{ are linearly independent in } U, \\ U \cap S &= \{q \in U : f_1(q) = \dots = f_k(q) = 0, f_{k+1}(q) > 0\}, \\ U \cap \partial S &= \{q \in U : f_1(q) = \dots = f_{k+1}(q) = 0\}. \end{aligned}$$

We define the continuous vector field on  $U \cap \partial S$

$$v_{\partial S}(q) := -\frac{\pi_{T_p^{\mathbb{H}^n} S} \nabla_{\mathbb{H}} f_{k+1}(q)}{|\pi_{T_p^{\mathbb{H}^n} S} \nabla_{\mathbb{H}} f_{k+1}(q)|}, \quad q \in U \cap \partial S$$

where  $\pi_{T_p^{\mathbb{H}^n} S}$  denotes the orthogonal projection on  $T_p^{\mathbb{H}^n} S$ . By Lemma 6.27 it is clear that  $v_{\partial S}$  does not depend on the choice of the set of the local defining functions  $f_1, \dots, f_{k+1}$  so that we can continuously extend  $v_{\partial S}$  on the whole  $O \cap \partial S$ .  $\square$

**Definition 6.31.** Under the assumptions and notation of Lemma 6.30, we call  $v_{\partial S}$  the *unit outward normal* to  $\partial S$ .

**Definition 6.32.** Let  $1 \leq k \leq n-1$ ,  $O \subseteq \mathbb{H}^n$  be an open set and  $S \subseteq \mathbb{H}^n$  be an orientable  $k$ -codimensional  $C_{\mathbb{H}}^1$  submanifold with boundary in  $O$ . Fix a choice of a continuous  $(2n+1-k)$ -vector field  $t_S^{\mathbb{H}}$  on  $O \cap S$  orienting  $O \cap S$  and extend  $t_S^{\mathbb{H}}$  continuously to  $O \cap \bar{S}$ . Let  $v_{\partial S}$  be the unit outward normal to  $\partial S$ . We define the *induced orientation on the boundary* as the one fixed by the continuous  $(2n-k)$ -vector field  $t_{\partial S}^{\mathbb{H}}$  on  $O \cap \partial S$  such that

$$v_{\partial S} \wedge t_{\partial S}^{\mathbb{H}} = t_S^{\mathbb{H}} \quad \text{on } O \cap \partial S.$$

We now turn to the case of critical codimension. In this case the role of the unit outward normal  $v_{\partial S}$  will be played, in some sense, either by the constant vector field  $T$  or by the constant vector field  $-T$ . We recall that  $S$  can either *lie above* or *lie below its boundary* according to the terminology introduced in Remark 6.24.

**Definition 6.33.** Let  $O \subseteq \mathbb{H}^n$  be an open set and  $S \subseteq \mathbb{H}^n$  be an orientable  $n$ -codimensional  $C_{\mathbb{H}}^1$  submanifold with boundary in  $O$ . Fix a choice of a continuous  $(n+1)$ -vector field  $t_S^{\mathbb{H}}$  on  $O \cap S$  determining an orientation on  $O \cap S$ , that we extend continuously on  $O \cap \bar{S}$ . We define the *induced orientation on the boundary*  $\partial S$  as the one determined by the continuous horizontal unit  $n$ -vector  $\tau_{\partial S}$  tangent to  $\partial S$  such that the following holds for every  $p \in O \cap \partial S$ : if  $S$  lies above its boundary around  $p$ , then

$$-T \wedge \tau_{\partial S}(p) = t_S^{\mathbb{H}}(p),$$

while if  $S$  lies below its boundary around  $p$  then

$$T \wedge \tau_{\partial S}(p) = t_S^{\mathbb{H}}(p).$$

### 6.3.2 Examples of $C_{\mathbb{H}}^1$ -regular submanifolds with boundary

In this subsection we present some examples of  $C_{\mathbb{H}}^1$ -regular submanifolds (coming from [61]) in the first Heisenberg group  $\mathbb{H}^1$ , where for simplicity we write  $X, Y$  in place of  $X_1, Y_1$ .

**Example 6.34.** The simplest examples of  $C_{\mathbb{H}}^1$ -regular submanifolds with boundary is given by vertical half-planes. As an example consider the set

$$P := \{(0, y, t) \in \mathbb{H}^1 : y \in \mathbb{R}, t > 0\}.$$

It is immediate to see that  $P$  is a  $C_{\mathbb{H}}^1$  submanifold with boundary and that  $\partial P = \{(0, y, 0) \in \mathbb{H}^1 : y \in \mathbb{R}\}$  is the integral line of the horizontal vector field  $Y$ .

We now provide two examples of compact  $C_{\mathbb{H}}^1$ -regular submanifold with boundary. Let us first state the following observation, that is well-known to experts.

**Remark 6.35.** A smooth surface in  $\mathbb{H}^1$  with the topology of the (closed) disk and with horizontal boundary has necessarily at least one characteristic<sup>3</sup> point. The existence of at least one characteristic point is due to topological reasons. Let us sketch a proof by contradiction; we thank B. Franchi for suggesting us this argument. Assume that the (closed) disk  $D$  has no characteristic points. For  $p \in D$  we define  $V(p) \in T_p D$  to be the orthogonal projection of the vector field  $T$  onto  $T_p D$  with respect to the left invariant inner product making  $X, Y, T$  orthonormal. We observe that the smooth vector field  $V$  has no zeros on  $D$ ; moreover, it can be easily seen that, for every  $p \in \partial D$ , the vector  $V(p)$  is parallel to the outward pointing unit normal  $\nu_{\partial D}(p) \in T_p D$  to  $\partial D$ . Therefore, possibly replacing  $V$  with  $-V$ ,  $V$  is outward pointing at every point of  $\partial D$  and, by the Poincaré-Hopf theorem (see for instance [131]), the Euler characteristic of  $D$  is 0, a contradiction.

Motivated by Remark 6.35, we conjecture that in  $\mathbb{H}^1$  there exists no  $C_{\mathbb{H}}^1$ -regular 2-dimensional submanifold with boundary with the disk topology. Let us provide examples of compact orientable  $C_{\mathbb{H}}^1$ -regular submanifolds with boundary of different topologies.

**Example 6.36.** Consider the curve  $\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2$  (also known as *lemniscate of Geronno*) given by

$$\gamma(t) = (\cos(t), \sin(t) \cos(t)), \quad t \in [0, 2\pi].$$

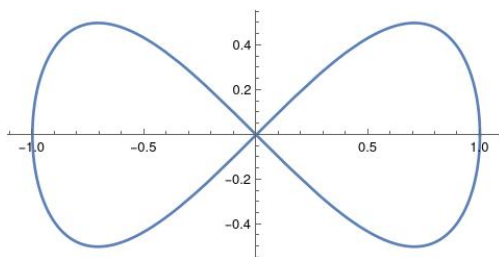


Figure 6.1: The lemniscate of Geronno.

<sup>3</sup>A point  $p$  of a hypersurface  $S \subset \mathbb{H}^n$  is called *characteristic* if  $TS = \text{span}(X_i, Y_i)_{1 \leq i \leq n}$ . Observe that a smooth hypersurface without characteristic points is automatically  $C_{\mathbb{H}}^1$ -regular.

Now let  $\Gamma : [0, 2\pi] \rightarrow \mathbb{H}^1$  be the *horizontal lift* of  $\gamma$  such that  $\Gamma(0) = (1, 0, 0)$  (for further details on horizontal lifts see [136, page 4]). The curve  $\Gamma$  is a 1-dimensional  $C_{\mathbb{H}}^1$ -regular submanifold (i.e., a  $C^1$ -regular horizontal curve) given by

$$\Gamma(t) = \left( \cos(t), \sin(t) \cos(t), \frac{1}{24}(-9 \sin(t) - \sin(3t)) \right), \quad t \in [0, 2\pi]$$

It is easy to check that  $\Gamma$  is a closed curve without self-intersections.

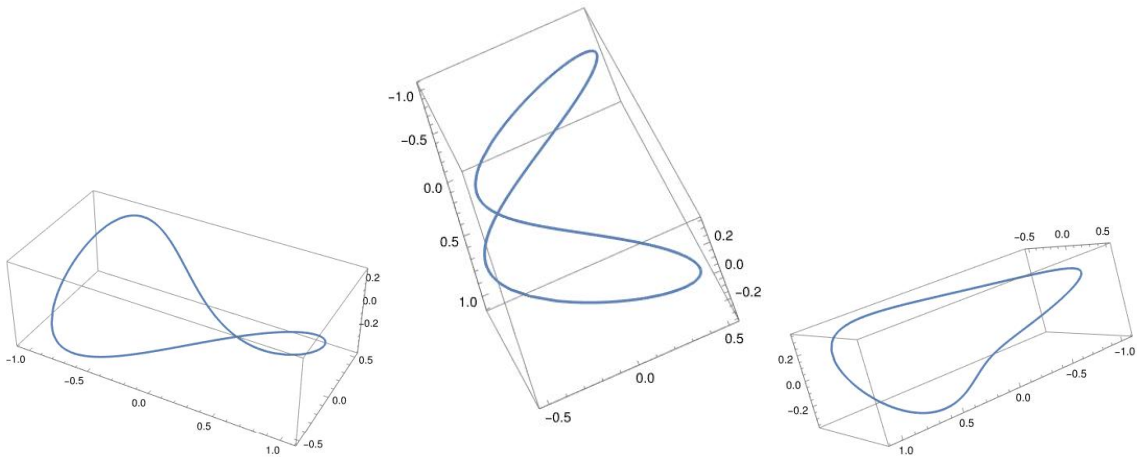


Figure 6.2: The horizontal lift of the lemniscate of Geroni seen from different perspectives.

