SCUOLA
NORMALE
SUPERIORE

Classe di Scienze

Corso di perfezionamento in
Matematica

XXXVII ciclo

Nonlocal phase transitions in codimension-one: diffuse approximation, stability and asymptotics

Settore Scientifico Disciplinare **MAT/05**

Candidato
dr. Michele Caselli

Relatori

Prof. Andrea Malchiodi
Prof. Joaquim Serra

Supervisione interna

Prof. Andrea Malchiodi

Anno accademico 2024–2025

Nonlocal phase transitions in codimension-one: diffuse approximation, stability and asymptotics

Michele Caselli

Abstract

This thesis explores the theory of fractional perimeter on closed Riemannian manifolds, with a focus on codimension-one phenomena and their approximation through diffuse interface models. A central theme is the existence and regularity of nonlocal minimal surfaces, which arise as critical points of the fractional perimeter, on closed manifolds. Despite their intrinsic nonlocality, which is an anomalous feature compared with the classical theory of minimal surfaces, these surfaces share many structural elements with classical minimal hypersurfaces. In several respects, particularly in the context of stability and finite Morse index, we aim to show that they exhibit improved compactness and regularity properties. These properties make nonlocal minimal surfaces highly suitable for min-max constructions since Morse theory for nonlocal minimal surfaces is, in some sense, as flawless as finite-dimensional Morse theory.

At the core of our existence result is the use of the fractional Allen-Cahn equation as a diffuse approximation of nonlocal minimal surfaces. This method has proved to be particularly effective in the context of min-max constructions, as it allows for the existence of many critical points with precise index and energy bounds.

Addressing the regularity theory of nonlocal minimal surfaces on manifolds requires several key ingredients: precise local estimates on the heat kernel of complete manifolds, rigidity results for stationary cones stable in $\mathbb{R}^n \setminus \{0\}$, and the development of the Caffarelli-Silvestre extension theory on closed manifolds. In this work, we develop (some of) these ingredients and utilize these technical tools to address different problems. For example, the local estimates for the heat kernel will be used both to deduce regularity, via a blow-up procedure, of finite Morse index nonlocal minimal surfaces arising as limits of the fractional Allen-Cahn equation and to characterize the asymptotics of the fractional Laplacian on noncompact manifolds as $s \to 0$.

~~*To my friends in Pisa*~~

~~*To my friends of the Complesso Polvani*~~

To my friends of the Lido Polvani

Acknowledgements

First and foremost, I am deeply grateful to my advisors, Andrea Malchiodi and Joaquim Serra, for their constant support and all the time and resources they invested in me during these years. They were able to guide me in finding my own path in mathematics without imposing one on me, without pressuring me, but with constant enthusiasm for what we do. I am grateful for your trust, encouragement, and support for my exploration needs.

I truly admire the approach to mathematics that I have seen from both of you: your genuine positivity and curiosity, your open-mindedness to new tools and ideas, your focus on enjoying the journey itself, and your balance with life outside mathematics. If I now look forward to being a mathematician, I owe this also to you, in that you showed me the bright side of academia, the one I could see myself being part of.

In this regard, I cannot miss the chance to thank Otis Chodosh for welcoming me to Stanford University in the spring of 2024. Your curiosity and indefatigable enthusiasm for mathematics have been genuinely inspiring. I have loved my time in California, both for the mathematics and for everything else, and I will always have good memories of my months at the Bay. I want to thank Nico Sigler (and Oliver), Jared Marx-Kuo, Shuli Chen, Henry Bosh, and Yujie Wu for making my time at Stanford wonderful.

A lot of what I understand now about mathematics I learned not from studying books or papers but from spending time with, and doing mathematics with, other people, often friends. I have had the privilege of traveling and working with exceptionally talented mathematicians. Many things I understand today carry with them the memory of a conversation or a blackboard session that stretched late into the night (any reference to Luca Gennaioli is purely coincidental). I have a modern and non-stereotypical view of being a mathematician, in that I strongly believe in sharing and collaboration, and that mathematics is better done together. I am incredibly thankful to my collaborators and friends Enric Florit-Simon, Luca Gennaioli, Mattia Freguglia, and Nicola Picenni, without whom this manuscript would certainly not exist. This shared journey of learning and discovery is what made me stick with mathematics, and I am grateful to all of you for this.

Moreover, I would also like to thank all the other fellow mathematicians with whom I have shared part of these years in Pisa and elsewhere. I owe great thanks to Pietro Capovilla, Camillo Brena, Alessandra Pluda, Riccardo Caniato, Federico Franceschini, Marco Badran, Davide Parise, Lorenzo Sarnataro, Albachiara Cogo, Matilde Gianocca, Filippo Gaia, and Gerard Orriols.

I am aware that this way of mine of doing mathematics—if not just the way I live—means that I have often been absent from many people I love. In the future, I will find a compromise between my drive for novelty and exploration, mathematics and the world, and my desire to be around the people I love. I am forever grateful to my family, my parents, my brother Gabriele, and Silvia, for their constant love and encouragement, even when I was afar (i.e., always). I am also immensely grateful to my girlfriend Gaia, who joined this journey of mine at a stormy time near its end, and quietly steadied me with constant patience and love.

Above all, the time came for the hardest goodbyes.

To my friends of the Lido Polvani: Alberto, Bruna, Eliseo, Fabio, Francesco, Giorgio, Marco, Marin, Pasquale, Pietro (again), Sofia, and Vittoria. This thesis is dedicated to you.

I hope you know how grateful I am for what we had in Pisa, and that it is the closest I have known to *home* in a long time. We shared moments and spaces like a family does, and for me, you are one.

Michele

Contents

Chapter 1

Introduction

1.1 Overview and Motivation

The theory of minimal surfaces has always occupied a central role within differential geometry and geometric analysis. In codimension one, classical minimal hypersurfaces are characterized by the vanishing of the mean curvature. On the other hand, the modern theory of minimal surfaces is rooted in the variational study of the area functional and is closely connected to the framework of geometric measure theory.

In the 1960s, Almgren laid the groundwork for a remarkably general existence theory of minimal surfaces: using a variational min–max method, he proved that in every ambient Riemannian manifold there exists at least one generalized solution in every codimension, namely a stationary integral varifold (see [Alm65, Corollary 15.2]). In codimension one, this program reached a higher level of regularity through subsequent works of Pitts, Schoen, and Simon ([Pit81; SS81]), who showed that in ambient dimension $n \leq 7$, such generalized solutions are classical smooth, embedded, closed minimal hypersurfaces, with singularities possibly appearing only in dimension $n > 7$. The combination of these results can be summarized as follows.

Theorem 1.1.1 ([Alm65; Pit81; SS81])**.** *Every closed n-dimensional Riemannian manifold contains at least one minimal hypersurface, smooth and embedded outside a set of Hausdorff dimension at most $n-8$.*

Some years later, Yau conjectured that a much stronger abundance result should hold. This conjecture—now theorem—was listed as Problem 88 in Yau's list of open problems in differential geometry [Yau82].

Theorem 1.1.2 (Yau's conjecture, [MN17a; Son23])**.** *Every closed 3-dimensional Riemannian manifold contains infinitely many smooth, closed, immersed minimal surfaces.*

This result was recently proved in full generality by Song [Son23] relying on the Almgren–Pitts framework. Song's breakthrough relies on techniques previously developed by Marques and Neves in [MN17a], and establishes somewhat an even stronger result: for $3 \leq n \leq 7$, every closed n-dimensional Riemannian manifold contains infinitely many smooth, closed, *embedded* minimal hypersurfaces.

While the Almgren-Pitts min-max theory is extremely powerful and has led to several deep results in minimal surfaces theory, alternative frameworks, particularly in codimension one, exist that allow for finer control over regularity and multiplicity, and can be better suited to detect the existence of several minimal surfaces.

1.1.1 Phase transitions and minimal surfaces

A central development in the realm of these alternative frameworks for the existence of minimal surfaces has been the introduction of diffuse interface approximations of the perimeter functional; most notably, the Allen–Cahn energy. The Allen-Cahn energy framework provides a PDE-based alternative to the classical min-max theory of existence of Almgren-Pitts [Alm62; Alm65; Pit81]. This energy is defined for every $\varepsilon > 0$ by

$$AC_\varepsilon(u) := \int_M \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}, \quad \text{where } W(u) := \frac{(1-u^2)^2}{4}.$$

In the case of $M = \mathbb{R}^n$, the seminal results of Modica and Mortola [MM77b; MM77a; Mod87] established the Γ-convergence of the Allen–Cahn energy AC_ε to the classical perimeter as $\varepsilon \to 0$. We refer to [Bra02] for an introduction to the notion of Γ-convergence. Since Γ-convergence implies convergence of minimizers, say with prescribed boundary conditions in a given domain, this result provides the first rigorous foundation for recovering minimal hypersurfaces in the limit as the scale parameter ε tends to zero. This framework has yielded alternative proofs of classical existence theorems also on Riemannian manifolds and has given rise to powerful min-max constructions in the diffuse setting, parallel to those developed for the area functional described above.

Indeed, using this approach together with some profound results on the regularity of stable solutions of the Allen-Cahn equation by Tonegawa-Wickramasekera [TW12], Guaraco in [Gua18] managed to prove the existence of a smooth embedded minimal hypersurface in ambient Riemannian manifolds, recovering a proof of Theorem 1.1.1. Guaraco's result is achieved by proving the existence of nontrivial critical points with Morse index at most one and a careful analysis of the limit as $\varepsilon \to 0$.

This Allen-Cahn program culminated in the celebrated result by Chodosh and Mantoulidis [CM20]. In this work, for $n = 3$, Chodosh and Mantoulidis establish the Multiplicity One and Index Lower Bound conjectures and are able to pass to the limit—for generic metrics—the infinitely many min-max critical points of the Allen-Cahn equation constructed in [GG18]. Among other things, the result in [CM20] implies the following.

Theorem 1.1.3 ([GG18; CM20])**.** *Let (M, g) be a closed 3-dimensional manifold with a generic metric. Then, for every positive integer $p \geq 1$ there exists in M a smooth, embedded minimal hypersurface Σ_p with $\operatorname{index}(\Sigma_p) = p$ and energy proportional (up to absolute constants) to $p^{1/3}$.*

This result provides an alternative proof of Yau's Conjecture (i.e., Theorem 1.1.2) for generic metrics, with extremely more precise information on the index and area growth of the constructed surfaces. We stress that this was the first "direct" (to say, constructive) proof of Yau's conjecture, since all the arguments in [Son23; MN17b; IMN18; GG18] are indirect and rely at some point on arguing by contradiction and supposing that there only exists finitely many mininal hypersurfaces.

Later, based on the work by Chodosh and Mantoulidis, Gaspar-Guaraco [GG19] proved the density of the separating limit surfaces for generic metrics in dimension 3, recovering the density and equidistribution of minimal hypersurfaces for generic metrics by Irie-Marques-Neves [IMN18].

1.1.2 The fractional perimeter and nonlocal interactions

Very recently, a new possible approach to the approximation of minimal surfaces appeared: that of nonlocal minimal surfaces. These objects are critical points (in a suitable sense) of the fractional perimeter. In the Euclidean space, for $s \in (0, 1)$, the fractional perimeter of a measurable set

$E \subset \mathbb{R}^n$ is defined as

$$\mathrm{Per}_s(E) := \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+s}}\, dxdy.$$

Since their first precise definition in the seminal work [CRS10] by Caffarelli, Roquejoffre, and Savin, much interest has been devoted to the study of nonlocal minimal surfaces and of fractional perimeters in general. Nonlocal minimal surfaces, at first sight less "natural" than other approximations of minimal surfaces, enjoy a long list of properties utterly analogous to those for classical minimal surfaces. We refer to [Dip20; Ser23] for a complete discussion of the similarities with the classical world.

Apart from a long list of similarities, the literature has recently pointed out a few striking differences with the world of classical minimal surfaces.

A first instance of these differences appeared in a work by Cinti-Serra-Valdinoci [CSV19]. In this paper, the authors prove that stable nonlocal minimal surfaces (say, in a ball of radius one) enjoy an interior uniform bound on their classical perimeter. This is in clear contrast with the classical case since many (arbitrarily close) parallel hyperplanes are a stable configuration for the perimeter with, clearly, no uniform bound on the area in any ball. This suggests that nonlocal minimal surfaces tend to become unstable when many sheets are close to each other. Concerning this heuristic, we describe a simple and instructive example in full detail in Proposition 2.2.14.

This feature has an analog for the fractional Allen-Cahn equation, as proved in [CCS21], where the authors prove that stable solutions of the fractional Allen-Cahn equation enjoy a uniform BV estimate.

Another remarkable result for the fractional Allen-Cahn equation with no parallel in the classical world is the improvement of flatness theorem proved by Dipierro-Serra-Valdinoci in [DSV20]. In this work, the authors show that entire solutions to the fractional Allen-Cahn equation with asymptotically flat level sets (say, in Hausdorff distance) are one-dimensional. This is in contrast with the classical Allen-Cahn equation since there exist solutions, constructed in [PKW13], concentrating on catenoids, which have arbitrary flat blow-downs in Hausdorff distance but are clearly not one-dimensional.

This feature represents a remarkable departure from the classical world and is in agreement with the fact that "nonlocal catenoids" have conical, nontrivial blow-downs. This means that the blow-down of the nonlocal catenoids is not a single plane with multiplicity two. These nonlocal catenoids were first constructed in [DPW18], where the authors proved that in $\mathbb{R}^3$ there exists a connected, embedded s-minimal surface of revolution whose blow-down is a nontrivial cone. Moreover, in the same work [DPW18] it is also proved that these nonlocal catenoids have infinite Morse index, in any reasonable sense.

These features suggest that sequences of s-minimal surfaces with uniformly bounded index should enjoy stronger compactness properties than classical minimal surfaces with bounded index. This turns out to be true. Indeed, in the recent work [CFS24b], which is part of this manuscript, the author, together with Florit-Simon and Serra, obtained a uniform BV estimate for finite index solutions of the fractional Allen-Cahn equation, extending the one of [CCS21] for stable solutions to the case of finite index. This result is Theorem 1.2.13 below.

These surprisingly strong estimates for nonlocal minimal surfaces of finite Morse index confer exceptional compactness and regularity properties to these objects. Thanks to these features and a classical min-max method, we establish a far-reaching existence result of infinitely many smooth nonlocal minimal surfaces in every n-dimensional closed Riemannian manifold for $n = 3, 4$. Actually, we establish the analogue of Yau's Conjecture (that is, Theorem 1.1.2) with the additional properties similar to Theorem 1.1.3 by Chodosh and Mantoulidis. Our result, which is the combination of Theorem 1.2.4 (existence) and Theorem 1.2.6 (regularity), holds for all metrics and not just generic ones. Our work suggests that s-minimal surfaces are an ideal class of objects on which to apply min-max methods, as they seem to prevent almost every pathology that arises for classical minimal surfaces (such as multiplicity and pinching).

From here, a natural question arises: can one take advantage of these exceptional compactness and regularity properties of s-minimal surfaces and send $s \to 1$ afterward to recover classical minimal surfaces? Surprisingly, the answer is affirmative.

To draw any conclusion about classical minimal surfaces using their fractional (nonlocal) counterparts, the first essential requirement is that the fractional perimeter converges to the classical perimeter as $s \to 1$. Indeed, the pointwise convergence for a fixed set is just a particular case of the so-called BBM formula (see [BBM01; Dáv02]). Moreover, it has also been shown in [ADM11] (see also [Pon04]) that the fractional perimeter converges to the classical perimeter also in the sense of Γ-convergence with respect to the L^1 convergence of sets (that corresponds to the flat convergence of their boundaries).

In the recent work [Cha+23], Chan, Dipierro, Serra, and Valdinoci push this convergence as $s \to 1$ to a much deeper level. In [Cha+23], the authors obtain robust curvature estimates and optimal sheet separation estimates, as $s \to 1$, for stable s-minimal surfaces in three dimensions. This result allows to send $s \to 1$ and pass to the limit *the supports* of the s-minimal surfaces to obtain standard minimal surfaces. Based on these ideas, Florit-Simon in [Flo24] was able to send $s \to 1$ for the surfaces we construct in Theorem 1.2.4 and recover a proof of the classical Yau's conjecture for generic metrics. Moreover, letting $s \to 1$, the author in [Flo24] recovers many other classical results in this field. For example, the author obtains a new proof (in dimension $n = 3$) for the Weyl law for the volume spectrum of [LMN18], and the existence, density, and equidistribution of (infinitely many) minimal surfaces of [IMN18; MNS19].

The result in [Flo24] leaves no doubt that the existence results for nonlocal minimal surfaces can be passed to the limit to recover those for classical minimal surfaces. This makes the approximation of minimal surfaces via their nonlocal counterparts an effective tool for approaching classical problems in the theory.

A glimpse of the codimension-two case

In codimension two, the goal of finding a diffuse approximation of the codimension two area that is well-behaved in the limit is an extremely challenging problem. For example, it is known that with the classical complex-valued Ginzburg-Landau model, one can produce nontrivial, stationary $(n-2)$-dimensional varifolds in ambient Riemannian manifolds. Nevertheless, it has been shown in [PS23] that, in general, these varifolds are not integral and that every density $\theta \in \{1\} \cup [2, \infty)$ on an $(n-2)$-plane can be realized as a limit of complex Ginzburg–Landau critical points.

In the groundbreaking work [PS21] by Pigati and Stern, the authors substitute the complex Ginzburg-Landau energy with the Yang-Mills-Higgs energy and can produce stationary *integral* $(n-2)$-varifolds in the limit. As a by-product, in codimension two, they obtain a new proof of Almgren's existence result of nontrivial, stationary integral $(n-2)$-dimensional varifolds in closed Riemannian manifolds. More recently, in [PPS24] it has been proved that the Yang-Mills-Higgs energy also Γ-converges to the classical area.

At this point, it is natural to ask if there is a natural codimension two analog of the fractional perimeter that reflects similar properties for stationary objects as the ones recently discovered for nonlocal minimal surfaces. To the author's knowledge, tentative notions of codimension ≥ 2 fractional masses appeared just in two very recent preprints: [Cic+24] for codimension $(n-1)$ and [MS23] in any codimension. In the same works, both notions are proved to converge pointwise, as $s \to 1^-$, to the standard Hausdorff measure of the correct dimension.

Recently, Serra in [Ser23, Section 5] suggested a notion of fractional s-mass for codimension

$k \in \{1, 2, \ldots, n-1\}$ smooth, multiplicity one submanifolds of $\mathbb{R}^n$ that are level sets of (regular values of) maps from $\mathbb{R}^n$ to $\mathbb{R}^k$. In our work [CFP24] jointly with Mattia Freguglia and Nicola Picenni, which is not technically part of this manuscript but is briefly described in Section 1.3.1, we study the case of codimension two in detail and with full generality. Following the idea in [Ser23, Section 5], in [CFP24] we introduce a notion of fractional s-mass for codimension two objects, and we prove that it is well defined for closed, oriented (not necessarily connected) codimension-two surfaces of locally constant multiplicity. Moreover, we prove its Γ-convergence with respect to the flat topology, as $s \to 1^-$, to the $(n-2)$-dimensional Hausdorff measure with multiplicity. Our study in [CFP24] is robust and suited to be extended naturally to ambient Riemannian manifolds and to every codimension. In particular, on Riemannian manifolds, our notion is well-defined for $(n-2)$-dimensional oriented boundaries with integer multiplicity, and this is the natural class to apply min-max methods on ambient Riemannian manifolds.

Compared to their local counterparts, we expect stationary sets for the fractional s-mass to exhibit enhanced regularity and compactness properties, much like the codimension-one scenario of the s-minimal surfaces described above.

1.1.3 The heat kernel on Riemannian manifolds

A central analytic ingredient in the study of nonlocal minimal surfaces on a Riemannian manifold M is the fractional Laplacian $(-\Delta)^{s/2}$ (that is, the operator which is a fractional power of the Laplace-Beltrami operator on M), whose definition—unlike its classical counterpart—relies in a delicate way on global properties of the manifold such as the heat kernel H_M or the spectrum of the Laplace-Beltrami operator. We refer the reader to the monographs [Cha84; Dav89; Ros97] for a classical introduction to the fractional Laplacian from the point of view of spectral theory.

To develop a fine theory in this context, one must simultaneously capture two distinct regimes: the short-time, local behavior, which should closely approximate the Euclidean heat kernel and ensures that the heat kernel is, infinitesimally, well-approximated by the Euclidean one; and the long-time, global behavior, which becomes essential on noncompact manifolds where geometry at infinity and volume growth can depart significantly from the flat setting.

The history of Gaussian upper bounds (and asymptotics) for the heat kernel $H_M(x, y, t)$ on manifolds is far from new. Here, with "Gaussian upper bounds," we mean estimates containing the factor

$$\exp\left(-\frac{d(x,y)^2}{Ct}\right),$$

where $d(x, y)$ is the geodesic distance between x and y. The first precise result for short times under general hypothesis is Varadhan's asymptotic formula in [Var67], which states that

$$-4t \log(H_M(x, y, t)) \to d(x, y)^2, \quad \text{as } t \to 0^+.$$

For positive times, Gaussian upper bounds were first established by Cheng-Li-Yau for manifolds of bounded geometry in [CLY81]. In this work, for every $\Lambda > 4$, the authors prove

$$H_M(x, y, t) \le C(x, \Lambda) t^{-n/2} \exp\left(-\frac{d(x,y)^2}{\Lambda t}\right), \quad \forall\, t \in [0, 1],$$

provided the sectional curvature of M is bounded between two constants. If one only assumes a Ricci curvature lower bound $\mathrm{Ric} \ge -K$, the Li-Yau parabolic Harnack inequality of [LY86] (see also [Li12, Chapter 13]) yields the same estimate and, in many cases, a matching lower bound. Lastly, Saloff-Coste showed that a volume-doubling condition together with a scale-invariant Poincaré inequality is both necessary and sufficient for two-sided Gaussian estimates, thus unifying and extending many preceding results [Sal95; Sal92]. A comprehensive discussion of such pointwise bounds—and of their interplay with volume growth, parabolic Harnack inequalities, and stochastic completeness—can be found in Grigor'yan's survey [Gri99b, Section 5].

In Chapter 3 we carry out a local analysis of the heat kernel for short times, both on closed and noncompact manifolds, somewhat of a different nature with respect to the classical results described above, since we do not assume any control on the manifold outside a fixed small ball. That is, our estimates are uniform with respect to the geometry of the manifold outside a fixed region. Once a geodesic ball on M (diffeomorphic to an Euclidean ball with quantitative control) is chosen, our estimates are uniform with respect to the geometry of M outside of this ball. Consequently, although our analysis is nonlocal, meaning that it always depends on the global geometry of M, even in its localized forms, the estimates we derive depend only on the local geometry and exhibit uniformity outside this chosen region. This type of control will be essential for carrying out blow-up procedures, where we will need to argue that the singular kernel of M related to the fractional Laplacian converges suitably to that of $\mathbb{R}^n$.

1.1.4 Noncompact manifolds: how the geometry at infinity affects the asymptotics of the fractional Laplacian as $s \to 0^+$

The study of the asymptotics as $s \to 0^+$ of the (relative) fractional perimeter

$$\mathrm{Per}_s(E,\Omega) := \frac{1}{2} \iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus\Omega^c\times\Omega^c} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dxdy \tag{1.1}$$

was initiated in [Dip+13], where the authors characterize the quantity $\lim_{s\to0^+} s\mathrm{Per}_s(E,\Omega)$, whenever it exists, for a bounded regular $\Omega \subset \mathbb{R}^n$. In the case of a bounded and smooth $E \subset \mathbb{R}^n$, the limit of the full (that is, (1.1) with $\Omega = \mathbb{R}^n$) fractional perimeter

$$\lim_{s\to0^+} s\mathrm{Per}_s(E) = \lim_{s\to0^+} s \iint_{E\times E^c} \frac{1}{|x-y|^{n+s}}\,dxdy$$

can be easily characterized using the Fourier transform and Plancherel's theorem. Indeed, up to a dimensional constant, by dominated convergence (see, for example, [DPV12, Proposition 3.4] or [MS02])

$$\lim_{s\to0^+} s\mathrm{Per}_s(E) = \lim_{s\to0^+} \int_{\mathbb{R}^n} |\xi|^s |\widehat{\chi_E}|^2\,d\xi = \int_{\mathbb{R}^n} |\widehat{\chi_E}|^2\,d\xi = |E|.$$

Nevertheless, since

$$\mathrm{Per}_s(E,\Omega) = \mathrm{Per}_s(E^c,\Omega),$$

the limit of the relative fractional perimeter cannot be just a volume but has to be a sort of volume invariant under complementation. For E unbounded (think, for example, of E being a cone), it turns out that this limit is affected by the tails at infinity of E. In [Dip+13], the authors address these problems in $\mathbb{R}^n$ and completely characterize the limit of the relative fractional perimeter.

Theorem 1.1.4 ([Dip+13])**.** *Suppose that* $\mathrm{Per}_{s_\circ}(E,\Omega) < +\infty$ *for some* $s_\circ \in (0,1)$*, and assume that the following limit exists*

$$\alpha(E) := \lim_{s\to0^+} s\int_{E\setminus B_1} \frac{1}{|y|^{n+s}}\,dy \in [0,\omega_{n-1}], \qquad \omega_{n-1} = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}).$$

Then, the limit $\lim_{s\to0^+} s\mathrm{Per}_s(E,\Omega)$ *exists and*

$$\lim_{s\to0^+} s\mathrm{Per}_s(E,\Omega) = (\omega_{n-1}-\alpha(E))|E\cap\Omega| + \alpha(E)|E^c\cap\Omega|. \tag{1.2}$$

Some years later, in [Car+22], the authors characterized this limit for the fractional Gaussian perimeter in the Gaussian space $(\mathbb{R}^n, (2\pi)^{-n/2}e^{-|x|^2/2}dx)$. The Gaussian space is a positively curved (with respect to the Bakry-Émery Ricci tensor) weighted manifold that enjoys the curvature-dimension condition $\mathrm{CD}(1,\infty)$. We refer the reader to [BGL14, Chapter 2] for a detailed introduction to this space.

Theorem 1.1.5 ([Car+22])**.** *Let $\Omega \subset \mathbb{R}^n$ be a regular domain and*

$$\mathrm{Per}_s^{\gamma}(E,\Omega) := \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus\Omega^c\times\Omega^c} |\chi_E(x)-\chi_E(y)|\mathcal{K}_s(x,y)\,d\gamma(x)d\gamma(y),$$

be the fractional Gaussian perimeter, where $\mathcal{K}_s(x,y)$ is defined as in (1.4) with on the right-hand side the heat kernel H_γ of the Gaussian space and

$$d\gamma(x) = \frac{1}{(2\pi)^{n/2}}e^{\frac{-|x|^2}{2}}dx.$$

Suppose that $\mathrm{Per}_{s_\circ}^{\gamma}(E,\Omega) < +\infty$ for some $s_\circ \in (0,1)$. Then

$$\lim_{s\to 0^+}\frac{1}{2}\mathrm{Per}_s^{\gamma}(E,\Omega) = \gamma(E)\gamma(E^c\cap\Omega) + \gamma(E\cap\Omega)\gamma(E^c\cap\Omega^c), \tag{1.3}$$

Remark 1.1.6. *Even though it seems that we have normalized the fractional perimeter differently in the left-hand sides of (1.2) and (1.3), this is only due to the normalization constant $s/2$ that we have put in front of the kernel (1.4) in the definition of the fractional Gaussian perimeter.*

Interestingly, the limit in the case of the Gaussian space has a different form with respect to the case of $\mathbb{R}^n$. Both the proofs in [Dip+13] and [Car+22] heavily use the fact that, in $\mathbb{R}^n$ and on the Gaussian space, respectively, the explicit form of the heat kernel is known, and a direct computation of the limit can be carried out.

In Chapter 6, which describes the results we have obtained in [CG24] jointly with Luca Gennaioli, we extend both these results by providing a complete characterization of this limit for essentially every Riemannian and weighted manifold. Our analysis reveals that the behaviors observed on $\mathbb{R}^n$ and in the Gaussian space represent the only two possible asymptotic behaviors. Indeed, the form of the limit depends only on whether the manifold (or the weighted manifold) has infinite or finite volume.

Furthermore, our results establish a connection between the limit of the relative fractional perimeter and the existence of bounded harmonic functions on the manifold. In fact, the asymptotic behavior of the fractional perimeter (and of the fractional Laplacian) turns out to be naturally related to the geometric properties of the manifold at infinity.

1.2 Main results

The remainder of this introduction is devoted to a detailed description of the main results of the thesis, some of which have been obtained by the author with colleagues and friends, to whom he is very grateful.

1.2.1 Fractional Sobolev spaces on Riemannian manifolds

In this section, we describe the results that we have obtained in [CFS24a], jointly written with Enric Florit-Simon and Joaquim Serra, which will be proved in Chapter 3.

In recent years, there has been significant development in the theory of nonlocal equations. The simplest example of a nonlocal operator on $\mathbb{R}^n$ is the fractional Laplacian

$$(-\Delta)^\sigma u(x) = P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2\sigma}}\, dy,$$

where $\sigma \in (0,1)$ and $u : \mathbb{R}^n \to \mathbb{R}$. Formally, this corresponds to the σ-th power of the usual Laplacian, and it is, therefore, an operator of order (of differentiation) 2σ. Another way to look at it is as the operator arising from the Euler-Lagrange equation of the functional

$$[v]^2_{H^\sigma(\mathbb{R}^n)} = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\, dxdy,$$

which involves a fractional Sobolev energy term. There are precise multiplicative constants that one should put in front of these objects, which will be given later, but we will omit them in this introduction for the sake of exposition.

In Chapter 3 we address how the fractional Sobolev space $H^\sigma(M) = W^{\sigma,2}(M)$ and the associated fractional Laplacian on M have a natural, canonical interpretation in the case where M is a closed Riemannian manifold. We provide several definitions for these objects and show them to be identical, which justifies their canonical nature. Moreover, we obtain fundamental properties for these objects thanks to a deeper study of their different definitions.

Let (M,g) be an n-dimensional, closed Riemannian manifold, with $n \geq 2$. For convenience and consistency throughout the work, we put $\sigma = s/2$, where $s \in (0,2)$. Let us start by giving the definition of the fractional Sobolev seminorm $H^{s/2}(M)$. The $H^{s/2}(M)$ seminorm can be defined in at least three equivalent ways:

(i) Using the *heat kernel* $H_M(p,q,t)$ of M, we can put

$$\mathcal{K}_s(p,q) := \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty H_M(p,q,t)\, \frac{dt}{t^{1+s/2}}, \tag{1.4}$$

and then define

$$[u]^2_{H^{s/2}(M)} := \iint_{M\times M} (u(p)-u(q))^2 \mathcal{K}_s(p,q)\, dV_p dV_q.$$

(ii) Following a *spectral approach*, we can set

$$[u]^2_{H^{s/2}(M)} = \sum_{k\geq 1} \lambda_k^{s/2} \langle u, \phi_k \rangle^2_{L^2(M)},$$

where $\{\phi_k\}_k$ is an orthonormal basis of eigenfunctions of the Laplace-Beltrami operator $(-\Delta_g)$ and $\{\lambda_k\}_k$ are the corresponding eigenvalues. For $s = 2$ this immediately recovers the usual $[u]^2_{H^1(M)}$ seminorm.

(iii) Considering a *Caffarelli-Silvestre type extension* (cf. [CS07; BGS15]). Namely, a degenerate-harmonic extension problem in one extra dimension, we can set

$$[u]^2_{H^{s/2}(M)} = \inf\left\{ \iint_{M\times\mathbb{R}_+} z^{1-s} |\widetilde{\nabla} U(p,z)|^2\, dV_p dz \ \text{ s.t. } \ U(x,0) = u(x) \right\}.$$

Here $\widetilde{\nabla}$ denotes the Riemannian gradient of the manifold $\widetilde{M} = M \times (0,\infty)$, with respect to the natural product metric $\widetilde{g} = g + dz \otimes dz$, and the infimum is taken over all U belonging to a suitable weighted Hilbert space $\widetilde{H}^1(\widetilde{M})$ (see Definition 3.2.3 for more details).

One of our main results is the following.

Theorem 1.2.1. *The definitions $(i), (ii), (iii)$ above coincide. To say: one of the seminorms is finite if and only if the other two are finite, and in this case, their values coincide.*

Definition (i) will allow us to control precisely the behaviour of the fractional Sobolev energy. See, for example, Lemma 3.4.17 and Corollary 3.4.18, which show that the fractional Sobolev energy is smooth with quantitative bounds under inner variations. For that, we will give precise quantitative estimates for the kernel $\mathcal{K}_s(p, q)$ (defined in (1.4)) and its derivatives, depending only on local quantities. In particular, we will show that it is comparable to $d(p, q)^{-(n+s)}$ if p and q are contained in a Riemannian ball with controlled geometry. We recall that $\mathcal{K}_s(p, q)$ reduces to $|x - y|^{-(n+s)}$ in the case $M = \mathbb{R}^n$ (up to a constant factor).

The estimates for $\mathcal{K}_s$ will follow from the corresponding estimates for the heat kernel H_M. Although somewhat standard in flavor, they are hard to find in the literature with this level of precision and generality, and we give an almost entirely self-contained account that we believe to be of independent interest. Moreover, our local estimates on the heat kernel hold for a general complete Riemannian manifold, without any control on the geometry of the manifold at infinity.

However, for reasons that will be clear in Chapter 3, these precise results are very technical and have many slightly different hypotheses. For this reason, the author believes that it would be pointless to list here the exact statements of all estimates on H_M and $\mathcal{K}_s$. In the following table, we give an overview of the estimates for the heat kernel H_M and the singular kernel $\mathcal{K}_s$ that we have obtained in [CFS24a] and are described in Chapter 3. In particular, the reader is advised to consult Theorem 3.4.6, which records several of the main results for $\mathcal{K}_s$, including an explicit asymptotic expansion for short distances.

	Heat kernel H_M	Singular kernel $\mathcal{K}_s$
Global comparability with $(\mathbb{R}^n, g)$ with a general metric	Lemma 3.4.7	Lemma 3.4.7
Short distance comparability	Lemma 3.4.8, Lemma 3.4.11	Lemma 3.4.13
Long distance estimates	Lemma 3.4.9, Lemma 3.4.10	Theorem 3.4.6, Lemma 3.4.14
Precise asymptotics	Proposition 3.4.12	Theorem 3.4.6

The extension and the monotonicity formula

The extension definition (iii) will be used to give a monotonicity formula for stationary points u of semilinear elliptic functionals, that is, of functionals of the form

$$\mathcal{E}(v) = [v]^2_{H^{s/2}(M)} + \int_M F(v)\, dV, \tag{1.5}$$

under the assumption that $F \geq 0$. More precisely, u needs only to be stationary for $\mathcal{E}(v)$ under inner variations; in particular, setting $F \equiv 0$ will give a monotonicity formula for nonlocal s-minimal surfaces, which we will define in a moment. Up to now, the result was known on $\mathbb{R}^n$ by [CRS10], [CC14], and [MSW19].

The general monotonicity formula we obtain is the following (see Section 3.2.1 for the precise definitions and notation).

Theorem 1.2.2 (Monotonicity formula)**.** *Let M be an n-dimensional, closed Riemannian manifold. Let $s \in (0,2)$ and $\mathcal{E}$ be as in (1.5), where F is any smooth nonnegative function. Let $u : M \to \mathbb{R}$ be stationary for $\mathcal{E}$ under inner variations, meaning that $\mathcal{E}(u) < \infty$ and for any smooth vector field X on M there holds $\frac{d}{dt}\big|_{t=0}\mathcal{E}(u \circ \psi_X^t) = 0$, where ψ_X^t is the flow of X at time t. For $(p_\circ, 0) \in \widetilde{M}$ and $R > 0$ define*

$$\Phi(R) := \frac{1}{R^{n-s}}\left(2\beta_s \int_{\widetilde{B}^+_R(p_\circ,0)} z^{1-s}|\widetilde{\nabla} U|^2 \, dV dz + \int_{B_R(p_\circ)} F(u)\, dV\right),$$

where U is the unique solution given by Theorem 3.2.4. Then, there exist constants $C = C(n)$ and $R_{\max} = R_{\max}(M, p_\circ) > 0$ with the following property: whenever $R_\circ \le R_{\max}$ and K is an upper bound for all the sectional curvatures of M in $B_{R_\circ}(p_\circ)$, then

$$R \mapsto \Phi(R)e^{C\sqrt{K}R} \quad \text{is non-decreasing for } \; R < R_\circ\,,$$

and the inequality

$$\Phi'(R) \ge -C\sqrt{K}\Phi(R) + \frac{s}{R^{n-s+1}}\int_{B_R(p_\circ)} F(u)\, dV + \frac{2\beta_s}{R^{n-s}}\int_{\partial^+\widetilde{B}^+_R(p_\circ,0)} z^{1-s}\langle \widetilde{\nabla} U, \widetilde{\nabla} d\rangle^2 \, d\widetilde{\sigma}$$

holds for all $R < R_\circ$, with $d(\cdot) = d_{\widetilde{g}}((p_\circ, 0), \cdot)$ the distance function on $\widetilde{M}$.
Moreover, in the particular case where $M = \mathbb{R}^n$, $F \equiv 0$, $s \in (0,1)$, and $u = \chi_E - \chi_{E^c}$ is a stationary set for the fractional s-perimeter, there holds

$$\Phi'(R) = \frac{2\beta_s}{R^{n-s}}\int_{\partial^+\widetilde{B}^+_R(p_\circ,0)} z^{1-s}\langle \widetilde{\nabla} U, \widetilde{\nabla} d\rangle^2 \, dx dz \ge 0\,,$$

which shows that Φ is nondecreasing and that it is constant if and only if E is a cone.

Remark 1.2.3. *It will follow from the proof that the radius $R_{\max}$ in Theorem 1.2.2 can be taken to be $R_{\max} = \mathrm{inj}_M(p_\circ)/4$. Moreover, since M is compact $R_{\max}$ is uniformly bounded below as $R_{\max}(M, p_\circ) \ge \mathrm{inj}_M/4$, for all $p_\circ \in M$.*

1.2.2 Yau's conjecture for nonlocal minimal surfaces

In this section, we describe the results that we have obtained in [CFS24b], jointly written with Enric Florit-Simon and Joaquim Serra, which will be proved in detail in Chapter 4.

In [CFS24b] we introduce nonlocal minimal (hyper)surfaces—in the spirit of Caffarelli, Roquejoffre and Savin [CRS10]—on closed Riemannian manifolds and we develop their existence and regularity theory. A main purpose of the work [CFS24b] is to prove that nonlocal minimal surfaces are an ideal class of objects on which to apply min-max methods (as they seem to prevent almost every pathology that arises for classical minimal surfaces, such as multiplicity and loss of topology), as well as to approximate classical minimal surfaces.

On this second point, let us emphasize that nonlocal minimal surfaces approximate classical minimal ones as the fractional parameter $s \in (0,1)$ converges to 1. The recent results in [Cha+23] show, among other things, uniform curvature estimates and optimal sheet separation (of order

$\sqrt{1-s}$) for stable nonlocal s-minimal surfaces in a three-dimensional Euclidean setting as $s \to 1$, which implies their multisheeted convergence towards smooth classical minimal surfaces. Based on the ideas in [Cha+23], recently Florit-Simon in [Flo24] was able to send $s \to 1$ for the surfaces we construct in [CFS24b] and recover the classical Yau conjecture for generic metrics. Moreover, sending $s \to 1$, the author in [Flo24] recovers many other classical results in this field. For example, the author obtains a new proof (in dimension $n = 3$) for the Weyl law for the volume spectrum of [LMN18], and the existence, density, and equidistribution of (infinitely many) minimal surfaces of [IMN18; MNS19].

This leaves no doubt that existence results for fractional minimal surfaces can be passed to the limit to recover the ones for classical minimal surfaces. This method resembles in some ways the Allen-Cahn approximation in [Gua18; GG18; GG19; CM20], but presents several advantages (some of which are discussed in [Cha+23], and some of which will become evident in this work).

We obtain surprisingly strong estimates for finite Morse index nonlocal minimal surfaces that do not hold for classical minimal surfaces. These estimates confer finite Morse index nonlocal minimal surfaces exceptional compactness and regularity properties, thanks to which we establish far-reaching existence and regularity results, including a nonlocal analog of Yau's conjecture on the existence of infinitely many minimal (geometrically distinct) minimal surfaces on three-manifolds. Let us give a quick selection/highlights of our results here:

(i) Any closed manifold of dimension $n \geq 3$ contains infinitely many nonlocal minimal (hyper)surfaces (i.e., the nonlocal analog of Yau's conjecture holds). More precisely, given $s \in (0,1)$, for every $\mathfrak{p} \in \mathbb{N}$ there exists an s-minimal surface with Morse index $\leq \mathfrak{p}$ and fractional perimeter comparable to $\mathfrak{p}^{s/n}$. These surfaces are smooth in low dimensions and smooth away from a closed lower-dimensional set in every dimension. We stress that our result holds for *every* metric and not just generic ones.

(ii) For $n \in \{3,4\}$ and $s \in (0,1)$ sufficiently close to 1 (the limit case $s = 1$ formally corresponds to classical minimal surfaces), the following holds:

- Any smooth (embedded) s-minimal hypersurface of finite Morse index in $\mathbb{R}^n$ must be a hyperplane.
- In a closed n-dimensional manifold M^n, any sequence of smooth s-minimal surfaces with uniformly bounded Morse index automatically satisfies uniform curvature and sheet separation estimates. As a consequence, any such sequence has a subsequence that converges smoothly and with multiplicity one to a (smooth) submanifold. In particular, if all the elements of the sequence are homeomorphic to the same topological space $\mathbb{X}$ then the limit is also homeomorphic to $\mathbb{X}$.

Thanks to their exceptional compactness and regularity properties, Morse theory for nonlocal minimal surfaces is in some sense as "flawless" as finite-dimensional Morse theory, at least from the functional analysis (i.e., compactness) perspective. It goes without saying that this is in remarkable contrast to the situation for classical minimal surfaces with respect to the area functional.

Since we believe that our work may be of interest to readers without prior knowledge of nonlocal elliptic equations, we aim to provide an accessible and largely self-contained presentation in Chapter 4. Moreover, we have spared no effort in making our proofs as efficient as possible.

One of our main results establishes the existence of infinitely many s-minimal surfaces on every closed manifold.

Theorem 1.2.4 (Fractional Yau-type result)**.** *Let (M^n, g) be an n-dimensional, closed Riemannian manifold, with $n \geq 2$. Fix $s_0 \in (0,1)$ and let $s \in (s_0, 1)$. Then, for every integer $\mathfrak{p} \geq 1$, there exists an s-minimal surface $\Sigma^{\mathfrak{p}} = \partial E^{\mathfrak{p}}$ with Morse index at most $\mathfrak{p}$ and fractional perimeter*

$$C^{-1}\mathfrak{p}^{s/n} \leq (1-s)\mathrm{Per}_s(E^{\mathfrak{p}}) \leq C\mathfrak{p}^{s/n},$$

for some $C = C(M, s_0) > 1$. In particular, M contains infinitely many s-minimal surfaces. Moreover, these surfaces are viscosity solutions to the NMS (i.e., Nonlocal Minimal Surface) equation (see Proposition 4.3.1), and satisfy the structural properties (1.7)-(1.8) *in Proposition 1.2.17.*

The regularity of the constructed surfaces depends on the classification of stable s-minimal cones (an open subset $E \subset \mathbb{R}^n$ is said to be a cone if E is an open set and $\lambda E = E$ for all $\lambda > 0$).

Definition 1.2.5. *Given $s \in (0,1)$, we define the critical dimension n_s^* as the minimum dimension $n \geq 3$ such that there exists a smooth and stable s-minimal cone in $\mathbb{R}^n \setminus \{0\}$ which is not a hyperplane.*

By [CCS20] and [Cha+23], $n_s^* \geq 5$ for all $s \in (s_0, 1]$, where $s_0 \in (0,1)$ is a universal constant. It is conjectured that, in fact, $n_s^* = 8$ for all s sufficiently close to 1. For $n = 8$, the Simons cone $E = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 < x_5^2 + x_6^2 + x_7^2 + x_8^2\} \subset \mathbb{R}^8$, which is a minimizer in the classical case $s = 1$, is easily shown to be stable for all $s \in (s_0, 1)$, for some $s_0 < 1$ sufficiently close to 1, so that $n_s^* \leq 8$ in this case[1].

We now state a regularity result for these surfaces, which will be proved in Section 4.5.

Theorem 1.2.6 (Size of the singular set)**.** *For $n \geq 3$, the surfaces $\{\Sigma^{\mathfrak{p}}\}_{\mathfrak{p} \in \mathbb{N}}$ of Theorem 1.2.4 are smooth submanifolds outside of a closed set* $\mathrm{sing}(\Sigma^{\mathfrak{p}})$ *of Hausdorff dimension at most $n - n_s^*$. In particular,* $\mathrm{sing}(\Sigma^{\mathfrak{p}}) = \varnothing$ *if $n < n_s^*$ (and this holds for $n = 3, 4$ and s close to 1, since $n_s^* \geq 5$). Moreover, in the case $n = n_s^*$ the set* $\mathrm{sing}(\Sigma^{\mathfrak{p}})$ *is discrete.*

The surfaces in Theorem 1.2.4 will be constructed as limits as $\varepsilon \to 0^+$ of solutions to the fractional Allen-Cahn equation on M. We emphasize that—in sharp contrast to the case of classical minimal surfaces—the Allen-Cahn approximation does not really play a crucial role in our construction. We use it so that we are able to apply standard min-max existence results of critical points (like those in the book by Ghoussoub [Gho93]). What really makes our construction easier, in comparison with the classical case $s = 1$, are the very strong a priori estimates satisfied by finite Morse index s-minimal surfaces for $s < 1$; see Remark 1.2.10. The corresponding analog estimates are satisfied by Allen-Cahn solutions with bounded index, which allows us to send $\varepsilon \to 0$ without difficulty. In contrast, in the classical case, this passage to the limit is really delicate: one is forced to use varifold convergence, and then multiplicity and neck-pinching situations need to be ruled out. This requires generic metric assumption and has only been done for $n = 3$ in [CM20].

For $3 \leq n < n_s^*$, we prove a strong regularity and separation result for s-minimal surfaces, which are limits of Allen-Cahn solutions with bounded index (as in our case), and which will be proved in Section 4.5.4.

Definition 1.2.7 (Family of Allen-Cahn limits)**.** *A surface $\Sigma \subset M$ is said to belong to the class $\mathcal{A}_m(M)$ if $\Sigma = \partial E$ and there exists a sequence of functions $u_j : M \to (-1, 1)$ which are solutions*

[1] More generally, the natural generalization of the Simons cone to higher (even) n has been shown in [FS20a] to be stable in dimension $n \geq 14$ for any $s \in (0,1)$. In particular, $n_s^* \leq 14$ for any $s \in (0,1)$, and the definition of n_s^* as a minimum is justified.

to the Allen-Cahn equation (2.7) on M, with Morse index $m(u_j) \le m$ for all j, and parameters $\varepsilon_j \to 0$, such that $u_j \to u_0 := \chi_E - \chi_{E^c}$ in $L^1(M)$.

Theorem 1.2.8 (Uniform regularity and separation)**.** *Let $s \in (0,1)$ and $3 \le n < n_s^*$. Let (M^n, g) be an n-dimensional, closed Riemannian manifold satisfying the flatness assumption $\mathrm{FA}_3(M, g, p, 1, \varphi)$ around p (see Definition 3.4.1). Assume that $\partial E \in \mathcal{A}_m(M)$.*
Then ∂E is a $C^{1,\alpha}$ hypersurface for some $\alpha \in (0,1)$, with uniform regularity and separation estimates around p. That is, there exists a radius $R = R(n,s,m) > 0$ such that, after a rotation, $\varphi^{-1}(\partial E) \cap \big(\mathcal{B}_R^{n-1}(0) \times [-R, R]\big)$ is the graph of a ***single*** *function $f : \mathcal{B}_R^{n-1}(0) \times \{0\} \to [-R, R]$ inside the chart, and*

$$\|f\|_{C^{1,\alpha}(B_R^{n-1} \times \{0\})} \le C(n,s,m)\,.$$

As an immediate application of Theorem (1.2.8) and Arzelà-Ascoli Theorem we obtain:

Corollary 1.2.9. *Let (M^n, g) be a closed Riemannian manifold, and let $s \in (0,1)$ and $3 \le n < n_s^*$. Then, every sequence $\Sigma_k = \partial E_k \in \mathcal{A}_m(M)$ admits a subsequence converging in C^1 to some Σ_∞. In particular, if all elements Σ_k of the sequence are homeomorphic to the same topological space $\mathbb{X}$, then the limit Σ_∞ is also homeomorphic to $\mathbb{X}$.*

We now make an important remark.

Remark 1.2.10. *Define the class $\mathcal{A}'_m(M)$ consisting of surfaces $\Sigma = \partial E \subset M$ such that there exists a sequence of s-minimal surfaces $\Sigma_j = \partial E_j$ of class C^2, with Morse index at most m for all j, such that $E_j \to E$ in $L^1(M)$. Then, the result of Theorem 1.2.8 would also hold for surfaces in $\mathcal{A}'_m(M)$ (and in particular for surfaces which are a priori known to be C^2), with a similar proof but with several technical modifications.*

We conclude this section with a technical remark about dimension $n = 2$, which can be skipped on a first reading.

Remark 1.2.11. *In dimension $n = 2$, the analogous regularity result to $n \in \{3,4\}$ does not hold in general, and the surfaces of Theorem 1.2.4 can be singular. Indeed, for $n = 2$, the tangent cones to the sets in $\mathcal{A}_m(M)$ can be nontrivial, and this cannot be ruled out. For example (compare with Definition 1.2.5 of the critical dimension), for $n = 2$ the cross $\{xy > 0\} \subset \mathbb{R}^2$ is an s-minimal surface in the plane that is expected to be stable in $\mathbb{R}^2 \setminus \{0\}$ for s close to 1. This is in accordance with the classical case of the area, as min-max solutions to the Allen-Cahn equation on surfaces produce (in general) smooth immersed geodesics; see the introduction of [MSS24] for a discussion on the singularities developed by min-max methods on surfaces.*

However, for $n = 2$ and s close to 0, it follows from our Theorem 1.2.20 that nonlocal phenomena prevent the existence of such nontrivial tangent cones that are stable in $\mathbb{R}^2 \setminus \{0\}$. In particular, for $n = 2$ and s close to 0, this implies that the s-minimal surfaces given by Theorem 1.2.4 are smooth and are a finite union of closed embedded curves. We refer to Section 1.2.4 for more details.

In the range s close to 1 and $n = 2$, the same regularity result to $n \in \{3,4\}$ would hold for limits of stable solutions of the fractional Allen-Cahn equation (or for stable s-minimal surfaces that are of class C^2). Instead of arguing using the classification of stable cones smooth outside the origin, which is not true for $n = 2$ as we discussed above, one would argue that Lemma 4.5.16 is also true in the case when $n = 2$ and $E \subset \mathbb{R}^2$ is a cone which is the limit of stable Allen-Cahn solutions. Indeed, for $n \ge 3$ the extra dimensions in the proof of Lemma 4.5.16 are essentially *used to reduce to the case of almost stability (see Definition 4.2.1) in a ball centered on the spine of the cone. For $n = 2$, if we assume stability in the first place, the proof still goes through.*

Existence of min-max solutions to Allen-Cahn

In Section 4.1.1, we exhibit in a simple manner the existence of critical points of the Allen-Cahn energy (2.6) on M, employing a min-max theorem as in [GG18]. Then, we prove lower and upper bounds for the energies of the constructed solutions. The complete statement of our result is the following.

Theorem 1.2.12 (Existence of min-max Allen-Cahn solutions)**.** *Let (M^n, g) be an n-dimensional, closed Riemannian manifold, and fix $s_0 \in (0, 1)$. Let $\mathfrak{p} \geq 1$ be a natural number (the number of min-max parameters) and $s \in (s_0, 1)$. Then, there exists $\varepsilon_{\mathfrak{p}} > 0$ (depending also on M and s) such that for all $\varepsilon \in (0, \varepsilon_{\mathfrak{p}})$, there exists a solution $u_{\varepsilon,\mathfrak{p}}$ to the Allen-Cahn equation (2.7) on M with Morse index $m(u_{\varepsilon,\mathfrak{p}}) \leq \mathfrak{p}$. Moreover, there exists $C > 1$ depending only on M and s_0 such that*

$$C^{-1}\mathfrak{p}^{s/n} \leq (1-s)\,\mathcal{E}^{\varepsilon,s}_M(u_{\varepsilon,\mathfrak{p}}) \leq C\mathfrak{p}^{s/n}\,. \tag{1.6}$$

After proving this result, our main goal will be to show that, for fixed $\mathfrak{p}$, as $\varepsilon \to 0$ a subsequence of the $u_{\varepsilon,\mathfrak{p}}$ converges in a strong sense to a fractional minimal surface $\Sigma^{\mathfrak{p}} = \partial E^{\mathfrak{p}} \subset M$, meaning in particular that

$$\mathrm{Per}_s(E^{\mathfrak{p}}) = \lim_{\varepsilon\to 0} \mathcal{E}^{\varepsilon,s}_M(u_{\varepsilon,\mathfrak{p}})\,.$$

Together with the bound given by (1.6), we get for every $\mathfrak{p} \in \mathbb{N}$ a fractional minimal surface $\Sigma^{\mathfrak{p}} = \partial E^{\mathfrak{p}}$ with fractional perimeter $\mathrm{Per}_s(E^{\mathfrak{p}}) \sim \mathfrak{p}^{s/n}$. This perimeter growth shows that the family of surfaces $\{\Sigma^{\mathfrak{p}}\}_{\mathfrak{p}\in\mathbb{N}}$ necessarily forms an infinite set, thus proving the fractional Yau's conjecture.

For this reason, a large part of Chapter 4 is devoted to studying the properties of solutions to the Allen-Cahn equation with a uniform upper bound on their Morse index.

Estimates for finite Morse index solutions to Allen-Cahn

In Section 4.2, we prove several estimates for finite Morse index solutions to the Allen-Cahn equation. In order to quantify the dependence of the constants in the estimates on the geometry of the ambient manifold precisely, the notion of "local flatness assumption" will be very useful (this quantification will be important when we perform blow-up arguments). We will precisely introduce this notion in Definition 3.4.1.

Here, as in the rest of the thesis, $\mathcal{B}_R(0)$ denotes the Euclidean ball of radius R centered at $0 \in \mathbb{R}^n$, and $B_R(p)$ denotes the metric ball on M of radius R and center $p \in M$.

One of the main results of this thesis is the following estimate, to be proved in Section 4.2.3.

Theorem 1.2.13 (BV estimate)**.** *Let M be a closed n-dimensional Riemannian manifold for which $\mathrm{FA}_2(M, g, R, p, \varphi)$ holds—see Definition 3.4.1. Let $s \in (0,1)$ and $u : B_R(p) \to (-1,1)$ be a solution of the Allen-Cahn equation (2.7) in $B_R(p) \subset M$ with parameter ε, and with Morse index $m_{B_R(p)}(u) \le m$. Then*

$$\int_{B_{R/2}(p)} |\nabla u| dx \le C R^{n-1},$$

for some $C = C(n, s, m) > 0$ independent of ε.

Remark 1.2.14. *Our proof of Theorem 1.2.13 gives a control on the behavior of the constant $C(n,s,m)$ as $s \uparrow 1$. More precisely, for fixed $s_\circ \in (0,1)$ we have $C(n,s,m) \le C(n,s_\circ,m)/(1-s)$ for all $s \in (s_\circ, 1)$. In view of the results from [Cha+23], the sharp asymptotic for s close to 1 is expected to be $C(n,s,m) \le C(n,s_\circ,m)/(1-s)^{1/2}$. See also the example in Section 2.2.3.*

Another important result is a bound on the Sobolev and potential parts of the energies, obtained in Section 4.2.4.

Theorem 1.2.15 (Energy estimate)**.** *Let $u : M \to (-1,1)$ be a solution of (2.7) in $B_R(p) \subset M$ with parameter ε and Morse index $m_{B_R(p)}(u) \le m$. Suppose that $\mathrm{FA}_2(M, g, R, p, \varphi)$ holds —see Definition 3.4.1. Then*

$$\mathcal{E}^{\mathrm{Sob}}_{B_{R/2}(p)}(u) \le C R^{n-s},$$

and there exists $\varepsilon_0 = \varepsilon_0(n,s,m)$ such that for $\varepsilon < \varepsilon_0$

$$\mathcal{E}^{\mathrm{Pot}}_{B_{R/2}(p)}(u) \le C \left(\frac{\varepsilon}{R}\right)^{\beta} R^{n-s},$$

where $C = C(n,s,m)$ and $\beta := \min\left(\frac{1-s}{2}, s\right) > 0$.

In Section 4.2.4 we prove the following result, which will give, among other things, that the level sets of Allen-Cahn solutions converge to the limit (hyper)surfaces in the Hausdorff distance of sets.

Proposition 1.2.16 (Density estimates)**.** *Let $u : M \to (-1,1)$ be a solution of (2.7) in $B_R(p) \subset M$ with Morse index $m_{B_R(p)}(u) \le m$, and suppose that $\mathrm{FA}_2(M, g, R, p, \varphi)$ holds —see Definition 3.4.1. Then, there exist positive constants ω_0, C_0 and ε_0, depending only on n, s, and m, such that the following holds: whenever $\varepsilon \le \varepsilon_0$, $R \ge C_0 \varepsilon$ and*

$$R^{-n} \int_{B_R(p)} |1 + u_\varepsilon| \le \omega_0 \qquad \left(\text{respectively,} \quad R^{-n} \int_{B_R(p)} |1 - u_\varepsilon| \le \omega_0\right),$$

then

$$\left\{u_\varepsilon \ge -\tfrac{9}{10}\right\} \cap B_{R/2}(p) = \varnothing \qquad \left(\text{respectively,} \quad \left\{u_\varepsilon \le \tfrac{9}{10}\right\} \cap B_{R/2}(p) = \varnothing\right).$$

Convergence results

In Section 4.3, the estimates we have just stated are used to show the convergence, as $\varepsilon \to 0$, of solutions of (2.7) to a limit interface. The precise statement of our convergence result is the following.

Theorem 1.2.17. *(Convergence to interface as $\varepsilon \to 0$). Fix $s \in (0,1)$. Let u_{ε_j} be a sequence of solutions of (2.7) on M with parameters $\varepsilon_j \to 0$ and Morse index $m(u_{\varepsilon_j}) \le m$. Then, there exist a subsequence, still denoted by u_{ε_j}, and a nonlocal s-minimal surface $\Sigma = \partial E$ with Morse index at most m, such that*

$$u_{\varepsilon_j} \xrightarrow{H^{s/2}} u_0 = \chi_E - \chi_{E^c}.$$

In particular $\mathcal{E}_M^{\rm Sob}(u_{\varepsilon_j}) \to {\rm Per}_s(E) = \mathcal{E}_M^{\rm Sob}(u_0)$ and $\mathcal{E}_M^{\rm Pot}(u_{\varepsilon_j}) \to 0 = \mathcal{E}_M^{\rm Pot}(u_0)$.
In addition, up to changing E on a set of measure zero, we have

$$\begin{aligned} {\rm int}(E) &\supseteq \Big\{p \in M : \liminf_{r\downarrow 0} \tfrac{|E\cap B_r(p)|}{|B_r(p)|} = 1\Big\}, \qquad (1.7)\\ M \setminus \overline{E} &\supseteq \Big\{p \in M : \limsup_{r\downarrow 0} \tfrac{|E\cap B_r(p)|}{|B_r(p)|} = 0\Big\},\\ \Sigma &= \Big\{p \in M : \tfrac{|E\cap B_r(p)|}{|B_r(p)|} \in [\delta, 1-\delta] \quad \forall\, r \in (0, r_p), \ \ \text{for some } r_p > 0\Big\}, \qquad (1.8) \end{aligned}$$

where $\delta = \delta(n,s,m) \ll 1$ and $\Sigma = \partial E$ represents the topological boundary of E. Moreover, for all given $c \in (-1,1)$

$$d_{\rm H}(\{u_{\varepsilon_j} \ge c\}, E) \to 0\,, \quad \text{as } j \to \infty,$$

where $d_{\rm H}$ denotes the standard Hausdorff distance between subsets of M.

As explained in Section 1.2.2, this result combined with Theorem 1.2.12 will give Theorem 1.2.4.

Regularity in low dimensions

Sections 4.5.1–4.5.5 are devoted to proving the uniform regularity and separation estimate in low dimensions of Theorem 1.2.8, as well as the result of Theorem 1.2.6 on the size of the singular set in higher dimensions.

First, Sections 4.5.1 and 4.5.2 define and describe the properties of blow-ups of s-minimal surfaces, in particular when they are the limits of Allen-Cahn solutions with bounded index.

Then, in Section 4.5.3 it is shown that such blow-ups converge to a single hyperplane in $\mathbb{R}^n$, under the assumption that stable s-minimal cones in $\mathbb{R}^n$ are flat; that is, when $n < n_s^*$ is less than the critical dimension of Definition 1.2.5. This classification result for blow-ups is used in Section 4.5.4 to prove Theorem 1.2.8. The proof is done by a blow-up and contradiction strategy to show that the surfaces are flat at some fixed scale, and an improvement of flatness theorem[2] which holds for all nonlocal minimal surfaces that are viscosity solutions of the zero nonlocal mean curvature equation, a criticality condition much weaker than minimality.

Finally, a dimension-reduction argument combined with the previous strategy allows to prove Theorem 1.2.6 for all n.

[2]This improvement of flatness theorem was proved on $\mathbb{R}^n$ in the seminal article [CRS10] which first defined nonlocal minimal surfaces, and the version of it on manifolds has been recently proved in [Moy25].

Bernstein and De Giorgi type results

Section 4.5.6 establishes the validity of the "finite Morse index versions" of the nonlocal De Giorgi and Bernstein conjectures, once again under the assumption of the classification of stable cones. This represents a remarkable departure from the behavior of classical minimal surfaces and of solutions to the classical (local) Allen-Cahn equation with a bounded index.

The proof of both results uses the same strategy as the proof, in Section 4.5.3, of the fact that blow-up limits of s-minimal surfaces satisfying a certain list of properties, which are in particular satisfied by limits of Allen-Cahn, need to be half-spaces.

The Bernstein conjecture (today theorem) states that graphical complete minimal hypersurfaces must be hyperplanes in low dimensions. See [Alm66; Ber15; CL24; CP79; FS80; Pog81; Sim68] for related generalizations to the classes of minimizing and stable hypersurfaces.

In Section 4.5.6 we establish the following.

Theorem 1.2.18 (Finite index nonlocal Bernstein)**.** *Let $s \in (0,1)$ and $3 \le n < n_s^*$, where n_s^* is the critical dimension (see Definition 1.2.5).*
Then, any finite Morse index s-minimal surface in $\mathbb{R}^n$ of class C^2 is a half-space.

Under the assumption of stability (Morse index zero) the previous theorem was established in [CCS21] and in the case of minimizers if follows from [CRS10].

The De Giorgi conjecture is a famous related statement about certain entire solutions to the Allen-Cahn equation being one-dimensional or equivalently about their level sets being hyperplanes in low dimensions. See [AAC01; AC00; FS20b; GG98; Sav09; CCS21] for related previous results in the minimizing and stable cases.

In Section 4.5.6, we also show:

Theorem 1.2.19 (Finite index nonlocal De Giorgi)**.** *Let $s \in (0,1)$ and $3 \le n < n_s^*$, where n_s^* is the critical dimension (see Definition 1.2.5).*
Then, every finite Morse index solution u of $(-\Delta)^{s/2}u + W'(u) = 0$ in $\mathbb{R}^n$ is a 1D layer solution, namely, $u(x) = \phi(e \cdot x)$ for some $e \in \mathbb{S}^{n-1}$ and increasing function $\phi : \mathbb{R} \to (-1,1)$.

Under the assumption of stability (Morse index zero), the previous theorem was established in [CCS21], and for minimizers, it followed from [CRS10; DSV20].

1.2.3 A classification result for stable s-minimal cones in the plane

This section describes the results obtained in [Cas25], which will be proved in Chapter 5, where we show that for s close to zero, half-planes are the only s-minimal cones in $\mathbb{R}^2$ that are stable in $\mathbb{R}^2 \setminus \{0\}$. Here, by an s-minimal cone, we mean an open cone $E \subset \mathbb{R}^2$ that is an s-minimal surface (that is, a stationary set for the s-perimeter under inner variations; see Definition 2.2.5).

This result is purely nonlocal since it is in direct contrast with both the classical case (formally $s = 1$) and the regime where s is close to 1, where the cross $\mathbf{X} = \{xy > 0\}$ is a nontrivial (i.e., not a half-plane) stationary cone in the plane that is expected to be stable for inner variations, in any reasonable sense.

Nevertheless, for s close to zero, the cross $\mathbf{X}$ is unstable in $\mathbb{R}^2 \setminus \{0\}$, and has infinite index by Corollary 1.2.21. Our proof relies on the behavior of the best constant in Hardy's inequality for the $H^\sigma(\mathbb{R})$ seminorm as $\sigma \downarrow 1/2$.

Classification results for s-minimal cones in $\mathbb{R}^n$ have been previously proved in different ranges of s and n and with various hypotheses on the cone, such as minimality in compact subsets or stability. Before stating the main result in [Cas25] precisely, let us recall the previous literature on this problem. Even though slightly different notions of stability have been used in the literature—see Subsection 5.1.1 for a complete discussion about this—the known classification results for s-minimal cones in $\mathbb{R}^n$ can be summarized as follows.

The table below has to be read in this way: for a cone $E \subset \mathbb{R}^n$, the hypotheses in each row imply that E is a half-space.

	Class of cones:	**Range of s:**
[SV13a; SV13b]	Minimizing in $\mathbb{R}^2$	$\forall\, s \in (0,1)$
[CSV19]	Stable by rearrangements[3] in $\mathbb{R}^2$	$\forall s \in (0,1)$
[CCS20]	Stable and smooth in $\mathbb{R}^3 \setminus \{0\}$	s close to 1
[Cha+23]	Stable and smooth in $\mathbb{R}^4 \setminus \{0\}$	s close to 1
[CV13]	Minimizing in $\mathbb{R}^n$, for $2 \le n \le 7$	s close to 1

Let us stress that the results in [SV13a; SV13b] provide, for all $s \in (0,1)$, the classification of cones in $\mathbb{R}^2$ minimizing the s-perimeter in compact sets, in accordance with the classical case. These results do not imply that stable s-minimal cones in $\mathbb{R}^2$ are flat. In fact, for the notion of stability that we consider in this work (Definition 2.2.10 below), which is the most natural one induced by inner variations and also used in similar contexts like stationary varifolds, this fact is not even believed to be true for all $s \in (0,1)$. Indeed, for inner variations, the cross $\mathbf{X}$ is expected to be a stable s-minimal cone in $\mathbb{R}^2$ for s close to 1, again in accordance with the classical case.

For s close to zero, the situation could differ from that of the classical perimeter. For example, in [DPW18, Theorem 4], for s close to zero, the authors construct a non-flat s-minimal cone in $\mathbb{R}^7$ that is smooth and stable in $\mathbb{R}^7 \setminus \{0\}$. This is in contrast with the case of the classical perimeter since, by a celebrated result by Simons [Sim68], for $3 \le n \le 7$, the only cones in $\mathbb{R}^n$ that are smooth and stable in $\mathbb{R}^n \setminus \{0\}$ are the hyperplanes. We refer to [Che69, Chapter 9] for a simplified exposition of Simons' result and to [CG18, Theorem 1.16] for a modern presentation.

In [Cas25], for small s, we proved the first classification result for stable s-minimal cones in $\mathbb{R}^2 \setminus \{0\}$ in direct contrast with the case of the classical perimeter or the regime s close to 1. The precise statement of our main result is as follows.

Theorem 1.2.20. *There exists $s_\circ \in (0,1/2)$ with the following property. Let $s \in (0,s_\circ)$ and $E \subset \mathbb{R}^2$ be an s-minimal cone stable in $\mathbb{R}^2 \setminus \{0\}$ (see Definition 2.2.10). Then E is a half-plane.*

Moreover, using the fact that s-minimal cones with finite Morse index outside the origin are stable outside the origin (which is a trivial observation in the classical case of the perimeter, but not entirely trivial for s-minimal cones), we deduce that the conclusion of Theorem 1.2.20 also holds for cones of finite Morse index.

[3] The notion of stability by rearrangements, which is Definition A in Subsection 5.1.1, is an ad-hoc notion of stability developed to get rid of cross-like singularities directly from the definition.

Corollary 1.2.21. *The classification of Theorem 1.2.20 holds for s-minimal cones of finite Morse index in* $\mathbb{R}^2 \setminus \{0\}$ *(see Definition 2.2.13).*

Here and in the rest of this thsis, by finite Morse index, we mean with respect to the notion Definition 2.2.13 introduced in [CFS24b; Flo24].

1.2.4 Min-max curves and model singularities

On a closed Riemannian manifold (M^n, g), the volume spectrum, introduced by Gromov, is a sequence of geometric invariants $\{\omega_p(M,g)\}_{p\in\mathbb{N}}$ called p-widths, which can be thought of as a nonlinear analog of the spectrum of the Laplacian. We refer to [MSS24, Section 2.2] or [CM23, Section 2] for the precise definition of the p-widths. These p-widths play a crucial role in the theory of minimal hypersurfaces. In ambient dimension $3 \le n \le 7$, each p-width equals the weighted area of the union of disjoint, connected, smooth, closed, embedded minimal hypersurfaces. These hypersurfaces can be chosen to satisfy a bound on their Morse index, meaning that the sum of the Morse indices of the connected components is at most p.

For $n = 2$, the situation is different as min-max methods on surfaces typically only produce stationary geodesic networks with regular support up to finitely many points (e.g., [Pit74]), making the standard index control techniques ineffective. In this direction, Chodosh and Mantoulidis recently achieved a significant breakthrough in [CM23], showing that on surfaces, the p-widths are achieved by finite unions of closed immersed geodesics rather than simply geodesic nets.

In the case of geodesics on surfaces, the regularity of these objects cannot be improved, meaning that even for generic metrics, the min-max scheme will produce immersed geodesics that are not embedded. At every self-intersection point, the tangent cone to these geodesics consists of a finite union of distinct lines intersecting transversely, which cannot be ruled out.

On the other hand, in this case of ambient dimension two, these multiple-junction model singularities (where the tangent cone consists of a finite union of distinct lines) are the only obstruction to the complete regularity of these geodesic nets arising from a min-max scheme. If one knew that there are no points of multiple junctions, then the net would be a finite union of disjoint, closed, embedded geodesics. This is even true in higher dimensions under suitable hypotheses. Indeed, by a deep result by Wickramasekera [Wic14], for $3 \le n \le 7$, any stationary, stable on its regular part codimension 1 varifold without multiple-junctions is smooth (to say, it is supported on a finite union of disjoint, smooth, embedded, connected hypersurfaces).

Let us now turn to the implications of this work to "fractional geodesics". Recall that, similarly to the terminology used for sets of finite perimeter (e.g., [Mag12, Part II]), an s-minimal surface is, to be precise, a set $E \subset M$ with finite s-perimeter and zero first variation (Definition 2.2.5 below). Nevertheless, with a bit of abuse of the notion, we often refer to just its boundary ∂E as "the" surface, which is a codimension one object.

The main result of this section—that is Theorem 1.2.20—together with the ones in sections 4.5.1–4.5.5 (see also Remark 1.2.11), implies that the situation for the fractional analog of the volume spectrum is, for s small and $n = 2$, drastically different from the classical one of geodesics. We refer to Chapter 4 and [Flo24] for the precise definition of the fractional widths $\{\ell_{s,p}(M,g)\}_{p\in\mathbb{N}}$ and for the proof that these are indeed attained by s-minimal surfaces E_p^s with Morse index at most p on M, in the sense of Definition 2.2.13 above. Moreover, by Proposition 4.3.1, these surfaces are slightly more than stationary for inner variations: they are viscosity solutions of the NMS (i.e., Nonlocal Minimal Surface) equation.

Since having Morse index at most p is a property that is stable under blow-up, by the monotonicity formula for s-minimal surfaces (see Theorem 1.2.2 or [MSW19, Lemma 6.2]) and the BV estimate in the finite Morse index case [Flo24, Theorem 5.4] we have that, for every $x \in \partial E^s_p$, any blow-up of E^s_p around x is a cone in $\mathbb{R}^n$ of finite Morse index in $\mathbb{R}^n \setminus \{0\}$. Thus, if the ambient Riemannian manifold is two-dimensional and $s \in (0, s_\circ)$, where $s_\circ$ is the constant of Theorem 1.2.20, every such cone is a half-plane. This fact, together with the improvement of flatness theorem for viscosity solutions of the NMS equation in [CRS10] (see also Theorem 4.5.14), implies that ∂E^s_p has a unique flat tangent cone at every point. Then, one can deduce that ∂E^s_p is smooth by arguing exactly as in the proof of Theorem 1.2.8. We refer the reader to Section 4.5 for all the details regarding this blow-up procedure.

Hence, our classification result, Theorem 1.2.20, implies the following.

Theorem 1.2.22. *Let (M^2, g) be a Riemannian surface and $s \in (0, s_\circ)$, where $s_\circ$ is the one given by Theorem 1.2.20. Then the fractional widths $\{\ell_{s,p}(M^2, g)\}_{p\in\mathbb{N}}$ are attained by smooth s-minimal surfaces, which are a finite union of smooth embedded curves.*

1.2.5 Asymptotics as $s \to 0^+$ of the fractional perimeter on Riemannian manifolds

In this section, we present the results obtained in [CG24], jointly written with Luca Gennaioli, which will be the content of Chapter 6.

In this work, we completely describe the limiting behavior of the relative fractional perimeter on (essentially) any Riemannian or weighted manifold, showing that only two asymptotic regimes can occur: the Euclidean-type limit for infinite-volume manifolds and the Gaussian-type limit for finite-volume ones. We refer to Section 1.1.4 for more details on the literature on this problem.

Moreover, our work reveals a link between the limiting asymptotics of the relative fractional perimeter as $s \to 0^+$ and the existence of bounded harmonic functions on the manifold.

