HOMOGENIZATION OF PERIODIC MULTI-DIMENSIONAL STRUCTURES: THE LINEARLY ELASTIC/PERFECTLY PLASTIC CASE

Nadia Ansini and François Bille Ebobisse

Abstract. In this paper we study the asymptotic behaviour via Γ-convergence of some integral functionals F_ε which model some multi-dimensional structures and depend explicitly on the linearized strain tensor. The functionals F_ε are defined in particular classes of functions with bounded deformation while the limit problem is set in the usual framework of Sobolev spaces or $BD(\Omega)$. We also construct an example of such functionals showing that under some special assumptions we can have non local effects.

1 Introduction

In recent years there has been an increasing interest in the description of media whose microscopic behaviour takes into account lower dimensional or multi-dimensional structures and can be modeled by suitable integral functionals with respect to periodic measures (see [9], [22], [1], [6], [4]). Zhikov studied in [22] the homogenization of functionals F_ε defined as

$$F_\varepsilon(u,\Omega) = \int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla u\Big) d\mu_\varepsilon$$

on $C^\infty(\Omega; \mathbf{R}^m)$, where μ_ε is defined by $\mu_\varepsilon(B) := \varepsilon^n \mu(\frac{1}{\varepsilon}B)$ with μ a fixed 1-periodic Radon measure and f is a Borel function 1-periodic in the first variable (see also Braides and Chiadò Piat [9] for the case $\mu = \chi_E$ with E periodic, and Bouchitté, Buttazzo and Seppecher [6] for relaxation results in the case of general μ). On the other hand, following the approach of Ambrosio, Buttazzo and Fonseca [1], which is somehow complementary to the "smooth approach" described above, Ansini, Braides and Chiadò Piat studied in [4] the asymptotic behaviour of energy functionals concentrated on periodic multi-dimensional structures, of the form

$$F_\varepsilon(u,\Omega) = \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dDu}{d\mu_\varepsilon}\Big) d\mu_\varepsilon\,.$$

In this case the problem is set in the framework of Sobolev spaces $\mathrm{W}^{1,p}_{\mu_\varepsilon}(\Omega;\mathbf{R}^m)$ with respect to the measure μ_ε of [1]. We recall that $\mathrm{W}^{1,p}_{\mu_\varepsilon}(\Omega;\mathbf{R}^m)$ is the space of functions $u \in \mathrm{L}^p(\Omega;\mathbf{R}^m)$ whose distributional derivative is a measure absolutely continuous with respect to μ_ε with p-summable density $dDu/d\mu_\varepsilon$. A homogenization theorem for F_ε has been proved under a standard growth condition of order p on f and a notion of p-homogenizability introduced for the measure μ (see [4] Theorem 3.5).

In the context of linear elasticity or perfect plasticity, in place of considering energies depending on the deformation gradient Du, it is more appropriate to consider energy functionals depending explicitly on the *linearized strain tensor* Eu. Our goal in this paper is to study the asymptotic behaviour of functionals of the type

$$F_\varepsilon(u,\Omega) = \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dEu}{d\mu_\varepsilon}\Big)\, d\mu_\varepsilon$$

defined in a particular class of functions with bounded deformation denoted by $\mathrm{LD}^p_{\mu_\varepsilon}(\Omega)$ (introduced in Section 3). More precisely, $\mathrm{LD}^p_{\mu_\varepsilon}(\Omega)$ is the space of functions $u \in \mathrm{L}^p(\Omega;\mathbf{R}^n)$, whose deformation tensor Eu is a measure absolutely continuous with respect to μ_ε with p-summable density $dEu/d\mu_\varepsilon$. Using both classical and fine properties of functions with bounded deformation and the same assumptions as in [4] with a modified definition of 'p-homogenizable measure', we prove in the first part of the paper, a homogenization theorem (Theorem 5.1). Precisely, we show the existence of the Γ-limit of the functionals F_ε with respect to L^p-convergence in the Sobolev space $\mathrm{W}^{1,p}(\Omega;\mathbf{R}^n)$, and with respect to L^1-convergence in $BD(\Omega)$ (the space of functions with bounded deformation in Ω, that is the space of functions $u \in \mathrm{L}^1(\Omega;\mathbf{R}^n)$ whose deformation tensor Eu is a Radon measure with finite total variation in Ω, see [2]). We show that the Γ-limit admits an integral representation

$$F_{\mathrm{hom}}(u,\Omega) = \int_\Omega f_{\mathrm{hom}}(Eu)\, dx$$

in $\mathrm{W}^{1,p}(\Omega;\mathbf{R}^n)$; moreover, if f is convex then

$$F_{\mathrm{hom}}(u,\Omega) = \int_\Omega f_{\mathrm{hom}}(\mathcal{E}u)\, dx + \int_\Omega f^\infty_{\mathrm{hom}}\Big(\frac{dE^s u}{d|E^s u|}\Big)\, d|E^s u|$$

in $BD(\Omega)$, where $\mathcal{E}u$ is the density of the absolutely continuous part and $E^s u$ is the singular part of Eu with respect to the Lebesgue measure; f_{hom} is described by an asymptotic formula and f^∞_{hom} denotes the recession function of f_{hom} (see (5)).

In the second part of this paper we show that when the scaling argument leading to the functionals F_ε does not apply, non local effects can arise. More precisely, we consider functionals of the type

$$F^\gamma_\varepsilon(u,\Omega) = \varepsilon^\gamma \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dEu}{d\mu_\varepsilon}\Big)\, d\mu_\varepsilon\,,$$

which in the previous approach tend to the null functional when $\gamma > 0$, and we construct an explicit example showing that, with a suitable choice of γ, μ_ε and of the convergence with respect to which the Γ-limit is computed, we have a limit functional of a non local nature.

2 Notation and preliminaries

In the sequel $\mathbb{M}^{n\times n}$ stands for the space of $n\times n$ matrices and $\mathbb{M}^{n\times n}_{sym}$ for the space of $n\times n$ symmetric matrices. The letter c will stand for an arbitrary fixed strictly-positive constant independent of the parameters under consideration, whose value may vary from line to line. The symbols $(\cdot,\cdot)$ and $|\cdot|$ stand for the Euclidean scalar product and the Euclidean norm. The Hausdorff k-dimensional measure and the Lebesgue measure in $\mathbf{R}^n$ are denoted by $\mathcal{H}^k$ and $\mathcal{L}^n$ respectively. We write $|E|$ for the Lebesgue measure $\mathcal{L}^n$ of E. We recall that for any two vectors a and b in $\mathbf{R}^n$, the symmetric product $a\odot b$ is the symmetric $n\times n$ matrix defined by $a\odot b=\frac{1}{2}(a\otimes b+b\otimes a)$, where $\otimes$ denotes the tensor product. Ω is a bounded open subset of $\mathbf{R}^n$; we denote by $\mathcal{A}(\Omega)$ the family of all open subsets of Ω.

Given a matrix-valued measure μ on Ω, we adopt the notation $|\mu|$ for its total variation (see Federer [14]). The measure $\mu \llcorner F$ is defined by $(\mu\llcorner F)(B)=\mu(B\cap F)$. We write $\mu << \lambda$ to mean that the measure μ is absolutely continuous with respect to the positive measure λ. We denoted by $\frac{d\mu}{d\lambda}$ the Radon-Nikodym derivative of μ with respect to λ.

$\mathrm{L}^p_\lambda(\Omega;\mathbf{R}^N)$ stands for the usual Lebesgue space of p-summable $\mathbf{R}^N$-valued functions with respect to λ. If $u\in \mathrm{L}^1(\Omega;\mathbf{R}^n)$ then Du denotes its distributional gradient. We say that $u\in \mathrm{L}^1(\Omega;\mathbf{R}^n)$ is a *function of bounded variation*, and we write $u\in BV(\Omega;\mathbf{R}^n)$, if all its distributional first derivatives D_iu_j are Radon measure with finite total variation in Ω; we denote by Du the $\mathbb{M}^{n\times n}$-valued measure whose entries are D_iu_j.

We will use the following notion of Sobolev space with respect to a measure λ, which is a finite Borel positive measure on Ω, introduced by Ambrosio, Buttazzo and Fonseca [1]

$$\mathrm{W}^{1,p}_\lambda(\Omega;\mathbf{R}^n)=\Big\{u\in \mathrm{L}^p(\Omega;\mathbf{R}^n):\ u\in BV(\Omega;\mathbf{R}^n),\ Du<<\lambda,\ \frac{dDu}{d\lambda}\in \mathrm{L}^p_\lambda(\Omega;\mathbb{M}^{n\times n})\Big\}$$

for all $1\le p\le +\infty$.

Let $u\in \mathrm{L}^1(\Omega;\mathbf{R}^n)$, and let Eu be the symmetric part of the distributional gradient of u; *i.e.*,

$$Eu:=E_{ij}u,\qquad E_{ij}u:=\frac{1}{2}(D_iu_j+D_ju_i).$$

The space $\mathrm{LD}(\Omega)$ is defined as the set of all functions $u\in \mathrm{L}^1(\Omega;\mathbf{R}^n)$ such that $E_{ij}u\in \mathrm{L}^1(\Omega)$ for any $i,j=1,...,n$.