Now if we consider the submanifold  $\Sigma$  given from the lift (in the vertical direction) of height<sup>4</sup>  $1/3$ , i.e.,

$$\Sigma(t, s) = \left( \cos(t), \sin(t) \cos(t), \frac{1}{24}(-9 \sin(t) - \sin(3t)) + s \right), \quad t \in [0, 2\pi], s \in [0, 1/3],$$

we obtain an example of compact 2-dimensional  $C_{\mathbb{H}}^1$ -regular submanifold with boundary. In this case, the boundary has two connected components: the horizontal curve  $\Gamma$ , and a vertical translation (and thus horizontal itself) of  $\Gamma$ .

<sup>4</sup>We still get a set without self-intersections if we replace  $1/3$  with any positive number less than  $2/3$ .

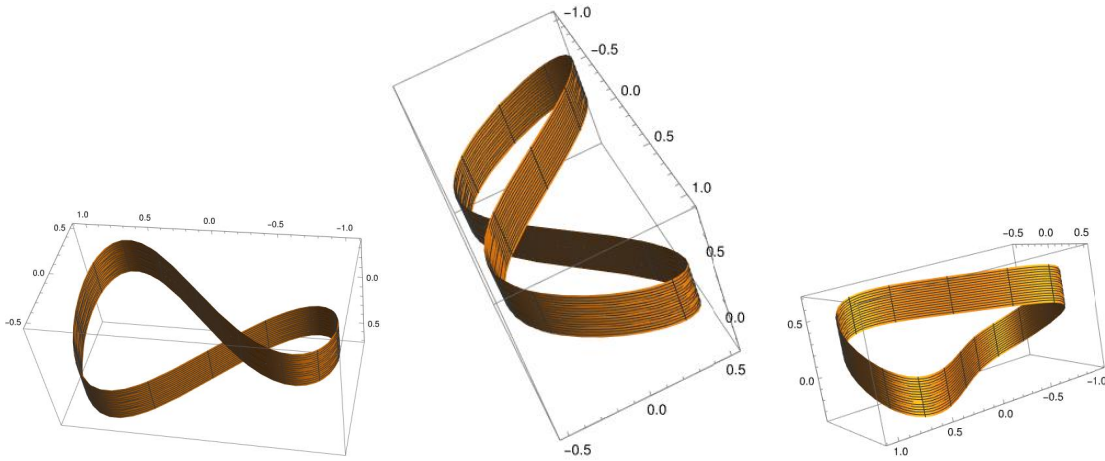


Figure 6.3: The compact  $C_{\mathbb{H}}^1$ -regular submanifold  $\Sigma$  obtained by lifting the lemniscate of Gerono seen from different perspectives.

**Example 6.37.** For  $R > r > 0$  consider the torus  $\mathcal{T}$  parametrized by

$$\mathfrak{T}(u, v) = ((R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u), \quad u, v \in [0, 2\pi].$$

The above torus is obtained by revolving a circle of radius  $r$  in the  $xt$ -plane around the circle of radius  $R$  around the  $t$ -axis;  $\mathcal{T}$  has no characteristic points. As one can see from [20, Lemma 7.1], the characteristic foliation on an horizontal torus is filled either with periodic trajectories, or with everywhere dense trajectories, depending only on the choice of the radii  $r, R$ . For example one can prove that by choosing  $r = 1$  and  $R = \sqrt{1 + n^{2/3}}$  with  $n \in \mathbb{N}$  the characteristic foliation<sup>5</sup> on the horizontal torus is filled with periodic trajectories (that is, closed horizontal curves) which spiral  $n$  times around the torus. In Figure 6.4 we can see one of such curves for  $n = 2$ .

<sup>5</sup>The characteristic foliation is defined by the unique horizontal direction that is tangent to  $\mathcal{T}$ .

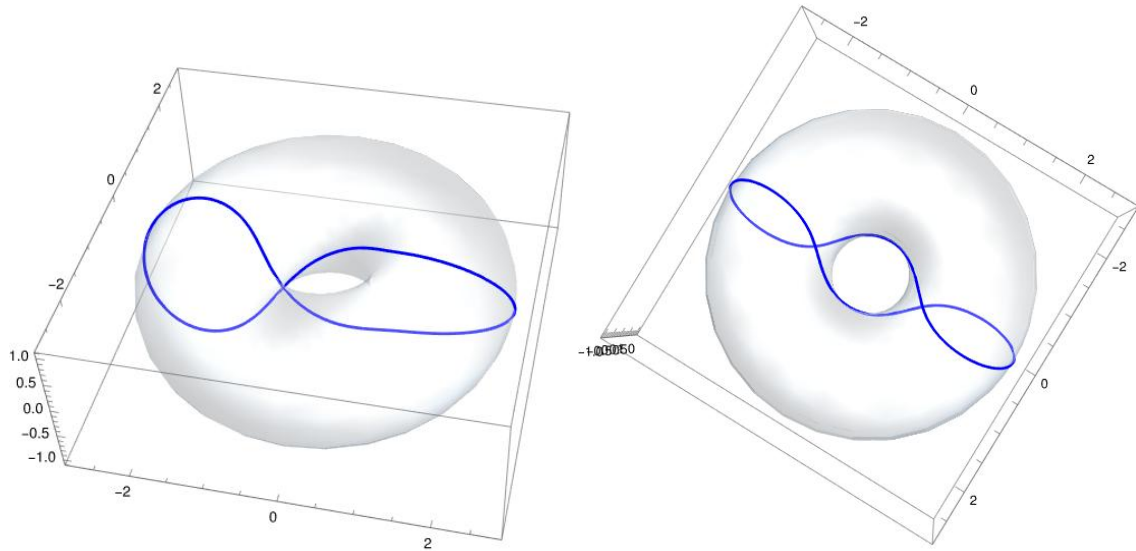


Figure 6.4: Two views of a closed smooth horizontal curve on  $\mathcal{T}$  which spirals twice around the torus.

Then revolving a given periodic trajectory  $\sigma$  of the characteristic foliation around the  $t$ -axis with an angle small enough to not have any self-intersection (in Figure 6.5 such angle is  $\frac{\pi}{12}$ ), and recalling that  $\text{span}(X, Y)$  and  $\mathcal{T}$  are invariant under rotations around the  $t$ -axis, we obtain a surface  $S$  on the torus (and therefore a 2-dimensional  $C_{\mathbb{H}}^1$ -regular submanifold) which is bounded by two horizontal smooth curves. In conclusion  $S$  is also an orientable 2-dimensional  $C_{\mathbb{H}}^1$ -submanifold with boundary<sup>6</sup>.

<sup>6</sup>As in Example 6.36, the boundary of  $S$  is given by two connected components:  $\sigma$  and the other horizontal curve obtained by rotating  $\sigma$  (with a small enough angle) around the  $t$ -axis.

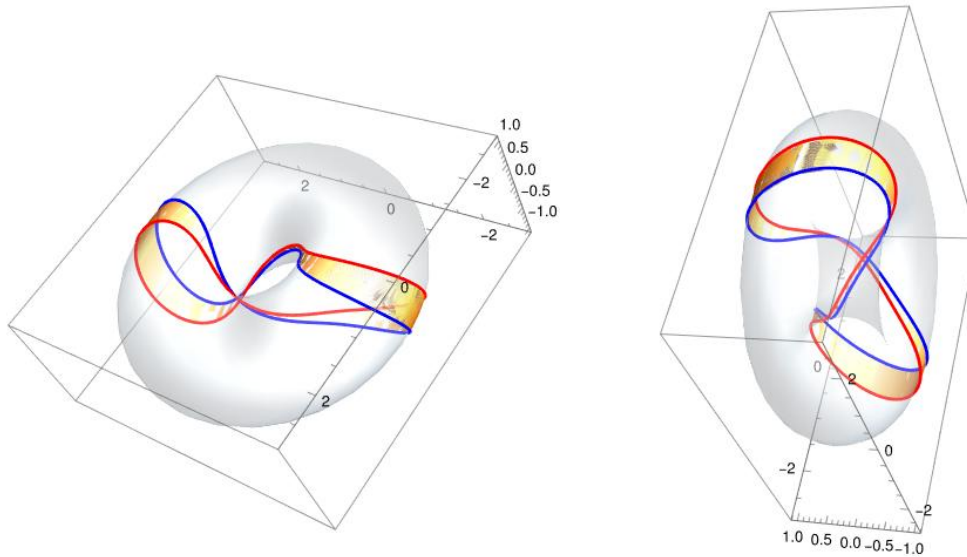


Figure 6.5: The compact  $C^1_{\mathbb{H}}$ -regular submanifold with boundary  $S$  (in yellow) obtained by revolving the curve in Figure 6.4 around the  $t$ -axis with an angle of  $\frac{\pi}{12}$ . The two connected components of the boundary are the original curve (in blue) and the curve rotated by  $\frac{\pi}{12}$  (in red).