Infinite volume asymptotics

Given a set $E \subset M$, our analysis is based on the study of the following quantity

$$\theta_E(p) := \lim_{s\to 0^+} \int_{E\setminus B_R(p)} \mathcal{K}_s(x,p)\,d\mu(x), \tag{1.9}$$

where

$$\mathcal{K}_s(x,y) := \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty H_M(x,y,t)\,\frac{dt}{t^{1+s/2}}, \tag{1.10}$$

and H_M is the heat kernel of M as described in Section 2.1. The quantity analogous to (1.9) on $\mathbb{R}^n$ was previously studied in [Dip+13], where the authors deal with the study of the fractional s-perimeter as $s \to 0^+$. In the case of $M = \mathbb{R}^n$, the limit in (1.9) does not depend on p (whenever it exists); therefore, θ_E is a constant function.

One of the main observations of this work is that θ_E is always a harmonic function on M, with values in $[0,1]$, and in general can be nonconstant if M does not satisfy the L^∞-Liouville property (see Definition 6.1.2). Moreover, for $E \equiv M$, the function θ_M encodes the asymptotics of the fractional Laplacian as $s \to 0^+$ on every complete (M, g) (see Theorem 1.2.25).

The precise statement is as follows.

Theorem 1.2.23. *Let (M,g) be a complete Riemannian manifold with $\mu(M)=+\infty$, and let $E\subset M$ be a measurable set. Then*

(i) If for some $R>0$ and every $p\in M$, the following limit exists

$$\theta_E(p) := \lim_{s\to 0^+}\int_{E\setminus B_R(p)} \mathcal{K}_s(x,p)d\mu(x)\in[0,1]\,, \tag{1.11}$$

then it is independent of the choice of R, and $\theta_E : M\to[0,1]$ is a bounded harmonic function on M.

(ii) For $R>0$ and $p\in M$ the limit (1.11) with $E=M$ always exists, does not depend on the choice of R, and equals

$$\theta_M(p) = \lim_{t\to\infty}\int_M H_M(p,x,t)\,d\mu(x)\,. \tag{1.12}$$

In particular, we see that $\theta_M : M\to[0,1]$ is always a bounded harmonic function on M.

Remark 1.2.24. *Unless otherwise stated, when we will say "assume θ_E exists" we intend that the limit in (1.11) exists for some (thus any) $R>0$ and every $p\in M$. As we will prove in Lemma 6.2.5 if M has the L^∞-Liouville property and $\theta_E(p)$ exists for some $p\in M$ then, it exists for all $q\in M$ and the values coincide. Let us stress that, on manifolds with L^∞-Liouville property, the limit does not need to exist, but if it does not exist at some point, then it does not exist everywhere. For example, even on $\mathbb{R}^n$ in [Dip+13, Example 2.8], the authors exhibit a set for which the limit $\theta_E(x)$ does not exist at every point $x\in\mathbb{R}^n$. On the other hand, on a general M without the L^∞-Liouville property, we believe that $\theta_E(\cdot)$ could exist for some $p\in M$ and fail to exist for some $q\neq p$. We refer to [CG24, Subsection 7.1] for a more detailed discussion of this phenomenon.*

Next, we discuss the asymptotics of the fractional Laplacian and how it relates to θ_M above. Note that in well-behaved ambient spaces, one would expect (as happens on $\mathbb{R}^n$) that the fractional $(s/2)$-Laplacian tends to the identity as $s\to 0^+$. With the following result, we show that this is not true on general Riemannian manifolds and that the harmonic function θ_M encodes how this limit differs from the identity.

Theorem 1.2.25. *Let M be a complete Riemannian manifold with $\mu(M)=+\infty$, and let θ_M be as above. Let also $s_\circ\in(0,2)$ and $u\in H^{s_\circ/2}(M)\cap L^\infty(M)$ with bounded support. Then, as $s\to 0^+$ there holds*

$$(-\Delta)^{s/2}_{\mathrm{Si}}u \xrightarrow{a.e.} \theta_M u, \tag{1.13}$$

where $(-\Delta)^{s/2}_{\mathrm{Si}}$ is the singular integral fractional Laplacian (3.1).

With this result, we can easily deduce the following result, which is a generalization of [MS02] for $p=2$ for very general Riemannian manifolds.

Theorem 1.2.26. *Let M be a complete Riemannian manifold with $\mu(M)=+\infty$, and let $s_\circ\in(0,1)$. Then, for every $u\in H^{s_\circ/2}(M)\cap L^\infty(M)$ with bounded support there holds*

$$\lim_{s\to 0^+}\frac{1}{2}[u]^2_{H^{s/2}(M)} = \int_M u^2\theta_M\,d\mu\,.$$

Actually, it follows from the proof of Theorem 1.2.26 that the convergence in (1.13) also holds strongly in $L^2(M)$. Moreover, just taking $u = \chi_E$ in Theorem 1.2.26 gives:

Corollary 1.2.27. *Let M be a complete Riemannian manifold with $\mu(M) = +\infty$. Let $E \subset M$ be bounded with $\mathrm{Per}_{s_\circ}(E) < +\infty$ for some $s_\circ \in (0,1)$. Then*

$$\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E) = \int_E \theta_M \, d\mu.$$

With this result, we make an interesting observation regarding a Riemannian manifold constructed by Pinchover in [Pin95]. This manifold satisfies the L^∞-Liouville property (see Definition 6.1.2), but it is not stochastically complete (see Definition 2.1.1), and we show in Example 6.2.4 that it satisfies $\theta_M \equiv 0$. Consequently, there exist complete Riemannian manifolds where the mass of the heat kernel escapes so rapidly that the asymptotic of the fractional Laplacian not only differs from the identity but becomes identically zero.

As a corollary of the above results, we are able to obtain the asymptotics of the relative fractional perimeter as $s \to 0^+$ in an extremely general setting, generalizing both the existing results [Dip+13] for $\mathbb{R}^n$ and [Car+22] for the Gaussian space. In particular, with Theorems 1.2.28 and 1.2.30, we show that these two known behaviors of the asymptotics, the one of $\mathbb{R}^n$ and the one of the Gaussian space, are also the only two possible in this general setting.

Theorem 1.2.28 (Infinite volume asymptotics)**.** *Let M be a complete, stochastically complete Riemannian manifold with $\mu(M) = +\infty$ and with the L^∞-Liouville property (see Definition 6.1.2). Let $\Omega \subset M$ be an open, bounded, connected set with Lipschitz boundary. Let also $E \subset M$ be a set with $\mathrm{Per}_{s_\circ}(E, \Omega) < +\infty$, for some $s_\circ \in (0,1)$, and such that θ_E exists (see (1.11)). Then*

(i) The quantity $\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E,\Omega)$ exists and

$$\begin{aligned}\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E,\Omega) &= (1-\theta_E)\mu(E\cap\Omega) + \theta_E \mu(E^c \cap \Omega)\\ &= \theta_{M\setminus E}\mu(E\cap\Omega) + \theta_E\mu(E^c\cap\Omega)\,.\end{aligned}$$

(ii) Conversely, if the limit $\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E,\Omega)$ exists and $\mu(E\cap\Omega) \neq \mu(E^c\cap\Omega)$, then the limit in (1.11) exists and there holds

$$\theta_E = \frac{\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E,\Omega) - \mu(E\cap\Omega)}{\mu(E^c\cap\Omega) - \mu(E\cap\Omega)}\,.$$

Lastly, if $\mu(E\cap\Omega) = \mu(E^c\cap\Omega)$ then the limit $\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E,\Omega)$ always exists and

$$\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E,\Omega) = \mu(E\cap\Omega) = \mu(E^c\cap\Omega)\,.$$

Remark 1.2.29. *Without the assumption of stochastic completeness of M, the situation can be different. The example we describe in Example 6.2.4 has L^∞-Liouville property, is not stochastically complete and satisfies $\lim_{s\to 0^+} \mathrm{Per}_s(E) = 0$ for every regular $E \subset M$.*

Finite volume asymptotics

Next, we have our main result on the asymptotics in the case of finite volume.

Theorem 1.2.30 (Finite volume asymptotics)**.** *Let M be a complete Riemannian manifold with $\mu(M) < +\infty$, and let $\Omega \subset M$ be an open and connected set with Lipschitz boundary. Let $E \subset M$ be a set with $\mathrm{Per}_{s_\circ}(E,\Omega) < +\infty$, for some $s_\circ \in (0,1)$. Then the limit $\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E,\Omega)$ exists and*

$$\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E,\Omega) = \frac{1}{\mu(M)}\Big(\mu(E)\mu(E^c\cap\Omega) + \mu(E\cap\Omega)\mu(E^c\cap\Omega^c)\Big).$$

Moreover, the same result holds for a weighted manifold of finite volume.

This theorem, in the case of weighted manifolds, recovers the main result in [Car+22] for the Gaussian space.

1.3 Further results: codimension-two

In what follows, we provide a concise overview of additional research conducted during the author's Ph.D program that, while not directly aligned with the central theme of this thesis (that is, problems in codimension-one), represents a significant part of the author's research during these years. We summarize the key results [CFP24; CFP25] and direct the reader to the original publications for detailed discussions of these problems and their associated literature.

1.3.1 Nonlocal approximation of the area in codimension two

In [CFP24], jointly written with Mattia Freguglia and Nicola Picenni, we introduce and study a fractional notion of area for codimension two surfaces in $\mathbb{R}^n$ or in a closed Riemannian manifold M, which we call the *fractional s-mass* and we denote it by $\mathbb{M}_s$. In analogy with the fractional perimeter, our notion provides a fractional counterpart to the classical $(n-2)$-dimensional Hausdorff measure.

Let $\Sigma \subset \mathbb{R}^n$ be an oriented $(n-2)$-dimensional surface with locally constant integer multiplicity, and write

$$\Sigma := d_1\Sigma_1 \cup \cdots \cup d_m\Sigma_m,$$

where $m \geq 1$, $d = (d_1,\ldots,d_m) \in \mathbb{N}_+^m$, and $\Sigma_1,\ldots,\Sigma_m$ are closed, connected, oriented, $(n-2)$-dimensional surfaces of class C^2. For $s \in (0,1)$ consider the follwing class of maps

$$\mathfrak{F}_s(\Sigma) := \Big\{ u \in C^1(\mathbb{R}^n \setminus \Sigma; \mathbb{S}^1) \,:\, [u]^2_{H^{\frac{1+s}{2}}(\mathbb{R}^n)} < +\infty \text{ and } |\deg(u,\gamma)| = |d_1 \operatorname{link}(\gamma,\Sigma_1) + \cdots + d_m \operatorname{link}(\gamma,\Sigma_m)| \text{ for any } \gamma \in L(\Sigma) \Big\},$$

where

$$L(\Sigma) := \big\{ \gamma \subset \mathbb{R}^n \setminus \Sigma \,:\, \gamma \text{ is a bi-Lipschitz image of } \mathbb{S}^1 \big\},$$

and

$$[u]^2_{H^{\frac{1+s}{2}}(\mathbb{R}^n)} = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+1+s}}\,dxdy$$

is the fractional Sobolev energy of exponent $\frac{1+s}{2}$. The family $\mathfrak{F}_s(\Sigma)$ is the class of smooth maps *linking* with Σ, and can be thought of as prescribing standard-vortex singularities around Σ with the suitable degree.

With this notation, we define the s-mass of Σ as

$$\mathbb{M}_s(\Sigma) := \min_{u\in\mathfrak{F}_s^w(\Sigma)} [u]^2_{H^{\frac{1+s}{2}}(\mathbb{R}^n)},$$

where here $\mathfrak{F}_s^w(\Sigma)$ is the class of maps $u : \mathbb{R}^n \to \mathbb{S}^1$ that *weakly linking* with Σ. The precise definition of $\mathfrak{F}_s^w(\Sigma)$ may seem a bit cumbersome at first, and we do not include it here, but it is the closure of $\mathfrak{F}_s(\Sigma)$ in a suitable topology that is essentially $\bigcap_{\sigma\in(0,\frac{1+s}{2})} H^\sigma_{\mathrm{loc}}(\mathbb{R}^n)$.

Our main result establishes that the s-mass Γ-converges to the classical $(n-2)$-dimensional Hausdorff measure with multiplicity as $s \to 1^-$, after appropriate rescaling.

Theorem 1.3.1 ([CFP24])**.** *Let $\Sigma \subset \mathbb{R}^n$ be as above. Then*

$$\Gamma - \lim_{s\to 1^-} (1-s)^2 \mathbb{M}_s(\Sigma) = \frac{2\pi\omega_{n-1}}{n} \sum_{i=1}^m d_i \mathcal{H}^{n-2}(\Sigma_i),$$

where the Γ-limit is taken with respect to the flat topology of boundaries in $\mathbb{R}^n$. Moreover, this convergence can be localized in every open set $\Omega \subset \mathbb{R}^n$ (see [CFP24, Remark 1.5]).

Here, the flat norm of boundaries is defined as

$$\mathbf{F}(\Sigma) = \inf\{\mathbb{M}(T) : T \text{ is an integral } (n-1)\text{-current such that } \partial T = \Sigma\}.$$

Remark 1.3.2. *It would be possible to define the s-mass as the minimum of the fractional Sobolev energy over the class*

$$\mathfrak{F}_s^J(\Sigma) := \left\{u : \mathbb{R}^n \to \mathbb{S}^1 : [u]^2_{H^{\frac{1+s}{2}}(\mathbb{R}^n)} < +\infty, \text{ and } \star Ju = \pi\Sigma\right\},$$

where Ju is the Jacobian of u (see [BM21, Section 8.1]) and $\star Ju$ is the $(n-2)$-current associated to the Jacobian (see [ABO03, Remark 3.3]).

This, in principle, might lead to a different notion of fractional mass, since the inclusion $\mathfrak{F}_s^w(\Sigma) \subseteq \mathfrak{F}_s^J(\Sigma)$ is standard, but the opposite inclusion seems to be more delicate. Here, we chose to stick to the definition proposed in [Ser23], so we do not address this issue.

However, our Γ-convergence result applies also to this alternative notion, because every map $u \in \mathfrak{F}_s^J(\Sigma)$ can be approximated by a map $u' \in \mathfrak{F}_s(\Sigma')$, with a possibly different singular set Σ', in such a way that both the distance in the fractional Sobolev space between u and u' and the flat distance between Σ and Σ' are small (in any fixed bounded set). This approximation allows to reduce the liminf inequality to the setting of Theorem 1.3.1.

Remark 1.3.3. *Our notion is robust and adaptable to more general settings. We refer to [CFP24, Section 5.1] and [CFP24, Section 5.2] for a discussion of the extensions to ambient Riemannian manifolds and higher codimensions, respectively.*

1.3.2 Gamma-convergence of the p-energy of sphere-valued Sobolev maps

In [CFP25], jointly written with Mattia Freguglia and Nicola Picenni, we study the asymptotic behavior of sequences of sphere-valued Sobolev maps $u_p \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^{n+m}; \mathbb{S}^{n-1})$ as $p \in (n-1, n)$

approaches n from below, subject to a uniform bound on their rescaled p-energies

$$(n-p)\int |\nabla u_p|^p\,dx \tag{1.14}$$

Our main result provides a general compactness theorem for the Jacobians of these maps. Moreover, we prove Γ-convergence of the energy in (1.14) to the mass of the limit current of the Jacobians. Our main theorem is in complete analogy with the results for the Ginzburg-Landau energy in general codimension obtained by Alberti, Baldo, and Orlandi in [ABO05], which extended the earlier work in codimension two by Jerrard and Soner [JS02].

As a corollary of our analysis in the case of fixed boundary conditions, we recover several previous results, including one by Hardt and Lin on the convergence of the energy densities of p-energy minimizing maps with fixed boundary conditions, as p approaches n. We also obtain an "oriented version" of this convergence for the Jacobians of these maps.

Our approach is completely variational and applies to general sequences satisfying the natural energy growth (1.14) without minimality or stationarity assumptions. For this reason, we believe that the techniques developed in this work may apply to a broader class of problems involving singular limits; in particular in the zero-dimensional case (see Theorem [CFP25, Theorem 4.3]), where we provide a direct and elementary, yet highly nontrivial, approach to the compactness and liminf inequality.

Our main result reads as follows.

Theorem 1.3.4 ([CFP25])**.** *Let $n \geq 2$ and $m \geq 0$ be integers, and let $\Omega \subset \mathbb{R}^{n+m}$ be a bounded open set with Lipschitz boundary. For a (discrete) sequence $p \in (n-1, n)$ let $u_p \in W^{1,p}(\Omega; \mathbb{S}^{n-1})$ be maps such that*

$$\limsup_{p\to n^-} (n-p)\int_\Omega |\nabla u_p|^p\,dx < +\infty.$$

Then the following statements hold.

(i) (Compactness and Γ-liminf inequality) There exists an integral m-boundary Σ in Ω such that, up to a subsequence, the Jacobians $\{\star Ju_p\}$ converge to $\frac{\omega_{n-1}}{n}\Sigma$ in the flat topology of Ω and (along this subsequence) it holds that

$$\liminf_{p\to n^-} (n-p)\int_\Omega |\nabla u_p|^p\,dx \geq (n-1)^{\frac{n}{2}}\omega_{n-1}\mathbb{M}_\Omega(\Sigma).$$

(ii) (Γ-limsup inequality) For every integral m-boundary Σ on Ω and every $p \in (n-1, n)$ there exists maps $u_p \in W^{1,p}(\Omega; \mathbb{S}^{n-1})$ such that $\star Ju_p \to \gamma_n\Sigma$ as $p \to n^-$ in the flat topology of Ω and

$$\limsup_{p\to n^-} (n-p)\int_\Omega |\nabla u_p|^p\,dx \leq (n-1)^{\frac{n}{2}}\omega_{n-1}\mathbb{M}_\Omega(\Sigma).$$

The proof of the compactness and Γ-liminf inequality takes up most of the work and is divided into two macrosteps. First, we address the case of dimension zero $m = 0$, that is when the model singularities are isolated points. Then, we extend the result to general dimension and codimension using the zero-dimensional case. Although the extension to arbitrary dimensions (Section 5 in [CFP25]) follows the techniques of deformation onto grids developed in [ABO05] (which in turn were inspired by the deformation theorem of [FF60]), the core novelty of our work lies in the zero-dimensional setting (see Theorem [CFP25, Theorem 4.3]).

The following result is an immediate corollary of the previous theorem.

Corollary 1.3.5 ([CFP25])**.** *Let $n \geq 2$ and $m \geq 0$ be integers, and let $\Omega \subset \mathbb{R}^{n+m}$ be a bounded open set with Lipschitz boundary. For a (discrete) sequence $p \in (n-1, n)$, let Σ_p be an integral m-boundary and set*

$$\mathcal{E}_p^{\min}(\Sigma_p, \Omega) := \inf\left\{ \int_\Omega |\nabla v|^p \,:\, v \in W^{1,p}(\Omega; \mathbb{S}^{n-1}),\ \star Jv = \tfrac{\omega_{n-1}}{n}\Sigma_p \right\}.$$

If there holds

$$\limsup_{p \to n^-} (n-p)\mathcal{E}_p^{\min}(\Sigma_p, \Omega) < +\infty,$$

then there exists an integral m-boundary Σ in Ω such that, up to a subsequence, $\Sigma_p \to \Sigma$ in the flat topology of Ω and (along this subsequence) it holds that

$$\liminf_{p \to n^-} (n-p)\mathcal{E}_p^{\min}(\Sigma_p, \Omega) \geq (n-1)^{\frac{n}{2}} \omega_{n-1} \mathbb{M}_\Omega(\Sigma).$$

A remarkable feature of our analysis is that we work in a setting where the energy bound (1.14) does not give a uniform control on the mass of the m-dimensional Jacobians $\star Ju_p$. Indeed, even for p fixed, the Jacobian $\star Ju_p$ may have an infinite mass even when the p-energy of u_p is finite. This departs significantly from the diffuse frameworks such as the Yang-Mills-Higgs and Ginzburg-Landau, where the energy bound provides some control (uniform and logarithmic, respectively) on the mass of the Jacobians. We refer to [CFP25, Section 1.2.1] for a detailed discussion about this feature.

As a corollary, we deduce both a new proof and an "oriented version" (Corollary 1.3.7 below) of a result by Hardt and Lin concerning the convergence of energy densities of p-energy minimizing maps as $p \to n^-$.

Theorem 1.3.6 (Theorem 3.1 in [Lin11])**.** *Let $n \geq 2$, $m \geq 0$ be integers, $p \in (n, 1-n)$ and μ_p be the normalized energy densities*

$$\mu_p := \frac{n-p}{\omega_n (n-1)^{p/2}} |\nabla u_p(x)|^p dx \llcorner \Omega \tag{1.15}$$

associated with maps $u_p \in W^{1,p}(\Omega; \mathbb{S}^{n-1})$ that minimize the p-energy in Ω with fixed boundary datum $g \in W^{1,n-1}(\partial\Omega; \mathbb{S}^{n-1})$. Then, as $p \to n^-$, a subsequence of μ_p weakly converges to a Radon measure μ such that $\mathrm{supp}(\mu) = \mathrm{supp}(\Sigma)$ and $\mu(\Omega) = \mathbb{M}_\Omega(\Sigma)$, where Σ is an integral area-minimzing current in Ω with $\partial\Sigma = \star Jg$.

Specifically, we show that for such minimizers, the m-dimensional Jacobians converge to an integral m-current that is area-minimizing in a suitable cobordism class determined by the boundary conditions. We refer to [ABO05, Section 2.7] or [CFP25, Section 5.3] for the precise definition of cobordant currents.

Corollary 1.3.7 ([CFP25]). *Let $n \geq 2$, $m \geq 1$ be integers, and let $\Omega \subset \mathbb{R}^{n+m}$ be a bounded Lipschitz domain. Let $g \in W^{1-1/n,n}(\partial\Omega; \mathbb{S}^{n-1})$ be a map, fix $u \in W^{1,n}(\Omega; \mathbb{R}^n)$ with trace equal to g on $\partial\Omega$, and let $\Sigma_y := u^{-1}(y)$ be a regular level set of u with $|y| < 1$. Let $u_p \in W^{1,p}(\Omega; \mathbb{S}^{n-1})$ be p-energy minimizing maps with $u_p|_{\partial\Omega} = g$, for $p \in (n-1, n)$.*

Then, as $p \to n^-$, a subsequence of $\star J u_p$ converges (in the flat topology of $\mathbb{R}^{n+m}$) to an integer rectifiable m-current Σ that is area-minimizing among rectifiable m-currents cobordant with Σ_y in $\overline{\Omega}$, and

$$\lim_{p \to n^-} (n-p) \int_\Omega |\nabla u_p|^p \, dx = (n-1)^{\frac{n}{2}} \omega_{n-1} \mathbb{M}_{\mathbb{R}^{n+m}}(\Sigma).$$

Chapter 2

Preliminaries and notation

2.1 The heat kernel on Riemannian manifolds

In what follows, we let (M, g) be a complete (possibly noncompact) Riemannian manifold that we often denote by just M.

Let $H_M(x, y, t) : M \times M \times (0, \infty) \to \mathbb{R}_+$ denote the heat kernel of a Riemannian manifold (M, g). This is defined as the minimal, positive fundamental solution to the heat equation

$$\partial_t u - \Delta_g u = 0, \quad \text{on } M,$$

subject to the initial condition that $u(t, \cdot) \to \delta_y$ as $t \to 0^+$, in the sense of distributions. Here, Δ_g is the Laplace-Beltrami operator associated with the metric g, and δ_y denotes the Dirac delta distribution centered at the point $y \in M$, with respect to the volume form dV_g of M.

The term minimal refers to the following comparison property: if $v : (0, \infty) \times M \to \mathbb{R}$ is any other positive solution of the heat equation that satisfies $v(t, \cdot) \to \delta_y$ in the distributional sense as $t \to 0^+$, then

$$H_M(x, y, t) \leq v(x, t)$$

for all $x \in M$, $t > 0$. That is, the heat kernel provides the least such solution and is unique under these conditions. This minimality is a consequence of the maximum principle and the theory of fundamental solutions for parabolic equations. For a detailed discussion of this property, see [Gri09, Section 9.1], and we refer to [Li12, Theorem 12.4] for a detailed construction of this minimal solution on general complete (not necessarily compact) manifolds.

Definition 2.1.1 (Stochastic completeness)**.** *We call a Riemannian manifold (M, g) stochastically complete if, for every $t > 0$ and for every $p \in M$*

$$\int_M H_M(x, p, t)\, dV_x = 1. \tag{2.1}$$

For equivalent definitions of stochastic completeness, one can refer to the manuscript [Gri09] or to the more recent [GIM20] and [Gri+23].

Lemma 2.1.2. *Let M be a complete Riemannian manifold, then for every $p \in M$*

$$\mathcal{M}(t, p) = \int_M H_M(x, p, t)\, dV_x \quad \textit{is nonincreasing in } t.$$

Proof. The proof is an easy consequence of the semigroup property. Indeed, for $t > s$ we can write

$$H_M(z,p,t) = \int_M H_M(z,x,t-s)H_M(x,p,s)\,dV_x.$$

Integrating in dV_z, using Fubini's theorem and the fact that $\int_M H_M(z,x,t-s)dV_x \le 1$ we get

$$\int_M H_M(z,p,t)dV_z \le \int_M H_M(x,p,s)dV_x,$$

which is the thesis. □

Note that, because of Lemma 2.1.2, being stochastically complete is equivalent to the fact that (2.1) holds for one single time $t = t_\circ > 0$.

2.2 Fractional minimal surfaces

Let H_M be the heat kernel of M, in the sense of the previous section. A quantity that will play a key role in this work is the singular kernel $\mathcal{K}_s : M \times M \to (0,+\infty)$ associated to the fractional Laplacian on M, that is

$$\mathcal{K}_s(p,q) = \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty H_M(p,q,t)\frac{dt}{t^{1+s/2}}. \tag{2.2}$$

Definition 2.2.1. *For $s \in (0,2)$, we define the fractional Sobolev seminorm $[u]_{H^{s/2}(M)}$ as*

$$[u]^2_{H^{s/2}(M)} := \iint_{M\times M} (u(p)-u(q))^2\mathcal{K}_s(p,q)\,dV_p dV_q\,. \tag{2.3}$$

The associated functional space $H^{s/2}(M)$ is

$$H^{s/2}(M) = \{u \in L^2(M) \,:\, [u]^2_{H^{s/2}(M)} < \infty\}\,,$$

and is a Hilbert space with norm given by $\|u\|^2_{H^{s/2}(M)} = \|u\|^2_{L^2(M)} + [u]^2_{H^{s/2}(M)}$.

Definition 2.2.2. *For $s \in (0,1)$, the fractional perimeter (or s-perimeter) of a measurable set $E \subset M$ is defined as*

$$\mathrm{Per}_s(E) := [\chi_E]^2_{H^{s/2}(M)} = 2\iint_{E\times E^c} \mathcal{K}_s(x,y)\,dV_p dV_q,$$

where $\mathcal{K}_s$ is defined in (2.2).

Remark 2.2.3. *If the manifold M is replaced by the Euclidean space $\mathbb{R}^n$ then*

$$\begin{aligned}\mathcal{K}_s(x,y) &= \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty H_{\mathbb{R}^n}(x,y,t)\frac{dt}{t^{1+s/2}} \\ &= \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty \left(\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}\right)\frac{dt}{t^{1+s/2}} = \frac{\alpha_{n,s}}{|x-y|^{n+s}}\,,\end{aligned}$$

where

$$\alpha_{n,s} = \frac{2^s\Gamma\left(\frac{n+s}{2}\right)}{\pi^{n/2}|\Gamma(-s/2)|} = \frac{s2^{s-1}\Gamma\left(\frac{n+s}{2}\right)}{\pi^{n/2}\Gamma(1-s/2)}. \tag{2.4}$$

Hence, we recover (up to a constant) the usual notion of fractional s-perimeter

$$\mathrm{Per}_s(E) = \iint_{E\times E^c} \frac{1}{|x-y|^{n+s}}\,dxdy,$$

introduced by Caffarelli, Roquejoffre, and Savin in [CRS10].

The s-perimeter also has a natural localized version in a bounded open set $\Omega \subset M$, in the same spirit of the localized fractional Sobolev spaces $H^s(\Omega)$. This is of use because, for example, one would like to say that a hyperplane in $\mathbb{R}^n$ is an s-minimal surface (see Definition 2.2.5 below) even though a half-space has infinite s-perimeter for Definition 2.2.2.

Definition 2.2.4. *For $s \in (0,1)$, the fractional perimeter (or s-perimeter) of a measurable set $E \subset M$ in a bounded, open set Ω is defined as*

$$\mathrm{Per}_s(E,\Omega) = \iint_{M\times M\setminus\Omega^c\times\Omega^c} |\chi_E(p)-\chi_E(q)|^2\mathcal{K}_s(p,q)\,dV_p dV_q.$$

Note that for $\Omega = M$ we recover Definition 2.2.2. Moreover, it follows directly from its definition that the previous notion of relative s-perimeter satisfies the following properties.

- $\mathrm{Per}_s(E,\Omega) = \mathrm{Per}_s(E^c,\Omega)$ for every (measurable) $E \subset M$.
- If $E \subset \Omega$ or $E^c \subset \Omega$ then $\mathrm{Per}_s(E,\Omega) = \mathrm{Per}_s(E)$, where $\mathrm{Per}_s(E)$ is the s -perimeter on the entire manifold M as in Definition 2.2.2.
- Let $\Omega_1, \Omega_2 \subset M$ with $\mu(\Omega_1\cap\Omega_2) = 0$. Then $\mathrm{Per}_s(E,\Omega_1\cup\Omega_2) \geq \mathrm{Per}_s(E,\Omega_1) + \mathrm{Per}_s(E,\Omega_2)$.
- Let $E_1, E_2 \subset M$ with $\mu(E_1\cap E_2) = 0$. Then $\mathrm{Per}_s(E_1\cup E_2,\Omega) \leq \mathrm{Per}_s(E_1,\Omega) + \mathrm{Per}_s(E_2,\Omega)$.

Definition 2.2.5 (s-minimal surface)**.** *Given $\mathcal{U} \subset M$ open, a set E is said to be an s-minimal surface in $\mathcal{U}$ if for every bounded Lipschitz domain $\Omega \Subset \mathcal{U}$ we have $\mathrm{Per}_s(E,\Omega) < +\infty$ and for every C^1 vector field X with $\mathrm{supp}(X) \Subset \Omega$ there holds*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Per}_s(\phi_t^X(E),\Omega) = 0,$$

where $\phi_t^X : \Omega \to \Omega$ is the flow of X at time $t > 0$.

By the first variation formula (e.g, [Fig+15, Theorem 6.1] for $M = \mathbb{R}^n$ or [Flo24, Proposition 5.1] for closed manifolds), if E is an s-minimal surface in $\mathcal{U}$ and $\partial E \cap \mathcal{U}$ is of class C^2 then

$$P.V.\!\int_M (\chi_{E^c}(y) - \chi_E(y))\mathcal{K}_s(x,y)\,d\mu(y) = 0, \qquad \forall\, x \in \partial E \cap \mathcal{U}.$$

The left-hand side is denoted by $H_s^E(x)$ and called *nonlocal mean curvature* of E at x. We refer to the beginning of Section 3.1 for a discussion on how the principal value must be understood in Riemannian manifolds. For $M = \mathbb{R}^n$, the first-variation formula takes the usual form

$$P.V.\!\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{n+s}}\,dy = 0, \qquad \forall\, x \in \partial E \cap \mathcal{U}. \tag{2.5}$$

Remark 2.2.6. *Inspecting the proof of the first variation formula in [Fig+15; Flo24], it follows that if ∂E is C^2 just in a neighborhood of $x \in \partial E$ and is globally Lipschitz, then* (2.5) *holds at x.*

2.2.1 The fractional Allen-Cahn energy

Definition 2.2.7 (Fractional Allen-Cahn energy)**.** *Let $s \in (0,2)$ and $\varepsilon > 0$. Given $v : M \to \mathbb{R}$, we define the fractional Allen-Cahn (abbr. A-C) energy of v on the open set $\Omega \subseteq M$ as*

$$\mathcal{E}_\Omega(v) := \mathcal{E}^{\mathrm{Sob}}_\Omega(v) + \mathcal{E}^{\mathrm{Pot}}_\Omega(v), \tag{2.6}$$

where

$$\mathcal{E}^{\mathrm{Sob}}_\Omega(v) := \frac{1}{4}\iint_{M\times M\setminus\Omega^c\times\Omega^c}(v(p)-v(q))^2\mathcal{K}_s(p,q)dV_p dV_q, \quad \mathcal{E}^{\mathrm{Pot}}_\Omega(v) := \varepsilon^{-s}\int_\Omega W(v)\,dx\,,$$

and $W(v) = \frac{1}{4}(1-v^2)^2$ is the standard quartic double-well potential with wells at ± 1. We will sometimes denote $\mathcal{E}_\Omega$ by $\mathcal{E}^{\varepsilon,s}_\Omega$ or $\mathcal{E}^{\varepsilon}_\Omega$ if we want to stress the dependence of the energy from ε and/or s.

Note that, with this definition of the Allen-Cahn energy, we have

$$\mathcal{E}_\Omega(\chi_E - \chi_{E^c}) = \mathcal{E}^{\mathrm{Sob}}_\Omega(\chi_E - \chi_{E^c}) = \mathrm{Per}_s(E,\Omega)\,,$$

and

$$\mathcal{E}_{\Omega_1\cup\Omega_2}(v) \le \mathcal{E}_{\Omega_1}(v) + \mathcal{E}_{\Omega_2}(v)\,.$$

The double-well potential penalizes functions that are not identical to ±1, and that is why one expects to find nonlocal s-minimal surfaces as the limits of critical points of this energy when $\varepsilon \to 0$.

A function $u : M \to \mathbb{R}$ is a critical point of $\mathcal{E}_\Omega$ if and only if it solves the fractional Allen-Cahn equation

$$(-\Delta)^{s/2}u + \varepsilon^{-s}W'(u) = 0 \quad \text{in } \Omega\,. \tag{2.7}$$

Here $(-\Delta)^{s/2}$ is the fractional Laplacian on (M,g), and it can be represented as (see Section 3.1 for details)

$$(-\Delta)^{s/2}u(p) = P.V.\int_M (u(p)-u(q))\mathcal{K}_s(p,q)\,dV_q\,.$$

We also have a definition of Morse index, related to the second variation of the energy.

Proposition 2.2.8 (Second variation)**.** *Let $\Omega \subset M$ be an open set. Let $u \in H^{s/2}(M)$ be a critical point of $\mathcal{E}_\Omega$. Then, given $\xi \in C^1_c(\Omega)$, the second variation of $\mathcal{E}_\Omega$ at u is given by*

$$\mathcal{E}''_\Omega(u)[\xi,\xi] = \frac{1}{4}\iint_{(M\times M)\setminus(\Omega^c\times\Omega^c)}|\xi(p)-\xi(q)|^2\mathcal{K}_s(p,q)\,dV_p\,dV_q + \varepsilon^{-s}\int_\Omega W''(u)\xi^2\,dV\,. \tag{2.8}$$

Definition 2.2.9 (Morse index)**.** *Let $\Omega \subset M$ an open set, and let $u \in H^{s/2}(M)$ be a critical point of $\mathcal{E}_\Omega$. The Morse index of u in Ω, denoted by $m_\Omega(u)$, is defined as the maximum dimension among all linear subspaces $\mathcal{L} \subset C^1_c(\Omega) \subset H^{s/2}(M)$ such that $\mathcal{E}''_\Omega(u)$ is negative definite on $\mathcal{L}$. Moreover, we say that u is stable in Ω if $m_\Omega(u) = 0$.*

2.2.2 Stability and Morse index

We also precisely state the notion of stability that we will use in this work.

Definition 2.2.10 (Stability)**.** *Let $\mathcal{U} \subset M$ be an open set and E be an s-minimal surface in $\mathcal{U}$ (see Definition 2.2.5). We say that E is stable in $\mathcal{U}$ if: for every bounded Lipschitz domain $\Omega \Subset \mathcal{U}$,* *for every C^2 vector field X with* $\operatorname{supp}(X) \Subset \Omega$ *we have*

$$\delta^2 \mathrm{Per}_s(E,\Omega)[X] := \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\phi_t^X(E),\Omega) \geq 0, \tag{2.9}$$

where $\phi_t^X : \Omega \to \Omega$ is the flow of X at time $t > 0$.

Remark 2.2.11. *By Corollary 3.4.18, if $\mathrm{Per}_s(E,\Omega) < +\infty$ and X is a vector field as in the hypothesis then the map $t \mapsto \mathrm{Per}_s(\phi_t^X(E),\Omega)$ is well-defined for all $t > 0$ and of class C^2. Thus, the previous definition of stability is meaningful without any a priori assumption on the regularity of E.*

If $M = \mathbb{R}^n$ and $\partial E \subset \mathbb{R}^n$ is smooth, the second variation can be written in a form very reminiscent of the second variation formula for classical minimal surfaces. This second variation formula for smooth s-minimal surfaces was proved in [DPW18; Fig+15], and has recently been generalized to ambient Riemannian manifolds in [Flo24].

Theorem 2.2.12. *Let E be an s-minimal surface in $\mathbb{R}^n$, and assume that ∂E is C^2 in some open set Ω. Then, for every $X \in C^2_c(\partial E \cap \Omega; \mathbb{R}^n)$, setting $\varphi := X \cdot \nu_{\partial E}$, we have*

$$\delta^2 \mathrm{Per}_s(E)[X] = \iint_{\partial E \times \partial E} \left(\frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{n+s}} - \frac{|\nu_{\partial E}(x) - \nu_{\partial E}(y)|^2}{|x-y|^{n+s}} \varphi(x)^2 \right) d\sigma_x d\sigma_y, \tag{2.10}$$

where $\nu_{\partial E}$ is the outer unit normal to ∂E. In particular, if E is stable in Ω, there holds

$$\iint_{\partial E \times \partial E} \frac{|\nu_{\partial E}(x) - \nu_{\partial E}(y)|^2}{|x-y|^{n+s}} \varphi(x)^2 \, d\sigma_x d\sigma_y \leq \iint_{\partial E \times \partial E} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{n+s}} \, d\sigma_x d\sigma_y,$$

for every $\varphi \in C^2_c(\Omega)$.

Let us also recall the definition of finite Morse index for smooth s-minimal surfaces as introduced in [CFS24b] or [Flo24].

Definition 2.2.13 (Morse index)**.** *Let $E \subset M$ be an s-minimal surface in Ω. Then, E is said to have Morse index at most m in Ω if: for every $(m+1)$ vector fields $X_1, \dots, X_{m+1}$ of class C^2 with compact support in Ω, there exists coefficients $a_1, \dots, a_{m+1}$ such that $a_1^2 + \ldots + a_{m+1}^2 = 1$ and*

$$\delta^2 \mathrm{Per}_s(E,\Omega)[a_1 X_1 + \ldots + a_{m+1} X_{m+1}] \geq 0,$$

where $\delta^2 \mathrm{Per}_s$ is defined as in (2.9).

2.2.3 An instructive example on stability

We now discuss a simple but instructive example to understand what stability and instability mean in the context of nonlocal minimal surfaces. Consider the set $E \subset \mathbb{R}^n$ defined by

$$E := \bigcup_{k \in \mathbb{Z}} \left\{ 2kC_\circ \sqrt{1-s} \leq x_n \leq (2k+1) C_\circ \sqrt{1-s} \right\}, \tag{2.11}$$

whose boundary consists of parallel hyperplanes at distance

$$d := C_\circ \sqrt{1-s}.$$

Clearly, by symmetry, $H_s^E(x) = 0$ for every $x \in \partial E$. Since ∂E is smooth, we have that E is an s-minimal surface. The crucial property regarding this set is that, depending on the value of $C_\circ$, E can be either stable or unstable in the unit cube.

Proposition 2.2.14. *Let E be as in (2.11) and $Q = [-1,1]^n$. Then, for every $s \in (9/10, 1)$*

(i) E is stable in Q for $C_\circ \gg 1$ large (depending only on n).

(ii) E is unstable in Q for $C_\circ \ll 1$ small (depending only on n).

Proof. In what follows, $C, c > 0$ denote dimensional constants where, in general, C is big and c is small.

Set $\Sigma := \partial E = \cup_{i\in\mathbb{Z}}\Sigma_i$, where each Σ_i has the induced orientation from E, and we denote by N_i the outer unit normal to Σ_i from E. Moreover, we call $\Sigma_1, \dots, \Sigma_m$ the hyperplanes that intersect Q, that is

$$\bigcup_{i=1}^{m} \Sigma_i \cap Q = \Sigma \cap Q.$$

We want to show that E is stable provided we take $C_\circ$ large, that is (recall Theorem 2.2.12)

$$\iint_{\Sigma\times\Sigma} \frac{|N(x)-N(y)|^2}{|x-y|^{n+s}} f^2(x)\, d\sigma_x d\sigma_y \le \iint_{\Sigma\times\Sigma} \frac{|f(x)-f(y)|^2}{|x-y|^{n+s}}\, d\sigma_x d\sigma_y, \qquad \forall\, f \in C_c^\infty(\Sigma\cap Q). \tag{2.12}$$

Hence, to show (i), we estimate the left-hand side from above.

For $i = 1, 2, \dots m$ let $J(i) = \{\, j \,:\, j \ge 1\,,\, j - i \text{ odd}\}$. We have

$$\begin{aligned}
\iint_{\Sigma\times\Sigma} \frac{|N(x)-N(y)|^2}{|x-y|^{n+s}} f^2(x)\, d\sigma_x d\sigma_y &= \sum_{i,j} \iint_{\Sigma_i\times\Sigma_j} \frac{|N_i(x)-N_j(y)|^2}{|x-y|^{n+s}} f^2(x)\, d\sigma_x d\sigma_y \\
&= \sum_{i=1}^{m} \sum_{j\in J(i)} \iint_{\Sigma_i\times\Sigma_j} \frac{4}{|x-y|^{n+s}} f(x)^2\, d\sigma_x d\sigma_y \\
&= 4\sum_{i=1}^{m} \sum_{j\in J(i)} \int_{\Sigma_i\cap Q} f(x)^2 \left(\int_{\Sigma_j} \frac{d\sigma_y}{|x-y|^{n+s}} \right) d\sigma_x,
\end{aligned}$$

where we have used that $\operatorname{supp}(f) \Subset Q$.

Write the coordinates in $\mathbb{R}^n$ as $x = (\widetilde{x}, x_n)$, and for $x \in \Sigma_i$ set $d_{ij} := \operatorname{dist}(\Sigma_i, \Sigma_j) = d|i-j|$. We get, for every $j \in J(i)$, that

$$\begin{aligned}
\int_{\Sigma_j} \frac{d\sigma_y}{|x-y|^{n+s}} = \int_{\mathbb{R}^{n-1}} \frac{d\widetilde{y}}{(|\widetilde{x}-\widetilde{y}|^2 + |x_n-y_n|^2)^{\frac{n+s}{2}}} &= \int_{\mathbb{R}^{n-1}} \frac{d\widetilde{y}}{(|\widetilde{x}-\widetilde{y}|^2 + d_{ij}^2)^{\frac{n+s}{2}}} \\
&= \frac{1}{d_{ij}^{1+s}} \int_{\mathbb{R}^{n-1}} \frac{d\widetilde{z}}{(|\widetilde{z}|^2+1)^{\frac{n+s}{2}}} = \frac{C}{d_{ij}^{1+s}},
\end{aligned} \tag{2.13}$$

where we have made the substitution $\widetilde{x} - \widetilde{y} = d_{ij}\widetilde{z}$. Thus

$$\begin{aligned}
\iint_{\Sigma\times\Sigma} \frac{|N(x)-N(y)|^2}{|x-y|^{n+s}} f^2(x)\, d\sigma_x d\sigma_y &= 4\sum_{i=1}^{m} \sum_{j\in J(i)} \int_{\Sigma_i} f(x)^2 \left(\int_{\Sigma_j} \frac{d\sigma_y}{|x-y|^{n+s}} \right) d\sigma_x \\
&= C\sum_{i=1}^{m} \sum_{j\in J(i)} \frac{1}{d_{ij}^{1+s}} \int_{\Sigma_i} f(x)^2\, d\sigma_x
\end{aligned}$$

$$= \frac{C}{d^{1+s}} \sum_{i=1}^{m} \sum_{j \in J(i)} \frac{1}{|i-j|^{1+s}} \int_{\Sigma_i} f^2 \, d\sigma.$$

Clearly, for every $i = 1, 2, \ldots, m$ and $s \in (9/10, 1)$ there holds

$$c \le \sum_{j \in J(i)} \frac{1}{|i-j|^{1+s}} \le C, \tag{2.14}$$

for $C, c > 0$ absolute constants. This gives, for $s \in (9/10, 1)$ and $C_\circ \ge 1$ that

$$\begin{aligned}
\iint_{\Sigma \times \Sigma} \frac{|N(x) - N(y)|^2}{|x-y|^{n+s}} f^2(x) \, d\sigma_x d\sigma_y &\le \frac{C}{d^{1+s}} \sum_{i=1}^{m} \int_{\Sigma_i} f^2 \, d\sigma \\
&= \frac{C}{C_\circ^{1+s}(1-s)^{\frac{1+s}{2}}} \sum_{i=1}^{m} \int_{\Sigma_i} f^2 \, d\sigma \,.
\end{aligned}$$

By the fractional Poincaré inequality for the $H^{\frac{1+s}{2}}(\mathbb{R}^{n-1})$-seminorm (see, for example, Theorem 1 and Remark 1 in [BBM02]), applied to each restriction $f|_{\Sigma_i} \in C_c^\infty(\Sigma_i \cap Q)$, we have

$$\int_{\Sigma_i} f^2 \, d\sigma \le C(1-s) \iint_{\Sigma_i \times \Sigma_i} \frac{|f(x) - f(y)|^2}{|x-y|^{n+s}} d\sigma_x d\sigma_y. \tag{2.15}$$

Then

$$\begin{aligned}
\iint_{\Sigma \times \Sigma} \frac{|N(x) - N(y)|^2}{|x-y|^{n+s}} f^2(x) \, d\sigma_x d\sigma_y &\le \frac{C(1-s)^{\frac{1-s}{2}}}{C_\circ} \sum_{i=1}^{m} \iint_{\Sigma_i \times \Sigma_i} \frac{|f(x) - f(y)|^2}{|x-y|^{n+s}} d\sigma_x d\sigma_y \\
&\le \frac{C(1-s)^{\frac{1-s}{2}}}{C_\circ} \iint_{\Sigma \times \Sigma} \frac{|f(x) - f(y)|^2}{|x-y|^{n+s}} d\sigma_x d\sigma_y \,,
\end{aligned}$$

which, since $(1-s)^{\frac{1-s}{2}} \to 1$ as $s \to 1^-$, implies stability (2.12) if $C_\circ \ge C$ for some dimensional $C > 0$. This concludes the proof of (i).

Now, we show (ii). We want to choose $C_\circ$ small to make (2.12) fail. With no loss of generality, assume $C_\circ \le 1/100$.

On the one hand, following the same lines above using the lower bound in (2.14) and choosing each $f|_{\Sigma_i} = \varphi \in C_c^\infty(\Sigma_i \cap Q)$ equal to the minimizer of the fractional Poincaré inequality (2.15) (this is the first eigenfunction of the fractional $\frac{1+s}{2}$-Laplacian in $\mathbb{R}^{n-1}$, with zero Dirichlet boundary condition outside $[-1, 1]^{n-1}$) one gets for the left-hand side of the stability inequality (2.12) that

$$\begin{aligned}
&\iint_{\Sigma \times \Sigma} \frac{|N(x) - N(y)|^2}{|x-y|^{n+s}} \varphi^2(x) \, d\sigma_x d\sigma_y \\
&\qquad \ge \frac{c(1-s)^{\frac{1-s}{2}}}{C_\circ^{1+s}} \sum_{i=1}^{m} \iint_{\Sigma_i \times \Sigma_i} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{n+s}} d\sigma_x d\sigma_y \ge \frac{c\, m}{C_\circ^{1+s}} \mathcal{E}(\varphi),
\end{aligned} \tag{2.16}$$

where we have set $\mathcal{E}(\varphi) := [\varphi]^2_{H^{\frac{1+s}{2}}(\mathbb{R}^{n-1})}$.

On the other hand, for the right-hand side of the stability inequality

$$\begin{aligned}
&\iint_{\Sigma\times\Sigma} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\, d\sigma_x d\sigma_y \\
&= \sum_{i=1}^{m} \iint_{\Sigma_i\times\Sigma_i} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\, d\sigma_x d\sigma_y + \sum_{i\neq j} \iint_{\Sigma_i\times\Sigma_j} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\, d\sigma_x d\sigma_y \\
&= m\mathcal{E}(\varphi) + \sum_{i\neq j} \iint_{\Sigma_i\times\Sigma_j} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\, d\sigma_x d\sigma_y. \qquad (2.17)
\end{aligned}$$

Moreover

$$\iint_{\Sigma_i\times\Sigma_j} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\, d\sigma_x d\sigma_y \le 2\int_{\Sigma_i\cap Q} \left(\int_{\Sigma_j} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\, d\sigma_y \right) d\sigma_x$$

and, similarly to (2.13), for fixed i and $j\neq i$ we have

$$\begin{aligned}
\int_{\Sigma_j} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\, d\sigma_y &= \int_{\mathbb{R}^{n-1}} \frac{|\varphi(\widetilde{x},x_n)-\varphi(\widetilde{y},y_n)|^2}{(|\widetilde{x}-\widetilde{y}|^2+d_{ij}^2)^{\frac{n+s}{2}}}\, d\widetilde{y} \\
&\le \int_{\mathbb{R}^{n-1}} \frac{\min\{C, |\widetilde{x}-\widetilde{y}|^2\}}{(|\widetilde{x}-\widetilde{y}|^2+d^2|i-j|^2)^{\frac{n+s}{2}}}\, d\widetilde{y} \\
&= \frac{C}{d^{1+s}} \int_{\mathbb{R}^{n-1}} \frac{\min\{1, d^2|\widetilde{z}|^2\}}{(|\widetilde{z}|^2+|i-j|^2)^{\frac{n+s}{2}}}\, d\widetilde{z}.
\end{aligned}$$

Claim 1. There holds

$$\int_{\mathbb{R}^{n-1}} \frac{\min\{1, d^2|\widetilde{z}|^2\}}{(|\widetilde{z}|^2+|i-j|^2)^{\frac{n+s}{2}}}\, d\widetilde{z} \le C\min\left\{ \frac{d^{1+s}}{1-s}, \frac{1}{|i-j|^{1+s}} \right\}. \qquad (2.18)$$

Proof of Claim 1. We have

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} \frac{\min\{1, d^2|\widetilde{z}|^2\}}{(|\widetilde{z}|^2+|i-j|^2)^{\frac{n+s}{2}}}\, d\widetilde{z} &\le d^2\int_{B_{1/d}} \frac{|\widetilde{z}|^2}{(|\widetilde{z}|^2+1)^{\frac{n+s}{2}}}\, d\widetilde{z} + \int_{B_{1/d}^c} \frac{1}{(|\widetilde{z}|^2+1)^{\frac{n+s}{2}}}\, d\widetilde{z} \\
&= Cd^2 \int_1^{1/d} \frac{1}{\rho^{n+s-2}}\rho^{n-2}\, d\rho + C\int_{1/d}^{\infty} \frac{1}{\rho^{n+s}}\rho^{n-2}\, d\rho \\
&\le d^2 \frac{C}{d^{1-s}(1-s)} + Cd^{1+s} \le \frac{Cd^{1+s}}{1-s}.
\end{aligned}$$

On the other hand, bounding trivially $\min\{1, d^2|\widetilde{z}|^2\}\le 1$ we get arguing exactly as in (2.13)

$$\int_{\mathbb{R}^{n-1}} \frac{\min\{1, d^2|\widetilde{z}|^2\}}{(|\widetilde{z}|^2+|i-j|^2)^{\frac{n+s}{2}}}\, d\widetilde{z} \le \int_{\mathbb{R}^{n-1}} \frac{1}{(|\widetilde{z}|^2+|i-j|^2)^{\frac{n+s}{2}}}\, d\widetilde{z} = \frac{C}{|i-j|^{1+s}}.$$

These two last inequalities prove (2.18). $\square$

Lastly, by the very definition of d and since $s \in (9/10, 1)$ note that

$$\frac{d^{1+s}}{1-s} = \frac{C_\circ^{1+s}}{(1-s)^{\frac{1-s}{2}}} \le 2C_\circ \,.$$

Hence, putting together the estimates above

$$\begin{aligned}\sum_{i=1}^m \sum_{j\neq i} \iint_{\Sigma_i\times\Sigma_j} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\, d\sigma_x d\sigma_y &\le 2\sum_{i=1}^m \sum_{j\neq i} \int_{\Sigma_i\cap Q} \left(\int_{\Sigma_j} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\, d\sigma_y \right) d\sigma_x \\ &\le \frac{C}{d^{1+s}} \sum_{i=1}^m \sum_{j\neq i} \min\left\{ C_\circ, \frac{1}{|i-j|^{1+s}} \right\} \\ &\le \frac{Cm}{d^{1+s}} \sum_{j=1}^\infty \min\left\{ C_\circ, \frac{1}{j^{3/2}} \right\}.\end{aligned}$$

Moreover, as $C_\circ \to 0^+$ we have $\sum_{j=1}^\infty \min\left\{C_\circ, \frac{1}{j^{3/2}}\right\} \to 0$. Indeed

$$\begin{aligned}\sum_{j=1}^\infty \min\left\{ C_\circ, \frac{1}{j^{3/2}} \right\} &= \sum_{1\le j\le C_\circ^{-2/3}} \min\left\{ C_\circ, \frac{1}{j^{3/2}} \right\} + \sum_{j\ge C_\circ^{-2/3}} \min\left\{ C_\circ, \frac{1}{j^{3/2}} \right\} \\ &= C_\circ^{1/3} + \sum_{j\ge C_\circ^{-1/3}} \frac{1}{j^{3/2}} \to 0 \,.\end{aligned}$$

Let us denote by $o(1)$ anything that tends to zero as $C_\circ \to 0^+$ uniformly in $s \in (9/10, 1)$. With this notation, we have shown

$$\sum_{i=1}^m \sum_{j\neq i} \iint_{\Sigma_i\times\Sigma_j} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\, d\sigma_x d\sigma_y \le \frac{Cm}{d^{1+s}} o(1) \,. \tag{2.19}$$

Now, suppose by contradiction that E was stable for $C_\circ$ arbitrarily small. Then stability (2.12) would hold for $f = \varphi$. Putting together the estimates (2.16), (2.17) and (2.19) it would imply

$$\frac{c\, m}{C_\circ^{1+s}} \mathcal{E}(\varphi) \le m\mathcal{E}(\varphi) + \frac{Cm}{d^{1+s}} o(1) \,,$$

or, multiplying by $C_\circ^{1+s}(1-s)$ and recalling $d = C_\circ\sqrt{1-s}$, equivalently

$$c(1-s)\mathcal{E}(\varphi) \le C_\circ^{1+s}(1-s)\mathcal{E}(\varphi) + C(1-s)^{\frac{1-s}{2}} o(1) \,.$$

But since $c \le (1-s)\mathcal{E}(\varphi) \le C$ for every $s \in (9/10, 1)$ (as φ is the minimizer in the fractional Poincaré inequality (2.15)), sending $C_\circ \to 0$ gives $0 < c \le 0$, contradiction. Hence, φ is an unstable direction for E for $C_\circ$ small, depending only on n, which concludes the proof of (ii).

□

Chapter 3

Fractional Sobolev spaces on Riemannian manifolds

This chapter describes the results obtained in [CFS24a]. The present chapter addresses how the fractional Sobolev energy $H^\sigma(M) = W^{\sigma,2}(M)$ and the associated fractional Laplacian on M have a natural, canonical interpretation in the case where M is a closed Riemannian manifold. We give several definitions for these objects and show that they coincide, which justifies their canonical nature.

3.1 The fractional Laplacian

Taking inspiration from the case of $\mathbb{R}^n$, in this section, we give several equivalent definitions for the fractional Laplacian $(-\Delta)^{s/2}$ on a closed Riemannian manifold, with $s \in (0,2)$. On (say) smooth functions, there are at least three different natural definitions of the fractional Laplacian.

(*i*) The *singular integral* definition

$$\begin{aligned}(-\Delta)^{s/2}_{\mathrm{Si}} u(p) &= P.V.\, \frac{1}{|\Gamma(-s/2)|} \int_M (u(p) - u(q)) \mathcal{K}_s(p,q)\, dV_q \\ &:= \lim_{\varepsilon \to 0} \int_M (u(p) - u(q)) \mathcal{K}_s^\varepsilon(p,q)\, dV_q,\end{aligned} \tag{3.1}$$

where here $\mathcal{K}_s(p,q) : M \times M \to \mathbb{R}$ denotes the singular kernel given by (2.2) and $\mathcal{K}_s^\varepsilon(p,q)$ is the natural regularization

$$\mathcal{K}_s^\varepsilon(p,q) = \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty H_M(p,q,t) e^{-\varepsilon^2/4t} \frac{dt}{t^{1+s/2}}. \tag{3.2}$$

(*ii*) Following the *Bochner definition* of the fractional Laplacian

$$(-\Delta)^{s/2}_{\mathrm{B}} u = \frac{1}{\Gamma(-s/2)} \int_0^\infty (e^{t\Delta} u - u) \frac{dt}{t^{1+s/2}}. \tag{3.3}$$

(*iii*) By *spectral theory*, one can set

$$(-\Delta)^{s/2}_{\mathrm{Spec}} u := \sum_{k \geq 1} \lambda_k^{s/2} \langle u, \phi_k \rangle_{L^2(M)} \phi_k$$

where $\{\phi_k\}_k$ is an orthonormal basis of eigenfunctions of the Laplace-Beltrami operator $(-\Delta_g)$ and $\{\lambda_k\}_k$ are the corresponding eigenvalues. Note that for $s = 2$ this gives the usual Lalpacian.

Let us briefly comment on our choice of the "natural" regularization $\mathcal{K}_s^\varepsilon(p,q)$ that we have used to define the principal value in (3.1). First, we will see in the proof of (3.14) that this approximation naturally appears in the computation. This is because $\mathcal{K}_s^\varepsilon(p,q)$ is directly related to the fractional Poisson kernel $\mathbb{P}_{\widetilde{M}}(p,q,z)$ of $\widetilde{M} := M \times (0,+\infty)$ by the formula

$$\mathcal{K}_s^\varepsilon(p,q) = \mathbb{P}_{\widetilde{M}}(p,q,\varepsilon)\varepsilon^{-s}\,,$$

and the fractional Poisson kernel is the fundamental solution of the Caffarelli-Silvestre extension problem. Moreover, if the compact manifold M is replaced by the Euclidean space $\mathbb{R}^n$ then

$$\mathcal{K}_s^\varepsilon(x,y) = \frac{s/2}{\Gamma(1-s/2)}\int_0^\infty H_{\mathbb{R}^n}(x,y,t)e^{-\varepsilon^2/4t}\frac{dt}{t^{1+s/2}} = \frac{\alpha_{n,s}}{(|x-y|^2+\varepsilon^2)^{\frac{n+s}{2}}}\,,$$

which is arguably a very natural regularization of $\frac{\alpha_{n,s}}{|x-y|^{n+s}}$, and is easily seen to give the same notion of principal value that one would get by integrating $\mathcal{K}_s(x,y)$ against a function on $\mathbb{R}^n \setminus B_\varepsilon(y)$ and then taking $\varepsilon \to 0^+$. This is also true on a Riemannian manifold, and actually, many other desingularizations of the (singular) kernel $\mathcal{K}_s(p,q)$ are possible and give the same notion of principal value under mild hypotheses.

Proposition 3.1.1. *Let (M,g) be a closed, n-dimensional Riemannian manifold, and let $\{\mathcal{K}_s^\varepsilon\}_{\varepsilon>0}$ be a family of nonnegative kernels defined on $L^\infty(M)$. Assume that the following hold:*

- *The $\mathcal{K}_s^\varepsilon$ converge uniformly to $\mathcal{K}_s(p,\cdot)$ on compact subsets of $M \setminus \{p\}$, as $\varepsilon \to 0^+$.*
- *There exist some $r = r(p) > 0$ and some chart parametrization $\varphi : \mathcal{B}_r \subset \mathbb{R}^n \to M$ with $\varphi(0) = p$ such that:*

  (i) The flatness assumptions $\mathrm{FA}_2(M,g,r,p,\varphi)$ are satisfied (see Definition 3.4.1).

  (ii) Setting $\widetilde{\mathcal{K}}_s^\varepsilon(y) := \mathcal{K}_s^\varepsilon(p,\varphi(y))$, there is some positive constant C such that

$$\widetilde{\mathcal{K}}_s^\varepsilon(y) \le \frac{C}{|y|^{n+s}}, \quad \forall\, y \in \mathcal{B}_{r(p)}, \tag{3.4}$$

  and moreover, the symmetry condition

$$\left|\widetilde{\mathcal{K}}_s^\varepsilon(y) - \widetilde{\mathcal{K}}_s^\varepsilon(-y)\right| \le \frac{C}{|y|^{n+s-1}} \tag{3.5}$$

  is satisfied.

Then, for every $f \in C^\infty(M)$, the limit

$$\lim_{\varepsilon\to 0}\int_M (f(p)-f(q))\mathcal{K}_s^\varepsilon(p,q)\,dV_q$$

exists and is independent of the family $\mathcal{K}_s^\varepsilon$. In particular, any such family gives the same value for (3.1) as the choice in (3.2).

Proof. Recall that $\varphi^{-1}(p) = 0$. Given $0 < \delta < r(p)$, we can write

$$\int_M (f(p) - f(q))\mathcal{K}_s^\varepsilon(p,q)\, dV_q = \int_{M\setminus\varphi(\mathcal{B}_\delta)} (\dots)\, dV_q + \int_{\varphi(\mathcal{B}_\delta)} (\dots)\, dV_q\,.$$

By the dominated convergence theorem, the first term on the RHS converges to

$$\int_{M\setminus\varphi(\mathcal{B}_\delta)} (f(p) - f(q))\mathcal{K}_s(p,q)\, dV_q\,.$$

Therefore

$$\begin{aligned}\limsup_{\varepsilon\to 0}\Big|\int_M (f(p) - f(q))\mathcal{K}_s^\varepsilon(p,q)\, dV_q - \int_{M\setminus\varphi(\mathcal{B}_\delta)} (f(p) - f(q))\mathcal{K}_s(p,q)\, dV_q\Big| \\ \leq \limsup_{\varepsilon\to 0}\Big|\int_{\varphi(\mathcal{B}_\delta)} (f(p) - f(q))\mathcal{K}_s^\varepsilon(p,q)\, dV_q\Big|.\end{aligned}$$

In other words, to conclude our desired result, it suffices to show that $\int_{\varphi(\mathcal{B}_\delta)}(f(p)-f(q))K_s^\varepsilon(p,q)\, dV_q$ is bounded independently of ε and moreover can be made arbitrarily small by choosing δ small enough.

To check this, we start by changing variables using the coordinates given by φ, leading to

$$\int_{\varphi(\mathcal{B}_\delta)} (f(p) - f(q))\mathcal{K}_s^\varepsilon(p,q)\, dV_q = \int_{\mathcal{B}_\delta} (f(\varphi(0)) - f(\varphi(y)))\widetilde{\mathcal{K}}_s^\varepsilon(y)\sqrt{|g|}(y)\, dy\,.$$

Defining $h(y) := (f(\varphi(0)) - f(\varphi(y)))\sqrt{|g|}(y)$, which verifies that $h(y) = y\cdot\nabla h(0) + O(|y|^2)$, and using the symmetry of the Lebesgue measure under the transformation $y \mapsto (-y)$, we can then compute

$$\begin{aligned}\Big|\int_{\varphi(\mathcal{B}_\delta)} (f(p) - f(q))\mathcal{K}_s^\varepsilon(p,q)\, dV_q\Big| &= \Big|\int_{\mathcal{B}_\delta} h(y)\widetilde{\mathcal{K}}_s^\varepsilon(y)\, dy\Big| \\ &\leq \Big|\int_{\mathcal{B}_\delta} y\cdot\nabla h(0)\widetilde{\mathcal{K}}_s^\varepsilon(y)\, dy\Big| + C\int_{\mathcal{B}_\delta} |y|^2\widetilde{\mathcal{K}}_s^\varepsilon(y)\, dy \\ &= \Big|\frac{1}{2}\int_{\mathcal{B}_\delta} y\cdot\nabla h(0)\widetilde{\mathcal{K}}_s^\varepsilon(y)\, dy + \frac{1}{2}\int_{\mathcal{B}_\delta} (-y)\cdot\nabla h(0)\widetilde{\mathcal{K}}_s^\varepsilon(-y)\, dy\Big| \\ &\quad + C\int_{\mathcal{B}_\delta} |y|^2\widetilde{\mathcal{K}}_s^\varepsilon(y)\, dy \\ &= \frac{1}{2}\Big|\int_{\mathcal{B}_\delta} y\cdot\nabla h(0)(\widetilde{\mathcal{K}}_s^\varepsilon(y) - \widetilde{\mathcal{K}}_s^\varepsilon(-y))\, dy\Big| + C\int_{\mathcal{B}_\delta} |y|^2\widetilde{\mathcal{K}}_s^\varepsilon(y)\, dy\end{aligned}$$

Using the assumptions on the kernel, we conclude that

$$\begin{aligned}\Big|\int_{\varphi(\mathcal{B}_\delta)} (f(p) - f(q))\mathcal{K}_s^\varepsilon(p,q)\, dV_q\Big| &\leq C\int_{\mathcal{B}_\delta} |y\cdot\nabla h(0)|\frac{1}{|y|^{n+s-1}}\, dy + C\int_{\mathcal{B}_\delta} |y|^2\frac{1}{|y|^{n+s}}\, dy \\ &\leq C\int_{\mathcal{B}_\delta} \frac{1}{|y|^{n+s-2}}\, dy \\ &\leq C\delta^{2-s}\,.\end{aligned}$$

Since $s \in (0,2)$, this quantity can be made arbitrarily small by choosing δ small enough, independently of ε. This concludes the proof of our result. □

Remark 3.1.2. *As discussed above, this covers the case of removing a geodesic ball $B_\varepsilon(p)$ in the corresponding definition of the fractional Laplacian as an integral and then sending $\varepsilon \to 0$, as in the usual Euclidean definition of a principal value integral. Indeed, this corresponds to considering $\mathcal{K}_s^\varepsilon := \mathcal{K}_s(p,q)\chi_{M\setminus B_\varepsilon(p)}$ in the proposition above. Another reasonable desingularization could be*

$$\mathcal{K}_s^\varepsilon(p,q) = \frac{s/2}{\Gamma(1-s/2)} \int_\varepsilon^\infty H_M(p,q,t)\frac{dt}{t^{1+s/2}}\,.$$

The fact that both of these choices for the families $\{\mathcal{K}_s^\varepsilon\}_\varepsilon$, as well as the choice (3.2), satisfy the hypotheses in Proposition 3.1.1, can be easily seen using the results that will appear in the next section. More precisely, they follow from the combination of Remark 3.4.3 and (the proof of) estimate (3.38) from Theorem 3.4.6. The latter shows that conditions (3.4) and (3.5) hold directly for the kernel $\mathcal{K}_s$ thanks to precise estimates on the heat kernel H_M, and it is then simple to see that they hold for the regularisations $\mathcal{K}_s^\varepsilon$ as well.

Proposition 3.1.3. *Let (M,g) be a complete, stochastically complete Riemannian manifold. Then*

(i) For $u \in C_c^\infty(M)$, the singular integral $(-\Delta)^{s/2}_{\mathrm{Si}}u$ (defined in (3.1)) and the Bochner $(-\Delta)^{s/2}_{\mathrm{B}}u$ (defined in (3.3)) fractional Laplacian coincide.

(ii) For $u \in L^2(M)$ they coincide as distributions.

Proof. Let $u \in C_c^\infty(M)$. Expressing the solution $P_t u := e^{t\Delta}u$ to the heat equation in terms of the initial datum as

$$P_t u(p) = \int_M u(q)H_M(p,q,t)\,dV_q\,,$$

and using that $\int_M H_M(p,q,t)\,dV_q = 1$ gives, for every $\varepsilon > 0$, that

$$\frac{1}{\Gamma(-s/2)}\int_0^\infty (P_t u - u)\frac{e^{-\varepsilon^2/4t}dt}{t^{1+s/2}} = \int_M (u-u(q))\mathcal{K}_s^\varepsilon(p,q)\,dV_q\,. \tag{3.6}$$

Since u is smooth, letting $\varepsilon \to 0^+$ gives convergence of both integrals pointwise everywhere and

$$\frac{1}{\Gamma(-s/2)}\int_0^\infty (P_t u - u)\frac{dt}{t^{1+s/2}} = P.V.\int_M (u-u(q))\mathcal{K}_s(p,q)\,dV_q\,,$$

and this proves (i).

Now to show (ii) take $u \in L^2(M)$ and $\varphi \in C^\infty(M)$. Multiply (3.6) by φ and integrate over M to get

$$\frac{1}{\Gamma(-s/2)}\int_M\int_0^\infty (P_t u - u)\varphi\frac{e^{-\varepsilon^2/4t}}{t^{1+s/2}}\,dtdV = \iint_{M\times M}(u(p)-u(q))\varphi(p)\mathcal{K}_s^\varepsilon(p,q)\,dV_q dV_p\,.$$

Note that since $\varepsilon > 0$ is fixed and positive, neither of the two integrals above is singular, and they are both absolutely convergent. Hence, we can exchange the other of integration in both integrals.

For the left-hand-side using that P_t is self adjoint in $L^2(M)$ we get

$$\begin{aligned}\frac{1}{\Gamma(-s/2)}\int_M\int_0^\infty (P_tu-u)\varphi\frac{e^{-\varepsilon^2/4t}}{t^{1+s/2}}\,dtdV &= \frac{1}{\Gamma(-s/2)}\int_0^\infty \frac{e^{-\varepsilon^2/4t}}{t^{1+s/2}}\langle P_tu-u,\varphi\rangle_{L^2}\,dt\\ &= \frac{1}{\Gamma(-s/2)}\int_0^\infty \frac{e^{-\varepsilon^2/4t}}{t^{1+s/2}}\langle P_t\varphi-\varphi,u\rangle_{L^2}\,dt\\ &= \int_M\left(\frac{1}{\Gamma(-s/2)}\int_0^\infty (P_t\varphi-\varphi)\frac{e^{-\varepsilon^2/4t}}{t^{1+s/2}}\,dt\right)u\,dV\,.\end{aligned}$$

Regarding the right-hand side, since $\mathcal{K}_s^\varepsilon(p,q)$ is symmetric

$$\iint_{M\times M}(u(p)-u(q))\varphi(p)\mathcal{K}_s^\varepsilon(p,q)\,dV_qdV_p = \iint_{M\times M}(\varphi(p)-\varphi(q))u(p)\mathcal{K}_s^\varepsilon(p,q)\,dV_qdV_p\,.$$

Thus

$$\int_M\left(\frac{1}{\Gamma(-s/2)}\int_0^\infty (P_t\varphi-\varphi)\frac{e^{-\varepsilon^2/4t}}{t^{1+s/2}}\,dt\right)u\,dV = \int_M\left(\int_M(\varphi(p)-\varphi(q))\mathcal{K}_s^\varepsilon(p,q)\,dV_q\right)u(p)dV_p\,,$$

and letting $\varepsilon\to 0^+$ and using (i) proves (ii). □

3.2 The Caffarelli-Silvestre Extension

Definition 3.2.1. *We define the weighted Sobolev space*

$$\widetilde{H}^1(\mathbb{R}^n\times(0,\infty)) = H^1(\mathbb{R}^n\times(0,\infty),z^{1-s}dxdz)$$

as the completion of $C_c^\infty(\mathbb{R}^n\times[0,\infty))$ with the norm

$$\|U\|^2_{\widetilde{H}^1} := \|U\|^2_{L^2(\mathbb{R}^n\times(0,\infty),z^{1-s}dxdz)} + \|\widetilde{D}U\|^2_{L^2(\mathbb{R}^n\times(0,\infty),z^{1-s}dxdz)}\,,$$

where $\widetilde{D}U = \left(\frac{\partial U}{\partial x^1},\dots,\frac{\partial U}{\partial x^n},\frac{\partial U}{\partial z}\right)$ denotes the Euclidean gradient in $\mathbb{R}^{n+1}$. This is a Hilbert space with the natural inner product that induces the norm above. It is a known fact that any $U\in\widetilde{H}^1(\mathbb{R}^n\times(0,\infty))$ has a well defined trace in $L^2(\mathbb{R}^n)$ that we denote by $U(x,\cdot)$.

The following essential result by Caffarelli and Silvestre [CS07] shows that fractional powers of the Laplacian on $\mathbb{R}^n$ can be realized as a Dirichlet-to-Neumann map via an extension problem.

Theorem 3.2.2 ([CS07])**.** *Let $s\in(0,2)$ and $u\in H^{s/2}(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)$. Then, there is a unique solution $U=U(x,z):\mathbb{R}^n\times[0,+\infty)\to\mathbb{R}$ among functions in $\widetilde{H}^1(\mathbb{R}^n\times(0,\infty))$ to the problem*

$$\begin{cases}\mathrm{div}_{\mathbb{R}^{n+1}}(z^{1-s}\widetilde{D}U)=0\,, & \text{on } \mathbb{R}^n\times(0,\infty)\\ U(x,0)=u(x) & \text{for } x\in\mathbb{R}^n\,,\end{cases}\tag{3.7}$$

and it satisfies

$$\lim_{z\to 0^+} z^{1-s}\frac{\partial U}{\partial z}(x,z) = -\beta_s^{-1}\,(-\Delta)^{s/2}u(x)\,,\tag{3.8}$$

where Δ denotes the Laplacian on $\mathbb{R}^n$ and β_s is a positive constant that depends only on s.

In [CS07], three different proofs of this fact are presented, but each of these proofs relies on some additive structure of the base space. It was proved by Stinga in [Sti10] that the unique solution to (3.7) verifying (3.8) admits the explicit representation

$$U(p,z) = \frac{z^s}{2^s\Gamma(s/2)} \int_0^\infty P_t u(p)\, e^{-\frac{z^2}{4t}} \frac{dt}{t^{1+s/2}}\,, \tag{3.9}$$

which expresses U in terms of the solution to the heat equation $P_t u$. The proof of this fact does not strongly rely on the additive structure of $\mathbb{R}^n$ and extends to the case of Riemannian manifolds.

First, let us define the weighted Sobolev spaces for the extension on compact manifolds.