We say that $u\in \mathrm{L}^1(\Omega;\mathbf{R}^n)$ is a *function with bounded deformation*, and we write $u\in BD(\Omega)$, if $E_{ij}u$ is a Radon measure with finite total variation in Ω for any $i,j=1,...,n$. For every $u\in BD(\Omega)$ we consider the Radon-Nikodym decomposition of Eu, with respect to the Lebesgue measure $\mathcal{L}^n$, into a singular part E^su and an absolutely continuous part $E^au=\mathcal{E}u\,\mathcal{L}^n$, with density $\mathcal{E}u=\frac{dEu}{d\mathcal{L}^n}$. We say that $x\in\Omega$ belongs to J_u, the jump set of u, if and only if there exist a unit normal $\nu\in S^{n-1}$ and two vectors a and b in $\mathbf{R}^n$ such that

$$\lim_{\rho\to 0^+}\frac{1}{\rho^n}\int_{B^+_\rho(x,\nu)}|u(y)-a|\,dy=0$$

$$\lim_{\rho\to 0^+}\frac{1}{\rho^n}\int_{B^-_\rho(x,\nu)}|u(y)-b|\,dy=0$$

where $B^\pm_\rho(x,\nu)=\{y\in B_\rho(x)\ :\ (y-x,\pm\nu)>0\}$ and $B_\rho(x)$ is the open ball of center x and radius ρ. The triplet (a,b,ν) is uniquely determined up to a change of sign of ν and a permutation of (a,b). For every $x \in J_u$ we define $u^+(x) = a$, $u^-(x) = b$ and $\nu_u(x) = \nu$. The singular part $E^s u$ can be written as the sum of $E^s u \llcorner J_u$ and of $E^s u \llcorner (\Omega \setminus J_u)$; the first part, called the *jump part*, can be represented by

$$E^s u \llcorner J_u = (u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u \tag{1}$$

while the second part, called the *Cantor part*, vanishes on any Borel set which is σ-finite with respect to $\mathcal{H}^{n-1}$ (see [2] Remark 4.2 and Proposition 4.4). We call *intermediate topology* on $BD(\Omega)$ that defined by the distance

$$\|u - v\|_{\mathrm{L}^1(\Omega;\mathbf{R}^n)} + ||Eu|(\Omega) - |Ev|(\Omega)|. \tag{2}$$

If $u \in \mathrm{L}^1(\Omega;\mathbf{R}^n)$ is such that $E_{ij}u \in \mathrm{L}^p(\Omega)$ for any $i,j = 1,...,n$ and Ω has a locally Lipschitz boundary, then we have Korn's inequality for all $1 < p < +\infty$

$$\sum_{i,j=1}^{n} \int_\Omega |D_i u_j(x)|^p \, dx \le c \int_\Omega \Big(|u(x)|^p + |Eu(x)|^p \Big) \, dx\,; \tag{3}$$

hence, this space is none other than $\mathrm{W}^{1,p}(\Omega;\mathbf{R}^n)$ (see Chapter 1, Section 1 in [20]). For a general exposition of the theory of functions of bounded deformation we refer to [18], [19], [16], [17], [5], [21], [20], [2].

If $u \in \mathrm{L}^1(\Omega;\mathbf{R}^n)$, we denote by $\tilde{u}$ the *precise representative* of u, whose components are defined by

$$\tilde{u}_i(x) = \limsup_{\rho \to 0^+} \fint_{B_\rho(x)} u_i(y) \, dy\,. \tag{4}$$

Let $f : \mathbf{R}^k \to [0,+\infty]$ be a convex function. We define the recession function f^∞ of f as

$$f^\infty(\xi) = \lim_{t \to \infty} \frac{f(t\xi)}{t} \qquad \text{for every } \xi \in \mathbf{R}^k\,. \tag{5}$$

It is well-known (see, for instance, [11]) that this limit exists, and defines a convex, subadditive and positively homogeneous of degree one function.

We recall the definition of De Giorgi's Γ-*convergence* in L^p spaces. Given a family of functionals $F_j : \mathrm{L}^p(\Omega;\mathbf{R}^n) \to [0,+\infty]$, $j \in \mathbf{N}$, for $u \in \mathrm{L}^p(\Omega;\mathbf{R}^n)$, we define

$$\Gamma(\mathrm{L}^p)\text{-}\liminf_{j \to +\infty} F_j(u) = \inf\Big\{ \liminf_{j \to +\infty} F_j(u_j) \ : \ u_j \stackrel{\mathrm{L}^p}{\to} u \Big\},$$

and

$$\Gamma(\mathrm{L}^p)\text{-}\limsup_{j \to +\infty} F_j(u) = \inf\Big\{ \limsup_{j \to +\infty} F_j(u_j) \ : \ u_j \stackrel{\mathrm{L}^p}{\to} u \Big\};$$

if these two quantities coincide then their common value is called the Γ-*limit* of the sequence (F_j) at u, and is denoted by $\Gamma(\mathrm{L}^p)\text{-}\lim_{j \to +\infty} F_j(u)$. It is easy to check that $l = \Gamma(\mathrm{L}^p)\text{-}\lim_{j \to +\infty} F_j(u)$ if and only if

(a) for every sequence (u_j) converging to u we have

$$l \le \liminf_{j \to +\infty} F_j(u_j);$$

(b) there exists a sequence (u_j) converging to u such that

$$l \geq \limsup_{j\to+\infty} F_j(u_j).$$

We say that (F_ε) $\Gamma(\mathrm{L}^p)$-converges to l at u as $\varepsilon \to 0$ if for every sequence of positive numbers (ε_j) converging to 0^+ there exists a subsequence (ε_{j_k}) for which we have

$$l = \Gamma(\mathrm{L}^p)\text{-}\lim_{k\to+\infty} F_{\varepsilon_{j_k}}(u).$$

We recall that the Γ-upper and lower limits defined above are L^p-lower semicontinuous functions. For a comprehensive study of Γ-convergence we refer to [12] and [8], while a detailed analysis of some of its applications to homogenization theory can be found in [10].

3 The space $\mathrm{LD}^p_\lambda(\Omega)$

In this section we define the analog of $\mathrm{W}^{1,p}_\lambda(\Omega;\mathbf{R}^n)$ when the gradient is replaced by the linearized strain tensor.

Definition 3.1 *Let λ be a finite Borel positive measure on the open set $\Omega \subset \mathbf{R}^n$, and let $1 \leq p \leq +\infty$. We define the space*

$$\mathrm{LD}^p_\lambda(\Omega) = \left\{ u \in \mathrm{L}^p(\Omega;\mathbf{R}^n) : \ u \in BD(\Omega), \ Eu << \lambda, \ \frac{dEu}{d\lambda} \in \mathrm{L}^p_\lambda(\Omega;\mathbb{M}^{n\times n}_{sym}) \right\}. \tag{6}$$

We will use the notation $\mathrm{LD}_\lambda(\Omega)$ instead of $\mathrm{LD}^1_\lambda(\Omega)$.

Proposition 3.2 (i) *The spaces $\mathrm{LD}^p_\lambda(\Omega)$ and $\mathrm{LD}^p_{\lambda'}(\Omega)$ coincide whenever $|\lambda-\lambda'|(\Omega\setminus B) = 0$ for some $\mathcal{H}^{n-1}$-negligible Borel subset B of Ω.*
(ii) *The measure λ in Definition 3.1 can always be assumed concentrated on a Borel set where its $(n-1)$-dimensional upper density is finite.*

PROOF. Point (i) easily follows from the fact that BD functions do not charge $\mathcal{H}^{n-1}$-negligible sets (see Remark 3.3 in [2]). Point (ii) follows from Remark 2.3 in [1]. □

In the following proposition we prove a Leibniz-type formula for the densities with respect to a measure λ. This formula will be used in the proof of the fundamental estimate, Proposition 5.3.

Proposition 3.3 *If $u \in \mathrm{LD}^p_\lambda(\Omega)$, $v \in \mathrm{W}^{1,\infty}_\lambda(\Omega)$ and $\tilde{u} \odot \frac{dDv}{d\lambda} \in \mathrm{L}^1_\lambda(\Omega;\mathbb{M}^{n\times n})$ then $uv \in \mathrm{LD}^p_\lambda(\Omega)$, and*

$$\frac{dE(uv)}{d\lambda} = \tilde{v}\frac{dEu}{d\lambda} + \tilde{u} \odot \frac{dDv}{d\lambda}. \tag{7}$$

PROOF. By definition, functions in LD^p_λ have bounded deformation. Using the characterization of the spaces $BV(\Omega)$ and $BD(\Omega)$ by means of one-dimensional sections (see Proposition 3.2 in [2]) we have

$$u^\xi_y \in BV(\Omega^\xi_y), \qquad v_{y,\xi} \in BV(\Omega^\xi_y) \qquad \mathcal{H}^{n-1}\text{-}a.e. \quad y \in \Omega^\xi$$

where

$$u^\xi_y(t) = u^\xi(y+t\xi) = (u(y+t\xi),\xi), \qquad v_{y,\xi}(t) = v(y+t\xi) \qquad \forall t \in \Omega^\xi_y.$$

Hence by the chain rule formula for BV functions (see [7] Section 1.8, [3] Theorem 3.93 and Example 3.94) we have

$$(uv)^\xi_y = u^\xi_y\, v_{y,\xi} \in BV(\Omega^\xi_y)$$

and

$$D(u^\xi_y v_{y,\xi}) = \tilde{v}_{y,\xi} Du^\xi_y + \tilde{u}^\xi_y Dv_{y,\xi} \qquad \mathcal{H}^{n-1} - a.e. \quad y \in \Omega^\xi.$$

By Proposition 3.2 in [2] and by the structure theorem for BV functions (see [7] Section 1.8), we can prove that $uv \in BD(\Omega)$ and

$$(Euv\xi,\xi) = (\tilde{v}Eu\xi,\xi) + (\tilde{u} \odot Dv\xi,\xi) \qquad \forall \xi \in \mathbf{R}^n.$$

By choosing $\xi = \xi_i + \xi_j$, where $\xi_1,\dots,\xi_n$ is a basis of $\mathbf{R}^n$, we get

$$E(uv) = \tilde{v}Eu + \tilde{u} \odot Dv. \tag{8}$$

Since the measures in the left hand-side of (8) are absolutely continuous with respect to λ with densities in $\mathrm{L}^p_\lambda(\Omega;\mathbb{M}^{n\times n}_{sym})$, we finally get $uv \in \mathrm{LD}^p_\lambda(\Omega)$ and (7) is proved. $\square$

Remark 3.4 Note that in (7) it is necessary to consider the precise representatives of u and v, since the measure λ may take into account also sets of zero Lebesgue measure.

4 Choice of the measure and some examples

Let μ be a non-zero positive Radon measure on $\mathbf{R}^n$ which is 1-periodic; i.e.,

$$\mu(B+e_i) = \mu(B)$$

for all Borel subsets B of $\mathbf{R}^n$ and for all $i = 1,\dots,n$. We will assume the normalization

$$\mu([0,1)^n) = 1\,. \tag{9}$$

For all $\varepsilon > 0$ we define the ε-periodic positive Radon measure μ_ε by

$$\mu_\varepsilon(B) = \varepsilon^n\,\mu\Big(\frac{1}{\varepsilon}B\Big) \tag{10}$$

for all Borel sets B. Note that by (9) the family (μ_ε) converges locally weakly* in the sense of measures to the Lebesgue measure as $\varepsilon \to 0$.