One can obtain more examples of  $C^1_{\mathbb{H}}$ -regular submanifolds with boundary starting from different curves, as Figure 6.6 shows.

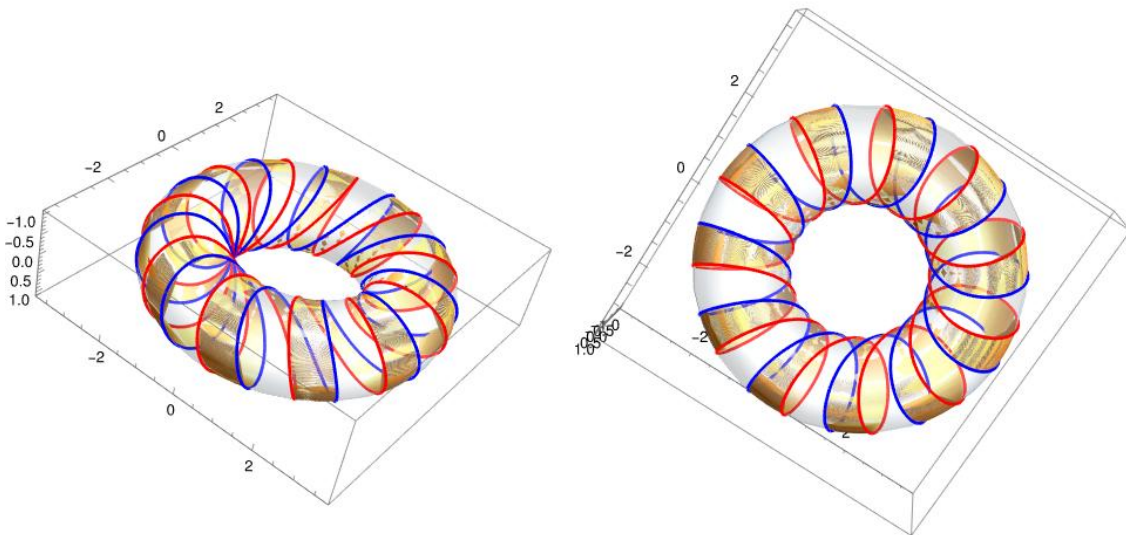


Figure 6.6: A compact  $C^1_{\mathbb{H}}$ -regular submanifold obtained as the one in Figure 6.5 starting from a curve that spirals 11 times around the torus.

## 6.4 Approximation of submanifolds with boundary

In this section we show that submanifolds with boundary can be locally approximated by  $C^1$  submanifolds with boundary. The approximation is in the sense of weak convergence

of currents, which are locally well defined because every submanifold is locally orientable; however, the careful reader will notice that the convergence is also in the sense of local Hausdorff distance and of the  $C_{\mathbb{H}}^1$  topology. Note also that, because of Lemma 6.22, low dimensional  $C_{\mathbb{H}}^1$ -submanifold with boundary are Euclidean  $C^1$ -submanifold with boundary; therefore, in this section we deal only with low codimensional and critical codimensional  $C_{\mathbb{H}}^1$ -submanifold with boundary. We start with the case of non-critical codimension  $k \leq n - 1$ .

**Lemma 6.38.** *Let  $1 \leq k \leq n - 1$  be an integer,  $O \subset \mathbb{H}^n$  an open set,  $S$  an oriented  $k$ -codimensional  $C_{\mathbb{H}}^1$  submanifold with boundary in  $O$  such that  $S$  and  $\partial S$  have locally finite measures<sup>7</sup> on  $O$ . Then for each  $p \in O \cap \partial S$  there exist an open set  $U$  with  $p \in U$  and a sequence  $(S_h)_{h \in \mathbb{N}}$  of oriented smooth and  $C_{\mathbb{H}}^1$  regular submanifolds of codimension  $k$  with smooth and  $C_{\mathbb{H}}^1$  boundary in  $O$  and locally finite measures in  $O$  such that*

$$\llbracket S_h \rrbracket(\omega) \xrightarrow{h \rightarrow +\infty} \llbracket S \rrbracket(\omega) \quad \text{for every } \omega \in \mathcal{D}_{\mathbb{H}}^{2n+1-k}(U) \quad (6.17)$$

and

$$\llbracket \partial S_h \rrbracket(\alpha) \xrightarrow{h \rightarrow +\infty} \llbracket \partial S \rrbracket(\alpha) \quad \text{for every } \alpha \in \mathcal{D}_{\mathbb{H}}^{2n-k}(U). \quad (6.18)$$

*Proof.* By Theorem 6.19, for every  $p \in \partial S \cap O$  there exist a neighbourhood  $U \subset O$  and  $f_1, \dots, f_{k+1} \in C_{\mathbb{H}}^1(U)$  such that, writing  $f = (f_1, \dots, f_{k+1})$  and  $\hat{f} = (f_1, \dots, f_k)$ ,

$$\begin{aligned} \nabla_{\mathbb{H}} f_1, \dots, \nabla_{\mathbb{H}} f_{k+1} &\text{ are linearly independent in } U \\ U \cap S &= \{q \in U : \hat{f}(q) = 0, f_{k+1}(q) > 0\} \\ U \cap \partial S &= \{q \in U : f(q) = 0\}. \end{aligned}$$

We also introduce the  $C_{\mathbb{H}}^1$ -regular submanifold  $S' = \{q \in U : f_1(q) = \dots = f_k(q) = 0\} \supset \bar{S}$ . By Remark 6.21 we can also assume that  $f_1, \dots, f_{k+1}$  are smooth on  $U \setminus S'$ ; we define

$$\begin{aligned} S'_h &:= \{q \in U : \hat{f}(q) = (\tfrac{1}{h}, 0, \dots, 0)\} \\ S_h &:= \{q \in U : \hat{f}(q) = (\tfrac{1}{h}, 0, \dots, 0), f_{k+1}(q) > 0\}. \end{aligned}$$

We observe that  $S_h$  are  $k$ -codimensional smooth and  $C_{\mathbb{H}}^1$ -regular submanifolds with smooth and  $C_{\mathbb{H}}^1$ -regular boundary  $\partial S_h = \{q \in U : \hat{f}(q) = (\tfrac{1}{h}, 0, \dots, 0), f_{k+1}(q) = 0\}$ .

We claim that, possibly reducing  $U$ , (6.17) holds. By Theorem 6.6 and the Implicit Function Theorem [100, Lemma 2.10], up to reducing  $U$  there exist an orthogonal splitting  $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ , an open subset  $E \subset \mathbb{W}$  and maps  $\phi, \phi_h : E \rightarrow \mathbb{V}$  such that  $S' = \text{gr}_{\phi}$ ,  $S'_h = \text{gr}_{\phi_h}$  and  $\phi_h$  are smooth. Let  $E_h, E_{\infty} \subset E$  be such that  $S_h = \text{gr}_{\phi_h|_{E_h}}$  and  $S = \text{gr}_{\phi|_{E_{\infty}}}$ ; notice that  $E_h, E_{\infty}$  are open. Reasoning as in the proof of Lemma 6.14 we observe that, possibly reducing  $E$  and  $U$ , for every  $\omega \in \mathcal{D}_{\mathbb{H}}^{2n+1-k}(U)$

<sup>7</sup>The measures on  $S, \partial S$  are of course the Hausdorff measures on  $S, \partial S$  of the appropriate dimensions that are (respectively)  $2n + 1 - k, 2n - k$  (in the low dimensional case  $k \geq n + 1$ ),  $Q - k, Q - k - 1$  (in the low-codimensional and non-critical case  $k \leq n - 1$ ) or  $n + 2, n$  (in the critical case  $k = n$ ).