Definition 3.2.3. *We define the weighted Sobolev space*

$$\widetilde{H}^1(\widetilde{M}) = \widetilde{H}^1(M \times (0,\infty))$$

as the completion of $C_c^\infty(M \times [0,\infty))$ with the norm

$$\|U\|^2_{\widetilde{H}^1} := \|TU\|^2_{L^2(M)} + \|\widetilde{\nabla} U\|^2_{L^2(\widetilde{M}, z^{1-s} dV dz)}\,, \tag{3.10}$$

where $TU = U(\cdot, 0)$ is the trace of U and $\widetilde{\nabla} U = (\nabla U, U_z)$ denotes the gradient in $M \times (0,+\infty)$. This is a Hilbert space with the natural inner product that induces the norm above. Moreover, basically by definition, any $U \in \widetilde{H}^1(\widetilde{M})$ leaves a trace in $L^2(M \times \{0\})$.

The weighted Sobolev space $\widetilde{H}^1(\widetilde{M})$, defined as the completion of $C_c^\infty(M\times[0,\infty))$ with respect to the norm (3.10), can be concretely realized as a space of functions U in $L^2_{\mathrm{loc}}(\widetilde{M})$ having weak derivatives $\widetilde{\nabla} U$ in the same weighted space. Indeed, it follows by the fundamental theorem of calculus and Hölder's inequality (see (3.35) in the proof of Lemma 3.3.2) that

$$\int_{M\times(0,R)} |U|^2 \, dV dz \le CR \int_M |TU|^2 \, dV + CR^s \int_{M\times(0,R)} |\widetilde{\nabla} U|^2 z^{1-s} \, dV dz,$$

for every $U \in C_c^\infty(M \times [0,\infty))$ and $R > 0$. This inequality easily implies that every Cauchy sequence of smooth functions converges to an actual function in $L^2_{\mathrm{loc}}(\widetilde{M})$ and the limit is well-defined pointwise almost everywhere.

We have the following essential result, analogous to the classical one for $\mathbb{R}^n$ in [CS07], which shows that the fractional power of the Laplacian on M can be realized as a Dirichlet-to-Neumann map via an extension problem.

Theorem 3.2.4. *Let (M^n, g) be a closed Riemannian manifold, let $s \in (0,2)$ and $u : M \to \mathbb{R}$ be smooth. Consider the product manifold $\widetilde{M} = M \times (0,+\infty)$ endowed with the natural product metric. Then, there is a unique solution $U : M \times (0,\infty) \to \mathbb{R}$ among functions in $\widetilde{H}^1(\widetilde{M})$ to*

$$\begin{cases} \widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla} U) = 0 & \text{in } \ \widetilde{M}\,, \\ U(p,0) = u(p) & \text{for } \ p \in \partial\widetilde{M} = M\,, \end{cases} \tag{3.11}$$

given by (3.9), and it satisfies

$$[u]^2_{H^{s/2}(M)} = 2\beta_s \int_{\widetilde{M}} |\widetilde{\nabla} U|^2 z^{1-s} \, dV dz\,, \tag{3.12}$$

where

$$[u]^2_{H^{s/2}(M)} := \iint_{M\times M} (u(p) - u(q))^2 \mathcal{K}_s(p,q) \, dV_p dV_q,$$

and
$$\beta_s := \frac{2^{s-1}\Gamma(s/2)}{\Gamma(1-s/2)}\,. \tag{3.13}$$
Moreover
$$\lim_{z\to 0^+} z^{1-s}\frac{\partial U}{\partial z}(p,z) = -\beta_s^{-1}(-\Delta)^{s/2}u(p)\,, \tag{3.14}$$
where the fractional Laplacian on the right-hand side is defined by either (3.3) *or* (3.1) *(since these coincide on smooth functions).*

Proof. Note that functions in $\widetilde{H}^1(\widetilde{M})$ leave a well-defined trace on $M\times\{0\}$ (that is, there exists a continuous trace operator with respect to the norm on $\widetilde{H}^1(\widetilde{M})$). Then, the fact that a solution among functions in $\widetilde{H}^1(\widetilde{M})$ exists follows by direct minimization of the associated energy $v\mapsto \int_{\widetilde{M}} |\widetilde{\nabla} v|^2 z^{1-s}dV dz$ over $\widetilde{H}^1(\widetilde{M})$. Since the energy is convex, the solution is also unique.

From here we divide the proof in three steps.

Step 1. We show that the (unique) solution $U\in \widetilde{H}^1(\widetilde{M})$ to (3.11) is given by (3.9).

Making the identification $T(M\times(0,+\infty))\simeq TM\times(0,+\infty)$ we have
$$\begin{aligned}\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla} U) &= \operatorname{div}_g(z^{1-s}\nabla U) + \partial_z(z^{1-s}U_z)\\ &= z^{1-s}\Delta U + (1-s)z^{-s}U_z + z^{1-s}U_{zz}\\ &= z^{1-s}(\Delta U + \frac{1-s}{z}U_z + U_{zz})\,.\end{aligned}$$
Thus, in order to prove that U solves $\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla} U)=0$ we show that U (weakly) solves
$$\mathcal{L}U := \Delta U + \frac{1-s}{z}U_z + U_{zz} = 0\,. \tag{3.15}$$
Define
$$G(z,t) := \frac{1}{2^s\Gamma(s/2)}\frac{z^s e^{-\frac{z^2}{4t}}}{t^{1+s/2}}\,,$$
so that (3.9) rewrites simply as
$$U(\cdot,z) = \int_0^\infty (P_t u)G(z,t)\,dt\,. \tag{3.16}$$
It can be easily checked that G satisfies
$$-G_t + \frac{1-s}{z}G_z + G_{zz} = 0\,, \tag{3.17}$$
and also
$$\lim_{t\to 0^+}\sup_{[z_1,z_2]} G(\cdot,t) = 0\,, \qquad \lim_{t\to+\infty}\sup_{[z_1,z_2]} G(\cdot,t) = 0\,,$$
for every $0<z_1<z_2$. Moreover, from the definition of G and the fact that u is smooth, we see that the integral on the right-hand side of (3.16) is absolutely convergent in $\widetilde{H}^1(\widetilde{M})$. Hence $U\in\widetilde{H}^1(\widetilde{M})$.

Now we check that U weakly solves the desired problem. Let $\varphi\in C_c^\infty(\widetilde{M})$, $K:=\operatorname{supp}(\varphi)$ and

$z_1, z_2 \in (0, +\infty)$ such that $K \Subset M \times [z_1, z_2]$. Let also

$$\mathcal{L}^*\varphi := \Delta\varphi + \partial_z\left(\frac{1-s}{z}\varphi\right) + \varphi_{zz}\,.$$

This is the formal adjoint of the operator in (3.15). Clearly $\mathcal{L}^*\varphi$ still has compact support in $K \Subset \widetilde{M}$ and is smooth. Then

$$\int_{\widetilde{M}} U\mathcal{L}^*(\varphi)\,dVdz = \int_{\widetilde{M}}\int_0^\infty (P_t u)G(z,t)\mathcal{L}^*(\varphi)\,dtdVdz\,,$$

and we claim that this integral is absolutely convergent. Indeed

$$\begin{aligned}\int_{\widetilde{M}}\int_0^\infty |(P_t u)G(z,t)\mathcal{L}^*(\varphi)|\,dt\,dVdz &\le \|\mathcal{L}^*(\varphi)\|_{L^\infty}\int_0^\infty\int_K |P_t u||G(z,t)|\,dVdzdt\\ &\le C\left(\int_K |P_t u|^2|G(z,t)|^2 z^{1-s}\,dVdz\right)^{\frac{1}{2}}\left(\int_K \frac{1}{z^{1-s}}\,dVdz\right)^{\frac{1}{2}} < +\infty\,,\end{aligned}$$

since the integral in (3.16) is absolutely convergent in $\widetilde{H}^1(\widetilde{M})$. Hence we can exchange the order of integration and we get, integrating by parts in space many times

$$\begin{aligned}\int_{\widetilde{M}} U\mathcal{L}^*(\varphi)\,dVdz &= \int_0^\infty\left(\int_K (P_t u)G(z,t)\mathcal{L}^*(\varphi)\,dVdz\right)dt\\ &= \int_0^\infty\int_K\left(G(z,t)\Delta(P_t u) + (P_t u)\frac{1-s}{z}G_z(z,t) + (P_t u)G_{zz}(z,t)\right)\varphi\,dVdzdt\,.\end{aligned}$$

Since $P_t u$ is smooth and solves the heat equation, the first term equals to

$$\begin{aligned}\int_0^\infty\int_K G(z,t)\Delta(P_t u)\,dVdzdt &= \int_0^\infty\int_K G(z,t)\partial_t(P_t u)\,dVdzdt = \int_K\int_0^\infty G(z,t)\partial_t(P_t u)\,dtdVdz\\ &= \int_K (P_t u)G(z,t)\,dVdz\Big|_{0^+}^{\infty} - \int_K\int_0^\infty (P_t u)G_t(z,t)\,dtdVdz\,.\end{aligned}$$

The boundary terms vanish since

$$\begin{aligned}\left|\int_K (P_t u)G(z,t)\,dVdz\right| &\le \left(\int_M |P_t u|\,dV\right)\left(\int_{z_1}^{z_2}|G(z,t)|\,dz\right)\\ &\le |M|^{1/2}\|u\|_{L^2(M)}|z_2 - z_1|\sup_{[z_1,z_2]} G(\cdot,t) \to 0\,,\end{aligned}$$

both as $t \to \infty$ and as $t \to 0^+$. Hence, putting all together and using (3.17)

$$\int_{\widetilde{M}} \mathcal{L}(U)\varphi\,dVdz = \int_{\widetilde{M}} U\mathcal{L}^*(\varphi)\,dV_p dz = \int_K\int_0^\infty (P_t u)\left(-G_t + \frac{1-s}{z}G_z + G_z z\right)\varphi\,dVdzdt = 0.$$

Hence U given by (3.9) is a weak solution of (3.15), and by standard elliptic regularity it is also a classical solution.

Moreover, the fact that $U(\cdot, 0^+) = u$ follows by the explicit formula (3.9). Indeed, by a simple change of variable in the integral, we have

$$U(p,z)=\frac{1}{\Gamma(s/2)}\int_0^\infty (P_{z^2/4r}u)(p)\frac{e^{-r}}{r^{1-s}}dr\,,$$

and taking $z\to 0^+$ in this formula gives $U(\cdot,0^+)=u$. This concludes Step 1.

Step 2. Proof of (3.14).
Note that by the representation formula we just proved for U we have

$$\tau_z^s U(p):=\frac{U(p,z)-u(p)}{z^s}=\frac{1}{2^s\Gamma(s/2)}\int_0^\infty (P_tu(p)-u(p))e^{-\frac{z^2}{4t}}\frac{dt}{t^{1+s/2}}\,.$$

Moreover, by l'Hopital's rule

$$\lim_{z\to 0^+}\tau_z^s U=\lim_{z\to 0^+}s^{-1}z^{1-s}\frac{\partial U}{\partial z}\,.$$

Writing P_tu as the convolution against the heat kernel H_M of M we get

$$\tau_z^s U(p)=\frac{1}{2^s\Gamma(s/2)}\int_0^\infty\int_M H_M(p,q,t)(u(q)-u(p))\frac{e^{-\frac{z^2}{4t}}}{t^{1+s/2}}dV_q dt\,.$$

Since u is smooth, and since $\mathcal{K}_s^\varepsilon(p,q)$ (defined in (3.2)) is non-singular for $\varepsilon>0$, there holds

$$\int_M\frac{s/2}{\Gamma(1-s/2)}\int_0^\infty H_M(p,q,t)|u(q)-u(p)|\frac{e^{-\frac{z^2}{4t}}}{t^{1+s/2}}dtdV_q=\int_M|u(q)-u(p)|\mathcal{K}_s^z(p,q)\,dV_q<+\infty\,.$$

Thus the integral in $\tau_z^sU(p)$ is absolutely convergent, and we can exchange the order of integration to get

$$\tau_z^s U(p)=\frac{\Gamma(1-s/2)}{s2^{s-1}\Gamma(s/2)}\int_M (u(q)-u(p))\mathcal{K}_s^z(p,q)\,dV_q\,.$$

From here, by the very definition of the principal value

$$\begin{aligned}\lim_{z\to 0^+}z^{1-s}\frac{\partial U}{\partial z}=\lim_{z\to 0^+}s\cdot\tau_z^sU&=\lim_{z\to 0^+}\beta_s^{-1}\int_M(u(q)-u)\mathcal{K}_s^z(p,q)\,dV_q\\&=-\beta_s^{-1}\left(P.V.\int_M(u-u(q))\mathcal{K}_s(p,q)\,dV_q\right)\\&=-\beta_s^{-1}(-\Delta)^{s/2}u\,.\end{aligned}$$

This finishes Step 2.

Step 3. The convergence in (3.14) holds in $L^r(M)$ for every $r\in[1,\infty)$.
Since we have already proved pointwise convergence, it suffices to show that the sequence is dominated. In particular, we prove that for $z\le 1$ there holds

$$z^{1-s}|U_z(\cdot,z)|\le C, \tag{3.18}$$

where C depends on $\|\Delta u\|_{L^\infty}$, $\|u\|_{L^\infty}$ and s. The proof is a standard barrier argument very similar to the proof of Lemma 3.2.7. Consider $b(p,z):=u(p)-C(z^2-2z^s)$, for $C>0$ that will be chosen soon. Since

$$\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}b)=z^{1-s}(\Delta u-(4-2s)C)\,,$$

taking $C = \frac{1}{4-2s}\|\Delta u\|_{L^\infty} + 2\|u\|_{L^\infty}$, we see that b is a supersolution of (3.11) and that $U \le b$ on $M \times \{0,1\}$. Hence b is barrier for U, and by the maximum principle for $z \le 1$ we have

$$U(\cdot, z) \le u - \left(\frac{1}{4-2s}\|\Delta u\|_{L^\infty} + 2\|u\|_{L^\infty}\right)(z^2 - 2z^s) \le u + z^s\left(\frac{1}{2-s}\|\Delta u\|_{L^\infty} + 4\|u\|_{L^\infty}\right),$$

and this implies

$$\lim_{z\to 0^+} z^{1-s}U_z = \lim_{z\to 0^+} s\frac{U(\cdot,z) - u}{z^s} \le \frac{s}{2-s}\|\Delta u\|_{L^\infty} + 4s\|u\|_{L^\infty}\,.$$

Using $-b$ as a barrier for $-U$, complete analogously, one also gets the reverse inequality. Moreover, note that the function $V := z^{1-s}U_z$ solves $-(\Delta V + V_{zz}) + \frac{1-s}{z}V_z = 0$, thus by the maximum principle

$$\sup_{M\times(0,1]} |V| \le \max\left\{\sup_M V(\cdot, 0^+), \sup_M V(\cdot, 1)\right\} \le \max\left\{C(\|\Delta u\|_{L^\infty}, \|u\|_{L^\infty}, s), \sup_M V(\cdot, 1)\right\}.$$

But since $V(\cdot, 1) = U_z(\cdot, 1)$ we have by standard interior gradient estimates

$$|U_z(p, 1)| \le C \sup_{\widetilde{B}_{1/10}(p,1)} |U| \le C\|u\|_{L^\infty}\,,$$

for some absolute constant $C > 0$ independent of u. Putting everything together

$$\sup_{M\times(0,1]} |V| = \sup_{M\times(0,1]} z^{1-s}|U_z| \le C(\|\Delta u\|_{L^\infty}, \|u\|_{L^\infty}, s)\,,$$

and this concludes the proof of (3.18) and of Step 3.

We're left with proving (3.12). Integrating by parts, for every $\delta > 0$ we find that

$$\int_{M\times[\delta,\infty)} |\widetilde{\nabla} U|^2 z^{1-s}\, dV dz = \beta_s^{-1}\int_M U(\cdot, \delta)\delta^{1-s}U_z(\cdot, \delta)\, dV\,.$$

Letting now $\delta \to 0^+$, by (3.14), (3.18) and dominated convergence on the right-hand side

$$\int_{\widetilde{M}} |\widetilde{\nabla} U|^2 z^{1-s}\, dV dz = \beta_s^{-1}\int_M u(-\Delta)^{s/2}u\, dV = \frac{\beta_s^{-1}}{2}[u]^2_{H^{s/2}(M)}\,.$$

This concludes the proof.

□

3.2.1 Proof of the monotonicity formula (Theorem 1.2.2)

The monotonicity formula for minimizing s-minimal surfaces in $\mathbb{R}^n$ was proved in the seminal article [CRS10], and for Allen-Cahn type critical points, it was first obtained in [CC14]. In [MSW19], the monotonicity formula is shown to extend to stationary s-minimal surfaces. Here, we prove the analogous (local) monotonicity formula on a Riemannian manifold. The proof holds simultaneously for any s-minimal surface, that is, for any stationary point of the fractional perimeter regardless of second variation or regularity, and also for any stationary point of a semilinear elliptic functional with a nonnegative potential term, hence including the fractional

Allen-Cahn energy. For $r > 0$ and $p \in M$ denote

$$\begin{aligned} B_r(p) &= \left\{ q \in M \,:\, d_g(q,p) < r \right\}, \\ \widetilde{B}^+_r(p,0) &= \left\{ (q,z) \in \widetilde{M} \,:\, d_{\widetilde{g}}((q,z),(p,0)) < r \right\}, \\ \partial \widetilde{B}^+_r(p,0) &= \partial \left(\widetilde{B}^+_r(p,0) \right) \\ \partial^+ \widetilde{B}^+_r(p,0) &= \partial \widetilde{B}^+_r(p,0) \cap \{z > 0\}. \end{aligned} \tag{3.19}$$

In this section, since there will be no possible ambiguity, we will use ∇ instead of $\widetilde{\nabla}$ to denote the gradient in $\widetilde{M}$ with respect to the product metric.

Before proving the monotonicity formula of Theorem 1.2.2, we will need two preliminary lemmas from Riemannian geometry, which will allow us to bound the "Riemannian errors" in two formulas regarding the distance function.

Lemma 3.2.5. *Let (M^n, g) be an n-dimensional Riemannian manifold, $p \in M$, $R_0 < \mathrm{inj}_M(p)$ and let K be an upper bound for all the sectional curvatures in $B_{R_0}(p)$. Denote by d the distance function to the point p. Then, for all $R < \min\{R_0, \frac{1}{\sqrt{K}}\}$ there holds in $B_R(p)$:*

$$|\langle \nabla_V(d\nabla d), V\rangle - |V|^2| \leq \sqrt{K} R |V|^2 ,$$

for every vector field V on M.

Proof. We can compute

$$\begin{aligned} \langle V, \nabla_V(d\nabla d)\rangle &= \langle V, \langle V, \nabla d\rangle \nabla d\rangle + d\langle V, \nabla_V(\nabla d)\rangle \\ &= \langle V, \nabla d\rangle^2 + d\, \nabla^2 d(V,V). \end{aligned}$$

On the other hand, the Hessian Comparison theorem—see Lemma 7.1 in [CM11]—gives that

$$|d\,\nabla^2 d(V,V) - |V - \langle V, \nabla d\rangle \nabla d|^2| \leq d\sqrt{K}|V|^2$$

in $B_R(p)$, whenever $R < \min\{\mathrm{inj}_M(p), \frac{1}{\sqrt{K}}\}$. Moreover, since $|\nabla d|^2 = 1$, we also have that

$$|V - \langle V, \nabla d\rangle \nabla d|^2 = |V|^2 - 2\,\langle V, \nabla d\rangle^2 + \langle V, \nabla d\rangle^2 |\nabla d|^2 = |V|^2 - \langle V, \nabla d\rangle^2.$$

Hence

$$|d\,\nabla^2 d(V,V) + \langle V, \nabla d\rangle^2 - |V|^2| \leq d\sqrt{K}|V|^2 \leq R\sqrt{K}|V|^2$$

holds in $B_R(p)$, as long as $R < \min\{R_0, \frac{1}{\sqrt{K}}\}$, and this conludes the proof. ☐

Lemma 3.2.6. *Let (M^n, g) be an n-dimensional Riemannian manifold, $p \in M$, $R_0 < \mathrm{inj}_M(p)$ and let K be an upper bound for all the sectional curvatures in $B_{R_0}(p)$. Then, there exists $C = C(n) > 0$ such that, for all $R < R_0$, in $B_R(p)$ we have that*

$$|\mathrm{div}(d\nabla d) - n| \leq CKR^2 .$$

Proof. Fix $p \in M$, and denote $d(p,\cdot)$ just by $d(\cdot)$. Observe first that every geodesic σ with $\sigma(0) = p$ and contained in $B_{R_0}(p)$ is uniquely minimizing. For any $R < R_0$ and $x \in B_R(p)$, let $\gamma : [0,d] \to M$ be the normalized geodesic with $\gamma(0) = p$ and $\gamma(d) = x$. Note also that

$$\mathrm{div}(d\nabla d) = |\nabla d|^2 + d\Delta d = 1 + d\Delta d .$$

Consider $\dot{\gamma}(d) \in T_xM$, and complete it to an orthonormal basis $\{e_1 := \dot{\gamma}(d), e_2, \dots, e_n\}$ of T_xM. For $i = 2, 3, \dots, n$, let γ_i be the geodesic with $\gamma_i(0) = x$ and $\dot{\gamma}_i(0) = e_i$. We can compute

$$\Delta d(x) = \sum_{i=1}^{n} \nabla^2 d(x)(e_i, e_i) = \sum_{i=1}^{n} \frac{d^2}{ds^2}\bigg|_{s=0} (d \circ \gamma_i) = \sum_{i=2}^{n} \frac{d^2}{ds^2}\bigg|_{s=0} (d \circ \gamma_i)\,,$$

where we have used that $\frac{d^2}{ds^2}\Big|_{s=0} (d \circ \gamma) = \frac{d^2}{ds^2}\Big|_{s=0} (d(x) + s) = 0$.
Let J_i be the Jacobi field along γ with $J_i(0) = 0$ and $J_i(d) = e_i$, well defined by uniqueness of geodesics between endpoints. Denote by

$$I(X, Y) = \int_0^d \langle D_t X, D_t Y\rangle - \mathrm{Rm}(\dot{\gamma}, X, \dot{\gamma}, Y)\, dt$$

the index form associated to γ on $[0, d]$. Since γ is minimizing along all curves with the same endpoints, for every vector field X on $\gamma([0, d])$ orthogonal to $\dot{\gamma}$ and with $X(0) = 0$ and $X(d) = e_i$ we must have

$$0 \le I(J_i - X, J_i - X) = I(J_i, J_i) - 2I(J_i, X) + I(X, X)\,.$$

Since J_i is a Jacobi field, one can easily check that $I(J_i, X) = I(J_i, J_i)$, hence $I(J_i, J_i) \le J(X, X)$. Take $X(t) = \frac{t}{d} E_i(t)$, where $E_i(t)$ is the parallel transport of $e_i \in T_xM$ along γ. From the second variation formula for arc length we get

$$\begin{aligned}
\frac{d^2}{ds^2}\bigg|_{s=0} (d \circ \gamma_i) &= \int_0^d |D_t J_i|^2 - \mathrm{Rm}(\dot{\gamma}, J_i, \dot{\gamma}, J_i)\, dt = I(J_i, J_i) \\
&\le I(X, X) = \int_0^d |D_t X|^2 - \mathrm{Rm}(\dot{\gamma}, X, \dot{\gamma}, X)\, dt \\
&\le \int_0^d |D_t X|^2 + K|X|^2\, dt\,,
\end{aligned}$$

where we have used that $\sup_{p \in B_{R_0}} |\mathrm{Sec}_p| \le K$. Thus

$$\frac{d^2}{ds^2}\bigg|_{s=0} (d \circ \gamma_i) \le \int_0^d |D_t X|^2 + K|X|^2\, dt = \int_0^d \frac{1}{d^2} + K\frac{t^2}{d^2}\, dt = \frac{1}{d}\left(1 + K\frac{d^2}{3}\right).$$

Hence

$$d\Delta d = \sum_{i=2}^{n} \frac{d^2}{ds^2}\bigg|_{s=0} (d \circ \gamma_i) \le n - 1 + K\frac{n-1}{3} d^2\,,$$

or equivalently

$$|\mathrm{div}(d\nabla d)(x) - n| = |d(x)\Delta d(x) + 1 - n| \le K\frac{n-1}{3} d^2 \le K\frac{n-1}{3} R^2,$$

and this completes the proof with $C(n) = \frac{n-1}{3} > 0$. □

We can now prove the monotonicity formula.

Proof of Theorem 1.2.2. Since during the entire proof, the point $p_\circ \in M$ will be fixed, we will not specify the center of the balls in what follows, as this will always be $(p_\circ, 0)$ for balls inside $\widetilde{M}$ and $p_\circ$ for balls on M. We divide the proof into two steps.

Step 1. First, we show that if u is stationary for the energy $\mathcal{E}(v) = [v]^2_{H^{s/2}(M)} + \int_M F(v)$ under inner variations, then its Caffarelli-Silvestre extension U is stationary for the energy

$$U \mapsto 2\beta_s \int_{\widetilde{M}} z^{1-s} |\nabla U|^2 \, dV dz + \int_M F(U|_M) \, dV \,,$$

under inner variations on $\widetilde{M}$ given by vector fields Y on $\widetilde{M}$ such that $Y|_M$ is tangent to M.

Recall that the Caffarelli-Silvestre extension of u is given by (3.11).

Let Y be a vector field on $\widetilde{M}$ such that $Y|_M$ is tangent to M, and let ψ^t_Y denote its flow at time t. Let also V_t be the Caffarelli-Silvestre extension of $u \circ \psi^t_{Y|_M}$, for any $t \in \mathbb{R}$. By the minimality of the extension in the energy space, we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} 2\beta_s \int_{\widetilde{M}} z^{1-s} |\nabla(U \circ \psi^{-t}_Y)|^2 &= \lim_{t\to 0} \frac{1}{t} \left(2\beta_s \int_{\widetilde{M}} z^{1-s} |\nabla U|^2 - 2\beta_s \int_{\widetilde{M}} z^{1-s} |\nabla (U \circ \psi^t_Y)|^2 \right) \\
&\le \lim_{t\to 0} \frac{1}{t} \left(2\beta_s \int_{\widetilde{M}} z^{1-s} |\nabla U|^2 - 2\beta_s \int_{\widetilde{M}} z^{1-s} |\nabla V_t|^2 \right) \\
&= \lim_{t\to 0} \frac{[u]^2_{H^{s/2}(M)} - [u \circ \psi^t_Y]^2_{H^{s/2}(M)}}{t} \\
&= \frac{d}{dt}\Big|_{t=0} [u \circ \psi^{-t}_Y]^2_{H^{s/2}(M)} \,,
\end{aligned}$$

and likewise

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} 2\beta_s \int_{\widetilde{M}} z^{1-s} |\nabla(U \circ \psi^{-t}_Y)|^2 &= \lim_{t\to 0} \frac{1}{t} \left(2\beta_s \int_{\widetilde{M}} z^{1-s} |\nabla (U \circ \psi^{-t}_Y)|^2 - 2\beta_s \int_{\widetilde{M}} z^{1-s} |\nabla U|^2 \right) \\
&\ge \lim_{t\to 0} \frac{1}{t} \left(2\beta_s \int_{\widetilde{M}} z^{1-s} |\nabla V_{-t}|^2 - 2\beta_s \int_{\widetilde{M}} z^{1-s} |\nabla U|^2 \right) \\
&= \lim_{t\to 0} \frac{[u \circ \psi^{-t}_Y]^2_{H^{s/2}(M)} - [u]^2_{H^{s/2}(M)}}{t} \\
&= \frac{d}{dt}\Big|_{t=0} [u \circ \psi^{-t}_Y]^2_{H^{s/2}(M)} \,.
\end{aligned}$$

Hence

$$\frac{d}{dt}\Big|_{t=0} 2\beta_s \int_{\widetilde{M}} z^{1-s} |\nabla(U \circ \psi^{-t}_Y)|^2 = \frac{d}{dt}\Big|_{t=0} [u \circ \psi^{-t}_Y]^2_{H^{s/2}(M)}.$$

Since u is stationary for the energy $\mathcal{E}(v) = [v]^2_{H^{s/2}(M)} + \int_M F(v)\, dV$ under inner variations, this shows that U is stationary for the energy $U \mapsto 2\beta_s \int_{\widetilde{M}} z^{1-s} |\nabla U|^2 \, dV dz + \int_M F(U|_M)\, dV$ under inner variations on $\widetilde{M}$, with vector fields Y as above, and this concludes the first step.

Step 2. We now compute such an inner variation for a suitably chosen Y. First, the variation of the potential part of the energy is

$$\frac{d}{dt}\Big|_{t=0} \int_M F(u \circ \psi^{-t}_Y) \, dV = \frac{d}{dt}\Big|_{t=0} \int_M F(u) J_t(p) \, dV_p$$

$$= \int_M F(u)\mathrm{div}_g(Y|_M)\, dV. \tag{3.20}$$

The quantity $\mathrm{div}_g(Y|_M)$ will be estimated later. We now focus on computing the variation for the Sobolev part of the energy. Once again, we change variables in the integral using the flow ψ^t_Y, obtaining

$$\int_{\widetilde{M}} z^{1-s}|\nabla(U\circ\psi_Y^{-t})|^2\, dV dz = \int_{\widetilde{M}} (z\circ\psi^t_Y)^{1-s}|\nabla(U\circ\psi_Y^{-t})|^2\circ\psi^t_Y\, J_t(p,z)\, dV_p dz\,. \tag{3.21}$$

Now, we choose the vector field Y. We take $Y = \eta(d)d\nabla d$, where $d = d_{\widetilde{g}}((p_\circ, 0), \cdot\,)$ is the distance on $\widetilde{M}$ from the point $(p_\circ, 0)$ and $\eta = \eta_\delta$ is a single variable smooth function with $\eta \equiv 1$ on $[0, R]$, decreasing to zero on $[R, R+\delta]$, and $\eta \equiv 0$ on $[R+\delta, +\infty)$. Since the distance $d_{\widetilde{g}}((p_\circ, 0), \cdot\,)$ restricts to the distance $d_g(p_\circ, \cdot\,)$ on M when computed on points on $\widetilde{M}$ with $z = 0$, clearly $Y|_M$ is tangent to M. We want to exchange the order of derivation and integration in (3.21). Hence, we compute separately the three terms that will appear in doing so. For the first term, using that $d^2_{\widetilde{g}}((p, z),(p_\circ, 0)) = d^2_g(p, p_\circ) + z^2$ and the definition of Y we see that

$$\frac{d}{dt}\Big|_{t=0}(z\circ\psi^t_Y)^{1-s} = (1-s)z^{-s}\eta(d)z = (1-s)z^{1-s}\eta(d)\,.$$

As for the second term that will appear, a simple general computation—see for example the lines after Lemma 3.1 in [Gas20]—shows that

$$\frac{d}{dt}\Big|_{t=0}|\nabla(U\circ\psi_Y^{-t})|^2(\psi^t_Y(x)) = -2\langle\nabla_{\nabla U}Y, \nabla U\rangle\,.$$

Moreover, using the form chosen for Y we have that

$$\begin{aligned}
\langle\nabla_{\nabla U}Y, \nabla U\rangle &= \langle\nabla_{\nabla U}(\eta(d)d\nabla d), \nabla U\rangle\\
&= \langle\nabla U, \nabla\eta(d)\rangle\langle d\nabla d, \nabla U\rangle + \eta(d)\langle\nabla_{\nabla U}(d\nabla d), \nabla U\rangle\\
&= \eta'(d)\langle\nabla U, \nabla d\rangle\langle d\nabla d, \nabla U\rangle + \eta(d)\langle\nabla_{\nabla U}(d\nabla d), \nabla U\rangle\\
&= d\eta'(d)|\langle\nabla U, \nabla d\rangle|^2 + \eta(d)\langle\nabla_{\nabla U}(d\nabla d), \nabla U\rangle\,.
\end{aligned}$$

Notice that K is also an upper bound for all the sectional curvatures on $\widetilde{M}$ in $\widetilde{B}_{R_\circ}(p_\circ, 0)$ and that $\mathrm{inj}_M(p_\circ) = \mathrm{inj}_{\widetilde{M}}(p_\circ, 0)$. Thus, by Lemma 3.2.5 applied to $V = \nabla U$,

$$\langle\nabla_{\nabla U}Y, \nabla U\rangle = d\eta'(d)|\langle\nabla U, \nabla d\rangle|^2 + \eta(d)(1 + O(\sqrt{K}R))|\nabla U|^2$$

for all $R < \min\left\{R_\circ, \frac{1}{\sqrt{K}}\right\}$. Lastly, for the remaining factor in the integral, Lemma (3.2.6) gives that

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}J_t = \widetilde{\mathrm{div}}(Y) &= \eta'(d)d|\nabla d|^2 + \eta(d)\widetilde{\mathrm{div}}(d\nabla d)\\
&= d\eta'(d) + \eta(d)(n+1)(1 + O(\sqrt{K}R))\,,
\end{aligned}$$

in $B_R(p_\circ)$, for $R < \min\left\{R_\circ, \frac{1}{\sqrt{K}}\right\}$.

Now, analogously applying Lemma (3.2.6) on M instead of $\widetilde{M}$ to (3.20), we already find an estimation for the potential energy:

$$\frac{d}{dt}\Big|_{t=0}\int_M F(u\circ\psi_Y^{-t})\,dV = \int_M F(u)\big(d\eta'(d)+\eta(d)n(1+O(\sqrt{K}R))\big)\,dV\,.$$

Moreover, it follows from (the local version of) Bonnet-Myers' theorem that $R_\circ < R_{\max} := \mathrm{inj}_M(p_\circ)/4 < \min\left\{\mathrm{inj}_M(p_\circ), \frac{1}{\sqrt{K}}\right\}$, and this will be our final choice of $R_{\max}$ for the statement. From now on, we always consider $R < R_\circ \le R_{\max} = \mathrm{inj}_M(p_\circ)/4$.
Regarding the Sobolev part of the energy, exchanging differentiation and integration and substituting the estimates we have obtained so far gives:

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0}\int_{\widetilde{M}} z^{1-s}|\nabla(U\circ\psi_Y^{-t})|^2\\
&=\int_{\widetilde{M}}(1-s)z^{1-s}\eta(d)|\nabla U|^2 + z^{1-s}\big(-2d\eta'(d)|\langle\nabla U,\nabla d\rangle|^2 - 2\eta(d)(1+O(\sqrt{K}R))|\nabla U|^2\big)\\
&\quad+\int_{\widetilde{M}} z^{1-s}|\nabla U|^2\big(d\eta'(d)+\eta(d)(n+1)(1+O(\sqrt{K}R))\big)\,dV\,dz\\
&=(n-s)(1+O(\sqrt{K}R))\int_{\widetilde{B}^+_{R+\delta}} z^{1-s}|\nabla U|^2\eta(d) + \int_{\widetilde{B}^+_{R+\delta}\setminus\widetilde{B}^+_R} z^{1-s}d\eta'(d)\big(|\nabla U|^2 - 2|\langle\nabla U,\nabla d\rangle|^2\big).
\end{aligned}$$

Adding the expressions for the potential and Sobolev parts of the energy, we get

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}&\left(2\beta_s\int_{\widetilde{M}} z^{1-s}|\nabla(U\circ\psi_Y^{-t})|^2 + \int_M F(u\circ\psi_Y^{-t})\right)\\
&= (n-s)(1+O(\sqrt{K}R))2\beta_s\int_{\widetilde{B}^+_{R+\delta}}\eta(d)z^{1-s}|\nabla U|^2\\
&\quad+2\beta_s\int_{\widetilde{B}^+_{R+\delta}\setminus\widetilde{B}^+_R} d\eta'(d)z^{1-s}(|\nabla U|^2 - 2\langle\nabla U,\nabla d\rangle^2)\\
&\quad+n(1+O(\sqrt{K}R))\int_{B_{R+\delta}}\eta(d)F(u) + \int_{B_{R+\delta}\setminus B_R} d\eta'(d)F(u)\,.
\end{aligned}$$

By stationarity of u and Step 1 we know that the left-hand side is equal to 0 for every Y, thus the right-hand side vanishes for all $\eta=\eta_\delta$ defined as above. Since this holds for all $\delta>0$, we now let $\delta\searrow 0$ so that η_δ converges to the characteristic function of $[0,R]$. This gives (for a.e. $R\in(0,R_\circ)$)

$$\begin{aligned}
0 =&(n-s)(1+O(\sqrt{K}R))2\beta_s\int_{\widetilde{B}^+_R} z^{1-s}|\nabla U|^2 - 2R\beta_s\int_{\partial\widetilde{B}^+_R} z^{1-s}|\nabla U|^2 + 4R\beta_s\int_{\partial^+\widetilde{B}^+_R}(\partial_\nu U)^2\\
&+n(1+O(\sqrt{K}R))\int_{B_R}F(u) - R\int_{\partial B_R}F(u).
\end{aligned}$$

Rearranging the terms and multiplying by R^{-n+s-1}, we deduce that

$$\begin{aligned}
&-\frac{(n-s)}{R^{n-s+1}}\left(2\beta_s\int_{\widetilde{B}^+_R} z^{1-s}|\nabla U|^2 + \int_{B_R}F(u)\right) + \frac{1}{R^{n-s}}\left(2\beta_s\int_{\partial\widetilde{B}^+_R} z^{1-s}|\nabla U|^2 + \int_{\partial B_R}F(u)\right)\\
&\ge -\frac{C\sqrt{K}}{R^{n-s}}\left(2\beta_s\int_{\widetilde{B}^+_R} z^{1-s}|\nabla U|^2 + \int_{B_R}F(u)\right) + \frac{\beta_s}{R^{n-s}}\int_{\partial^+\widetilde{B}^+_R} z^{1-s}\langle\nabla U,\nabla d\rangle^2 + \frac{s}{R^{n-s+1}}\int_{B_R}F(u)\,,
\end{aligned}$$

for some absolute constant $C > 0$. In other words,

$$\Phi'(R) \geq -C\sqrt{K}\Phi(R) + \frac{\beta_s}{R^{n-s}} \int_{\partial^+ \widetilde{B}_R^+} z^{1-s} \langle \nabla U, \nabla d\rangle^2 + \frac{s}{R^{n-s+1}} \int_{B_R} F(u)\, dV\,,$$

and this implies, in particular, that

$$\frac{d}{dR}\Big(e^{C\sqrt{K}R}\Phi(R)\Big) \geq 0 \quad \text{for all} \quad R < R_\circ.$$

Lastly, in the case where $M = \mathbb{R}^n$, $F \equiv 0$, $s \in (0,1)$ and $u = \chi_E - \chi_{E^c}$ is a stationary set for the fractional s-perimeter, instead of the two bounds used above

$$\begin{aligned} \langle \nabla_{\nabla U}(d\nabla d), \nabla U\rangle &= (1 + O(\sqrt{K}R))|\nabla U|^2\,, \\ \widetilde{\operatorname{div}}(d\nabla d) &= (n+1)(1 + O(\sqrt{K}R))\,, \end{aligned}$$

given respectively by Lemmas 3.2.5 and 3.2.6, one has the equalities

$$\begin{aligned} \langle \nabla_{\nabla U}(d\nabla d), \nabla U\rangle_{\mathbb{R}^{n+1}} &= |\nabla U|^2\,, \\ \operatorname{div}_{\mathbb{R}^{n+1}}(d\nabla d) &= n+1\,, \end{aligned}$$

where U is the extension of $u = \chi_E - \chi_{E^c}$. Thus, following the proof one finds the exact expression

$$\Phi'(R) = \frac{2\beta_s}{R^{n-s}} \int_{\partial^+ \widetilde{B}_R^+(p_\circ,0)} z^{1-s} \langle \nabla U, \nabla d\rangle^2\, dxdz \geq 0\,.$$

In particular, Φ is constant if and only if $\langle \nabla U, \nabla d\rangle = 0$, that is, if and only if E is dilation-invariant for dilations with center at $p_\circ \in \mathbb{R}^n$. With this, we conclude the proof. □

3.2.2 Estimates for the extension problem

Lemma 3.2.7. *Let $s \in (0,2)$ and M satisfy the flatness assumption $\mathrm{FA}_2(M, g, 2, p, \varphi)$. Let also $U : \widetilde{B}_2^+(p,0) \to \mathbb{R}$ be any function solving*

$$\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla} U) = 0 \;\; in \;\; \widetilde{B}_2^+(p,0)\,,$$

and let u be its trace on $B_2(p)$. Assume also $U \in L^\infty(\widetilde{B}_2^+(p,0))$ and $u = 0$ in $B_{3/2}(p)$. Then there exists $C = C(n) > 0$ such that

$$z^{1-s}|\widetilde{\nabla} U(q,z)| \leq \frac{C}{s}\|U\|_{L^\infty(\widetilde{B}_{6/5}^+(p,0))}\,, \quad \forall (q,z) \in \widetilde{B}_1^+(p,0).$$

Proof. Let C denote a constant that depends only on n. This estimate is proved by a barrier argument. Let $\alpha, \beta > 0$ to be chosen later, and for $q_\circ \in B_{11/10}(p)$ define

$$b_{q_\circ}(q,z) := \frac{\alpha}{2}\big|\varphi^{-1}(q) - \varphi^{-1}(q_\circ)\big|^2 - \beta(z^2 - 2z^s)\,.$$

Denote by Δ_g the Laplace-Beltrami operator of (M,g). Then, by $\mathrm{FA}_2(M,g,2,p,\varphi)$, for $(x,z) \in \widetilde{B}_{6/5}^+$ there holds $|\Delta_g b_{q_\circ}(q,z)| \leq C\alpha$. Moreover

$$\big(\partial_{zz} + \tfrac{1-s}{z}\partial_z\big)z^2 = 2s \quad \text{and} \quad \big(\partial_{zz} + \tfrac{1-s}{z}\partial_z\big)z^s = 0\,.$$

Hence

$$\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla} b_{q_\circ}) = z^{1-s}\left(\Delta_g b_{q_\circ} + \partial_{zz} b_{q_\circ} + \frac{1-s}{z}\partial_z b_{q_\circ}\right) \le z^{1-s}(C\alpha - 2s\beta) \le 0\,,$$

provided we take $\beta = C\alpha/s$.

Since $U = 0$ in $B_{3/2}(p)\times\{0\}$ clearly $|U| \le b_{q_\circ}(\cdot, 0)$ in $B_{6/5}(p)\times\{0\}$. Moreover, for every $(x,z)\in\partial^+\widetilde{B}^+_{6/5}(p,0)$ there holds

$$b_{q_\circ}(q,z) \ge C\alpha - \beta(z^2 - 2z^s) \ge C\alpha \ge \|U\|_{L^\infty(\widetilde{B}^+_{6/5}(p,0))} \ge U(x,z)\,,$$

provided we choose $\alpha = C^{-1}\|U\|_{L^\infty(\widetilde{B}^+_{6/5}(p,0))}$.

Hence, with this choice of α and β, $|U| \le b_{q_\circ}$ on the full boundary $\partial\widetilde{B}^+_{6/5}(p,0)$. Since also U solves $\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}U) = 0$ in $\widetilde{B}^+_{6/5}(p,0)$, by the maximum principle we get

$$|U(q_\circ, z)| \le b_{q_\circ}(q_\circ, z) \le \frac{Cz^s}{s}\|U\|_{L^\infty(\widetilde{B}^+_{6/5}(p,0))}, \tag{3.22}$$

for $(q_\circ, z)\in\widetilde{B}^+_{11/10}(p,0)$. Moreover, by standard interior gradient estimates for uniformly elliptic equations, for all $(q,z)\in\widetilde{B}^+_1(p,0)$ we have

$$|\widetilde{\nabla}U(q,z)| \le \|\widetilde{\nabla}U\|_{L^\infty(\widetilde{B}_{z/100}(q,z))} \le \frac{C}{z}\|U\|_{L^\infty(\widetilde{B}_{z/50}(q,z))}. \tag{3.23}$$

In fact, it is well known that for uniformly elliptic equations the constant in the gradient estimates depends linearly on the ratio between the upper and lower ellipticity bounds (see, for example, [GT77, §15.3]). Since U solves $\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}U) = 0$ in $\widetilde{B}_{z/50}(q,z)$, and since

$$\sup_{(p,t)\in\widetilde{B}_{z/50}(q,z)} t^{1-s} \quad \text{and} \quad \inf_{(p,t)\in\widetilde{B}_{z/50}(q,z)} t^{1-s}$$

are comparable (i.e., their ratio is bounded by an absolute constant independent of z and s), standard gradient estimates give (3.23) with a dimensional constant on the right-hand side.

Lastly, since $\widetilde{B}^+_{z/50}(q,z)\subset\widetilde{B}^+_{11/10}(p,0)$, plugging (3.22) in (3.23) gives

$$|\widetilde{\nabla}U(q,z)| \le \frac{C}{s}z^{s-1}\|U\|_{L^\infty(\widetilde{B}^+_{6/5})}\,, \quad \forall\,(q,z)\in\widetilde{B}^+_1(p,0),$$

as desired. □

Lemma 3.2.8. *Let $s\in(0,2)$ and M satisfy the flatness assumption* $\mathrm{FA}_2(M,g,3,p,\varphi)$. *Let also $U:\widetilde{B}^+_3(p,0)\to\mathbb{R}$ be any function solving*

$$\begin{cases}\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}U) = 0 & \text{in } \widetilde{B}^+_3(p,0)\,,\\ U = u & \text{on } B_3(p)\times\{0\}.\end{cases}$$

Assume also that $U\in L^\infty(\widetilde{B}^+_3(p,0))$ and $u = 0$ in $B_2(p)$. Then

$$z^{1-s}|\widetilde{\nabla}\nabla U(q,z)| \le \frac{C}{s}\|U\|_{L^\infty(\widetilde{B}^+_2(p,0))}\,, \quad \forall(q,z)\in\widetilde{B}^+_1(p,0).$$

Proof. For simplicity, we just carry on the proof in full details for $M = \mathbb{R}^n$. The proof on a general manifold that satisfies $\mathrm{FA}_2(M, g, 3, p, \varphi)$ is identical working in coordinates given by φ^{-1}. We keep denoting by $\widetilde{\mathrm{div}}$ and $\widetilde{\nabla}$ the operators on the extended half-space $\mathbb{R}^{n+1}_+$.

For every $i = 1, 2, \dots, n$ consider the partial derivative $U_{x_i} := \frac{\partial U}{\partial x_i}$. This function solves $\widetilde{\mathrm{div}}(z^{1-s}\widetilde{\nabla} U_{x_i}) = 0$ in $\widetilde{B}^+_3(p, 0)$ and satisfies $U_{x_i}(\cdot, 0) = u_{x_i} = 0$ in $B_2(p)$. Thus, by Lemma 3.2.7 (rescaled) applied to U_{x_i} we get

$$z^{1-s}|\widetilde{\nabla} U_{x_i}(q, z)| \le \frac{C}{s}\|U_{x_i}\|_{L^\infty(\widetilde{B}^+_{9/5}(p,0))}, \quad \forall (q, z) \in \widetilde{B}^+_1(p, 0). \tag{3.24}$$

Claim. There holds

$$\|U_{x_i}\|_{L^\infty(\widetilde{B}^+_{9/5}(p,0))} \le C\|U\|_{L^\infty(\widetilde{B}^+_2(p,0))}.$$

This follows by a proof similar to the standard gradient estimates for uniformly elliptic PDEs. We denote by $C > 0$ a constant that depends only on n. Let $\mathcal{L}V := \widetilde{\mathrm{div}}(z^{1-s}\widetilde{\nabla} V)$ and let η be a smooth cutoff function such that $\chi_{\widetilde{B}^+_{9/5}(p,0)} \le \eta \le \chi_{\widetilde{B}^+_2(p,0)}$. We have

$$\begin{aligned}
\mathcal{L}(\eta^2 U^2_{x_i}) &= \eta^2\mathcal{L}(U^2_{x_i}) + U^2_{x_i}\mathcal{L}(\eta^2) + 2z^{1-s}\widetilde{\nabla}(\eta^2)\cdot\widetilde{\nabla}(U^2_{x_i}) \\
&= \eta^2 U_{x_i}\mathcal{L}(U_{x_i}) + 2\eta^2 z^{1-s}|\widetilde{\nabla} U_{x_i}|^2 + U^2_{x_i}\mathcal{L}(\eta^2) + 8z^{1-s}\eta U_{x_i}\widetilde{\nabla}\eta\cdot\widetilde{\nabla} U_{x_i} \\
&\ge 2\eta^2 z^{1-s}|\widetilde{\nabla} U_{x_i}|^2 + U^2_{x_i}\mathcal{L}(\eta^2) - 2\eta^2 z^{1-s}|\widetilde{\nabla} U_{x_i}|^2 - Cz^{1-s}|\widetilde{\nabla}\eta|^2 U^2_{x_i},
\end{aligned}$$

where in the last line we have used that $\mathcal{L}(U_{x_i}) = 0$ and Young's inequality. Hence

$$\mathcal{L}(\eta^2 U^2_{x_i}) \ge -\big(|\mathcal{L}(\eta^2)| + Cz^{1-s}\big)U^2_{x_i}. \tag{3.25}$$

Moreover, since U solves $\mathcal{L}(U) = 0$ we also have

$$\mathcal{L}(U^2) = 2U\mathcal{L}(U) + +2z^{1-s}|\widetilde{\nabla} U|^2 = 2z^{1-s}|\widetilde{\nabla} U|^2. \tag{3.26}$$

Putting together (3.25) and (3.26), for every $M > 0$, gives

$$\mathcal{L}(MU^2 + \eta^2 U^2_{x_i}) \ge U^2_{x_i}(2Mz^{1-s} - |\mathcal{L}(\eta^2)| - Cz^{1-s}).$$

It is easily checked that

$$|\mathcal{L}(\eta^2)| = |z^{1-s}\Delta(\eta^2) + (1-s)z^{-s}\cdot\partial_z(\eta^2)| \le Cz^{1-s}.$$

Thus

$$\mathcal{L}(MU^2 + \eta^2 U^2_{x_i}) \ge U^2_{x_i}(2Mz^{1-s} - Cz^{1-s}),$$

and taking $M = C > 0$ we get that $MU^2 + \eta^2 U^2_{x_i}$ is a subsolution in $\widetilde{B}^+_2(p, 0)$. Hence, by the maximum principle

$$\|U_{x_i}\|^2_{L^\infty(\widetilde{B}^+_{9/5}(p,0))} \le \sup_{\widetilde{B}^+_2(p,0)}(CU^2 + \eta^2 U^2_{x_i}) \le \sup_{\partial\widetilde{B}^+_2(p,0)}(CU^2 + \eta^2 U^2_{x_i}) = C\sup_{\partial\widetilde{B}^+_2(p,0)} U^2,$$

where in the last equality we have used that $\eta^2 U^2_{x_i} = 0$ on $\partial\widetilde{B}^+_2(p, 0)$ since $\eta = 0$ on $\partial^+\widetilde{B}^+_2(p, 0)$ and $U_{x_i} = 0$ on $B_2(p)$. The last inequality implies the claim.

Now we can easily conclude the proof. Using the claim in (3.24) we get

$$z^{1-s}|\widetilde{\nabla} U_{x_i}(q,z)| \le \frac{C}{s}\|U\|_{L^\infty(\widetilde{B}_2^+(p,0))}, \quad \forall (q,z) \in \widetilde{B}_1^+(p,0),$$

and the result follows by summing up this inequality for $i = 1, 2, \ldots, n$.

□

Lemma 3.2.9. *Let $s_0 \in (0,2)$, $s \in (s_0, 2)$. Consider the Riemannian manifold $(\mathbb{R}^n, g)$ with $\big(1-\frac{1}{100}\big)|v|^2 \le g_{ij}(x)v^iv^j \le \big(1+\frac{1}{100}\big)|v|^2$ and $\|g_{ij}\|_{C^{1,1}(\mathbb{R}^n)} \le 1$. Let also $u : \mathbb{R}^n \to [-1,1]$ and $U : \mathbb{R}^n \times \mathbb{R}_+ \to [-1,1]$ be the extension of u (in the sense of Theorem 3.2.4). Then*

$$\int_{\mathcal{B}_1^+(0,0)} |\widetilde{\nabla} U|^2 z^{1-s} dV dz \le C \iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\mathcal{B}_2^c\times\mathcal{B}_2^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\, dxdy,$$

with C depending only on n and s_0.

Proof. We proceed as in [CRS10, Proposition 7.1]. Assume without loss of generality that $\int_{\mathcal{B}_2} u(x)dx = 0$. Let $\xi : \mathbb{R}^n \to \mathbb{R}$ be a cutoff function such that $\xi = 1$ in $\mathcal{B}_{3/2}(0)$ and is compactly supported in $\mathcal{B}_2(0)$. Write $u = u\xi + u(1-\xi) = u_1 + u_2$ and $U = U_1 + U_2$.

On the one hand, since u_1 is compactly supported in $\mathcal{B}_2 := \mathcal{B}_2(0)$ and by Lemma 3.4.7 we have

$$\begin{aligned}
2\beta_s \int_{\mathbb{R}^{n+1}_+} z^{1-s}|\widetilde{\nabla} U_1|^2 dV dz &= \|u_1\|^2_{H^{s/2}(\mathbb{R}^n,g)} \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\mathcal{B}_2^c\times\mathcal{B}_2^c)} |u_1(x)-u_1(y)|^2 \mathcal{K}_s(x,y)\, dV_x dV_y \\
&\le C\alpha_{n,s} \iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\mathcal{B}_2^c\times\mathcal{B}_2^c)} \frac{|u(x)\xi(x)-u(y)\xi(y)|^2}{|x-y|^{n+s}} \xi^2(x)\, dxdy \\
&\le C\alpha_{n,s} \iint_{\mathcal{B}_2\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}} \xi^2(x)\, dxdy + C\alpha_{n,s} \iint_{\mathcal{B}_2\times\mathbb{R}^n} |u(x)|^2 \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{n+s}}\, dxdy \\
&\le C\alpha_{n,s} \iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\mathcal{B}_2^c\times\mathcal{B}_2^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\, dxdy + C\int_{\mathcal{B}_2} |u(x)|^2 dx.
\end{aligned}$$

Moreover, using the fractional Poincaré inequality (recall $\int_{\mathcal{B}_2} u(x)dx = 0$):

$$\begin{aligned}
\int_{\mathcal{B}_2} |u(x)|^2 dx &\le C\alpha_{n,s} \iint_{\mathcal{B}_2\times\mathcal{B}_2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\, dxdy \\
&\le C\alpha_{n,s} \iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\mathcal{B}_2^c\times\mathcal{B}_2^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\, dxdy =: I.
\end{aligned}$$

On the other hand (using again $\int_{\mathcal{B}_2} u(y)dy = 0$)

$$\int_{\mathbb{R}^n} \frac{u(x)^2}{(1+|x|^2)^{\frac{n+s}{2}}} dx = \iint \frac{(u(x)-u(y))^2}{(1+|x|^2)^{\frac{n+s}{2}}} \frac{\chi_{\mathcal{B}_2}(y)}{|\mathcal{B}_2|} dxdy - \iint \frac{u(y)^2}{(1+|x|^2)^{\frac{n+s}{2}}} \frac{\chi_{\mathcal{B}_2}(y)}{|\mathcal{B}_2|} dxdy \le CI,$$

and by Holder's inequality

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|^2)^{\frac{n+s}{2}}} dx \le \left(\int_{\mathbb{R}^n} \frac{|u(x)|^2}{(1+|x|^2)^{\frac{n+s}{2}}} dx\right)^{1/2} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{\frac{n+s}{2}}} dx\right)^{1/2} \le CI^{1/2}.$$

Claim. There is $C = C(n) > 0$ such that for $(x, z) \in \mathcal{B}_1^+$

$$z^{1-s}|\widetilde{\nabla} U_2(x,z)| \le C \int_{\mathbb{R}^n} \frac{|u_2(y)|}{(1+|y|^2)^{\frac{n+s}{2}}} dy \,. \tag{3.27}$$

We postpone the proof of this claim and first see how to conclude the proof of Lemma 3.2.9.

By the claim, if $(x, z) \in \mathcal{B}_1^+$ then

$$z^{1-s}|\widetilde{\nabla} U_2(x,z)| \le C \int_{\mathbb{R}^n} \frac{|u_2(y)|}{(1+|y|^2)^{\frac{n+s}{2}}} dy \le C \int_{\mathbb{R}^n} \frac{|u(y)|}{(1+|y|^2)^{\frac{n+s}{2}}} dy \le CI^{1/2}$$

But then the inequality

$$\begin{aligned}\int_{\mathcal{B}_1^+} z^{1-s}|\widetilde{\nabla} U_2|^2 dxdz &\le CI^{1/2} \int_{\mathcal{B}_1^+} |\widetilde{\nabla} U_2| dxdz \\ &\le CI^{1/2} \left(\int_{\mathcal{B}_1^+} z^{1-s}|\widetilde{\nabla} U_2|^2 dxdz \right)^{1/2} \left(\int_{\mathcal{B}_1^+} z^{s-1} dxdz \right)^{1/2},\end{aligned}$$

gives

$$\int_{\mathcal{B}_1^+} z^{1-s}|\widetilde{\nabla} U_2|^2 dxdz \le CI,$$

and the lemma follows.

It only remains to prove (3.27). Let $H_N(x, y, t)$ be the heat kernel of $N := (\mathbb{R}^n, g)$. By (3.9) the fractional Poisson kernel[1] $\mathbb{P}_N : \mathbb{R}^n \times \mathbb{R}^n \times (0, \infty) \to [0, \infty)$ of N can be represented as

$$\mathbb{P}_N(x, y, z) = \frac{z^s}{2^s \Gamma(s/2)} \int_0^\infty H_N(x, y, t) e^{-\frac{z^2}{4t}} \frac{dt}{t^{1+s/2}} \,,$$

and the solution U_2 to the extension problem with trace u_2 is $U_2(x, z) = \int_{\mathbb{R}^n} \mathbb{P}_N(x, y, z) u_2(y) \, dV_y$. Then, by Lemma 3.4.7 we have that $\mathbb{P}_N$ is comparable (up to dimensional constants) to the fractional Poisson kernel of $\mathbb{R}^n$ with its standard metric, that is

$$cs \frac{z^s}{(|x-y|^2 + z^2)^{\frac{n+s}{2}}} \le \mathbb{P}_N(x, y, z) \le Cs \frac{z^s}{(|x-y|^2 + z^2)^{\frac{n+s}{2}}} \,,$$

for some $C, c > 0$ dimensional. Hence, for every $(x, z) \in \widetilde{\mathcal{B}}_{6/5}^+$

$$|U_2(x, z)| \le Cs \int_{\mathbb{R}^n} \frac{|u_2(y)|}{(|x-y|^2 + z^2)^{\frac{n+s}{2}}} \, dy \le Cs \int_{\mathbb{R}^n \setminus \mathcal{B}_2} \frac{|u_2(y)|}{|x-y|^{n+s}} \, dy \,.$$

Since $x \in \mathcal{B}_{6/5}$ and $y \in \mathbb{R}^n \setminus \mathcal{B}_2$ there holds $|x - y| \ge \frac{1}{100}\sqrt{1 + |y|^2}$, and hence

$$\|U_2\|_{L^\infty(\mathcal{B}_{6/5}^+)} \le Cs \int_{\mathbb{R}^n} \frac{|u_2(y)|}{(1+|y|^2)^{\frac{n+s}{2}}} \, dy \,.$$

From here, the result follows directly by Lemma 3.2.7. □

[1] Which equals $\sigma_{n,s} \frac{z^s}{(|x-y|^2+z^2)^{\frac{n+s}{2}}}$ on $\mathbb{R}^n$ with its standard metric, for some normalization constant $\sigma_{n,s} > 0$.

Lemma 3.2.10. *Let $s_0 \in (0,1)$ and $s \in (s_0,1)$. Let M satisfy flatness assumptions* $\mathrm{FA}_1(M,g,1,p,\varphi)$. *Let also $U : \widetilde{B}_1^+(p,0) \to (-1,1)$ be any function solving*

$$\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla} U) = 0 \quad \text{in } \widetilde{B}_1^+(p,0), \tag{3.28}$$

and let u be its trace on $B_1(p)$. Then for all $\varrho > 0$, $R \geq 1$, $k \in \mathbb{R}$ and $q \in B_{1/2}(p)$ such that $B_{R\varrho}(q) \subset B_{3/4}(p)$,

$$\frac{\beta_s}{\varrho^{n-s}} \int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}|\widetilde{\nabla} U|^2\, dV dz \leq \frac{C}{R^s} + \frac{C}{1-s}\left(\frac{1}{\varrho^n}\int_{B_{R\varrho}(q)} |u+k|\, dV\right)^{1-s}\left(\frac{1}{\varrho^{n-1}}\int_{B_{R\varrho}(q)} |\nabla u|\, dV\right)^s,$$

where the constant C depends only on n and s_0.

Proof. Let us show that if U and U' are two different solutions $\widetilde{B}_1^+(p,0) \to (-1,1)$ of (3.28) with the same trace u on $B_{3/4}(p)$, then

$$\int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}\big(|\widetilde{\nabla} U|^2 - |\widetilde{\nabla} U'|^2\big)\, dV dz \leq C \int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}|\widetilde{\nabla} U'|^2\, dV dz, \tag{3.29}$$

whenever $R \geq 1$ and $B_{R\varrho}(q) \subset B_{3/4}(p)$.

Indeed, by Lemma 3.2.7 (rescaled) we have that $z^{1-s}\big|\widetilde{\nabla}(U-U')\big| \leq C$ in $\widetilde{B}_{1/2}^+(p,0)$. Thus, using that $\varrho \leq 3/4$ we obtain

$$\begin{aligned}
&\int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}\big(|\widetilde{\nabla} U|^2 - |\widetilde{\nabla} U'|^2\big)\, dV dz \\
&\qquad = \int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}\big(\widetilde{\nabla}(U-U')\big)\cdot\big(\widetilde{\nabla}(U+U')\big)\, dV dz \\
&\qquad \leq C\left(\int_{\widetilde{B}_\varrho^+(q,0)} z^{s-1} dV dz\right)^{\frac{1}{2}}\left(\int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}\big(|\widetilde{\nabla} U|^2 + |\widetilde{\nabla} U'|^2\big) dV dz\right)^{\frac{1}{2}} \\
&\qquad = C\frac{\varrho^{s/2}}{s^{s/2}}\left(\int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}\big(|\widetilde{\nabla} U|^2 + |\widetilde{\nabla} U'|^2\big) dV dz\right)^{\frac{1}{2}} \\
&\qquad \leq \frac{1}{2}\int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}\big(|\widetilde{\nabla} U|^2 - |\widetilde{\nabla} U'|^2\big)\, dV dz + C\int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}|\widetilde{\nabla} U'|^2\, dV dz,
\end{aligned}$$

and (3.29) follows.

Now let g_{ij} be the components of the metric in the coordinates φ^{-1}, $\eta \in C_c^\infty(\mathcal{B}_1)$ be a nonnegative smooth cut-off function satisfying $\eta \equiv 1$ in $\mathcal{B}_{3/4}$, and put $g'_{ij} = g_{ij}\eta + \delta_{ij}(1-\eta)$, a metric defined in the whole $\mathbb{R}^n$. Thanks to (3.29) it is enough to prove the lemma for the manifold $(\mathbb{R}^n, g')$ with $p = 0$ and with U replaced by the (unique!) bounded solution U' of (3.28) (with respect to the metric g') in all of $\mathbb{R}^n \times \mathbb{R}_+$. But in this case we can use Lemma 3.2.9 (rescaled) and obtain

$$\begin{aligned}\varrho^{s-n}\int_{\widetilde{B}_\varrho^+} z^{1-s}|\widetilde{\nabla}U'|^2\,dV dz &\le C\varrho^{s-n}\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\mathcal{B}_{2\varrho}^c\times\mathcal{B}_{2\varrho}^c)}\frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}dxdy\\
&\le 2C\varrho^{s-n}\iint_{\mathcal{B}_{R\varrho}\times\mathcal{B}_{R\varrho}\cup\mathcal{B}_\varrho\times\mathcal{B}_{R\varrho}^c}\frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}dxdy\\
&\le C\varrho^{s-n}\left(\iint_{\mathcal{B}_{R\varrho}\times\mathcal{B}_{R\varrho}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}dxdy+\frac{C\varrho^{n-s}}{R^s}\right),\end{aligned}$$

where we have used that

$$\iint_{\mathcal{B}_\varrho\times\mathcal{B}_{R\varrho}^c}\frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\,dxdy\le C\int_{\mathcal{B}_\varrho}\left(\int_{\mathcal{B}_{R\varrho}}\frac{1}{(|y|-\varrho)^{n+s}}dy\right)dx\le\frac{C\varrho^n}{(R\varrho)^s}.$$

We conclude using the interpolation inequality of Proposition A.1.1, since the modulus of the Euclidean gradient in $\mathbb{R}^{n+1}$ and the metric gradient $|\widetilde{\nabla}U|$ are comparable. □

3.3 Equivalence with the spectral definition

Recall the definition for the fractional Sobolev seminorm that we used in the previous section (that is, Definition 2.2.1). In the next result, we show that the fractional Sobolev seminorm can equivalently be expressed using the spectral or extension approaches.

Proposition 3.3.1. *Let $u\in H^{s/2}(M)$ (that is, the integral (2.3) is finite). Then, the fractional Sobolev seminorm (2.3) is equal to*

$$[u]^2_{H^{s/2}(M)}=2\sum_{k=1}^{\infty}\lambda_k^{s/2}\langle u,\phi_k\rangle^2_{L^2(M)}\tag{3.30}$$

and

$$[u]^2_{H^{s/2}(M)}=\inf_{v\in\widetilde{H}^1(\widetilde{M})}\left\{2\beta_s\int_{\widetilde{M}}|\widetilde{\nabla}v|^2z^{1-s}\,dVdz\;:\;v(\cdot,0)=u(\cdot)\;\text{in}\;L^2(M)\right\}.\tag{3.31}$$

Moreover, the conclusions of Theorem 3.2.4 also hold for u (with the exception of (3.14)), and the infimum in (3.31) is attained by the unique $U\in\widetilde{H}^1(\widetilde{M})$ given by Theorem 3.2.4. In particular, we also have that

$$[u]^2_{H^{s/2}(M)}=2\beta_s\int_{\widetilde{M}}|\widetilde{\nabla}U|^2z^{1-s}\,dVdz,\tag{3.32}$$

where β_s is the constant defined in (3.13).

Proof. **Step 1.** We show that (2.3) and (3.30) coincide for a function in $L^2(M)$.

Recall the regularised kernel $\mathcal{K}_s^\varepsilon$ defined in (3.2), which is bounded, symmetrical and increases monotonically to $\mathcal{K}_s$ as $\varepsilon\to 0$. By monotone convergence and these properties, for any function $u\in L^2(M)$ we can write

$$\begin{aligned}[u]^2_{H^{s/2}(M)}&=\iint_{M\times M}(u(p)-u(q))^2\mathcal{K}_s(p,q)\,dV_pdV_q\\
&=\lim_{\varepsilon\to0}\iint_{M\times M}(u(p)-u(q))^2\mathcal{K}_s^\varepsilon(p,q)\,dV_pdV_q\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0} 2 \iint_{M\times M} (u(p)-u(q))u(p)\mathcal{K}_s^\varepsilon(p,q)\,dV_p dV_q \\
&= \lim_{\varepsilon\to 0} 2 \int_M ((-\Delta)_\varepsilon^{s/2}u)(p)u(p)\mathcal{K}_s^\varepsilon(p,q)\,dV_p\,,
\end{aligned}
\tag{3.33}
$$

where we have set

$$
\begin{aligned}
((-\Delta)_\varepsilon^{s/2}u)(p) &:= \int_M (u(p)-u(q))\mathcal{K}_s^\varepsilon(p,q)\,dV_q \\
&= \frac{s/2}{\Gamma(1-s/2)} \int_M (u(p)-u(q)) \int_0^\infty H_M(p,q,t)e^{-\varepsilon^2/4t}\frac{dt}{t^{1+s/2}}\,dV_q \\
&= \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty (u(p)-P_t u(p))e^{-\varepsilon^2/4t}\frac{dt}{t^{1+s/2}}\,.
\end{aligned}
$$

Now, if $\{\phi_k\}_k$ is an orthonormal basis of $L^2(M)$ made of eigenfunctions for $(-\Delta)$, with eigenvalues

$$
0=\lambda_0<\lambda_1\le\lambda_2\le\ldots\le\lambda_k \xrightarrow{k\to\infty} +\infty\,,
$$

then they are also eigenfunctions for $(-\Delta)_\varepsilon^{s/2}$ with eigenvalues

$$
\lambda_{k,\varepsilon}^{s/2} := \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty (1-e^{-\lambda_k t})e^{-\varepsilon^2/4t}\frac{dt}{t^{1+s/2}}\,,
$$

which one sees immediately by applying the formula above for $(-\Delta)_\varepsilon^{s/2}$ to ϕ_k and using that $P_t\phi_k = e^{-\lambda_k t}\phi_k$. These eigenvalues are uniformly bounded in k (for a fixed $\varepsilon>0$) and increase monotonically to the $\lambda_k^{s/2}$ as $\varepsilon\to 0^+$.

Expanding $u=\sum_{k=0}^\infty a_k\phi_k$, with $a_k := \langle u,\phi_k\rangle_{L^2(M)}$, we deduce that

$$
(-\Delta)_\varepsilon^{s/2}u = \sum_{k=0}^\infty \lambda_{k,\varepsilon}^{s/2}\langle u,\phi_k\rangle_{L^2(M)}\phi_k\,.
$$

We remark that the expression makes sense since the $\lambda_{k,\varepsilon}^{s/2}$ are bounded uniformly in k (for a fixed ε), and thus the sum is absolutely convergent in $L^2(M)$. Using this fact, substituting into (3.33) gives that

$$
\begin{aligned}
[u]_{H^{s/2}(M)}^2 &= \lim_{\varepsilon\to 0} 2 \int_M ((-\Delta)_\varepsilon^{s/2}u)(p)u(p)\mathcal{K}_s^\varepsilon(p,q)\,dV_p \\
&= \lim_{\varepsilon\to 0} 2 \sum_{k=0}^\infty \lambda_{k,\varepsilon}^{s/2}a_k^2\,.
\end{aligned}
$$

Using again the monotone convergence theorem (for sums now), we deduce that

$$
[u]_{H^{s/2}(M)}^2 = 2\sum_{k=0}^\infty \lambda_k^{s/2}a_k^2,
$$

as desired.

Step 2. We show that U given by the representation formula (3.9), which was only used for smooth functions u, is valid for $u\in H^{s/2}(M)$ in general and moreover (3.32) still holds.

Fix $u \in H^{s/2}(M)$, and let U be defined through the representation formula (3.9). We will first show that U has finite $\widetilde{H}^1(\widetilde{M})$ energy, using the spectral expression (3.30) for the energy that we have just proved. Recall that if ϕ_k is an eigenfunction of $(-\Delta)$, then $P_t\phi_k = e^{-\lambda_k t}\phi_k$. Therefore, writing $u = \sum_{k=0}^{\infty} a_k\phi_k$, where $a_k := \langle u, \phi_k\rangle_{L^2(M)}$, we have that

$$\begin{aligned} U(p,z) &= \frac{z^s}{2^s\Gamma(s/2)}\int_0^\infty P_t u(p)\, e^{-\frac{z^2}{4t}}\frac{dt}{t^{1+s/2}} \\ &= \frac{z^s}{2^s\Gamma(s/2)}\sum_{k=0}^{\infty} a_k\phi_k(p)\int_0^\infty e^{-\lambda_k t-\frac{z^2}{4t}}\frac{dt}{t^{1+s/2}}\,. \end{aligned}$$

Then, we can compute (recall that ∇ denotes the gradient on M)

$$\nabla U(p,z) = \frac{z^s}{2^s\Gamma(s/2)}\sum_{k=1}^{\infty} a_k\nabla\phi_k(p)\int_0^\infty e^{-\lambda_k t-\frac{z^2}{4t}}\frac{dt}{t^{1+s/2}}\,,$$

and

$$\begin{aligned} \partial_z U(p,z) &= \frac{1}{2^s\Gamma(s/2)}\sum_{k=1}^{\infty} a_k\phi_k(p)\int_0^\infty e^{-\lambda_k t-\frac{z^2}{4t}}\Big(sz^{s-1} - \frac{z^{1+s}}{2t}\Big)\frac{dt}{t^{1+s/2}} \\ &= \frac{z^{s-1}}{2^s\Gamma(s/2)}\sum_{k=1}^{\infty} a_k\phi_k(p)\int_0^\infty e^{-\lambda_k t-\frac{z^2}{4t}}\Big(s - \frac{z^2}{2t}\Big)\frac{dt}{t^{1+s/2}}\,. \end{aligned}$$

Recall that the ϕ_i and ϕ_j are orthogonal in $L^2(M)$ and $H^1(M)$ seminorms for $i \neq j$, and that moreover $\int_M \phi_k^2 = 1$ and $\int_M |\nabla\phi_k|^2 = \lambda_k$ for every k. Then, given $z > 0$ we find that

$$\begin{aligned} \int_{M\times\{z\}} |\nabla U(p,z)|^2 dV_p &= \frac{z^{2s}}{2^{2s}\Gamma^2(s/2)}\sum_{k=1}^{\infty} a_k^2\Big(\int_0^\infty e^{-\lambda_k t-\frac{z^2}{4t}}\frac{dt}{t^{1+s/2}}\Big)^2\int_M |\nabla\phi_k|^2\,dV \\ &= \frac{z^{2s}}{2^{2s}\Gamma^2(s/2)}\sum_{k=1}^{\infty} \lambda_k a_k^2\Big(\int_0^\infty e^{-\lambda_k t-\frac{z^2}{4t}}\frac{dt}{t^{1+s/2}}\Big)^2 \\ &= \frac{z^{2s}}{2^{2s}\Gamma^2(s/2)}\sum_{k=1}^{\infty} \lambda_k^{1+s} a_k^2\Big(\int_0^\infty e^{-r-\frac{z^2\lambda_k}{4r}}\frac{dr}{r^{1+s/2}}\Big)^2, \end{aligned}$$

where in the last line we have performed the change of variables $r = \lambda_k t$.