In the sequel $f : \mathbf{R}^n \times \mathbb{M}^{n\times n} \to [0,+\infty)$ will be a fixed Borel function 1-periodic in the first variable and satisfying the growth condition of order $p \geq 1$: there exist $0 < \alpha \leq \beta$ such that

$$\alpha|A|^p \leq f(x,A) \leq \beta(1+|A|^p) \tag{11}$$

for all $x \in \mathbf{R}^n$ and $A \in \mathbb{M}^{n\times n}$.

For every bounded open set Ω, we define the functionals at scale $\varepsilon > 0$ as

$$F_\varepsilon(u,\Omega) = \begin{cases} \displaystyle\int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dEu}{d\mu_\varepsilon}\Big) d\mu_\varepsilon & \text{if } u \in \mathrm{LD}^p_{\mu_\varepsilon}(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{12}$$

Now we consider some additional assumptions on the measure μ, in order to prove the existence and the integral representation of the Γ-limit of the functionals F_ε as $\varepsilon \to 0$. In the sequel we will point out that these conditions are necessary and sufficient.

We assume:

(i) (*existence of cut-off functions*) there exist $K > 0$ and $\delta > 0$ such that for all $\varepsilon > 0$, for all pairs U, V of open subsets of $\mathbf{R}^n$ with $U \subset\subset V$, and $\mathrm{dist}\,(U, \partial V) \geq \delta\varepsilon$, and for all $u \in \mathrm{LD}^p_{\mu_\varepsilon}(V)$ there exists $\phi \in \mathrm{W}^{1,\infty}_{\mu_\varepsilon}(V)$ with $0 \leq \phi \leq 1$, $\phi = 1$ on U, $\phi = 0$ in a neighbourhood of ∂V, such that

$$\int_V \Big|\frac{dD\phi}{d\mu_\varepsilon} \odot \tilde{u}\Big|^p d\mu_\varepsilon \leq \frac{K}{(\mathrm{dist}\,(U,\partial V))^p} \int_{V\setminus U} |u|^p \, dx\,. \tag{13}$$

Such a ϕ will be called a *cut-off function between* U and V;

(ii) (*existence of periodic test-functions*) for all $i, j = 1, \ldots, n$, there exists $z_{ij} \in \mathrm{LD}^p_{\mu,\mathrm{loc}}(\mathbf{R}^n)$ such that $x \mapsto z_{ij}(x) - x_j e_i$ is 1-periodic.

Remark 4.1 Note that if μ is p-homogenizable in the sense of Definition 3.2 in [4]; *i.e.,* if there exists $z_i \in \mathrm{W}^{1,p}_{\mu,\mathrm{loc}}(\mathbf{R}^n)$ such that $x \mapsto z_i(x) - x_i$ is 1-periodic, then the functions $z_{ij} = z_j e_i$ trivially satisfy the condition (ii) above but the converse is not true.

Remark 4.2 Note that the Lebesgue measure trivially satisfies properties (i), (ii). Note that property (i) depends on μ and p.

We consider in our context the measure μ of Examples 3.1(a) and (b) in [4].

Example 4.3 (*Perfectly-rigid bodies connected by springs.*)
We consider

$$E = \{y \in \mathbf{R}^n : \exists i \in \{1,\ldots,n\} \text{ such that } y_i \in \mathbf{Z}\},$$

that is, the union of all the boundaries of cubes $Q_i = i + (0,1)^n$ with $i \in \mathbf{Z}^n$. E is an $(n-1)$-dimensional set in $\mathbf{R}^n$. We set

$$\mu(B) = \frac{1}{n}\mathcal{H}^{n-1}(B \cap E)$$

for all Borel sets B. For every $\varepsilon > 0$ we have

$$\mu_\varepsilon(B) = \frac{1}{n}\varepsilon\mathcal{H}^{n-1}(B \cap \varepsilon E)\,.$$

If $u \in \mathrm{LD}^p_{\mu_\varepsilon}(\Omega)$ then $Eu = 0$ on every connected component of each $\varepsilon Q_i \cap \Omega$, so in this case $\mathrm{LD}^p_{\mu_\varepsilon}(\Omega)$ consists of functions which are rigid displacements on these sets; *i.e.*, $u_i = R_i x + c_i$ on each $\varepsilon Q_i \cap \Omega$ with R_i a $n \times n$ skew symmetric matrix, and $c_i \in \mathbf{R}^n$. Hence by (1), we have

$$\frac{dEu}{d\mu_\varepsilon} = \frac{n}{\varepsilon}\frac{dEu}{d\mathcal{H}^{n-1}} = \frac{n}{\varepsilon}(u_i - u_j) \odot (i - j) \text{ on } \partial(\varepsilon Q_i) \cap \partial(\varepsilon Q_j) \cap \Omega\,.$$

In this case the functionals F_ε take the form

$$\varepsilon \int_{\Omega \cap \varepsilon E} g\Big(\frac{x}{\varepsilon}, \frac{1}{\varepsilon}\frac{dEu}{d\mathcal{H}^{n-1}}\Big) d\mathcal{H}^{n-1}.$$

Note that if Ω is bounded then $\mathrm{LD}^p_{\mu_\varepsilon}(\Omega) = \mathrm{LD}^\infty_{\mu_\varepsilon}(\Omega)$ for all p if the number of connected components of each $\Omega \cap \varepsilon Q_i$ is finite.

Comparing with Example 3.1(a) in [4], we get that $\mathrm{W}^{1,p}_{\mu_\varepsilon}(\Omega; \mathbf{R}^n) \subset \mathrm{LD}^p_{\mu_\varepsilon}(\Omega)$.

The measure μ satisfies the conditions (i) and (ii) for all $p \geq 1$. In fact, to prove (i) we consider the same cut-off function in Example 3.4(a) of [4]

$$\phi(x) = 1 - \Big(\frac{1}{C}\Big[\frac{1}{\varepsilon}\inf\{|x - y|_\infty : y \in U_\varepsilon\}\Big] \wedge 1\Big),$$

where fixed $\varepsilon > 0$, $U_\varepsilon = \bigcup\{\varepsilon Q_i :\ \varepsilon Q_i \cap U \neq \emptyset\}$, $|x - y|_\infty = \max_{1 \leq i \leq n} |x_i - y_i|$, and

$$C = \Big[\frac{1}{\varepsilon}\inf\Big\{|x - y|_\infty : x \in U_\varepsilon,\ y \in \partial V\Big\}\Big] - 2$$

(we denote $[t]$ the integer part of t). Note that $|dD\phi/d\mu_\varepsilon| \leq n/(C\varepsilon) \leq c/\mathrm{dist}\,(U, \partial V)$ for some constant c independent of U and V.

Interpreting $u^\pm$ as traces of Sobolev functions defined on each cube Q_i, we have

$$\Big(\int_{\partial Q_i} |u^\pm|^p d\mathcal{H}^{n-1}\Big)^{1/p} \leq c\|u\|_{\mathrm{W}^{1,p}(Q_i)},$$

hence by a scaling argument and by Korn's inequality (3)

$$\begin{aligned}\Big(\varepsilon\int_{\partial\varepsilon Q_i} |u^\pm|^p d\mathcal{H}^{n-1}\Big)^{1/p} &\leq c\Big(\int_{\varepsilon Q_i} |u|^p dx\Big)^{1/p} + \Big(\frac{1}{\varepsilon}\int_{\varepsilon Q_i} |Eu|^p dx\Big)^{1/p} \\ &= c\Big(\int_{\varepsilon Q_i} |u|^p dx\Big)^{1/p}\end{aligned}$$

where c depends only on the cube. If $p = 1$ we can apply the trace inequality in $\mathrm{LD(Q_i)}$

$$\int_{\partial Q_i} |u^\pm| d\mathcal{H}^{n-1} \leq c\int_{Q_i} |u| dx + |Eu|(Q_i)\,,$$

so we get

$$\varepsilon\int_{\partial\varepsilon Q_i} |u^\pm| d\mathcal{H}^{n-1} \leq c\int_{\varepsilon Q_i} |u| dx.$$

Hence for all $p \geq 1$

$$\varepsilon\int_{\partial\varepsilon Q_i} |u^\pm|^p\, d\mathcal{H}^{n-1} \leq c\int_{\varepsilon Q_i} |u|^p dx.$$

For two cubes

$$\varepsilon \int_{\partial\varepsilon Q_i \cap \partial\varepsilon Q_j} |\tilde{u}|^p \, d\mathcal{H}^{n-1} \leq \varepsilon \int_{\partial\varepsilon Q_i \cap \partial\varepsilon Q_j} (|u_i|^p + |u_j|^p) \, d\mathcal{H}^{n-1} \leq c \int_{\varepsilon Q_i \cup \varepsilon Q_j} |u|^p \, dx$$

so that

$$\begin{aligned} \int_V \Big| \frac{dD\phi}{d\mu_\varepsilon} \odot \tilde{u} \Big|^p d\mu_\varepsilon &\leq \frac{c^p \varepsilon}{\operatorname{dist}(U, \partial V)^p} \int_{(V \setminus U) \cap \varepsilon E \cap \operatorname{spt} D\phi} |\tilde{u}|^p \, d\mathcal{H}^{n-1} \\ &\leq 2n \frac{c^p}{\operatorname{dist}(U, \partial V)^p} \int_{V \setminus U} |u|^p \, dx \,. \end{aligned}$$

The proof of (i) is then complete. To verify (ii) we apply Remark 4.1 to Example 3.4(a) in [4] and take simply $z_{ij}(x) = [x_j] e_i$.