- (a)  $\Phi_h \xrightarrow{h \rightarrow +\infty} \Phi$  uniformly on  $E$ , where  $\Phi(\xi) := \xi \cdot \phi(\xi)$  and  $\Phi_h(\xi) := \xi \cdot \phi_h(\xi)$
- (b)  $\left(\nabla_{\mathbb{H}} \left(f_1 - \frac{1}{h}\right) \wedge \cdots \wedge \nabla_{\mathbb{H}} f_k\right) \circ \Phi_h \xrightarrow{h \rightarrow +\infty} \left(\nabla_{\mathbb{H}} f_1 \wedge \cdots \wedge \nabla_{\mathbb{H}} f_k\right) \circ \Phi$  uniformly on  $E$
- (c)  $t_{S_h}^{\mathbb{H}} \circ \Phi_h \xrightarrow{h \rightarrow +\infty} t_S^{\mathbb{H}} \circ \Phi$  uniformly on  $E$
- (d)  $\omega \circ \Phi_h \xrightarrow{h \rightarrow +\infty} \omega \circ \Phi$  uniformly on  $E$
- (e)  $\Delta_h \circ \Phi \xrightarrow{h \rightarrow +\infty} \Delta \circ \Phi$  uniformly on  $E$ , where  $\Delta, \Delta_h$  are as in Lemma 6.14
- (f) there exists  $C > 0$  such that  $\Delta_h \geq C$  and  $\Delta \geq C$ .

Moreover, we claim that

- (g)  $\chi_{E_h} \rightarrow \chi_{E_\infty}$  almost everywhere on  $E$ .

Statement (g) can be proved observing that

- if  $\xi \in E_\infty$ , then  $f_{k+1}(\Phi_h(\xi)) \rightarrow f_{k+1}(\Phi(\xi)) > 0$ , therefore  $\xi \in E_h$  for large  $h$
- $\mathcal{H}_E^{2n+1-k}(\partial E_\infty) = 0$ ; this follows from (6.7) and taking into account that  $\partial E_\infty = \Phi^{-1}(\partial S)$  and  $\mathcal{S}^{Q-k}(\partial S) = 0$
- if  $\xi \in E \setminus \overline{E_\infty}$ , then  $f_{k+1}(\Phi_h(\xi)) \rightarrow f_{k+1}(\Phi(\xi)) < 0$ , therefore  $\xi \notin E_h$  for large  $h$ .

Using (a)–(g) we deduce

$$\begin{aligned}
\lim_{h \rightarrow +\infty} \llbracket S_h \rrbracket(\omega) &= \lim_{h \rightarrow +\infty} \int_{E_h} \langle t_{S_h}^{\mathbb{H}}(\Phi_h(\xi)) | \omega(\Phi_h(\xi)) \rangle \frac{|\nabla_{\mathbb{H}} \left(f_1 - \frac{1}{h}\right) \wedge \cdots \wedge \nabla_{\mathbb{H}} f_k|}{\Delta_h}(\Phi_h(\xi)) d\xi \\
&= \int_{E_\infty} \langle t_S^{\mathbb{H}}(\Phi(\xi)) | \omega(\Phi(\xi)) \rangle \frac{|\nabla_{\mathbb{H}} f_1 \wedge \cdots \wedge \nabla_{\mathbb{H}} f_k|}{\Delta}(\Phi(\xi)) d\xi \\
&= \frac{1}{C_{n,k}} \int_{S \cap U} \langle t_S^{\mathbb{H}}(q) | \omega(q) \rangle d\mathcal{S}_d^{Q-k}(q) \\
&= \llbracket S \rrbracket(\omega),
\end{aligned}$$

as desired.

Statement (6.18) can be proved (possibly restricting  $U$ ) with the same argument as in Lemma 6.14 by writing  $\partial S, \partial S_h$  as intrinsic graphs of codimension  $k+1$ ; we omit the boring details.  $\square$

We now turn to the case of critical codimension  $k = n$ . Before proving Lemma 6.39, let us give an idea of the approximation technique that we are going to use. We seek to produce a family  $(S_h)_{h \in \mathbb{N}}$  of Euclidean  $C^1$  submanifolds with boundary  $C_{\mathbb{H}}^1$ -approximating  $S$ , with the additional condition that  $\partial S_h = \partial S$ . Recall that  $\partial S$  is an  $n$ -dimensional horizontal  $C^1$  submanifold. The approximation argument goes like this:

- construct a Euclidean  $C^1$  submanifold with boundary  $\tilde{S}$  such that  $\partial\tilde{S} = \partial S$ ,
- produce smooth approximations (in  $C^1_{\mathbb{H}}$ )  $\Sigma_h$  of  $S$ , without any concern for their boundaries,
- define  $S_j$  by interpolating between  $\tilde{S}$  (in a neighbourhood of  $\partial S$ ) and  $\Sigma_h$  (away from  $\partial S$ ).

This construction eventually provides the desired sequence  $(S_h)_{h \in \mathbb{N}}$ , which will allow us to prove Stokes' Theorem. In fact, let us recall that the structure of Rumin's differential operator in critical dimension  $D$  is given by

$$D\omega = d(\omega + v_\omega), \quad \omega \in \mathcal{D}_{\mathbb{H}}^n$$

where  $v_\omega$  is a suitable  $n$ -form, depending on the first-order derivatives of  $\omega$ , which is "vertical", in the sense that it annihilates horizontal  $n$ -vectors. By using the classical Stokes Theorem, we will get

$$\int_S D\omega = \lim_{h \rightarrow +\infty} \int_{S_h} D\omega = \lim_{h \rightarrow +\infty} \int_{S_h} d(\omega + v_\omega) = \int_{\partial S} \omega + v_\omega = \int_{\partial S} \omega,$$

by exploiting the fact that  $\partial S_h = \partial S$  is horizontal.

**Lemma 6.39.** *Let  $O \subset \mathbb{H}^n$  be an open set and  $S$  an oriented  $n$ -codimensional  $C^1_{\mathbb{H}}$  submanifold with boundary in  $O$  such that  $S, \partial S$  have locally finite measures on  $O$ . Then, for each  $p \in O \cap \partial S$  there exist an open set  $U$  with  $p \in U$  and a sequence  $(S_h)_{h \in \mathbb{N}}$  of  $C^1$  and  $C^1_{\mathbb{H}}$  oriented regular submanifolds of codimension  $n$  in  $U$  with boundary  $U \cap \partial S = U \cap \partial S_h$  for every  $h \in \mathbb{N}$  and with locally finite measures in  $O$  such that*

$$\llbracket S_h \rrbracket(\omega) \xrightarrow{h \rightarrow +\infty} \llbracket S \rrbracket(\omega) \quad \text{for every } \omega \in \mathcal{D}_{\mathbb{H}}^{n+1}(U). \quad (6.19)$$

*Proof.* Let  $p \in \partial S$  be fixed; denote by  $S'$  a  $C^1_{\mathbb{H}}$  submanifold as in Definition 6.18. We use an approximation argument which is split into 8 steps.

*Step 1:*  $S'$  as the level set of a  $C^1_{\mathbb{H}}$  regular function as well as a graph.

By Lemma 6.5, up to an isometry of  $\mathbb{H}^n$  there exist a neighbourhood  $U$  of  $p$ , a function  $f : \mathbb{H}^n \rightarrow \mathbb{R}^n$  and  $C_0 > 0, \delta_0 > 0$  such that

$$\left\{ \begin{array}{l} f \in C^1_{\mathbb{H}}(\mathbb{H}^n, \mathbb{R}^n), \\ f \in C^\infty(\mathbb{H}^n \setminus \{f = 0\}, \mathbb{R}^n), \\ |\nabla_{\mathbb{H}} f| \leq C_0 \text{ on } \mathbb{H}^n, \\ \hat{\nabla}_{\mathbb{H}} f(q) \geq \delta_0 \text{Id}_{n \times n} \forall q \in \mathbb{H}^n \text{ in the sense of quadratic forms,} \\ S' \cap U = \{q \in U : f(q) = 0\}. \end{array} \right. \quad (6.20)$$

We assume without loss of generality that  $S' = \{f = 0\} \subset \mathbb{H}^n$ ; by [158, Theorem 1.4],  $S'$  is the entire intrinsic graph  $S' = \text{gr}_\phi$  of a suitable  $\phi : \mathbb{W} \rightarrow \mathbb{V}$  where

$$\mathbb{W} := \{(0, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \equiv \mathbb{H}^n\}, \quad \mathbb{V} := \{(x, 0, 0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \equiv \mathbb{H}^n\}.$$

Moreover, possibly reducing  $U$  we can assume that  $U = U_{\mathbb{W}} \cdot U_{\mathbb{V}}$  for suitable open sets  $U_{\mathbb{W}} \subseteq \mathbb{W}$  and  $U_{\mathbb{V}} \subseteq \mathbb{V}$ .