We can argue analogously for $\partial_z U$, which leads to

$$\begin{aligned} \int_{M\times\{z\}} \big(\partial_z U(p,z)\big)^2 dV_p &= \frac{z^{2s-2}}{2^{2s}\Gamma(s/2)^2}\sum_{k=1}^{\infty} a_k^2\left(\int_0^\infty e^{-\lambda_k t-\frac{z^2}{4t}}\Big(s - \frac{z^2}{2t}\Big)\frac{dt}{t^{1+s/2}}\right)^2 \\ &= \frac{z^{2s-2}}{2^{2s}\Gamma(s/2)^2}\sum_{k=1}^{\infty} a_k^2\lambda_k^s\left(\int_0^\infty e^{-r-\frac{z^2\lambda_k}{4r}}\Big(s - \frac{z^2\lambda_k}{2r}\Big)\frac{dr}{r^{1+s/2}}\right)^2. \end{aligned}$$

Now, multiplying by z^{1-s} and integrating in z over $(0,\infty)$, and then performing the change of variables $z = \lambda_k^{-1/2}w$ (so that $z^2\lambda_k = w^2$), gives that

$$\iint_{M\times\mathbb{R}_+} |\nabla U(p,z)|^2 z^{1-s} dV_p\,dz = \frac{1}{2^{2s}\Gamma^2(s/2)}\sum_{k=1}^{\infty} \lambda_k^{1+s}a_k^2\int_0^\infty z^{1+s}\Big(\int_0^\infty e^{-r-\frac{z^2\lambda_k}{4r}}\frac{dr}{r^{1+s/2}}\Big)^2 dz$$

$$
\begin{aligned}
&= \frac{1}{2^{2s}\Gamma^2(s/2)} \sum_{k=1}^{\infty} \lambda_k^{s/2} a_k^2 \int_0^\infty w^{1+s} \Big(\int_0^\infty e^{-r-\frac{w^2}{4r}} \frac{dr}{r^{1+s/2}} \Big)^2 dw \\
&= 2c_1(s) \sum_{k=1}^{\infty} \lambda_k^{s/2} a_k^2 \\
&= c_1(s)[u]^2_{H^{s/2}(M)} ,
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\iint_{M\times\mathbb{R}_+} |\partial_z U(p,z)|^2 z^{1-s} dV_p \, dz \\
&\qquad = \frac{1}{2^{2s}\Gamma^2(s/2)} \sum_{k=1}^{\infty} \lambda_k^{s} a_k^2 \int_0^\infty z^{s-1} \left(\int_0^\infty e^{-r-\frac{z^2\lambda_k}{4r}} \left(s - \frac{z^2\lambda_k}{2r} \right) \frac{dr}{r^{1+s/2}} \right)^2 dz \\
&\qquad = \frac{1}{2^{2s}\Gamma^2(s/2)} \sum_{k=1}^{\infty} \lambda_k^{s/2} a_k^2 \int_0^\infty w^{s-1} \left(\int_0^\infty e^{-r-\frac{w^2}{4r}} \left(s - \frac{w^2}{2r} \right) \frac{dr}{r^{1+s/2}} \right)^2 dw \\
&\qquad = 2c_2(s) \sum_{k=1}^{\infty} \lambda_k^{s/2} a_k^2 \\
&\qquad = c_2(s)[u]^2_{H^{s/2}(M)} .
\end{aligned}
$$

Here, we have defined $c_1(s)$ and $c_2(s)$ implicitly as the corresponding constants (which depend only on s) resulting from the expression, and we have applied (3.30) in the last line in both computations.

Putting everything together, we get that

$$
\iint_{M\times\mathbb{R}_+} |\widetilde{\nabla} U(p,z)|^2 z^{1-s} \, dV_p dz = \big(c_1(s) + c_2(s)\big)[u]^2_{H^{s/2}(M)} .
$$

We could write the constant $\big(c_1(s)+c_2(s)\big)$ explicitly in terms of the resulting complicated integral expressions. On the other hand, thanks to (3.12) and (3.13) from Theorem 3.2.4 (which was proved only for smooth functions), we know that $c_1(s) + c_2(s) = (2\beta_s)^{-1}$. This proves (3.32) with U given by the representation formula (3.9).

In particular, we now know that U has finite energy for the extension problem. Moreover, arguing as in Step 1 of the proof of Theorem 3.2.4, it is simple to see that U has u as its trace in $L^2(M)$, and that it is a weak solution (meaning in duality with $C_c^\infty(\widetilde{M})$) to $\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla} U) = 0$. Let now $U_{\min} \in \widetilde{H}^1(\widetilde{M})$ be defined as the unique minimizer of (3.31). The fact that $U_{\min}$ exists follows by a standard lower-semicontinuity argument, just as at the beginning of the proof of Theorem 3.2.4, together with the fact that the space of competitors is not empty (which holds since, for example, U defined above, which has finite $\widetilde{H}^1(\widetilde{M})$ energy, is one such competitor). Clearly, $U_{\min}$ is also a weak solution of $\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla} U_{\min}) = 0$ with trace u.

Step 3. $U = U_{\min}$.

This follows directly from the uniqueness of weak solutions shown in Lemma 3.3.2, which we state as a separate result after the present proof.

With this, we conclude the proof of Proposition 3.3.1. □

Lemma 3.3.2. *(Uniqueness of weak solutions) Let $u \in L^2(M)$, and denote by $T : \widetilde{H}^1(\widetilde{M}) \to$*

$L^2(M)$ the trace operator. Then, there exists at most one solution $U \in \widetilde{H}^1(\widetilde{M})$ to the problem

$$\begin{cases} \widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}U) = 0, & \text{in } (C_c^\infty(M \times (0,\infty)))^*, \\ TU = u. \end{cases}$$

Proof. Suppose U_1 and U_2 are two such solutions and denote $V := U_1 - U_2$. By hypothesis $TV = 0$.

We claim that there exists a sequence $(V_k)_k \in C_c^\infty(M \times (0,\infty))$ such that

$$\int_{\widetilde{M}} |\widetilde{\nabla}V_k - \widetilde{\nabla}V|^2 z^{1-s}\, dV dz \to 0\,, \quad \text{as } k \to \infty\,. \tag{3.34}$$

The point here being that V_k is zero both in a neighborhood of $M \times \{0\}$ and in a neighborhood of infinity.

The proof is inspired by (a weighted version of) [Eva10, Section 5.5, Theorem 2]. By the definition of the space $\widetilde{H}^1(\widetilde{M})$ there exists a sequence $(U_k)_k \subset C^\infty(M \times [0,\infty))$ with (as $k \to \infty$)

$$\int_{\widetilde{M}} |\widetilde{\nabla}U_k - \widetilde{\nabla}V|^2 z^{1-s}\, dV dz \to 0\,, \quad \text{and } TU_k = U_k(\cdot, 0) \to 0 \text{ in } L^2(M)\,.$$

Note also that V is smooth in $M \times (0,\infty)$. Now, for every $(p,z) \in \widetilde{M}$, by the fundamental theorem of calculus and Holder's inequality

$$\begin{aligned} |U_k(p,z)|^2 &\le 2|U_k(p,0)|^2 + 2\left(\int_0^z |\widetilde{\nabla}U_k(p,y)|\, dy\right)^2 \\ &\le C|U_k(p,0)|^2 + Cz^s \int_0^z |\widetilde{\nabla}U_k(p,y)|^2 y^{1-s}\, dy\,, \end{aligned}$$

and integrating for $p \in M$ gives

$$\int_M |U_k(p,z)|^2\, dV_p \le C\int_M |U_k(\cdot,0)|^2 + Cz^s \int_M \int_0^z |\widetilde{\nabla}U_k(p,y)|^2 y^{1-s}\, dy\, dV_p\,. \tag{3.35}$$

Letting $k \to \infty$ we get

$$\int_M |V(\cdot,z)|^2\, dV \le Cz^s \int_M \int_0^z |\widetilde{\nabla}V|^2 y^{1-s}\, dy\, dV_p\,. \tag{3.36}$$

Now, for every $k \ge 10$, let $\eta_k \in C^\infty([0,+\infty))$ be a smooth cutoff function with $\eta = 0$ on $[0,1/k]$, $\eta = 1$ on $[2/k,\infty)$ and $|\eta'| \le Ck$. We claim that the sequence $V\eta_k = V(p,z)\eta_k(z) \in C_c^\infty(M \times (0,\infty))$ has the desired property. We have

$$\int_{\widetilde{M}} |\widetilde{\nabla}(V\eta_k) - \widetilde{\nabla}V|^2 z^{1-s}\, dV dz \le C\int_{\widetilde{M}} |\widetilde{\nabla}V|^2(1-\eta_k)^2 z^{1-s} + C\int_{\widetilde{M}} |V|^2|\eta_k'|^2 z^{1-s} =: I_{1,k} + I_{2,k}\,.$$

We estimate the two integrals separately.

For the first integral we have

$$I_{1,k} \le C\int_0^{2/k} \int_M |\widetilde{\nabla}V|^2 z^{1-s}\, dV dz \to 0\,,$$

as $k \to \infty$, since V has finite energy.

Moreover, by (3.36), we have regarding the second integral

$$\begin{aligned}
I_{2,k} &\le Ck^2 \int_0^{2/k} \int_M z^{1-s} |V|^2 \, dV dz \\
&\le Ck^2 \int_0^{2/k} z^{1-s} \left(z^s \int_M \int_0^z |\widetilde{\nabla} V|^2 y^{1-s} \, dy dV \right) dz \\
&\le Ck^2 \left(\int_0^{2/k} z \, dz \right) \left(\int_0^{2/k} \int_M |\widetilde{\nabla} V|^2 y^{1-s} \, dV dy \right) \\
&= C \int_0^{2/k} \int_M |\widetilde{\nabla} V|^2 y^{1-s} \, dV dy \to 0
\end{aligned}$$

as $k \to \infty$, again as V has finite energy.

Hence $V_k := V\eta_k$ has the desired property (3.34), and it can be used as a test function in the weak formulation in duality with $C_c^\infty(\widetilde{M})$. Multiplying $\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla} V) = 0$ by V_k, integrating on $\widetilde{M}$ and integrating by parts gives

$$\int_{\widetilde{M}} (\widetilde{\nabla} V \cdot \widetilde{\nabla} V_k) z^{1-s} \, dV dz = 0 \, .$$

Letting $k \to \infty$ and using (3.34) gives

$$\int_{\widetilde{M}} |\widetilde{\nabla} V|^2 z^{1-s} \, dV dz = 0 \, ,$$

hence V is constant, and then (since $TV = 0$) it must be $V \equiv 0$. Thus, $U_1 = U_2$ coincide, and the proof is complete. □

3.4 Local estimates

In order to quantify the dependence of the constants in the estimates on the geometry of the ambient manifold precisely, the notion of "local flatness assumption" will be very useful (this quantification will be important when we perform blow-up arguments). Let us introduce it below.

Here, as in the rest of the work, $\mathcal{B}_R(0)$ denotes the Euclidean ball of radius R centered at 0 of $\mathbb{R}^n$, and $B_R(p)$ denotes the metric ball on M of radius R and center p.

Definition 3.4.1 (Local flatness assumption)**.** *Let (M^n, g) be an n-dimensional Riemannian manifold and $p \in M$. For $R > 0$, we say that (M, g) satisfies the ℓ-th order flatness assumption at scale R around the point p, with parametrization φ, abbreviated as $\mathrm{FA}_\ell(M, g, R, p, \varphi)$, whenever there exists an open neighborhood V of p and a diffeomorphism*

$$\varphi : \mathcal{B}_R(0) \to V, \quad \text{with } \varphi(0) = p \, ,$$

such that, letting $g_{ij} = g\left(\varphi_*\left(\frac{\partial}{\partial x^i}\right), \varphi_*\left(\frac{\partial}{\partial x^j}\right)\right)$ be the representation of the metric g in the coordinates φ^{-1}, we have

$$\left(1 - \frac{1}{100}\right) |v|^2 \le g_{ij}(x) v^i v^j \le \left(1 + \frac{1}{100}\right) |v|^2 \quad \forall\, v \in \mathbb{R}^n \text{ and } \forall\, x \in \mathcal{B}_R(0) \, , \tag{3.37}$$

and

$$R^{|\alpha|}\left|\frac{\partial^{|\alpha|}g_{ij}(x)}{\partial x^\alpha}\right| \le \tfrac{1}{100} \quad \forall\alpha \text{ multi-index with } 1\le|\alpha|\le\ell, \text{ and } \forall x\in\mathcal{B}_R(0).$$

Definition 3.4.2 (Locally uniformly flat manifold)**.** *Let (M,g) be a complete Riemannian manifold. We say that M is locally uniformly flat if: for every $\ell\ge 0$ there exists $R_0>0$ for which $\mathrm{FA}_\ell(M,g,R_0,p,\varphi_p)$ is satisfied for all $p\in M$, where φ_p can be chosen to be the restriction of the exponential map*[2] *(of M) at p to the (normal) ball $\mathcal{B}_{R_0}(0)\subset T_pM\cong\mathbb{R}^n$.*

Remark 3.4.3. *Notice that every smooth closed manifold (M,g) is locally uniformly flat.*

Remark 3.4.4. *The notion above of local flatness is used in our results to stress the fact that once the local geometry of the manifold is controlled in the sense of Definition 3.4.1, then our estimates are independent of M. Interestingly, this makes our estimates of local nature even though the equation we deal with is nonlocal.*

Remark 3.4.5. *Throughout the paper, the following scaling properties will be used several times.*

(a) Given $M=(M,g)$ and $r>0$, we can consider the "rescaled manifold" $\widehat{M}=(M,r^2g)$. When performing this rescaling, the new heat kernel $H_{\widehat{M}}$ satisfies

$$H_{\widehat{M}}(p,q,t)=r^{-n}H_M(p,q,t/r^2)\,.$$

As a consequence, the "rescaled kernel" $\widehat{\mathcal{K}}_s$ defining the s-perimeter on $\widehat{M}$ satisfies

$$\widehat{\mathcal{K}}_s(p,q)=r^{-(n+s)}\mathcal{K}_s(p,q).$$

(b) Concerning the flatness assumption, it is easy to show that $\mathrm{FA}_\ell(M,g,R,p,\varphi)\Rightarrow\mathrm{FA}_\ell(M,g,R',p,\varphi)$ for all $R'<R$ and $\mathrm{FA}_\ell(M,g,R,p,\varphi)\Leftrightarrow\mathrm{FA}_\ell(M,r^2g,R/r,p,\varphi(r\,\cdot\,))$.

(c) Similarly, if $\mathrm{FA}_\ell(M,g,R,p,\varphi)$ holds, and $q\in\varphi(\mathcal{B}_R(0))$ is such that $\mathcal{B}_r(\varphi^{-1}(q))\subset\mathcal{B}_R(0)$, then $\mathrm{FA}_\ell(M,r^2g,R/r,q,\varphi_{\varphi^{-1}(q),r})$ holds, where $\varphi_{x,\rho}:=\varphi(x+\rho\,\cdot\,)$.

In all the sections, we will use the (standard) multi-index notation for derivatives. A multi-index $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ will be an n-tuple of nonnegative integers (in other words $\alpha\in\mathbb{N}^n$). We define

$$|\alpha|:=\alpha_1+\alpha_2+\cdots+\alpha_n.$$

For a function $f:\mathbb{R}^n\to\mathbb{R}$ is of class C^ℓ we shall use the notation

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}f:=\frac{\partial^{\alpha_1+\alpha_2+\cdots\alpha_n}f}{(\partial x^1)^{\alpha_1}(\partial x^2)^{\alpha_2}\cdots(\partial x^n)^{\alpha_n}}\,.$$

For $\alpha=0$, we put $\frac{\partial^{|0|}}{\partial x^0}f:=f$.

The next main theorem gives the precise behavior of the kernel around a point satisfying flatness assumptions, including an explicit approximation in coordinates.

Theorem 3.4.6. *Let (M,g) be a Riemannian n-manifold, not necessarily closed, $s\in(0,2)$ and let $p\in M$. Assume $\mathrm{FA}_\ell(M,g,R,p,\varphi)$ holds and denote $K(x,y):=\mathcal{K}_s(\varphi(x),\varphi(y))$.*

Given $x\in\mathcal{B}_R(0)$, let $A(x)$ denote the positive symmetric square root of the matrix $(g_{ij}(x))$ —g_{ij} being the metric in coordinates φ^{-1}— and, for $x,z\in\mathcal{B}_{R/2}(0)$, define

$$k(x,z):=K(x,x+z)\quad\text{and}\quad \widehat{k}(x,z):=k(x,z)-\frac{\alpha_{n,s}}{|A(x)z|^{n+s}}.$$

[2] That is $\varphi_p=(\exp_p\circ i)|_{\mathcal{B}_{R_0}(0)}$ for any isometric identification of $i:\mathbb{R}^n\to TM_p$

Then
$$\left|\widehat{k}(x,z)\right| \le R^{-1}\frac{C(n,s)}{|z|^{n+s-1}} \quad \textit{for all } x,z \in \mathcal{B}_{R/4}(0)\,, \tag{3.38}$$
and, for every multi-indices α, β with $|\alpha| + |\beta| \le \ell$, we have
$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial z^\beta}k(x,z)\right| \le \frac{C(n,s,\ell)}{|z|^{n+s+|\beta|}} \quad \textit{for all } x,z \in \mathcal{B}_{R/4}(0). \tag{3.39}$$
The constants $C(n,s)$ and $C(n,s,l)$ stay bounded for s away from 0 and 2.

Moreover, for all $x \in \mathcal{B}_{R/4}(0)$ and for all $q \in M \setminus \varphi(\mathcal{B}_R(0))$ we have
$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\mathcal{K}_s(\varphi(x),q)\right| \le \frac{C(n,\ell)}{R^{n+s}}\,, \tag{3.40}$$
and
$$\int_{M\setminus\varphi(\mathcal{B}_R(0))}\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\mathcal{K}_s(\varphi(x),q)\right|dV_q \le \frac{C(n,\ell)}{R^s}, \tag{3.41}$$
for every multi-index α with $|\alpha| \le \ell$.

3.4.1 Heat kernel estimates

To prove Theorem 3.4.6 we will need several preliminary lemmas studying the properties of the heat kernel of M. The first result compares locally the heat kernel $H_M(p,q,t)$ or the singular kernel $\mathcal{K}_s(p,q)$ on $\mathbb{R}^n$ endowed with a metric g with the standard ones on $\mathbb{R}^n$.

Lemma 3.4.7. *Let g be a metric on $\mathbb{R}^n$ such that $\frac{|v|^2}{4} \le g_{ij}(x)v^iv^j \le 4|v|^2$ and $|Dg_{ij}(x)| \le 1$ for all $x, v \in \mathbb{R}^n$. Denote $M := (\mathbb{R}^n, g)$ and let $\mathcal{K}_s$ be defined by (2.2). Then, there exist positive constants $c_i = c_i(n)$ for $1 \le i \le 6$ such that*
$$\frac{c_1}{t^{n/2}}e^{-\frac{|x-y|^2}{c_2 t}} \le H_M(x,y,t) \le \frac{c_3}{t^{n/2}}e^{-\frac{|x-y|^2}{c_4 t}}\,,$$
and
$$c_5\frac{\alpha_{n,s}}{|x-y|^{n+s}} \le \mathcal{K}_s(x,y) \le c_6\frac{\alpha_{n,s}}{|x-y|^{n+s}}\,,$$
for all $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,\infty)$.

Proof. The two-sided estimates for the heat kernel H_M follow directly from the classical parabolic estimates of Aronson [Aro67]. The second inequality follows by integrating the first one, from the definition (2.2) of $\mathcal{K}_s(x,y)$. ◻

The next lemma concerns the concentration of mass of the heat kernel.

Lemma 3.4.8. *Let (M^n, g) be a Riemannian manifold, $p \in M$, and assume* $\mathrm{FA}_0(M,g,1,p,\varphi)$ *holds. Then*
$$1 - Ce^{-c/t} \le \int_{\varphi(\mathcal{B}_{1/2}(0))} H_M(p,q,t)dV_q \le 1, \quad \textit{for all } t > 0,$$
with $C, c > 0$ depending only on n.

Proof. Put $H(x,y,t) := H_M(\varphi(x),\varphi(y),t)$. Let $g_{ij} \in C^0(\mathcal{B}_1(0))$ be the metric coefficients in the chart φ^{-1}, choose $\xi \in C^2_c(\mathcal{B}_1(0))$ such that $\chi_{\mathcal{B}_{3/4}(0)} \le \xi \le \chi_{\mathcal{B}_1(0)}$ and put $g'_{ij} := g_{ij}\xi + \delta_{ij}(1-\xi)$.

By assumption, we have $\left|g'_{ij}(x)v^iv^j - |v|^2\right| \le \frac{1}{100}|v|^2$ for all $x, v \in \mathbb{R}^n$. Morevoer, $g'_{ij} \equiv g_{ij}$ inside $\mathcal{B}_{3/4}(0)$. Consider the complete Riemannian manifold $M' := (\mathbb{R}^n, g')$ and let $H'(x,y,t)$ denote its associated heat kernel. Then, by Lemma 3.4.7 we have

$$\frac{c_1}{t^{n/2}} e^{-c_2|x-y|^2/t} \le H'(x,y,t) \le \frac{c_3}{t^{n/2}} e^{-c_4|x-y|^2/t}. \tag{3.42}$$

Let $\tau_* > 0$ be the largest number such that

$$h(t) := \frac{c_3}{t^{n/2}} e^{-\frac{c_4}{4^2 t}} \quad \text{is increasing on } (0, \tau_*).$$

This is a dimensional constant that depends only on n. Fix $\tau \in (0, \tau_*)$. Observe that, since $g' \equiv g$ in $\mathcal{B}_{1/4}(0)$, H, H' are both solutions of $\partial_t v = \Delta_g v$ in $\mathcal{B}_{1/4}(0) \times (0, \tau)$ and with initial condition δ_0 for $t = 0^+$. Moreover, by (3.42) we have

$$H'(0, \cdot, t) \le h(t) \le h(\tau), \quad \text{on } \partial\mathcal{B}_{1/4}(0) \times (0, \tau).$$

Hence $u(x,t) := \big(H'(0,x,t) - h(\tau)\big)^+$ is a subsolution to the heat equation in $\mathcal{B}_{1/4}(0) \times (0,\tau)$ and satisfies $u \le H$ on the parabolic boundary of $\mathcal{B}_{1/4}(0) \times (0,\tau)$. Thus, by the maximum principle $u \le H$ on $\mathcal{B}_{1/4}(0) \times (0,\tau)$, which in particular implies $H' - h(\tau) \le H$ on $\mathcal{B}_{1/4}(0) \times (0,\tau)$.

Integrating this pointwise inequality at $t = \tau$ gives

$$\begin{aligned}
\int_{\mathcal{B}_{1/4}(0)} H(0,x,\tau)\sqrt{|g|}\,dx &\ge \int_{\mathcal{B}_{1/4}(0)} H'(0,x,\tau)\sqrt{|g'|}\,dx - \frac{c_3}{\tau^{n/2}} e^{-\frac{c_4}{16\tau}} \\
&= 1 - \int_{\mathbb{R}^n \setminus \mathcal{B}_{1/4}(0)} H'(0,x,\tau)\sqrt{|g'|}\,dx - \frac{c_3}{\tau^{n/2}} e^{-\frac{c_4}{16\tau}},
\end{aligned}$$

where we have used that M' is stochastically complete in the last line. Lastly, by (3.42) we have

$$\int_{\mathbb{R}^n \setminus \mathcal{B}_{1/4}(0)} H'(0,x,\tau)\sqrt{|g'|}\,dx \le \int_{\mathbb{R}^n \setminus \mathcal{B}_{1/4}(0)} \frac{c_3}{t^{n/2}} e^{-c_4|x|^2/t} \big(1 + \tfrac{1}{100}\big)^{n/2} dx \le Ce^{-c/\tau}.$$

Finally, since also $h(\tau) \le Ce^{-c/\tau}$ (notice that we can "absorbe" $\tau^{-n/2}$ in $Ce^{-c/\tau}$ chosing $c > 0$ slightly smaller and a larger C), we obtain the desired estimate

$$1 - Ce^{-c/\tau} \le \int_{\mathcal{B}_{1/4}(0)} H(0,x,\tau)\sqrt{|g|}\,dx \le \int_{\varphi(\mathcal{B}_{1/2}(0))} H_M(p,q,\tau)dV_q, \quad \forall \tau \in (0,\tau_*).$$

The bound by above by 1 of the same quantity follows immediately using that H_M is a heat kernel, i.e. nonnegative and with total mass bounded by 1. $\square$

Lemma 3.4.9. *Under the same assumptions as in Theorem 3.4.6, for all $q \in M \setminus \varphi(\mathcal{B}_1(0))$ we have*

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha} H_M(\varphi(x), q, t)\right| \le Ce^{-c/t}, \quad \text{for } (x,t) \in \mathcal{B}_{1/2}(0) \times (0,\infty) \tag{3.43}$$

and for every multi-index α with $|\alpha| \le \ell$, with $C, c > 0$ depending only on n and ℓ.

Proof. Notice that $u(x,t) := H_M(\varphi(x), q, t)$ satisfies $u_t = Lu$, in $\mathcal{B}_1(0) \times (0,\infty)$ and $u \equiv 0$ at $t = 0$, where

$$Lu = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i}\left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x^j}\right) \tag{3.44}$$

is the Laplace-Beltrami operator with metric g.

Let us show that

$$|u| \le Ce^{-c/t} \quad \text{for } (x,t) \in \mathcal{B}_{3/4}(0) \times (0,\infty), \tag{3.45}$$

with $C, c > 0$ dimensional constants. This follows from the following standard probabilistic consideration. Fix $x_\circ \in \varphi(\mathcal{B}_{3/4}(0))$. By continuity of sample paths, the probability that a Brownian motion started at $q \in M \setminus \varphi(\mathcal{B}_1(0))$ hits $\varphi(\mathcal{B}_\delta(x_\circ))$ $(0 < \delta \ll 1)$ within time $\le t$ is less that the supremum among $q' \in \varphi(\partial\mathcal{B}_{8/9}(0))$ of the probability that a Brownian motion started at a point q' hits $\varphi(\mathcal{B}_\delta(x_\circ))$ within time $\le t$. This gives

$$u(x_\circ, t) \le \sup_{q' \in \varphi(\partial\mathcal{B}_{8/9}(0))} H_M(\varphi(x_\circ), q', t). \tag{3.46}$$

Now, we use (3.46), Lemma 3.4.8, and the parabolic Harnack inequality to show (3.45).

For fixed $q' \in \varphi(\partial\mathcal{B}_{8/9})$ set $v(x,t) := H_M(\varphi(x), q', t)$ and consider the rescaled $\widetilde{v}(x,t) := v(x_\circ + rx, t_\circ + r^2 t)$ for $r \in (0, 1/10)$. Then $\widetilde{v} \ge 0$ satisfies a (uniformly) parabolic equation in $\mathcal{B}_1(0) \times (0,1)$ with smooth coefficients (that only improve as r gets smaller). Thus, by the Harnack inequality for every $x \in \mathcal{B}_{1/2}(0)$ and $t \in (1/4, 1/2)$ we have

$$\widetilde{v}(x,t) \le C \inf_{\mathcal{B}_{1/2}(0)} \widetilde{v}(\,\cdot\,, 1) \le C\widetilde{v}(y,1),$$

for all $y \in \mathcal{B}_{1/2}(0)$. Integrating

$$\widetilde{v}(0,t) \le C \int_{\mathcal{B}_{1/2}(0)} \widetilde{v}(y,1)dy = C \int_{\mathcal{B}_{1/2}(0)} v(x_\circ + ry, t_\circ + r^2)dy = Cr^{-n} \int_{\mathcal{B}_{r/2}(x_\circ)} v(z, t_\circ + r^2)dz\,,$$

for some $C = C(n) > 0$. Thus, for all $t \in (t_\circ + r^2/4, t_\circ + r^2/2)$

$$v(x_\circ, t) \le Cr^{-n} \int_{\mathcal{B}_{r/2}(x_\circ)} v(z, t_\circ + r^2)dz\,.$$

But $\varphi(\mathcal{B}_{r/2}(x_\circ)) \subset M \setminus B_{1/10}(q')$ for every $q' \in \varphi(\partial\mathcal{B}_{8/9}(0))$. Then by Lemma 3.4.8 we get

$$v(x_\circ, t) \le Cr^{-n} \int_{M \setminus \varphi(\mathcal{B}_{1/10}(q'))} H_M(z, q', t_\circ + r^2)dz \le Cr^{-n} \exp\left(-\frac{c}{t_\circ + r^2}\right),$$

where $C, c > 0$ depend only on n. Now, for small times $t_\circ \le 1/100$ choosing $r^2 = t_\circ$ (together with the probabilistic argument above) gives the result, since one can absorb the term $r^{-n} = t_\circ^{-n/2}$ in the exponential up to slightly decreasing the value of c. For non-small times $t_\circ > 1/100$, one can just take $r = 1/10$ and obtain the upper bound by a constant as desired. This concludes the proof of (3.45).

Now, similarly to above, for all $r \in (0, 1/4)$ and $(x_\circ, t_\circ) \in \mathcal{B}_{1/2}(0) \times (0,\infty)$ the rescaled function $\overline{u}(x,t) = u(x_\circ + rx, t_\circ + r^2 t)$ satisfies a (uniformly) parabolic equation with smooth coefficients (since the bounds on every C^k norm of the coefficients only improve as r gets smaller) and, from (3.45), we have $|\overline{u}| \le Ce^{-c/t}$ in $\mathcal{B}_1 \times (0,1)$. Hence standard parabolic Schauder estimates give

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\,\overline{u}\right| \le Ce^{-c/t}, \quad \text{for } (x,t) \in \mathcal{B}_{1/2}(0) \times [1/2, 1)\,,$$

for every multi-index α with $|\alpha| \le \ell$, with $C > 0$ depending only on n and ℓ and $c > 0$ as above.

After scaling back the estimate above we obtain, for all $r \in (0, 1/4]$

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha} u(x_\circ, t_\circ + t)\right| \le C r^{-|\alpha|} e^{-c/r^2}, \quad \text{for } (x_\circ, t) \in \mathcal{B}_{1/2}(0) \times [r^2/2, r^2)\,.$$

Then, for "non-small" times $t_\circ \ge 1/16$ we notice that (3.43) follows taking $r = 1/4$. On the other hand, for small times $t_\circ \in (0, 1/16)$ we obtain (3.43) taking $r^2 = t_\circ$, bounding $r^{-|\alpha|}$ by $t_\circ^{-\ell/2}$, and absorbing (chosing $c > 0$ smaller and C larger) this negative power of $t_\circ$ in the exponential. □

Lemma 3.4.10. *Under the same assumptions as in Theorem 3.4.6, we have*

$$\int_{M \setminus \varphi(\mathcal{B}_1(0))} \left|\frac{\partial^{|\alpha|}}{\partial x^\alpha} H_M(\varphi(x), q, t)\right| dV_q \le C e^{-c/t}, \quad \textit{for } (x, t) \in \mathcal{B}_{1/2}(0) \times [0, \infty)$$

and for every multi-index α with $|\alpha| \le \ell$, with $C, c > 0$ depending only on n and ℓ.

Proof. The proof is similar to the one of Lemma 3.4.9. Let $\sigma : M \setminus \varphi(\mathcal{B}_1(0)) \to \{+1, -1\}$ be any measurable function to be chosen. Consider

$$u(x, t) := \int_{M \setminus \varphi(\mathcal{B}_1(0))} H_M(\varphi(x), q, t)\sigma(q) dV_q,$$

By Lemma 3.4.8 — since $H_M \ge 0$ and $\int_M H_M(p, q, t) dV_q \le 1$ — we obtain

$$|u(x, t)| \le \int_{M \setminus \varphi(\mathcal{B}_{1/4}(0))} H(\varphi(x), q, t) dV_q \le C \exp(-c/t), \qquad \forall (x, t) \in \mathcal{B}_{3/4}(0) \times [0, \infty).$$

Notice that in this estimate, C and c are positive dimensional constants (and in particular, they do not depend on the choice of σ). Also, by the superposition principle u satisfies $u_t = Lu$, in $\mathcal{B}_1(0) \times [0, \infty)$ and $u \equiv 0$ at $t = 0$, where L is as in (3.44).

Now proceeding exactly as in the proof of Lemma 3.4.9 we obtain that

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha} u\right| \le C \exp(-c/t), \quad \text{for } (x, t) \in \mathcal{B}_{1/2}(0) \times [0, \infty)$$

for $|\alpha| \le \ell$. Now, for any given α, x, and t, we can choose $\sigma : M \setminus \varphi(\mathcal{B}_1(0)) \to \{+1, -1\}$ so that

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} u(x, t) = \int_{M \setminus \varphi(\mathcal{B}_1(0))} \frac{\partial^{|\alpha|}}{\partial x^\alpha} H_M(\varphi(x), q, t)\sigma(q) dV_q = \int_{M \setminus \varphi(\mathcal{B}_1(0))} \left|\frac{\partial^{|\alpha|}}{\partial x^\alpha} H_M(\varphi(x), q, t)\right| dV_q,$$

and we are done. □

Lemma 3.4.11 (Localization principle)**.** *Let (M, g) and (M', g') be two Riemannian n-manifolds. Assume that both M and M' satisfy the flatness assumptions $\mathrm{FA}_\ell(M, g, 1, p, \varphi)$ and $\mathrm{FA}_\ell(M', g, 1, p', \varphi')$ respectively, and suppose that $g_{ij} \equiv g'_{ij}$ in $\mathcal{B}_1(0)$ in the coordinates induced by φ^{-1} and $(\varphi')^{-1}$.*

Then, letting $H(x, y, t) := H_M(\varphi(x), \varphi(y), t)$ and $H'(x, y, t) := H_{M'}(\varphi'(x), \varphi'(y), t)$, we have that the difference $(H - H')(x, y, t)$ is of class C^ℓ in $\mathcal{B}_{1/2}(0) \times \mathcal{B}_{1/2}(0) \times [0, \infty)$ and

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial^{|\beta|}}{\partial y^\beta} (H - H')(x, y, t)\right| \le C e^{-c/t} \quad \textit{for } (x, y, t) \in \mathcal{B}_{1/2}(0) \times \mathcal{B}_{1/2}(0) \times [0, \infty),$$

for every $|\alpha|+|\beta|\le \ell$, with $C,c>0$ depending only on n and ℓ.

Proof. Let us show that

$$|H-H'|\le Ce^{-c/t}\quad \text{for } (x,y,t)\in \mathcal{B}_{3/4}(0)\times\mathcal{B}_{3/4}(0)\times[0,\infty), \tag{3.47}$$

with $C,c>0$ dimensional constants.

Indeed, fix $x_\circ\in\mathcal{B}_{3/4}(0)$ and let us show first that we have

$$\big|(H-H')(x_\circ,y,t)\big|\le Ce^{-c/t}\quad \text{for all } y\in\mathcal{B}_{3/4}(0)\setminus\mathcal{B}_{1/8}(x_\circ)$$

Indeed, the L^∞ estimate of Lemma 3.4.9 —appropriately rescaled to have $\varphi(B_{1/8}(x_\circ))$ instead of $\varphi(\mathcal{B}_1(0))$— gives

$$H_M(\varphi(x_\circ),\varphi(y),t)\le Ce^{-c/t}\quad \text{for all } y\in\mathcal{B}_{3/4}(0)\setminus\mathcal{B}_{1/8}(x_\circ),$$

and the same estimate with H_M replaced by $H_{M'}$. Hence (3.47) follows using $|H-H'|\le H+H'$.

Now observing that for all $x_\circ$ as above $u(y,t):=(H-H')(x_\circ,y,t)$ solves the heat equation $(\partial_t-L_y)u=0$ with zero intial condition, (3.47) easily follows from the maximum principle.

Finally, the estimate for the higher derivatives follows from standard parabolic estimates, noticing that $u(x,y,t):=(H-H')(x,y,t)$ solves

$$\partial_t u=\frac{1}{2}(L_{x,y}u)$$

where

$$L_{x,y}u=\frac{1}{\sqrt{|g(x)|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g(x)|}g^{ij}(x)\frac{\partial}{\partial x^j}u\right)+\frac{1}{\sqrt{|g(y)|}}\frac{\partial}{\partial y^i}\left(\sqrt{|g(y)|}g^{ij}(y)\frac{\partial}{\partial y^j}u\right).$$

is the sum of the Laplace-Beltrami operators with respect to the variables x and y (or, equivalently, the Laplace-Beltrami operator with respect to the product metric in $\mathcal{B}_1(0)\times\mathcal{B}_1(0)$). □

Proposition 3.4.12. *Assume that $M=(\mathbb{R}^n,g)$ with g satisfying*

$$\frac{1}{2}|v|^2\le g_{ij}v^iv^j\le 2|v|^2,\qquad \textit{and}\qquad \left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}g_{ij}\right|\le 1\quad \textit{for all } |\alpha|\le\ell, \tag{3.48}$$

for some $\ell\ge 1$. For $x\in\mathbb{R}^n$ let $A(x)$ denote the (unique) positive definite symmetric square root of the matrix $g(x)=\{g_{ij}(x)\}_{ij}$, and define $h(z,x,t)$ by the identity

$$H_M(x,y,t)=\frac{1}{t^{n/2}}h\left(A(x)\Big(\frac{y-x}{\sqrt{t}}\Big),x,t\right).$$

Let also

$$h_\circ(x,z,t)=h_\circ(z):=\frac{1}{(4\pi)^{n/2}}e^{-|z|^2/4},\qquad \textit{and}\qquad \widehat{h}:=h-h_\circ.$$

Then, there are positive dimensional C and c such that

$$|\widehat{h}|\le C\min(1,\sqrt{t})e^{-c|z|^2}\qquad \textit{for all } (x,z,t)\in\mathbb{R}^n\times\mathbb{R}^n\times(0,\infty).$$

Moreover, we have

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial z^\beta}h\right| \le Ce^{-c|z|^2} \quad \textit{for all } (x,z,t)\in\mathbb{R}^n\times\mathbb{R}^n\times(0,1) \textit{ and } \alpha,\beta \textit{ with } |\alpha|+|\beta|\le \ell,$$

for positive constants C and c depending only on n and ℓ.

Proof. Since in the proof H_M will always refer to the heat kernel of M, we denote it just by H. First, notice that since $H(x,y,t)=H(y,x,t)$ we have

$$H(x,y,t)=\frac{1}{t^{n/2}}\,h\Big(A(x)\Big(\frac{y-x}{\sqrt{t}}\Big),x,t\Big)=\frac{1}{t^{n/2}}\,h\Big(A(y)\Big(\frac{x-y}{\sqrt{t}}\Big),y,t\Big).$$

Let $Lf := \frac{1}{\sqrt{|g|(x)}}\frac{\partial}{\partial x^i}\Big(\sqrt{|g|(x)}g(x)^{ij}\frac{\partial f}{\partial x^j}\Big)$ denote the Laplace-Beltrami operator (with respect to x). By the chain rule, direct computation shows

$$\begin{aligned} LH &= \frac{1}{t^{n/2+1}}\bigg(\sqrt{t}\Big(\frac{\partial_i\big(\sqrt{|g|}g^{ij}\big)}{\sqrt{|g|}}\Big)(x)A^l_j(y)\frac{\partial h}{\partial z^l}(\star)+g^{ij}(x)\big(A^k_iA^l_j\big)(y)\frac{\partial^2 h}{\partial z^k\partial z^l}(\star)\bigg),\\ \partial_t H &= \frac{1}{t^{n/2+1}}\bigg(-\frac{n}{2}h(\star)-\frac{1}{2}\frac{\partial h}{\partial z^l}(\star)\frac{(A(y)(x-y))^l}{\sqrt{t}}+t\partial_t h(\star)\bigg),\end{aligned}$$

where

$$(\star)\text{ means evaluated at }\Big(A(y)\Big(\frac{x-y}{\sqrt{t}}\Big),y,t\Big).$$

This leads to the equation for $h=h(z,y,t)$, where we denote $\partial_i := \frac{\partial}{\partial z^i}$ and $\partial_{ij} := \frac{\partial^2}{\partial z^i\partial z^j}$,

$$t\partial_t h = \overline{L}h := a^{ij}(z,y,t)\partial_{ij}h+\big(\sqrt{t}b^i(z,y,t)+\tfrac{z^i}{2}\big)\partial_i h+\frac{n}{2}h,$$

where

$$a^{ij}(z,y,t) := g^{kl}\Big(y+\sqrt{t}z\Big)\big(A^i_kA^j_l\big)(y)$$

and

$$b^i(z,y,t) := \Big(\frac{\partial_k\big(\sqrt{|g|}g^{kl}\big)}{\sqrt{|g|}}\Big)\Big(y+\sqrt{t}z\Big)A^i_l(y);$$

with initial condition:

$$h(z,y,0^+)=h_\circ(z)=\frac{1}{(4\pi)^{n/2}}e^{-|z|^2/4}.$$

(Notice that we defined h so that its initial condition is independent of y.)

We emphasize that, by the assumption (3.48), this equation is uniformly elliptic, and the derivatives of a^{ij}, b^i up to order ℓ in the variables z and y are uniformly bounded for times $t\in(0,T_\circ)$ by constants depending only on n and $T_\circ$.

Let us now compute an equation for $\widehat{h}=h-h_\circ$. Since

$$\delta^{ij}\partial_{ij}h_\circ+\frac{z^i}{2}\partial_i h+\frac{n}{2}h_\circ=0,$$

we obtain

$$
\begin{aligned}
t\partial_t \widehat{h} - \overline{L}\widehat{h} = \overline{L}h_\circ &= (a^{ij} - \delta^{ij})\partial_{ij}h_\circ + \sqrt{t}b^i\partial_i h_\circ \\
&= \big((a^{ij} - \delta^{ij})(z_i z_j - \delta_{ij}) - \sqrt{t}b^i \tfrac{z^j}{2}\delta_{ij}\big)h_\circ\,,
\end{aligned}
$$

and $\widehat{h}$ satisfies the initial condition

$$
\widehat{h}(z, y, 0^+) \equiv 0.
$$

Notice that (since by definition $A(y)$ is a square root of $g(y)$) we have, for all y

$$
g^{kl}(y)\big(A^i_k A^j_l\big)(y) = \delta^{ij}
$$

and hence, since g^{kl} is smooth,

$$
|a^{ij}(z, y, t) - \delta^{ij}| \le C\sqrt{t}
$$

Hence, we have

$$
|t\partial_t \widehat{h} - \overline{L}\widehat{h}| \le C(1 + |z|^2)\sqrt{t}\, h_\circ \tag{3.49}
$$

Let us now find some barrier allowing us to control $\widehat{h}$. We can use as barrier

$$
b(z, t) := \sqrt{t}e^{-(1/4-\kappa)|z|^2}
$$

Direct computation shows that, for $\sqrt{t} < \theta\kappa$ (so that $a^{ij}\delta_{ij} \ge n - C\theta\kappa$ and $|\delta^{ij} - a^{ij}|z^k z^l \delta_{ik}\delta_{jl} \le C\theta\kappa|z|^2$)

$$
\begin{aligned}
t\partial_t b - \overline{L}b &= \left(\frac{1}{2} - 4\big(\tfrac{1}{4} - \kappa\big)^2 (a^{ij})z^k z^l \delta_{ik}\delta_{jl} + 2(\tfrac{1}{4} - \kappa)\, a^{ij}\delta_{ij} + (\sqrt{t}b^i + \frac{z^i}{2})2(\tfrac{1}{4} - \kappa)z^j\delta_{ij} - \frac{n}{2}\right)b \\
&\ge \left(\frac{1}{2} + \big(\tfrac{1}{4} - \kappa\big)4\kappa|z|^2 - C\theta\kappa|z|^2 - C\kappa - C\theta\kappa|z|\right)b \\
&\ge \big(\tfrac{1}{4} + \tfrac{\kappa}{2}|z|^2\big)b \ge 0,
\end{aligned}
$$

provided we chose $\theta > 0$ and $\kappa > 0$ sufficiently small.

Since clearly $b \ge \sqrt{t}\, h_\circ$ we obtain that Cb is a supersolution of (3.49) for $\sqrt{t} < \theta\kappa$. This shows that $|\widehat{h}| \le Cb$ for all t small enough.

Notice that the estimate $|\widehat{h}| \le Cb$ (fixing $\kappa > 0$ and $\theta > 0$ small dimensional) shows, in particular, that

$$
|\widehat{h}(z, y, t)| \le C\sqrt{t}\exp(-c|z|^2) \tag{3.50}
$$

holds with $c > 0$ dimensional for all "small" times $t \in (0, \theta^2\kappa^2)$. On the other hand, for "non-small" times $t \ge \theta^2\kappa^2$, the standard heat kernel estimate (3.42) for H (which holds with c_i dimensional) immediately yields (3.50) with $\sqrt{t}$ replaced by 1.

In order to bound the derivatives of h with respect to z we notice we notice that, in logarithmic time $\tau = \log t$, the function $h(z, y, e^\tau)$ satisfies, for y fixed, a standard parabolic equation with smooth coefficients in the domain $\mathbb{R}^n \times (-\infty, 0)$. Then, thanks to (3.50), applying standard parabolic estimates in parabolic cylinders $\{|x - x_\circ| < 2, |\tau - \tau_\circ| < 2\}$ we easily obtain the claimed bounds for all partial derivatives of h with respect to z.

In order to show the regularity in y, one can then differentiate the equation with respect to y as many times as needed (the coefficients depend in a very smooth way also in y) and notice that the initial condition will be zero (since $h_\circ$ is independent of y). By standard parabolic regularity arguments (e.g., using a Duhamel-type formula to represent the solutions), we obtain the desired estimates. □

3.4.2 Singular kernel estimates

As a first consequence of Lemma 3.4.11 we have that the following local version of Lemma 3.4.7 above also holds.

Lemma 3.4.13. *Let $s_0 \in (0,2)$ and $s \in (s_0, 2)$. Let (M,g) be a Riemannian n-manifold and $p \in M$. Assume that $\mathrm{FA}_1(M,g,p,1,\varphi)$ holds. Then*

$$c_7 \frac{\alpha_{n,s}}{|x-y|^{n+s}} \le \mathcal{K}_s(\varphi(x),\varphi(y)) \le c_8 \frac{\alpha_{n,s}}{|x-y|^{n+s}}\,,$$

for all $x, y \in \mathcal{B}_{1/2}(0)$, where $c_7, c_8 > 0$ depends on n and s_0.

Proof. Take $\eta \in C_c^\infty(\mathcal{B}_1(0))$ with $\chi_{\mathcal{B}_{1/2}(0)} \le \eta \le \chi_{\mathcal{B}_1(0)}$ and let $g'_{ij} := g_{ij}\eta + (1-\eta)\delta_{ij}$. This is a metric on $\mathbb{R}^n$ with $g'_{ij} = g_{ij}$ in $\mathcal{B}_{1/2}(0)$. Denote by K_s, K'_s and H, H' the singular kernels and heat kernels of (M,g) and $(\mathbb{R}^n, g')$ respectively. Then, by Lemma 3.4.11 applied to the manifolds (M,g) and $(\mathbb{R}^n,g')$ we have, for $x,y \in \mathcal{B}_{1/4}(0)$:

$$\begin{aligned}\big|\mathcal{K}_s(\varphi(x),\varphi(y)) - \mathcal{K}'_s(x,y)\big| &\le \frac{s/2}{\Gamma(1-s/2)}\int_0^\infty \big|H(\varphi(x),\varphi(y),t) - H'(x,y,t)\big|\frac{dt}{t^{1+s/2}} \\ &\le \frac{Cs}{\Gamma(1-s/2)}\int_0^\infty e^{-c/t}\frac{dt}{t^{1+s/2}} \le C(2-s)\,,\end{aligned}$$

for some dimensional $C = C(n)$. Then, the result follows directly by Lemma 3.4.7 (and the explicit formula (2.4) for $\alpha_{n,s}$) for $x,y \in \mathcal{B}_{1/4}(0)$, and the conclusion also holds for $x, y \in \mathcal{B}_{1/2}(0)$ by a standard covering argument. □

Now, we have all the ingredients to give the proof of Theorem 3.4.6.

Proof of Theorem 3.4.6. Note that the statement is scaling invariant. Hence, with no loss of generality, we can (and do) assume that $R = 1$. Moreover, it suffices to consider the case $M = (\mathbb{R}^n, g)$, $p = 0$, $\varphi = \mathrm{id}$, and g_{ij} satisfying the assumptions of Proposition 3.4.12:

Indeed, similarly to the proof of Corollary 3.4.13, in the general case we can fix a radially nonincreasing cutoff function $\eta \in C_c^\infty(\mathcal{B}_1)$ such that $\eta \equiv 1$ in $\mathcal{B}_{7/8}$ and consider the "extended" metric $g'_{ij} := g_{ij}\eta + \delta_{ij}(1-\eta)$. Observe that (M,g) and $(\mathbb{R}^n, g')$ the assumptions of Lemma 3.4.11 with $M' = \mathbb{R}^n$ and $\varphi' = \mathrm{id}$. Let $H(x,y,t)$ and $H'(x,y,t)$ be defined as in Lemma 3.4.11.

Recall that, by definition, for all $x, y \in \mathcal{B}_1(0)$

$$K(x,y) = \mathcal{K}_s(\varphi(x),\varphi(y)) = c_s\int_0^\infty H_M(\varphi(x),\varphi(y),t)\frac{dt}{t^{1+s/2}} = c_s\int_0^\infty H(x,y,t)\frac{dt}{t^{1+s/2}}, \qquad (3.51)$$

where $c_s = \frac{s/2}{\Gamma(1-s/2)}$. Let likewise

$$K'(x,y) = c_s\int_0^\infty H'(x,y,t)\frac{dt}{t^{1+s/2}}.$$

Now, thanks to Lemma 3.4.11 we obtain, for all $x, y \in \mathcal{B}_{1/2}$:

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}(K-K')(x,y,t)\right| \le c_s\int_0^\infty \left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}(H-H')(x,y,t)\right|\frac{dt}{t^{1+s/2}} \le Cs\int_0^\infty e^{-c/t}\frac{dt}{t^{1+s/2}} \le C.$$

So, as claimed, we are left to prove the estimate for the $M = (\mathbb{R}^n, g)$, $p = 0$, $\varphi = \mathrm{id}$, and g_{ij} satisfying the assumptions of Proposition 3.4.12.

Recalling (3.51), notice that

$$k(x,z) = K(x,x+z) = c_s \int_0^\infty H(x,x+z,t)\frac{dt}{t^{1+s/2}} = c_s \int_0^\infty h\Big(\frac{A(x)z}{\sqrt{t}},x,t\Big)\frac{dt}{t^{n/2+1+s/2}}. \quad (3.52)$$

Also, recalling that $h_\circ(z) := (4\pi)^{-n/2}e^{-|z|^2/4}$, we have

$$\begin{aligned}\widehat{k}(x,z) = k(x,z) - \frac{\alpha_{n,s}}{|A(x)z|^{n+s}} &= c_s \int_0^\infty \bigg(h\Big(\frac{A(x)z}{\sqrt{t}},x,t\Big) - h_\circ\Big(\frac{A(x)z}{\sqrt{t}}\Big)\bigg)\frac{dt}{t^{n/2+1+s/2}}\\ &= c_s \int_0^\infty \widehat{h}\Big(\frac{A(x)z}{\sqrt{t}},x,t\Big)\frac{dt}{t^{n/2+1+s/2}},\end{aligned}$$

Therefore using the heat kernel estimates from Proposition 3.4.12 (and noticing $|A(x)z| \geq \frac{1}{\sqrt{2}}|z|$ for all x,z by assumption) we obtain

$$|\widehat{k}(x,z)| \leq c_s \int_0 \bigg|\widehat{h}\Big(\frac{A(x)z}{\sqrt{t}},x,t\Big)\bigg|\frac{dt}{t^{n/2+1+s/2}} \leq Cs\int_0^\infty \sqrt{t}\exp(-c|z|/\sqrt{t})\frac{dt}{t^{n/2+1+s/2}} = C|z|^{1-n-s}.$$

This proves (3.38). Similarly, the estimates (3.39) follow differentiating (3.52) and using the corresponding estimates for derivatives of the heat kernel from Proposition 3.4.12.

Finally, (3.40) and (3.41) follow analogously integrating the heat kernel estimates in Lemmas 3.4.9 and 3.4.10, respectively. □

The next property concerns the behavior of the kernel when the two points p and q are separated from each other.

Proposition 3.4.14. *Let (M,g) be a Riemannian n-manifold and $s \in (0,2)$. Assume that for some $p,q \in M$ both $\mathrm{FA}_\ell(M,g,1,p,\varphi_p)$ and $\mathrm{FA}_\ell(M,g,1,q,\varphi_q)$ hold, and suppose that $\varphi_p(\mathcal{B}_1(0)) \cap \varphi_q(\mathcal{B}_1(0)) = \varnothing$. Put $K_{pq}(x,y) := \mathcal{K}_s(\varphi_p(x),\varphi_q(y))$. Then*

$$\bigg|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}K_{pq}(x,y)\bigg| \leq C(n,\ell) \quad \textit{for all } |x| < \tfrac{1}{2} \textit{ and } |y| < \tfrac{1}{2},$$

whenever $|\alpha| + |\beta| \leq \ell$.

Proof. Let $H_*(x,y,t) := H_M(\varphi_p(x),\varphi_q(y),t)$. It follows from Lemma 3.4.9 that

$$\bigg|\frac{\partial^{|\alpha|}}{\partial x^\alpha}H_*(x,y,t)\bigg| \leq Ce^{-c/t},$$

for all $|x| < \frac{3}{4}$ and $|y| < \frac{3}{4}$, where C and c depend only on n, and $|\alpha|$.

We now use that (by the symmetry of the heat kernel in p and q), for each $x \in \mathcal{B}_{1/2}$ fixed, the function $u(y,t) := \frac{\partial^{|\alpha|}}{\partial x^\alpha}H_*(x,y,t)$ is solution of the heat equation $u_t = Lu$, in the ball $|y| < 1$, where L denotes the Laplace-Beltrami (with respect to y, in local coordinates). Since $|u| \leq Ce^{-c/t}$ in $\mathcal{B}_{3/4} \times (0,\infty)$, reasoning exactly as in the proof of Lemma 3.4.9 (only that now the spatial variables are y instead of x) we obtain

$$\bigg|\frac{\partial^{|\beta|}}{\partial y^\beta}u(y,t)\bigg| \leq Ce^{-c/t},$$

for some constants $C, c > 0$ depending only on n and $|\beta|$. This shows

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}H_*(x,y,t)\right| \le Ce^{-c/t}.$$

Then the proposition follows immediately after noticing that, by definition,

$$K_{pq}(x,y) = \frac{s/2}{\Gamma(1-s/2)}\int_0^\infty H_*(x,y,t)\frac{dt}{t^{1+s/2}},$$

and hence

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}K_{pq}(x,y)\right| = \left|\frac{s/2}{\Gamma(1-s/2)}\int_0^\infty \frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}H_*(x,y,t)\frac{dt}{t^{1+s/2}}\right| \le Cs\int_0^\infty \exp(-c/t)\frac{dt}{t^{1+s/2}} \le C\,,$$

for some constant $C > 0$ that depends only on n and ℓ, and this concludes the proof. □

3.4.3 The fractional Sobolev energy under inner variations

We next study how the fractional Sobolev energy behaves under inner variations. For this, we need first to study how the singular kernel $\mathcal{K}_s$ behaves when translating its arguments under the flow of a vector field.

Proposition 3.4.15. *Let M be a locally uniformly flat manifold (see Definition 3.4.2) and $s \in (0,2)$. Consider any smooth vector field X and fix points $p, q \in M$. Then, for every $\ell \ge 0$ the kernel satisfies*

$$\left|\frac{d^\ell}{dt^\ell}\right|_{t=0}\mathcal{K}_s(\phi_t^X(p),\phi_t^X(q))\Bigg| \le C(1+\mathcal{K}_s(p,q)),$$

for some constant $C = C(M, s, \ell, \|X\|_{C^\ell(M)})$ which stays bounded for s away from 0 and 2.

Proof. This follows from the estimates of Theorem 3.4.6, in particular by (3.39) and (3.40). We prove the result just for $\ell = 1$, as the general case just follows by induction by the very same arguments. Let $R = R(M) > 0$ be such that the flatness assumption $\mathrm{FA}_\ell(M, g, 16R, p, \varphi_p)$ holds for every $p \in M$; such an R exists since M is locally uniformly flat. We split in two cases.

Case 1: $q \in \varphi_p(\mathcal{B}_{4R}(0))$.
In this case, denoting $K(x,y) := \mathcal{K}_s(\varphi_p(x), \varphi_p(y))$ and $k(x,z) := K(x, x+z)$ as in Theorem 3.4.6, we have that

$$\mathcal{K}_s(\phi_t^X \circ \varphi_p(x), \phi_t^X \circ \varphi_p(y)) = K(\psi_p^t(x), \psi_p^t(y)) = k(\psi_p^t(x), \psi_p^t(y) - \psi_p^t(x))\,,$$

where ψ_p^t is the flow of $\xi = (\varphi_p)^* X$, i.e. the vector field $\xi = \xi_p$ with $\mathrm{supp}(\xi) \Subset \mathcal{B}_{16R}(0)$ such that $X \circ \varphi_p = (\varphi_p)_* \xi$. Then, for all $x, y \in \mathcal{B}_{2R}(0)$ we have:

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\mathcal{K}(\psi_p^t(x),\psi_p^t(y)) &= \frac{d}{dt}\bigg|_{t=0}k(\psi_p^t(x),\psi_p^t(y)-\psi_p^t(x)) \\ &= \frac{\partial k}{\partial x^\alpha}(x,y-x)\xi^\alpha(x) + \frac{\partial k}{\partial z^\alpha}(x,y-x)(\xi^\alpha(x)-\xi^\alpha(y))\,,\end{aligned}$$

where sum over repeated indices is assumed. Hence, by (3.39) of Theorem 3.4.6 we get

$$\left|\frac{d}{dt}\bigg|_{t=0}K(\psi_p^t(x),\psi_p^t(y))\right| \le \frac{C}{|y-x|^{n+s}}\|\xi\|_{L^\infty} + \frac{C}{|y-x|^{n+s+1}}\|D\xi\|_{L^\infty}|y-x|$$

$$\le \frac{C}{|y-x|^{n+s}} \le CK(x,y)$$

for some $C = C(n, s, \|\xi\|_{C^1})$, where in the last line we have also used Lemma 3.4.13. Finally, evaluating this inequality at $x = 0$ and $y = \varphi_p^{-1}(q)$ we obtain

$$\left|\frac{d}{dt}\right|_{t=0} \mathcal{K}_s(\phi_t^X(p), \phi_t^X(q))\Bigg| = \left|\frac{d}{dt}\right|_{t=0} K(\psi_p^t(0), \psi_p^t(y))\Bigg| \le CK(0,y) = C\mathcal{K}_s(p,q),$$

as wanted.

Case 2: $q \notin \varphi_p(\mathcal{B}_{4R}(0))$. Then $\mathrm{FA}_\ell(M, g, R, q, \varphi_q)$ holds and the sets $\varphi_p(\mathcal{B}_R(0))$ and $\varphi_q(\mathcal{B}_R(0))$ are disjoint. Hence, by Proposition 3.4.14 the kernel $K_{pq}(x,y) := \mathcal{K}_s(\varphi_p(x), \varphi_q(y))$ is smooth (with uniform estimates on all derivatives) in the domain $\mathcal{B}_{R/2}(0) \times \mathcal{B}_{R/2}(0)$. Hence

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} \mathcal{K}_s(\phi_t^X \circ \varphi_p(x), \phi_t^X \circ \varphi_q(x)) &= \frac{d}{dt}\bigg|_{t=0} K_{pq}(\psi_p^t(x), \psi_p^t(y)) \\ &= \frac{\partial K_{pq}}{\partial x^\alpha}(x,y)\xi_p^\alpha(x) + \frac{\partial K_{pq}}{\partial y^\alpha}(x,y)\xi_q^\alpha(y).\end{aligned}$$

Using Proposition 3.4.14 to bound the derivatives of K_{pq}, and then evaluating at $(x,y) = (0,0)$ gives

$$\left|\frac{d}{dt}\right|_{t=0} \mathcal{K}_s(\phi_t^X(p), \phi_t^X(q))\Bigg| \le \frac{C}{R^{n+s}}\,,$$

for some $C = C(n, s, \|\xi_p\|_{L^\infty}, \|\xi_q\|_{L^\infty})$.

Putting together the two cases above, we get

$$\left|\frac{d}{dt}\right|_{t=0} \mathcal{K}_s(\phi_t^X(p), \phi_t^X(q))\Bigg| \le C(1 + \mathcal{K}_s(p,q))\,,$$

for some $C = C(M, n, s, \|X\|_{C^1(M)})$ and concludes the proof.

□

We also record a version of Proposition 3.4.15, which depends only on local quantities:

Proposition 3.4.16. *Let (M, g) be a Riemannian manifold and $s \in (0,2)$. Assume that the flatness assumption $\mathrm{FA}_\ell(M, g, R, p, \varphi)$ holds, and let X be a C^ℓ vector field with $\mathrm{supp}(X) \Subset \varphi(\mathcal{B}_{R/4})$. Then, for every $x, y \in \mathcal{B}_{R/4}(0)$ we have*

$$\left|\frac{d^\ell}{dt^\ell}\right|_{t=0} \mathcal{K}_s(\phi_t^X(\varphi(x)), \phi_t^X(\varphi(y)))\Bigg| \le C\mathcal{K}_s(\varphi(x), \varphi(y)) \le C\frac{\alpha_{n,s}}{|x-y|^{n+s}}\,, \tag{3.53}$$

for some constant $C = C(n, s, \|X\|_{C^\ell(\varphi(\mathcal{B}_R))})$. Moreover, given $T > 0$ we have that

$$\left|\frac{d^\ell}{dt^\ell}\mathcal{K}_s(\phi_t^X(\varphi(x)), \phi_t^X(\varphi(y)))\right| \le C_T\mathcal{K}_s(\varphi(x), \varphi(y)) \le C_T\frac{\alpha_{n,s}}{|x-y|^{n+s}}, \quad \forall\, t \in [0,T], \tag{3.54}$$

where $C_T = C_T(n, s, T, \|X\|_{C^\ell(\varphi(\mathcal{B}_{R/4}))})$ and the constants stay bounded for s away from 0 and 2.

Proof. By scaling, we can assume $R = 1$. The second inequality in both (3.53) and (3.54) then follows from Lemma 3.4.13. As for the first inequality of (3.53), it follows from the proof of Case 1 in Proposition 3.4.15, since it only depends on local estimates for X. Finally, (3.54) can be deduced from (3.53). Indeed, note that for all $1 \le k \le \ell$ and $0 \le t \le T$

$$\begin{aligned} \left| \frac{d^k}{dt^k} \mathcal{K}_s(\phi_t^X(\varphi(x)), \phi_t^X(\varphi(y))) \right| &= \left| \frac{d^k}{dr^k} \right|_{r=0} \mathcal{K}_s(\phi_{t+r}^X(\varphi(x)), \phi_{t+r}^X(\varphi(y))) \Big| \\ &\le C_0 \mathcal{K}_s(\phi_t^X(\varphi(x)), \phi_t^X(\varphi(y))) \,, \end{aligned} \tag{3.55}$$

with $C_0 = C_0(n, s, \|X\|_{C^\ell(\varphi(\mathcal{B}_1))})$. Thus, we are only left with proving that

$$\mathcal{K}_s(\phi_t^X(\varphi(x)), \phi_t^X(\varphi(y))) \le C_T \mathcal{K}_s(\varphi(x), \varphi(y))$$

for some $C_T = C_T(n, s, T, \|X\|_{C^\ell(\varphi(\mathcal{B}_1))})$. But this follows itself from (3.55), with $k = 1$, since we can write the inequality as

$$\frac{d}{dt}\big[e^{-C_0 t} \mathcal{K}_s(\phi_t^X(\varphi(x)), \phi_t^X(\varphi(y)))\big] \le 0,$$

and integrating we find that

$$\mathcal{K}_s(\phi_t^X(\varphi(x)), \phi_t^X(\varphi(y))) \le e^{C_0 T} \mathcal{K}_s(\varphi(x), \varphi(y)),$$

for every $0 \le t \le T$. □

Proposition 3.4.15 can be used to bound time derivatives of the energy of "flown objects" by their energy at time zero. We show this for the fractional Sobolev energy:

Lemma 3.4.17. *Let $s \in (0, 2)$ and $v \in H^{s/2}(M)$ be a function with $|v| \le 1$. Let X be a smooth vector field and $v_t := v \circ \phi_{-t}^X$. Then, for all $T > 0$ there holds*

$$\sup_{0<t<T} \left| \frac{d^\ell}{dt^\ell} \mathcal{E}_M(v_t) \right| \le C\big(1 + \mathcal{E}_M(v)\big) \,,$$

for some constant $C = C(M, s, \ell, T, \|X\|_{C^\ell(M)})$ which stays bounded for s away from 0 and 2..

Proof. Let C denote a constant that depends only on M, s, ℓ, T and $\|X\|_{C^\ell(M)}$.

The idea of the proof is to change variables using the flow ϕ_t^X in the corresponding integrals defining the Allen-Cahn energy, and after that to exchange integration and differentiation. Let us start with the Sobolev part of the energy. Denoting by J_t the Jacobian of ϕ_t^X, we have

$$\begin{aligned} \frac{d^\ell}{dt^\ell} \mathcal{E}_M^{\rm Sob}(v_t) &= \frac{d^\ell}{dt^\ell} \iint |v(\phi_{-t}^X(p)) - v(\phi_{-t}^X(q))|^2 \mathcal{K}_s(p, q) \, dV_p \, dV_q \\ &= \frac{d^\ell}{dt^\ell} \iint |v(p) - v(q)|^2 \mathcal{K}_s(\phi_t^X(p), \phi_t^X(q)) \, J_t(p) J_t(q) \, dV_p \, dV_q \\ &= \iint |v(p) - v(q)|^2 \frac{d^\ell}{dt^\ell} \Big[\mathcal{K}_s(\phi_t^X(p), \phi_t^X(q)) \, J_t(p) J_t(q)\Big] dV_p \, dV_q. \end{aligned} \tag{3.56}$$

Since $0 < t < T$, the derivatives in time of the Jacobians J_t can be bounded by a constant C with the right dependencies. What remains in order to bound (3.56) by $C(1 + \mathcal{E}_M^{\rm Sob}(v))$ is to control the first k-th derivatives in time of $\mathcal{K}_s(\phi_t^X(p), \phi_t^X(q))$ by $C(1 + \mathcal{K}_s(p, q))$, for all $0 < t < T$. The main bound is given by Proposition 3.4.15, which gives for all $1 \le k \le \ell$:

$$\left| \frac{d^k}{dt^k} \mathcal{K}_s(\phi_t^X(p), \phi_t^X(q)) \right| = \left| \frac{d^k}{dr^k} \right|_{r=0} \mathcal{K}_s(\phi_{t+r}^X(p), \phi_{t+r}^X(q)) \Big| \le C\big(1 + \mathcal{K}_s(\phi_t^X(p), \phi_t^X(q))\big) \,. \tag{3.57}$$

Now, integrating this inequality for $k = 1$ similarly to how we proceeded in the proof of Lemma 3.4.16, we conclude that

$$\mathcal{K}_s(\phi_t^X(p), \phi_t^X(q)) \leq C(1 + \mathcal{K}_s(p, q)), \quad \text{for all } 0 < t < T\,.$$

We can now go back to (4.22) and apply the bounds that we just derived. We get that

$$\begin{aligned}
\left|\frac{d^\ell}{dt^\ell}\mathcal{E}_M^{\text{Sob}}(v_t)\right| &\leq \iint |v(p) - v(q)|^2 \frac{d^\ell}{dt^\ell}\Big[\mathcal{K}_s(\phi_t^X(p), \phi_t^X(q))\, J_t(p)J_t(q)\Big] dV_p\, dV_q \\
&\leq C \iint |v(p) - v(q)|^2 (1 + \mathcal{K}_s(p, q))\, dV_p dV_q \\
&\leq C(1 + \mathcal{E}_M^{\text{Sob}}(v))
\end{aligned}$$

for all $0 < t < T$, where C has the right dependencies.

The potential part of the energy is simpler to deal with. Indeed, we have

$$\frac{d^\ell}{dt^\ell}\mathcal{E}_M^{\text{Pot}}(v_t) = \frac{d^\ell}{dt^\ell}\int_M \varepsilon^{-s} W(v(\phi_{-t}^X(p))dV_p \;= \int_M \varepsilon^{-s} W(v(p))\frac{d^\ell}{dt^\ell} J_t(p) dV_p, \tag{3.58}$$

from which we directly conclude that

$$\left|\frac{d^\ell}{dt^\ell}\mathcal{E}_M^{\text{Pot}}(v_t)\right| \leq C\mathcal{E}_M^{\text{Pot}}(v),$$

finishing the proof. ∎

Corollary 3.4.18. *Let M be either a closed manifold or $\mathbb{R}^n$, and let $\Omega \subset M$ be a bounded, open set with regular boundary. Let $s \in (0, 1)$, E be a set with $\text{Per}_s(E, \Omega) < +\infty$ and X be a vector field of class C^k with $\text{supp}(X) \Subset \Omega$. Then, the map*

$$t \mapsto \text{Per}_s(\phi_t^X(E), \Omega)$$

is of class C^k for all $t > 0$.

Proof. The proof is almost identical to the one of Lemma 3.4.17 with $v = \chi_E$. Indeed, since (ϕ_t^X, ϕ_t^X) sends $M \times M \setminus \Omega^c \times \Omega^c$ to itself, one can do the same substitution in the integral and use Proposition 3.4.15 to bound the derivatives of $\mathcal{K}_s$ with itself. This results in

$$\begin{aligned}
\left|\frac{d^\ell}{dt^\ell}\text{Per}_s(\phi_t^X(E), \Omega)\right| &\leq \iint_{M\times M\setminus\Omega^c\times\Omega^c} |\chi_E(p) - \chi_E(q)|^2 \frac{d^\ell}{dt^\ell}\Big[\mathcal{K}_s(\phi_t^X(p), \phi_t^X(q))\, J_t(p)J_t(q)\Big] dV_p\, dV_q \\
&\leq C \iint_{M\times M\setminus\Omega^c\times\Omega^c} |\chi_E(p) - \chi_E(q)|^2 (1 + \mathcal{K}_s(p, q))\, dV_p dV_q \\
&\leq C\big(|M|^2 + \text{Per}_s(E, \Omega)\big),
\end{aligned}$$

for every $0 \leq \ell \leq k$, as desired.

In the case of $\mathbb{R}^n$, the same computation applies with the only difference that for the kernel of $\mathbb{R}^n$, there holds the "improved" estimate

$$\left|\frac{d^\ell}{dt^\ell}\frac{1}{|\phi_t^X(p) - \phi_t^X(q)|^{n+s}}\right| \leq \frac{C}{|p - q|^{n+s}},$$

without the $+C$ on the right-hand side that appeared in Proposition 3.4.15. Hence, in this case

$$\left|\frac{d^\ell}{dt^\ell}\mathrm{Per}_s(\phi_t^X(E),\Omega)\right| \leq C \iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus\Omega^c\times\Omega^c} \frac{|\chi_E(p)-\chi_E(q)|^2}{|p-q|^{n+s}}\,dpdq = C\mathrm{Per}_s(E,\Omega).$$

□

Lemma 3.4.17 has a local version, which comes from applying local estimates for the kernel instead.

Lemma 3.4.19. *Let M satisfy the flatness assumptions $\mathrm{FA}_\ell(M,g,p,R,\varphi)$. Let $s\in(0,2)$, $v\in H^{s/2}(M)$ be a function with $|v|\leq 1$, X be a C^ℓ vector field with $\mathrm{supp}(X)\Subset\varphi(\mathcal{B}_{R/2})$, and set $v_t := v\circ\phi_{-t}^X$. Then, for all $T>0$ there holds*

$$\sup_{0<t<T}\left|\frac{d^\ell}{dt^\ell}\mathcal{E}_{\varphi(\mathcal{B}_{R/2})}(v_t)\right| \leq C(1+\mathcal{E}_{\varphi(\mathcal{B}_{R/2})}(v))\,,$$

for some constant $C=C(s,\ell,T,\|X\|_{C^\ell(\varphi(\mathcal{B}_{R/2}))})$ which stays bounded for s away from 0 and 2.

Proof. We modify the proof of Lemma 3.4.17 accordingly. First, by scaling, it suffices to prove the Lemma in the case $R=1$. Since X is supported on $\varphi(\mathcal{B}_{1/2})$, the integrand in (4.22) is supported then on

$$\begin{aligned}(M\times M)\setminus(\varphi(\mathcal{B}_{1/2})^c\times\varphi(\mathcal{B}_{1/2})^c) = &\Big[(\varphi(\mathcal{B}_{2/3})\times\varphi(\mathcal{B}_{2/3}))\setminus(\varphi(\mathcal{B}_{1/2})^c\times\varphi(\mathcal{B}_{1/2})^c)\Big]\cup\\ &\cup\Big[(\varphi(\mathcal{B}_{1/2})\times(M\setminus\varphi(\mathcal{B}_{2/3})))\cup((M\setminus\varphi(\mathcal{B}_{2/3}))\times\varphi(\mathcal{B}_{1/2}))\Big]\,,\end{aligned}$$

so that

$$\begin{aligned}
\left|\frac{d^k}{dt^k}\mathcal{E}_M^{\mathrm{Sob}}(v_t)\right| &= \left|\frac{d^k}{dt^k}\mathcal{E}^{\mathrm{Sob}}_{\varphi(\mathcal{B}_{1/2})}(v_t)\right|\\
&= \Bigg|\iint_{(\varphi(\mathcal{B}_{2/3})\times\varphi(\mathcal{B}_{2/3}))\setminus(\varphi(\mathcal{B}_{1/2})^c\times\varphi(\mathcal{B}_{1/2})^c)} |v(p)-v(q)|^2\frac{d^k}{dt^k}\Big[\mathcal{K}_s(\phi_t^X(p),\phi_t^X(q))J_t(p)J_t(q)\Big]\,dV_p\,dV_q\\
&\quad + 2\iint_{\varphi(\mathcal{B}_{1/2})\times(M\setminus\varphi(\mathcal{B}_{2/3}))} |v(p)-v(q)|^2\frac{d^k}{dt^k}\Big[\mathcal{K}_s(\phi_t^X(p),q)J_t(p)\Big]\,dV_p\,dV_q\Bigg|\\
&\leq C\iint_{(\mathcal{B}_{2/3}\times\mathcal{B}_{2/3})\setminus(\mathcal{B}^c_{1/2}\times\mathcal{B}^c_{1/2})} |v(\varphi(x))-v(\varphi(y))|^2\left|\frac{d^k}{dt^k}\Big[\mathcal{K}_s(\varphi(\phi_t^X(x)),\varphi(\phi_t^X(y)))J_t(\varphi(x))J_t(\varphi(y))\Big]\right|dx\,dy\\
&\quad + C\iint_{\mathcal{B}_{1/2}\times(N\setminus\varphi(\mathcal{B}_{2/3}))} |v(\varphi(x))-v(q)|^2\left|\frac{d^k}{dt^k}\Big[\mathcal{K}_s(\varphi(\phi_t^X(x)),q)J_t(\varphi(x))\Big]\right|dx\,dV_q\,.
\end{aligned}$$

Bounding the derivatives in time of the Jacobians by a constant with the right dependencies, using (3.53) to bound the kernel in the first double integral, and using (3.41) to bound the integral in q in the second double integral by a constant, we conclude that

$$\left|\frac{d^k}{dt^k}\mathcal{E}^{\mathrm{Sob}}_{\varphi(\mathcal{B}_{1/2})}(v_t)\right| \leq C(1+\mathcal{E}^{\mathrm{Sob}}_{\varphi(\mathcal{B}_{1/2})}(v))\,.$$

Regarding the potential part of the energy, from the computation in (3.58), we readily find that

$$\left|\frac{d^\ell}{dt^\ell}\mathcal{E}^{\mathrm{Pot}}_{\varphi(\mathcal{B}_{1/2})}(v_t)\right| \leq C\mathcal{E}^{\mathrm{Pot}}_{\varphi(\mathcal{B}_{1/2})}(v)$$

where C has the right dependencies, which completes the proof. $\square$

Chapter 4

Yau's conjecture for nonlocal minimal surfaces

This chapter describes the results obtained in [CFS24b].