Example 4.4 (*Elastic media connected by springs*).
Let E be as in the previous example and let

$$\begin{aligned} \mu(B) &= \frac{1}{n+1} \Big(|B| + \mathcal{H}^{n-1}(E \cap B) \Big) \\ \mu_\varepsilon(B) &= \frac{1}{n+1} \Big(|B| + \varepsilon \mathcal{H}^{n-1}((\varepsilon E) \cap B) \Big) \,. \end{aligned}$$

In this case the functions in $\mathrm{LD}^p_{\mu_\varepsilon}(\Omega)$ are functions whose restriction to each $\varepsilon Q_i \cap \Omega$ belongs to $\mathrm{W}^{1,p}(\varepsilon Q_i \cap \Omega; \mathbf{R}^n)$ when $p > 1$ by the Korn's inequality (3) (we suppose that $\varepsilon Q_i \cap \Omega$ has a locally Lipschitz boundary) and to $\mathrm{LD}(\varepsilon \mathrm{Q_i} \cap \Omega)$ when $p = 1$, while the difference of the traces on both sides of $\partial(\varepsilon Q_i) \cap \partial(\varepsilon Q_j) \cap \Omega$ is p-summable for every $i, j \in \mathbf{Z}^n$. Hence if we compare our case with Example 3.1(b) in [4], we can conclude that $\mathrm{W}^{1,p}_{\mu_\varepsilon}(\Omega; \mathbf{R}^n) = \mathrm{LD}^p_{\mu_\varepsilon}(\Omega)$ if $p > 1$ and $\mathrm{W}^{1,1}_{\mu_\varepsilon}(\Omega; \mathbf{R}^n) \subset \mathrm{LD}_{\mu_\varepsilon}(\Omega)$ if $p = 1$. The functionals F_ε take the form

$$\frac{1}{n+1} \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{dEu}{dx}\Big) \, dx + \varepsilon \int_{\Omega \cap \varepsilon E} g\Big(\frac{x}{\varepsilon}, \frac{1}{\varepsilon} \frac{dEu}{d\mathcal{H}^{n-1}}\Big) d\mathcal{H}^{n-1}.$$

The measure μ satisfies conditions (i) and (ii) for all $p \geq 1$ by Example 3.4(b) in [4].

5 The homogenization theorem

The homogenization theorem for the functionals in (12) takes the following form.

Theorem 5.1 *Let μ be a measure which satisfies conditions* (i) *and* (ii) *in Section* 4, *and for every bounded open subset Ω of $\mathbf{R}^n$ let $F_\varepsilon(\cdot, \Omega)$ be defined on $\mathrm{L}^p(\Omega; \mathbf{R}^n)$ by* (12). *Then the Γ-limit*

$$F_{\mathrm{hom}}(u, \Omega) = \Gamma(\mathrm{L}^p)\text{-}\lim_{\varepsilon \to 0} F_\varepsilon(u, \Omega) \tag{14}$$

exists for all bounded open subsets Ω with Lipschitz boundary and for all $u \in \mathrm{L}^p(\Omega; \mathbf{R}^n)$; it can be represented on $\mathrm{W}^{1,p}(\Omega; \mathbf{R}^n)$ for $p \geq 1$ as

$$F_{\mathrm{hom}}(u, \Omega) = \int_\Omega f_{\mathrm{hom}}(Eu) \, dx \,, \tag{15}$$

where the homogenized integrand satisfies the asymptotic formula

$$f_{\rm hom}(A) = \lim_{k\to+\infty} \inf\Big\{\frac{1}{k^n}\int_{[0,k)^n} f\Big(x,\frac{dEu}{d\mu}\Big)d\mu : \\ u\in {\rm LD}^p_{\mu,{\rm loc}}(\mathbf{R}^n),\ u-Ax \quad k\text{-periodic}\Big\} \tag{16}$$

for all $A\in \mathbb{M}^{n\times n}_{sym}$.

Moreover, $F_{\rm hom}(u,\Omega) = +\infty$ *if* $p>1$ *and* $u\in {\rm L}^p(\Omega;\mathbf{R}^n)\setminus {\rm W}^{1,p}(\Omega;\mathbf{R}^n)$*, or if* $u\in {\rm L}^1(\Omega;\mathbf{R}^n)\setminus BD(\Omega)$ *when* $p=1$.

Furthermore, if f *is convex then the* Γ*-limit can be represented as*

$$F_{\rm hom}(u,\Omega) = \int_\Omega f_{\rm hom}(\mathcal{E}u)\,dx + \int_\Omega f^\infty_{\rm hom}\Big(\frac{dEu^s}{d|Eu^s|}\Big)d|Eu^s|$$

for all $u\in BD(\Omega)$ *when* $p=1$.

Remark 5.2 Note that we cannot replace the sets $[0,k)^n$ by the sets $(0,k)^n$ if $\mu([0,k)^n\setminus (0,k)^n)\neq 0$, see Remark 3.6 in [4].

Same examples and considerations of Remarks 3.7 and 3.8 in [4], applied to our case, show that condition (ii) for the measure μ and the assumption that Ω has a Lipschitz boundary are necessary to get a homogenization theorem. In fact, if condition (ii) fails then $f_{\rm hom}(A)=+\infty$ if $A\neq 0$; while if Ω does not have Lipschitz boundary then the equality (15) may not hold.

The following proposition is a usual tool to prove the existence of the Γ-limit and its integral representation (see [12] Chapter 18, [10] Chapter 11).

Proposition 5.3 (Fundamental Estimate) *For every* $\sigma>0$ *there exists* ε_σ *and* $M>0$ *such that for all* U,U',V *open subsets of* Ω *with* $U'\subset U$ *and* ${\rm dist}\,(U',V\setminus U)>0$*, for all* $\varepsilon<\varepsilon_\sigma{\rm dist}\,(U',V\setminus U)$ *and for all* $u\in {\rm LD}^p_{\mu_\varepsilon}(\Omega)$*,* $v\in {\rm LD}^p_{\mu_\varepsilon}(\Omega)$ *there exists a cut-off function between* U' *and* U*,* $\phi\in {\rm W}^{1,\infty}_{\mu_\varepsilon}(U\cup V)$*, such that*

$$F_\varepsilon(\phi u+(1-\phi)v,U'\cup V) \le (1+\sigma)(F_\varepsilon(u,U)+F_\varepsilon(v,V)) \\ +\frac{M}{({\rm dist}\,(U',V\setminus U))^p}\int_{(U\cap V)\setminus U'}|u-v|^p dx+\sigma\mu_\varepsilon((U\cap V)\setminus U'). \tag{17}$$

Proof. By taking (7) and condition (i) into account, the proof follows exactly that of Proposition 4.1 [4].

Proposition 5.4 *For every* $A\in\mathbb{M}^{n\times n}_{sym}$ *there exists* $z_A\in {\rm LD}^p_{\mu,{\rm loc}}(\mathbf{R}^n)$ *such that* z_A-Ax *is* 1*-periodic and satisfies*

$$\int_{[0,1)^n}\Big|\frac{dEz_A}{d\mu}\Big|^p d\mu\le c|A|^p. \tag{18}$$

Proof. Define $z_A=\sum_{i,j=1}^n A_{ij}z_{ij}$, where z_{ij} are as in condition (ii). Inequality (18) is then trivial. $\square$

We fix (ε_j) which goes to zero. We define

$$F'(u,U) = \Gamma(\mathrm{L}^p)\text{-}\liminf_{j\to+\infty} F_{\varepsilon_j}(u,U)$$

$$F''(u,U) = \Gamma(\mathrm{L}^p)\text{-}\limsup_{j\to+\infty} F_{\varepsilon_j}(u,U)$$

for all $u \in \mathrm{L}^p(\Omega;\mathbf{R}^n)$ and for all open subsets U of Ω.

Proposition 5.5 (Growth Condition) *We have for all open subsets U of Ω with $|\partial U| = 0$*

$$F''(u,U) \leq c\int_U (1+|Eu|^p)dx$$

for all $u \in \mathrm{W}^{1,p}(\Omega;\mathbf{R}^n)$ if $p>1$ and

$$F''(u,U) \leq c(|U| + |Eu|(U))$$

for all $u \in BD(\Omega)$ if $p=1$.

PROOF. This Growth Conditions can be obtained modifying the proof of Proposition 4.3 in [4]. In particular in Step 2 therein now we have to consider the affine functions $u_i(x) = A_i x + c_i$ for some $A_i \in \mathbb{M}^{n\times n}_{sym}$ and $c_i \in \mathbf{R}^n$, in Step 3 we just have to note that piecewise affine functions are dense in BD endowed with the intermediate topology (2) (see [20] Theorem 3.2 Chapter 2 Section 3). $\square$

Proposition 5.6 *There exists a subsequence of (ε_j) (not relabeled) such that for all open subsets U of Ω with $|\partial U| = 0$ there exists the Γ-limit*

$$\Gamma\text{-}\lim_{j\to+\infty} F_{\varepsilon_j}(u,U) = F(u,U)\,,$$

for all $u \in \mathrm{W}^{1,p}(\Omega;\mathbf{R}^n)$ if $p>1$ and for all $u \in BD(\Omega)$ if $p=1$. There exists a function $\varphi : \mathbb{M}^{n\times n} \to \mathbf{R}$ such that

$$F(u,U) = \int_U \varphi(Eu)dx$$

for all $u \in \mathrm{W}^{1,p}(\Omega;\mathbf{R}^n)$ if $p \geq 1$; moreover if f is convex

$$F(u,U) = \int_U \varphi\Big(\mathcal{E}u\Big)dx + \int_U \varphi^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big)d|E^s u|$$

for all $u \in BD(\Omega)$ if $p=1$.

PROOF. To prove the existence of the Γ-limit on $\mathrm{W}^{1,p}(\Omega;\mathbf{R}^n)$ for $p>1$ and $BD(\Omega)$ for $p=1$, and the integral representation of the Γ-limit

$$F(u,U) = \int_U \varphi(Du)dx$$

on $W^{1,p}(\Omega;\mathbf{R}^n)$ when $p\geq 1$, we repeat the proof of Proposition 4.4 [4] using Propositions 5.3 and 5.5. Moreover, we can prove that $\varphi(Du)=\varphi(Eu)$. In fact, let $w_j\to Ax$ be such that

$$F(Ax,\Omega)=\lim_{j\to+\infty}F_{\varepsilon_j}(w_j,\Omega)$$

and let $Rx+c$ be a rigid displacement, then

$$\begin{aligned} F(Ax+Rx+c,\Omega) &\leq \liminf_{j\to+\infty}F_{\varepsilon_j}(w_j+Rx+c,\Omega)\\ &= \lim_{j\to+\infty}F_{\varepsilon_j}(w_j,\Omega)=F(Ax,\Omega)\end{aligned}$$

so that $\varphi(A+R)\leq\varphi(A)$. The reverse inequality follows similarly, therefore for all R $(n\times n)$ skew-symmetric matrix

$$\varphi(A+R)=\varphi(A)$$

which implies $\varphi(B)=\varphi(\frac{B+B^T}{2})$ for any $B\in\mathbb{M}^{n\times n}$.