*Step 2:  $\pi(\partial S)$  as the level set of a  $C^1$  function  $\hat{g}$ .*

By Lemma 6.25, possibly reducing  $U$ , there exist  $\hat{g} \in C^1(\mathbb{R}^{2n}, \mathbb{R}^n)$  and  $\delta_1 > 0$  such that

$$\left\{ \begin{array}{l} \hat{g} \in C^1(\mathbb{R}^{2n}, \mathbb{R}^n), \\ |\nabla \hat{g}| \text{ is bounded on } \mathbb{R}^{2n}, \\ \nabla \hat{g} \circ \pi = \nabla_{\mathbb{H}} f \text{ on } \Sigma = U \cap \partial S, \\ \text{col}[\partial_{x_1} \hat{g} | \cdots | \partial_{x_n} \hat{g}](z) \geq \delta_1 \text{Id}_{n \times n} \forall z \in \mathbb{R}^{2n} \text{ in the sense of quadratic forms,} \\ \pi(\Sigma) \subseteq \{\hat{g} = 0\}. \end{array} \right.$$

*Step 3: Construction of a  $C^1$  and  $C_{\mathbb{H}}^1$  regular submanifold  $\tilde{S}$  which contains  $\Sigma = U \cap \partial S$ .*

Define  $g : \mathbb{H}^n \rightarrow \mathbb{R}^n$  by  $g(x, y, t) := \hat{g}(x, y)$ ; let  $\tilde{S} := \{g = 0\}$ . By Step 2 there exist  $C_1 > 0$  and  $\delta_1 > 0$  such that

$$\left\{ \begin{array}{l} |\nabla_{\mathbb{H}} g| \leq C_1 \text{ on } \mathbb{H}^n, \\ \Sigma = U \cap \partial S \subseteq \tilde{S}, \\ \widehat{\nabla}_{\mathbb{H}} g(q) = \text{col}[\partial_{x_1} \hat{g} | \cdots | \partial_{x_n} \hat{g}](q) \geq \delta_1 \text{Id}_{n \times n} \forall q \in \mathbb{H}^n \text{ in the sense of quadratic forms,} \\ \nabla_{\mathbb{H}} g = \nabla_{\mathbb{H}} f \text{ on } \Sigma. \end{array} \right. \quad (6.21)$$

Using [158, Theorem 1.4] and the fact that  $g \in C^1(\mathbb{H}^n, \mathbb{R}^n)$  one obtains

$$\left\{ \begin{array}{l} \tilde{S} \text{ is a } C_{\mathbb{H}}^1 \text{ regular submanifold and Euclidean } C^1 \text{ submanifold,} \\ U \cap \tilde{S} = U \cap (\pi(\Sigma) \times \mathbb{R}) \subset \mathbb{R}^{2n} \times \mathbb{R}^n \equiv \mathbb{H}^n, \\ \tilde{S} = \text{gr}_{\tilde{\phi}} \text{ for some globally defined } \tilde{\phi} \in C^1(\mathbb{W}, \mathbb{V}). \end{array} \right.$$

*Step 4: Construction of  $C_{\mathbb{H}}^1$  regular submanifolds  $S'_h$  approximating  $S'$ .*

We fix a non-negative kernel  $K \in C_c^\infty(U(0, 1))$  such that  $\int K d\mathcal{L}^{2n+1} = 1$  and we define

$$K_r(p) := \frac{1}{r^{2n+2}} K(\delta_{1/r}(p)).$$

For every  $h \in \mathbb{N}$  we also define  $\psi_h \in C_c^\infty(\Sigma_{3/h})$  (where by  $\Sigma_\varepsilon$  we denote the  $\varepsilon$ -neighbourhood of  $\Sigma$ ) such that  $\psi_h \equiv 1$  on  $\Sigma_{2/h}$  and  $0 \leq \psi_h \leq 1$ . Define

$$\tilde{f}_h := \psi_h(f \star K_{r_h}) + (1 - \psi_h)f,$$

where the positive parameters  $r_h \leq 1/h$  will be fixed later. Observe that

$$\nabla_{\mathbb{H}} \tilde{f}_h = (\nabla_{\mathbb{H}} \psi_h)(f \star K_{r_h} - f) + \psi_h(\nabla_{\mathbb{H}} f \star K_{r_h}) + (1 - \psi_h) \nabla_{\mathbb{H}} f.$$

Since  $(f \star K_r - f) \xrightarrow{r} 0$  uniformly on  $U$ , using (6.20) and choosing  $r_h$  small enough one gets

$$\begin{cases} |\nabla_{\mathbb{H}} \tilde{f}_h| \leq 2C_0 + 1, \\ \text{col}[X_1 \tilde{f}_h | \cdots | X_n \tilde{f}_h](q) \geq \frac{\delta_0}{2} \text{Id}_{n \times n} \quad \forall q \in \mathbb{H}^n \text{ in the sense of quadratic forms,} \\ \|f \star K_{r_h} - f\|_{C^0(U)} \leq \frac{1}{h^2}. \end{cases} \quad (6.22)$$

Since  $\nabla_{\mathbb{H}} g = \nabla_{\mathbb{H}} f$  on  $\Sigma$  we have

$$\|\nabla_{\mathbb{H}}(g - f)\|_{C^0(\Sigma_{3/h})} = o(1) \text{ as } h \rightarrow +\infty;$$

this, together with the fact that  $g = f = 0$  on  $\Sigma$ , implies that

$$\|g - f\|_{C^0(\Sigma_{3/h})} = o\left(\frac{1}{h}\right) \text{ as } h \rightarrow +\infty.$$

Using (6.22), for small enough  $r_h$  we find

$$\|g - \tilde{f}_h\|_{C^0(\Sigma_{2/h})} = o\left(\frac{1}{h}\right). \quad (6.23)$$

For every  $h \in \mathbb{N}$  we fix a cut-off function  $\chi_h \in C_c^\infty(\Sigma_{2/h})$  such that

$$\begin{cases} 0 \leq \chi_h \leq 1, \\ \chi_h \equiv 1 \text{ on } \Sigma_{1/h}, \\ |\nabla_{\mathbb{H}} \chi_h| \leq C_2 h, \end{cases} \quad (6.24)$$

for a suitable  $C_2 > 0$  not depending on  $j$ . Let  $v_h := (\frac{2}{h^2}, 0, \dots, 0) \in \mathbb{R}^n$  and define

$$f_h := \chi_h g + (1 - \chi_h)(\tilde{f}_h - v_h).$$

We observe that

$$\nabla_{\mathbb{H}} f_h = \nabla_{\mathbb{H}} \chi_h (g - \tilde{f}_h + v_h) + \chi_h \nabla_{\mathbb{H}} g + (1 - \chi_h) \nabla_{\mathbb{H}} \tilde{f}_h.$$

Because of (6.23) and (6.24) one has

$$\|\nabla_{\mathbb{H}} \chi_h (g - \tilde{f}_h + v_h)\|_{C^0(\Sigma_{2/h})} \leq C_2 h \left( o\left(\frac{1}{h}\right) + \frac{1}{h^2} \right) = o(1)$$

while

$$\nabla_{\mathbb{H}} \chi_h (g - \tilde{f}_h + v_h) \equiv 0 \text{ out of } \Sigma_{2/h}.$$

Therefore, if  $h$  is large enough we get by (6.21) and (6.22) that for suitable  $C_3 > 0$  and  $\delta_3 > 0$

$$\begin{cases} |\nabla_{\mathbb{H}^n} f_h| \leq C_3 \text{ on } \mathbb{H}^n, \\ \text{col}[X_1 f_h] \cdots [X_n f_h](q) \geq \delta_3 \text{Id}_{n \times n} \quad \forall q \in \mathbb{H}^n \text{ in the sense of quadratic forms.} \end{cases} \quad (6.25)$$

Therefore, the level set  $S'_h := \{f_h = 0\}$  is a  $C^1_{\mathbb{H}}$  regular submanifold and, by [158, Theorem 1.4], it is also the intrinsic Lipschitz graph of a globally defined function  $\phi_h : \mathbb{W} \rightarrow \mathbb{V}$ .

*Step 5:  $U \cap S'_h$  is a Euclidean  $C^1$  submanifold and  $\phi_h \in C^1(U_{\mathbb{W}}, \mathbb{V})$ .*

Let  $q \in S'_h$ , i.e.,  $f_h(q) = 0$ . If  $q \in \Sigma_{2/h}$ , then  $\psi_h \equiv 1$  in a neighbourhood of  $q$  and (again in a neighbourhood of  $q$ )

$$f_h = \chi_h g + (1 - \chi_h)(f \star K_{r_h} - v_h).$$

In particular,  $f_h$  is  $C^1$  regular in a neighbourhood of  $q$ , hence  $S'_h$  is of class  $C^1$  in a neighbourhood of  $q$ . Instead, if  $q \in U \setminus \Sigma_{2/h}$ , then  $\chi_h \equiv 0$  in a neighbourhood of  $q$  and

$$0 = f_h(q) = \tilde{f}_h(q) - v_h = \psi_h(q)(f \star K_{r_h})(q) + (1 - \psi_h(q))f(q) - v_h,$$

i.e.,

$$f(q) = v_h + \psi_h(q) \left( f(q) - (f \star K_{r_h})(q) \right).$$

Using (6.22)

$$|f(q)| \geq \|v_h\| - \|f - f \star K_{r_h}\|_{C^0(U)} \geq \frac{2}{h^2} - \frac{1}{h^2} > 0,$$

hence  $f(q) \neq 0$  and, by (6.20),  $f$  is  $C^\infty$  in a neighbourhood of  $q$  where one also has

$$f_h = \tilde{f}_h - v_h = \psi_h(f \star K_{r_h}) + (1 - \psi_h)f - v_h.$$

It follows that  $f_h$  is  $C^\infty$  smooth in a neighbourhood of  $q$ ; in particular,  $U \cap S'_h$  is a  $C^1$  Euclidean submanifold and  $\phi_h$  is  $C^1$  on  $U_{\mathbb{W}}$ .