4.1 Existence of min-max solutions to Allen-Cahn

4.1.1 Min-max procedure and existence

In this section, we exhibit in a simple manner the existence of critical points of the Allen-Cahn energy (2.6) on M, employing a min-max theorem as in [GG18]. Then, we prove lower and upper bounds for the energies of the constructed solutions and with these ingredients we deduce Theorem 1.2.12.

The solutions in Theorem 1.2.12 are obtained using an equivariant min-max procedure, based on the construction in [GG18] and the min-max theorems of [Gho93], [Gho91] and [LS88]. Since the topology of $H^{s/2}(M)$ is trivial, this is done by exploiting the $\mathbb{Z}_2$-symmetry of the functional $\mathcal{E}^{\varepsilon}_{M}$. Indeed, we consider the family $\mathcal{F}_{\mathfrak{p}}$ of all sets $A \subset H^{s/2}(M)\backslash\{0\}$ which are continuous odd images of $\mathfrak{p}$-spheres:

$$\mathcal{F}_{\mathfrak{p}} := \big\{A = f(\mathbb{S}^{\mathfrak{p}}) \ : \ f \in C^0(\mathbb{S}^p; H^{s/2}(M)\backslash\{0\}) \ \text{ and } \ f(-x) = -f(x) \ \forall \ x \in \mathbb{S}^{\mathfrak{p}}\big\}.$$

Remark 4.1.1. *This min-max family has been chosen for simplicity, but other min-max families can be considered; see the seminal article [LS88] by Lazer-Solimini, as well as the discussion in Remark 3.7 of [GG18]. In particular, one can obtain solutions in Theorem 1.2.12, which also satisfy lower bounds for their (extended) Morse indices, and such that the corresponding min-max families come from a topological index. We nevertheless remark that a growth for the (proper!) index of the solutions is already implied in our case, by combining the lower energy bound in Theorem 1.2.12 with the upper energy bounds in Theorem 1.2.15 (which will be proved later).*

For fixed ε, the min-max value of the family $\mathcal{F}_{\mathfrak{p}}$ is defined as

$$c_{\varepsilon,\mathfrak{p}} := \inf_{A\in\mathcal{F}_{\mathfrak{p}}} \sup_{u\in A} \mathcal{E}^{\varepsilon}_{M}(u). \tag{4.1}$$

Note that, defining $T(u) := \max\{-1, \min\{u, +1\}\}$ the truncation of u between the values ±1, we have that $|T(u)|(x) \leq 1$ for all $x \in M$ and $\mathcal{E}^{\varepsilon}_{M}(T(u)) \leq \mathcal{E}^{\varepsilon}_{M}(u)$. Hence

$$c_{\varepsilon,\mathfrak{p}} = \inf_{A\in\mathcal{F}_{\mathfrak{p}}} \sup_{u\in A} \mathcal{E}^{\varepsilon}_{M}(u) = \inf_{A\in\widetilde{\mathcal{F}}_{\mathfrak{p}}} \sup_{u\in A} \mathcal{E}^{\varepsilon}_{M}(u)\,,$$

where
$$\widetilde{\mathcal{F}}_{\mathfrak{p}} = \{A \in \mathcal{F}_{\mathfrak{p}} : |u| \leq 1 \ \text{ for all } \ u \in A\}.$$
This shows that we can consider, in the arguments that follow, that the functions in the min-max sets have absolute values pointwise bounded by one. The proof of Theorem 1.2.12 relies on the existence result given by the min-max scheme and the following bound on the min-max values.

Theorem 4.1.2. *Let (M^n, g) be a compact Riemannian manifold, $s_0 \in (0,1)$ and $s \in (s_0, 1)$. Then, for every $\mathfrak{p} \in \mathbb{N}$ there exists $\varepsilon_{\mathfrak{p}} > 0$, depending on M, s and $\mathfrak{p}$, such that the min-max values (4.1) satisfy*
$$\frac{C^{-1}}{1-s}\mathfrak{p}^{s/n} \leq c_{\varepsilon,\mathfrak{p}} \leq \frac{C}{1-s}\mathfrak{p}^{s/n}\,, \quad \text{for all } \varepsilon \in (0, \varepsilon_{\mathfrak{p}})\,, \tag{4.2}$$
for some constant $C = C(M, s_0)$.

Proof. The proof of this is contained in Sections 4.1.2 and 4.1.3 below, which deal with the lower bound and upper bound, respectively. □

To apply the existence result we need the energy $\mathcal{E}_M^\varepsilon$ to satisfy the Palais-Smale condition along appropriately bounded sequences, and this is addressed by the next lemma.

Lemma 4.1.3. *Let $\varepsilon > 0$ and $s \in (0,1)$. Suppose that $(u_k)_k \subset H^{s/2}(M)$ is a sequence of functions satisfying $|u_k| \leq 1$, $|\mathcal{E}_M^\varepsilon(u_k)| \leq C$, and $d\mathcal{E}_M^\varepsilon(u_k) \to 0$ strongly in $H^{s/2}(M)$. Then, there is a subsequence of $(u_k)_k$ converging strongly in $H^{s/2}(M)$.*

Proof. The proof is an adaptation of Proposition 2.25 in [BR14]. We just prove the statement for $\varepsilon = 1$, as exactly the same proof works for every fixed $\varepsilon > 0$. The boundedness of the energies $\mathcal{E}_M(u_k)$ gives the convergence
$$u_k \to u \ \text{ in } L^2(M), \ \text{ and } \ u_k \rightharpoonup u \ \text{ weakly in } H^{s/2}(M),$$
of a subsequence, that we do not relabel, to some $u \in H^{s/2}(M)$. To upgrade the convergence to be in the strong sense, we use the particular form of the functional. First, note that given $v \in H^{s/2}(M)$ we have
$$\begin{aligned} d\mathcal{E}_M^\varepsilon(u)[v] &= \frac{1}{4}\iint (u(p)-u(q))(v(p)-v(q))\mathcal{K}_s(p,q)\,dV_p dV_q + \int W'(u)v\,dV \\ &= \lim_{k\to\infty}\frac{1}{4}\iint (u_k(p)-u_k(q))(v(p)-v(q))\mathcal{K}_s(p,q)\,dV_p dV_q + \int W'(u_k)v\,dV \\ &= \lim_{k\to\infty} d\mathcal{E}_M^\varepsilon(u_k)[v] = 0\,, \end{aligned}$$
where we used $|u_k| \leq 1$ to pass to the limit in the term $\int W'(u_k)v$. In other words, u is a critical point of $\mathcal{E}_M^\varepsilon$. From this we deduce that
$$\begin{aligned} 0 &= \lim_{k\to\infty}\left(d\mathcal{E}_M^\varepsilon(u_k)[u_k-u] - d\mathcal{E}_M^\varepsilon(u)[u_k-u]\right) \\ &= \mathcal{E}_M^{\mathrm{Sob}}(u_k-u) + \int (W'(u_k)-W'(u))(u_k-u)\,dV, \end{aligned}$$
and since the second term tends to zero, the first term must do so as well. This proves that $u_k \to u$ strongly in $H^{s/2}(M)$ and concludes the proof. □

Theorem 4.1.4. *For every $\mathfrak{p} \in \mathbb{N}$, there exists $\varepsilon_{\mathfrak{p}} > 0$ such that: for all $\varepsilon \in (0, \varepsilon_{\mathfrak{p}})$ there exists $u_{\varepsilon,\mathfrak{p}} \in H^{s/2}(M)$ which is a critical point of $\mathcal{E}^{\varepsilon}_M$ with $\mathcal{E}^{\varepsilon}_M(u_{\varepsilon,\mathfrak{p}}) = c_{\varepsilon,\mathfrak{p}}$ and Morse index $m(u_{\varepsilon,\mathfrak{p}}) \le \mathfrak{p}$ (see Definition 2.2.9).*

Proof. First, note that $\widetilde{\mathcal{F}}_{\mathfrak{p}}$ is an invariant $\mathfrak{p}$-dimensional homotopic family without boundary, in the sense of Section 3 in [Gho91] with $B = \varnothing$. Moreover, for every $\varepsilon > 0$, $\mathcal{E}^{\varepsilon}_M$ satisfies the Palais-Smale condition along appropriately bounded sequences (see Lemma 4.1.3 above) and $d^2\mathcal{E}^{\varepsilon}_M$ is a Fredholm operator on critical points. Then, [Gho91, Corollary 13] applied with $B = \varnothing$ (see also [Gho91, Theorem 4]) implies that there exists a critical point $u_{\varepsilon,\mathfrak{p}}$ for $\mathcal{E}^{\varepsilon}_M$ at energy level $c_{\varepsilon,\mathfrak{p}}$ and with Morse index $m(u_{\varepsilon,\mathfrak{p}}) \le \mathfrak{p}$.

There is only one detail that we have to address: [Gho91, Corollary 13] would apply directly to a complete connected Banach manifold X on which $\mathbb{Z}_2$ acted freely. In our case, we want to consider $X = H^{s/2}(M)$, together with the action $x \mapsto -x$ under which $\widetilde{\mathcal{F}}_{\mathfrak{p}}$ is an invariant $\mathfrak{p}$-dimensional homotopic family, but which is not free since it maps the point 0 to itself. This is not an issue for the following reason. By the upper bound for the min-max values (4.2) (that we will prove in Section 4.1.3 below), we have that $c_{\varepsilon,\mathfrak{p}} \le C(s, \mathfrak{p}, n)$ for every $\varepsilon \in (0, \varepsilon_{\mathfrak{p}})$, for some $\varepsilon_{\mathfrak{p}} = \varepsilon_{\mathfrak{p}}(M, s, \mathfrak{p}) > 0$. On the other hand, the Allen-Cahn energy of the zero function tends to infinity as $\varepsilon \searrow 0$, which shows that there is $\varepsilon_{\mathfrak{p}} > 0$ such that, for all $\varepsilon \in (0, \varepsilon_{\mathfrak{p}})$, every min-max sequence is uniformly separated from $0 \in H^{s/2}(M)$. Hence, for a small $r > 0$ the min-max result [Gho91, Corollary 13] can be applied to $X = H^{s/2}(M) \setminus \{\|u\|_{H^{s/2}} < r\}$ on which the action $x \mapsto -x$ is free. □

Hence, to obtain Theorem 1.2.12 we are only left with proving the lower and upper bounds in (4.2). The analog bounds in the case of min-max families of hypersurfaces and their areas were first proved by Gromov in [Gro88]. In [Gut09], Guth gave an elegant new proof of the result by Gromov on which the proof of our bounds is partially based. See also [MN17b] for an adaptation of the proof by Guth to the setting of closed manifolds, as well as [GG18] for the adaptation to the classical Allen-Cahn case. Nevertheless, our proof of the bounds is closer to (and also simpler than) those in [Gut09] or [MN17b], thanks to the fact that sets of finite perimeter embed naturally[1] in the same function space $H^{s/2}(M)$ as their fractional Allen-Cahn approximations, as long as $s \in (0, 1)$.

4.1.2 Lower bound

In the proofs of both the upper and lower bounds, we will make use of the following simple fact.

Lemma 4.1.5. *Let (M^n, g) be a closed, n-dimensional Riemannian manifold. Then, there exist positive constants C_0, C_1, C_2 depending only on M such that: for every $\mathfrak{p} \in \mathbb{N}$ there exist N disjoint balls $B_r(q_1), \dots B_r(q_N)$ with*

$$C_1\,\mathfrak{p} \le N \le C_2\,\mathfrak{p}\,,$$
$$r = C_0\,\mathfrak{p}^{-1/n}\,,$$

and

$$\bigcup_{i \le N} B_{3r}(q_i) = M\,.$$

Proof. Since M is compact, by a comparison argument there exists a constant $c = c(M) > 1$ such that

$$c^{-1} r^n \le \mathcal{H}^n(B_r(q)) \le c\, r^n\,, \quad \text{for all } q \in M\,,\ r < \mathrm{inj}(M)\,. \tag{4.3}$$

[1] Via functions of the form $\chi_E - \chi_{E^c}$.

We claim that, for $C_0 = \mathrm{inj}(M)/3$, the statement holds. In fact, with this choice we have $r = C_0\,\mathfrak{p}^{-1/n} \le \mathrm{inj}(M)/3$ for all $\mathfrak{p} \in \mathbb{N}$. Consider the cover $\bigcup_{q\in M} B_r(q)$ of M, and let $B_r(q_1), \dots, B_r(q_N)$ be a maximal disjoint collection. Then, by maximality $\bigcup_{i\le N} B_{3r}(q_i) = M$. Moreover, comparing volumes

$$c^{-1}r^nN \le \sum_{i\le N}\mathcal{H}^n(B_r(q_i)) \le \mathrm{Vol}(M) \le \sum_{i\le N}\mathcal{H}^n(B_{3r}(q_i)) \le c3^nr^nN\,,$$

which by the choice of r implies

$$C_1\,\mathfrak{p} := \frac{\mathrm{Vol}(M)}{c3^nC_0^n}\,\mathfrak{p} \le N \le \frac{c\mathrm{Vol}(M)}{C_0^n}\,\mathfrak{p} =: C_2\,\mathfrak{p}\,.$$

□

The proof of the lower bound depends on the next two lemmas.

Lemma 4.1.6. *Let $\{B_r(q_i)\}_{i=1}^{\mathfrak{p}}$ be a family of $\mathfrak{p}$ balls on M. Then, given any $A \in \mathcal{F}_{\mathfrak{p}}$, there exists some $u \in A$ such that*

$$\int_{B_r(q_i)} u = 0 \quad \text{for all} \quad i = 1, \dots, \mathfrak{p}.$$

Proof. This is simply a consequence of the Borsuk-Ulam theorem. Indeed, let A be the continuous odd image of $f : \mathbb{S}^{\mathfrak{p}} \to H^{s/2}(M) \setminus \{0\}$. Define the (odd) function

$$g : H^{s/2}(M) \to \mathbb{R}^{\mathfrak{p}} \quad \text{by} \quad g(u) := \left(\int_{B_r(q_1)} u, \dots, \int_{B_r(q_{\mathfrak{p}})} u\right).$$

Then $g \circ f : \mathbb{S}^{\mathfrak{p}} \to \mathbb{R}^{\mathfrak{p}}$ is an odd, continuous map, and by the Borsuk-Ulam theorem there exists $a \in \mathbb{S}^{\mathfrak{p}}$ with $g \circ f(a) = 0$. Hence, taking $u = f(a)$ finishes the proof. □

For the next lemma, it will be convenient to define the local part of the Sobolev energy as follows. Recall

$$\begin{aligned}\mathcal{E}_\Omega^{\mathrm{Sob}}(v) &= \frac{1}{4}\iint_{M\times M\setminus\Omega^c\times\Omega^c}(v(p)-v(q))^2\mathcal{K}_s(p,q)dV_pdV_q\\ &= \frac{1}{4}\iint_{\Omega\times\Omega}(v(p)-v(q))^2\mathcal{K}_s(p,q)dV_pdV_q + \frac{1}{2}\iint_{\Omega\times\Omega^c}(v(p)-v(q))^2\mathcal{K}_s(p,q)dV_pdV_q\,.\end{aligned}$$

Then, we set

$$\mathcal{E}^{\mathrm{Sob}}|_\Omega(v) := \frac{1}{4}\iint_{\Omega\times\Omega}(v(p)-v(q))^2\mathcal{K}_s(p,q)dV_pdV_q\,.$$

Moreover, we also denote by

$$\mathrm{Per}_s|_\Omega(E) := \mathcal{E}^{\mathrm{Sob}}|_\Omega(\chi_E - \chi_{E^c})$$

the local part of the nonlocal perimeter. We stress that $\mathrm{Per}_s|_\Omega(E) \le \mathrm{Per}_s(E,\Omega)$, with equality iff $E\cap\Omega = \emptyset$ or $E^c\cap\Omega = \emptyset$.

With this notation, let us recall the fractional isoperimetric inequality (which is implied, actually equivalent, to the fractional Sobolev inequality, for example, in [FS08b]). Let $s \in (0,1)$.

Then is $c_{iso} = c_{iso}(n, s) > 0$ such that for every $E \subset \mathcal{B}_1$

$$\mathrm{Per}_s|_{\mathcal{B}_1}(E) \geq c_{iso} \min \left\{ \frac{|E|}{|\mathcal{B}_1|}, \frac{|\mathcal{B}_1 \setminus E|}{|\mathcal{B}_1|} \right\}^{\frac{n-s}{n}}, \tag{4.4}$$

and actually if $s \in (s_0, 1)$ then $c_{iso} \geq \frac{c'}{1-s}$ for some $c' = c'(n, s_0)$.

Lemma 4.1.7. *Let $s_0 \in (0,1)$. Then, there exists a constant $c_0 = c_0(n, s_0) > 0$ such that the following holds. Let $q \in M$, and assume that the flatness hypothesis $\mathrm{FA}_1(M, g, q, 2R_0, \varphi)$ holds. Given $s \in (s_0, 1)$, there exists $\varepsilon_0 = \varepsilon_0(n, s) > 0$ such that: for every $r \leq R_0$ and $\varepsilon \leq \varepsilon_0 r$, given any $u \in H^{s/2}(M)$ with $|u| \leq 1$ and $\int_{B_r(q)} u = 0$ there holds*

$$\mathcal{E}^\varepsilon|_{B_r(q)}(u) \geq \frac{c_0}{1-s} r^{n-s}.$$

Proof. Let $\psi : B_{R_0(q)} \to \mathbb{R}^n$ be normal coordinates at q, and let g_{ij} denote the metric in these coordinates. It is not difficult to show that $\mathrm{FA}_1(M, g, q, 2R_0, \varphi)$ implies

$$\frac{1}{C}\mathrm{id} \leq (g_{ij})(x) \leq C\mathrm{id} \quad \text{in } \mathcal{B}_{R_0}$$

for some $C > 1$ depending only on n.

Let $v = u \circ \psi^{-1}$. By assumption we have $\int_{\mathcal{B}_r} v\sqrt{|g|}\, dx = 0$, and $|v| \leq 1$. This implies

$$-1 + 1/C \leq \fint_{\mathcal{B}_r} v \leq +1 - 1/C\,.$$

By Lemma 3.4.13

$$\begin{aligned}\mathcal{E}^{\mathrm{Sob}}|_{B_r(q)}(u) &+ \varepsilon^{-s} \int_{B_r(q)} W(u)\, dV \\ &\geq \frac{1}{C}\bigg(\frac{1}{4}\iint_{\mathcal{B}_r\times\mathcal{B}_r} \frac{|v(x)-v(y)|^2}{|x-y|^{n+s}}\, dxdy + r^{-s}(\varepsilon/r)^{-s}\int_{\mathcal{B}_r} W(v)\, dx\bigg).\end{aligned} \tag{4.5}$$

We claim that the right-hand side of (4.5) is bounded below by $\frac{c_{iso}}{4C^2} r^{n-s}$, provided $\varepsilon/r \leq \varepsilon_0(n, s)$, where c_{iso} is the constant in the fractional isoperimetric inequality (4.4).

To prove this lower bound, by scaling invariance we may assume without loss of generality $r = 1$. We argue by contradiction. Suppose there exist sequences $\varepsilon_k \downarrow 0$ and $v_k \in H^{s/2}(\mathcal{B}_1)$ with $|v_k| \leq 1$, $\fint_{\mathcal{B}_1} v_k \in [-1 + 1/C,\, 1 - 1/C]$, but such that

$$\frac{1}{4}\iint_{\mathcal{B}_1\times\mathcal{B}_1} \frac{|v_k(x)-v_k(y)|^2}{|x-y|^{n+s}}\, dxdy + \varepsilon_k^{-s}\int_{\mathcal{B}_1} W(v_k)\, dx < \frac{c_{iso}}{4C}. \tag{4.6}$$

In particular, $\|v_k\|^2_{H^{s/2}(\mathcal{B}_1)}$ is uniformly bounded in k and $\int_{\mathcal{B}_1} W(v_k) \to 0$. Hence, up to subsequences (that we do not relabel), we have that $v_k \rightharpoonup v_\infty$ weakly in $H^{s/2}(\mathcal{B}_1)$ and $v_k \to v$ almost everywhere. Then, by Fatou's Lemma, $\int_{\mathcal{B}_1} W(v_\infty) = 0$ and therefore $|v_\infty| = 1$ almost everywhere. Moreover

$$\fint_{\mathcal{B}_1} v_\infty \in [-1 + 1/C,\, 1 - 1/C]\,.$$

This implies that $v_\infty = \chi_E - \chi_{E^c}$ for some set $E \subset \mathcal{B}_1$ with $\frac{1}{|\mathcal{B}_1|}\min\{|E|, |\mathcal{B}_1 \setminus E|\} \geq \frac{1}{2C}$. By the lower-semicontinuity of the Sobolev seminorm, the fractional isoperimetric inequality (4.4) and

(4.6) we get

$$\frac{c_{iso}}{2C} \le \mathrm{Per}_s|_{\mathcal{B}_1}(E) \le \frac{1}{4}\liminf_{k\to\infty} \iint_{\mathcal{B}_1\times\mathcal{B}_1} \frac{|v_k(x)-v_k(y)|^2}{|x-y|^{n+s}}\,dxdy \le \frac{c_{iso}}{4C},$$

contradiction.

Going back to (4.5), we have therefore proved that there exists $\varepsilon_0 = \varepsilon_0(n,s) > 0$ such that, for every $\varepsilon \le \varepsilon_0 r$ and every u as in the statement

$$\mathcal{E}^\varepsilon|_{B_r(q)}(u) \ge \frac{c_{iso}}{4C^2} r^{n-s}\,.$$

Since the constant $c_{iso} = c_{iso}(n,s)$ for the fractional isoperimetric inequality satisfies that $c_{iso} \ge \frac{c'(n,s_0)}{1-s}$ for some $c' = c'(n,s_0) > 0$, we conclude the desired result. □

We can now give the proof of the lower bound.

Proof of Theorem 4.1.2 (part 1). The lower bound in (4.2) follows, in a simple manner, from the lemmas above: Given $\mathfrak{p} \in \mathbb{N}$, by Lemma 4.1.5 find $N \ge C_1\mathfrak{p}$ disjoint balls $\{B_r(q_i)\}_{i=1}^N$ in M with radius $r = C_0\,\mathfrak{p}^{-1/n}$. Moreover, up to taking C_1 bigger and C_0 smaller, we can also assume that $r < R_0$, where R_0 is such that $\mathrm{FA}_1(M,g,q,2R_0,\varphi)$ holds for all $q \in M$ (see Remark 3.4.3). Given any $A \in \mathcal{F}_\mathfrak{p}$, by Lemma 4.1.6 there exists $u \in A$ such that

$$\int_{B_r(q_i)} u = 0 \quad \text{for all} \quad i = 1,\dots,N.$$

Then, by Lemma 4.1.7, for $\varepsilon \le \varepsilon_0 r$ we have

$$\mathcal{E}^\varepsilon|_{B_r(q_i)}(u) \ge \frac{c_0}{1-s} r^{n-s} \quad \text{for all} \quad i = 1,\dots N,$$

which by the choice of r implies

$$\mathcal{E}^\varepsilon_M(u) \ge \sum_{i=1}^N \mathcal{E}^\varepsilon|_{B_r(q_i)}(u) \ge N\frac{c_0}{1-s} r^{n-s} \ge \frac{C^{-1}}{1-s}\mathfrak{p}^{s/n}\,,$$

for some constant C that depends only on M and s_0. Since we have found such a $u \in A$ given any $A \in \mathcal{F}_p$, we deduce the desired lower bound. □

4.1.3 Upper bound and proof of Theorem 1.2.12

While the lower bound required finding a function with high energy inside every admissible set $A \in \mathcal{F}_\mathfrak{p}$, the upper bound requires finding a single admissible set A such that all its elements have "low" energy. We will explicitly construct a continuous odd map $f : \mathbb{S}^\mathfrak{p} \to H^{s/2}(M)\setminus\{0\}$ so that all the elements in $A = f(\mathbb{S}^\mathfrak{p})$ have controlled energy. These functions will be of the form $\chi_E - \chi_{E^c}$, for some set $E \subset M$, and our task is then to bound the fractional perimeter of these sets.

The intuition behind the construction is as follows. Imagine embedding M into some Euclidean space $\mathbb{R}^m$. Suppose for the sake of exposition that we are in the limiting case $s = 1$, meaning that we have to find a family of hypersurfaces with a small total perimeter. A natural way to do so would be to assign a hypersurface $\Sigma_a \subset M$ to each $a = (a_0,\dots,a_p) \in \mathbb{S}^\mathfrak{p}$ oddly and continuously as follows. Consider the polynomial $P_a(z) = a_0 + a_1 z + \dots + a_\mathfrak{p} z^\mathfrak{p}$, its $\mathfrak{p}$ real roots are in continuous correspondence with its $(\mathfrak{p}+1)$ coefficients, and they remain the same changing a with $-a$. Let $\alpha_1, \dots, \alpha_\ell$ be a list of the real roots (at most $\mathfrak{p}$) of P_a. We then define Σ_a as the intersection of $M \subset \mathbb{R}^m$ with the at most $\mathfrak{p}$ parallel hyperplanes $\{x_m = \alpha_1\}, \dots, \{x_m = \alpha_\ell\}$. However, this construction alone is not sufficient to obtain the desired upper bound on the energy. Consider, for example, $M = \mathbb{S}^n \subset \mathbb{R}^{n+1}$, so that the desired energy bound would be of the form $\mathcal{H}^{n-1}(\Sigma_a) \le C\mathfrak{p}^{1/n}$. Using the procedure described above, the intersection of $\mathfrak{p}$ parallel hyperplanes with $\mathbb{S}^n$ can have an area of the order $\mathcal{H}^{n-1}(\Sigma_a) \sim \mathfrak{p}\mathcal{H}^{n-1}(\mathbb{S}^{n-1})$ if the $\mathfrak{p}$ hyperplanes are all very close to the equator $\mathbb{S}^{n-1} \subset \mathbb{S}^n$. Hence, we cannot expect a growth like $\mathfrak{p}^{1/n}$ without modifying the argument.

To fix this construction, essentially one first divides M in $\mathfrak{p}$ patches $Q_1, Q_2, \dots, Q_\mathfrak{p}$ of comparable volume $\mathrm{Vol}(Q_i) \sim \mathrm{Vol}(M)/\mathfrak{p}$, and then defines Σ_a as the full boundary of this partition $\bigcup_i \partial Q_i$, plus the intersection of each of the hyperplanes $\{x_m = \alpha_1\}, \dots, \{x_m = \alpha_\ell\}$ just with one single patch Q_j from the ones they intersect, and not with the whole M. This process will be continuous for the fractional perimeter and will give the desired upper bound.

The proof of the upper bound in (4.2) goes as follows.

Proof of Theorem 4.1.2 (part 2). By Lemma 4.1.5 (with $\mathfrak{p}$ replaced by k), for every $k \ge 1$ there exist $N \le C_2 k$ disjoint balls $B_r(q_1), \dots, B_r(q_N)$ of radius $r = C_0 k^{-1/n}$ and with $\bigcup_{i \le N} B_{3r}(q_i) = M$. Moreover, recalling the proof of Lemma 4.1.5, the bounds (4.3) hold for such an r.

Now, given $(a_0, a_1, \dots, a_\mathfrak{p}) \in \mathbb{S}^\mathfrak{p}$ consider the polynomial $P_a(z) = a_0 + a_1 z + \dots + a_p z^\mathfrak{p}$ and name $\{\alpha_1, \dots, \alpha_\ell\}$ its real roots in increasing order, so that $\ell \le \mathfrak{p}$. In $\mathbb{R}^n$ consider the set

$$E := \bigcup_{i=1}^{N} \mathcal{B}_{3r}(3r(2i+1), 0, \dots, 0)\,;$$

these are N aligned balls, tangent to each other, with centers on the x_1-axis. Now we split the set E into two disjoint subsets $E = E^+ \cup E^-$. Given the real roots $\{\alpha_1, \dots, \alpha_\ell\}$, assign the set $E \cap \{x_1 \le \alpha_1\}$ to E^+ if $P_a(z) \ge 0$ for all $z \in (-\infty, \alpha_1]$, and else assign it to E^- if $P_a(z) \le 0$ for all $z \in (-\infty, \alpha_1]$. Then, analogously assign $E \cap \{\alpha_1 < x_1 \le \alpha_2\}$ to E^+ if $P_a(z) \ge 0$ for all $z \in (\alpha_1, \alpha_2]$, and assign it to E^- if $P_a(z) < 0$ for $z \in (\alpha_1, \alpha_2]$. Repeat this procedure until E is divided into the two subsets E^+ and E^-. Note that there are at most $\mathfrak{p}$ transitions[2] between E^+ and E^-, and thus E^+ has perimeter at most $|\partial E^+| \le N|\partial \widetilde{B}_{3r}| + (6r)^{n-1}\mathfrak{p}$. Now, basically we want to do the same on the balls $\{B_{3r}(q_i)\}_{i=1}^N$ on M identifying $B_{3r}(q_i)$ with $\mathcal{B}_{3r}(3r(2i+1), 0, \dots, 0)$ via the exponential map, that we consider as a map

$$\exp_{q_i} : \mathcal{B}_{3r}(3r(2i+1), 0, \dots, 0) \to B_{3r}(q_i)\,.$$

In order to do so, we first have to make the covering $\{B_{3r}(q_i)\}_{i=1}^N$ of M disjoint. For this purpose, for all $1 \le i \le N$ we consider

$$Q_i := B_{3r}(q_i) \setminus \bigcup_{j \le i-1} B_{3r}(q_j)\,,$$

and note that $\{Q_i\}_{i=1}^N$ is a disjoint partition of M with $Q_i \subset B_{3r}(q_i)$. Let $u_a : M \to \{+1, -1\}$ be defined as

$$u_a(q) := \begin{cases} +1 & \text{if } q \in Q_i \text{ and } (\exp_{q_i})^{-1}(q) \in E^+, \\ -1 & \text{if } q \in Q_i \text{ and } (\exp_{q_i})^{-1}(q) \in E^-. \end{cases}$$

[2] This corresponds to the case when $P_a(z)$ has $\mathfrak{p}$ distinct real roots in the interval $(0, 6rN)$.

Set $\Sigma_a := \partial\{u_a = 1\}$. By interpolation (use for example Proposition A.1.1 applied to a covering of M with small enough balls) there exists $C = C(M, s_0)$ so that

$$
\begin{aligned}
\mathcal{E}^{\mathrm{Sob}}_M(u_a) &= \frac{1}{4}[u_a]^2_{H^{s/2}(M)} = \mathrm{Per}_s(\{u_a = 1\}) \\
&\le \frac{C}{1-s}\mathcal{H}^{n-1}(\Sigma_a)^s \ \mathrm{Vol}(\{u_a = 1\}) \le \frac{C}{1-s}\mathcal{H}^{n-1}(\Sigma_a)^s.
\end{aligned} \tag{4.7}
$$

Moreover, by (4.3) we have

$$
\begin{aligned}
\mathcal{H}^{n-1}(\Sigma_a) &\le C(N|\partial\mathcal{B}_{3r}| + r^{n-1}\mathfrak{p}) \\
&\le C(kr^{n-1} + \mathfrak{p}r^{n-1}) = C(k^{1/n} + \mathfrak{p}k^{-1+1/n})\,,
\end{aligned}
$$

for all $k \ge 1$, with C that depends only on M. Clearly, the optimal value of the right-hand side is attained for $k = \mathfrak{p}$ and gives $\mathcal{H}^{n-1}(\Sigma_a) \le C\mathfrak{p}^{1/n}$. This, together with (4.7), implies

$$
\mathcal{E}^{\mathrm{Sob}}_M(u_a) \le \frac{C}{1-s}\mathfrak{p}^{s/n}\,, \tag{4.8}
$$

for all $a \in \mathbb{S}^{\mathfrak{p}}$, with C depending only on M and s_0.

From the definition of $E^{\pm}$, it is clear that $u_{-a}(x) = -u_a(x)$. Hence, to conclude that $a \mapsto u_a$ is an element of $\mathcal{F}_{\mathfrak{p}}$ we are left to show that it is continuous for the strong $H^{s/2}(M)$ topology. This easily follows by interpolation. Indeed, for every $a, b \in \mathbb{S}^{\mathfrak{p}}$, by Proposition A.1.1 (again applied to a covering of M with small balls) we have

$$
\begin{aligned}
[u_a - u_b]^2_{H^{s/2}(M)} &\le \frac{C}{1-s}[u_a - u_b]^s_{BV(M)}\|u_a - u_b\|^{1-s}_{L^1(M)} \\
&\le \frac{C}{1-s}\Big(\mathcal{H}^{n-1}(\Sigma_a) + \mathcal{H}^{n-1}(\Sigma_b)\Big)^s\|u_a - u_b\|^{1-s}_{L^1(M)} \le \frac{C\mathfrak{p}^{s/n}}{1-s}\|u_a - u_b\|^{1-s}_{L^1(M)},
\end{aligned}
$$

where we have used that $\sup_{a\in\mathbb{S}^{\mathfrak{p}}}\mathcal{H}^{n-1}(\Sigma_a) \le C\mathfrak{p}^{1/n}$. From here, continuity follows since $a \mapsto u_a$ is continuous in $L^1(M)$ by construction.

Together with the bound (4.8), which holds for all $a \in \mathbb{S}^{\mathfrak{p}}$, this concludes the proof. □

One should compare the simplicity of this construction, with the one in [GG18] for the classical Allen-Cahn equation and the classical perimeter; in that case, to define the $\mathfrak{p}$-sweepouts with the correct interface, one has to consider functions that are compositions of:

(*i*) The solution to the 1-dimensional Allen-Cahn equation with parameter $\varepsilon > 0$.

(*ii*) A "modified" distance function, measuring the distance to hyperplanes $\{x_1 \le c\}$ (which play a similar role to those in our construction), but also accounting for the complex parts of the roots of the polynomials in order to smooth out the cancellations of the leaves.

Using this composition of functions is necessary in the classical Allen-Cahn case to regularize the construction, as characteristic functions of sets of finite perimeter do not belong to $H^1(M)$, while they do belong to $H^{s/2}(M)$ for $s < 1$.

Remark 4.1.8. *Notice that for every fixed $\mathfrak{p}$, both the proofs of the lower bound and the upper bound in (4.2) rely on the fact that there is the same "critical scale" $r = C\mathfrak{p}^{-1/n}$ in the construction. This is given by dividing M in $N \sim \mathfrak{p}$ disjoint patches of volume of order $\sim 1/\mathfrak{p}$. The lower bound shows that, given any $A \in \mathcal{F}_{\mathfrak{p}}$, there is one element of A that has zero average - see Lemma 4.1.6*

- *in each of these patches, and in particular this element has energy uniformly bounded from below of order $\mathfrak{p}^{s/n}$. On the other hand, the upper bound shows that this configuration, i.e. making the transitions take place in $N \sim \mathfrak{p}$ disjoint patches that cover M, is also (of the order of) the best that one can achieve.*

As a consequence of the above results, we deduce our complete existence result.

Proof of Theorem 1.2.12. The statement follows from combining the existence result of Theorem 4.1.4 and the bounds on the min-max values given by Theorem 4.1.2. □

4.2 Estimates for Allen-Cahn solutions with bounded Morse index

4.2.1 Finite Morse index and almost stability

For critical points of local functionals, it is well known that having Morse index bounded by m implies stability in one out of every $m+1$ disjoint open sets. In the nonlocal case this is not the case anymore, but in one of the sets we will be able to obtain a weaker, quantitative lower bound on the second derivative which we will refer to as *almost stability*.

Definition 4.2.1 (Almost stability)**.** *Let $\Omega \subset M$ open and $u : M \to (-1,1)$ be a critical point of $\mathcal{E}_\Omega$. Given $\Lambda \geq 0$, we say that u is Λ-almost stable in Ω if*

$$\mathcal{E}''_\Omega(u)[\xi,\xi] \geq -\Lambda \|\xi\|^2_{L^1(\Omega)}, \quad \forall\, \xi \in C^1_c(\Omega).$$

Lemma 4.2.2 (Finite Morse index and almost stability)**.** *Let $u : M \to (-1,1)$ be a solution of the Allen-Cahn equation $(-\Delta)^{s/2}u + \varepsilon^{-s}W'(u) = 0$ on M with Morse index at most m (see Definition 2.2.9, with $\Omega = M$). Consider a collection $\mathcal{U}_1,\dots,\mathcal{U}_{m+1} \subset M$ of $(m+1)$ disjoint open sets at positive distance from eachother, and set*

$$\Lambda := m \max_{i\neq j} \sup_{\mathcal{U}_i\times\mathcal{U}_j} \mathcal{K}_s(p,q) < +\infty.$$

Then, there is (at least) one set $\mathcal{U}_k$ among $\mathcal{U}_1,\dots,\mathcal{U}_{m+1}$ such that u is Λ-almost stable in $\mathcal{U}_k$, that is

$$\mathcal{E}''(u)[\xi,\xi] \geq -\Lambda \|\xi\|^2_{L^1(\mathcal{U}_k)}, \quad \forall\, \xi \in C^1_c(\mathcal{U}_k).$$

Proof. We prove the Lemma just for $m = 1$ for the sake of clarity, the proof goes on exactly the same for general m. Let $\xi_1 \in C^1_c(\mathcal{U}_1)$ and $\xi_2 \in C^1_c(\mathcal{U}_2)$. Testing the second variation of the Allen-Cahn energy, explicitly in (2.8), with linear combinations of ξ_1 and ξ_2 gives

$$\mathcal{E}''(u)[a\xi_1 + b\xi_2] = a^2\mathcal{E}''(u)[\xi_1,\xi_1] + b^2\mathcal{E}''(u)[\xi_2,\xi_2] - 2ab \iint_{\mathcal{U}_1\times\mathcal{U}_2} \xi_1(p)\xi_2(q)\mathcal{K}_s(p,q)\,.$$

Since $\mathcal{K}_s(p,q) \leq \Lambda$ for all $(p,q) \in \mathcal{U}_1 \times \mathcal{U}_2$, the interaction term can be bounded as

$$\begin{aligned} -2ab \iint_{\mathcal{U}_1\times\mathcal{U}_2} \xi_1(p)\xi_2(q)\mathcal{K}_s(p,q) &\leq 2ab\Lambda \|\xi_1\|_{L^1(\mathcal{U}_1)}\|\xi_2\|_{L^1(\mathcal{U}_2)} \\ &\leq a^2\Lambda\|\xi_1\|^2_{L^1(\mathcal{U}_1)} + b^2\Lambda\|\xi_2\|^2_{L^1(\mathcal{U}_2)}. \end{aligned}$$

Hence

$$\mathcal{E}''(u)[a\xi_1+b\xi_2] \le a^2\Big(\underbrace{\mathcal{E}''(u)[\xi_1,\xi_1]+\Lambda\|\xi_1\|^2_{L^1(\mathcal{U}_1)}}_{=:F_1(\xi_1)}\Big)+b^2\Big(\underbrace{\mathcal{E}''(u)[\xi_2,\xi_2]+\Lambda\|\xi_2\|^2_{L^1(\mathcal{U}_2)}}_{=:F_2(\xi_2)}\Big). \qquad (4.9)$$

We want to show that either $F_1(\xi_1)\ge 0$ for all $\xi_1\in C^1_c(\mathcal{U}_1)$ or $F_2(\xi_2)\ge 0$ for all $\xi_2\in C^1_c(\mathcal{U}_2)$. Suppose neither of these two holds; then there would exist $\xi_1\in C^1_c(\mathcal{U}_1)$, $\xi_2\in C^1_c(\mathcal{U}_2)$ such that $F_1(\xi_1)<0$ and $F_2(\xi_2)<0$. However, this would imply that (4.9) is negative for all $(a,b)\neq(0,0)$, contradicting the Morse index of u being at most one. $\square$

4.2.2 Control of the potential part by the Sobolev part

By Theorem 3.2.4, notice that u is a solution to the Allen-Cahn equation

$$(-\Delta)^{s/2}u+\varepsilon^{-s}W'(u)=0 \quad \text{on } M\,,$$

if and only if the Caffarelli-Silvestre extension U, i.e. the unique solution to (3.11), solves

$$\begin{cases}\widetilde{\operatorname{div}}(z^{1-s}\nabla U)=0\,, & \text{in } \widetilde{M}\\ \lim_{z\to0^+} z^{1-s}U_z(\,\cdot\,,z)=\beta_s^{-1}\varepsilon^{-s}W'(u(\cdot))\,, & \text{on } M.\end{cases} \qquad (4.10)$$

For the next results, recall the notation for balls in the extended manifold (3.19). We begin with an auxiliary lemma.

Lemma 4.2.3. *Let $s\in(0,1)$, (M,g) satisfy the flatness assumption $\mathrm{FA}_2(M,g,2,p,\varphi)$, and let $\eta\in C^2_c(\mathcal{B}_{3/4}(0))$ be a cutoff function with $\eta=1$ in $\mathcal{B}_{1/2}(0)$. Define $\eta_0=\varphi\circ\eta$ and let $\widetilde{\eta}$ solve*

$$\begin{cases}\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}\widetilde{\eta})=0 & \text{in } \widetilde{B}^+_1(p,0)\,,\\ \widetilde{\eta}=0 & \text{on } \partial^+\widetilde{B}_1(p,0)\,,\\ \widetilde{\eta}=\eta_0 & \text{on } B_1(p)\times\{0\}\,.\end{cases}$$

Then, for all $q\in B_{3/4}(p)$ there holds that

$$\beta_s\left|\big(-z^{1-s}\partial_z\widetilde{\eta}\big)(q,0^+)\right|\le C \quad \text{and} \quad \beta_s\int_{\widetilde{B}^+_1(p,0)} z^{1-s}|\widetilde{\nabla}\widetilde{\eta}|^2 dV\,dz\le C\,, \qquad (4.11)$$

for some dimensional $C=C(n)>0$.

Proof. Let $U_0\in\widetilde{H}^1(M\times(0,\infty))$—see Definition 3.2.3—be the unique Caffarelli-Silvestre extension of η_0 (considered as defined on M extended by zero outside $B_1(p)$), in the sense of Theorem 3.2.4. Since U_0 and $\widetilde{\eta}$ are two different solutions of $\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}U)=0$ with the same trace on $B_1(p)$, by Lemma 3.2.10 (rescaled) we have that $\beta_s\, z^{1-s}|\widetilde{\nabla}(U_0-\widetilde{\eta})|\le C$ in $\widetilde{B}^+_{3/4}(p)$ for some dimensional C. Hence in $B_{3/4}(p)$ there holds

$$\beta_s\left|\big(-z^{1-s}\partial_z\widetilde{\eta}\big)(\,\cdot\,,0^+)\right|\le\beta_s\left|\big(-z^{1-s}\partial_z U_0\big)(\,\cdot\,,0^+)\right|+C=\left|(-\Delta)^{s/2}\eta_0\right|+C,$$

where we have used Theorem 3.2.4 in the last equality. Now, a dimensional bound for the $\left|(-\Delta)^{s/2}\eta_0\right|$ follows, for $q\in B_{3/4}(p)$, by writing

$$(-\Delta)^{s/2}\eta_0(q)=\int_{B_1(p)}(\eta_0(q)-\eta_0(r))\mathcal{K}_s(q,r)\,dV_r+\int_{M\setminus B_1(p)}(\eta_0(q)-\eta_0(r))\mathcal{K}_s(q,r)\,dV_r$$

and using Lemma 3.4.13 and (3.41) of Proposition 3.4.6 respectively to bound these two integrals. This concludes the proof of the first estimate in (4.11). The second estimate follows from the first one just by integrating by parts and using $\mathrm{FA}_2(M,g,2,p,\varphi)$. ☐

Lemma 4.2.4. *Let $R \in (0,1]$, and assume M satisfies flatness assumption $\mathrm{FA}_2(M,g,2R,p,\varphi)$. Let $\varepsilon > 0$, $s \in (0,1)$ and $u : M \to (-1,1)$ be a solution of the Allen-Cahn equation*

$$(-\Delta)^{s/2}u + \varepsilon^{-s}W'(u) = 0 \tag{4.12}$$

in $B_R(p)$, that is Λ-almost stable in $B_R(p)$, in the sense of Definition 4.2.1, for $\Lambda \le \Lambda_1/R^{n+s}$. Then, there exists a positive constant $C = C(n,\Lambda_1)$ such that, for all $a \in (0,1]$

$$R^{s-n}\mathcal{E}^{\mathrm{Pot}}_{B_{R/2}(p)}(u) \le C\left(\frac{\beta_s}{a}R^{s-n}\int_{\widetilde{B}^+_R(p,0)} z^{1-s}|\widetilde{\nabla}U|^2 + a + (\varepsilon/R)^s R^{s-n}\mathcal{E}^{\mathrm{Pot}}_{B_R(p)}(u)\right).$$

In particular, since $|u| \le 1$, for $a = 1$ we have

$$R^{s-n}\mathcal{E}^{\mathrm{Pot}}_{B_{R/2}(p)}(u) \le C\left(\beta_s R^{s-n}\int_{\widetilde{B}^+_R(p,0)} z^{1-s}|\widetilde{\nabla}U|^2 + 1\right).$$

Proof. Notice that, as $\varepsilon > 0$ is arbitrary, the statement is scaling invariant. Indeed, if $u : B_R(p) \to (-1,1)$ is a Λ-almost stable solution of (4.12) with parameter ε, then, on the rescaled manifold $(M, R^{-2}g)$, u is an $(R^{n+s}\Lambda)$-almost stable solution of (4.12) in $B_1(p)$ with parameter ε replaced by ε/R, and Λ replaced by $R^{n+s}\Lambda \le \Lambda_1$ since $R \le 1$ by hypothesis. Hence, we can assume $R = 1$.

In what follows, C denotes a general constant that depends only on n. To compare the potential energies on M with the Sobolev energies in the extended manifold, we need a well-chosen cutoff function $\widetilde{\eta}$ defined on the extended manifold $\widetilde{M}$. To this aim, let $\widetilde{\eta}$ solve

$$\begin{cases} \widetilde{\mathrm{div}}(z^{1-s}\widetilde{\nabla}\widetilde{\eta}) = 0 & \text{in } \widetilde{B}^+_1(p,0)\,, \\ \widetilde{\eta} = 0 & \text{on } \partial^+\widetilde{B}_1(p,0)\,, \\ \widetilde{\eta} = \eta_0 & \text{on } B_1(p)\times\{0\}\,, \end{cases}$$

where $\eta_0 = \varphi \circ \eta \in C^2_c(B_1(p))$ and η is a fixed cutoff function with $\eta = 1$ in $\mathcal{B}_{2/3}(0)$ and $\eta = 0$ outside $\mathcal{B}_{3/4}(0)$. First, since $\mathrm{FA}_2(M,g,2,p,\varphi)$ holds, by the estimates of Lemma 4.2.3 we have for all $q \in B_1(p)$

$$\beta_s\left|\big(-z^{1-s}\partial_z\widetilde{\eta}\big)(q,0^+)\right| \le C \quad \text{and} \quad \beta_s\int_{\widetilde{B}^+_1(p,0)} z^{1-s}|\widetilde{\nabla}\widetilde{\eta}|^2 \le C\,,$$

for some dimensional constant $C = C(n)$. Note also that $|\widetilde{\eta}| \le 1$. Then

$$\begin{aligned}\mathcal{E}^{\mathrm{Pot}}_{B_{1/2}(p)} &= \frac{\varepsilon^{-s}}{4}\int_{B_{1/2}(p)}(1-u^2)^2 \le \frac{\varepsilon^{-s}}{4}\int_{B_1(p)}(1-u^2)^2\eta_0^2 \\ &= \frac{1}{4}\left(\varepsilon^{-s}\int_{B_1(p)} u^2(1-u^2)^2\eta_0^2 + \varepsilon^{-s}\int_{B_1(p)}(1-u^2)^3\eta_0^2\right) =: \frac{1}{4}(I+J)\,.\end{aligned}$$

On the one hand by (4.12) and the divergence theorem

$$
\begin{aligned}
I &\le \int_{B_1(p)} \varepsilon^{-s} u^2(1-u^2)\eta_0^2 = \int_{B_1(p)} u\eta_0^2(-\Delta)^{s/2}u \\
&= \beta_s \int_{B_1(p)} u\eta_0^2\big(-z^{1-s}U_z\big)(\,\cdot\,,0^+) \\
&= \beta_s \int_{\widetilde{B}_1^+(p,0)} \widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}U\,U\widetilde{\eta}^2) \\
&= \beta_s \int_{\widetilde{B}_1^+(p,0)} z^{1-s}(|\widetilde{\nabla}U|^2\widetilde{\eta}^2 + 2\widetilde{\eta}U\widetilde{\nabla}\widetilde{\eta}\cdot\widetilde{\nabla}U) \\
&\le \beta_s \int_{\widetilde{B}_1^+(p,0)} z^{1-s}\Big((1+\tfrac{1}{a})|\widetilde{\nabla}U|^2\widetilde{\eta}^2 + aU^2|\widetilde{\nabla}\widetilde{\eta}|^2\Big) \\
&\le \frac{C\beta_s}{a}\int_{\widetilde{B}_1^+(p,0)} z^{1-s}|\widetilde{\nabla}U|^2 + Ca\,,
\end{aligned}
$$

where β_s is the constant defined in (3.13) (see also Proposition 3.3.1) and we have used (4.10) and Young's inequality in the second to last line.

On the other hand, since $W''(u) = 3u^2-1$ and u is Λ-almost stable

$$
\begin{aligned}
J &= \int_{B_1(p)} \varepsilon^{-s}(1-3u^2+2u^2)\big((1-u^2)\eta_0\big)^2 \\
&\le \mathcal{E}^{\mathrm{Sob}}_{B_1(p)}((1-u^2)\eta_0) + \Lambda\bigg(\int_{B_1(p)} \big|(1-u^2)\eta_0\big|\bigg)^2 + 2I \\
&\le \underbrace{\frac{\beta_s}{4}\int_{\widetilde{B}_1^+(p,0)} z^{1-s}\big|\widetilde{\nabla}\big((1-U^2)\widetilde{\eta}\big)\big|^2}_{=:J_1} + C(\Lambda_1,n)\underbrace{\int_{B_1(p)}(1-u^2)^2\eta_0^2}_{=:J_2} + 2I.
\end{aligned}
$$

Here we have bounded $\mathcal{E}^{\mathrm{Sob}}((1-u^2)\eta_0)$ by J_1 since the former is the infimum of $\frac{\beta_s}{4}\int z^{1-s}|\widetilde{\nabla}V|^2$ over all the extensions V of $(1-u^2)\eta$, and $(1-U^2)\widetilde{\eta}$ is one such extension. Now, since $\widetilde{\eta}\equiv 0$ on $\partial^+\widetilde{B}_1(p,0)$ and $\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}\widetilde{\eta}) = 0$ we have

$$
\begin{aligned}
J_1 &= \frac{\beta_s}{4}\int_{\widetilde{B}_1^+(p,0)} z^{1-s}\Big(4U^2|\widetilde{\nabla}U|^2\widetilde{\eta}^2 + \frac{1}{2}\widetilde{\nabla}\big((1-U^2)^2\big)\cdot\widetilde{\nabla}(\widetilde{\eta}^2) + (1-U^2)^2|\widetilde{\nabla}\widetilde{\eta}|^2\Big) \\
&= \frac{\beta_s}{4}\bigg(4\int_{\widetilde{B}_1^+(p,0)} z^{1-s}U^2|\widetilde{\nabla}U|^2\widetilde{\eta}^2 - \int_{B_1(p)} z^{1-s}(1-u^2)^2\eta_0\partial_z\widetilde{\eta} + \\
&\qquad - \int_{\widetilde{B}_1^+(p,0)}(1-U^2)^2\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\eta}\widetilde{\nabla}\widetilde{\eta}) + \int_{\widetilde{B}_1^+(p,0)}(1-U^2)^2|\widetilde{\nabla}\widetilde{\eta}|^2\bigg) \\
&= C\beta_s\int_{\widetilde{B}_1^+(p,0)} z^{1-s}U^2|\widetilde{\nabla}U|^2\widetilde{\eta}^2 + \int_{B_1(p)}(1-u^2)^2\eta_0\big(-\beta_s z^{1-s}\partial_z\widetilde{\eta}\big)(\,\cdot\,,0^+) \\
&\le C\beta_s\int_{\widetilde{B}_1^+(p,0)} z^{1-s}|\widetilde{\nabla}U|^2 + C\varepsilon^s\mathcal{E}^{\mathrm{Pot}}_{B_1(p)}\,,
\end{aligned}
$$

and also

$$
J_2 = \int_{B_1(p)}(1-u^2)^2\eta^2 \le C\varepsilon^s\mathcal{E}^{\mathrm{Pot}}_{B_1(p)}.
$$

Thus

$$J = J_1 + CJ_2 + 2I \le C \left(\frac{\beta_s}{a} \int_{\widetilde{B}_1^+(p,0)} z^{1-s} |\widetilde{\nabla} U|^2 + a + \varepsilon^s \mathcal{E}^{\mathrm{Pot}}_{B_1(p)} \right),$$

for some $C = C(n, \Lambda_1)$. Combining the above estimates yields the result. □

4.2.3 The BV estimate for finite index solutions

The aim of this subsection is to prove Theorem 1.2.13.

Proposition 4.2.5 (Almost stability ⇒ BV)**.** *Let $p \in M$, $s_0 \in (0,1)$, $s \in (s_0, 1)$ and assume that M satisfies the flatness assumption $\mathrm{FA}_2(M, g, 1, p, \varphi)$. Let $u : M \to (-1,1)$ be a solution of $(-\Delta)^{s/2} u + \varepsilon^{-s} W'(u) = 0$ which is Λ-almost stable in $B_1(p) \subset M$ (see Definition 4.2.1).*

Then, there exist constants Λ_0 and C, depending only on n and s_0, such that: if $\Lambda \le \Lambda_0$ then

$$\int_{B_{1/4}(p)} |\nabla u| \, dV \le \frac{C}{1-s}.$$

Remark 4.2.6. *The blow up rate $(1-s)^{-1}$ as $s \nearrow 1$ is not expected to be sharp, but $(1-s)^{-1/2}$ is; see Proposition 2.2.14.*

To prove Proposition 4.2.5 we will need two lemmas.

Lemma 4.2.7. *Let $p \in M$ and assume M satisfies the local flatness assumption $\mathrm{FA}_2(M, g, 1, p, \varphi)$. Let $X \in \mathrm{Vect}_c(B_{3/4}(p))$, and denote by $X^* : \mathcal{B}_1(0) \to \mathbb{R}^n$ the pullback $X^* := \varphi^* X$ of X via the chart φ^{-1}. Let $\varepsilon > 0$, $s \in (0,1)$ and $u \in H^{s/2}(M)$ be with $|u| \le 1$. Then*

$$\mathcal{E}''(u)[\nabla_X u, \nabla_X u] \le C \left(\beta_s \int_{\widetilde{B}^+_{3/4}(p,0)} |\widetilde{\nabla} U|^2 z^{1-s} dV dz + \int_{B_{3/4}(p)} \varepsilon^{-s} W(u) dV \right),$$

where $C = C\big(n, \|X^*\|_{C^2(\mathcal{B}_1)}\big)$ and U is the extension of u.

Proof. Denote by ψ_t^X the flow of X at time t and $u_t := u \circ \psi_t^X$. By Lemma 3.4.17 the map $t \mapsto \mathcal{E}(u_t)$ is smooth in a neighborhood of the origin. Hence

$$\mathcal{E}''(u)[\nabla_X u, \nabla_X u] = \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{E}(u_t) = \lim_{t \to 0} \frac{\mathcal{E}(u_t) + \mathcal{E}(u_{-t}) - 2\mathcal{E}(u)}{t^2}. \tag{4.13}$$

Let $\widetilde{X}$ be any smooth extension of X in $\widetilde{B}^+_{3/4}(p,0)$ with support compactly contained in $\widetilde{B}^+_{3/4}(p,0)$ (in particular, $\widetilde{X}$ vanishes in a neighborhood of $\partial^+ \widetilde{B}^+_{3/4}(p,0)$) and such that $\widetilde{X}^{n+1} \equiv 0$. This last condition implies that, if $\widetilde{\psi}^t$ is the flow of $\widetilde{X}$, $\widetilde{\psi}^t$ leaves invariant the z component in the extended manifold $\widetilde{M}$.

To bound the increment above we split the energy $\mathcal{E}$ in its Sobolev part and potential part. For the Sobolev part, by the minimality of the extension in the energy space

$$\mathcal{E}^{\mathrm{Sob}}(u_t) = \frac{\beta_s}{4} \int_{\widetilde{M}} z^{1-s} |\widetilde{\nabla} \widetilde{u}_t|^2 \, dV dz \le \frac{\beta_s}{4} \int_{\widetilde{M}} z^{1-s} |\widetilde{\nabla} U_t|^2 \, dV dz,$$

where β_s is the constant in Theorem 3.2.4. Here $\widetilde{u}_t$ is the extension of u_t and $U_t := U \circ \widetilde{\psi}^{-t}$. We emphasize that, with our current notation, U_t is not the extension of u_t, but instead the translation of U via $\widetilde{\psi}^t$ in the extended manifold $\widetilde{M}$. Denote

$$I(t) := \int_{\widetilde{M}} |\widetilde{\nabla} U_t|^2 z^{1-s} dV dz\,.$$

We then have

$$\begin{aligned}
\lim_{t\to 0} \frac{\mathcal{E}^{\mathrm{Sob}}(u_t) + \mathcal{E}^{\mathrm{Sob}}(u_{-t}) - 2\mathcal{E}^{\mathrm{Sob}}(u)}{t^2} &\leq \frac{\beta_s}{4}\left(\lim_{t\to 0}\frac{I(t)+I(-t)-2I(0)}{t^2}\right)\\
&= \frac{\beta_s}{4}\frac{d^2}{dt^2}\Big|_{t=0} I(t)\\
&= \frac{\beta_s}{4}\frac{d^2}{dt^2}\Big|_{t=0}\int_{\widetilde{B}^+} |\widetilde{\nabla} U_t|^2 z^{1-s} dV dz\,,
\end{aligned}$$

Now, since M satisfies local flatness assumption $\mathrm{FA}_2(M, g, 1, p, \varphi)$, setting $\widetilde{\varphi}(x, z) = (\varphi(x), z)$, $\widetilde{\Omega} := \widetilde{\varphi}^{-1}(\widetilde{B}^+_{3/4}(p, 0))$, $\phi_t := \widetilde{\varphi}^{-1} \circ \widetilde{\psi}^t \circ \widetilde{\varphi}$, and $\widetilde{U} := U \circ \widetilde{\varphi}$, we have

$$\begin{aligned}
I(t) &= \int_{\widetilde{B}^+_{3/4}(p,0)} |\widetilde{\nabla}(U\circ\widetilde{\psi}^{-t})|^2 z^{1-s} dV dz = \int_{\widetilde{\Omega}} \widetilde{g}^{ij}\partial_i(\widetilde{U}\circ\phi_{-t})\partial_j(\widetilde{U}\circ\phi_{-t}) z^{1-s}\sqrt{|g|}\,dxdz\\
&= \int_{\widetilde{\Omega}} \widetilde{g}^{ij}((\partial_k\widetilde{U})\circ\phi_{-t})\partial_i\phi^k_{-t}\,((\partial_l\widetilde{U})\circ\phi_{-t})\partial_j\phi^l_{-t} z^{1-s}\sqrt{|g|}\,dxdz\\
&= \int_{\phi_{-t}(\widetilde{\Omega})} (\partial_k U)(\partial_l U)\Big(\widetilde{g}^{ij}\partial_i\phi^k_{-t}\partial_j\phi^l_{-t}\sqrt{|g|}\Big)\circ\phi_t(\phi^{n+1}_t)^{1-s}\,d\phi^1_t\wedge\cdots\wedge d\phi^{n+1}_t\\
&= \int_{\widetilde{\Omega}} (\partial_k U)(\partial_l U) z^{1-s}\Big(\widetilde{g}^{ij}\,(\partial_i\phi^k_{-t}\partial_j\phi^l_{-t}\sqrt{|g|}\Big)\circ\phi_t\,|D\phi_t|\,dxdz
\end{aligned}$$

Hence

$$I''(0) = \int_{\widetilde{\Omega}} (\partial_k U)(\partial_l U) z^{1-s}\frac{\partial^2 F^{kl}}{\partial t^2}(0, x, z)\,dxdz\,,$$

where

$$F^{kl}(t, \cdot, \cdot) := \Big(\widetilde{g}^{ij}\partial_i\phi^k_{-t}\partial_j\phi^l_{-t}\sqrt{|\widetilde{g}|}\Big)\circ\phi_t\,|D\phi_t|.$$

Since $\phi : (0, \infty) \times \widetilde{\Omega} \to \mathbb{R}^{n+1}$ is the flow of $(X^*, 0)$, together with the flatness assumption, a direct computation shows that

$$\left\|\frac{\partial^2 F^{kl}}{\partial t^2}(0, \cdot)\right\|_{L^\infty(\widetilde{\Omega})} \leq C\big(n, \|X^*\|_{C^2(\mathcal{B}_1)}\big).$$

Thus

$$I''(0) \leq C\int_{\widetilde{\Omega}} \sum_{k=1}^{n+1} |\partial_k U|^2 z^{1-s}\,dxdz \leq C\int_{\widetilde{B}^+_{3/4}(p,0)} |\widetilde{\nabla} U|^2 z^{1-s}\,dV dz\,,$$

where $C = C\big(n, \|X^*\|_{C^2(\mathcal{B}_1)}\big)$ and we have used the flatness assumption to compare the Euclidean metric on $\mathbb{R}^{n+1}$ to the one on $\widetilde{M}$.

Similarly, for the potential part of the energy

$$\lim_{t\to 0}\frac{\mathcal{E}^{\mathrm{Pot}}(u_t)+\mathcal{E}^{\mathrm{Pot}}(u_{-t})-2\mathcal{E}^{\mathrm{Pot}}(u)}{t^2}=\frac{d^2}{dt^2}\bigg|_{t=0}\mathcal{E}^{\mathrm{Pot}}(u_t)\,.$$

Arguing as in the last part of the proof of Lemma 3.4.17 (the one regarding the potential part of the energy, with $\ell = 2$) we have

$$\frac{d^2}{dt^2}\bigg|_{t=0}\mathcal{E}^{\mathrm{Pot}}(u_t)\le C\mathcal{E}^{\mathrm{Pot}}_{B_{3/4}(p)}(u)=C\int_{B_{3/4}(p)}\varepsilon^{-s}W(u)\,dV\,,$$

where $C>0$ depends only on $\|X^*\|_{C^2(\mathcal{B}_1)}$ since by direct computation for the Jacobian

$$\left\|\frac{\partial^2}{\partial t^2}\left(\sqrt{|g|}|D\psi_t|\right)(0,\,\cdot\,)\right\|_{L^\infty}\le C\big(\|X^*\|_{C^2(\mathcal{B}_1)}\big)\,.$$

This, together with (4.13) and the bound for $I''(0)$, concludes the proof. □

Lemma 4.2.8. *Let $n\ge 2$, $p\in M$, $s_0\in(0,1)$, $s\in(s_0,1)$ and assume that M satisfies the flatness assumption $\mathrm{FA}_2(M,g,R,p,\varphi)$. Then, there exist Λ_0 and C_0, depending only on n and s_0, such that the following holds. Let $u:M\to(-1,1)$ be a solution of $(-\Delta)^{s/2}u+\varepsilon^{-s}W'(u)=0$ which is Λ-almost stable in $B_R(p)\subset M$ for $\Lambda\le\Lambda_0/R^{n+s}$ (see Definition 4.2.1). Then, for every $\delta>0$*

$$R^{1-n}\int_{B_{R/2}(p)}|\nabla u|dV\le\frac{C_0}{(1-s)\delta}+\delta R^{1-n}\int_{B_R(p)}|\nabla u|dV\,.$$

Proof. Since the statement is scaling invariant, as the constant C_0 does not depend on ε, we can assume $R=1$. See the beginning of the proof of Lemma 4.2.4 for details on the scaling.

We show that there exists a constant $C_0=C_0(n,s_0)>0$ such that, for any given $\delta>0$, there holds

$$\|\nabla u\|_{L^1(B_{1/2}(p))}\le\frac{C}{(1-s)\delta}+\delta\|\nabla u\|_{L^1(B_1(p))}\,.$$

In particular, this C does not depend on ε.

Let X be a vector field compactly supported in $B_{3/4}(p)$ to be chosen later, and let us denote $B:=B_1(p)$ during this proof. Let also $\nabla_X u:=\langle X,\nabla u\rangle$. Since the second variation (2.8) is continuous with respect to the $H^{s/2}(M)$ topology, by density we can test the almost stability inequality with any $\xi\in H^{s/2}(M)$. Testing the almost stability inequality with $\xi=|\nabla_X u|\in H^{s/2}(M)$ gives

$$0\le\mathcal{E}''_B(u)[|\nabla_X u|,|\nabla_X u|]+\Lambda\|\nabla_X u\|^2_{L^1(B)}.$$

On the other hand

$$\mathcal{E}''_B(u)[|\nabla_X u|,|\nabla_X u|]=\mathcal{E}''_B(u)[\nabla_X u,\nabla_X u]-4\int_B\int_B(\nabla_X u)_+(p)(\nabla_X u)_-(q)\mathcal{K}_s(p,q)dV_p dV_q,$$

thus we find that

$$4\int_B\int_B(\nabla_X u)_+(p)(\nabla_X u)_-(q)\mathcal{K}_s(p,q)dV_p dV_q\le\mathcal{E}''_B(u)[\nabla_X u,\nabla_X u]+\Lambda\|\nabla_X u\|^2_{L^1(B)}\,.$$

Moreover, by Lemma 4.2.7 and Lemma 4.2.4 respectively, we have

$$\begin{aligned}\mathcal{E}_B''(u)[\nabla_X u, \nabla_X u] &\le C\left(\beta_s \int_{\widetilde{B}^+_{3/4}(p,0)} |\widetilde{\nabla} U|^2 z^{1-s} dV dz + \mathcal{E}^{\mathrm{Pot}}_{B_{3/4}(p)}(u)\right)\\ &\le C\left(\beta_s \int_{\widetilde{B}^+_{1}(p,0)} |\widetilde{\nabla} U|^2 z^{1-s} dV dz + 1\right),\end{aligned}$$

for some $C = C(n, \|\xi\|_{C^2(\mathcal{B}_1(0))}, \Lambda_0)$, where $\xi^i = X^i \circ \varphi$ and Λ_0 will be chosen shortly depending only on n and s_0. Hence

$$4\int_B\int_B (\nabla_X u)_+(p)(\nabla_X u)_-(q)\mathcal{K}_s(p,q) dV_p dV_q \le C\left(\beta_s \int_{\widetilde{B}^+} |\widetilde{\nabla} U|^2 z^{1-s} dV dz + 1\right) + \Lambda \|\nabla_X u\|^2_{L^1(B)}.$$

Now, since by Lemma 3.4.13 there holds $\mathcal{K}_s(p,q) \ge c_0 > 0$ for all $(p,q) \in B \times B$, for some constant $c_0 = c_0(n, s_0) > 0$, we have

$$\begin{aligned}\|(\nabla_X u)_+\|_{L^1(B)} \|(\nabla_X u)_-\|_{L^1(B)} &= \int_B \int_B (\nabla_X u)_+(p)(\nabla_X u)_-(q) dV_p dV_q\\ &\le \frac{1}{c_0}\int_B \int_B (\nabla_X u)_+(p)(\nabla_X u)_-(q)\mathcal{K}_s(p,q) dV_p dV_q\,.\end{aligned}$$

Also

$$\begin{aligned}\|(\nabla_X u)_+\|_{L^1(B)} - \|(\nabla_X u)_-\|_{L^1(B)} &= \int_B \nabla_X u\, dV \triangleq \int_B \langle \nabla u, X\rangle\, dV\\ &= \int_B \mathrm{div}(uX) - u\mathrm{div}(X)\, dV = \int_{\partial B} u\langle X, N\rangle\, d\sigma - \int_B u \mathrm{div}(X)\, dV\,,\end{aligned}$$

where N is the outer unit normal vector field to ∂B. Then, since $|u| \le 1$

$$\left|\|(\nabla_X u)_+\|_{L^1(B)} - \|(\nabla_X u)_-\|_{L^1(B)}\right| \le \|X\|_{L^\infty(B)} + \|\mathrm{div}(X)\|_{L^\infty(B)} \le C\left(\|\xi\|_{C_1(\mathcal{B}_1(0))}\right).$$

Hence, we get

$$\begin{aligned}\|\nabla_X u\|^2_{L^1(B)} &= \left(\|(\nabla_X u)_+\|_{L^1(B)} + \|(\nabla_X u)_-\|_{L^1(B)}\right)^2\\ &= \left(\|(\nabla_X u)_+\|_{L^1(B)} - \|(\nabla_X u)_-\|_{L^1(B)}\right)^2 + 4\|(\nabla_X u)_+\|_{L^1(B)}\|(\nabla_X u)_-\|_{L^1(B)}\\ &\le C\beta_s \int_{\widetilde{B}^+} |\widetilde{\nabla} U|^2 z^{1-s} dV dz + C + \frac{\Lambda}{c_0}\|\nabla_X u\|^2_{L^1(B)}.\end{aligned}$$

Thus, for $\Lambda \le \frac{1}{2c_0} =: \Lambda_0$ we obtain

$$\|\nabla_X u\|^2_{L^1(B)} \le C\beta_s \int_{\widetilde{B}^+} |\widetilde{\nabla} U|^2 z^{1-s} dV dz + C.$$

Moreover, by Lemma 3.2.10 with $R = 1$, $k = 0$ we have

$$\beta_s \int_{\widetilde{B}^+} |\widetilde{\nabla} U|^2 z^{1-s} dV dz \le \frac{C}{1-s}(1 + \|\nabla u\|_{L^1(B)}).$$

Thus, for every $\delta > 0$ by Young's inequality

$$\begin{aligned}\|\nabla_X u\|_{L^1(B)} &\le C + C\sqrt{\frac{1}{1-s}(1+\|\nabla u\|_{L^1(B)})}\\ &\le \frac{C}{(1-s)\delta} + \delta\|\nabla u\|_{L^1(B)}.\end{aligned}$$

Now, let η be a smooth cutoff compactly supported in $B_{3/4}(p)$ and with $\eta \equiv 1$ on $B_{1/2}(p)$. Choosing $X = \eta\frac{\partial}{\partial x^i}$ above and summing up from $i = 1$ to $i = n$, together with (3.37), gives

$$\|\nabla u\|_{L^1(B_{1/2}(p))} \le \frac{C}{(1-s)\delta} + \delta\|\nabla u\|_{L^1(B_1(p))},$$

for some $C = C(n, s_0)$, and this concludes the proof.

□

To conclude the proof of the BV-estimate for almost stable sets, that is Proposition 4.2.5, let us state an abstract (but very useful) result due to Leon Simon that we will need at the end of the proof. We include the proof of this result in the appendix.

Lemma 4.2.9. *Let $\beta \in \mathbb{R}$, $M_\circ > 0$ and $\mathcal{S} : \mathfrak{B} \to [0, +\infty)$ be a nonnegative function defined on the family $\mathfrak{B}$ of open balls contained in the Euclidean ball $B_1(0) \subset \mathbb{R}^n$ that is subadditive for finite unions, meaning that whenever $B \subset \bigcup_i B_i$ a finite union then $\mathcal{S}(B) \le \sum_i \mathcal{S}(B_i)$. Then, there exists a constant $\delta_\circ = \delta_\circ(n, \beta) > 0$ such that, if*

$$r^\beta \mathcal{S}(B_{r/4}(x_0)) \le \delta_\circ r^\beta \mathcal{S}(B_r(x_0)) + M_\circ \quad \text{whenever} \quad B_r(x_0) \subset B_1(0),$$

then

$$\mathcal{S}(B_{1/4}(0)) \le CM_\circ\,,$$

for some constant $C = C(n, \beta) > 0$.

Remark 4.2.10. *The standard situation where this lemma is of use is when one can obtain, for some $\theta \in (0, 1)$ and $C > 0$, an inequality like*

$$\|\nabla u\|_{L^q(B_1)} \le C + C\|\nabla u\|^\theta_{L^q(B_4)}.$$

Indeed, if this holds, then Young's inequality gives, for every $\delta > 0$, that

$$\begin{aligned}\|\nabla u\|_{L^q(B_1)} &\le C + \delta\|\nabla u\|_{L^q(B_4)} + C(\delta, \theta)\\ &= \delta\|\nabla u\|_{L^q(B_4)} + C_1(\delta, \theta)\,.\end{aligned}$$

Then, just by scaling and translating for every $B_r(x) \subset \mathbb{R}^n$ we get

$$r^{q-n}\|\nabla u\|_{L^q(B_r(x))} \le \delta r^{q-n}\|\nabla u\|_{L^q(B_{4r(x)})} + C_1(\delta, \theta)\,.$$

From here, choosing $\delta = \delta_\circ(n, q-n)$ the one of Lemma 4.2.9 one concludes a uniform bound

$$\|\nabla u\|_{L^q(B_{1/2})} \le CC_1(\delta_\circ, \theta).$$

We can now give the proof of Proposition 4.2.5.