Let us prove the integral representation of the Γ-limit on $BD(\Omega)$ whenever f is convex. We consider the functional defined in $L^1_{\mathrm{loc}}(\Omega;\mathbf{R}^n)$

$$G(u)=\begin{cases}\displaystyle\int_\Omega\varphi(Eu)\,dx & \text{if } u\in C^1(\Omega;\mathbf{R}^n)\\ +\infty & \text{otherwise,}\end{cases}$$

and we introduce

$$\overline{G}(u,U)=\inf\Big\{\liminf_{h\to+\infty}G(u_h,U):u_h\in C^1(\Omega;\mathbf{R}^n)\quad u_h\to u\quad\text{in}\quad L^1_{\mathrm{loc}}(\Omega;\mathbf{R}^n)\Big\}$$

the relaxed functional of G. It is well known that φ is convex and it is easy to check that $\varphi(A)\geq c|A|$ for every $A\in\mathbb{M}^{n\times n}_{sym}$, hence by the lower semicontinuity and relaxation theorems for functionals of measures (see for instance [15], [11]), we obtain

$$\overline{G}(u,U)=\int_U\varphi(\mathcal{E}u)\,dx+\int_U\varphi^\infty\Big(\frac{dE^su}{d|E^su|}\Big)d|E^su|$$

for every $u\in BD(\Omega)$ (see [20] Section 5). Since $F(\cdot,U)\leq G(\cdot,U)$ in $W^{1,1}(\Omega;\mathbf{R}^n)$, by the lower semicontinuity of the Γ-limit we obtain

$$F(u,U)\leq\int_U\varphi(\mathcal{E}u)\,dx+\int_U\varphi^\infty\Big(\frac{dE^su}{d|E^su|}\Big)d|E^su|$$

for all $u\in BD(\Omega)$. The reverse inequality is obtained by a convolution argument. In fact we consider $U_k=\{x\in U:d(x,\partial U)>\frac{1}{k}\}$, ρ_k with $\operatorname{spt}\rho_k\subset B(0,\frac{1}{k})$ and $u_k=u*\rho_k$. For $y\in B(0,\frac{1}{k})$ and k large enough we have that $U_k\subset y+U$.
Since $F(\cdot,U)$ is convex for all $U\in\mathcal{A}(\Omega)$ and $F(u^y,U_k)\leq F(u,U)$ with $u^y(x)=u(x-y)$, by Jensen's inequality

$$F(u*\rho_k,U_k)\leq F(u,U).$$

On the other hand, we also have

$$\lim_{k\to+\infty}F(u_k,U_k)=\overline{G}(u,U)$$

hence we can conclude that

$$F(u,U)=\int_U \varphi(\mathcal{E}u)\,dx+\int_U \varphi^\infty\Big(\frac{dE^s u}{d|E^s u|}\Big)d|E^s u|$$

as desired. □

Proposition 5.7 (Homogenization Formula) *For all $A\in \mathbb{M}^{n\times n}_{sym}$ there exists the limit in* (16) *and we have $\varphi(A)=f_{\rm hom}(A)$.*

Proof. It can be obtain repeating the proof of the Proposition 4.5 of [4] but defining

$$g_k(A)=\inf\Big\{\frac{1}{k^n}\int_{(0,k)^n} f(x,\frac{dEu}{d\mu})d\mu : u\in {\rm LD}^p_{\mu,{\rm loc}}(\mathbf{R}^n),\ \ u-Ax\ \ k\text{-periodic}\Big\}$$

for all $A\in \mathbb{M}^{n\times n}_{sym}$ and $k\in\mathbf{N}$. □

Proof of Theorem 5.1. It remains to check the coercivity of the Γ-limit. By the growth condition on f and a comparison argument, it is enough to prove this for $f(A)=|A|^p$. We know that the Γ-limit $F_{\rm hom}$ exists for all $u\in {\rm L}^p(\Omega;\mathbf{R}^n)$ and for all sets R in the countable family $\mathcal{R}$ of all finite unions of open rectangles of Ω with rational vertices, in this case $F_{\rm hom}$ is also convex. For all U', $U\in\mathcal{A}(\Omega)$ such that $U'\subset\subset U$ there exists $R\in\mathcal{R}$ such that $U'\subset\subset R\subset\subset U$. Reasoning as in the previous proof, for $y\in B(0,\frac{1}{k})$ and k large enough we have that $R\subset y+U$ hence

$$F_{\rm hom}(u_k,R)\le F'(u,U)$$

and

$$\liminf_{k\to+\infty} F_{\rm hom}(u_k,U')\le F'(u,U) \tag{19}$$

with $u_k=u*\rho_k$ (see [12] Chapter 23).

It will be enough then to prove that $f_{\rm hom}(A)\ge c|A|^p$. In fact for any $u\in {\rm L}^p(\Omega;\mathbf{R}^n)\setminus {\rm W}^{1,p}(\Omega;\mathbf{R}^n)$ when $p>1$ by (19)

$$F'(u,U)\ge c\liminf_{k\to+\infty}\int_{U'}|Du_k|^p\,dx$$

by the arbitrarity of U', we get $F_{\rm hom}(u,U)=+\infty$. Similarly, if $p=1$ for all $u\in {\rm L}^1(\Omega;\mathbf{R}^n)\setminus BD(\Omega)$ we have $|Eu|(\Omega)=+\infty$, let $\Omega'\subset\subset\Omega$ we get by (19) that

$$F'(u,\Omega)\ge c\liminf_{k\to+\infty}|Eu_k|(\Omega')$$

by arbitrarity of Ω' we obtain $F_{\rm hom}(u,\Omega)=+\infty$.

Since $f_{\rm hom}$ is positively homogeneous of degree p, to prove that $f_{\rm hom}(A)\ge c|A|^p$, it is sufficient to check that $f_{\rm hom}(A)\ne 0$ if $A\ne 0$. To this aim, let $u_\varepsilon\to Ax$ be such that $F_\varepsilon(u_\varepsilon,(0,1)^n)\to f_{\rm hom}(A)$. If $f_{\rm hom}(A)=0$ then by a “Poincaré-type” inequality for BD functions (Proposition 2.3 Chapter 2 of [20]), by Hölder's inequality and a scaling argument we obtain that

$$\begin{aligned} 0=f_{\rm hom}(A) &= \lim_{\varepsilon\to0}\int_{(0,1)^n}\Big|\frac{dEu_\varepsilon}{d\mu_\varepsilon}\Big|^p\,d\mu_\varepsilon\\ &\ge \lim_{\varepsilon\to0} c\Big(\int_{(0,1)^n}|u_\varepsilon-Ru_\varepsilon|\,dx\Big)^p\end{aligned}$$

where the constant c depends only on Ω and Ru_ε is a rigid displacement. Hence $Ru_\varepsilon\to Ax$ in ${\rm L}^1$, and we get a contradiction because A is a symmetric matrix. □

6 Non local effects

Theorem 5.1 shows the $\Gamma(\mathrm{L}^p)$-convergence of the functionals F_ε to F_{hom} in $\mathrm{W}^{1,p}(\Omega;\mathbf{R}^n)$ and that the Γ-limit is local; in fact we have represented F_{hom} as the integration over Ω of a local density of energy of the form $f_{\mathrm{hom}}(Eu)$.

Now, if we consider

$$F_\varepsilon^\gamma(u,\Omega)=\varepsilon^\gamma\int_\Omega f\Big(\frac{dEu}{d\mu_\varepsilon}\Big)\,d\mu_\varepsilon$$

then $\Gamma(\mathrm{L}^p)$-$\lim_{\varepsilon\to 0}F_\varepsilon^\gamma(u,\Omega)=0$ on $\mathrm{W}^{1,p}(\Omega;\mathbf{R}^n)$, when $\gamma>0$. In this case, however, no coerciveness result may hold for sequences (u_ε) with $\sup_{\varepsilon>0}F_\varepsilon^\gamma(u_\varepsilon,\Omega)<+\infty$ in any norm.

We will show with an example that a more complex notion of convergence may have to be introduced and that the Γ-limit functionals may be of a non-local nature.

Let $\Omega=\omega\times(0,1)$ be a 'cylindrical' domain where ω is a connected open subset of $\mathbf{R}^2$.

We define εD_i to be a two dimensional disk centered at $x_i=(\varepsilon i_1+\frac{\varepsilon}{2},\varepsilon i_2+\frac{\varepsilon}{2})$ of radius $\varepsilon/4$

$$\varepsilon E_i^{\,2}=\varepsilon D_i\times(0,1)\qquad\qquad \varepsilon E^2=\bigcup_{i\in I_\varepsilon}\varepsilon E_i^{\,2}$$

where $i=(i_1,i_2)\in I_\varepsilon=\{i\in\mathbf{Z}^2:\varepsilon E_i^{\,2}\subset\Omega\}$,

$$\varepsilon E^1=\Omega\setminus\varepsilon E^2.$$

We call $E=D_0\times(0,1)$.

We consider the measures

$$\mu_\varepsilon(B)=\varepsilon\mathcal{H}^2(B\cap\partial\varepsilon E^2)$$

and the functionals

$$F_\varepsilon^\gamma(u,\Omega)=\varepsilon^\gamma\int_\Omega\Big|\frac{dEu}{d\mu_\varepsilon}\Big|^2\,d\mu_\varepsilon\,.$$

Note that, up to normalization, μ_ε is the same measure of Example 4.3.