*Step 6: Construction of the approximating submanifolds  $S_h$ .*

Let  $\pi_{\mathbb{W}} : \mathbb{H}^n \rightarrow \mathbb{W}$  be the projection  $\pi_{\mathbb{W}}(x, y, t) := (x, y, t) \cdot (-x, 0, 0)$ ; in particular,  $\pi_{\mathbb{W}}(U) = \pi_{\mathbb{W}}(U_{\mathbb{W}} \cdot U_{\mathbb{V}}) = U_{\mathbb{W}}$ . Recall that  $S \subseteq S' = \text{gr}_\phi$ ,  $S'_h = \text{gr}_{\phi_h}$  and  $\tilde{S} = \text{gr}_{\tilde{\phi}}$ . The open set  $U_{\mathbb{W}}^+ := \pi_{\mathbb{W}}(U \cap S) \subseteq U_{\mathbb{W}}$  satisfies

$$U_{\mathbb{W}} \cap \partial U_{\mathbb{W}}^+ = \pi_{\mathbb{W}}(\Sigma).$$

The submanifolds  $S_h$  defined for  $h \in \mathbb{N}$  by  $S_h := \text{gr}_{\phi_h|_{U_{\mathbb{W}}^+}}$  are relatively open subsets of  $S'_h$ , hence they are both  $C^1_{\mathbb{H}}$  regular and Euclidean  $C^1$  submanifold.

*Step 7:  $U \cap S_h$  is a  $C^1$  manifold with boundary and  $U \cap \partial S_h = U \cap \partial S$ .*

For  $h \in \mathbb{N}$  and  $i = 1, 2, 3$  we define  $\Delta_h^i := \pi_{\mathbb{W}}(\Sigma_{i/h})$ ;  $\Delta_h^i$  are open neighbourhoods of

$\pi_{\mathbb{W}}(\Sigma)$ . We observe that  $\phi_h \equiv \tilde{\phi}$  on  $\Delta_h^1$ : this follows upon noticing that  $f_h \equiv g$  on  $\Sigma_{1/h}$ , hence

$$\Sigma_{1/h} \cap \text{gr}_{\phi_h} = \Sigma_{1/h} \cap S'_h = \Sigma_{1/h} \cap \tilde{S} = \Sigma_{1/h} \cap \text{gr}_{\tilde{\phi}}.$$

This implies that  $\phi_h \equiv \tilde{\phi}$  on  $\Delta_h^1$  as well as the fact that the  $U \cap S_h$  is a classical  $C^1$  submanifold with boundary  $U \cap \partial S_h = U \cap \partial S$ .

We observe in passing that

$$\phi_h \equiv \tilde{\phi}_h \text{ on } U_{\mathbb{W}}^+ \setminus \Delta_h^3, \text{ where } \tilde{\phi}_h \in C^\infty(\mathbb{W}, \mathbb{V}) \text{ is such that } \{f = v_h\} = \text{gr}_{\tilde{\phi}_h}. \quad (6.26)$$

*Step 8:*  $\llbracket S_h \rrbracket(\omega) \xrightarrow{h \rightarrow +\infty} \llbracket S \rrbracket(\omega)$  for every  $\omega \in \mathcal{D}_{\mathbb{H}}^{n+1}(U)$ .

Let  $\Phi_h(w) := w \cdot \phi_h(w)$ ,  $w \in U_{\mathbb{W}}$ , be the graph map associated with the  $C^1$  map  $\phi_h$ . Let  $\omega \in \mathcal{D}_{\mathbb{H}}^{n+1}(U)$  and  $k \in \mathbb{N}$  be fixed. Later we will let  $k \rightarrow +\infty$  after we let  $h \rightarrow +\infty$  so we can suppose  $h > k$ . We use Theorem 6.6 to get

$$\llbracket S_h \rrbracket(\omega) = \int_{U_{\mathbb{W}}^+} \langle t_{S_h}^{\mathbb{H}}(\Phi_h(\xi)) | \omega(\Phi_h(\xi)) \rangle \frac{|\nabla_{\mathbb{H}}(f_h)_1 \wedge \cdots \wedge \nabla_{\mathbb{H}}(f_h)_n|}{|\det(\widehat{\nabla}_{\mathbb{H}} f_h)|}(\Phi_h(\xi)) d\xi$$

where we used  $(f_h)_\ell$  to denote the  $\ell$ -th component of  $f_h$ . We split the integral as

$$\begin{aligned} \llbracket S_h \rrbracket(\omega) &= \int_{U_{\mathbb{W}}^+ \setminus \Delta_k^3} \langle t_{S_h}^{\mathbb{H}}(\Phi_h(\xi)) | \omega(\Phi_h(\xi)) \rangle \frac{|\nabla_{\mathbb{H}}(f_h)_1 \wedge \cdots \wedge \nabla_{\mathbb{H}}(f_h)_n|}{|\det(\widehat{\nabla}_{\mathbb{H}} f_h)|}(\Phi_h(\xi)) d\xi \\ &\quad + \int_{\Delta_k^3} \langle t_{S_h}^{\mathbb{H}}(\Phi_h(\xi)) | \omega(\Phi_h(\xi)) \rangle \frac{|\nabla_{\mathbb{H}}(f_h)_1 \wedge \cdots \wedge \nabla_{\mathbb{H}}(f_h)_n|}{|\det(\widehat{\nabla}_{\mathbb{H}} f_h)|}(\Phi_h(\xi)) d\xi \end{aligned}$$

We first focus on the second integral. Using the properties in (6.25) we obtain that the area factor  $\frac{|\nabla_{\mathbb{H}}(f_h)_1 \wedge \cdots \wedge \nabla_{\mathbb{H}}(f_h)_n|}{|\det(\widehat{\nabla}_{\mathbb{H}} f_h)|}(\Phi_h(\xi))$  is bounded by a positive constant only depending on  $\delta_3$  and  $C_3$  while  $|\langle t_{S_h}^{\mathbb{H}}(\Phi_h(\xi)) | \omega(\Phi_h(\xi)) \rangle|$  is bounded on  $\Delta_k^3$  by  $\|\omega\|_{C^0(\Sigma_{3/k})} < +\infty$ . This means that

$$\int_{\Delta_k^3} \langle t_{S_h}^{\mathbb{H}}(\Phi_h(\xi)) | \omega(\Phi_h(\xi)) \rangle \frac{|\nabla_{\mathbb{H}}(f_h)_1 \wedge \cdots \wedge \nabla_{\mathbb{H}}(f_h)_n|}{|\det(\widehat{\nabla}_{\mathbb{H}} f_h)|}(\Phi_h(\xi)) d\xi = O(\mathcal{L}^{n+1}(\Delta_k^3)).$$

We now focus on the first integral. On  $U \setminus \Sigma_{3/k}$  we recall that  $f_h$  is defined (since  $\psi_h \equiv 0$  on  $U \setminus \Sigma_{3/k}$  because  $h > k$ ) as

$$f_h = f - v_h.$$

It follows that the horizontal derivatives of  $f_h$  coincide with the horizontal derivatives of  $f$  on  $U \setminus \Sigma_{3/k}$ . Since  $\Phi_h \xrightarrow{h \rightarrow +\infty} \Phi$  uniformly on  $U_{\mathbb{W}}^+ \setminus \Delta_k^3$  the latter implies that

$$\frac{|\nabla_{\mathbb{H}}(f_h)_1 \wedge \cdots \wedge \nabla_{\mathbb{H}}(f_h)_n|}{|\det(\widehat{\nabla}_{\mathbb{H}} f_h)|} \circ \Phi_h \xrightarrow{h \rightarrow +\infty} \frac{|\nabla_{\mathbb{H}} f_1 \wedge \cdots \wedge \nabla_{\mathbb{H}} f_n|}{|\det(\widehat{\nabla}_{\mathbb{H}} f)|} \circ \Phi \text{ uniformly on } U_{\mathbb{W}}^+ \setminus \Delta_k^3$$