Proof of Proposition 4.2.5. Let Λ_0 and C_0 be the constants given by Lemma 4.2.8. Fix any Euclidean ball $\mathcal{B}_r(x) \subset \mathcal{B}_{3/4}(0)$. Consider the subadditive function (defined on the family of

Euclidean balls contained in $\mathcal{B}_{3/4}(0)$)

$$\mathcal{S}(\mathcal{B}_r(x)) := \int_{\varphi(\mathcal{B}_r(x))} |\nabla u| \, dV \, .$$

Notice that $\mathrm{FA}_2(M, g, 1, p, \varphi)$ implies $B_{4r/5}(\varphi(x)) \subset \varphi(\mathcal{B}_r(x))$ and $\varphi(\mathcal{B}_{1/8}(x)) \subset B_{r/5}(\varphi(x))$. Hence, by Lemma 4.2.8 applied with $R = 4r/5$, for every $\delta > 0$ and $\mathcal{B}_r(x) \subset \mathcal{B}_{3/4}(0)$ we have

$$r^{1-n}\mathcal{S}(\mathcal{B}_{r/8}(x)) \leq \delta r^{1-n}\mathcal{S}(\mathcal{B}_r(x)) + \frac{C}{(1-s)\delta} \, ,$$

for some $C = C(n, s_0) > 0$. Using Lemma 4.2.9, taking δ the one given by the lemma with $\beta = 1 - n$, $\rho = 3/4$, $\theta = 1/8$, we find that

$$S(\mathcal{B}_{3/8}(0)) = \int_{\varphi(\mathcal{B}_{3/8}(0))} |\nabla u| \, dV \leq \frac{C}{1-s} \, ,$$

where C depends only on n and s_0. In particular, since $B_{1/4}(p) \subset \varphi(\mathcal{B}_{3/8}(0))$ this concludes the proof. □

Now, we will prove the full BV-estimate by iteratively reducing to the almost-stable case thanks to a covering lemma, which is inspired by the proof of Proposition 2.6 in [FZ24].

In the following lemma we denote by $\mathcal{Q}_r(x) \subset \mathbb{R}^n$ the (hyper)cube of center x and side r.

Lemma 4.2.11. *Let $n \geq 1$, $m \geq 0$, $\theta \in (0, 1)$, $D_0 > 0$ and $\beta > 0$. Let $\mathcal{S} : \mathfrak{B} \to [0, +\infty)$ be a subadditive*[3] *function defined on the family $\mathfrak{B}$ of the (hyper)cubes contained in $\mathcal{Q}_1(0) \subset \mathbb{R}^n$, such that*

(i) $\sup_{\{x \,:\, \mathcal{Q}_r(x) \in \mathfrak{B}\}} \mathcal{S}(\mathcal{Q}_r(x)) \to 0$ *as* $r \to 0$.

(ii) Whenever $\mathcal{Q}_r(x_0), \mathcal{Q}_r(x_1), \ldots, \mathcal{Q}_r(x_m) \subset \mathcal{Q}_1(0)$ are $(m + 1)$ disjoint cubes of the same side at pairwise distance at least $D_0 r$, then

$$\exists\, i \in \{0, 1, \ldots, m\} \;\; \textit{such that} \;\; \mathcal{S}(\mathcal{Q}_{\theta r}(x_i)) \leq r^\beta M_0 \, .$$

Then

$$\mathcal{S}(\mathcal{Q}_{1/2}(0)) \leq C M_0 \, ,$$

for some $C = C(n, \theta, m, \beta, D_0) > 0$.

Proof. Let $\rho = 2^{-k}$, for a fixed integer $k > 1$, and consider the regular partition of $\mathcal{Q}_\theta(0)$ into 2^{kn} cubes of sidelength $\theta\varrho$. Let us call $\mathfrak{F}_1 := \{\mathcal{Q}^1_i\}$ the family of cubes in this partition. In this way, clearly $\#\mathfrak{F}_1 \leq \rho^{-n}$. Let x^1_i denote the center of the cube $\mathcal{Q}^1_i$ and, for every $\lambda > 0$ and cube $\mathcal{Q}$ of side r, let $\lambda\mathcal{Q}$ be the cube with the same center and side λr.

Now, we split the family $\mathfrak{F}_1$ as $\mathfrak{F}_1 = \mathfrak{G}_1 \cup \mathfrak{B}_1$ into the families of *good* and *bad* cubes as follows. Start by considering $\mathcal{Q}^1_1$, if there holds

$$\mathcal{S}(\mathcal{Q}^1_1) \leq M_0 \rho^\beta \tag{4.14}$$

then it is considered a good cube, we assign it to $\mathfrak{G}_1$ and we remove it from $\mathfrak{F}_1$. On the other hand, if $\mathcal{Q}^1_1$ does not satisfy (4.14), then we assign it to the bad cubes $\mathfrak{B}_1$ and remove it from

[3] Meaning subadditive for finite unions of (hyper)cubes.

$\mathfrak{F}_1$. Moreover, if this happens, also all the cubes $\mathcal{Q} \in \mathfrak{F}_1$ such that the distance of $\frac{1}{\theta}\mathcal{Q}$ from $\frac{1}{\theta}\mathcal{Q}_1^1$ is less than $D_0\rho$ are considered bad as well, so they are assigned to $\mathfrak{B}_1$ and removed from $\mathfrak{F}_1$. Importantly, this last rule (of labeling bad the cubes nearby a bad cube) is applied only to the cubes that are still in $\mathfrak{F}_1$. Once a cube is classified as good and placed in $\mathfrak{G}_1$, it is no longer reclassified in later steps.

By a simple count, there are at most $(2 + 2D_0 + 4\sqrt{n}/\theta)^n$ such cubes. We continue this procedure of splitting $\mathfrak{F}_1$ in good cubes and bad cubes until there are no cubes left.

By property (ii), we may have assigned cubes to the bad set $\mathfrak{B}_1$ at most at m steps. Since at each of these steps we removed at most $(2 + 2D_0 + 4\sqrt{n}/\theta)^n$ cubes, this means that $\#\mathfrak{B}_1 \leq m(2 + 2D_0 + 4\sqrt{n}/\theta)^n =: N_0$.

Regarding the good set $\mathfrak{G}_1$, we know it contains at most $\#\mathfrak{F}_1 \leq \rho^{-n}$ cubes since this is just the total number of cubes in the cover. Moreover, by construction in every $\mathcal{Q} \in \mathfrak{G}_1$ we have

$$\mathcal{S}(\mathcal{Q}) \leq M_0\rho^\beta .$$

Hence

$$\mathcal{S}(\mathcal{Q}_\theta(0)) \leq \sum_{\mathcal{Q}\in\mathfrak{G}_1} \mathcal{S}(\mathcal{Q}) + \sum_{\mathcal{Q}\in\mathfrak{B}_1} \mathcal{S}(\mathcal{Q}) \leq M_0\rho^{\beta-n} + \sum_{\mathcal{Q}\in\mathfrak{B}_1} \mathcal{S}(\mathcal{Q}) .$$

The argument continues iteratively under the same scheme, on the union of the at most N_0 bad cubes that are in $\mathfrak{B}_1$. Consider the partition $\mathfrak{F}_2 := \{\mathcal{Q}_i^2\}$ of the cubes in $\mathfrak{B}_1$ obtained splitting each cube into 2^{kn} smaller cubes of side $\theta\rho^2$. Notice that $\#\mathfrak{F}_2 \leq N_0\rho^{-n}$. Now assign cubes in $\mathfrak{F}_2$ to the good cubes $\mathfrak{G}_2$ or bad cubes $\mathfrak{B}_2$ as before: starting from $\mathcal{Q}_1^2$, assign it to $\mathfrak{G}_2$ if

$$\mathcal{S}(\mathcal{Q}_1^2) \leq M_0\rho^{2\beta} ,$$

and then remove it from $\mathfrak{F}_2$. Else, if this is not the case we assign $\mathcal{Q}_1^2$ to the bad cubes $\mathfrak{B}_2$ and remove it, together with all the cubes $\mathcal{Q} \in \mathfrak{F}_2$ such that $\frac{1}{\theta}Q$ is at distance less than $D_0\rho^2$ from $\frac{1}{\theta}\mathcal{Q}_1^2$. Continue the procedure until there are no cubes left in $\mathfrak{F}_2$. By property (ii) again, exactly the same argument as in the first part shows that $\mathfrak{F}_2$ contains at most $N_0 = m(2+2D_0+4\sqrt{n}/\theta)^n$ cubes assigned to the bad set, that is $\#\mathfrak{B}_2 \leq N_0$. This produces a partition $\mathfrak{F}_2 = \mathfrak{G}_2 \cup \mathfrak{B}_2$, and we get

$$\sum_{\mathcal{Q}\in\mathfrak{B}_1} \mathcal{S}(\mathcal{Q}) \leq \sum_{\mathcal{Q}\in\mathfrak{G}_2} \mathcal{S}(\mathcal{Q}) + \sum_{\mathcal{Q}\in\mathfrak{B}_2} \mathcal{S}(\mathcal{Q}) \leq N_0M_0\rho^{2\beta-n} + \sum_{\mathcal{Q}\in\mathfrak{B}_2} \mathcal{S}(\mathcal{Q}) .$$

Iterating this argument, after k steps we have always $\#\mathfrak{B}_k \leq N_0$, and in particular by (i) and subadditivity

$$\mathcal{S}(\mathfrak{B}_k) \leq \sum_{\mathcal{Q}\in\mathfrak{B}_k} \mathcal{S}(\mathcal{Q}) \to 0 ,$$

since each $\mathcal{Q} \in \mathfrak{B}_k$ has side $\theta\rho^k \to 0$. Thus, the set of the points belonging to infinitely many bad families is $\mathcal{S}$-negligible. Hence[4]

$$\begin{aligned}\mathcal{S}(\mathcal{Q}_\theta(0)) &\leq M_0\rho^{\beta-n} + N_0M_0\rho^{2\beta-n} + N_0M_0\rho^{3\beta-n} + \dots \\ &\leq N_0M_0\rho^{\beta-n}\sum_{j\geq 0}\rho^{j\beta} = \frac{N_0}{\rho^n(\rho^{-\beta}-1)}M_0 .\end{aligned} \tag{4.15}$$

[4]Note that we could also have stopped the exhaustion process when the error in the tail of (4.15) is less than the constant on the right-hand side, and we would have obtained the estimate with two times this constant.

Now notice that $\mathcal{Q}_{1/2}(0)$ can be covered, for some $\xi = \xi_n$ dimensional constant, by $\xi_n \theta^{-n}$ many cubes of side $\theta/10$ such that the cube with the same center and side $1/10$ still is contained in $\mathcal{Q}_1(0)$. Since property (ii) is translation invariant, covering $\mathcal{Q}_{1/2}(0)$ in such a way gives

$$\mathcal{S}(\mathcal{Q}_{1/2}(0)) \le \frac{\xi_n \theta^{-n} N_0}{\rho^n(\rho^{-\beta}-1)} M_0 = \frac{\xi_n \theta^{-n} m(2D_0 + 3\sqrt{n}/\theta)^n}{\rho^n(\rho^{-\beta}-1)} M_0 \,,$$

and as this holds for every $\rho = 2^{-k}$, just choosing any fixed k gives the desired estimate.

□

Theorem 4.2.12. *Suppose that M satisfies the flatness assumption* $\mathrm{FA}_2(M, g, 1, p, \varphi)$*, in the sense of Definition 3.4.1. Let $s_0 \in (0,1)$, $s \in (s_0, 1)$ and $u : M \to (-1,1)$ be a solution of the Allen-Cahn equation (2.7) in $B_1(p) \subset M$ with parameter ε, and with Morse index $m_{B_1(p)}(u) \le m$. Then*

$$\int_{B_{1/2}(p)} |\nabla u| dx \le \frac{C}{1-s},$$

for some constant $C = C(n, s_0, m)$.

Proof. For a set $E \subset \mathbb{R}^n$ denote by $\lambda E := \{\lambda y \,:\, y \in E\}$. Consider the subadditive function[5]

$$\mathcal{S}(\mathcal{Q}) := \int_{\varphi\left(\frac{1}{2\sqrt{n}}\mathcal{Q}\right)} |\nabla u| \, dV \,,$$

defined on the cubes $\mathcal{Q} \subset \mathcal{Q}_1(0)$.

Claim. $\mathcal{S}$ satisfies properties (i) and (ii) of Lemma 4.2.11 with $M_0 = C/(1-s)$, $\beta = n-1$, $\theta = 1/8$, and D_0 depending only on n, s_0 and m.

Proof of the claim. The first property is clear from the definition of $\mathcal{S}$, since u is smooth. The second property is a consequence of the Morse index of u being at most m.

Indeed, let $\mathcal{Q}_r(x_0), \mathcal{Q}_r(x_1), \dots, \mathcal{Q}_r(x_m) \subset \mathcal{Q}_1(0)$ be $(m+1)$ disjoint cubes of the same side at pairwise distance at least $D_0 r$, and let $q_i := \varphi(x_i)$. Then, since $\mathcal{B}_{r/2}(x_i) \subset \mathcal{Q}_r(x_i)$ by Lemma 4.2.2 and Lemma 3.4.13 for at least one $\ell \in \{1, \dots m\}$, the inequality

$$\mathcal{E}''(u)[\xi, \xi] \ge -\frac{C_1 m}{(D_0 r/2)^{n+s}} \|\xi\|^2_{L^1(B_{r/2}(q_\ell))}$$

holds for all $\xi \in C^\infty_c(B_{r/2}(q_\ell))$, for some $C_1 = C_1(n, s_0)$. That is, u is a Λ-almost stable solution (in the sense of Definition 4.2.1) in $B_{r/2}(q_\ell)$ with $\Lambda = \frac{C_1 m}{(D_0 r/2)^{n+s}}$.

Note that, in this case, on the rescaled manifold $\widehat{M} := (M, (2/r)^2 g)$ we have that u is a $\Lambda (r/2)^{n+s}$-almost stable solution of $(-\Delta)^{s/2} u + (2\varepsilon/r)^{-s} W'(u) = 0$ in $\widehat{B}_1(q_\ell)$, and the flatness assumption $\mathrm{FA}_2(M, (2/r)^2 g, 1, q_\ell, \varphi_{x_\ell, r/2})$ holds.

Let Λ_0 be the constant given by Proposition 4.2.5. Then, there exists $D_0 = D_0(n, s_0, m) > 0$ so that u is a Λ-almost stable solution of the Allen-Cahn equation in $\widehat{B}_1(q_\ell)$ with

$$\Lambda = \frac{C_1 m}{(D_0 r/2)^{n+s}} \le \Lambda_0,$$

[5]The factor $\frac{1}{2\sqrt{n}}$ inside $\varphi\left(\frac{1}{2\sqrt{n}}\mathcal{Q}\right)$ is needed to have $\frac{1}{2\sqrt{n}}\mathcal{Q} \subset \mathcal{B}_{1/2}(0)$ for $\mathcal{Q} \subset \mathcal{Q}_1(0)$ in order to apply Lemma 3.4.13.

for D_0 sufficiently large. Hence, by Proposition 4.2.5 we get

$$\int_{\widehat{B}_{1/4}(q_\ell)} |\nabla u|_{\widehat{g}}\, d\widehat{V} \le \frac{C}{1-s},$$

and, since $\varphi\big(\mathcal{Q}_{\frac{r}{16\sqrt{n}}}(x_\ell)\big) \subset B_{r/8}(q_\ell)$, scaling back this inequality on M gives

$$\mathcal{S}(\mathcal{Q}_{r/8}(x_\ell)) \le \int_{B_{r/8}(q_\ell)} |\nabla u|\, dV \le \frac{C}{1-s} r^{n-1},$$

for some $C = C(n, s_0, m)$, and this concludes the proof of the claim.

Hence, all the hypothesis of Lemma 4.2.11 are satsfied, and we get

$$\mathcal{S}(\mathcal{Q}_{1/2}(0)) = \int_{\varphi\left(\mathcal{Q}_{\frac{1}{4\sqrt{n}}}(0)\right)} |\nabla u|\, dV \le \frac{C}{1-s}.$$

Now, the fact that the BV-estimate holds in $B_{1/2}(p)$ follows by $\mathrm{FA}_2(M, g, 1, p, \varphi)$ and a standard covering argument, and this concludes the proof. □

As a corollary, simply by scaling we immediately get Theorem 1.2.13.

Proof of Theorem 1.2.13. Since the flatness assumption $\mathrm{FA}_2(M, g, R, p, \varphi)$ holds, the rescaled manifold $\widehat{M} := (M, R^{-2}g)$ satisfies $\mathrm{FA}_2(M, R^{-2}g, 1, p, \varphi_{0,R})$. Hence, Theorem 4.2.12 gives

$$\int_{B_{1/2}(p)} |\nabla u|_{\widehat{g}}\, d\widehat{V} \le \frac{C}{1-s},$$

for some $C = C(n, s_0, m)$. Scaling back this inequality on M gives the result.

□

4.2.4 Density estimate and energy decay

Now we have all the tools to prove the density estimate of Proposition 1.2.16.

Proof of Proposition 1.2.16. Since the statement is scaling-invariant, we prove the result just for $R = 1$. In what follows, $C, c > 0$ denote constants depending only on n,s, and m that can change from line to line, and, in general, C is big and c is small.

We argue by contradiction, suppose that $\int_{B_1(p)} |1+u_\varepsilon| \le \omega_0$ and that $\{u_\varepsilon \ge -\frac{9}{10}\} \cap B_{1/2}(p) \ne \varnothing$, for some $1 \ge C_0\varepsilon$. The constant $C_0 = C_0(n, s, m) > 0$ that will be chosen during the proof.

First, by continuity of u_ε and by taking $\omega_0 < |B_{1/2}(p)|$, there will be a point $q \in B_{1/2}(p)$ for which $|u_\varepsilon(q)| \le \frac{9}{10}$.

Claim. There exists $\alpha = \alpha(n, s) \in (0, 1)$ such that

$$[u_\varepsilon]_{C^\alpha(B_{\varepsilon/3}(q))} \le C\varepsilon^{-\alpha}, \quad \text{for all } \varepsilon \le 1/10.$$

Indeed, let $\eta \in C_c^\infty(\mathcal{B}_2(0))$ be a cutoff function with $\chi_{\mathcal{B}_{3/2}(0)} \le \eta \le \chi_{\mathcal{B}_2(0)}$. Then, the function $\widetilde{u}(x) := u_\varepsilon(\varphi(\varphi^{-1}(q) + \varepsilon x))\eta(x)$ is well defined on the whole $\mathbb{R}^n$, since $\widetilde{u}$ depends only on the values of u_ε in $\varphi(\mathcal{B}_{2\varepsilon}(\varphi^{-1}(q))) \subset \varphi(B_1)$. Now, by the flatness assumption $\mathrm{FA}_2(M, g, 1, p, \varphi)$ we have that $\widetilde{u}$ satisfies $|L\widetilde{u}| \le C$ in $\mathcal{B}_1(0)$, where the kernel of L satisfies (A.1) by Proposition

3.4.6—see in particular inequality (3.38). Hence, by Lemma A.1.2 we have $[\widetilde{u}]_{C^\alpha(\mathcal{B}_{1/2}(0))} \le C$, thus $[u_\varepsilon]_{C^\alpha(B_{\varepsilon/3}(q))} \le C\varepsilon^{-\alpha}$ as desired.

Then, for all ε sufficiently small (depending on n and s), we have $|u_\varepsilon| \le \frac{19}{20}$ in the ball $B_\varepsilon(q)$. Using that $W(u) = \frac{1}{4}(1-u^2)^2$, we deduce

$$\varepsilon^{s-n} \cdot \varepsilon^{-s} \int_{B_\varepsilon(q)} W(u_\varepsilon)\, dV \ge c_n > 0$$

for some dimensional constant c_n.

Let now U be the Caffarelli-Silvestre extension of u_ε in $\widetilde{M} = M \times (0,\infty)$. The previous lower bound on the potential energy in $B_\varepsilon(q)$ leads to

$$\varepsilon^{s-n}\widetilde{\mathcal{E}}_{\widetilde{B}^+_\varepsilon}(U) = \varepsilon^{s-n}\left(\frac{\beta_s}{2}\int_{\widetilde{B}^+_\varepsilon(q,0)} z^{1-s}|\nabla U|^2\, dV dz + \varepsilon^{-s}\int_{B_\varepsilon(q)} W(u_\varepsilon)\, dV\right) \ge c_n.$$

By the monotonicity formula of Theorem 1.2.2, for $\lambda \in (0,1)$, we deduce that

$$\frac{\widetilde{\mathcal{E}}_{\widetilde{B}^+_\rho}(U)}{\rho^{n-s}} \ge c\frac{\widetilde{\mathcal{E}}_{\widetilde{B}^+_{\lambda\rho}}(U)}{(\lambda\rho)^{n-s}}, \qquad \forall\, \rho \in (0, R_{\mathrm{mon}}),$$

where R_{mon} is the radius given by the monotonicity formula and can be taken to be $R_{\mathrm{mon}} = \mathrm{inj}_M(q)/4$—see Remark 1.2.3—and thus by hypothesis is $R_{\mathrm{mon}} \ge 1/8$. Subtracting $\rho^{s-n}\widetilde{\mathcal{E}}_{\widetilde{B}^+_{\lambda\rho}}(U)$ to both sides gives

$$\frac{\widetilde{\mathcal{E}}_{\widetilde{B}^+_\rho \setminus \widetilde{B}^+_{\lambda\rho}}(U)}{\rho^{n-s}} \ge c\big(1-\lambda^{n-s}\big)\frac{\widetilde{\mathcal{E}}_{\widetilde{B}^+_{\lambda\rho}}(U)}{(\lambda\rho)^{n-s}} \ge c\big(1-\lambda^{n-s}\big)c_n, \tag{4.16}$$

provided $\lambda\rho \ge \varepsilon$ and $\rho < 1/8$.

Now let $\ell \ge 0$ be an integer that will be chosen later sufficiently large, depending only on m, n, and s. For $k \in \{\ell, \ell+1 \dots, \ell+m\}$, cosider the annuli

$$A_k := B_{\frac{1}{2^{2k+3}}}(q) \setminus B_{\frac{1}{2^{2k+4}}}(q), \quad \text{and} \quad \widetilde{A}^+_k := \widetilde{B}^+_{\frac{1}{2^{2k+3}}}(q,0) \setminus \widetilde{B}^+_{\frac{1}{2^{2k+4}}}(q,0).$$

These are $(m+1)$ disjoint annuli at pairwise distance at least $2^{-2(m+\ell+1)}$. Since u has Morse index at most m by hypothesis, by Lemma 4.2.2 there is one of these annuli, say A_k, where u is Λ-almost stable in A_k with

$$\Lambda = m \max_{i\neq j} \sup_{A_i \times A_j} \frac{C}{d(x,y)^{n+s}} \le Cm2^{2(m+\ell+1)(n+s)}.$$

Set $\rho_k := 2^{-(2k+4)}$, and note that ρ_k this is the width of A_k. By Lemma 4.2.4 (and a simple covering argument), for every $a \in (0,1)$, we have

$$\begin{aligned}\rho_k^{s-n}\mathcal{E}^{\mathrm{Pot}}_{B_{\frac{7}{8}\frac{1}{2^{2k+3}}} \setminus B_{\frac{5}{8}\frac{1}{2^{2k+3}}}}(u_\varepsilon) &\le \frac{C}{a}\rho_k^{s-n}\int_{\widetilde{A}^+_k} z^{1-s}|\nabla U|^2\, dV dz + Ca + C\varepsilon^s\rho_k^{-n}\mathcal{E}^{\mathrm{Pot}}_{A_k}(u_\varepsilon)\\ &\le \frac{C}{a}\rho_k^{s-n}\int_{\widetilde{A}^+_k} z^{1-s}|\nabla U|^2\, dV dz + Ca + C\varepsilon^s.\end{aligned}$$

By (4.16) applied with $\lambda = 5/7$ and $\rho = \frac{7}{8}2^{-(2k+3)}$, and using that

$$\widetilde{B}^+_{\frac{7}{8}\frac{1}{2^{2k+3}}} \setminus \widetilde{B}^+_{\frac{5}{8}\frac{1}{2^{2k+3}}} \subset \widetilde{A}^+_k,$$

this implies

$$\begin{aligned}
c &\leq \left(\tfrac{7}{8}2^{-(2k+3)}\right)^{s-n} \widetilde{\mathcal{E}}_{\widetilde{B}^+_{\frac{7}{8}\frac{1}{2^{2k+3}}} \setminus \widetilde{B}^+_{\frac{5}{8}\frac{1}{2^{2k+3}}}}(U) \\
&\leq C\rho_k^{s-n} \int_{\widetilde{A}^+_k} z^{1-s} |\nabla U|^2 \, dV dz + C\rho_k^{s-n} \mathcal{E}^{\mathrm{Pot}}_{B_{\frac{7}{8}\frac{1}{2^{2k+3}}} \setminus B_{\frac{5}{8}\frac{1}{2^{2k+3}}}}(u_\varepsilon) \\
&\leq C\rho_k^{s-n} \int_{\widetilde{A}^+_k} z^{1-s} |\nabla U|^2 \, dV dz + \frac{C}{a}\rho_k^{s-n} \int_{\widetilde{A}^+_k} z^{1-s} |\nabla U|^2 \, dV dz + Ca + C\varepsilon^s \\
&\leq C\rho_k^{s-n} \int_{\widetilde{A}^+_k} z^{1-s} |\nabla U|^2 \, dV dz + \frac{c}{4} + \frac{c}{4},
\end{aligned}$$

provided we take a and ε sufficiently small (depending only on m, n and s). Hence, absorbing the last two terms to the left and using Lemma 3.2.10 with $k = 1$, we get

$$\begin{aligned}
\frac{c}{2} &\leq C\rho_k^{s-n} \int_{\widetilde{B}^+_{2\rho_k}} z^{1-s} |\nabla U|^2 \, dV dz \\
&\leq \frac{C}{r^s} + C\left(\rho_k^{-n} \int_{B_{2r\rho_k}(q)} |1 + u_\varepsilon|\right)^{1-s} \left(\rho^{1-n} \int_{B_{2r\rho_k}(q)} |\nabla u_\varepsilon|\right)^s,
\end{aligned}$$

for all $r \geq 1$ provided $2r\rho_k \leq 1/2$. Here we have also used that $\widetilde{A}^+_k \subset \widetilde{B}^+_{2\rho_k}$ in the first line.

Choosing $r = (c/4C)^{-1/s}$, we can absorb the first term to the left in the last inequality. After doing so, here we have to choose ℓ large (depending only on m, n and s) in order to have, with the previous choice of r, that

$$2r\rho_k = \frac{2r}{2^{2k+4}} \leq \frac{2r}{2^{2\ell+4}} \leq 1/2.$$

With these choices

$$\frac{c}{4} \leq C\omega_0^{1-s} \left(\int_{B_{1/2}(q)} |\nabla u_\varepsilon|\right)^s,$$

and by the BV-estimate of Theorem 1.2.13 we reach a contradiction if the density ω_0 is too small. This concludes the proof. □

From the proof of the proposition above, we can extract an auxiliary result. This fact will be useful in the proof of Proposition 4.2.15 below.

Proposition 4.2.13. *Let $u : M \to (-1, 1)$ be a solution of (2.7) in $B_R(p) \subset M$ with Morse index $m_{B_R(p)}(u) \leq m$, and suppose that M satisfies the flatness assumption $\mathrm{FA}_2(M, g, R, p, \varphi)$. Then, there exist positive constants C_0 and ε_0, depending only on n, s, and m, such that the following holds: whenever $\varepsilon \leq \varepsilon_0$ and $R \geq C_0\varepsilon$, if for some $q \in B_{R/2}(p)$ we have $|u_\varepsilon(q)| \leq \frac{9}{10}$ then*

$$\int_{B_{R/2}(q)} |\nabla u_\varepsilon| \, dV \geq c_0 R^{n-1}, \tag{4.17}$$

for some $c_0 = c_0(n, s, m)$.

Proof. It follows by simply repeating the proof of Proposition 1.2.16 above, from when we found a point $q \in B_{1/2}(p)$ with $|u_\varepsilon(q)| \le \frac{9}{10}$ to the very last line, and using that $|u_\varepsilon| \le 1$ to estimate the density $\int_{B_{1/2}(q)} |1 + u_\varepsilon|$ from above instead of using the bound ω_0. $\square$

Lemma 4.2.14. *Let $s \in (0,1)$, $p \in M$, and assume that* $\mathrm{FA}_2(M, g, p, R, \varphi)$ *holds—see Definition* (3.4.1)*. Let $u_\varepsilon : M \to (-1,1)$ be a solution of (2.7) in $B_R(p)$. Then, there exist positive constants $C = C(n,s)$ such that, if $\varepsilon < 1$ and $1 - |u_\varepsilon| \le \frac{1}{10}$ in $\varphi(\mathcal{B}_R(0))$, then*

$$0 \le 1 - |u_\varepsilon| \le C(\varepsilon/R)^s \quad \textit{in } \varphi(\mathcal{B}_{R/2}(0)).$$

Proof. Since the statement is scaling-invariant, we assume $R = 1$. Suppose in addition that $\frac{9}{10} \le u_\varepsilon \le 1$ in $\varphi(\mathcal{B}_1(0))$; the case $-1 \le u_\varepsilon \le -\frac{9}{10}$ can be reduced to the previous by the even symmetry of W (i.e., replacing u_ε by $-u_\varepsilon$). Then, since u_ε solves (2.7) we see that $v := 1 - u_\varepsilon$ satisfies

$$Lv := (-\Delta)^{s/2} v + \frac{1}{2\varepsilon^s} v \le 0, \quad \text{in } \varphi(\mathcal{B}_1(0)).$$

Now we simply build a barrier from above for v. Fix a smooth function $\xi_\circ \in C^\infty(\mathbb{R}^n)$ such that $\chi_{\mathcal{B}_{1/2}(0)} \le 1 - \xi_\circ \le \chi_{\mathcal{B}_{3/4}(0)}$ and consider the function $\xi := \xi_\circ \circ \varphi^{-1}$ defined on M, considered to be identically 1 outside $\varphi(\mathcal{B}_1(0))$. Since $\mathrm{FA}_2(M, g, p, 1, \varphi)$ holds, we have that $\mathrm{FA}_2(M, g, q, 1/10, \varphi_{\varphi^{-1}(q), 1/10})$ holds for every $q \in \varphi(\mathcal{B}_{3/4}(0))$ (see (c) in Remark 3.4.5). Then, for every $q \in \varphi(\mathcal{B}_{3/4}(0))$ we have

$$\begin{aligned} \left|(-\Delta)^{s/2}\xi\right|(q) &\le \int_M |\xi(q) - \xi(p)| \mathcal{K}_s(p,q)\, dV_p \\ &\le C \int_{B_{1/10}(q)} \mathrm{dist}(p,q) \mathcal{K}_s(p,q)\, dV_p + 2 \int_{M \setminus B_{1/10}(q)} \mathcal{K}_s(p,q)\, dV_p \le C_0, \end{aligned}$$

for some $C_0 > 0$ that depends only on n and s. The last estimate follows, respectively for the two integrals, from Lemma 3.4.13 and Theorem 3.4.6 (in particular by (3.41)). Then, using that $\xi \ge 0$, we have

$$L(\xi + 2C_0\varepsilon^s) = (-\Delta)^{s/2}\xi + \frac{1}{2\varepsilon^s}(\xi + 2C_0\varepsilon^s) \ge -C_0 + C_0 = 0, \quad \text{in } \varphi(\mathcal{B}_{3/4}(0)).$$

Since $\xi + 2C_0\varepsilon^s > 1 \ge v$ in $M \setminus \varphi(\mathcal{B}_{3/4}(0))$ we get, by the maximum principle, that $v \le \xi + 2C_0\varepsilon^s$ in $\varphi(\mathcal{B}_{3/4}(0))$. Hence, using that $\xi \equiv 0$ in $\varphi(\mathcal{B}_{1/2}(0))$, we have shown that $1 - u_\varepsilon \le 2C_0\varepsilon^s$ in $\varphi(\mathcal{B}_{1/2}(0))$, as desired. $\square$

The following proposition shows the quantitative convergence to zero, as $\varepsilon \searrow 0$, of the potential energy of finite index solutions to the A-C equation (2.7). The statement and proof are inspired by those of Proposition 6.2 in [CCS21], which deals with stable solutions of the fractional Allen-Cahn equation in $\mathbb{R}^n$. We, moreover, simplify the proof in [CCS21], using the lower bound (4.17) on the BV norm that we have obtained as a by-product of (the proof of) Proposition 1.2.16.

Proposition 4.2.15. *Let $s \in (0,1)$, $p \in M$, and assume that the flatness assumption* $\mathrm{FA}_2(M, g, p, R, \varphi)$ *holds. Let $u_\varepsilon : M \to (-1,1)$ be a solution of* (2.7) *in $B_R(p) \subset M$ with Morse index $m_{B_R(p)}(u_\varepsilon) \le m$. Then, there exist constants C and ε_0, depending only on n, s, and m, such that for all $\varepsilon \le \varepsilon_0$:*

$$\varepsilon^{-s} \int_{B_{R/2}(p)} W(u_\varepsilon)\, dV \le C R^{n-s} (\varepsilon/R)^\beta\,,$$

where $\beta := \min\left(\frac{1-s}{2}, s\right) > 0$.

Proof. Given $q \in B_{R/2}(p)$, let

$$r_q := \max(\min\left(\tfrac{R}{16}, \tfrac{1}{2}\mathrm{dist}(q, \{|u| \le \tfrac{9}{10}\})\right), C_0\varepsilon),$$

where $C_0 > 0$ is a large constant, depending only on n and s, to be chosen later.

Observe that, if $\frac{R}{16} \le C_0\varepsilon$ then

$$(\varepsilon/R)^{-s}\int_{B_{R/2}(p)} W(u_\varepsilon)\, dV \le (16C_0)^s(\max_{[-1,1]}W)|B_{R/2}(p)| \le CR^n \le CC_0^\beta(\varepsilon/R)^\beta R^n.$$

Thus, we may (and do) assume that $\frac{R}{16} > C_0\varepsilon$. In particular, $r_q \in \left[C_0\varepsilon, \frac{R}{16}\right]$ for all $q \in B_{R/2}(p)$.

Claim. For some constant $c = c(n,s,m) > 0$, there holds

$$\int_{B_{4r_q}(q)} |\nabla u_\varepsilon| \ge c(r_q)^{n-1} \quad \text{whenever } r_q < \tfrac{R}{16}. \tag{4.18}$$

Indeed, if $r_q < \frac{R}{16}$ then necessarily $\frac{R}{16} \ge \frac{1}{2}\mathrm{dist}(q, \{|u| \le \frac{9}{10}\})$, otherwise we would obtain

$$r_q = \max(\min\left(\tfrac{R}{16}, \tfrac{1}{2}\mathrm{dist}(q, \{|u| \le \tfrac{9}{10}\})\right), C_0\varepsilon) = \max(\tfrac{R}{16}, C_0\varepsilon) \ge \tfrac{R}{16},$$

which contradicts that $r_q < \frac{R}{16}$. Hence $\frac{R}{16} \ge \frac{1}{2}\mathrm{dist}(q, \{|u| \le \frac{9}{10}\})$ and

$$r_q = \max(\tfrac{1}{2}\mathrm{dist}(q, \{|u| \le \tfrac{9}{10}\}), C_0\varepsilon) \ge \tfrac{1}{2}\mathrm{dist}(q, \{|u| \le \tfrac{9}{10}\}).$$

Thus, since also $q \in B_{R/2}(p)$, there exists $q' \in \{|u_\varepsilon| \le \frac{9}{10}\} \cap B_{\frac{R}{2}+\frac{R}{16}}(p)$ such that $\mathrm{dist}(q, q') \le 2r_q$. Then, choosing ε_0 small and C_0 big (depending only on n, s and m) according to the constants in Proposition 1.2.16, by (4.17) we have

$$\int_{B_{4r_q}(q)} |\nabla u_\varepsilon| \ge \int_{B_{2r_q}(q')} |\nabla u_\varepsilon| \ge c(2r_q)^{n-1},$$

and the claim is proved.

We now produce a covering of $B_{R/2}(p)$ by some of the balls $\{B_{r_q}(q)\}_{q \in B_{R/2}(p)}$ as follows. Given $k \le -5$, let $X_k := \{q \in B_{R/2}(p) \,:\, r_q \in (2^k R, 2^{k+1}R]\}$ and let $\mathcal{J}_k$ be a discrete index set such that $\{q_j^k\}_{j \in \mathcal{J}_k}$ forms a maximal subset of X_k with the property that the balls $B_{r(q_j^k)/4}(q_j^k)$ are pairwise disjoint, where we denote $r(q_j^k) := r_{q_j^k}$. By construction of X_k, it follows that

$$X_k \subset \bigcup_{j \in \mathcal{J}_k} B_{r(q_j^k)}(q_j^k),$$

and that the family of enlarged balls

$$\left\{B_{4r(q_j^k)}(q_j^k)\right\}_{j \in \mathcal{J}_k}$$

has (dimensional) finite overlapping.[6] Note also that since $R/16 > C_0\varepsilon$ we have $\lfloor \log_2(C_0\varepsilon/R) \rfloor \le$

[6]That is, every point $q \in \left\{B_{4r(q_j^k)}(q_j^k)\right\}_{j \in \mathcal{J}_k}$ belongs to at most $N = N(n)$ of these balls. This is easy to check: if q belongs to N of such balls, we would have the existence of N points q_j^k in $B_{4 \cdot R2^{k+1}}(p)$ such that the balls

-5 and, by construction, the union of the sets X_k when k runs on $\{\lfloor \log_2(C_0\varepsilon/R)\rfloor \le k \le -5\}$ covers all of $B_{R/2}(p)$.

Now, on the one hand, by the BV-estimate of Theorem 1.2.13 we have $\int_{B_{3R/4}(p)} |\nabla u_\varepsilon| \le CR^{n-1}$, and this yelds

$$\#\mathcal{J}_k \le C2^{-k(n-1)}\,, \tag{4.19}$$

for all $k \le -5$. Indeed, this follows using that the balls $\big\{B_{4r(q_j^k)}(q_j^k)\big\}_{j\in\mathcal{J}_k}$ have finite overlap and are all contained in $B_{3/4R}(p)$. Indeed, when $k < -5$ then $r(q_j^k) < \frac{R}{16}$ and hence all the balls satisfy (4.18) and are strictly contained in $B_{3/4R}(p)$ by construction, while for $k = -5$ the radius of the balls is at least $\frac{R}{16}$ so their number must be bounded.

On the other hand, for any given $\alpha \in [0, 2s]$, we claim that Lemma 4.2.14 yields

$$\fint_{B_{r_q}(q)} W(u_\varepsilon)\,dV = \fint_{B_{r_q}(q)} \frac{1}{4}(1-u_\varepsilon^2)^2\,dV \le C\Big(\frac{\varepsilon}{r_q}\Big)^\alpha,$$

where $\fint$ denotes the integral average. Indeed, note that if $r_q = C_0\varepsilon$ the previous estimate is trivial, while if $r_q > C_0\varepsilon$ then $r_q = \frac{1}{2}\mathrm{dist}(q, \{|u| \le \frac{9}{10}\})$ and hence we may apply Lemma 4.2.14 (recall that $r_q \ge C_0\varepsilon \ge \varepsilon$) in $B_{2r_q}(q)$ to get the desired bound.

Therefore, choosing $\alpha := \min\big(\frac{1+s}{2}, 2s\big) \in (0,1)$ we obtain—using (4.19)—that

$$\begin{aligned}
(\varepsilon/R)^{-s}\int_{B_{R/2}(p)} W(u_\varepsilon)\,dV &\le C \sum_{k=\lfloor \log_2(C_0\varepsilon/R)\rfloor}^{-5} \sum_{j\in\mathcal{J}_k} (\varepsilon/R)^{-s}\int_{B_{r(q_j^k)}(q_j^k)} W(u_\varepsilon)\,dV \\
&\le C \sum_{k=\lfloor \log_2(C_0\varepsilon/R)\rfloor}^{-5} \sum_{j\in\mathcal{J}_k} (\varepsilon/R)^{-s}\Big(\frac{\varepsilon}{r_{q_j^k}}\Big)^\alpha r_{q_j^k}^n \\
&\le C \sum_{k=\lfloor \log_2(C_0\varepsilon/R)\rfloor}^{-5} (\varepsilon/R)^{-s}\Big(\frac{\varepsilon}{2^kR}\Big)^\alpha (2^{k+1}R)^n(\#\mathcal{J}_k) \\
&\le C \sum_{k=\lfloor \log_2(C_0\varepsilon/R)\rfloor}^{-5} (\varepsilon/R)^{\alpha-s}R^n 2^{k(n-\alpha)}2^{-k(n-1)} \\
&\le CR^n(\varepsilon/R)^{\alpha-s}\sum_{k=-\infty}^{-5}(2^k)^{1-\alpha} \\
&\le CR^n(\varepsilon/R)^\beta,
\end{aligned}$$

as we wanted to show. □

4.3 Letting $\varepsilon \to 0$: convergence to an s-minimal surface

With the estimates for Allen-Cahn solutions of Section 4.2 at hand, we can finally prove Theorem 1.2.17.

Proof of Theorem 1.2.17. We split the proof according to the different statements in the theorem.

Step 1. Convergence in $H^{s/2}(M)$.

$B_{\frac{1}{4}R2^k}(q_j^k)$ are disjoint and contained in $B_{9\cdot R2^k}(p)$. Then, comparing the volumes and using that $\mathrm{FA}_2(M,g,p,R,\varphi)$ holds gives a dimension bound on N.

Since M is compact, there is a small radius $R = R(M) > 0$ so that the flatness assumption $\mathrm{FA}_2(M, g, R, p, \varphi_p)$ holds for every $p \in M$; see Remark 3.4.3. We can then apply the BV-estimate of Theorem 1.2.13 to get a bound on the BV norm $[u_{\varepsilon_j}]_{BV(B_{R/2}(p))}$ independently of $p \in M$. For any $\sigma \in (0, 1)$, the interpolation result of Proposition A.1.1 together with the comparability between $\mathcal{K}_\sigma(\varphi_p(x), \varphi_p(y))$ and $\frac{1}{|x-y|^{n+\sigma}}$ (see Lemma 3.4.13) gives then the bound

$$\iint_{B_{R/2}(p)\times B_{R/2}(p)} |u_{\varepsilon_j}(p) - u_{\varepsilon_j}(q)|^2 \mathcal{K}_\sigma(p, q)\, dV_p\, dV_q \le C(n, \sigma),$$

valid for any $p \in M$. Combining this with (3.41) of Theorem 3.4.6 (with $\alpha = 0$), we see that

$$\iint_{B_{R/4}(p)\times M} |u_{\varepsilon_j}(p) - u_{\varepsilon_j}(q)|^2 \mathcal{K}_\sigma(p, q)\, dV_p\, dV_q \le C(n, \sigma),$$

which after covering M with finitely many such balls of radius $R/4$ shows that

$$\|u_{\varepsilon_j}\|_{H^{\sigma/2}(M)} \le C(M, \sigma).$$

In particular, we can choose some fixed $\sigma > s$. Then, the (standard) compactness of the inclusion[7] $H^{\sigma/2}(M) \hookrightarrow H^{s/2}(M)$ shows that a subsequence converges strongly in $H^{s/2}(M)$ to a limit function $u_0 \in H^{s/2}(M)$. Moreover, after extracting a further subsequence (that we do not relabel), we also assume that the convergence holds almost everywhere on M.

Step 2. Convergence of $\mathcal{E}_M^{\mathrm{Pot}}(u_{\varepsilon_j})$ to zero and structure of u_0.

Again as in Step 1, covering M with a finite number of balls of radius R so that $\mathrm{FA}_2(M, g, R, p, \varphi_p)$ holds for all $p \in M$, applying Proposition 4.2.15 to each ball of the covering we get (for j large) that

$$\mathcal{E}_M^{\mathrm{Pot}}(u_{\varepsilon_j}) \le C(M, s, m)\varepsilon_j^{\beta}, \tag{4.20}$$

which shows that $\mathcal{E}_M^{\mathrm{Pot}}(u_{\varepsilon_j}) \to 0$ as $j \to \infty$ (since then $\varepsilon_j \to 0$).

The fact that the limit function is of the form $u_0 = \chi_E - \chi_{E^c}$ for a set $E \subset M$ follows; since we just proved that $\varepsilon_j^{-s} \int_M W(u_{\varepsilon_j}) \to 0$ as $j \to \infty$, of course $\int_M W(u_{\varepsilon_j}) \to 0$ also. By Fatou's lemma, we deduce that $\int_M W(u_0) = 0$, which shows that the limit u_0 can only take the values ± 1. Hence $u_0 = \chi_E - \chi_{E^c}$ for some measurable set $E \subset M$, which is actually a set of finite perimeter since the u_{ε_j} satisfy a uniform BV estimate. The fact that (1.7)-(1.8) hold, after choosing the representative of E for which every point of E with density 1 belongs to its interior and every point of density 0 belongs to its complement, follows from the convergence in $L^1(M)$ and the density estimate of Proposition 1.2.16. In particular, (1.8) holds with $\delta = \omega_0/(2\omega_n)$, where ω_0 is the constant of Proposition 1.2.16 and ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Step 3. Convergence of the level sets to ∂E in the Hausdorff distance.

This is a direct consequence of Lemma 4.2.14 and the density estimate in Proposition 1.2.16. Fix $c \in (-1, 1)$. Arguing by contradiction, assume that there exist a small $r > 0$ and points $p_j \in \{u_{\varepsilon_j} \ge c\}$, $q_j \in E$ with $d(p_j, q_j) \ge r$ and either $B_{r/2}(p_j) \cap E = \varnothing$ or $B_{r/2}(q_j) \cap \{u_{\varepsilon_j} \ge c\} = \varnothing$. The proof of the two cases is almost identical; we carry on the full details just for the first case.

Assume that $B_{r/2}(p_j) \cap E = \varnothing$ for all j. By compactness, for a subsequence, there is $p_* \in M$ such that $p_j \to p_*$ and $B_{r/4}(p_*) \subset E^c$. This implies (up to subsequences that we do not relabel)

[7]The compactness of this inclusion is well known on balls of $\mathbb{R}^n$. This immediately gives one way of showing it for compact manifolds as well, after covering them with a finite number of small coordinate balls and using the same estimations for the kernel as in the present proof.

that $\lim_{j\to\infty} u_{\varepsilon_j} = -1$ a.e. in $B_{r/4}(p_*)$. By the density estimate of Proposition 1.2.16 we get that

$$u_{\varepsilon_j} \le -9/10 \quad \text{in } B_{r/8}(p_*),$$

for all j sufficiently large, and with Lemma 4.2.14 this implies $u_{\varepsilon_j} \le -1 + C(\varepsilon_j/r)^s$ in $B_{r/16}(p_*)$. This contradicts $u_{\varepsilon_j}(p_j) \ge c > -1$ for j sufficiently large.

If $B_{r/2}(q_j) \cap \{u_{\varepsilon_j} \ge c\} = \varnothing$ instead, we have $q_j \to q_*$ up to subsequnces and $|B_{r/4}(q_*) \cap E| = 0$ since $\{u_{\varepsilon_j} \ge c\} \to E$ in L^1. Arguing as above we would get $u_{\varepsilon_j} \le -9/10$ in $B_{r/8}(q_*)$, and this contradicts the fact that $u_{\varepsilon_j}(q_j) \to +1$ as $j \to \infty$.

Step 4. The limit set E is an s-minimal surface.

Claim. Let $X \in \mathrm{Vect}(M)$ and $\phi_t : M \to M$ denote its flow at time $t > 0$. Denote by $u_{\varepsilon_j,t}(p) := u_{\varepsilon_j}(\phi_{-t}(p))$. Then

$$\frac{d^\ell}{dt^\ell}\mathcal{E}_M^{\mathrm{Sob}}(u_{\varepsilon_j,t}) \to \frac{d^\ell}{dt^\ell}\mathrm{Per}_s(\phi_t(E)), \quad \text{and} \quad \frac{d^\ell}{dt^\ell}\mathcal{E}_M^{\mathrm{Pot}}(u_{\varepsilon_j,t}) \to 0, \quad \text{as } j \to \infty. \tag{4.21}$$

Proof of the claim. Changing variables with the flow ϕ_t, and denoting its Jacobian at time t as J_t, gives that

$$\begin{aligned}\frac{d^\ell}{dt^\ell}\mathcal{E}_M^{\mathrm{Sob}}(u_{\varepsilon_j,t}) &= \frac{d^\ell}{dt^\ell}\iint |u_{\varepsilon_j}(\phi_{-t}(p)) - u_{\varepsilon_j}(\phi_{-t}(q))|^2 \mathcal{K}_s(p,q)dV_p\,dV_q \\ &= \iint |u_{\varepsilon_j}(p) - u_{\varepsilon_j}(q)|^2 \frac{d^\ell}{dt^\ell}\Big[\mathcal{K}_s(\phi_t(p),\phi_t(q))\,J_t(p)J_t(q)\Big]dV_p\,dV_q,\end{aligned} \tag{4.22}$$

and likewise

$$\frac{d^\ell}{dt^\ell}\mathrm{Per}_s(\phi_t(E)) = \iint |u_0(p) - u_0(q)|^2 \frac{d^\ell}{dt^\ell}\Big[\mathcal{K}_s(\phi_t(p),\phi_t(q))\,J_t(p)J_t(q)\Big]dV_p\,dV_q.$$

We can rewrite the first expression as

$$\frac{d^\ell}{dt^\ell}\mathcal{E}_M^{\mathrm{Sob}}(u_{\varepsilon_j,t}) = \iint |u_{\varepsilon_j}(p) - u_{\varepsilon_j}(q)|^2 \mathcal{K}_s(p,q)\frac{\frac{d^\ell}{dt^\ell}\Big[\mathcal{K}_s(\phi_t(p),\phi_t(q))\,J_t(p)J_t(q)\Big]}{\mathcal{K}_s(p,q)}dV_p\,dV_q.$$

Since $u_{\varepsilon_j} \to u_0$ in $H^{s/2}(M)$ by Step 1, we immediately see that

$$A_j := |u_{\varepsilon_j}(p) - u_{\varepsilon_j}(q)|^2\mathcal{K}_s(p,q) \to |u_0(p) - u_0(q)|^2\mathcal{K}_s(p,q) =: A \quad \text{in } L^1(M \times M).$$

On the other hand (3.57) shows that the fixed function

$$B := \frac{\frac{d^\ell}{dt^\ell}\Big[\mathcal{K}_s(\phi_t(p),\phi_t(q))\,J_t(p)J_t(q)\Big]}{\mathcal{K}_s(p,q)}$$

belongs to $L^\infty(M \times M)$. Therefore, $A_jB \to AB$ in $L^1(M \times M)$ as well, which gives the first part of the claim.

For the second part of the claim, which regards the derivatives of the potential energy, we change variables once again with the flow ϕ_t, finding that

$$\frac{d^\ell}{dt^\ell}\mathcal{E}_M^{\mathrm{Pot}}(v_t) = \frac{d^\ell}{dt^\ell}\int \varepsilon^{-s}W(v(\phi_{-t}(p)))dV_p = \int \varepsilon^{-s}W(v(p))\frac{d^\ell}{dt^\ell}J_t(p)dV_p.$$

Bounding the derivatives of the Jacobian (in absolute value) by a constant, we deduce that

$$\left|\frac{d^\ell}{dt^\ell}\mathcal{E}_M^{\mathrm{Pot}}(v_t)\right| \le C\mathcal{E}_M^{\mathrm{Pot}}(v).$$

Combining this with (4.20), we conclude that

$$\frac{d^\ell}{dt^\ell}\mathcal{E}_M^{\mathrm{Pot}}(u_{\varepsilon_j,t}) \to 0, \quad \text{as } j \to \infty,$$

as desired. □

Now that we have shown the claim, the fact that E is stationary for the fractional perimeter follows applying the claim (with $l = 1$ and $t = 0$) and the stationarity of u_{ε_j} for the Allen-Cahn energy. Indeed

$$\frac{d}{dt}\Big|_{t=0}\mathrm{Per}_s(\phi_t(E)) = \lim_{j\to\infty}\frac{d}{dt}\Big|_{t=0}\left[\mathcal{E}_M^{\mathrm{Sob}}(u_{\varepsilon_j,t}) + \mathcal{E}_M^{\mathrm{Pot}}(u_{\varepsilon_j,t})\right] = 0.$$

Step 5. E has Morse index at most m (see Definition 2.2.13).

To check this, consider $(m+1)$ vector fields $X_0, ..., X_m$ of class C^∞ on M.

Letting $a := (a_0, a_1, ..., a_m) \in \mathbb{R}^{m+1}$ and $X[a] = a_0X_0 + ... + a_mX_m$, we can define the quadratic form $Q_{\varepsilon_j}(a) := \frac{d^2}{dt^2}\Big|_{t=0}\mathcal{E}(u_{\varepsilon_j}\circ\psi_{X[a]}^{-t})$, which we can write as $Q_{\varepsilon_j}(a) = Q_{\varepsilon_j}^{kl}a_ka_l$ for some coefficients $Q_{\varepsilon_j}^{kl}$. From (4.21) and the polarization identity for a quadratic form, it is immediate to see that $Q_{\varepsilon_j}^{kl} \to Q_0^{kl}$ as $j\to\infty$, where $Q_0(a) := \frac{d^2}{dt^2}\Big|_{t=0}\mathrm{Per}_s(\psi_{X[a]}^{t}(E)) = Q_0^{kl}a_ka_l$.
Now, since the u_{ε_j} have Morse index $\le m$, by definition we know that for every j there must exist some $a^j \in \mathbb{S}^m$ such that

$$Q_{\varepsilon_j}(a^j) = \frac{d^2}{dt^2}\Big|_{t=0}\mathcal{E}(u_{\varepsilon_j}\circ\psi_{X[a^j]}^{-t}) \ge 0\,;$$

the convergence of the coefficients $Q_{\varepsilon_j}^{kl}$ to Q_0^{kl} then immediately shows that $Q_0(a) \ge 0$ for some $a \in \mathbb{S}^m$ as well. □

4.3.1 Allen-Cahn limits are viscosity solutions

The rest of this section is devoted to proving that the sets constructed as limits of solutions to the Allen-Cahn equation (which were shown to be critical points of the fractional perimeter under inner variations) are actually viscosity solutions to the NMS equation.

Proposition 4.3.1. *Let $s \in (0,1)$ and assume that u_{ε_j} are solutions to the A-C equation (2.7) on M, with parameters $\varepsilon_j \to 0$ and Morse index $m(u_{\varepsilon_j}) \le m$, and that $u_{\varepsilon_j} \to u_0 := \chi_E - \chi_{E^c}$ in $H^{s/2}(M)$. Then ∂E is a viscosity solution of the NMS equation.*

That is: whenever $p \in \partial E$, and $\varphi : \mathcal{B}_{\varrho_\circ}(0) \to V$ is a diffeomorphism from $\mathcal{B}_{\varrho_\circ}$ to an open neighborhood $V \subset M$ of p satisfying $\varphi(0) = p$ and $V^+ := \varphi(\mathcal{B}_{\varrho_\circ}^+) \subset E$ (where we denote $\mathcal{B}_r^+ := \mathcal{B}_r \cap \{x_n > 0\}$) we have

$$\lim_{r\downarrow 0}\int_{M\setminus B_r(p)}(\chi_F - \chi_{F^c})(q)\mathcal{K}_s(p,q)\,dV_q \le 0, \qquad \textit{for } F = V^+ \cup (E\setminus V).$$

Let us recall some preliminary facts.

Lemma 4.3.2 ([PSV13]). *There exists a unique increasing function $v_\circ : \mathbb{R} \to (-1,1)$ with $v_\circ(0) = 0$ that solves $(-\Delta)^s(v_\circ) + W'(v_\circ) = 0$ in $\mathbb{R}$.*

Remark 4.3.3. *Let A be any symmetric positive definite matrix with $A_{in} = \delta_{in}$, $1 \le i \le n$. Defining $v_{\varepsilon,\tau}(x) = v_\circ\big(\varepsilon^{-1}(x_n - \tau)\big)$ (where $v_\circ$ is the function of Lemma 4.3.2) we have*

$$\alpha_{n,s} \int_{\mathbb{R}^n} \frac{\big(v_{\varepsilon,\tau}(x) - v_{\varepsilon,\tau}(y)\big)}{|A(x-y)|^{n+s}} |A| dy + \varepsilon_j^{-s} W'(v_{\varepsilon,\tau}) = 0,$$

where $|A|$ denotes the determinant of A

Remark 4.3.4. *We will implicitly use the following fact many times. Let $\varphi : \mathcal{B}_{r_\circ}(0) \to M$ be a diffeomorphism onto its image with $\varphi(0) = p$, and let $F \subset M$ be a measurable set. Then, for $s \in (0,1)$ the limit*

$$\lim_{r \downarrow 0} \int_{M \setminus B_r(p)} (\chi_F - \chi_{F^c})(q) \mathcal{K}_s(p,q)\, dV_q$$

exists if and only if the limit

$$\lim_{r \downarrow 0} \int_{M \setminus \varphi(\mathcal{B}_r(0))} (\chi_F - \chi_{F^c})(q) \mathcal{K}_s(p,q)\, dV_q$$

exists, and if they do exist they coincide. This is not due to cancellations and can be seen as follows: for r sufficiently small (so that $\mathrm{FA}_1(M,g,p,r,\varphi)$ holds), by Lemma 3.4.13 we can estimate

$$\begin{aligned}\Big| \int_{M \setminus B_r(p)} (\chi_F - \chi_{F^c})(q) \mathcal{K}_s(p,q)\, dV_q - \int_{M \setminus \varphi(\mathcal{B}_r(0))} (\chi_F - \chi_{F^c})(q) \mathcal{K}_s(p,q)\, dV_q \Big| \\ \le C \int_{B_r(p) \triangle \varphi(\mathcal{B}_r(0))} \frac{1}{d(q,p)^{n+s}}\, dV_q \le \frac{C}{r^{n+s}} Vol\big(B_r(p) \triangle \varphi(\mathcal{B}_r(0))\big) \to 0,\end{aligned}$$

as $r \to 0^+$, since $Vol\big(B_r(p) \triangle \varphi(\mathcal{B}_r(0))\big) = O(r^{n+1})$ for small r.

Proof of Proposition 4.3.1. We suppose by contradiction that for some p and $\varphi : \mathcal{B}_{\varrho_\circ} \to V$ as in the statement of the proposition we had

$$\lim_{r \downarrow 0} \int_{M \setminus B_r(p)} (\chi_F - \chi_{F^c})(q) \mathcal{K}_s(p,q)\, dV_q \ge 2\delta > 0, \qquad \text{for } F = V^+ \cup (E \setminus V). \tag{4.23}$$

Our goal is to obtain a contradiction.

Let us make the following useful observation that we will use several times throughout the proof. Let $\psi := \mathcal{B}_\varrho \to W \subset V$ be another diffeomorphism with $\psi(0) = p$ such that $V^+ \cap W \subset \psi(\mathcal{B}_\rho^+)$. Put $G = \psi(\mathcal{B}_\varrho^+) \cup (E \setminus \psi(\mathcal{B}_\varrho))$; then

$$(\chi_G - \chi_{G^c})(q) \ge (\chi_F - \chi_{F^c})(q) \quad \text{for all } q \in M.$$

Hence, the integral (4.23) only grows when replacing F by G. In particular, this applies to "restrictions of domain", such as $\psi = \varphi|_{\mathcal{B}_\varrho}$ for any $\varrho < \varrho_\circ$.

Step 1. Setting $x = (x', x_n)$, we claim that we can replace F by

$$F_t := \varphi(\{x \in \mathcal{B}_{\varrho_\circ} : x_n > t|x'|^2\}) \cup (E \setminus V)$$

in (4.23), for $t > 0$ sufficiently small, provided that we also replace 2δ by δ. In fact, this is a consequence of the fact that

$$f(t) := \lim_{r\downarrow 0} \int_{M\setminus B_r(p)} (\chi_{F_t} - \chi_{F_t^c})(q)\mathcal{K}_s(p,q)\,dV_q$$

is continuous in t. Since $f(0) \geq 2\delta$, for $t > 0$ sufficiently small, we will still have $f(t) \geq \delta > 0$. We now prove that f is continuous.

Fix $0 < \sigma < t$, ϱ sufficiently small so that $\mathrm{FA}_1(M, g, p, 2\varrho, \varphi)$ holds (here we use the observation at the beginning of the proof regarding the domain restriction) and $r \ll \varrho$. Let

$$S := F_\sigma \setminus F_t \subset V = \varphi(\mathcal{B}_\varrho).$$

We have

$$\begin{aligned}
&\left| \int_{M\setminus B_r(p)} (\chi_{F_\sigma} - \chi_{F_\sigma{}^c})(q)\mathcal{K}_s(p,q)\,dV_q - \int_{M\setminus B_r(p)} (\chi_{F_t} - \chi_{F_t{}^c})(q)\mathcal{K}_s(p,q)\,dV_q \right| \\
&\qquad = 2\int_{V\setminus B_r(p)} \chi_S(q)\mathcal{K}_s(p,q)\,dV_q \\
&\qquad = 2\int_{\mathcal{B}_\varrho\setminus\varphi^{-1}(B_r(p))} \chi_{\varphi^{-1}(S)}(z)\mathcal{K}_s(p,\varphi(z))|J\varphi|\,dz \\
&\qquad \leq C\int_{\mathcal{B}_\varrho\setminus\mathcal{B}_{r/2}} \frac{\chi_{\varphi^{-1}(S)}(z)}{|z|^{n+s}}\,dz, \qquad (4.24)
\end{aligned}$$

where we have computed the integral in coordinates φ^{-1} and we have used Lemma 3.4.13 to estimate the kernel $\mathcal{K}_s(p,\varphi(z)) = \mathcal{K}_s(\varphi(0),\varphi(z))$. By the very definition of S, for $0 < R < \varrho$, it follows that $\mathcal{H}^{n-1}(\varphi^{-1}(S)\cap\partial\mathcal{B}_R) \leq CR^{n-2}\cdot C|t-\sigma|R^2 = C|t-\sigma|R^n$. Hence, by polar coordinates

$$\begin{aligned}
\int_{\mathcal{B}_\varrho\setminus\mathcal{B}_{r/2}} \frac{\chi_{\varphi^{-1}(S)}(z)}{|z|^{n+s}}\,dz &= \int_{r/2}^{\varrho} \frac{1}{R^{n+s}}\mathcal{H}^{n-1}(\varphi^{-1}(S)\cap\partial\mathcal{B}_R)\,dR \\
&\leq C|t-\sigma|\int_{r/2}^{\varrho} R^{-s}\,dR = \frac{C}{1-s}|t-\sigma|(\varrho^{1-s} - (r/2)^{1-s}).
\end{aligned}$$

Thus, letting $r \to 0^+$ in (4.24) gives

$$|f(t) - f(\sigma)| \leq C|t-\sigma|\varrho^{1-s}.$$

In particular, f is continuous and this concludes Step 1.

Next, fix $t = t_\circ > 0$ small and choose "Fermi coordinates" adapted to the hypersurface $\Gamma := \varphi(\{x_n = t_\circ|x'|^2\})$ around p. More precisely: there exists a diffeomorphism $\psi : \mathcal{B}_{\varrho_1} \to W = \psi(\mathcal{B}_{\varrho_1})$, with $\psi(0) = p$ and $W \subset V$ open neighborhood of p, such that for all $x \in \mathcal{B}_{\varrho_1}$,

$$d(\psi(x),\Gamma) = \begin{cases} x_n & \text{if } x_n \geq 0 \\ -x_n & \text{if } x_n \leq 0 \end{cases}$$

and $\psi(\mathcal{B}_{\varrho_1}^+) = W \cap \varphi(\mathcal{B}_{\varrho_\circ} \cap \{x_n > t|x'|^2\})$.

Moreover, since $G := \psi(\mathcal{B}_{\varrho_1}^+) \cup (E \setminus W)$ contains F_t we have

$$\lim_{r\downarrow 0} \int_{M\setminus B_r(p)} (\chi_G - \chi_{G^c})(q)K_s(p,q)\,dV_q \geq \delta > 0\,. \qquad (4.25)$$

Also, by construction we have

$$\psi(\mathcal{B}_{\varrho_1} \cap \{x_n \geq -c|x'|^2\}) \subset E,$$

where $c > 0$ depends on $t_\circ$.

Step 2. We now perform a key computation in coordinates. Let us now choose a smooth cutoff function $\eta : \mathbb{R}^n \to \mathbb{R}_+$ satisfying $\chi_{\mathcal{B}_1} \leq \eta \leq \chi_{\mathcal{B}_2}$ and put

$$\eta_\varrho(x) := \eta(x/\varrho) \quad \text{and} \quad \bar{\eta}_\varrho := \eta_\varrho \circ \psi^{-1}. \tag{4.26}$$

Let $K(x,y) = K_s(\varphi(x), \varphi(y))$ be the expression in coordinates ψ^{-1} of the kernel $K_s(p,q)$ for $p,q \in W$, that is, for $x,y \in \mathcal{B}_1$. Let $g_{ij} : \mathcal{B}_{\varrho_1} \to \mathbb{R}^{n^2}$ denote the components of the metric in the coordinates ψ^{-1}. Since ψ^{-1} are Fermi coordinates, we have

$$g_{ni} = g_{in} = \delta_{ni}, \quad 1 \leq i \leq n. \tag{4.27}$$

This will be crucially used later.

Fix $\varrho \in (0, \varrho_1/2)$ small to be chosen later. By (4.25), we have for $G_\varrho := \psi(\mathcal{B}_\varrho^+) \cup (E \setminus \psi(\mathcal{B}_\varrho))$ and $H := \{x_n > 0\} \subset \mathbb{R}^n$

$$\begin{aligned}\lim_{r\downarrow 0} \int_{\mathcal{B}_{2\varrho}\setminus\mathcal{B}_r} (\chi_H - \chi_{H^c})(y) K(0,y) \eta_\varrho(y) \sqrt{|g|}(y)\, dy &= \lim_{r\downarrow 0} \int_{M\setminus B_r(p)} (\chi_{G_\varrho} - \chi_{G_\varrho^c})(q) K_s(p,q) \bar{\eta}_\varrho(q)\, dV_q \\ &\geq \delta - \int_{M\setminus\psi(\mathcal{B}_\varrho)} (\chi_E - \chi_{E^c})(q) K_s(p,q)(1-\bar{\eta}_\varrho)(q)\, dV_q.\end{aligned}$$

Notice that $\mathcal{B}_r(0)$ is not the same as $\varphi^{-1}(B_r(p))$, however as it will become clear from the proof below, the limits as $r \downarrow 0$ of the corresponding integrals give the same value.

Let us also write

$$K(x,y)\sqrt{|g|}(y) = \frac{\alpha_{n,s}}{|A(x)(x-y)|^{n+s}} \sqrt{|g|}(x) + \widehat{K}(x,y),$$

where $A(x)$ is the nonnegative definite symmetric square root of the matrix $(g_{ij}(x))$. Notice that, thanks to (4.27), we have $A_{ni}(x) = A_{in}(x) = \delta_{ni}$ for all $1 \leq i \leq n$. Also, by Proposition 3.4.6 the kernel $\widehat{K}(x,y)$ is not singular, in the sense that

$$\big|\widehat{K}(x,y)\big| \leq C(1 + |x-y|^{-n-s+1}).$$

We thus have

$$\begin{aligned}\alpha_{n,s} \lim_{r\downarrow 0} \int_{\mathcal{B}_{\varrho_1}\setminus\mathcal{B}_r} \frac{(\chi_H - \chi_{H^c})\eta_\varrho(y)}{|A(0)(0-y)|^{n+s}} \sqrt{|g|}(0) dy \geq \delta &- \int_{M\setminus\psi(\mathcal{B}_\varrho)} (\chi_E - \chi_{E^c})(q) K_s(p,q)(1-\bar{\eta}_\varrho)(q)\, dV_q \\ &- \int_{\mathcal{B}_{\varrho_1}} (\chi_H - \chi_{H^c})(y) \widehat{K}(0,y) \eta_\varrho(y)\, dy\end{aligned} \tag{4.28}$$

Let us now recall the assumption that $u_{\varepsilon_j} \to u_0 = \chi_E - \chi_{E^c}$, and let us define for x in a neighbourhood of 0

$$f_j(x) = -\int_{M\setminus\psi(\mathcal{B}_\varrho)} u_{\varepsilon_j}(q) K(\psi(x), q)(1-\bar{\eta}_\varrho)(q)\, dV_q - \int_{\mathcal{B}_{2\varrho}} (u_{\varepsilon_j} \circ \psi)(y) \widehat{K}(x,y) \eta_\varrho(y)\, dy$$

and

$$f_\infty(x) = -\int_{M\setminus\psi(\mathcal{B}_\varrho)} (\chi_E-\chi_{E^c})(q)K(\psi(x),q)(1-\bar\eta_\varrho)(q)\,dV_q - \int_{\mathcal{B}_{2\varrho}} (\chi_E-\chi_{E^c})(\psi(y))\widehat{K}(x,y)\eta_\varrho(y)\,dy$$

Define also

$$I(\varrho) := \int_{\mathcal{B}_{2\varrho}} (\chi_E - \chi_{E^c})(\psi(y))\widehat{K}(0,y)\eta_\varrho(y)\,dy - \int_{\mathcal{B}_{2\varrho}} (\chi_H - \chi_{H^c})(y)\widehat{K}(0,y)\eta_\varrho(y)\,dy$$

Fixing $\varrho > 0$ small enough, we will have $|I(\varrho)| \le \delta/4$. Then, is not difficult to show (using the kernel bounds of Proposition 3.4.6 and of Lemma 3.4.13) that $f_j(x) \to f_\infty(x)$ uniformly for all x in a neighborhood of 0, and that f_∞ is continuous in a neighborhood of 0. As a consequence, we have $|f_j(x) - f_\infty(0)| < \delta/4$ for all $x \in \mathcal{B}_{r_\circ}(0)$ and $j \ge j_\circ$, for some $j_\circ$.

On the other hand, recall that $(-\Delta)^s u_{\varepsilon_j} + \varepsilon_j^{-s} W'(u_{\varepsilon_j}) = 0$ in M. Hence, in particular

$$\lim_{r\downarrow 0}\int_{M\setminus B_r(\psi(x))} \big(u_{\varepsilon_j}(\psi(x)) - u_{\varepsilon_j}(q)\big)K_s(p,q)\,dV_q + \varepsilon_j^{-s}W'(u_{\varepsilon_j}(\psi(x))) = 0$$

for all $x \in \mathcal{B}_{r_\circ}(0)$. Proceeding similarly the previous equation can be rewritten as

$$\alpha_{n,s}\int_{\mathcal{B}_\varrho} \frac{\big((u_{\varepsilon_j}\circ\psi)(x) - (u_{\varepsilon_j}\circ\psi)(y)\big)\eta_\varrho(y)}{|A(x)(x-y)|^{n+s}}\sqrt{|g|}(x)\,dy + \varepsilon_j^{-s}W'(u_{\varepsilon_j}(\psi(x))) = f_j(x)\,. \qquad (4.29)$$

Notice also that (4.28) can be rewritten as

$$\alpha_{n,s}\lim_{r\downarrow 0}\int_{\mathcal{B}_{2\varrho}\setminus\mathcal{B}_r} \frac{(\chi_H - \chi_{H^c})\eta_\varrho(y)}{|A(0)(0-y)|^{n+s}}\sqrt{|g|}(0)\,dy \ge \delta + f_\infty(0) + I(\varrho)\,.$$

We now define $v_{\varepsilon,\tau}(x) = v_\circ\big(\varepsilon^{-1}(x_n-\tau)\big)$, where $v_\circ : \mathbb{R} \to (-1,1)$ is the function from Lemma 4.3.2. In view of Remark 4.3.3, we have for $x \in \mathcal{B}_{r_\circ}(0)$, j large, and $|\tau|$ sufficiently small,

$$\alpha_{n,s}\int_{\mathcal{B}_{2\varrho}} \frac{\big(v_{\varepsilon_j,\tau}(x) - v_{\varepsilon_j,\tau}(y)\big)\eta_\varrho(y)}{|A(x)(x-y)|^{n+s}}\sqrt{|g|}(x)\,dy + \varepsilon_j^{-s}W'(v_{\varepsilon_j,\tau}) \le \frac{\delta}{4}.$$

This implies that whenever $x \in \mathcal{B}_{r_\circ}$, j sufficiently large, and $|\tau|$ sufficiently small

$$\alpha_{n,s}\int_{\mathcal{B}_{2\varrho}} \frac{\big(v_{\varepsilon_j,\tau}(x) - v_{\varepsilon_j,\tau}(y)\big)\eta_\varrho(y)}{|A(x)(x-y)|^{n+s}}\sqrt{|g|}(x)\,dy + \varepsilon_j^{-s}W'(v_{\varepsilon_j,\tau}) \le f_j(x) - \frac{\delta}{4}. \qquad (4.30)$$

In other words, we have shown that $v_{\varepsilon_j,\tau}$ is a strict subsolution of (4.29).