In this case $\mathrm{LD}^2_{\mu_\varepsilon}(\Omega)$ consists of functions which are rigid displacements on the sets εE^1 and εE^2; *i.e.*, $u\in\mathrm{LD}^2_{\mu_\varepsilon}(\Omega)$ if and only if there exist $a_i,b_i,c,d\in\mathbf{R}^3$ such that

$$u=c\wedge x+d\quad\text{ on }\quad\varepsilon E^1$$
$$u=a_i\wedge x+b_i\quad\text{ on }\quad\varepsilon E_i^{\,2}$$

for each $i\in I_\varepsilon$. We use the notation $x=(x_\alpha,x_3)\in\mathbf{R}^3$, $x_\alpha=(x_1,x_2)$.

Hence

$$\frac{dEu}{d\mu_\varepsilon}=\frac{1}{\varepsilon}\frac{dEu}{d\mathcal{H}^2}=\frac{1}{\varepsilon}(c\wedge x+d-a_i\wedge x-b_i)\odot\nu\text{ on }\partial(\varepsilon E_i^2)\,.$$

Definition 6.1 *Let $u_\varepsilon\in\mathrm{LD}^2_{\mu_\varepsilon}(\Omega)$. We say that u_ε converges to $(u_1,u_2)\in\mathrm{L}^2(\Omega;\mathbf{R}^3)\times\mathrm{L}^2(\Omega;\mathbf{R}^3)$ if and only if*

$$\lim_{\varepsilon\to 0}\int_{\varepsilon E^1}|u_\varepsilon-u_1|^2\,dx\;=0\tag{20}$$

$$\lim_{\varepsilon\to 0}\int_{\varepsilon E^2}|u_\varepsilon-u_2|^2\,dx\;=0\,.\tag{21}$$

We will study the Γ-limit F of F_ε^γ with respect to the convergence introduced in Definition 6.1 (see Theorem 6.4). The domain of F will be the set of pairs (u_1, u_2) such that u_1 is a rigid displacement and u_2 is in the space $\mathcal{U}$ of functions whose 'vertical sections are rigid displacements', introduced in the following proposition.

Let us define, for all $\eta > 0$, $T_\eta^k = Q_\eta^k \times (0,1)$ where $Q_\eta^k = k + (0,\eta)^2$ with $k = (k_1, k_2) \in J = \{k \in \mathbf{Z}^2 : T_\eta^k \cap \Omega \neq \emptyset\}$.

Proposition 6.2 *Let $u_\varepsilon \in \mathrm{LD}^2_{\mu_\varepsilon}(\Omega)$ and $u_2 \in \mathrm{L}^2(\Omega; \mathbf{R}^3)$.*

$$\lim_{\varepsilon \to 0} \int_{\varepsilon E^2} |u_\varepsilon - u_2|^2 \, dx = 0$$

if and only if $u_2 \in \mathcal{U}$ where

$$\begin{aligned} \mathcal{U} \; = \Big\{ & v \in \mathrm{L}^2(\Omega; \mathbf{R}^3) : \forall \eta > 0 \; \exists J \subset \mathbf{Z}^2 \textit{ and } \exists \, A^k \wedge x + B^k \textit{ on } T_\eta^k \quad \forall k \in J \textit{ such that} \\ & \bigcup_{k \in J} T_\eta^k \cap \Omega = \Omega \quad \textit{and} \quad \sum_{k \in J} \int_{T_\eta^k \cap \Omega} |v(x) - A^k \wedge x - B^k|^2 \, dx \leq o(\eta) \Big\} . \end{aligned}$$

Proof. Let $u_\varepsilon \in \mathrm{LD}^2_{\mu_\varepsilon}(\Omega)$, by definition $u_\varepsilon = a_{\varepsilon,i} \wedge x + b_{\varepsilon,i}$ on εE_i^2. Let $h \in \mathbf{N}$ and $\eta > 0$ such that $\eta = h\varepsilon$, we extend $a_{\varepsilon,i} \wedge x + b_{\varepsilon,i}$ to T_η^k for each $i \in I_k = \{i \in \mathbf{Z}^2 : \varepsilon E_i^2 \subset T_\eta^k\}$, hence we can construct a rigid displacement on T_η^k

$$A_\varepsilon^k \wedge x + B_\varepsilon^k = \frac{1}{h^2} \sum_{i \in I_k} a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} \, .$$

Let us suppose that u_ε satisfies condition (21),

$$\begin{aligned} & \int_{T_\eta^k \cap \varepsilon E^2} \Big| u_2(x) - A_\varepsilon^k \wedge x - B_\varepsilon^k \Big|^2 dx \\ \leq \; & c \Big(\sum_{j \in I_k} \int_{\varepsilon E_j^2} \Big| u_2(x) - a_{\varepsilon,j} \wedge x - b_{\varepsilon,j} \Big|^2 dx \\ & + \sum_{j \in I_k} \int_{\varepsilon E_j^2} \Big| a_{\varepsilon,j} \wedge x + b_{\varepsilon,j} - \frac{1}{h^2} \sum_{i \in I_k} a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} \Big|^2 dx \Big) . \end{aligned} \tag{22}$$

Let us estimate the last term in (22)

$$\begin{aligned} & \sum_{j \in I_k} \int_{\varepsilon E_j^2} \Big| a_{\varepsilon,j} \wedge x + b_{\varepsilon,j} - \frac{1}{h^2} \sum_{i \in I_k} a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} \Big|^2 dx \\ \leq \; & c \Big(\sum_{j \in I_k} \int_{\varepsilon E_j^2} \Big| a_{\varepsilon,j} \wedge x + b_{\varepsilon,j} - u_2(x) \Big|^2 dx \\ & + \sum_{j \in I_k} \int_{\varepsilon E_j^2} \Big| \frac{1}{h^2} \sum_{i \in I_k} a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} - u_2(x + x_i - x_j) \Big|^2 dx \\ & + \sum_{j \in I_k} \int_{\varepsilon E_j^2} \Big| \frac{1}{h^2} \sum_{i \in I_k} u_2(x) - u_2(x + x_i - x_j) \Big|^2 dx \Big) . \end{aligned}$$

For each $x \in \varepsilon E_j^2$ we have that $x + x_i - x_j \in \varepsilon E_i^2$, hence with a change of coordinates we get

$$
\begin{aligned}
& \sum_{j\in I_k} \int_{\varepsilon E_j^2} \Big| a_{\varepsilon,j} \wedge x + b_{\varepsilon,j} - \frac{1}{h^2} \sum_{i\in I_k} a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} \Big|^2 dx \\
\leq \ & c\Big(\sum_{j\in I_k} \int_{\varepsilon E_j^2} \Big| a_{\varepsilon,j} \wedge x + b_{\varepsilon,j} - u_2(x) \Big|^2 dx \\
& + \sum_{i,j\in I_k} \frac{1}{h^2} \int_{\varepsilon E_i^2} \Big| a_{\varepsilon,i} \wedge (x + x_j - x_i) + b_{\varepsilon,i} - u_2(x) \Big|^2 dx \\
& + \sum_{i,j\in I_k} \frac{1}{h^2} \int_{\varepsilon E_j^2} \Big| u_2(x) - u_2(x + x_i - x_j) \Big|^2 dx \Big) \\
\leq \ & c\Big(\sum_{i\in I_k} \int_{\varepsilon E_i^2} \Big| a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} - u_2 \Big|^2 dx \\
& + \sum_{i,j\in I_k} \frac{1}{h^2} \int_{\varepsilon E_i^2} \Big| a_{\varepsilon,i} \wedge (x_j - x_i) \Big|^2 dx \\
& + \sum_{i,j\in I_k} \frac{1}{h^2} \int_{\varepsilon E_j^2} \Big| u_2(x) - u_2(x + x_i - x_j) \Big|^2 dx \Big) .
\end{aligned}
\tag{23}
$$

Now if we denote Λ the set of all translations of the type $x_i - x_j$ with $i, j \in I_k$ we get that

$$
\begin{aligned}
& \sum_{i,j\in I_k} \frac{1}{h^2} \int_{\varepsilon E_j^2} \Big| u_2(x) - u_2(x + x_i - x_j) \Big|^2 dx \\
\leq \ & \sum_{\tau\in\Lambda} \frac{1}{h^2} \sum_{r\in C(k)} \int_{T_\eta^r} |u_2(x) - u_2(x+\tau)|^2 \, dx
\end{aligned}
\tag{24}
$$

where $C(k) = \{(k_1, k_2), (k_1 \pm 1, k_2), (k_1, k_2 \pm 1), (k_1 \pm 1, k_2 \pm 1)\}$.

Since $|\Lambda| = c\,h^2$, by (24) we have

$$
\begin{aligned}
& \sum_{k\in J} \sum_{i,j\in I_k} \frac{1}{h^2} \int_{\varepsilon E_j^2} \Big| u_2(x) - u_2(x + x_i - x_j) \Big|^2 dx \\
\leq \ & c \sum_{\tau\in\Lambda} \frac{1}{h^2} \| u_2(\cdot) - u_2(\cdot + \tau) \|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^3)} \\
\leq \ & c \sup_{|\tau|\leq\sqrt{2}\eta} \| u_2(\cdot) - u_2(\cdot + \tau) \|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^3)} \, .
\end{aligned}
\tag{25}
$$

Let us consider the cubes $Q^j_{\varepsilon,i} = (\varepsilon i + (0,1)^2) \times (\varepsilon j + (0,\varepsilon))$ for $i \in I_\varepsilon$, and $j \in J_\varepsilon = \{j \in \mathbf{Z} : Q^j_{\varepsilon,i} \cap \varepsilon E_i^2 \neq \emptyset\}$. Since $u_2 \in \mathrm{L}^2(\Omega; \mathbf{R}^3)$, we can assume that there exists a sequence $(u_{\varepsilon,2})$ which is constant on each $Q^j_{\varepsilon,i}$ such that

$$
\lim_{\varepsilon\to 0} \int_\Omega |u_2 - u_{\varepsilon,2}|^2 \, dx = \lim_{\varepsilon\to 0} \sum_{i\in I_\varepsilon} \sum_{j\in J_\varepsilon} \int_{Q^j_{\varepsilon,i}\cap\Omega} |u_2 - u_{\varepsilon,2,i,j}|^2 \, dx = 0
\tag{26}
$$

where $u_{\varepsilon,2,i,j}$ is the value of $(u_{\varepsilon,2})$ on $Q^j_{\varepsilon,i}$.