(where with  $f_\ell$ ,  $1 \leq \ell \leq n$ , we denoted the  $\ell$ -th component of  $f$ ) and

$$\langle t_{S_h}^{\mathbb{H}} \circ \Phi_h | \omega \circ \Phi_h \rangle \xrightarrow{h \rightarrow +\infty} \langle t_S^{\mathbb{H}} \circ \Phi | \omega \circ \Phi \rangle \text{ uniformly on } U_{\mathbb{W}}^+ \setminus \Delta_k^3,$$

in the same fashion as in the proof of Lemma 6.14. Then we obtain

$$\begin{aligned}
& \lim_{h \rightarrow +\infty} \int_{U_{\mathbb{W}}^+ \setminus \Delta_k^3} \langle t_{S_h}^{\mathbb{H}}(\Phi_j(\xi)) | \omega(\Phi_h(\xi)) \rangle \frac{|\nabla_{\mathbb{H}}(f_h)_1 \wedge \cdots \wedge \nabla_{\mathbb{H}}(f_h)_n|}{|\det(\widehat{\nabla}_{\mathbb{H}} f_h)|} (\Phi_h(\xi)) d\xi \\
&= \int_{U_{\mathbb{W}}^+ \setminus \Delta_k^3} \langle t_S^{\mathbb{H}}(\Phi(\xi)) | \omega(\Phi(\xi)) \rangle \frac{|\nabla_{\mathbb{H}} f_1 \wedge \cdots \wedge \nabla_{\mathbb{H}} f_n|}{|\det(\widehat{\nabla}_{\mathbb{H}} f)|} (\Phi(\xi)) d\xi \\
&= \llbracket S \rrbracket(\omega) - \int_{\Delta_k^3} \langle t_S^{\mathbb{H}}(\Phi(\xi)) | \omega(\Phi(\xi)) \rangle \frac{|\nabla_{\mathbb{H}} f_1 \wedge \cdots \wedge \nabla_{\mathbb{H}} f_n|}{|\det(\widehat{\nabla}_{\mathbb{H}} f)|} (\Phi_j(\xi)) d\xi.
\end{aligned}$$

As before we can estimate

$$\int_{\Delta_k^3} \langle t_S^{\mathbb{H}}(\Phi(\xi)) | \omega(\Phi(\xi)) \rangle \frac{|\nabla_{\mathbb{H}} f_1 \wedge \cdots \wedge \nabla_{\mathbb{H}} f_n|}{|\det(\widehat{\nabla}_{\mathbb{H}} f)|} (\Phi_h(\xi)) d\xi = O(\mathcal{L}^{n+1}(\Delta_k^3))$$

so that, for every  $k \in \mathbb{N}$ ,

$$\limsup_{h \rightarrow +\infty} |\llbracket S_h \rrbracket(\omega) - \llbracket S \rrbracket(\omega)| \leq O(\mathcal{L}^{n+1}(\Delta_k^3)).$$

Letting  $k \rightarrow +\infty$  and using the fact that  $O(\mathcal{L}^{n+1}(\Delta_k^3)) \rightarrow \mathcal{L}^{n+1}(\pi_{\mathbb{W}}(\Sigma)) = 0$  we obtain

$$\llbracket S_h \rrbracket(\omega) \xrightarrow{h \rightarrow +\infty} \llbracket S \rrbracket(\omega)$$

for every  $\omega \in \mathcal{D}_{\mathbb{H}}^{n+1}(U)$ . □

**Remark 6.40.** By a standard approximating procedure, the submanifolds  $S_h$  of Lemma 6.39 can be chosen to with the extra property that each  $S_h$  (to be understood without its boundary) is actually  $C^\infty$  smooth.

## 6.5 Proof of Stokes' Theorem

We are ready to prove our main result. Observe that Theorem 6.1 is an immediate consequence of the following theorem.

**Theorem 6.41.** *Let  $1 \leq k \leq 2n + 1$  be an integer,  $O \subset \mathbb{H}^n$  be an open set and  $S \subset O$  be an orientable  $k$ -codimensional  $C_{\mathbb{H}}^1$  regular submanifold with boundary<sup>8</sup> such that  $S$  and  $\partial S$  have locally finite measures on  $O$ . Then*

$$\llbracket S \rrbracket(d_c \omega) = \llbracket \partial S \rrbracket(\omega) \quad \text{for every } \omega \in \mathcal{D}_{\mathbb{H}}^{2n-k}(O)$$

or, equivalently,  $\partial_c \llbracket S \rrbracket = \llbracket \partial S \rrbracket$ .

*Proof.* The low-dimensional case, i.e., when  $k \geq n + 1$ , is an easy consequence of the classical Stokes' Theorem; we then consider the low-codimensional case and assume hereafter that  $k \leq n$ . We first define a suitable open covering of  $O \cap \bar{S}$  as follows. For each

<sup>8</sup>The careful reader will notice that Theorem 6.41 holds also when  $O \cap \partial S$  is empty.

$p \in S$  we use Lemma 6.14 to find an open neighbourhood  $U_p^1 \subset O$  and a sequence  $(S_h)_{h \in \mathbb{N}}$  of smooth and  $C_{\mathbb{H}}^1$  submanifolds of codimension  $k$  contained in  $U_p^1$  such that

$$\llbracket S_h \rrbracket(\omega) \xrightarrow{h \rightarrow +\infty} \llbracket S \rrbracket(\omega) \quad \text{for every } \omega \in \mathcal{D}_{\mathbb{H}}^{2n+1-k}(U_p^1).$$

For each  $p \in O \cap \partial S$  we can use Lemma 6.38 (if  $k \leq n - 1$ ) or Lemma 6.39 (if  $k = n$ ) to find an open neighbourhood  $U_p^2$  of  $p$  and a sequence  $(S_h)_{h \in \mathbb{N}}$  of  $C^1$  and  $C_{\mathbb{H}}^1$   $k$ -codimensional submanifolds with  $C^1$  boundary in  $U_p^2$  such that

$$\begin{aligned} \llbracket S_h \rrbracket(\omega) &\xrightarrow{h \rightarrow +\infty} \llbracket S \rrbracket(\omega) && \text{for every } \omega \in \mathcal{D}_{\mathbb{H}}^{2n+1-k}(U_p^2) \\ \llbracket \partial S_h \rrbracket(\alpha) &\xrightarrow{h \rightarrow +\infty} \llbracket \partial S \rrbracket(\alpha) && \text{for every } \alpha \in \mathcal{D}_{\mathbb{H}}^{2n-k}(U_p^2). \end{aligned}$$

We extract from the families  $(U_p^1)_{p \in S}$  and  $(U_p^2)_{p \in O \cap \partial S}$  countable (or, possibly, finite) subfamilies  $(U_i^1)_{i \in \mathbb{N}}$  and  $(U_i^2)_{i \in \mathbb{N}}$  such that  $O \cap \bar{S} \subset \bigcup_{i \in \mathbb{N}} U_i^1 \cup \bigcup_{i \in \mathbb{N}} U_i^2$ . We also fix a partition of the unity, i.e., functions  $\zeta_i^j \in C_c^\infty(U_i^j)$ ,  $i \in \mathbb{N}$ ,  $j \in \{1, 2\}$ , such that

$$0 \leq \zeta_i^j \leq 1 \quad \text{and} \quad \sum_{\substack{i \in \mathbb{N} \\ j=1,2}} \zeta_i^j = 1 \text{ on } \bar{S}. \quad (6.27)$$

It is not restrictive to assume that the covering  $(U_i^1)_i \cup (U_i^2)_i$  of  $O \cap \bar{S}$  is locally finite, so that the sum in (6.27) is well defined.