Step 3. We now reach the desired contradiction. Fix now $\theta \in \big(0, \frac{1}{100}\big)$ sufficiently small (to be chosen) and let

$$\xi_\theta(t) := \begin{cases} -1+\theta & \text{if } t \in [-1,-1+\theta], \\ t & \text{if } t \in [-1+\theta, 1-\theta], \\ 1-\theta & \text{if } t \in [1-\theta, 1]. \end{cases}$$

By the Hausdorff convergence of the level sets of u_{ε_j} which we have proved in Step 3 at page 109, for any $t \in [-1+\theta, 1-\theta]$ the set $\{x \in \mathcal{B}_{2\varrho} : (u_{\varepsilon_j}\circ\varphi) \ge t\}$ converges in Hausdorff distance towards $\psi^{-1}(E) \supset \{x \in \mathcal{B}_{2\varrho} \ :\ x_n \le -c|x'|^2\}$. Hence, for every fixed $\tau > 0$ we have, for all j

sufficiently large,

$$\xi_\theta \circ u_{\varepsilon_j} \circ \psi \geq \xi_\theta \circ v_{\varepsilon_j,\tau} \quad \text{in } \overline{\mathcal{B}_{2\varrho}}\,. \tag{4.31}$$

Let us define

$$\tau_j := \min \left\{ \tau \in \mathbb{R} \mid (4.31) \text{ holds for } j \right\}.$$

Notice that by definition of τ_j there is $x_j \in \overline{\mathcal{B}_{2\varrho}} \cap \{|u_{\varepsilon_j}| \leq 1-\theta\} \cap \{|v_{\varepsilon_j,\tau}| \leq 1-\theta\}$. By the previous Hausdorff convergence property of level sets, it must be $x_j \to 0$ and $\tau_j \to 0$ as $j \to \infty$.

Let us show that, if θ is chosen sufficiently small, we have

$$u_{\varepsilon_j} \circ \psi \geq v_{\varepsilon_j,\tau_j} \quad \text{in } \mathcal{B}_{r_\circ/2}\,. \tag{4.32}$$

Indeed, thanks to (4.29)-(4.30) the difference

$$w := u_{\varepsilon_j} \circ \psi - v_{\varepsilon_j,\tau_j}$$

satisfies

$$\mathcal{L}w(x) := \alpha_{n,s} \int_{\mathcal{B}_\varrho} \frac{\big(w(x)-w(y)\big)\eta_\varrho(y)}{|A(x)(x-y)|^{n+s}} \sqrt{|g|}(x)\,dy \geq \frac{\delta}{4} + \varepsilon_j^{-s}\big(W'(v_{\varepsilon_j,\tau_j}) - W'(u_{\varepsilon_j}\circ\psi)\big)(x) \quad \text{in } \mathcal{B}_{r_\circ}. \tag{4.33}$$

Notice that since (4.31) holds for $\tau = \tau_j$ we have $w = v_{\varepsilon_j,\tau_j} - (u_{\varepsilon_j} \circ \varphi) \geq -\theta$ in $\mathcal{B}_{2\varrho}$.

Assume now by contradiction that $\inf_{\mathcal{B}_{r_\circ/2}} w < 0$. Recall (4.26) and define

$$\overline{\eta}_t = -\theta + t\eta_{r_\circ/2}\,.$$

and let $t_* \in [0,\theta)$ be the supremum of the $t \geq 0$ such that $w \geq \overline{\eta}_t$ in $\mathcal{B}_{2\varrho}$. By construction there exists $x_* \in \mathcal{B}_{r_\circ}$ such that

$$(w - \overline{\eta}_{t_*})(x_*) = 0 \quad \text{while} \quad w - \overline{\eta}_{t_*} \geq 0 \quad \text{in } \mathcal{B}_{2\varrho}.$$

Now evaluating the integro-differential operator $\mathcal{L}$ (whose kernel is supported in $B_{2\varrho}$; see (4.33)) at the point x_* we obtain

$$C\theta \geq \mathcal{L}\overline{\eta}_{t_*}(x_*) \geq \mathcal{L}w(x_\circ) \geq \frac{\delta}{4} + \varepsilon_j^{-s}\big(W'(v_{\varepsilon_j,\tau_j}) - W'(u_{\varepsilon_j}\circ\varphi)\big)(x_\circ) \geq \frac{\delta}{4},$$

Notice that $W'' > 0$ in the interval $[u_{\varepsilon_j} \circ \varphi(x_\circ), v_{\varepsilon_j,\tau_j}(x_\circ)]$ because (4.31) holds for $\tau = \tau_j$, and hence either $u_{\varepsilon_j} \circ \varphi(x_\circ) \geq 1-\theta$ or $v_{\varepsilon_j,\tau_j}(x_\circ) \leq -1+\theta$. Therefore, choosing $\theta > 0$ sufficiently small so that $C\theta < \delta/4$ we reach a contradiction. Hence, we have proved that $w \geq 0$ and (4.32) holds.

Finally, take j large so that $x_j \in \mathcal{B}_{r_\circ/4}$ (recall that $x_j \to 0$ as $j \to \infty$). Using that $w \geq 0$ in $\mathcal{B}_{r_\circ/2}$, $w(x_j) = 0$, and $w \geq -\theta$ in $\mathcal{B}_{2\varrho} \setminus \mathcal{B}_{r_\circ}$ and evaluating $\mathcal{L}w$ at the point $x_j \in \mathcal{B}_{r_\circ/4}$ we obtain, similarly as before

$$C(r_\circ)\theta \geq -\alpha_{n,s} \int_{\mathcal{B}_\varrho} \frac{w(y)\eta_\varrho(y)}{|A(x_j)(x_j-y)|^{n+s}} \sqrt{|g|}(x_j)\,dy = \mathcal{L}w(x_j) \geq \frac{\delta}{4}.$$

Choosing $\theta > 0$ sufficiently small, we obtain a contradiction, and this completes the proof. □

The "surfaces" Σ belonging to the class $\mathcal{A}_m(M)$ enjoy some additional properties compared to those already described in Theorem 1.2.17 and Proposition 4.3.1. We record them in the following remark.

Remark 4.3.5. *Every $\Sigma = \partial E \in \mathcal{A}_m(M)$ also satisfies that if $\mathrm{FA}_2(M, g, R, p, \varphi)$ is satisfied, then the following hold:*

(1) ***BV and energy estimate.*** *For some $C = C(n, s, m) > 0$ there holds*

$$\mathrm{Per}(E; B_{R/2}(p)) \leq CR^{n-1} \quad \text{and} \quad \mathrm{Per}_s(E; B_{R/2}(p)) \leq CR^{n-s}.$$

(2) ***Density estimate.*** *For some positive constant ω_0, which depends only on n, s and m, we have that if $R^{-n}|E \cap B_R(p)| \leq \omega_0$ then $|E \cap B_{R/2}(p)| = 0$.*

Indeed, by Definition 1.2.7 of $\mathcal{A}_m(M)$ we can find a sequence u_{ε_j}, made of A-C solutions with Morse index $\leq m$ and parameters $\varepsilon_j \to 0$, converging to E in $L^1(M)$, and also in $H^{s/2}(M)$ thanks to Theorem 1.2.17. Then, property (1) follows from the lower semicontinuity of the BV norm under L^1 convergence and the convergence of Sobolev energies under strong $H^{s/2}$ convergence, together with the fact that the u_{ε_j} satisfy uniform BV and Sobolev estimates themselves by Theorems 1.2.13 and 1.2.15. Similarly, property (2) follows from the L^1 convergence and the density estimates of Proposition 1.2.16 satisfied by the u_{ε_j} themselves.

4.4 The Yau conjecture for nonlocal minimal surfaces

We can now combine the existence and convergence results in the previous sections to prove the Yau conjecture for nonlocal minimal surfaces.

Proof of Theorem 1.2.4. Fix $\mathfrak{p} \in \mathbb{N}$. Theorem 1.2.12 gives the existence, for all $\varepsilon \in (0, \varepsilon_{\mathfrak{p}})$, of a solution $u_{\varepsilon,\mathfrak{p}}$ to the fractional Allen-Cahn equation with Morse index $m(u_{\varepsilon,p}) \leq \mathfrak{p}$ and energy bounds

$$C^{-1}\mathfrak{p}^{s/n} \leq (1-s)\mathcal{E}_M^{\varepsilon}(u_{\varepsilon,\mathfrak{p}}) \leq C\mathfrak{p}^{s/n}. \tag{4.34}$$

Thanks to the convergence result in Theorem 1.2.17, we can find a subsequence $\{\varepsilon_j\}_j$ such that the $u_{\varepsilon_j,\mathfrak{p}}$ converge in $H^{s/2}(M)$ to a limit function

$$u_{0,\mathfrak{p}} = \chi_{E^{\mathfrak{p}}} - \chi_{M \setminus E^{\mathfrak{p}}},$$

where $\partial E^{\mathfrak{p}}$ is an s-minimal surface and $\partial E^{\mathfrak{p}} \in \mathcal{A}_{\mathfrak{p}}(M)$ by definition. Moreover, by (4.34) and the strong convergence of the Allen-Cahn energies stated in Theorem 1.2.17, we deduce that the fractional perimeter of $E^{\mathfrak{p}}$ satisfies the bounds

$$C^{-1}\mathfrak{p}^{s/n} \leq (1-s)\mathrm{Per}_s(E^{\mathfrak{p}}) \leq C\mathfrak{p}^{s/n}.$$

In particular, the fractional perimeter of the $E^{\mathfrak{p}}$ goes to infinity as $\mathfrak{p} \to \infty$, whence we conclude that the family $\{E^{\mathfrak{p}}\}_{\mathfrak{p}\in\mathbb{N}}$ is infinite. □

Remark 4.4.1. *We remark that, unlike in [GG18] or [MN17b], due to the strong convergence as $\varepsilon \to 0$ there is no multiplicity phenomenon in the following sense. Two sets $E^{\mathfrak{p}}$ and $E^{\mathfrak{p}'}$ as in the proof of Theorem 1.2.4 above, respectively limits of the sequences $u_{\varepsilon,\mathfrak{p}}$ and $u_{\varepsilon,\mathfrak{p}'}$ as $\varepsilon \to 0$, for which*

$$\lim_{\varepsilon\to 0}(1-s)\mathcal{E}_M^{\varepsilon}(u_{\varepsilon,\mathfrak{p}}) \neq \lim_{\varepsilon\to 0}(1-s)\mathcal{E}_M^{\varepsilon}(u_{\varepsilon,\mathfrak{p}'}),$$

are necessarily distinct. Hence, their boundaries correspond to geometrically distinct s-minimal surfaces. This does not prevent, for example, that $E^{\mathfrak{p}} = E^{\mathfrak{p}+1}$ for some $\mathfrak{p}$ since the above bounds do not prevent that $(1-s)\mathrm{Per}_s(E^{\mathfrak{p}}) = (1-s)\mathrm{Per}_s(E^{\mathfrak{p}+1})$.

4.5 Regularity and rigidity results

4.5.1 Blow-up procedure

The goal of this subsection is to explicitly show how to perform blow-ups of (sequences of) s-minimal surfaces around points with flatness assumptions, proving strong convergence results for the blow-up sequence to a Euclidean limit surface in a manner similar to Section 4.3.

Definition 4.5.1 (**Blow-up sequence**). *Let $(M_j, g^{(j)})$ be a sequence of closed manifolds of dimension n, and let $p_j \in M_j$ be points such that M_j satisfies the flatness assumption $\mathrm{FA}_3(M_j, g^{(j)}, 1, p_j, \varphi_j)$. Suppose in addition that $g^{(j)}_{kl}(0) = \delta_{kl}$, i.e. that the metric of M_j with respect to the chart φ_j^{-1} at the point 0 is the Euclidean metric.*

For each j, let ∂E_j be an s-minimal surface in M_j, satisfying uniform BV estimates in the sense that there is some C_0 independent of j such that

$$Per(\varphi_j^{-1}(E_j); B_r(x)) \le C_0 r^{n-1} \quad \text{for all } x \in \mathcal{B}_{1/2} \text{ and } r \in (0, 1/4)\,,$$

where we put $\varphi_j^{-1}(E_j) := \{y \in \mathcal{B}_1 : \varphi_j(y) \in E_j\}$.
Given $r_j \searrow 0$, a sequence of subsets of $\mathbb{R}^n$ of the form

$$F_j := \frac{1}{r_j}\varphi_j^{-1}(E_j) \subset \mathcal{B}_{1/r_j} \subset \mathbb{R}^n$$

(for some M_j, p_j, E_j as above) is called a blow-up sequence.

Remark 4.5.2. *F_j is a blow-up sequence if and only if there exist $(\widehat{M}_j, \widehat{g}^{(j)})$, $\widehat{p}_j \in \widehat{M}_j$, and $R_j \nearrow \infty$ such that*

- $\mathrm{Per}(F_j; B_r(x)) \le C_0 r^{n-1}$ *for all $x \in \mathcal{B}_{R_j/2}$ and $r \in (0, R_j/4)$;*
- $\mathrm{FA}_3(\widehat{M}_j, \widehat{g}^{(j)}, R_j, \widehat{p}_j, \widehat{\varphi}_j)$ *holds and $\widehat{g}^{(j)}_{kl}(0) = \delta_{kl}$, where $\widehat{g}^{(j)}_{kl} = \widehat{g}^{(j)}((\widehat{\varphi}_j)_*(e_k), (\widehat{\varphi}_j)_*(e_l))$ denotes the metric in coordinates;*
- *For each j there is an s-minimal surface $\partial \widehat{E}_j$ in $\widehat{M}_j$ such that $F_j = \widehat{\varphi}_j^{-1}(E_j)$.*

Proof of the remark. This follows from putting $\widehat{M}_j = M_j$, $\widehat{p}_j = p_j$, $\widehat{g}^{(j)} = \frac{1}{r_j^2} g^{(j)}$ and $R_j = \frac{1}{r_j}$ in Definition 4.5.1 and considering the scaling properties stated in Remark 3.4.5. □

We record some auxiliary results. The notation $K_{\widehat{M}_j}$ will be used instead of K_s when we want to explicit which manifold the kernel K_s is being considered on.

Proposition 4.5.3. *Let $F_j \subset \mathbb{R}^n$ be a blow-up sequence, with associated $(\widehat{M}_j, \widehat{g}^{(j)})$, $\widehat{E}_j \subset \widehat{M}_j$ and $R_j \to \infty$ as in Remark 4.5.2. The following hold:*

(i) The components $\widehat{g}^{(j)}_{kl}$ of the metric of $\widehat{M}_j$ (using the chart parametrization $\widehat{\varphi}_j$) converge locally uniformly to the Euclidean ones, in the sense that given $R_0 > 0$,

$$\sup_{x \in \mathcal{B}_{R_0}} \left| \widehat{g}^{(j)}_{kl}(x) - \delta_{kl} \right| \to 0 \quad \text{as } j \to \infty.$$

(ii) The kernel $K_{\widehat{M}_j}$ converges locally uniformly to the Euclidean one, in the sense that given $R_0 > 0$,

$$\sup_{(x,y) \in \mathcal{B}_{R_0} \times \mathcal{B}_{R_0}} \left| \frac{K_{\widehat{M}_j}(\widehat{\varphi}_j(x), \widehat{\varphi}_j(y))}{\frac{\alpha_{n,s}}{|x-y|^{n+s}}} - 1 \right| \to 0 \quad \text{as } j \to \infty.$$

Proof. The first part follows from the definition of the flatness assumptions and the fact that $R_j \to \infty$.

As for part *(ii)*, it is a consequence of Proposition 3.4.6. Precisely, it follows from putting $R = R_j$ and $z = y - x$ in (3.38) of Proposition 3.4.6. $\square$

Lemma 4.5.4. *Let $F_j \subset \mathbb{R}^n$ be a blow-up sequence, with associated $(\widehat{M}_j, \widehat{g}^{(j)})$, $\widehat{E}_j \subset \widehat{M}_j$ and $R_j \to \infty$ as in Remark 4.5.2. Put*

$$K_j(x,y) := K_{\widehat{M}_j}(\widehat{\varphi}_j(x), \widehat{\varphi}_j(y))$$

and (for a fixed $\rho < R_j/4$)

$$\mathrm{Per}_s^{(j)}(F_j; \mathcal{B}_\rho) := \frac{1}{4} \iint_{(B_{R_j} \times B_{R_j}) \setminus (B_\rho^c \times B_\rho^c)} |u_j(x) - u_j(y)|^2 K_j(x,y) \sqrt{g^{(j)}}(x) \sqrt{g^{(j)}}(y)\, dx\, dy\,,$$

where $u_j := \chi_{F_j} - \chi_{F_j^c}$.
Given a vector field $X \in C_c^\infty(B_\rho; \mathbb{R}^n)$, define $X_j := (\widehat{\varphi}_j)_* X$ and extend it by zero to a vector field on $\widehat{M}_j$. The following hold:

(1) *Let $0 \le \ell \le 3$. Then*

$$\left| \frac{d^\ell}{dt^\ell} \Big(\mathrm{Per}_s^{\widehat{M}_j}(\psi^t_{X_j}(\widehat{E}_j); \widehat{\varphi}_j(\mathcal{B}_\rho)) - \mathrm{Per}_s^{(j)}(\psi^t_X(F_j); \mathcal{B}_\rho) \Big) \right| \le \frac{C_X}{R_j^s}\,.$$

(2) *If $\chi_{F_j} \to \chi_F$ in $H^{s/2}_{\mathrm{loc}}(\mathbb{R}^n)$, then for $0 \le \ell \le 2$*

$$\frac{d^\ell}{dt^\ell} \mathrm{Per}_s^{(j)}(\psi^t_X(F_j); \mathcal{B}_\rho) \to \frac{d^\ell}{dt^\ell} \mathrm{Per}_s^{\mathbb{R}^n}(\psi^t_X(F); \mathcal{B}_\rho)\,.$$

Proof. We begin by proving (1). Let $v_j^t := \chi_{\psi^t_{X_j}(\widehat{E}_j)} - \chi_{\psi^t_{X_j}(\widehat{E}_j^c)}$ and $u_j^t := \chi_{\psi^t_X(F_j)} - \chi_{\psi^t_X(F_j^c)}$. By splitting the domain of the corresponding integral and then passing to coordinates, we can write

$$\begin{aligned}
\mathrm{Per}_s^{\widehat{M}_j}(\psi^t_{X_j}(\widehat{E}_j); \widehat{\varphi}_j(\mathcal{B}_\rho)) &= \frac{1}{4} \iint_{(\widehat{\varphi}_j(\mathcal{B}_{R_j}) \times \widehat{\varphi}_j(\mathcal{B}_{R_j})) \setminus (\widehat{\varphi}_j(\mathcal{B}_\rho)^c \times \widehat{\varphi}_j(\mathcal{B}_\rho)^c)} |v_j^t(p) - v_j^t(q)|^2\, K_{\widehat{M}_j}(p,q)\, dV_p\, dV_q \\
&\quad + \frac{1}{2} \iint_{\widehat{\varphi}_j(\mathcal{B}_\rho) \times \widehat{\varphi}_j(\mathcal{B}_{R_j})^c} |v_j^t(p) - v_j^t(q)|^2\, K_{\widehat{M}_j}(p,q)\, dV_p\, dV_q \\
&= \frac{1}{4} \iint_{(\mathcal{B}_{R_j} \times \mathcal{B}_{R_j}) \setminus (\mathcal{B}_\rho^c \times \mathcal{B}_\rho^c)} |u_j^t(x) - u_j^t(y)|^2 K_j(x,y) \sqrt{g^{(j)}(x))} \sqrt{g^{(j)}(y))}\, dx\, dy \\
&\quad + \frac{1}{2} \iint_{\widehat{\varphi}_j(\mathcal{B}_\rho) \times \widehat{\varphi}_j(\mathcal{B}_{R_j})^c} |v_j^t(p) - v_j^t(q)|^2\, K_{\widehat{M}_j}(p,q)\, dV_p\, dV_q \\
&= \mathrm{Per}_s^{(j)}(\psi^t_X(F_j); \mathcal{B}_\rho) + \frac{1}{2} \iint_{\widehat{\varphi}_j(\mathcal{B}_\rho) \times \widehat{\varphi}_j(\mathcal{B}_{R_j})^c} |v_j^t(p) - v_j^t(q)|^2\, K_{\widehat{M}_j}(p,q)\, dV_p\, dV_q\,.
\end{aligned}$$

From this computation, changing variables with the flow as in (4.22) and then passing to coordinates in the first variable we can compute

$$\left|\frac{d^\ell}{dt^\ell}\Big(\mathrm{Per}_s^{\widehat{M}_j}(\psi^t_{X_j}(\widehat{E}_j);\widehat{\varphi}_j(\mathcal{B}_\rho)) - \mathrm{Per}_s^{(j)}(\psi^t_X(F_j);\mathcal{B}_\rho)\Big)\right| =$$

$$\begin{aligned}
&= \frac{1}{2}\left|\frac{d^\ell}{dt^\ell}\iint_{\widehat{\varphi}_j(\mathcal{B}_\rho)\times\widehat{\varphi}_j(\mathcal{B}_{R_j})^c} |v_j^t(p)-v_j^t(q)|^2\, K_{\widehat{M}_j}(p,q)\,dV_p\,dV_q\right| \\
&= \frac{1}{2}\left|\iint_{\widehat{\varphi}_j(\mathcal{B}_\rho)\times\widehat{\varphi}_j(\mathcal{B}_{R_j})^c} |v_j(p)-v_j(q)|^2\,\frac{d^\ell}{dt^\ell}\Big[K_{\widehat{M}_j}(\psi^t_{X_j}(p),q)\,J_t(p)\Big]\,dV_p\,dV_q\right| \\
&\le C\iint_{\mathcal{B}_\rho\times\widehat{\varphi}_j(\mathcal{B}_{R_j})^c}\left|\frac{d^\ell}{dt^\ell}\Big[K_{\widehat{M}_j}(\widehat{\varphi}_j(\psi^t_X(x)),q)\,J_t(p)\Big]\right|dx\,dV_q\,.
\end{aligned}$$

Bounding the derivatives in time of the Jacobian $J_t(p)$ by a constant, and using (3.41) with $R = R_j$ to bound the integral in q, we conclude the result in (1).

To see (2), let $R > \rho$ and put $f_j(t) := \mathrm{Per}_s^{(j)}(\psi^t_X(F_j);\mathcal{B}_\rho)$. Changing variables with the flow (as above) and splitting the domain of the integral, for $R < R_j$ we can write

$$\begin{aligned}
&\frac{d^\ell}{dt^\ell}\mathrm{Per}_s^{(j)}(\psi^t_X(F_j);\mathcal{B}_\rho) \\
&= \frac{1}{4}\iint_{(\mathcal{B}_R\times\mathcal{B}_R)\setminus(\mathcal{B}_\rho^c\times\mathcal{B}_\rho^c)} |u_j(x)-u_j(y)|^2\frac{d^\ell}{dt^\ell}\Big[K_j(\psi^t_X(x),\psi^t_X(y))\sqrt{g^{(j)}}(\psi^t_X(x))\sqrt{g^{(j)}}(\psi^t_X(y))J_t(x)J_t(y)\Big]\,dx\,dy \\
&\quad+\frac{1}{2}\iint_{\mathcal{B}_\rho\times(\widehat{\varphi}_j(\mathcal{B}_{R_j})\setminus\widehat{\varphi}_j(\mathcal{B}_R))} |u_j(x)-u_j(\varphi^{-1}(q))|^2\frac{d^\ell}{dt^\ell}\Big[K_{\widehat{M}_j}(\widehat{\varphi}_j(\psi^t_X(x)),q)\sqrt{g^{(j)}}(\psi^t_X(x))J_t(x)\Big]\,dx\,dV_q\,.
\end{aligned}$$

Let $0\le\ell\le 3$. Thanks to the flatness assumptions and (3.41) of Proposition 3.4.6, we can bound

$$\begin{aligned}
&\left|\iint_{\mathcal{B}_\rho\times(\widehat{\varphi}_j(\mathcal{B}_{R_j})\setminus\widehat{\varphi}_j(\mathcal{B}_R))} |u_j(x)-u_j(\varphi^{-1}(q))|^2\frac{d^\ell}{dt^\ell}\Big[K_{\widehat{M}_j}(\widehat{\varphi}_j(\psi^t_X(x)),q)\sqrt{g^{(j)}}(\psi^t_X(x))J_t(x)\Big]\,dx\,dV_q\right| \\
&\qquad\le C\iint_{\mathcal{B}_\rho\times(\widehat{M}_j\setminus\widehat{\varphi}_j(\mathcal{B}_R))}\left|\frac{d^\ell}{dt^\ell}\Big[K_{\widehat{M}_j}(\widehat{\varphi}_j(\psi^t_X(x)),q)\sqrt{g^{(j)}}(\psi^t_X(x))J_t(x)\Big]\right|dx\,dV_q \\
&\qquad\le\frac{C}{R^s}\,. \qquad (4.35)
\end{aligned}$$

On the other hand, by the flatness assumptions and (3.54) in Proposition 3.4.16 we have that, for $t\in(-T,T)$ and j large enough so that $R < R_j/4$,

$$\begin{aligned}
&\iint_{(\mathcal{B}_R\times\mathcal{B}_R)\setminus(\mathcal{B}_\rho^c\times\mathcal{B}_\rho^c)} |u_j(x)-u_j(y)|^2\left|\frac{d^\ell}{dt^\ell}\Big[K_j(\psi^t_X(x),\psi^t_X(y))\sqrt{g^{(j)}}(\psi^t_X(x))\sqrt{g^{(j)}}(\psi^t_X(y))J_t(x)J_t(y)\Big]\right|dx\,dy \\
&\qquad\le C_T\iint_{\mathcal{B}_R\times\mathcal{B}_R}|u_j(x)-u_j(y)|^2\frac{\alpha_{n,s}}{|x-y|^{n+s}}\,dx\,dy \\
&\qquad\le C_T\,,
\end{aligned}$$

where in the last line we combined the fact that F_j has bounded classical perimeter in $B_{R_j/4}$ with the interpolation result in Proposition A.1.1.

This shows that the functions $\frac{d^\ell}{dt^\ell} f_j(t)$ are locally uniformly bounded for $0 \le \ell \le 3$; in particular, for $0 \le \ell \le 2$ we deduce that the $\frac{d^\ell}{dt^\ell} f_j(t)$ are locally uniformly bounded and moreover have a uniform modulus of continuity, thus by Arzelà-Ascoli they subsequentially converge locally uniformly. By standard single-variable calculus, to conclude our desired result it then suffices to show that $f_j(t)$ converges pointwise to $g(t) := \mathrm{Per}_s^{\mathbb{R}^n}(\psi_X^t(F); \mathcal{B}_\rho)$, since then the first two derivatives of the limit function $g(t)$ will be the limits of the derivatives of the $f_j(t)$. We shall now prove the pointwise convergence result.

Denote $u^t := \chi_{\psi_X^t(F)} - \chi_{\psi_X^t(F^c)}$. We can then write

$$\begin{aligned} g(t) &= \mathrm{Per}_s^{\mathbb{R}^n}(\psi_X^t(F); \mathcal{B}_\rho) \\ &= \frac{1}{4} \iint_{(\mathcal{B}_R \times \mathcal{B}_R) \setminus (\mathcal{B}_\rho^c \times \mathcal{B}_\rho^c)} |u(x) - u(y)|^2 \frac{\alpha_{n,s}}{|\psi_X^t(x) - \psi_X^t(y)|^{n+s}} J_t(x) J_t(y) \, dx \, dy \\ &\quad + \frac{1}{2} \iint_{\mathcal{B}_\rho \times \mathcal{B}_R^c} |u(x) - u(y)|^2 \frac{\alpha_{n,s}}{|\psi_X^t(x) - y|^{n+s}} J_t(x) \, dx \, dy \, . \end{aligned}$$

Clearly

$$\iint_{\mathcal{B}_\rho \times \mathcal{B}_R^c} |u(x) - u(y)|^2 \frac{\alpha_{n,s}}{|\psi_X^t(x) - y|^{n+s}} J_t(x) \, dx \, dy \to 0 \quad \text{as } R \to \infty \, ,$$

since the integrand is absolutely integrable by (3.41) in Proposition 3.4.6. Together with (4.35), given $\varepsilon > 0$, we deduce that there exists an $R > \rho$ (depending only on ρ and ε) such that the aforementioned terms are both smaller than $\varepsilon/2$ for all j large enough. From this fact and a simple triangle inequality, we find that

$$\begin{aligned} &\left| \mathrm{Per}_s^{(j)}(\psi_X^t(F_j); \mathcal{B}_\rho) - \mathrm{Per}_s^{\mathbb{R}^n}(\psi_X^t(F); \mathcal{B}_\rho) \right| \\ &\le \frac{1}{4} \iint_{(\mathcal{B}_R \times \mathcal{B}_R) \setminus (\mathcal{B}_\rho^c \times \mathcal{B}_\rho^c)} |u_j(x) - u_j(y)|^2 \cdot \\ &\qquad \cdot \left| K_j(\psi_X^t(x), \psi_X^t(y)) \sqrt{g^{(j)}}(\psi_X^t(x)) \sqrt{g^{(j)}}(\psi_X^t(y)) - \frac{\alpha_{n,s}}{|\psi_X^t(x) - \psi_X^t(y)|^{n+s}} \right| J_t(x) J_t(y) \, dx \, dy \\ &+ \frac{1}{4} \Big| \iint_{(\mathcal{B}_R \times \mathcal{B}_R) \setminus (\mathcal{B}_\rho^c \times \mathcal{B}_\rho^c)} \Big(|u_j(x) - u_j(y)|^2 - |u(x) - u(y)|^2 \Big) \frac{\alpha_{n,s}}{|\psi_X^t(x) - \psi_X^t(y)|^{n+s}} J_t(x) J_t(y) \, dx \, dy \Big| \\ &\quad + \varepsilon \, . \end{aligned}$$

Regarding the first term, thanks to Proposition 4.5.3 and (3.54) in Proposition 3.4.16 it can be bounded as follows:

$$\begin{aligned} &\iint_{(\mathcal{B}_R \times \mathcal{B}_R) \setminus (\mathcal{B}_\rho^c \times \mathcal{B}_\rho^c)} |u_j(x) - u_j(y)|^2 \cdot \\ &\qquad \cdot \left| K_j(\psi_X^t(x), \psi_X^t(y)) \sqrt{g^{(j)}}(\psi_X^t(x)) \sqrt{g^{(j)}}(\psi_X^t(y)) - \frac{\alpha_{n,s}}{|\psi_X^t(x) - \psi_X^t(y)|^{n+s}} \right| J_t(x) J_t(y) \, dx \, dy \\ &= \iint_{(\mathcal{B}_R \times \mathcal{B}_R) \setminus (\mathcal{B}_\rho^c \times \mathcal{B}_\rho^c)} |u_j(x) - u_j(y)|^2 \frac{\alpha_{n,s}}{|\psi_X^t(x) - \psi_X^t(y)|^{n+s}} \cdot \\ &\qquad\qquad \cdot \left| \frac{K_j(\psi_X^t(x), \psi_X^t(y)) \sqrt{g^{(j)}}(x) \sqrt{g^{(j)}}(y)}{\frac{\alpha_{n,s}}{|\psi_X^t(x) - \psi_X^t(y)|^{n+s}}} - 1 \right| J_t(x) J_t(y) \, dx \, dy \end{aligned}$$

$$\leq o_j(1) \iint_{B_R\times B_R} |u_j(x)-u_j(y)|^2 \frac{\alpha_{n,s}}{|\psi^t_X(x)-\psi^t_X(y)|^{n+s}}\,dx\,dy$$
$$\leq o_j(1)\, C_T \iint_{\mathcal{B}_R\times\mathcal{B}_R} |u_j(x)-u_j(y)|^2 \frac{\alpha_{n,s}}{|x-y|^{n+s}}\,dx\,dy\,,$$

where $o_j(1)\to 0$ as $j\to\infty$.

This implies that the whole expression goes to zero since the factor $\iint_{\mathcal{B}_R\times\mathcal{B}_R} |u_j(x)-u_j(y)|^2 \frac{\alpha_{n,s}}{|x-y|^{n+s}}\,dx\,dy$ can be bounded by a constant independent of j: indeed, for j large enough so that $R<R_j/4$, the F_j satisfy uniform perimeter estimates in $\mathcal{B}_R$ by assumption (see Remark 4.5.2), and thus also uniform fractional energy estimates by interpolation (see Proposition A.1.1).

As for the second term, we can write

$$\left| \iint_{(\mathcal{B}_R\times\mathcal{B}_R)\setminus(\mathcal{B}^c_\rho\times\mathcal{B}^c_\rho)} \Big(|u_j(x)-u_j(y)|^2-|u(x)-u(y)|^2\Big) \frac{\alpha_{n,s}}{|\psi^t_X(x)-\psi^t_X(y)|^{n+s}} J_t(x)J_t(y)\,dx\,dy \right| =$$
$$= \left| \iint_{(\mathcal{B}_R\times\mathcal{B}_R)\setminus(\mathcal{B}^c_\rho\times\mathcal{B}^c_\rho)} \Big(|u_j(x)-u_j(y)|^2-|u(x)-u(y)|^2\Big) \frac{\alpha_{n,s}}{|x-y|^{n+s}} \cdot \right.$$
$$\left. \cdot \frac{\frac{\alpha_{n,s}}{|\psi^t_X(x)-\psi^t_X(y)|^{n+s}} J_t(x)J_t(y)}{\frac{\alpha_{n,s}}{|x-y|^{n+s}}}\,dx\,dy \right| . \tag{4.36}$$

Since $u_j\to u$ in $H^{s/2}_{\mathrm{loc}}(\mathbb{R}^n)$ by assumption, one immediately sees that

$$A_j(x,y) := \Big(|u_j(x)-u_j(y)|^2-|u(x)-u(y)|^2\Big)\frac{\alpha_{n,s}}{|x-y|^{n+s}} \to 0 \quad \text{in } L^1_{\mathrm{loc}}\,.$$

On the other hand,

$$B(x,y) := \frac{\frac{\alpha_{n,s}}{|\psi^t_X(x)-\psi^t_X(y)|^{n+s}} J_t(x)J_t(y)}{\frac{\alpha_{n,s}}{|x-y|^{n+s}}}$$

is a fixed function in L^∞_{loc} by (3.54) in Proposition 3.4.16. Thus $A_jB\to 0$ in L^1_{loc}, and this means that (4.36) goes to 0 as $j\to\infty$ as well.

Putting everything together, we deduce that

$$\limsup_{j\to\infty} \left| \mathrm{Per}^{(j)}_s(\psi^t_X(F_j);\mathcal{B}_\rho) - \mathrm{Per}^{\mathbb{R}^n}_s(\psi^t_X(F);\mathcal{B}_\rho)\right| \leq \varepsilon\,;$$

since ε was arbitrary, we conclude that

$$f_j(t) = \mathrm{Per}^{(j)}_s(\psi^t_X(F_j);\mathcal{B}_\rho) \to \mathrm{Per}^{\mathbb{R}^n}_s(\psi^t_X(F);\mathcal{B}_\rho) = g(t)\,.$$

As explained before, this gives the desired result. □

The main result of this section is the following:

Theorem 4.5.5 (Convergence to a limit)**.** *Let $F_j\subset\mathbb{R}^n$ be a blow-up sequence. Then, there exists a Euclidean s-minimal surface $F\subset\mathbb{R}^n$ such that a subsequence of the $v_j := \chi_{F_j}-\chi_{(\mathbb{R}^n\setminus F_j)}$ converges to $v:=\chi_F-\chi_{(\mathbb{R}^n\setminus F)}$ in $H^{s/2}_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. We divide the proof in two steps.

Step 1. Convergence to a limit set F.

Fix a radius R. For j large enough so that $R < R_j/4$, the F_j satisfy a uniform BV estimate in $\mathcal{B}_R$, as indicated in the third bullet of Proposition 4.5.3. As in Step 1 of the proof of Theorem 1.2.17 (see page 108), a bound on the BV norm implies that a subsequence of the $v_j = \chi_{F_j} - \chi_{F_j^c}$ converges strongly in $H^{s/2}(\mathcal{B}_R)$ norm. Iterating the same reasoning on increasingly large balls and using a diagonal selection argument, we can find a subsequence (still denoted by v_j) converging in each of the norms $H^{s/2}(\mathcal{B}_k)$, $k \in \mathbb{N}$, to a limit function $v = \chi_F - \chi_{F^c}$.

Step 2. Proof that F is stationary for the fractional perimeter.
Fix an arbitrary Euclidean vector field $X \in C_c^\infty(\mathcal{B}_\rho)$, for some $\rho > 0$, and let ψ_X^t denote its flow at time t.
Since the F_j are a blow-up sequence, let $\widehat{E}_j \subset \widehat{M}_j$ and $R_j \to \infty$ be those given by Remark 4.5.2. For j large enough so that $\rho < R_j$, define $X_j = (\widehat{\varphi}_j)_*(X)$; extending it by 0, we obtain a vector field X_j defined on all of $\widehat{M}_j$. Since $\partial \widehat{E}_j$ is an s-minimal surface in $\widehat{M}_j$, $\frac{d}{dt}\big|_{t=0} \mathrm{Per}_s^{\widehat{M}_j}(\psi_{X_j}^t(\widehat{E}_j); \widehat{\varphi}_j(\mathcal{B}_\rho)) = 0$. Lemma 4.5.4 gives then that

$$\left| \frac{d}{dt}\bigg|_{t=0} \mathrm{Per}_s^{\mathbb{R}^n}(\psi_X^t(F); \mathcal{B}_\rho) \right| = \lim_{j\to\infty} \left| \frac{d}{dt}\bigg|_{t=0} \mathrm{Per}_s^{(j)}(\psi_X^t(F_j); \mathcal{B}_\rho) \right| \leq \lim_{j\to\infty} \frac{C_X}{R_j^s} = 0$$

as desired. □

We will next prove that the convergence in the theorem also holds in the Hausdorff distance sense. First, we show that the assumptions in Definition 4.5.1 imply uniform density estimates.

Lemma 4.5.6. *Let (M, g) be a closed n-dimensional manifold satisfying the flatness assumption* $\mathrm{FA}_3(M, g, R, p, \varphi)$. *Suppose in addition that $g_{kl}^{(j)}(0) = \delta_{kl}$, i.e. the metric of M with respect to the chart φ^{-1} at the point 0 is the Euclidean metric. Let E be an s-minimal surface in M, satisfying a uniform BV estimate in the sense that there is some C_0 such that*

$$\mathrm{Per}(\varphi^{-1}(E); \mathcal{B}_r(x)) \leq C_0 r^{n-1} \quad \textit{for all } x \in \mathcal{B}_{R/2} \textit{ and } r \in (0, R/4)\,.$$

Then there exists a positive constant $\omega_0 = \omega_0(n, s, C_0)$ such that if

$$r^{-n}|E \cap B_r(q)| \leq \omega_0$$

for some $q \in \varphi(\mathcal{B}_{R/2})$ and $r \in (0, R/8)$, then

$$|E \cap B_{r/2}(q)| = 0\,.$$

Proof. Notice that since the statement is scaling-invariant, it suffices to prove it for $R = 1$. We also recall that the stationarity of E implies that it satisfies the monotonicity formula of Theorem 1.2.2 (with potential $F \equiv 0$). Observe also that up to modifying E on a set of measure zero we can assume that its topological boundary coincides with its essential boundary. We then proceed as follows:

Step 1. Positive density of the extended energy at every boundary point.
Since $\varphi^{-1}(E)$ is a set of finite perimeter in $\mathcal{B}_{1/2}$, De Giorgi's structure theorem for sets of finite perimeter gives that if $x \in \partial\varphi^{-1}(E) \cap \mathcal{B}_{1/2}$ is in the reduced boundary, given $r_j \to 0$ the sequence of sets $H_j = r_j^{-1}(\varphi^{-1}(E) - x)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a half-space H passing through 0.

For a fixed x as above, defining $M_j = M$, $E_j = E$, $p_j = \varphi(x)$, $r_j = \frac{1}{j}$, and $\varphi_j(y) = \varphi(x + A(x)y)$, where $A(x)$ is a matrix chosen so that the metric of M is the identity at 0 in the coordinates given by φ_j, the associated $F_j := \frac{1}{r_j}\varphi_j^{-1}(E) \subset \mathbb{R}^n$ are a blow-up sequence (in the sense of

Definition 4.5.1). Thus, by Theorem 4.5.5 they converge in $L^1_{\rm loc}$ to a limit F. On the other hand, $F_j = A(x)^{-1}\frac{1}{r_j}(\varphi^{-1}(E)-x) = A(x)^{-1}H_j$, so that in fact $F = A(x)^{-1}H$ and thus it is also a hyperplane passing through 0.

Let N_j denote the rescaled manifold $(M, r_j^{-2}g)$, and write $u_j = \chi_E - \chi_{E^c}$, viewed as a function on N_j. Write U_j for its Caffarelli-Silvestre extension to $N_j \times \mathbb{R}_+$, and V for the Caffarelli-Silvestre extension of $\chi_F - \chi_{F^c}$ to $\mathbb{R}^n \times \mathbb{R}_+$. By the lower semicontinuity of the extended Sobolev energy under a blow-up, seen for example arguing as in Step 2 in the proof of Lemma 4.5.16, we have

$$\liminf_{j\to\infty} \int_{\widetilde{B}_1^{N_j}(p_j,0)} |\widetilde{\nabla} U_j(p,z)|^2 z^{1-s}\, dV_p dz \geq \int_{\widetilde{\mathcal{B}}_1} |\widetilde{D}V(x,z)|^2 z^{1-s}\, dx dz = c(n,s) > 0\,. \tag{4.37}$$

If U denotes the Caffarelli-Silvestre extension of $u = \chi_E - \chi_{E^c}$ (viewed as a function on M) to $M \times \mathbb{R}_+$, by scaling (recall that $N_j = (M, r_j^{-2}g)$) the inequality (4.37) can be written as

$$\liminf_{j\to\infty} \frac{1}{r_j^{n-s}} \int_{\widetilde{B}^M_{r_j}(\varphi(x),0)} |\widetilde{\nabla} U(p,z)|^2 z^{1-s}\, dV_p dz \geq c(n,s) > 0\,.$$

In words, we have found that E has extended energy density uniformly bounded from below by a constant $c(n,s)$ at $p = \varphi(x)$, for every reduced boundary point p as above. On the other hand, the reduced boundary is dense in the essential boundary, as one can see, for example, by the isoperimetric inequality. Then, since we have shown that the above lower bound holds at all reduced boundary points $p \in \partial E$, by the upper semicontinuity of the extended energy density (proved as in case of classical minimal surfaces, using the monotonicity formula of Theorem 1.2.2) it actually holds at every $p \in \partial E$.

Step 2. Conclusion.
Assume that there are $q \in \varphi(\mathcal{B}_{1/2})$ and $r \in (0, 1/8)$ such that $r^{-n}|E \cap B_r(q)| \leq \omega_0$ but $|E \cap B_{r/2}(q)| > 0$; if ω_0 is small enough, then automatically also $|E^c \cap B_{r/2}(q)| > 0$. By the isoperimetric inequality, this implies that $\partial E \cap B_{r/2}(q) \neq \emptyset$ as well.

Let $p \in \partial E \cap B_{r/2}(q)$. We can now argue as in the proof of Proposition 1.2.16, which showed density estimates in the case of solutions of the Allen-Cahn equation. First, a uniform lower bound on the density holds in our case for all $\rho \in (0, R_{\rm mon})$, thanks to the combination of Step 1 and the monotonicity formula. We then apply the interpolation Lemma 3.2.10, after which the BV estimate assumption allows us to conclude the argument as in the proof of Proposition 1.2.16. □

Proposition 4.5.7. *In the conclusions of Theorem 4.5.5, the convergence also holds locally in the Hausdorff distance sense.*

Proof. Let E_j be as in Remark 4.5.2, so that $F_j = \widehat{\varphi}_j^{-1}(E_j)$. Applying Lemma 4.5.6 and the flatness assumption on the metric we find that the F_j satisfy density estimates in $\mathcal{B}_{R_j/16}$, with $R_j \to \infty$. The local convergence in the Hausdorff distance follows then arguing by contradiction, simply due to local L^1 convergence to F and the density estimates. □

4.5.2 Properties of blow-ups of Allen-Cahn limits

We define the class of all surfaces which are blow-up limits of sets in $\mathcal{A}_m$ (recall Definitions 1.2.7 and 4.5.1).

Definition 4.5.8. *A set $\partial F \subset \mathbb{R}^n$ is said to be in the class $\mathcal{A}_m^{Blow-up}$ if it is a blow-up limit of sets in $\mathcal{A}_m$. This means that there exist $\Sigma_j = \partial E_j \in \mathcal{A}_m(M_j)$ and $r_j \to 0$ such that the associated $F_j = r_j^{-1}\varphi_j^{-1}(E_j)$ are a blow-up sequence converging to F in $L^1_{\rm loc}(\mathbb{R}^n)$ as $j \to \infty$ (by Theorem*

4.5.5 and Proposition 4.5.7, the convergence can then be upgraded to be in $H^{s/2}_{\mathrm{loc}}(\mathbb{R}^n)$ and locally in the Hausdorff distance sense).

Remark 4.5.9. *Since $\partial E_j \in \mathcal{A}_m(M_j)$, the assumption in Definition 4.5.1 that the sets F_j satisfy the classical perimeter estimates is automatically satisfied if the rest of assumptions are, thanks to (1) in Remark 4.3.5.*

We now prove a precise almost-stability inequality for sets in $\mathcal{A}_m^{Blow-up}$, which will be used in the next section. We begin by showing its counterpart for Allen-Cahn solutions.

Lemma 4.5.10. *Assume that M satisfies the flatness assumptions $\mathrm{FA}_1(M, g, 1, p, \varphi)$, and let $u_\varepsilon : M \to \mathbb{R}$ be a solution to the Allen-Cahn equation in $\varphi(\mathcal{B}_1)$ with Morse index at most m. Let $A_1, ..., A_{m+1} \subset \varphi(\mathcal{B}_{1/2})$ be $(m+1)$ open sets, with pairwise distances denoted by $D_{ij} := \mathrm{dist}(A_i, A_j)$, and for every $1 \le i < j \le m+1$ choose any positive weights $\lambda_{ij} > 0$. Then, in at least one of the A_i there holds that*

$$\mathcal{E}''(u_\varepsilon)[\xi, \xi] \ge -C\|\xi\|^2_{L^1(A_i)} \left(\sum_{j<i} \frac{1}{\lambda_{ji}} D_{ij}^{-(n+s)} + \sum_{j>i} \lambda_{ij} D_{ij}^{-(n+s)} \right) \quad \forall\, \xi \in C^1_c(A_i)\,,$$

for some $C = C(n, s, m)$.

Proof. The statement is a more precise version of Lemma 4.2.2, and the proof proceeds similarly. Using (2.8), we compute the second variation at u_ε for linear combinations of $m+1$ test functions ξ_i, supported each in the corresponding A_i, getting

$$\begin{aligned}
&\mathcal{E}''(u_\varepsilon)[a_1\xi_1 + a_2\xi_2 + \ldots + a_{m+1}\xi_{m+1}, a_1\xi_1 + a_2\xi_2 + \ldots + a_{m+1}\xi_{m+1}] \\
&\quad = a_1^2 \mathcal{E}''(u)[\xi_1, \xi_1] + \ldots + a_{m+1}^2 \mathcal{E}''(u)[\xi_{m+1}, \xi_{m+1}] \\
&\qquad + 2a_1a_2 \iint_{A_1\times A_2} (\xi_1(p) - \xi_1(q))(\xi_2(p) - \xi_2(q)) K_s(p, q)\, dV_p\, dV_q \\
&\qquad + \ldots \\
&\qquad + 2a_m a_{m+1} \iint_{A_m\times A_{m+1}} (\xi_m(p) - \xi_m(q))(\xi_{m+1}(p) - \xi_{m+1}(q)) K_s(p, q)\, dV_p\, dV_q\,.
\end{aligned}$$

Thanks to the flatness assumptions and Lemma 3.4.13, we have that $K_s(\varphi(x), \varphi(y)) \le \frac{C}{|x-y|^{n+s}}$, for some $C = C(n, s)$ and for $(\varphi(x), \varphi(y)) \in A_i \times A_j$. Recall that the supports of ξ_i and ξ_j are the disjoint subsets $A_i, A_j \subset \varphi_j(\mathcal{B}_{1/2})$. Then, the term containing the double integral over $A_i \times A_j$ with $i < j$ can be bounded as follows:

$$\begin{aligned}
2a_ia_j &\iint_{A_i\times A_j} (\xi_i(p) - \xi_i(q))(\xi_j(p) - \xi_j(q)) K_s(p, q)\, dV_p dV_q \\
&= -2a_ia_j \iint_{A_i\times A_j} \xi_i(p)\xi_j(q) K(p, q)\, dV_p\, dV_q \\
&\le 2|a_ia_j| C D_{ij}^{-(n+s)} \|\xi_i\|_{L^1(A_i)} \|\xi_j\|_{L^1(A_j)} \\
&\le \lambda_{ij} a_i^2 C D_{ij}^{-(n+s)} \|\xi_i\|^2_{L^1(A_i)} + \frac{C}{\lambda_{ij}} a_j^2 D_{ij}^{-(n+s)} \|\xi_j\|^2_{L^1(A_j)}\,,
\end{aligned}$$

where we have applied Young's inequality in the last line. Substituting this into the second variation expression gives

$$\mathcal{E}''(u)[a_1\xi_1 + a_2\xi_2 + \ldots + a_{m+1}\xi_{m+1}, a_1\xi_1 + a_2\xi_2 + \ldots + a_{m+1}\xi_{m+1}] \leq \sum_{i=1}^{m+1} a_i^2 \left[\mathcal{E}''(u)[\xi_i,\xi_i] + C\|\xi_i\|^2_{L^1(A_i)} \left(\sum_{j<i} \frac{1}{\lambda_{ji}} D_{ij}^{-(n+s)} + \sum_{j>i} \lambda_{ij} D_{ij}^{-(n+s)}\right)\right].$$

The condition that the Morse index is at most m implies that the expression cannot be < 0 for all $(a_1, \ldots, a_{m+1}) \neq 0$. Hence, we find that there must exist some i such that

$$\mathcal{E}''(u)[\xi_i,\xi_i] \geq -C\|\xi_i\|^2_{L^1(A_i)} \left(\sum_{j<i} \frac{1}{\lambda_{ji}} D_{ij}^{-(n+s)} + \sum_{j>i} \lambda_{ij} D_{ij}^{-(n+s)}\right)$$

holds for all $\xi_i \in C^1_c(A_i)$, and this concludes the proof. □

From this, we will obtain the desired almost-stability inequality for blow-up sets.

Lemma 4.5.11. *Let $F \in \mathcal{A}_m^{Blow-up}$. Let $X_1, X_2, ..., X_{m+1}$ be smooth vector fields on $\mathbb{R}^n$ with disjoint compact supports $A_1, A_2, ..., A_{m+1}$, and denote $D_{k\ell} := \mathrm{dist}(A_k, A_\ell)$. For $1 \leq i < \ell \leq m+1$, choose positive weights $\lambda_{i\ell} > 0$. Then, for at least one of the i (depending on F) we have that*

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Per}_s(\psi^t_{X_i}(F); A_i) \geq -C\|X_i\|^2_{L^\infty} \mathrm{diam}(A_i)^{2(n-1)} \left(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} D_{i\ell}^{-(n+s)} + \sum_{\ell>i} \lambda_{i\ell} D_{i\ell}^{-(n+s)}\right), \tag{4.38}$$

where $C = C(n, s, m)$.

Proof. Since $F \in \mathcal{A}_m^{Blow-up}$, from the definition and proceeding as in Remark 4.5.2 there exist $(\widehat{M}_j, \widehat{g}^{(j)})$, $\widehat{p}_j \in \widehat{M}_j$, and $R_j \nearrow \infty$ satisfying the assumptions in the Remark and $\widehat{E}_j \in \mathcal{A}_m(\widehat{M}_j)$ such that the associated $F_j = \widehat{\varphi}_j^{-1}(\widehat{E}_j)$ converge to F in the appropriate sense. Fix one such j; since $\widehat{E}_j \in \mathcal{A}_m(\widehat{M}_j)$, by definition there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ of solutions to Allen-Cahn on $\widehat{M}_j$, with parameters $\varepsilon_k \to 0$, converging to $\chi_{\widehat{E}_j} - \chi_{\widehat{E}_j^c}$ in $L^1(\widehat{M}_j)$ as $k \to \infty$. By Lemma 4.5.10, given k we can find an index $i(k)$, $1 \leq i(k) \leq m+1$, such that the inequality in the Lemma is true for u_k on $\widehat{\varphi}_j(A_{i(k)}) \subset \widehat{M}_j$. We select an index i so that the inequality is valid for a whole subsequence of the u_k (which we do not relabel), so that

$$\mathcal{E}''(u_k)[\xi_i,\xi_i] \geq -C\|\xi_i\|^2_{L^1(\widehat{\varphi}_j(A_i))} \left(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} \widehat{D}_{i\ell}^{-(n+s)} + \sum_{\ell>i} \lambda_{i\ell} \widehat{D}_{i\ell}^{-(n+s)}\right)$$

for all $\xi_i \in C^1_c(\widehat{\varphi}_j(A_i))$ and $k \in \mathbb{N}$. Here $\widehat{D}_{i\ell} = \mathrm{dist}(\widehat{\varphi}_j(A_i), \widehat{\varphi}_j(A_\ell))$.
Put $X_{i,j} := (\widehat{\varphi}_j)_* X_i$, and extend it by zero outside its domain of definition to a vector field on all of $\widehat{M}_j$. Selecting $\xi_i = \nabla_{X_{i,j}} u_k$, we arrive at

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{E}(u_k \circ \psi^{-t}_{X_{i,j}}) &= \mathcal{E}''(u_k)[\nabla_{X_{i,j}} u_k, \nabla_{X_{i,j}} u_k] \\ &\geq -C\|\nabla_{X_{i,j}} u_k\|^2_{L^1(\widehat{\varphi}_j(A_i))} \left(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} \widehat{D}_{i\ell}^{-(n+s)} + \sum_{\ell>i} \lambda_{i\ell} \widehat{D}_{i\ell}^{-(n+s)}\right).\end{aligned} \tag{4.39}$$

Thanks to the BV estimate of Theorem 1.2.13 and the flatness assumption on the metric, we can bound

$$\|\nabla_{X_{i,j}} u_k\|^2_{L^1(\widehat{\varphi}_j(A_i))} \le C\|X_{i,j}\|^2_{L^\infty} \mathrm{diam}(\widehat{\varphi}_j(A_i))^{2(n-1)} \le C\|X_i\|^2_{L^\infty} \mathrm{diam}(A_i)^{2(n-1)} ,$$

and also

$$\widehat{D}_{i,l} = \mathrm{dist}(\widehat{\varphi}_j(A_i), \widehat{\varphi}_j(A_l)) \le C\mathrm{dist}(A_i, A_l) = CD_{i,l} .$$

Substituting this into (4.39), and using that by (4.21) there holds

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{E}(u_k \circ \psi^{-t}_{X_{i,j}}) \to \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s^{\widehat{M}_j}(\psi^t_{X_{i,j}}(\widehat{E}_j); \widehat{\varphi}_j(A_i)) \quad \text{as } k \to \infty ,$$

we obtain that

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0} &\mathrm{Per}_s^{\widehat{M}_j}(\psi^t_{X_{i,j}}(\widehat{E}_j); \widehat{\varphi}_j(A_i)) \\ &\ge -C\|X_i\|^2_{L^\infty} \mathrm{diam}(A_i)^{2(n-1)} \left(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} D_{i\ell}^{-(n+s)} + \sum_{\ell>i} \lambda_{i\ell} D_{i\ell}^{-(n+s)} \right) .\end{aligned}$$

On the other hand, by Lemma 4.5.4 we have that

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s^{\mathbb{R}^n}(\psi^t_{X_i}(F); A_i) &= \lim_j \frac{d^2}{dt^2}\bigg|_{t=0} \Big[\mathrm{Per}_s^{(j)}(\psi^t_{X_i}(F_j); A_i)\Big] \\ &= \lim_j \frac{d^2}{dt^2}\bigg|_{t=0} \Big[\mathrm{Per}_s^{\widehat{M}_j}(\psi^t_{X_{i,j}}(\widehat{E}_j); \widehat{\varphi}_j(A_i))\Big],\end{aligned}$$

which then proves (4.38). □

4.5.3 Classification of blow-up limits

The main result of this section is the following classification result:

Theorem 4.5.12 (Classification result). *Let $s \in (0,1)$ and $3 \le n < n_s^*$. Let $\mathcal{F}$ be any family of sets of $\mathbb{R}^n$ satisfying the following properties:*

(1) ***Stationarity.*** *Every set $E \in \mathcal{F}$ is an s-minimal surface, in the sense of Definition 2.2.5.*

(2) ***BV estimate.*** *There is $C = C(n,s) > 0$ such that for every $E \in \mathcal{F}$, $x \in \mathbb{R}^n$, and $R > 0$ we have*

$$\mathrm{Per}(E; B_R(x)) \le CR^{n-1} .$$

(3) ***Viscosity solution of the NMS equation.*** *If $x_0 \in \partial E$ and E admits an interior (resp. exterior) tangent ball at x_0, then*

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|y - x_0|^{n+s}}\, dy \le 0, \qquad (resp.\ \ge 0).$$

(4) ***Almost-stability in one out of $(m+1)$ disjoint sets.*** *There exists some (fixed) $m \in \mathbb{N}$ such that the following holds. Let $X_1, X_2, ..., X_{m+1}$ be smooth vector fields with disjoint compact supports $A_1, A_2, ..., A_{m+1}$, and denote $D_{kl} := \mathrm{dist}(A_k, A_l)$. For $1 \le i < l \le m+1$, choose positive*

weights $\lambda_{il} > 0$. Then, given $E \in \mathcal{F}$, for at least one of the i (depending on E) we have that

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Per}_s(\psi^t_{X_i}(E); A_i) \geq -C\|X_i\|^2_{L^\infty} \mathrm{diam}(A_i)^{2(n-1)} \Big(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} D_{i\ell}^{-(n+s)} + \sum_{\ell>i} \lambda_{i\ell} D_{i\ell}^{-(n+s)}\Big)\,, \quad (4.40)$$

where $C = C(\mathcal{F})$.

(5) ***Completeness under scalings and $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ limits.*** *If $E \in \mathcal{F}$, then any translation, dilation and rotation of E is in $\mathcal{F}$ as well. Moreover, if E_i is a sequence of elements of $\mathcal{F}$ and $E_i \to E_\infty$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $E_\infty \in \mathcal{F}$ as well.*

(6) ***Cones with $n-2$ translation-invariant directions are half-spaces.*** *If $E \in \mathcal{F}$ is a cone and there is a linear $(n-2)$-dimensional subspace $L \subset \mathbb{R}^n$ such that $E + x = E$ for all $x \in L$, then ∂E must be a hyperplane.*

Then, every $E \in \mathcal{F}$ which is not equal (up to null sets) to $\mathbb{R}^n$ or $\varnothing$ must be a half-space.

An important property follows from (1) and (2) above:

Lemma 4.5.13. *Let $\mathcal{F}$ be a family of sets of $\mathbb{R}^n$ satisfying properties (1) and (2) in Theorem 4.5.12. Then, any set $E \in \mathcal{F}$ also satisfies density estimates, meaning that there exists a positive constant $\omega_0 = \omega_0(n, s, C_0)$ such that if*

$$R^{-n}|E \cap B_R(q)| \leq \omega_0$$

for some $q \in \varphi(\mathcal{B}_{1/2})$ and $R \in (0, 1/8)$, then

$$|E \cap B_{R/2}(q)| = 0\,.$$

Moreover, if $E_i \in \mathcal{F}$ and $E_i \to E_\infty$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then they also converge to E_∞ locally in the Hausdorff distance sense.

Proof. Same as for Lemma 4.5.6 and Proposition 4.5.7. □

We will need the following result, which is obtained by combining the $C^{1,\alpha}$ improvement of flatness theorem in [CRS10] and the $C^{1,\alpha}$-to-C^∞ bootstrap result for nonlocal minimal graphs in [BFV14].

Theorem 4.5.14 ([BFV14; CRS10])**.** *Let $s \in (0, 1)$. Then, there exists $\sigma > 0$, depending on n and s, such that the following holds: let $E \subset \mathbb{R}^n$ and $x \in \partial E$, and assume that*

(i) The set E is a viscosity solution of the NMS equation in $\mathcal{B}_r(x)$, in the sense of Proposition 4.3.1.

(ii) The boundary ∂E is included in a σ-flat cylinder in $\mathcal{B}_r(x)$, that is

$$\partial E \cap \mathcal{B}_r(x) \subset \{y \in \mathbb{R}^n : |e \cdot (y - x)| \leq \sigma r\},$$

for some direction $e \in \mathbb{S}^{n-1}$.

Then ∂E is a C^∞ graph in the direction e in $\mathcal{B}_{r/2}(x)$, with uniform estimates. In particular, its second fundamental form $\mathrm{II}_{\partial E}$ satisfies

$$\sup_{y \in \partial E \cap \mathcal{B}_{r/2}(x)} |\mathrm{II}_{\partial E}|(y) \leq \frac{C}{r}\,, \quad (4.41)$$

with $C = C(n, s)$.

We will also need the following intuitive lemma, to be read as "cones with finite Morse index are stable outside the origin", and which will be proved after Theorem 4.5.12.

Lemma 4.5.15. *Let $E \subset \mathbb{R}^n$ be an s-minimal cone with $\mathrm{Per}_s(E; \mathcal{B}_1(0)) < +\infty$. Assume that E satisfies property (4) in the statement of Theorem 4.5.12. Then E is stable in $\mathbb{R}^n \setminus \{0\}$.*

We will now prove Theorem 4.5.12.

Proof of Theorem 4.5.12. Let E_∞ be a blow-down limit of E, i.e., a limit of a sequence $E_i = \frac{1}{r_i}E$, with $r_i \to \infty$ (by property (5), such a limit exists, and it is also a member of $\mathcal{F}$). Then E_∞ is a cone: Let U, U_∞ and U_i denote the Caffarelli-Silvestre extensions of $u := \chi_E - \chi_{E^c}$, $u_\infty := \chi_{E_\infty} - \chi_{E^c_\infty}$ and $u_i := \chi_{E_i} - \chi_{E^c_i}$, respectively. Using the notation $\Phi_V(r) := r^{s-n}\int_{\tilde{B}^+_r(0,0)} z^{1-s}|\nabla V(x,z)|^2 dx\, dz$, by convergence of the extended energies[8] and scaling we have that

$$\Phi_{U_\infty}(r) = \lim_i \Phi_{U_i}(r) = \lim_i \Phi_U(rr_i)\,.$$

By the monotonicity of Φ_E, which we know since E is an s-minimal surface by property (1) and thus satisfies Theorem 1.2.2, the limit $\lim_{R\to\infty}\Phi_U(R)$ exists, and by property (2) and the interpolation result in Lemma 3.2.10 it is a finite constant. The equality above then shows that $\Phi_{U_\infty}(r)$ is equal to this constant independently of r. Since E_∞ is also an s-minimal surface (by properties (1) and (5)), the last paragraph in Theorem 1.2.2 gives that E_∞ is a cone.

We will now prove that E_∞ is in fact a hyperplane; by the local Hausdorff convergence of the $E_i = \frac{1}{r_i}E$ to E_∞ (see Lemma 4.5.13 above), E then satisfies the hypotheses of Theorem 4.5.14 for every $r > 0$, and therefore by (4.41) E then needs to be a hyperplane as well (since $\mathrm{II}_{\partial E}$ vanishes).

First, Lemma 4.5.15 states that E_∞ is stable outside the origin. If it also were smooth outside the origin, the assumption that $3 \le n < n^*_s$ would imply that ∂E_∞ is a hyperplane and we would finish the proof.
If, arguing by contradiction, there is instead some point $x_1 \neq 0$ where E_∞ is not smooth, we need to apply a dimension reduction argument: blowing up around x_1, we obtain a new cone $E_1 \in \mathcal{F}$ which is now translation invariant along some direction; after a rotation, we can write $E_1 = \widetilde{E}_1 \times \mathbb{R}$, and this is allowed by property (5).

We claim that E_1 cannot be smooth outside the origin. First, if that were the case, E_1 would be a hyperplane, since $3 \le n < n^*_s$. Now, the blow-up rescalings of E_∞ around x_1 converge locally in the Hausdorff distance sense to E_1 by Lemma 4.5.13, and they are viscosity solutions of the NMS equation by property (3). If E_1 were indeed a hyperplane, the assumption in the improvement of flatness Theorem 4.5.14 would be satisfied for the blow-up rescalings of E_∞ around x_1 (for large enough indices in the sequence). Thus E_∞ would be smooth in a neighborhood of x_1, a contradiction.

Now that we know that E_1 is not smooth outside the origin, we can iterate the argument with this new cone: Since $E_1 = \widetilde{E}_1 \times \mathbb{R}$ is not smooth outside the origin, there is some point $x_2 \in \mathbb{R}^{n-1} \setminus \{0\}$ where $\widetilde{E}_1$ is not smooth. Hence, we can blow up again around $(x_2, 0)$ and obtain a new cone E_2, which is now translation invariant with respect to two orthogonal directions. After a rotation, $E_2 = \widetilde{E}_2 \times \mathbb{R}^2$. Moreover, E_2 cannot be smooth outside the origin, by the same improvement of flatness argument we applied to E_1.
Iterating this reasoning $n-2$ times, we end up with a cone that is translation invariant in $n-2$ orthogonal directions, i.e., of the form $\widetilde{E} \times \mathbb{R}^{n-2}$ after a rotation, and which is not smooth outside the origin. This is not possible by property (6), and therefore we reach a contradiction. □

[8]Follows easily from convergence in $H^{s/2}_{\mathrm{loc}}(\mathbb{R}^n)$. The latter is proved, thanks to property (2), as in Step 1 in the proof of Theorem 4.5.5.

We now give the proof of Lemma 4.5.15.

Proof of Lemma 4.5.15. Consider an annular region of the form $A_0 = B_1 \setminus B_{R_0}$ with $0 < R_0 < 1$, centered at the origin. It suffices to show that E is stable in A_0, by the arbitrariness of R_0 and the dilation invariance of E.

The strategy is the following. Let X be a vector field supported on the annulus A_0. Let $A_1, ..., A_{m+1}$ be $(m+1)$ rescaled copies of A_0 of the form $A_i = RA_{i-1} = R^i A_0$, with $R > 0$ sufficiently large so that they are disjoint. Likewise, consider the $(m+1)$ rescaled vector fields $X_i := R^i X(x/R^i)$, which are supported in the respective A_i. Since E satisfies property (4) in the statement of Theorem 4.5.12, we know that the almost-stability inequality (4.40) will hold in at least one of the A_i. Moreover, since E is dilation invariant, we will be able to translate this information back into A_0, and taking R arbitrarily large we will find that E is actually stable on A_0 and conclude the proof.

Define $u := \chi_E - \chi_{E^c}$, and let ψ_X^t denote the flow of the vector field X at time t. Observe that $u_i^t := \chi_{\psi_{X_i}^t(E)} - \chi_{\psi_{X_i}^t(E)^c}$ is the composition of $u = \chi_E - \chi_{E^c}$ with the flow of $X_i = R^i X(x/R^i)$, which is given by $\psi_{X_i}^t = R^i \psi_X^t(x/R^i)$. By the dilation-invariance of the cone E, and hence of u_∞, we have that

$$u_i^t(x) = u(R^i \psi_X^t(x/R^i)) = u(\psi_X^t(x/R^i)) = u^t(x/R^i);$$

the scaling property of the fractional Sobolev energy then gives

$$\mathrm{Per}_s(\psi_{X_i}^t(E); A_i) = \mathcal{E}_{A_i}^{\mathrm{Sob}}(u_i^t) = \mathcal{E}_{A_i}^{\mathrm{Sob}}(u^t(x/R^i)) = R^{i(n-s)} \mathcal{E}_{A_0}^{\mathrm{Sob}}(u^t) = R^{i(n-s)} \mathrm{Per}_s(\psi_X^t(E); A_0)\,,$$

so that in particular

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\psi_{X_i}^t(E); A_i) = R^{i(n-s)} \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\psi_X^t(E); A_0)\,. \tag{4.42}$$

Now, by assumption, we know that the almost-stability inequality (4.40) will be satisfied in one of the A_i. Combined with (4.42), we obtain that

$$R^{i(n-s)} \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\psi_X^t(E); A_0) \geq -C \|X_i\|_{L^\infty}^2 \mathrm{diam}(A_i)^2 \Big(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} D_{i\ell}^{-(n+s)} + \sum_{\ell>i} \lambda_{i\ell} D_{i\ell}^{-(n+s)}\Big)\,,$$

where λ_{ij} are positive weights and $D_{ij} = \mathrm{dist}(A_i, A_j)$. We can bound

$$\|X_i\|_{L^\infty}^2 \mathrm{diam}(A_i)^{2(n-1)} = \|X\|_{L^\infty}^2 (R^i)^2 (R^i \mathrm{diam}(A_0))^{2(n-1)} \leq C_{X,A_0} R^{2ni}\,.$$

We also observe that

$$D_{i,l} = \mathrm{dist}(R^i A_0, R^l A_0) \geq c R^{\max\{i,l\}}$$

for some small c, for all R sufficiently large depending on A_0.

Substituting into the inequality we obtained and dividing both sides by $R^{i(n-s)}$, we get

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\psi_X^t(E); A_0) &\geq -C_{X,A_0} R^{i(n+s)} \Big(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} R^{-i(n+s)} + \sum_{\ell>i} \lambda_{i\ell} R^{-\ell(n+s)}\Big) \\ &= -C_{X,A_0} \Big(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} + \sum_{\ell>i} \lambda_{i\ell} R^{-(\ell-i)(n+s)}\Big)\,.\end{aligned}$$

Now, choosing the positive weights as $\lambda_{ij} = R^{\frac{n+s}{2}}$ for every pair $i < j$, we obtain

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\psi_X^t(E); A_0) \geq -C_{X,A_0}\Big(\sum_{\ell<i} R^{-\frac{n+s}{2}} + \sum_{\ell>i} R^{-(\ell-i-\frac{1}{2})(n+s)}\Big),$$

so that all the powers of R become strictly negative. Letting $R \to \infty$, we deduce that

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\psi_X^t(E); A_0) \geq 0$$

as desired. □

We will, in particular, apply Theorem 4.5.12 to the class $\mathcal{A}_m^{Blow-up}$. The next lemma proves that property (6) in the assumptions holds for this class. In the case of stable critical points, the next result is the nonlocal analogue of an idea of Tonegawa-Wickramasekera [TW12], in turn based on previous ideas of Schoen and Simon.

Lemma 4.5.16. *Let $n \geq 3$. Assume that some nontrivial cone $E \subset \mathbb{R}^n$ belongs to $\mathcal{A}_m^{Blow-up}$ and is of the form $\widetilde{E} \times \mathbb{R}^{n-2}$ for some cone $\widetilde{E} \subset \mathbb{R}^2$. Then, ∂E is a hyperplane.*

Proof. We divide the proof into three steps.

Step 1. We show the following claim: assume that $\mathrm{FA}_1(M, g, 1, p, \varphi)$ holds, and $u : M \to (-1, 1)$ is a solution of Allen-Cahn with parameter $\varepsilon \in (0, 1)$ in $B_1(p)$ that is Λ-almost stable in $B_1(p)$ (see Definition 4.2.1) with $\Lambda \leq \Lambda_0$, where Λ_0 is the constant in Proposition 4.2.5. Let $U : M \times \mathbb{R}_+ \to (-1, 1)$ be the Caffarelli-Silvestre extension of u.

Then, for some constant $C = C(n, s) > 0$ we have

$$\int_{\widetilde{B}^+_{1/2}(p,0)} \mathcal{A}^2(U)\, dV\, z^{1-s} dz \leq C,$$

where

$$\mathcal{A}^2(U) := \big(|\nabla^2 U|^2 - |\nabla|\nabla U||^2\big)\chi_{\{|\nabla u|>0\}} = \Big(|\nabla^2 U|^2 - \nabla^2 U\Big(\tfrac{\nabla U}{|\nabla U|}, \tfrac{\nabla U}{|\nabla U|}\Big)\Big)\chi_{\{|\nabla u|>0\}} \geq 0\,. \quad (4.43)$$

Here $\nabla^2 U$ denotes the "horizontal" Hessian of $U(\cdot, z)$—i.e. for z fixed—with respect to g.