So by (21)) we get

$$\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\sum_{j\in J_\varepsilon}\int_{Q^j_{\varepsilon,i}\cap\varepsilon E^2_i}|u_\varepsilon-u_{\varepsilon,2,i,j}|^2\,dx=0\,. \tag{27}$$

Note that the L^2-norm on the set $\mathcal{R}$ of rigid displacements is equivalent to the norm on $\mathcal{R}$

$$\|a\wedge x+b\|_{\mathcal{R}}=(|a|^2+|b|^2)^{1/2}\,,$$

hence by (27)

$$\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\sum_{j\in J_\varepsilon}\varepsilon^3|a_{\varepsilon,i}|^2+\varepsilon^3|b_{\varepsilon,i}-u_{\varepsilon,2,i,j}|^2=0$$

which implies that

$$\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\varepsilon^2|a_{\varepsilon,i}|^2=0 \tag{28}$$

and

$$\sum_{i\in I_\varepsilon}\varepsilon^2|b_{\varepsilon,i}|^2\le c \tag{29}$$

for each $\varepsilon>0$ small enough.

Since $|x_j-x_i|\le\eta$, by the equivalence of the norms we have

$$\begin{aligned}\sum_{i,j\in I_k}\frac{1}{h^2}\int_{\varepsilon E^2_i}\Big|a_{\varepsilon,i}\wedge(x_j-x_i)\Big|^2\,dx &\le c\sum_{i,j\in I_k}\frac{\varepsilon^2}{h^2}\eta^2|a_{\varepsilon,i}|^2\\ &= c\eta^2\sum_{i\in I_k}\varepsilon^2|a_{\varepsilon,i}|^2\,. \end{aligned}\tag{30}$$

Note that $\sum_{k\in J}\sum_{i\in I_k}=\sum_{i\in I_\varepsilon}$.

Now we insert (30) into (23) and, summing up all the corresponding estimates obtained for different indices $k\in J$, by (25) we get

$$\begin{aligned}&\sum_{k\in J}\sum_{j\in I_k}\int_{\varepsilon E^2_j}\Big|a_{\varepsilon,j}\wedge x+b_{\varepsilon,j}-\frac{1}{h^2}\sum_{i\in I_k}a_{\varepsilon,i}\wedge x+b_{\varepsilon,i}\Big|^2\,dx \\ \le\;& c\Big(\sum_{i\in I_\varepsilon}\int_{\varepsilon E^2_i}\Big|a_{\varepsilon,i}\wedge x+b_{\varepsilon,i}-u_2(x)\Big|^2\,dx+\eta^2\sum_{i\in I_\varepsilon}\varepsilon^2|a_{\varepsilon,i}|^2\\ &+\sup_{|\tau|\le\sqrt{2}\eta}\|u_2(\cdot)-u_2(\cdot+\tau)\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^3)}\Big)\,.\end{aligned}\tag{31}$$

Finally, we sum up the estimates (22) for $k\in J$ and insert (31); by (21) and (28) we get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sum_{k\in J}\int_{T^k_\eta\cap\varepsilon E^2}\Big|u_2-A^k_\varepsilon\wedge x-B^k_\varepsilon\Big|^2\,dx\\ \le\;& c\sup_{|\tau|\le\sqrt{2}\eta}\|u_2(\cdot)-u_2(\cdot+\tau)\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^3)}\,.\end{aligned}\tag{32}$$

On the other hand it is easy to see by (28) and (29) that there exists $A^k\wedge x+B^k$ such that

$$\lim_{\varepsilon\to 0}\int_{T^k_\eta}\Big|A^k_\varepsilon\wedge x+B^k_\varepsilon-A^k\wedge x-B^k\Big|^2\,dx=0$$

for each $k \in J$, hence by (32) we can conclude that $u_2 \in \mathcal{U}$.

Conversely, if $u_2 \in \mathcal{U}$ then $\varepsilon E^2 = \cup_{k\in J} T^k_\eta \cap \varepsilon E^2$ and we have rigid displacements $A^k \wedge x + B^k$ on each T^k_η.

We define

$$a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} = (A^k \wedge x + B^k)_{|\varepsilon E^2_i}$$

for each $i \in I_k$. Hence

$$\sum_{i\in I_\varepsilon} \int_{\varepsilon E^2_i} \Big|a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} - u_2(x)\Big|^2 dx = \sum_{k\in J} \int_{T^k_\eta \cap \varepsilon E^2} \Big|A^k \wedge x + B^k - u_2(x)\Big|^2 dx$$

and by definition of $\mathcal{U}$

$$\begin{aligned} &\lim_{\varepsilon\to 0} \sum_{k\in J} \int_{T^k_\eta \cap \varepsilon E^2} \Big|A^k \wedge x + B^k - u_2(x)\Big|^2 dx \\ = \;& \sum_{k\in J} |E| \int_{T^k_\eta \cap \Omega} \Big|A^k \wedge x + B^k - u_2(x)\Big|^2 dx \le o(\eta)\,. \end{aligned} \tag{33}$$

By (33), passing to the limit as $\eta \to 0$, we get

$$\lim_{\varepsilon\to 0} \sum_{i\in I_\varepsilon} \int_{\varepsilon E^2_i} \Big|a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} - u_2(x)\Big|^2 dx = 0\,.$$

□

Remark 6.3 Note that, since u_ε are rigid displacements, by (20) it is easy to see that u_1 is a rigid displacement.

For simplicity, we will denote

$$F(u_1, u_2; \Omega) = \Gamma\text{-}\lim_{\varepsilon\to 0} F^\gamma_\varepsilon(u_1, u_2; \Omega)$$

for $(u_1, u_2) \in \mathcal{R} \times \mathcal{U}$. We will continue to write $F^\gamma_\varepsilon(u, \Omega)$ for $u \in \mathrm{LD}^2_{\mu_\varepsilon}(\Omega)$.

Theorem 6.4 *For $\gamma = 2$ the functionals F^γ_ε Γ-converge as $\varepsilon \to 0$ to*

$$F(u_1, u_2; \Omega) = c_1 \int_\Omega |(u_1)_\alpha - (u_2)_\alpha|^2 \, dx + c_2 \int_\Omega |(u_1)_3 - (u_2)_3|^2 \, dx$$

on $\mathcal{R}\times\mathcal{U}$ with respect to the convergence introduced in Definition 6.1, *where $c_1 = \frac{3}{8}\pi$, $c_2 = \frac{\pi}{4}$.*

Proof. By the invariance of the functionals with respect to translations of rigid displacements and by Remark 6.3 we can always assume without loss of generality that $u_\varepsilon = u_1$ on εE^1.

Let us call

$$\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} = u_1 - a_{\varepsilon,i} \wedge x - b_{\varepsilon,i}$$

hence

$$F^\gamma_\varepsilon(u_\varepsilon, \Omega) = \varepsilon^{\gamma-1} \sum_{i\in I_\varepsilon} \int_{\partial\varepsilon E^2_i} \Big|(\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i}) \odot \nu\Big|^2 d\mathcal{H}^2\,.$$

Fix $x_3 \in (0,1)$, we can find the following equality

$$
\begin{aligned}
& 4\varepsilon \int_{\partial \varepsilon D_i} \Big|(\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i}) \odot \nu\Big|^2 d\mathcal{H}^1 - 16 \int_{\varepsilon D_i} \Big|\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i}\Big|^2 dx_\alpha \\
= \; & \frac{\pi}{2}\varepsilon^2 \Big(\Big| (\fint_{\varepsilon D_i} \alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \; dx_\alpha)_1 \Big|^2 + \Big| (\fint_{\varepsilon D_i} \alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \; dx_\alpha)_2 \Big|^2 \Big) \\
& + \frac{\pi}{64}\varepsilon^4 ((\alpha_{\varepsilon,i})_1^2 + (\alpha_{\varepsilon,i})_2^2 + 2(\alpha_{\varepsilon,i})_3^2) .
\end{aligned}
$$

Hence, if we integrate also in x_3, we get

$$
\begin{aligned}
& 4\varepsilon \int_{\partial \varepsilon E_i^2} \Big|(\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i}) \odot \nu\Big|^2 d\mathcal{H}^2 - 16 \int_{\varepsilon E_i^2} \Big|\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i}\Big|^2 dx \\
= \; & \int_0^1 \frac{\pi}{2}\varepsilon^2 \Big(\Big| (\fint_{\varepsilon D_i} \alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \; dx_\alpha)_1 \Big|^2 + \Big| (\fint_{\varepsilon D_i} \alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \; dx_\alpha)_2 \Big|^2 \Big) dx_3 \\
& + \frac{\pi}{64}\varepsilon^4 ((\alpha_{\varepsilon,i})_1^2 + (\alpha_{\varepsilon,i})_2^2 + 2(\alpha_{\varepsilon,i})_3^2) .
\end{aligned} \tag{34}
$$

But

$$
\lim_{\varepsilon \to 0} \sum_{i \in I_\varepsilon} \int_{\varepsilon E_i^2} \Big| (\fint_{\varepsilon D_i} \alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \; dx_\alpha)_h \Big|^2 dx = \lim_{\varepsilon \to 0} \sum_{i \in I_\varepsilon} \int_{\varepsilon E_i^2} \Big| (\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i})_h \Big|^2 dx
$$

for each $h = 1, 2, 3$, and

$$
\frac{\pi}{2}\varepsilon^2 \Big| (\fint_{\varepsilon D_i} \alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \, dx_\alpha)_h \Big|^2 = 8 \int_{\varepsilon D_i} \Big| (\fint_{\varepsilon D_i} \alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \; dx_\alpha)_h \Big|^2 dx_\alpha \, ;
$$

hence,

$$
\begin{aligned}
& \lim_{\varepsilon \to 0} \sum_{i \in I_\varepsilon} \int_0^1 \frac{\pi}{2}\varepsilon^2 \Big| (\fint_{\varepsilon D_i} \alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \, dx_\alpha)_h \Big|^2 dx_3 \\
= \; & 8 \lim_{\varepsilon \to 0} \sum_{i \in I_\varepsilon} \int_{\varepsilon E_i^2} \Big| (\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i})_h \Big|^2 dx .
\end{aligned} \tag{35}
$$