Let  $\omega \in \mathcal{D}_{\mathbb{H}}^{2n-k}(O)$  be fixed; for every  $i \in \mathbb{N}$  and  $j = 1, 2$  denote by  $(S_h)_h$  the  $C^1$  submanifolds approximating  $S$  (in the sense of currents) in  $U_i^j$ . Using Lemma 6.12 we get

$$\begin{aligned} \llbracket S \rrbracket(d_c(\zeta_i^j \omega)) &= \lim_{h \rightarrow \infty} \llbracket S_h \rrbracket(d_c(\zeta_i^j \omega)) = \lim_{h \rightarrow \infty} \int_{S_h} d_c(\zeta_i^j \omega) \\ &\stackrel{*}{=} \lim_{h \rightarrow \infty} \int_{\partial S_h} \zeta_i^j \omega = \lim_{h \rightarrow \infty} \llbracket \partial S_h \rrbracket(\zeta_i^j \omega) = \llbracket \partial S \rrbracket(\zeta_i^j \omega), \end{aligned} \quad (6.28)$$

where the equality marked by  $*$  follows from the classical Stokes' Theorem if  $k \leq n - 1$  (when  $d_c = d$  is the classical exterior differentiation) while, if  $k = n$ , it follows from Stokes' Theorem and Theorem 2.44 upon observing that

$$\begin{aligned} \int_{S_h} D(\zeta_i^j \omega) &= \int_{S_h} d(\zeta_i^j \omega - \theta \wedge L^{-1}((d(\zeta_i^j \omega))_{\mathfrak{h}_1})) \\ &= \int_{\partial S_h} \zeta_i^j \omega - \theta \wedge L^{-1}((d(\zeta_i^j \omega))_{\mathfrak{h}_1}) = \int_{\partial S} \zeta_i^j \omega, \end{aligned}$$

the last equality following from the fact that  $\partial S_h$  equals  $\partial S$  and it is tangent to the horizontal distribution. Finally,

$$\partial_c \llbracket S \rrbracket(\omega) = \sum_{\substack{i \in \mathbb{N} \\ j=1,2}} \partial_c \llbracket S \rrbracket(\zeta_i^j \omega) = \sum_{\substack{i \in \mathbb{N} \\ j=1,2}} \llbracket S \rrbracket(d_c(\zeta_i^j \omega)) = \sum_{\substack{i \in \mathbb{N} \\ j=1,2}} \llbracket \partial S \rrbracket(\zeta_i^j \omega) = \llbracket \partial S \rrbracket(\omega)$$

and the proof is accomplished.  $\square$

**Remark 6.42.** Observe that the same proof of Theorem 6.41 holds also for  $(n + 1)$ -dimensional  $C^1$ -submanifolds with boundary (possibly with non-empty characteristic set<sup>9</sup> on the interior of the submanifold) such that  $\partial S$  is horizontal and  $S, \partial S$  have locally finite measures in  $\mathbb{H}^n$ . In fact, for every  $\omega \in \mathcal{D}_{\mathbb{H}}^n$  we have, by the classical Stokes' Theorem,

$$\int_S D\omega = \int_S d(\omega - \theta \wedge L^{-1}(d\omega)_{\mathfrak{h}_1}) = \int_{\partial S} (\omega - \theta \wedge L^{-1}(d\omega)_{\mathfrak{h}_1}) = \int_{\partial S} \omega,$$

the last equality coming from the fact that  $\partial S$  is horizontal.

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<sup>9</sup>Recall that for a  $C^1$ -regular submanifold  $S \subseteq \mathbb{H}^n$  of codimension  $1 \leq k \leq n$  we say that  $p \in S$  is a *characteristic point* of  $S$  if  $T_p S \subseteq H_p \mathbb{H}^n$ . The set of characteristic points of  $S$  is called *characteristic set*.

# A Low regularity contact theory

What follows in this Appendix is joint work (in progress) with Antoine Julia, Patrick Massot, and Anne Vaugon.

As we said in the Introduction, every contact manifold can be endowed with a Carnot-Carathéodory distance and therefore with a sub-Riemannian structure and we want to understand how much biLipschitz<sup>1</sup> maps between contact manifolds “remember” of the contact structures. In the Riemannian case there is a huge body of literature regarding the analogous problem (see for instance [97, 148, 149]). For example, every self-homeomorphism of a smooth manifold two or three dimensional is homotopic to a diffeomorphism (see for instance [133]). However this is not always the case in higher dimension; for instance there exist self-homeomorphisms of smooth manifolds (although with possibly large biLipschitz constant) that are *not* homotopic to a diffeomorphism: one of the most famous example is the four dimensional one produced in [87]. Examples like the previous one are the reason why we aim to prove results only regarding 3-dimensional contact manifolds. Before properly stating what we want to prove, let us recall some basic definitions and results. For a comprehensive introduction to contact geometry we refer the reader to [92].

**Definition A.1.** Let  $M$  be a manifold of odd dimension  $2n + 1$ . A *contact structure* is a maximally non-integrable hyperplane field  $\xi = \ker(\alpha) \subseteq TM$ , that is, the defining differential 1-form  $\alpha$  is required to satisfy

$$\alpha \wedge (d\alpha)^n \neq 0$$

(meaning that it vanishes nowhere). Such a 1-form is called a *contact form*. The pair  $(M, \xi)$  is called a *contact manifold*.

**Remark A.2.** Given a contact manifold  $(M, \xi)$  we can choose a smooth scalar product  $g$  on  $\xi$ . The latter allows to define a Carnot-Carathéodory distance on  $M$ ; in this case we say that  $(M, \xi, g)$  is a *contact sub-Riemannian manifold*.

**Definition A.3.** Let  $(M, \xi)$  and  $(M', \xi')$  be contact manifolds. We say that a diffeomorphism  $\psi : M \rightarrow M'$  is a *contactomorphism* if  $T\psi(\xi) = \xi'$ , where  $T\psi : TM \rightarrow TM'$  denotes the

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<sup>1</sup>With respect to the Carnot-Carathéodory metrics.

differential of  $f$ . If  $\xi = \ker(\alpha)$  and  $\xi' = \ker(\alpha')$ , this is equivalent to saying that  $\alpha'$  and  $f^*\alpha$  determine the same hyperplane field, and hence equivalent to the existence of a nowhere zero function  $\lambda : M \rightarrow \mathbb{R} \setminus \{0\}$  such that  $f^*\alpha' = \lambda\alpha$ .

**Example A.4.** Heisenberg groups are an example of contact manifolds with the contact form given by

$$\theta = dt + \frac{1}{2} \sum_{j=1}^n (y_j dx_j - x_j dy_j).$$

Observe that  $H\mathbb{H}^n = \ker(\theta)$ .

Actually, Heisenberg groups  $(\mathbb{H}^n, H\mathbb{H}^n)$  are the local model for every contact manifold, as stated by the following result, known as Darboux's theorem (see for instance [92, Theorem 2.5.1] for a proof).

**Theorem A.5.** *Let  $(M, \xi)$  be a contact manifold of dimension  $2n + 1$ . Then for every  $p \in M$  there exist a neighbourhood  $U$  of  $p$  and a contactomorphism  $\psi$  between  $U \subseteq (M, \xi)$  and  $\psi(U) \subseteq (\mathbb{H}^n, H\mathbb{H}^n)$ .*

How sub-Riemannian and contact structures interact is an interesting open question, see for instance [130, Section 12]. We aim to prove the following result.

**Conjecture A.6.** *Let  $(M, \xi, g)$  and  $(M', \xi', g')$  be two contact 3D sub-Riemannian manifolds. There exists  $\varepsilon_0 > 0$  such that, if there exists a  $(1 + \varepsilon)$ -biLipschitz<sup>2</sup> map between  $M$  and  $M'$  with  $\varepsilon < \varepsilon_0$  then there exists a contactomorphism between  $(M, \xi)$  and  $(M', \xi')$ .*

However, Conjecture A.6 seems quite difficult so we are more confident to prove in the near future the following weaker result.

**Theorem<sup>?</sup> A.7.** *Let  $(M, \xi, g)$  and  $(M', \xi', g')$  be two contact 3D sub-Riemannian manifolds. There exists  $\varepsilon_0 > 0$  such that, if there exists a  $(1 + \varepsilon)$ -biLipschitz<sup>3</sup> map between  $M$  and  $M'$  with  $\varepsilon < \varepsilon_0$  then  $M$  and  $M'$  are both tight or both overtwisted.*

The dichotomy tight/overtwisted for contact manifolds was introduced in [69], here we only recall the main definitions.

**Definition A.8.** We say that a contact manifold is *overtwisted* if it contains an overtwisted disc; we say that a contact manifold is *tight* if it does not contain an overtwisted disk. We say that an embedded disc  $\Delta$  in a contact manifold  $(M, \xi)$  is an *overtwisted disc* if  $T_p\Delta = \xi_p$  for every  $p \in \partial\Delta$ .

<sup>2</sup>With respect to the Carnot-Carathéodory distances.

<sup>3</sup>With respect to the Carnot-Carathéodory distances.

The proof of Theorem<sup>7</sup> A.7 is still work in progress so we omit the details, but the idea is the following. We assume  $(M, \xi)$  is overtwisted; let  $\Delta$  be an overtwisted disk  $\Delta \subseteq M$ . We assume the existence of a Carnot-Carathéodory biLipschitz map  $\psi : M \rightarrow M'$  with constant sufficiently close to 1. Then we take some Darboux charts (given by Theorem A.5) so we can look at the problem, locally, in  $\mathbb{H}^1$ . We are confident that the rigidity results proved in [15] for biLipschitz Carnot-Carathéodory maps with constant sufficiently close to 1 from  $\mathbb{H}^1$  to  $\mathbb{H}^1$  (also known as quasi-isometries<sup>4</sup> of  $\mathbb{H}^1$ ) will allow us to prove that  $\psi(\Delta)$  is close, *in some sense*, to an overtwisted disk.

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<sup>4</sup>Let us also recall that in [35] the authors proved that isometries between sub-Riemannian manifolds are smooth in any open sets where the sub-Riemannian structure is equiregular; see also [94, 105].

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