If we pass to the limit in (34), by (35) we obtain

$$
\begin{aligned}
& \liminf_{\varepsilon \to 0} \sum_{i \in I_\varepsilon} \varepsilon \int_{\partial \varepsilon E_i^2} \Big|(\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i}) \odot \nu\Big|^2 d\mathcal{H}^2 \\
\geq \; & 6 \lim_{\varepsilon \to 0} \sum_{i \in I_\varepsilon} \int_{\varepsilon E_i^2} \Big| (\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i})_1 \Big|^2 + \Big| (\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i})_2 \Big|^2 dx \\
& + 4 \lim_{\varepsilon \to 0} \sum_{i \in I_\varepsilon} \int_{\varepsilon E_i^2} \Big| (\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i})_3 \Big|^2 dx \\
& + \lim_{\varepsilon \to 0} \frac{\pi}{64} \sum_{i \in I_\varepsilon} \varepsilon^4 ((\alpha_{\varepsilon,i})_1^2 + (\alpha_{\varepsilon,i})_2^2 + 2(\alpha_{\varepsilon,i})_3^2) .
\end{aligned} \tag{36}
$$

For every sequence u_ε converging to (u_1, u_2) in the sense of Definition 6.1, by (28) we have that

$$
\lim_{\varepsilon \to 0} \frac{\pi}{64} \sum_{i \in I_\varepsilon} \varepsilon^4 ((\alpha_{\varepsilon,i})_1^2 + (\alpha_{\varepsilon,i})_2^2 + 2(\alpha_{\varepsilon,i})_3^2) = 0 \tag{37}
$$

so we insert (37) into (36) to find that

$$\liminf_{\varepsilon\to 0}\varepsilon^2\int_\Omega\Big|\frac{dEu_\varepsilon}{d\mu_\varepsilon}\Big|^2 d\mu_\varepsilon \;\geq\; 6|E|\int_\Omega|(u_1)_\alpha-(u_2)_\alpha|^2\,dx + 4|E|\int_\Omega|(u_1)_3-(u_2)_3|^2\,dx\,. \tag{38}$$

By the arbitrarity of u_ε, choosing $\gamma=2$

$$\Gamma\text{-}\liminf_{\varepsilon\to 0}F_\varepsilon{}^2(u_1,u_2;\Omega)\geq F(u_1,u_2;\Omega)\,. \tag{39}$$

Now we consider

$$u_\varepsilon=(c\wedge x+d)\,\chi_{\varepsilon E^1}+(a\wedge x+b)\,\chi_{\varepsilon E^2}$$

obviously it converges to $(c\wedge x+d,a\wedge x+b)$, and we call $\alpha\wedge x+\beta=(a-c)\wedge x+(b-d)$.

In this case

$$8\int_{\varepsilon E_i^2}\Big|(\alpha\wedge x+\beta)_h\Big|^2\,dx=\int_0^1\frac{\pi}{2}\varepsilon^2\Big|(\fint_{\varepsilon D_i}\alpha\wedge x+\beta\,dx_\alpha)_h\Big|^2\,dx_3+\frac{\pi}{128}\varepsilon^4\alpha_3^2$$

for $h=1,2$, hence by (34)

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\varepsilon\int_{\partial\varepsilon E_i^2}\Big|(\alpha\wedge x+\beta)\odot\nu\Big|^2 d\mathcal{H}^2\\
\leq\;&6\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\int_{\varepsilon E_i^2}\Big|(\alpha\wedge x+\beta)_1\Big|^2+\Big|(\alpha\wedge x+\beta)_2\Big|^2\,dx\\
&+4\lim_{\varepsilon\to 0}\sum_{i\in I_\varepsilon}\int_{\varepsilon E_i^2}\Big|(\alpha\wedge x+\beta)_3\Big|^2\,dx+c\lim_{\varepsilon\to 0}\varepsilon^2|\alpha|^2\\
=\;&6|E|\int_\Omega\Big|(\alpha\wedge x+\beta)_\alpha\Big|^2\,dx+4|E|\int_\Omega\Big|(\alpha\wedge x+\beta)_3\Big|^2\,dx\,.
\end{aligned}\tag{40}$$

By (38) and (40) we get

$$\lim_{\varepsilon\to 0}\varepsilon^2\int_\Omega\Big|\frac{dEu_\varepsilon}{d\mu_\varepsilon}\Big|^2 d\mu_\varepsilon \;=\; 6|E|\int_\Omega\Big|(\alpha\wedge x+\beta)_\alpha\Big|^2\,dx + 4|E|\int_\Omega\Big|(\alpha\wedge x+\beta)_3\Big|^2\,dx\,. \tag{41}$$

Now we fix $\eta>0$ and consider $u_1\in\mathcal{R}$ and v_2^η such that $v^\eta_{2|T^k_\eta}=A^k\wedge x+B^k$ with $k\in J$. By (41) we get

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}F^2_\varepsilon(u_1\,\chi_{\varepsilon E^1}+v_2^\eta\,\chi_{\varepsilon E^2},\Omega)\\
\leq\;&\sum_{k\in J}\limsup_{\varepsilon\to 0}F^2_\varepsilon(u_1\,\chi_{\varepsilon E^1}+(A^k\wedge x+B^k)\,\chi_{\varepsilon E^2},T^k_\eta\cap\Omega)\\
=\;&\sum_{k\in J}6|E|\int_{T^k_\eta\cap\Omega}\Big|(u_1(x)-A^k\wedge x-B^k)_\alpha\Big|^2\,dx\\
&+\sum_{k\in J}4|E|\int_{T^k_\eta\cap\Omega}\Big|(u_1(x)-A^k\wedge x-B^k)_3\Big|^2\,dx\\
=\;&6|E|\int_\Omega\Big|(u_1(x)-v_2^\eta(x))_\alpha\Big|^2\,dx+4|E|\int_\Omega\Big|(u_1(x)-v_2^\eta(x))_3\Big|^2\,dx
\end{aligned}\tag{42}$$

If $u_2 \in \mathcal{U}$ then for all $\eta > 0$ there exists v_2^η as above such that $\|u_2 - v_2^\eta\|_{\mathrm{L}^2(\Omega;\mathbf{R}^3)} \le o(\eta)$, since the Γ-upper limit is L^2-lower semicontinuous if we denote

$$F_2''(u_1, u_2; \Omega) = \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon{}^2(u_1, u_2; \Omega)$$

by (42) we get

$$\begin{aligned} F_2''(u_1, u_2; \Omega) &\le \liminf_{\eta\to 0} F_2''(u_1, v_2^\eta; \Omega) \\ &\le \liminf_{\eta\to 0} 6|E| \int_\Omega \Big|(u_1(x) - v_2^\eta(x))_\alpha\Big|^2 dx \\ &\qquad +4|E| \int_\Omega \Big|(u_1(x) - v_2^\eta(x))_3\Big|^2 dx \\ &= 6|E| \int_\Omega \Big|(u_1(x) - u_2(x))_\alpha\Big|^2 dx + 4|E| \int_\Omega \Big|(u_1(x) - u_2(x))_3\Big|^2 dx\,. \end{aligned}$$

It follows that given $(u_1, u_2) \in \mathcal{R} \times \mathcal{U}$

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon{}^2(u_1, u_2; \Omega) \le F(u_1, u_2; \Omega)$$

so that by (39)

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon{}^2(u_1, u_2; \Omega) = F(u_1, u_2; \Omega)$$

as desired. □

If u_ε converges to (u_1, u_2) in the sense of Definition 6.1 then u_ε converges weakly in $\mathrm{L}^2(\Omega; \mathbf{R}^3)$ to $(1-c)u_1 + cu_2$ where $c = |E|$. If we define the energy

$$F(u, \Omega) := \inf_{\substack{u=(1-c)u_1+cu_2 \\ (u_1,u_2)\in\mathcal{R}\times\mathcal{U}}} F(u_1, u_2; \Omega)$$

by Theorem 6.4

$$F(u, \Omega) = \inf_{r\in\mathcal{R}} \Big(\tilde{c_1} \int_\Omega |r_\alpha - u_\alpha|^2 \, dx + \tilde{c_2} \int_\Omega |r_3 - u_3|^2 \, dx\Big)$$

where $\tilde{c_1} = c_1/c^2$ and $\tilde{c_2} = c_2/c^2$, which explains the non local nature of our limit.

Remark 6.5 Let us consider, up to normalization, the same measure of Example 4.4

$$\widetilde{\mu_\varepsilon}(B) = \Big(|B| + \varepsilon\mathcal{H}^2(B \cap \partial\varepsilon E^2)\Big)$$

and the functionals

$$\widetilde{F_\varepsilon^2}(u, \Omega) = \varepsilon^2 \int_\Omega \Big|\frac{dEu_\varepsilon}{d\widetilde{\mu_\varepsilon}}\Big|^2 d\tilde{\mu}_\varepsilon\,.$$

In this case by Theorem 6.4 we can deduce that the $\Gamma\text{-}\limsup_{\varepsilon\to 0} \widetilde{F_\varepsilon^2}(u_1, u_2; \Omega)$ is finite for $(u_1, u_2) \in \mathcal{R} \times \mathcal{U}$.

In fact, since $\mathrm{LD}^2_{\mu_\varepsilon}(\Omega; \mathbf{R}^3) \subset \mathrm{LD}^2_{\widetilde{\mu_\varepsilon}}(\Omega; \mathbf{R}^3)$, given $(u_1, u_2) \in \mathcal{R} \times \mathcal{U}$ we have

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} \widetilde{F_\varepsilon^2}(u_1, u_2; \Omega) \le \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon^2(u_1, u_2; \Omega)\,.$$

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Nadia Ansini
SISSA/ISAS
Via Beirut 4, 34014 Trieste, Italy

François Bille Ebobisse
Dipartimento di Matematica
Università di Pisa
Via Buonarroti 2, 56127 Pisa, Italy