Indeed since u is Λ-almost stable, for all $\xi \in C^1(\widetilde{B}^+_1(p, 0))$ with support contained in $\overline{\widetilde{B}^+_{3/4}(p, 0)}$ and trace ξ_0 on $z = 0$, we have

$$\begin{aligned}\widetilde{\mathcal{E}}''_1(U)[\xi, \xi] &= \beta_s \int_{\widetilde{B}^+_1} z^{1-s}|\nabla\xi|^2\, dV dz + \varepsilon^{-s}\int_{B_1} W''(u)\xi_0^2\, dV \\ &\geq \mathcal{E}''_{B_1}(u)[\xi_0, \xi_0] \geq -\Lambda\|\xi_0\|^2_{L^1(B_1)},\end{aligned}$$

where $\widetilde{B}^+_1$ and B_1 are brief notations for $\widetilde{B}^+_1(p, 0)$ and $B_1(p)$. Testing the above almost stability inequality with a test function that is a product $\xi = A\eta$, we obtain

$$-\varepsilon^{-s}\int_{B_1} W''(u)A^2\eta^2\, dV \leq \beta_s \int_{\widetilde{B}^+_1} z^{1-s}A^2|\widetilde{\nabla}\eta|^2 + z^{1-s}\widetilde{\nabla}A \cdot \widetilde{\nabla}(A\eta^2)\, dV dz + \Lambda\Big(\int_{B_1} |A\eta|\, dV\Big)^2.$$

Integrating by parts the second term on the right-hand side (similarly to the proof of [CS22, Theorem 1.3]) we get

$$\int_{B_1} \Big(\beta_s A(z^{1-s}\partial_z A(\cdot,0^+)) - \varepsilon^{-s}W''(u)A^2\Big)\eta^2\,dV$$
$$\le \beta_s \int_{\widetilde{B}_1^+} \Big(z^{1-s}A^2|\widetilde{\nabla}\eta|^2 - A\eta^2\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}A)\Big)\,dV\,dz + \Lambda\Big(\int_{B_1}|A\eta|\,dV\Big)^2. \quad (4.44)$$

Since u solves the Allen-Cahn equation and by (3.14) we have that

$$\beta_s(z^{1-s}\partial_z U(\cdot,0^+)) - \varepsilon^{-s}W'(u) = 0, \quad \text{on } M.$$

Taking the horizontal gradient ∇ of this equation we get

$$\beta_s(z^{1-s}\partial_z\nabla U(\cdot,0^+)) - \varepsilon^{-s}W''(u)\nabla u = 0,$$

and computing the scalar product with ∇u gives

$$\beta_s(z^{1-s}\partial_z\nabla U(\cdot,0^+))\cdot\nabla u - \varepsilon^{-s}W''(u)|\nabla u|^2 = 0.$$

Observe that

$$\beta_s(z^{1-s}\partial_z\nabla U(\cdot,0^+))\cdot\nabla u = \frac{\beta_s}{2}(z^{1-s}\partial_z|\nabla U|^2)(\cdot,0^+) = \beta_s|\nabla U|(z^{1-s}\partial_z|\nabla U|)(\cdot,0^+).$$

Hence, choosing $A = |\nabla U|$ makes the left-hand side of (4.44) vanish. Hence, for this choice of A we obtain

$$\beta_s\int_{\widetilde{B}_1^+}\Big(z^{1-s}|\nabla U|^2|\widetilde{\nabla}\eta|^2 - |\nabla U|\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}|\nabla U|)\eta^2\Big)dV\,dz + \Lambda\Big(\int_{B_1}|\nabla u|\eta\,dV\Big)^2 \ge 0. \quad (4.45)$$

Claim. For some constant $C>0$ that depends only on M we have

$$|\nabla U|\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}|\nabla U|) \ge z^{1-s}\big(|\nabla^2 U|^2 - |\nabla|\nabla U||^2 - C|\nabla U|^2\big)\chi_{\{|\nabla u|>0\}}.$$

Indeed, keeping for a moment the notation $A = |\nabla U|$ we have

$$\begin{aligned}
A\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}A) &= z^{1-s}A\Delta A + A\partial_z(z^{1-s}\partial_z A)\\
&= \Big(\frac{1}{2}\Delta(A^2) - |\nabla A|^2\Big)z^{1-s} + A\partial_z(z^{1-s}\partial_z A)\\
&= \Big(\nabla U\cdot\nabla(\Delta U) + |\nabla^2 U|^2 + \operatorname{Ric}(\nabla U,\nabla U) - |\nabla|\nabla U||^2\Big)z^{1-s} + |\nabla U|\partial_z(z^{1-s}\partial_z|\nabla U|),
\end{aligned}$$

where we have used Bochner's identity on M in the last line. Moreover, by the equation defining the extension we know $z^{1-s}\Delta U = -\partial_z(z^{1-s}\partial_z U)$, hence

$$z^{1-s}\nabla U\cdot\nabla(\Delta U) = -\nabla U\cdot\partial_z(z^{1-s}\partial_z\nabla U).$$

Also, explicit computation shows that

$$\begin{aligned}|\nabla U|\partial_z(z^{1-s}\partial_z|\nabla U|) &= |\nabla U|\partial_z\Big(\frac{z^{1-s}\partial_z(\frac{1}{2}|\nabla U|^2)}{|\nabla U|}\Big)\\ &= \nabla U\partial_z\big(z^{1-s}\cdot\partial_z\nabla U)\big) + z^{1-s}\Big(|\partial_z\nabla U|^2 - \big(\partial_z|\nabla U|\big)^2\Big)\\ &\geq \nabla U\cdot\partial_z(z^{1-s}\partial_z\nabla U).\end{aligned}$$

Hence, estimating $\mathrm{Ric}(\nabla U,\nabla U)\geq -C_M|\nabla U|^2$, we deduce that

$$\begin{aligned}&A\widetilde{\mathrm{div}}(z^{1-s}\widetilde{\nabla}A)\\ &\quad\geq z^{1-s}\Big(|\nabla^2 U|^2 - |\nabla|\nabla U||^2 - C_M|\nabla U|^2\Big) + |\nabla U|\partial_z(z^{1-s}\partial_z|\nabla U|) - \nabla U\cdot\partial_z(z^{1-s}\partial_z\nabla U)\\ &\quad\geq z^{1-s}\Big(|\nabla^2 U|^2 - |\nabla|\nabla U||^2 - C_M|\nabla U|^2\Big),\end{aligned}$$

and the claim is proved.

Inserting the claim in (4.45), we reach

$$\int_{\widetilde{B}_1^+} z^{1-s}\mathcal{A}^2(U)\eta^2 dV dz \leq \beta_s\int_{\widetilde{B}_1^+}|\nabla U|^2 z^{1-s}(|\widetilde{\nabla}\eta|^2 + C\eta^2)dV dz + \Lambda\Big(\int_{B_1}|\nabla u||\eta|\,dV\Big)^2.$$

From this we can conclude the claim in Step 1. Fix a cutoff function satisfying $\chi_{\widetilde{B}^+_{1/2}}\leq\eta\leq\chi_{\widetilde{B}^+_{3/4}}$. By Lemma 3.2.10 with $R=1, k=0$ and Young's inequality we have that

$$\beta_s\int_{\widetilde{B}_1^+}|\nabla U|^2 z^{1-s}(|\widetilde{\nabla}\eta|^2 + C\eta^2)\,dV dz \leq C + C\left(\int_{B_{3/4}}|\nabla u|\,dV\right)^s \leq C + Cs\int_{B_{3/4}}|\nabla u|\,dV,$$

thus

$$\int_{\widetilde{B}^+_{1/2}} z^{1-s}\mathcal{A}^2(U)\,dV dz \leq C + C\int_{B_{3/4}}|\nabla u|\,dV + \Lambda\Big(\int_{B_{3/4}}|\nabla u|\,dV\Big)^2.$$

Moreover, since u is Λ-almost stable in B_1 with $\Lambda\leq\Lambda_0$, by Proposition 4.2.5 and a simple covering argument we get a uniform bound on the BV-norm of u in $B_{3/4}$, and this proves the claim in Step 1.

Step 2. Recall that, as recorded in Remark 4.5.2, for $E\in\mathcal{A}_m^{Blow-up}$ we have sequences of:

- closed manifolds (M_j, g^{M_j});
- points $p_j\in M_j$ and scales $R_j\uparrow\infty$ for which $\mathrm{FA}_3(M_j, g^{M_j}, R_j, p_j, \varphi_j)$ holds and $g^{M_j}_{k\ell}(0)=\delta_{k\ell}$.
- solutions of Allen-Cahn $u_j : M_j\to(-1,1)$ with parameters $\varepsilon_j\downarrow 0$ and Morse index bounded by m such that $(u_j\circ\varphi_j)\to u_\circ := \chi_E - \chi_{E^c}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Let $U_j : \widetilde{M}_j\to(-1,1)$ be the extensions of u_j and and observe that $U_j\rightharpoonup U_\circ$ weakly in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$, where $U_\circ$ is the (unique, bounded) Caffarelli-Silvestre extension of $u_\circ$ to $\mathbb{R}^{n+1}_+$. Actually, thanks to Theorem 4.5.5, one could prove local strong convergence in the weighted Hilbert space $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+; |z|^{1-s}dxdz)$, although (much rougher) weak L^1_{loc} will suffice here.

Notice also that in the local coordinates φ_j^{-1} we will have $g^{M_j}_{k\ell}\to\delta_{k\ell}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$, since $R_j\to\infty$. Hence, by standard elliptic estimates $U_j\circ\widetilde{\varphi}_j\to U_\circ$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n\times(0,+\infty))$ (up to subsequence), where $\widetilde{\varphi}_j(x,z) = (\varphi_j(x), z)$.

Now, for all $j \gg 1$ sufficiently large, take $m+1$ disjoint balls $\{B_1(\varphi_j(3iD_0e_3))\}_{i=0}^m$, where $D_0 \geq 1$ is a constant that will be chosen later. Since $\mathrm{FA}_3(M_j, g^{M_j}, R_j, p_j, \varphi_j)$ holds, by Lemma 4.2.2 and Lemma 3.4.13 for at least one $\ell \in \{0, \dots m\}$ we have that u_j is Λ-almost stable in $B_1(\varphi_j(3\ell D_0 e_3))$ for $\Lambda = mC(n,s)/D_0^{n+s}$. Hence, we can choose $D_0 = D_0(m,n,s)$ such that u_j is Λ_0-almost stable in this ball, where Λ_0 is the constant in Proposition 4.2.5. We may assume, without loss of generality (up to translation and a subsequence), that it this ball is actually $B_1(\varphi_1(0))$, and then Step 1 gives

$$\int_{\varphi_j(\mathcal{B}_{1/2}^+)} \mathcal{A}^2(U_j)\, dV_j\, z^{1-s} dz \leq C.$$

Passing to the limit as $j \to \infty$ and using that $U_j \circ \varphi_j \to U_\circ$ in C^2_{loc}, for every $\delta < 1/10$ we obtain

$$\int_{\mathcal{B}_{1/4}^+ \cap \{z > \delta\}} \mathcal{A}^2(U_\circ)\, dx\, z^{1-s} dz \leq C, \tag{4.46}$$

for a constant $C > 0$ independent of δ.

Step 3. We show the following claim: if $E \subset \mathbb{R}^n$ is a nontrivial cone of the form $\widetilde{E} \times \mathbb{R}^{n-2}$ for some (nontrivial) cone $\widetilde{E} \subset \mathbb{R}^2$, then

$$\lim_{\delta \to 0^+} \int_{\mathcal{B}_{1/4}^+ \cap \{z > \delta\}} \mathcal{A}^2(U_\circ)\, dx\, z^{1-s} dz = +\infty.$$

This will finish the proof proving that E is a half-space, otherwise we would contradict (4.46).

Let us denote by $x = (\widetilde{x}, y) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$ the variables of $\mathbb{R}^n$. Since E is invariant for translations of the type $(0, y)$ for $y \in \mathbb{R}^{n-2}$, it easily follows that $u_\circ = \chi_E - \chi_{E^c}$ depends only on $\widetilde{x}$. That is

$$u_\circ(\widetilde{x}, y) = u_\circ(\widetilde{x}, 0).$$

Since $U_\circ$ is the (unique, bounded) Caffarelli-Silvestre extension of $u_\circ$ to $\mathbb{R}^{n+1}_+$, it follows that also $U_\circ$ depends only on $(\widetilde{x}, z)$. This can be seen as follows. Let $U'_\circ(\widetilde{x}, z)$ be the unique extension of $u_\circ(\cdot, 0) : \mathbb{R}^2 \to (-1, 1)$ to $\mathbb{R}^3_+$. Then $U'_\circ$ clearly solves the extension problem also on $\mathbb{R}^{n+1}_+$ and since the solution is unique, we obtain $U'_\circ = U_\circ$.

Moreover, since E is a cone we see that $U_\circ$ is 0-homogeneous, and hence $\mathcal{A}(U_\circ)$ is homogeneous of degree -2. Then, since $\mathcal{A}(U_\circ)(x, z) = \mathcal{A}(U_\circ)(\widetilde{x}, z)$ we have

$$\int_{\mathcal{B}_{1/4}^+ \cap \{z > \delta\}} \mathcal{A}^2(U_\circ)\, z^{1-s} dx dz \geq c \int_Q \int_{\{z > \delta\}} \mathcal{A}^2(U_\circ)\, z^{1-s} dz d\widetilde{x},$$

where $Q := Q_{1/4\sqrt{n+1}}$ is a cube of side $1/4\sqrt{n+1}$ in $\mathbb{R}^2$ centered at 0. Substituting in the last integral $(\widetilde{x}, z) = (\delta x', \delta z')$ and using that $\mathcal{A}(U_\circ)$ is homogeneous of degree -2 we get

$$\begin{aligned}\int_Q \int_{\{z > \delta\}} \mathcal{A}^2(U_\circ)\, z^{1-s} dz d\widetilde{x} &= (\delta^{-2})^2 \cdot \delta^{1-s} \cdot \delta^3 \int_{\frac{1}{\delta}Q} \int_{\{z > 1\}} \mathcal{A}^2(U_\circ)\, (z')^{1-s} dz' dx' \\ &\geq \frac{1}{\delta^s} \int_Q \int_{\{z > 1\}} \mathcal{A}^2(U_\circ)\, (z')^{1-s} dz' dx'.\end{aligned}$$

It is easily seen that

$$\int_Q \int_{\{z > 1\}} \mathcal{A}^2(U_\circ)\, (z')^{1-s} dz' dx' > 0,$$

since the integrand cannot be identically zero as E is nontrivial. Hence

$$\lim_{\delta\to 0^+}\int_{\mathcal{B}^+_{1/4}\cap\{z>\delta\}} \mathcal{A}^2(U_\circ)\, z^{1-s} dx dz \geq \lim_{\delta\to 0^+}\frac{1}{\delta^s}\int_Q\int_{\{z>1\}} \mathcal{A}^2(U_\circ)\,(z')^{1-s}dz'dx' = +\infty.$$

This concludes the proof.

□

The properties proved so far for the family $\mathcal{F}=\mathcal{A}_m^{Blow-up}$ (see Definition 4.5.8) show that it satisfies the hypotheses of Theorem 4.5.12, whence we deduce

Corollary 4.5.17 (**Blow-ups of limit surfaces of Allen-Cahn are hyperplanes**). *Let $s\in(0,1)$ and $3\leq n<n_s^*$. Then, any nonempty ∂F in $\mathcal{A}_m^{Blow-up}$ is a hyperplane.*

Proof. It follows from applying Theorem 4.5.12 to the class $\mathcal{F}=\mathcal{A}_m^{Blow-up}$. Properties (1) and (5) in the assumptions of Theorem 4.5.12 follow immediately from (the proof of) Theorem 4.5.5. Property (2) follows from (1) in Remark 4.3.5 and the lower semicontinuity of the BV seminorm under L^1-convergence. Properties (4) and (6) have been proved, respectively, in Lemma 4.5.11 and Lemma 4.5.16. Finally, property (3) —namely that blow-ups are viscosity solutions of the NMS equation in $\mathbb{R}^n$— follows easily from Proposition 4.3.1 and the convergence of boundaries in Hausdorff distance under a blow-up (Proposition 4.5.7), using the convergence of the kernels in Proposition 4.5.3 part (ii). See [CRS10] and [CS11]. □

4.5.4 Uniform regularity and separation: proof of Theorem 1.2.8

In this section we will prove Theorem 1.2.8, which stated that sets in $\mathcal{A}_m(M)$, i.e. the limits of Allen-Cahn solutions on M with index at most m, are smooth with *uniform regularity* and *separation* estimates in low dimensions.

We will need the following improvement of flatness theorem for sets which are viscosity solutions of the NMS equation in a Riemannian manifold, proved in [Moy25] more generally assuming boundedness of the nonlocal mean curvature, and which extends the result in [CRS10] to the setting of ambient Riemannian manifolds.

Theorem 4.5.18 ([Moy25]). *Let $s\in(0,1)$ and $0<\alpha<s$. Then, there exists $\sigma>0$, depending on n,s and α, such that the following holds. Let (M,g) be an n-dimensional Riemannian manifold. Take $p\in M$, and assume that the flatness assumption* $\mathrm{FA}_1(M,g,r,p,\varphi)$ *holds. Let $E\subset M$ with $p\in\partial E$, and assume that*

(i) The set E is a viscosity solution of the NMS equation in $\varphi(B_r(0))$, in the sense of Proposition 4.3.1.

(ii) The boundary $\varphi^{-1}(\partial E)$ is included in a σ-flat cylinder in $\mathcal{B}_r(0)$, that is

$$\varphi^{-1}(\partial E)\cap\mathcal{B}_r(0)\subset\{|e\cdot x|\leq\sigma r\},$$

for some direction $e\in\mathbb{S}^{n-1}$.

Then $\varphi^{-1}(\partial E)$ is a single $C^{1,\alpha}$ graph in the direction e in $\mathcal{B}_{r/2}(0)$, with uniform estimates.

Proof of Theorem 1.2.8. We will first show that E is trapped (in the coordinates given by φ^{-1}) in a very flat cylinder, as recorded in the next claim:

Claim. Let $\sigma>0$. Then there exists a uniform constant $R_\sigma=R_\sigma(m,s,\sigma)$ and a unit vector $e\in\mathbb{S}^{n-1}$ such that

$$-\sigma R_\sigma\leq y\cdot e\leq\sigma R_\sigma \quad \text{for all } y\in\varphi^{-1}(\partial E)\cap\mathcal{B}_{R_\sigma}. \tag{4.47}$$

Proof of the Claim. Fix $\sigma > 0$; the proof will be by contradiction and blow-up. Let $R_j = 1/j$. If the Claim were false, then for every $j \in \mathbb{N}$ there would exist closed manifolds M_j satisfying the flatness assumptions $\text{FA}_3(M_j, g, 1, p_j, \varphi_j)$, and some sets $E_j \in \mathcal{A}_m(M_j)$ so that $p_j \in \partial E_j$ but such that (4.47) is not satisfied for any unit vector (with E_j, R_j and φ_j in place of E, R_σ and φ).

Consider, then, the blow-up sequence $F_j = \frac{1}{R_j}\varphi_j^{-1}(E_j)$. By Proposition 4.5.7, a subsequence of the F_j converges (in particular) locally in the Hausdorff distance sense to a limit set $F \in \mathcal{A}_m^{Blow-up}$. Moreover, since $0 \in F_j$, we see that $0 \in F$ as well.

Now, from the classification result of Corollary 4.5.17, we know that ∂F is, in fact, a hyperplane passing through the origin. The local Hausdorff convergence of the $\partial F_j = \frac{1}{R_j}\varphi_j^{-1}(\partial E_j)$ to the hyperplane ∂F implies then that the condition

$$-\sigma \le y \cdot e \le \sigma \quad \text{for all } y \in \tfrac{1}{R_j}\varphi_j^{-1}(\partial E_j) \cap \mathcal{B}_1$$

will be satisfied for all j large enough in the subsequence and for e the normal vector to the limit hyperplane. Rescaling this condition by a factor R_j, we obtain exactly that (for j large) the E_j satisfy (4.47) with E_j, R_j and φ_j, contradiction. This finishes the proof of the claim. □

Now that the claim is known to be true, choosing σ in it to be the constant in Theorem 4.5.18 (recall that sets in $\mathcal{A}_m(M)$ are viscosity solutions of the NMS equation by Proposition 4.3.1, and that our notion of viscosity solution in Proposition 4.3.1 is equivalent to the one used in [Moy25] to obtain Theorem 4.5.18), we obtain that $\varphi^{-1}(\partial E) \cap \mathcal{B}_{R_\sigma/2}$ is a single graph with uniform $C^{1,\alpha}$ estimates.

□

4.5.5 Dimension reduction – Proof of Theorem 1.2.6

This section proves Theorem 1.2.6. We will call singular points those points $x \in \partial E$ where ∂E cannot be described as a $C^{1,\alpha}$ graph around x, and we will denote by $\text{sing}(\partial E)$ or $\text{sing}(E)$ the (closed) set of all the singular points of ∂E. We state here a more general result about regularity for s-minimal surfaces, which are limits of Allen-Cahn, and immediately show how it proves Theorem 1.2.6.

Theorem 4.5.19. *Let $s \in (0,1)$. Let (M, g) be a closed Riemannian manifold of dimension $n \ge 3$, and let $\partial E \in \mathcal{A}_m(M)$. Then:*

- *If $n < n_s^*$, then ∂E is a C^∞ hypersurface.*
- *If $n = n_s^*$, ∂E is a C^∞ hypersurface outside of a discrete set.*
- *If $n > n_s^*$, then ∂E is a C^∞ hypersurface outside of a closed set of Hausdorff dimension at most $n - n_s^*$.*

We readily deduce:

Proof of Theorem 1.2.6. The surfaces $\Sigma^{\mathfrak{p}} = \partial E^{\mathfrak{p}}$ in Theorem 1.2.4 belong to $\mathcal{A}_{\mathfrak{p}}(M)$ by construction (see Section 4.4 for the proof of Theorem 1.2.4). Therefore, Theorem 4.5.19 applies to them, which gives Theorem 1.2.6. □

Theorem 4.5.19 will be proved after two preliminary lemmas.

Lemma 4.5.20. *Let $\partial E \in \mathcal{A}_m^{Blow-up}$ and let $x \in \operatorname{sing}(\partial E)$. Choose $r_j \to 0$; then, the blow-up sequence $\frac{1}{r_j}(E-x)$ converges in L^1_{loc} and locally in the Hausdorff distance sense to a singular cone $C_\infty \in \mathcal{A}_m^{Blow-up}$ which is stable in $\mathbb{R}^n \setminus \{0\}$. If, moreover, x is an accumulation point of $\operatorname{sing}(\partial E)$, then r_j can be chosen so that C_∞ has a singular point on ∂B_1 (thus an entire line of singular points).*

Proof. Recall that $\mathcal{A}_m^{Blow-up}$ satisfies the properties in the statement of Theorem 4.5.12, see the proof of Corollary 4.5.17. The convergence of the $F_j := \frac{1}{r_j}(E-x)$ to a cone C_∞ in the appropriate sense follows then as in the beginning of the proof of Theorem 4.5.12, and Lemma 4.5.15 gives the stability of C_∞ outside the origin.

If C_∞ were non-singular (i.e. if C_∞ were a half-space), then the Hausdorff convergence of the $F_j = \frac{1}{r_j}(E-x)$ to C_∞ on $\mathcal{B}_1$ would imply that the assumption of the improvement of flatness result of Theorem 4.5.14 is satisfied by E on a small ball centered at x. Hence ∂E would be a $C^{1,\alpha}$ hypersurface around x, and this would contradict the assumption that x is a singular point. Moreover, in case x is a limit point of a sequence $x_j \in \operatorname{sing}(\partial E)$, choosing $r_j := \operatorname{dist}(x, x_j)$ the $F_j = \frac{1}{r_j}(E-x)$ have singular points at 0 and at $\frac{1}{r_j}(x_j - x) \in \partial\mathcal{B}_1$. Selecting a subsequence j_k such that the x_{j_k} converge to a limit point $x' \in \partial\mathcal{B}_1$, the improvement of flatness argument above shows that the limit cone of the F_{j_k} must have a singular point at $x' \in \partial\mathcal{B}_1$. □

Lemma 4.5.21. *Let $C \subset \mathbb{R}^n$ be a cone in $\mathcal{A}_m^{Blow-up}$, with $n \geq n_s^*$. Then $\mathcal{H}^t(\operatorname{sing}(C)) = 0$ for all $t > n - n_s^*$. Moreover, in the case $n = n_s^*$, C is smooth outside the origin.*

Proof. Fix $t > n - n_s^*$, and assume for contradiction that $\mathcal{H}^t(\operatorname{sing}(C)) > 0$ (or that C is not smooth outside the origin in the case $n = n_s^*$).

Claim. If $n > n_s^*$, there exists $x \in \operatorname{sing}(C) \cap \partial\mathcal{B}_1$ such that, blowing up around x, we find a cone of the form $\widetilde{C} \times \mathbb{R}$ (up to a rotation) with $\mathcal{H}^{t-1}(\operatorname{sing}(\widetilde{C})) > 0$.

Proof of the claim. Since we are assuming that $\mathcal{H}^t(\operatorname{sing}(C)) > 0$, there must exist some point $x \in \operatorname{sing}(C) \cap \partial\mathcal{B}_1$ of positive $\mathcal{H}^t_\infty$-density, in the sense that (with the appropriate constant normalization) there exists a sequence $r_j \to 0$ such that $\mathcal{H}^t_\infty(\operatorname{sing}(C) \cap \mathcal{B}_{r_j}(x)) \geq r_j^t$ for all j. Consider the blow-up sequence $C_j = \frac{1}{r_j}(C-x)$; by Lemma 4.5.20, a subsequence will converge locally in the Hausdorff distance sense to a limit cone C_∞ which (since C is itself already a cone) is of the form $C_\infty = \widetilde{C}_\infty \times \mathbb{R}$, up to performing a rotation. Assume by contradiction that $\mathcal{H}^{t-1}(\operatorname{sing}(\widetilde{C}_\infty)) = 0$, or equivalently that $\mathcal{H}^t(\operatorname{sing}(C_\infty)) = 0$.
Now, given any fixed finite cover by open sets of $\operatorname{sing}(C_\infty) \cap \bar{\mathcal{B}}_1$, for j large enough the $\operatorname{sing}(C_j) \cap \mathcal{B}_1$ are also contained in the cover: otherwise, we would have a subsequence $y_j \in \operatorname{sing}(C_j)$ converging to some $y \in (C_\infty \setminus \operatorname{sing}(C_\infty)) \cap \bar{\mathcal{B}}_1$, so that by Hausdorff convergence the C_j would be contained (for j large enough) in an arbitrarily flat piece of slab around the y_j (thanks to the regularity of C_∞ at y); by the improvement of flatness result of Theorem 4.5.14, the y_j would be regular points as well, a contradiction. By arbitrariness of the finite open cover of $\operatorname{sing}(C) \cap \bar{\mathcal{B}}_1$, the assumption that $\mathcal{H}^t(\operatorname{sing}(\widetilde{C}_\infty)) = 0$ and the definition of $\mathcal{H}^t_\infty$ lead us to deduce that $\mathcal{H}^t_\infty(\operatorname{sing}(C_j) \cap \mathcal{B}_1)$ converges to zero. Scaling back (recall that $C_j = \frac{1}{r_j}(C-x)$), we find that for some j large enough $\mathcal{H}^t_\infty(\operatorname{sing}(C) \cap \mathcal{B}_{r_j}(x)) \leq \frac{1}{2} r_j^t$, a contradiction with how x was chosen. □

With the claim at hand, the proof now continues as follows.
In the case $n > n_s^*$, since we assumed $t > n - n_s^*$, the claim can be easily further iterated up to $(n - n_s^*)$ times. This leads to the existence, in the class $\mathcal{A}_m^{Blow-up}$, of a cone of the form $\widetilde{C} \times \mathbb{R}^{n-n_s^*}$ with $\widetilde{C} \subset \mathbb{R}^{n_s^*}$ and $\mathcal{H}^{t-(n-n_s^*)}(\operatorname{sing}(\widetilde{C})) > 0$. In particular, $\widetilde{C}$ is not smooth outside the origin.
In the case $n = n_s^*$, we are already assuming by contradiction that $\widetilde{C} := C \subset \mathbb{R}^{n_s^*}$ is not smooth

outside the origin.
The rest of the proof is now common for both cases. Let $y \in \mathbb{R}^{n_s^*}$ be such that $y \in \mathrm{sing}(\widetilde{C}) \cap \partial \mathcal{B}_1$. Blowing up around $(y, 0) \in \mathbb{R}^n$, we obtain a new cone that is translation invariant with respect to an additional orthogonal direction. Moreover, this new cone will not be smooth outside the origin either, since otherwise the definition of n_s^* would imply that it is a half-space, and then the Hausdorff convergence and the improvement of flatness result in Theorem 4.5.14 would give that $\widetilde{C}$ is smooth around y. Iterating this argument, we obtain in the end a cone in $\mathcal{A}_m^{Blow-up}$ which is translation invariant with respect to $n-2$ directions and which is not a half-space by the improvement of flatness argument we have repeatedly been using. Lemma 4.5.16 then gives a contradiction, concluding the proof. □

Proof of Theorem 4.5.19. Let $\partial E \in \mathcal{A}_m(M)$, with M of dimension $n \geq 3$. We distinguish between the three cases depending on n_s^*:

- Assume $n_s^* > n$. At every $p \in \partial E$, the flatness assumptions $\mathrm{FA}_3(M, g, R_0, p, \varphi_p)$ will be satisfied for some $R_0 > 0$ (recall (d) in Remark 3.4.5), so that we can apply Theorem 1.2.8 after scaling and conclude the $C^{1,\alpha}$ regularity (in fact, with quantitative estimates) of ∂E around p.

- Assume $n_s^* < n$. Fix any $t > n - n_s^*$; by Theorem 4.5.5 and the arguments in Lemma 4.5.20, given any $q \in \mathrm{sing}(E)$ we can blow up around q and find a cone C_q. Applying Lemma 4.5.21, we deduce that $\mathcal{H}^t(\mathrm{sing}(C_q)) = 0$.
Now, assume for contradiction that $\mathcal{H}^t(\mathrm{sing}(E)) > 0$. We can then apply the same argument as in the Claim in the proof of Lemma 4.5.21, but with $\mathrm{sing}(E)$ instead of $\mathrm{sing}(C) \cap \partial \mathcal{B}_1$; this shows the existence of a point $q \in \mathrm{sing}(E)$ such that, blowing-up around q, we would find a cone with $\mathcal{H}^t(\mathrm{sing}(C_q)) > 0$, thus reaching a contradiction.

- Assume $n_s^* = n$. Suppose that $q \in \mathrm{sing}(E)$ is an accumulation point. By Theorem 4.5.5 and the arguments in Lemma 4.5.20, we can blow up around q and find a cone C_q which is not smooth outside the origin. Lemma 4.5.21 then gives a contradiction.

This proves that E is $C^{1,\alpha}$ outside of a set of the desired size. The fact that $C^{1,\alpha}$ s-minimal surfaces are smooth (C^∞) is proved in [FS25]. □

4.5.6 The De Giorgi and Bernstein conjectures in the finite Morse index case – proof of Theorems 1.2.19 and 1.2.18

We will now first prove Theorem 1.2.19. We will need the following result, which is a consequence of an improvement of flatness theory for phase transitions in the "genuinely nonlocal" regime, meaning that the order s of the operator is strictly less than 1.

Theorem 4.5.22 (Theorem 1.2 in [DSV20]). *Let $n \geq 2$, $s \in (0,1)$, and $W(u) = \frac{1}{4}(1-u^2)^2$. Let $u : \mathbb{R}^n \to (-1,1)$ be a solution of $(-\Delta)^{s/2} u + W'(u) = 0$ in $\mathbb{R}^n$.*

Assume that there exists a function $a : (1, \infty) \to (0, 1]$ such that $a(R) \to 0$ as $R \to +\infty$ and such that, for all $R > 0$, we have

$$\{e_R \cdot x \leq -a(R)R\} \subset \left\{u \leq -\tfrac{4}{5}\right\} \subset \left\{u \leq \tfrac{4}{5}\right\} \subset \{e_R \cdot x \leq a(R)R\} \quad \text{in } \mathcal{B}_R\,, \tag{4.48}$$

for some $e_R \in \mathbb{S}^{n-1}$ which may depend on R.

Then, $u(x) = \phi(e \cdot x)$ for some direction $e \in \mathbb{S}^{n-1}$ and an increasing function $\phi : \mathbb{R} \to (-1,1)$.

Proof of Theorem 1.2.19. Let $u : \mathbb{R}^n \to (-1, 1)$ be a finite Morse index solution of the Allen-Cahn equation with parameter $\varepsilon = 1$. For every $R > 0$ we introduce the blow-down rescalings

$$u_R(x) := u(Rx).$$

These are solutions of the Allen-Cahn equation with parameter $\varepsilon = 1/R$ and the same Morse index as u.

By the strong convergence result of Theorem 1.2.17 (whose proof on $\mathbb{R}^n$ is identical to the closed-manifold case, and a diagonal argument to get convergence in each of the balls $\mathcal{B}_k$ for $k \in \mathbb{N}$), there exists a sequence $R_j \to \infty$ and an s-minimal surface $E \subset \mathbb{R}^n$ such that

$$u_{R_j} \longrightarrow u_E := \chi_E - \chi_{E^c} \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n).$$

In particular, E belongs to the class $\mathcal{A}_m(\mathbb{R}^n)$ (see Definition 1.2.7). The properties in the hypotheses of Theorem 4.5.12 can be proved for the class $\mathcal{A}_m(\mathbb{R}^n)$, exactly as they were proved for the class $\mathcal{A}_m^{Blow-up}$ as recorded in Corollary 4.5.17. In fact, all necessary results have been stated with local assumptions, other than the use of the kernel $K_s(x, y)$, which becomes $\alpha_{n,s}|x - y|^{-n-s}$ on $\mathbb{R}^n$, and in fact, several proofs could be simplified due to working on $\mathbb{R}^n$.

Applying Theorem 4.5.12 to the class $\mathcal{A}_m(\mathbb{R}^n)$ we deduce that E must be a half-space. Moreover, the local convergence in the Hausdorff distance of the level sets of u_{R_j} to ∂E—see Theorem 1.2.17—shows that both $\{u_{R_j} \le -4/5\}$ and $\{u_{R_j} \le 4/5\}$ converge (in Hausdorff distance) in $\mathcal{B}_1$ to a half-plane. Rescaling back to u gives that (4.48) is eventually satisfied in $\mathcal{B}_{R_j}$. Then, Theorem 4.5.22 gives u is a one-dimensional solution. □

We turn now to the proof of Theorem 1.2.18, the finite Morse index analog for s-minimal surfaces of class C^2 of the Bernstein conjecture. We recall that this result is false for classical minimal surfaces, since, for example, the catenoid in $\mathbb{R}^3$ is a complete minimal surface with Morse index 1, and that even under the assumption of stability, this result is only known up to dimension 6 despite stable classical minimal cones being known to be hyperplanes up to dimension 7 [Sim68]. See the Introduction for more details.

To prove Theorem 1.2.18, we will again apply the classification result of Theorem 4.5.12. For that reason, we introduce the following definition.

Definition 4.5.23. *We say that a set $E \subset \mathbb{R}^n$ belongs to the class $\mathcal{A}'_m(\mathbb{R}^n)$ if there exists a sequence of sets $E_j \subset \mathbb{R}^n$ with $E_j \to E$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ such that:*

(i) the boundaries ∂E_j are $(n-1)$-dimensional manifolds of class C^2;

(ii) E_j are s-minimal surfaces in $\mathbb{R}^n$ with Morse index $\le m$ in the weak sense.

Proposition 4.5.24. *Let $n \ge 3$. Then the family $\mathcal{A}'_m(\mathbb{R}^n)$ satisfies the properties in the hypotheses of Theorem 4.5.12.*

Proof. Let E be an s-minimal surface of class C^2 and Morse index at most m in $\mathbb{R}^n$. Then:

- By [Flo24, Theorem 5.4] a uniform BV estimate holds, that is

$$\text{Per}(E, B_R) \le CR^{n-1},$$

  for some $C = C(n, s.m) > 0$.

- The surface E is a viscosity solution of the NMS equation, since it is stationary for inner-variations and of class C^2 (in particular, its nonlocal mean curvature is well defined and equal to 0 at every point $x \in \partial E$).

- The almost-stability inequality (4) is proved exactly like the one in Lemma 4.5.10 considering the formula for the second variation of the fractional perimeter and test functions $\xi_i = X_i \cdot \nu_{\partial E}$ instead, where $\nu_{\partial E}$ denotes the outer normal vector; see [Flo24, Lemma 5.8].

If we then consider E to be any element of $\mathcal{A}'_m(\mathbb{R}^n)$, not necessarily of class C^2, by definition we can approximate it with E_j satisfying the above. This readily shows that E inherits the properties of the E_j, which proves (1)-(5) for the class $\mathcal{A}'_m(\mathbb{R}^n)$. Lastly, property (6), concerning the classification of cones in $\mathcal{A}'_m(\mathbb{R}^n)$ with $n-2$ directions of translation invariance, is proved in Lemma 4.5.25 below.

□

Lemma 4.5.25. *Let $n \geq 3$. Assume that some nontrivial cone $E \subset \mathbb{R}^n$ belongs to $\mathcal{A}'_m(\mathbb{R}^n)$ (see Definition 4.5.23) and is of the form $\widetilde{E} \times \mathbb{R}^{n-2}$, for some cone $\widetilde{E} \subset \mathbb{R}^2$. Then, ∂E is a hyperplane.*

Proof. Let E_i be the sequence of smooth s-minimal surfaces with Morse index $\leq m$ such that $E_j \to E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Denote by ν_i the respective outer unit normals to ∂E_i.

Let $X_i : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth extension of ν_i with $|X_i| \leq 1$, and let $\varphi \in C^2_c(B_3)$ be a smooth cutoff function with $\varphi = 1$ in B_2. Consider the $(m+1)$ vector fields with disjoint compact support

$$X_{i,k} := X_i(x)\varphi(x - 100ke_3), \quad k = 1, 2, \ldots, m+1.$$

By the almost-stability property in one out of $(m+1)$ disjoint sets, E_i will be almost stable in one of these balls. With no loss of generality (up to translations and a subsequence), we may assume that this holds for the first ball (relative to $k = 1$), that is

$$\delta^2 \mathrm{Per}_s(E_i; B_3)[X_{i,1}] \geq -C,$$

for some constant $C = C(n,s) > 0$.

By the second variation formula (i.e., Theorem 2.2.12), this writes as

$$\int_{\partial E_i} \int_{\partial E_i} \frac{|\nu_i(x) - \nu_i(y)|^2}{|x-y|^{n+s}} \varphi(x)^2 \leq \int_{\partial E_i} \int_{\partial E_i} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{n+s}} + C.$$

Observe that

$$\partial E_i \times \partial E_i \setminus B_3^c \times B_3^c = (\partial E_i \cap B_3 \times \partial E_i \cap B_3) \cup (\partial E_i \cap B_3 \times \partial E_i \cap B_3^c) \cup (\partial E_i \cap B_3^c \times \partial E_i \cap B_3),$$

hence

$$\begin{aligned} \iint_{\partial E_i \times \partial E_i} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{n+s}} &= \iint_{\partial E_i \times \partial E_i \setminus B_3^c \times B_3^c} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{n+s}} \\ &\leq 3 \int_{\partial E_i \cap B_3} \int_{\partial E_i} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{n+s}} \\ &\leq 3 \int_{\partial E_i \cap B_3} d\mathcal{H}^{n-1}(x) \int_{\partial E_i} \frac{\min\{C|x-y|^2, 1\}}{|x-y|^{n+s}} d\mathcal{H}^{n-1}(y). \end{aligned}$$

Letting $A_j := B_{2^{j+1}}(x) \setminus B_{2^j}(x)$ we have

$$\int_{\partial E_i} \frac{\min\{C|x-y|^2, 1\}}{|x-y|^{n+s}} d\mathcal{H}^{n-1}(y) \leq C \sum_{j \in \mathbb{Z}} \frac{\min\{2^{2j}, 1\}}{2^{j(n+s)}} \mathcal{H}^{n-1}(\partial E_i \cap B_{2^{j+1}})$$

$$\leq C \sum_{j\in\mathbb{Z}} \frac{\min\{2^{2j},1\}}{2^{j(n+s)}} 2^{(j+1)(n-1)} \leq C(n,s),$$

where in the second inequality we have used the uniform BV estimate for the sets E_i. Hence, using again the BV estimate for the set E_i, we have

$$\iint_{\partial E_i\times\partial E_i} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}} \leq C\mathcal{H}^{n-1}(\partial E_i\cap B_3) \leq C.$$

Plugging this estimate into the stability inequality

$$\int_{\partial E_i}\int_{\partial E_i\cap B_2} \frac{|\nu_i(x)-\nu_i(y)|^2}{|x-y|^{n+s}} \leq \int_{\partial E_i}\int_{\partial E_i\cap B_3} \frac{|\nu_i(x)-\nu_i(y)|^2}{|x-y|^{n+s}}\varphi(x)^2 \leq C.$$

Now fix $\delta>0$. The last inequality implies

$$\int_{\partial E_i\cap B_2}\int_{\partial E_i\cap B_2} \frac{|\nu_i(x)-\nu_i(y)|^2}{(\delta^2+|x-y|^2)^{\frac{n+s}{2}}} \leq C,$$

for a constant $C>0$ independent of δ.

Let ν be the outer unit normal to ∂E; this is defined everywhere away from the spine of the cone, which is an $\mathcal{H}^{n-1}$ negligible set. We claim that

$$\int_{\partial E\cap B_1}\int_{\partial E\cap B_1} \frac{|\nu(x)-\nu(y)|^2}{(\delta^2+|x-y|^2)^{\frac{n+s}{2}}} \leq \liminf_{i\to\infty} \int_{\partial E_i\cap B_2}\int_{\partial E_i\cap B_2} \frac{|\nu_i(x)-\nu_i(y)|^2}{(\delta^2+|x-y|^2)^{\frac{n+s}{2}}}. \tag{4.49}$$

Proof of the claim. Let $\Psi: B_2\times B_2\to(0,+\infty)$ denote the kernel

$$\Psi(x,y) := \frac{1}{(\delta^2+|x-y|^2)^{\frac{n+s}{2}}},$$

and let $\widetilde{\Psi}: B_2\times B_2\to(0,+\infty)$ be a function with $0\leq\widetilde{\Psi}\leq\Psi$, $\widetilde{\Psi}=0$ outside $B_{3/2}\times B_{3/2}$ and $\widetilde{\Psi}=\Psi$ on $B_1\times B_1$. Such a function can be easily constructed by multiplying Ψ by a smooth cutoff function. We have

$$\begin{aligned}\int_{\partial E_i\cap B_2}\int_{\partial E_i\cap B_2} & |\nu_i(x)-\nu_i(y)|^2\widetilde{\Psi}(x,y)\\ &= 2\int_{\partial E_i\cap B_2}\int_{\partial E_i\cap B_2} (1-\nu_i(x)\cdot\nu_i(y))\widetilde{\Psi}(x,y)\\ &= 2\int_{\partial E_i\cap B_2}\int_{\partial E_i\cap B_2} \widetilde{\Psi}(x,y) - 2\int_{\partial E_i\cap B_2}\int_{\partial E_i\cap B_2} \nu_i(x)\cdot\nu_i(y)\widetilde{\Psi}(x,y),\end{aligned}$$

Applying the divergence theorem twice (here we use that the boundary terms vanish since $\widetilde{\Psi}=0$ on $\partial(B_2\times B_2)$)

$$\int_{\partial E_i\cap B_2}\int_{\partial E_i\cap B_2} \nu_i(x)\cdot\nu_i(y)\widetilde{\Psi}(x,y) = \int_{E_i\cap B_2}\int_{E_i\cap B_2} \sum_{\ell=1}^n \frac{\partial^2\widetilde{\Psi}}{\partial x_\ell\partial y_\ell}\,dxdy,$$

which, by the convergence $E_i \to E$ in $L^1_{\rm loc}(\mathbb{R}^n)$, converges (as $i \to \infty$) to

$$\int_{E\cap B_2}\int_{E\cap B_2}\sum_{\ell=1}^n \frac{\partial^2 \widetilde{\Psi}}{\partial x_\ell \partial y_\ell}\,dxdy = \int_{\partial E\cap B_2}\int_{\partial E\cap B_2} \nu(x)\cdot\nu(y)\widetilde{\Psi}(x,y).$$

Moreover, since $\widetilde{\Psi}$ is smooth, it is standard that

$$\int_{\partial E\cap B_2}\int_{\partial E\cap B_2}\widetilde{\Psi}(x,y) \le \liminf_{i\to\infty}\int_{\partial E_i\cap B_2}\int_{\partial E_i\cap B_2}\widetilde{\Psi}(x,y).$$

Hence

$$\begin{aligned}
\liminf_{i\to\infty} &\int_{\partial E_i\cap B_2}\int_{\partial E_i\cap B_2} |\nu_i(x)-\nu_i(y)|^2\widetilde{\Psi}(x,y)\\
&\ge 2\int_{\partial E\cap B_2}\int_{\partial E\cap B_2}\widetilde{\Psi}(x,y) - 2\int_{\partial E\cap B_2}\int_{\partial E\cap B_2}\nu(x)\cdot\nu(y)\widetilde{\Psi}(x,y)\\
&= \int_{\partial E\cap B_2}\int_{\partial E\cap B_2}|\nu(x)-\nu(y)|^2\widetilde{\Psi}(x,y).
\end{aligned}$$

Since $\Psi \ge \widetilde{\Psi}$ in $B_2\times B_2$ and $\widetilde{\Psi}=\Psi$ on $B_1\times B_1$, this inequality implies (4.49). □

Hence, letting also $\delta \to 0^+$, we get

$$\int_{\partial E\cap B_1}\int_{\partial E\cap B_1}\frac{|\nu(x)-\nu(y)|^2}{|x-y|^{n+s}} \le C. \tag{4.50}$$

Now we show, by contradiction, that this condition cannot be satisfied unless E is a half-space (in which case the left-hand side vanishes). Assume that $E=\widetilde{E}\times\mathbb{R}^{n-2}$ is not a half-plane. Then, $\partial\widetilde{E}$ must contain at least two non-aligned rays $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$, which are half-lines in $\mathbb{R}^2$. Set

$$\Sigma_1 := \widetilde{\Sigma}_1\times\mathbb{R}^{n-2}, \quad \Sigma_2 := \widetilde{\Sigma}_2\times\mathbb{R}^{n-2}.$$

Since $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ are not aligned, we have that

$$|\nu(x)-\nu(y)|^2 \equiv c > 0, \quad \forall\,(x,y)\in\Sigma_1\times\Sigma_2,$$

which, together with (4.50), gives

$$\int_{\Sigma_1\cap B_1}\int_{\Sigma_2\cap B_1}\frac{1}{|x-y|^{n+s}} \le \frac{1}{c}\int_{\partial E\cap B_1}\int_{\partial E\cap B_1}\frac{|\nu(x)-\nu(y)|^2}{|x-y|^{n+s}} \le C. \tag{4.51}$$

Write

$$\int_{\Sigma_1\cap B_1}\int_{\Sigma_2\cap B_1}\frac{1}{|x-y|^{n+s}} \ge \int_{\Sigma_1\cap B_{1/10}} d\mathcal{H}^{n-1}(y)\int_{\Sigma_2\cap B_1}\frac{1}{|x-y|^{n+s}}d\mathcal{H}^{n-1}(x).$$

Let $\theta\in(0,\pi)$ be the angle between Σ_1 and Σ_2, and let $\tau := \sin(\theta)$ if $\theta\in(0,\pi/2]$ and $\tau=1$ if $\theta\in(\pi/2,\pi)$. For every $y\in\Sigma_1\cap B_{1/10}$, by the coarea formula on Σ_2 we have

$$\int_{\Sigma_2\cap B_1}\frac{1}{|x-y|^{n+s}}d\mathcal{H}^{n-1}(x) = \int_{\tau|\widetilde{y}|}^2 \frac{1}{r^{n+s}}\mathcal{H}^{n-2}(\partial B_r(y)\cap\Sigma_2\cap B_1)\,dr$$

$$\geq \int_{\tau|\widetilde{y}|}^{1/2} \frac{1}{r^{n+s}} \mathcal{H}^{n-2}(\partial B_r(y) \cap \Sigma_2)\, dr.$$

Clearly, for $r \in (\tau|\widetilde{y}|, 1/2)$ we have

$$\mathcal{H}^{n-2}(\partial B_r(y) \cap \Sigma_2) \geq \frac{1}{100}\omega_{n-2}(r - \tau|\widetilde{y}|)^{n-2}.$$

Hence

$$\int_{\Sigma_2 \cap B_1} \frac{1}{|x-y|^{n+s}} d\mathcal{H}^{n-1}(x) \geq c \int_{\tau|\widetilde{y}|}^{1/2} \frac{(r-\tau|\widetilde{y}|)^{n-2}}{r^{n+s}}\, dr = \frac{c}{|\widetilde{y}|^{1+s}} \int_1^{\frac{1}{2\tau|\widetilde{y}|}} \frac{(t-1)^{n-2}}{t^{n+s}}\, dt,$$

where we have made the substitution $r = \tau|\widetilde{y}|t$ in the last equality.

Since $\tau|\widetilde{y}| \leq 1/10$ we have that

$$\int_1^{\frac{1}{2\tau|\widetilde{y}|}} \frac{(t-1)^{n-2}}{t^{n+s}}\, dt \geq \int_1^{20} \frac{(t-1)^{n-2}}{t^{n+s}}\, dt = c.$$

Let $Q_{\frac{1}{10\sqrt{n}}}$ be a cube of side $\frac{1}{10\sqrt{n}}$ rotated so that its projection on the first $\mathbb{R}^2$ factor has sides parallel (and orthogonal) to Σ_1. Putting everything together, we have shown

$$\begin{aligned}
\int_{\Sigma_1 \cap B_1} \int_{\Sigma_2 \cap B_1} \frac{1}{|x-y|^{n+s}} &\geq c \int_{\Sigma_1 \cap B_{1/10}} \frac{1}{|\widetilde{y}|^{1+s}} d\mathcal{H}^{n-1}(y) \\
&\geq c \int_{\Sigma_1 \cap Q_{\frac{1}{10\sqrt{n}}}} \frac{1}{|\widetilde{y}|^{1+s}} d\mathcal{H}^{n-1}(y) \\
&= c \int_0^{\frac{1}{10\sqrt{n}}} \frac{1}{\xi^{1+s}}\, d\xi = +\infty,
\end{aligned}$$

which contradicts (4.51).

□

Chapter 5

Classification of stable s-minimal cones in the plane for small s

5.1 Classification of cones

5.1.1 On the different notions of stability

In recent years, two different notions of stability for s-minimal surfaces have been used in the literature [CSV19; CCS20]. We refer to [CCS20, Section 2] for a discussion on these two notions of stability and why a second notion (other than the natural one coming from inner variations) was developed. For the sake of clarity, we recall their major difference here and the role played in this work.

First, we have the natural notion of stability that comes from inner variations of the set, which we call just *stability* and is our Definition 2.2.10. This is the usual notion of stability, which is also used in the context of stationary varifolds.

Secondly, we have a stronger notion of stability coming from *"outer rearrangements"* of the set.

Definition A (Definition 1.6 in [CSV19]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and E be a set with $\mathrm{Per}_s(E;\Omega) < +\infty$. Then, E is said to be stable by rearrangements in Ω if, for every vector field $X \in C^2_c(\Omega;\mathbb{R}^n)$ there holds*

$$\liminf_{t\to 0^+} \frac{1}{t^2}\big(\mathrm{Per}_s(\phi^X_t(E)\cap E,\Omega) - \mathrm{Per}_s(E,\Omega)\big) \geq 0,$$

and

$$\liminf_{t\to 0^+} \frac{1}{t^2}\big(\mathrm{Per}_s(\phi^X_t(E)\cup E,\Omega) - \mathrm{Per}_s(E,\Omega)\big) \geq 0,$$

where $\phi^X_t : \Omega \to \Omega$ is the flow of X at time $t > 0$.

In [CSV19], this notion is called just "stability", but we believe this terminology to be slightly misleading for the reason that follows. While for sets with C^2 boundary in Ω and $s \in (0,1]$, Definition A is known to be equivalent to Definition 2.2.10 (see [CCS20, Remark 3.2] for a proof of this fact), for singular objects they do not coincide in general. In particular, stability by rearrangements (Definition A) allows to infinitesimally break the topology of the set E, while classical stability by inner variations does not. We emphasize that the notion of stability by rearrangements was developed ad-hoc to get rid of cross-like singularities directly from the definition.

Remark 5.1.1. *In the case of the classical perimeter (formally $s = 1$), these two notions of stability are indeed different, since the cross $\boldsymbol{X}$ is stable in $\mathbb{R}^2$ but is not stable by rearrangements in $\mathbb{R}^2$. Nevertheless, since $\boldsymbol{X}$ is smooth outside the origin, $\boldsymbol{X}$ is stable for both notions in $\mathbb{R}^2 \setminus \{0\}$.*

Observe that our Definition 2.2.10 is weaker than Definition A. Indeed, assume that E is stable by rearrangements in Ω. For every $X \in C^2_c(\Omega; \mathbb{R}^n)$, by the elementary inequality

$$\mathrm{Per}_s(\phi^X_t(E), \Omega) + \mathrm{Per}_s(E, \Omega) \geq \mathrm{Per}_s(\phi^X_t(E) \cap E, \Omega) + \mathrm{Per}_s(\phi^X_t(E) \cup E, \Omega),$$

it follows that

$$\liminf_{t \to 0^+} \frac{1}{t^2} \big(\mathrm{Per}_s(\phi^X_t(E), \Omega) - \mathrm{Per}_s(E, \Omega)\big) \geq 0. \tag{5.1}$$

By Corollary 3.4.18, the map $t \mapsto \mathrm{Per}_s(\phi^X_t(E), \Omega)$ is of class C^2 for all $t > 0$. Hence, the limits in

$$a_1 := \frac{d}{dt}\bigg|_{t=0} \mathrm{Per}_s(\phi^X_t(E), \Omega) \quad \text{and} \quad a_2 := \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\phi^X_t(E), \Omega)$$

exist. It follows that $a_1 = 0$ (assuming the opposite, for some $X \in C^2_c(\Omega; \mathbb{R}^n)$ one would get the $\liminf$ in (5.1) to be $-\infty$), and consequently

$$\frac{a_2}{2} = \lim_{t \to 0^+} \frac{1}{t^2} \big(\mathrm{Per}_s(\phi^X_t(E), \Omega) - \mathrm{Per}_s(E, \Omega) - t a_1\big) = \lim_{t \to 0^+} \frac{1}{t^2} \big(\mathrm{Per}_s(\phi^X_t(E), \Omega) - \mathrm{Per}_s(E, \Omega)\big) \geq 0.$$

Thus, E is stable in Ω for Definition 2.2.10.

Remark 5.1.2. *To be precise, in [CCS20] the authors take (5.1) as their definition of stability. Even though this seems to be slightly different from the notion of stability used in this work (Definition 2.2.10), since $t \mapsto \mathrm{Per}_s(\phi^X_t(E), \Omega)$ is of class C^2 for $X \in C^2_c(\Omega; \mathbb{R}^n)$, by the argument above the two notions are equivalent without any additional hypotheses on the set E.*

To explain the surprising part of this work further, let us start by recalling the precise statement of the two available results in the literature on the classification of s-minimal cones in $\mathbb{R}^2$.

Theorem 5.1.3 ([SV13a; SV13b])**.** *Let $s \in (0, 1)$ and $E \subset \mathbb{R}^2$ be an s-minimal cone minimizing the s-perimeter in compact sets of $\mathbb{R}^2$. Then, E is a half-plane.*

Theorem 5.1.4 (Corollary 1.16 in [CSV19])**.** *Let $s \in (0, 1)$ and $E \subset \mathbb{R}^2$ be an s-minimal cone that is stable by rearrangements in $\mathbb{R}^2$ (i.e., for Definition A for every $\Omega \Subset \mathbb{R}^2$). Then, E is a half-plane.*

Since every set that minimizes the s-perimeter in compact sets is stable by rearrangements, we see that Theorem 5.1.4 is stronger than the previous Theorem 5.1.3. The key point for which our result is, in the range s small, more surprising than Theorem 5.1.4 is the following: Theorem 5.1.4 assumes stability (by rearrangement) also around the vertex of the cone, while our Theorem 1.2.20 assumes stability just in $\mathbb{R}^2 \setminus \{0\}$. This is a major difference since Theorem 5.1.4 does not hold only assuming that E is stable by rearrangements in $\mathbb{R}^2 \setminus \{0\}$. Actually, this classification of cones in the plane stable in $\mathbb{R}^2 \setminus \{0\}$ is not even expected to be true for all $s \in (0, 1)$, since the cross $\mathbf{X}$ is expected to be stable in $\mathbb{R}^2 \setminus \{0\}$ for the s-perimeter and s close to 1, in accordance with the case of the classical perimeter.

In this regard, our classification result Theorem 1.2.20 for cones in the plane stable in $\mathbb{R}^2 \setminus \{0\}$ is of purely nonlocal nature and represents a remarkable difference from the theory of the classical perimeter.

5.1.2 Hardy's inequality and the BV estimate for small s

Our proof relies on the Hardy inequality for the $H^\sigma(\mathbb{R})$ seminorm and, specifically, on the asymptotic behavior of its optimal constant as $\sigma \downarrow 1/2$. The sharp constant in this inequality has been established in [FS08a] (equation (1.6)), and it is also stated in [CCS20, Theorem 3.3]. We will also use the fact that radially symmetric functions in $C^2_c(\mathbb{R} \setminus \{0\})$ nearly saturate Hardy's inequality; this is proved in Section 3.3 of [FS08a].

Theorem 5.1.5 (Hardy's inequality)**.** *Let $n \geq 1$, $\sigma \in (0,1)$ and $u \in H^\sigma_0(\mathbb{R}^n \setminus \{0\})$. Then*

$$\mathcal{H}_{n,\sigma} \int_{\mathbb{R}^n} \frac{u^2}{|x|^{2\sigma}}\, dx \leq c_{n,\sigma} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\sigma}}\, dxdy,$$

where

$$\mathcal{H}_{n,\sigma} = 2^{2\sigma-1}(\sigma - n/2)^2 \frac{\Gamma(n/4+\sigma/2)^2}{\Gamma(n/4-\sigma/2+1)^2},$$

and

$$c_{n,\sigma} = 2^{2\sigma-1}\pi^{-n/2}\frac{\Gamma(n/2+\sigma)}{\Gamma(2-\sigma)}\sigma(1-\sigma).$$

Moreover, the inequality is saturated by radial functions, that is: for every $\varepsilon > 0$ there exists a radial function $\xi(x) = \xi(|x|) \in C^2_c(\mathbb{R}^n \setminus \{0\})$ such that

$$(\mathcal{H}_{n,\sigma} + \varepsilon) \int_{\mathbb{R}^n} \frac{\xi(x)^2}{|x|^{2\sigma}}\, dx \geq c_{n,\sigma} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{n+2\sigma}}\, dxdy.$$

In particular, for $n = 1$, $s \in (0,1/2)$ and $\sigma = \frac{1+s}{2}$ the constant reads as

$$\mathcal{H}_{1,\frac{1+s}{2}} = s^2 2^{s-2}\frac{\Gamma(\frac{2+s}{4})^2}{\Gamma(\frac{4-s}{4})^2},$$

and, by elementary properties of the Gamma function, it is easily checked that

$$\frac{\mathcal{H}_{1,\frac{1+s}{2}}}{c_{1,\frac{1+s}{2}}} \leq \frac{Cs^2}{C\frac{1+s}{2}(1-\frac{1+s}{2})} \leq Cs^2,$$

for some absolute $C > 0$ and every $s \in (0,1/2)$.

Summarizing, taking the ξ relative to $\varepsilon = \mathcal{H}_{1,\frac{1+s}{2}}$ in the saturation statement above, for every $s \in (0,1/2)$ there exists an even function $\xi \in C^2_c(\mathbb{R} \setminus \{0\})$ such that

$$\int_{\mathbb{R}} \frac{\xi(x)^2}{|x|^{1+s}}\, dx \geq \frac{1}{Cs^2} \iint_{\mathbb{R}\times\mathbb{R}} \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\, dxdy.$$

Lastly, for the same ξ, this directly implies

$$\int_0^\infty \frac{\xi(x)^2}{x^{1+s}}\, dx \geq \frac{1}{Cs^2} \int_0^\infty \int_0^\infty \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\, dxdy. \tag{5.2}$$

The BV-estimate for small s

It is known [CSV19; Flo24] that stable s-minimal surfaces enjoy a uniform interior BV-estimate, and that the same holds for stable solutions of the fractional Allen-Cahn equation [CCS21; CFS24b].

The results in these references only control the dependence of the constant from s as $s \to 1^-$. In this work, we need the same type of BV-estimate with a control of the constant as $s \to 0$.

Theorem 5.1.6. *Let $R > 0$, $s \in (0, 1/2)$, and $E \subset \mathbb{R}^n$ be an s-minimal surface which stable in $B_R(x)$ and such that ∂E is C^2 in $B_R(x)$. Then*

$$\mathrm{Per}(E, B_{R/2}(x)) \le \frac{C}{s},$$

for some dimensional constant $C > 0$.

We first recall a useful interpolation inequality for the s-perimeter that accounts for the dependence on s both for $s \to 1^-$ and $s \to 0$. The inequality in the form we use here is not explicitly stated anywhere; it is written throughout the lines of the proof of Theorem 3.1 in [Tho24], or it follows from [Flo24, Lemma 3.13] and Young's inequality.

Lemma 5.1.7. *Let $s \in (0, 1)$, $R > 0$, and E be a set with locally finite perimeter. Then*

$$\mathrm{Per}_s(E, B_R(x)) \le C \left(\frac{R^{1-s}}{1-s} \mathrm{Per}(E, B_{5R}(x)) + \frac{R^{n-s}}{s} \right),$$

for some dimensional constant $C > 0$.

With this inequality, we can deduce the BV-estimate for small s.

Proof of Theorem 5.1.6. Similarly to [CCS20], the theorem follows by inspection of the proof of Theorem 1.7 in [CSV19], taking care of the explicit dependence of the constants as $s \to 0$. For the sake of clarity, we rewrite here the crucial estimates in the proof of Theorem 1.7 in [CSV19], with the precise dependence of all constants on s, as $s \to 0$. In the proof that follows $C > 0$ is a dimensional constant that can change from line to line.

Since the statement is scaling and translation invariant, we can assume $R = 1$ and $x = 0$. Since E is stable in B_1, by Theorem 1.9 in [CSV19] applied to the kernel $K(z) = 1/|z|^{n+s}$, we get

$$\mathrm{Per}(E, B_1) \le C \left(1 + \sqrt{\mathrm{Per}_s(E, B_4)} \right). \tag{5.3}$$

Moreover, by Lemma 5.1.7 applied with $R = 4$ we have

$$\mathrm{Per}_s(E, B_4) \le C \left(\frac{1}{1-s} \mathrm{Per}(E, B_{20}) + \frac{1}{s} \right) \le C \left(\mathrm{Per}(E, B_{20}) + \frac{1}{s} \right),$$

where we have also used that $s \le 1/2$. Thus, by (5.3) and Young's inequality we get

$$\begin{aligned} \mathrm{Per}(E, B_1) \le C \left(1 + \sqrt{\mathrm{Per}(E, B_{20}) + \frac{1}{s}} \right) &\le C \left(1 + \delta \mathrm{Per}(E, B_{20}) + \frac{\delta}{s} + \frac{1}{\delta} \right) \\ &= C \left(1 + \frac{\delta}{s} + \frac{1}{\delta} \right) + \delta \mathrm{Per}(E, B_{20}), \end{aligned}$$

for every $\delta > 0$. From here, arguing exactly as the end of the proof of [CSV19, Theorem 1.7] or [CFS24b, Proposition 3.14], choosing δ smaller than a dimensional constant $\delta_\circ = \delta_\circ(n) > 0$ and a covering argument one concludes the uniform bound

$$\mathrm{Per}(E, B_{1/2}) \le C \left(1 + \frac{\delta_\circ}{s} + \frac{1}{\delta_\circ} \right) \le \frac{C}{s}.$$

$\square$

5.1.3 Classification of stable s-minimal cones in the plane for small s

It is well known that, for sets with C^2 boundary, the nonlocal mean curvature can be expressed as a boundary integral (see, for example, the introduction of [Cab+18] or [Cir+18]). Precisely

$$H_s^E(x) = \frac{2}{s} P.V.\int_{\partial E} \frac{(y-x)\cdot \nu_{\partial E}(y)}{|y-x|^{n+s}}\, d\sigma(y), \tag{5.4}$$

where $\nu_{\partial E}$ denotes the outer unit normal to ∂E.

In the proof of our main result, that is Theorem 1.2.20, we will need to use this fact for a cone in $\mathbb{R}^2$, which is not globally C^2. The validity of this representation formula (5.4) at some $x \in \partial E$ does not really require the smoothness of ∂E everywhere, but just smoothness in a neighborhood of x and mild regularity of ∂E outside this neighborhood. Since the author could not find any precise statement of this fact, and since we will crucially need it, we state and prove it here.

Lemma 5.1.8. *Let $s \in (0,1)$, and let $E \subset \mathbb{R}^n$ be a set that is a finite union of open domains with Lipschitz boundary. Let $x \in \partial E$ and assume that ∂E is C^2 in a neighborhood of x. Then $H_s^E(x)$ can be expressed as in (5.4).*

Proof. Fix $\varepsilon > 0$ small. As the statement is invariant by translations, we can assume $x = 0 \in \partial E$. Observe that

$$\operatorname{div}\left(\frac{y}{|y|^{n+s}}\right) = -\frac{s}{|y|^{n+s}}.$$

We want to apply the divergence theorem to the vector field $F \in C^1(\mathbb{R}^n \setminus B_\varepsilon; \mathbb{R}^n)$ defined by $F(y) = y|y|^{-(n+s)}$, but this has not compact support. Nevertheless, since $|F| = o(|y|^{1-n})$ as $|y| \to +\infty$, it is a classical fact that the divergence theorem can be applied (e.g., [Ser83, Corollary 5.2]). Hence, by the divergence theorem, we can infer

$$\int_{\partial(E\setminus B_\varepsilon)} \frac{y\cdot\nu(y)}{|y|^{n+s}}\, d\sigma(y) = \int_{E\setminus B_\varepsilon} \operatorname{div}\left(\frac{y}{|y|^{n+s}}\right) dy = -s\int_{E\setminus B_\varepsilon} \frac{1}{|y|^{n+s}}\, dy,$$

where ν is the outer unit normal to $\partial(E \setminus B_\varepsilon)$. Similarly

$$\int_{\partial(E^c\setminus B_\varepsilon)} \frac{y\cdot\widetilde{\nu}(y)}{|y|^{n+s}}\, d\sigma(y) = -s\int_{E^c\setminus B_\varepsilon} \frac{1}{|y|^{n+s}}\, dy,$$

where $\widetilde{\nu}$ is the outer unit normal to $\partial(E^c \setminus B_\varepsilon)$. Note that

$$\partial(E\setminus B_\varepsilon) = (\partial E\setminus B_\varepsilon)\cup(\partial B_\varepsilon\cap E^c), \quad \text{and } \partial(E^c\setminus B_\varepsilon) = (\partial E^c\setminus B_\varepsilon)\cup(\partial B_\varepsilon\cap E),$$

and that $\widetilde{\nu} = -\nu$ on the shared part of the boundary $\partial E \setminus B_\varepsilon = \partial E^c \setminus B_\varepsilon$. Thus, subtracting the two equalities above

$$\begin{aligned} s\int_{\mathbb{R}^n\setminus B_\varepsilon} \frac{\chi_{E^c}(y)-\chi_E(y)}{|y|^{n+s}}\, dy &= \int_{\partial(E\setminus B_\varepsilon)} \frac{y\cdot\nu(y)}{|y|^{n+s}}\, d\sigma(y) - \int_{\partial(E^c\setminus B_\varepsilon)} \frac{y\cdot\widetilde{\nu}(y)}{|y|^{n+s}}\, d\sigma(y) \\ &= 2\int_{\partial E\setminus B_\varepsilon} \frac{y\cdot\nu(y)}{|y|^{n+s}}\, d\sigma(y) - \int_{\partial B_\varepsilon\cap E^c} \frac{d\sigma(y)}{|y|^{n+s-1}} + \int_{\partial B_\varepsilon\cap E} \frac{d\sigma(y)}{|y|^{n+s-1}}. \end{aligned}$$

Moreover

$$-\int_{\partial B_\varepsilon \cap E^c} \frac{d\sigma(y)}{|y|^{n+s-1}} + \int_{\partial B_\varepsilon \cap E} \frac{d\sigma(y)}{|y|^{n+s-1}} = \frac{1}{\varepsilon^{n+s-1}} \big(\mathcal{H}^{n-1}(\partial B_\varepsilon \cap E) - \mathcal{H}^{n-1}(\partial B_\varepsilon \cap E^c)\big),$$

and it easily follows from the hypothesis that ∂E is C^2 in a neighborhood of 0 that

$$\mathcal{H}^{n-1}(\partial B_\varepsilon \cap E) - \mathcal{H}^{n-1}(\partial B_\varepsilon \cap E^c) = O(\varepsilon^n).$$

Thus

$$-\int_{\partial B_\varepsilon \cap E^c} \frac{d\sigma(y)}{|y|^{n+s-1}} + \int_{\partial B_\varepsilon \cap E} \frac{d\sigma(y)}{|y|^{n+s-1}} = O(\varepsilon^{1-s}) \to 0, \quad \text{as } \varepsilon \to 0^+.$$

Letting $\varepsilon \to 0^+$ above and diving both sides by s gives

$$P.V.\!\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y|^{n+s}}\, dy = \frac{2}{s} P.V.\!\int_{\partial E} \frac{y \cdot \nu_{\partial E}(y)}{|y|^{n+s}}\, d\sigma(y),$$

which is what we wanted to prove. □

Let us fix the notation that we will use for cones in the plane. For a cone $E \subset \mathbb{R}^2$, we write $\Sigma = \partial E$ and observe that Σ is a union of half-lines from the origin. Write

$$E = \bigcup_{i=1}^{N} E_i\,, \quad \Sigma = \bigcup_{i=1}^{2N} \Sigma_i\,, \quad \partial E_i = \{\Sigma_i, \Sigma_{i+1}\},$$

with the convention that $\Sigma_{2N+1} = \Sigma_1$. Here E_i are disjoint conical sectors from the origin, that is $\lambda E_i = E_i$ for every $\lambda > 0$, Σ_i are rays from the origin with the induced orientation from E_i, and the number N could be $+\infty$ in general, but will be finite in our proof.

We also denote by θ_i^j the counterclockwise angle from Σ_i and Σ_j.

Lemma 5.1.9. *In the notation above, there is $c > 0$ such that for every $x \in \Sigma_j$ there holds*

$$\int_{\Sigma_i} \frac{1}{|x-y|^{2+s}}\, d\sigma(y) \geq \frac{c}{|x|^{1+s}(1-\cos(\theta_i^j))^{1+s}}.$$

Proof. We have

$$\begin{aligned}
\int_{\Sigma_i} \frac{1}{|x-y|^{2+s}}\, d\sigma(y) &= \int_{\Sigma_i} \frac{d\sigma(y)}{(|x|^2+|y|^2-2\cos(\theta_i^j)|x||y|)^{\frac{2+s}{2}}} \\
&= \int_0^\infty \frac{dz}{(|x|^2+z^2-2\cos(\theta_i^j)|x|z)^{\frac{2+s}{2}}} = \frac{1}{|x|^{1+s}} \int_0^\infty \frac{dt}{(1+t^2-2t\cos(\theta_i^j))^{\frac{2+s}{2}}},
\end{aligned}$$

where we have substituted $z = t|x|$ in the last line. Moreover

$$\begin{aligned}
\int_0^\infty \frac{dt}{(1+t^2-2t\cos(\theta_i^j))^{\frac{2+s}{2}}} &= \int_0^\infty \frac{dt}{((t-1)^2+2t(1-\cos(\theta_i^j)))^{\frac{2+s}{2}}} \\
&\geq \int_{1/2}^{3/2} \frac{dt}{((t-1)^2+2t(1-\cos(\theta_i^j)))^{\frac{2+s}{2}}} \geq \int_{1/2}^{3/2} \frac{dt}{((t-1)^2+3(1-\cos(\theta_i^j)))^{\frac{2+s}{2}}} \\
&= \int_{-1/2}^{1/2} \frac{dt}{((t-1)^2+3(1-\cos(\theta_i^j)))^{\frac{2+s}{2}}} \geq \int_{1/2}^{3/2} \frac{dt}{(t^2+3(1-\cos(\theta_i^j)))^{\frac{2+s}{2}}}
\end{aligned}$$

$$\geq \frac{1}{(1-\cos(\theta_i^j))^{1+s}} \int_{-1/10}^{1/10} \frac{dt}{(t^2+3)^{\frac{2+s}{2}}} \geq \frac{c}{(1-\cos(\theta_i^j))^{1+s}}.$$

This concludes the proof. $\square$

Now, we have all the ingredients to prove our main result Theorem 1.2.20. In the proof, we plug in the stability inequality a radial test function that nearly saturates Hardy's inequality on $(0,\infty)$, in the sense that (5.2) holds.

Proof of Theorem 1.2.20. By Theorem 5.1.6, that is the BV-estimate for stable s-minimal surfaces for $s \in (0,1/2)$, and a standard covering argument we get that

$$2N = \mathrm{Per}(E, B_2 \setminus B_1) \leq \frac{C}{s}.$$

Hence $\Sigma = \partial E$ is a finite number of rays from the origin, whose number is bounded by

$$2N \leq \frac{C}{s}. \tag{5.5}$$

Recall the stability inequality (2.10), and let ν_i be the outer unit normal to Σ_i from E_i. For the left hand side, for every $\varphi \in C_c^2(\mathbb{R}^2 \setminus \{0\})$, we have

$$\begin{aligned}\iint_{\Sigma\times\Sigma} \frac{|\nu_\Sigma(x)-\nu_\Sigma(y)|^2}{|x-y|^{2+s}}\varphi(x)^2\, d\sigma(x)d\sigma(y) &= \sum_{i,j} \iint_{\Sigma_i\times\Sigma_j} \frac{|\nu_i(x)-\nu_j(y)|^2}{|x-y|^{2+s}}\varphi(x)^2\, d\sigma(x)d\sigma(y)\\ &= 2\sum_{i\neq j}(1-(-1)^{i+j}\cos(\theta_i^j)) \iint_{\Sigma_i\times\Sigma_j} \frac{\varphi(x)^2}{|x-y|^{2+s}}\, d\sigma(x)d\sigma(y).\end{aligned}$$

By Lemma 5.1.9 we can estimate

$$\iint_{\Sigma_i\times\Sigma_j} \frac{\varphi(x)^2}{|x-y|^{2+s}}\, d\sigma(x)d\sigma(y) \geq \frac{c}{(1-\cos(\theta_i^j))^{1+s}} \int_{\Sigma_j} \frac{\varphi(x)^2}{|x|^{1+s}}\, dx.$$

Now, taking $\varphi(x) = \xi(|x|)$ with ξ saturating Hardy's inequality on $(0,\infty)$ as in (5.2), gives

$$\int_{\Sigma_j} \frac{\xi(x)^2}{|x|^{1+s}}\, d\sigma(x) \geq \frac{1}{Cs^2} \iint_{\Sigma_j\times\Sigma_j} \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\, d\sigma(x)d\sigma(y),$$

thus

$$\begin{aligned}&\iint_{\Sigma\times\Sigma} \frac{|\nu_\Sigma(x)-\nu_\Sigma(y)|^2}{|x-y|^{1+s}}\xi(|x|)^2\, d\sigma(x)d\sigma(y)\\ &\qquad\geq \frac{c}{s^2}\sum_{j=1}^{2N}\left(\iint_{\Sigma_j\times\Sigma_j} \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\, d\sigma(x)d\sigma(y)\right) \sum_{1\leq i\leq 2N,\, i\neq j} \frac{1-(-1)^{i+j}\cos(\theta_i^j)}{(1-\cos(\theta_i^j))^{1+s}}.\\ &\qquad= \frac{c}{s^2}\left(\int_0^\infty\int_0^\infty \frac{|\xi(x)-\xi(y)|^2}{|x-y|^{2+s}}\, dxdy\right)\sum_{j=1}^{2N}\sum_{\{1\leq i\leq 2N,\, i\neq j\}} \frac{1-(-1)^{i+j}\cos(\theta_i^j)}{(1-\cos(\theta_i^j))^{1+s}}.\end{aligned} \tag{5.6}$$

Claim. There exists $s_\circ < 1/2$ sufficiently small with the following property. For every

$s \in (0, s_\circ)$, there exists $j \in \{1, 2, \ldots, 2N\}$ such that

$$\sum_{\{1 \le i \le 2N,\, i \ne j\}} \frac{1 - (-1)^{i+j} \cos(\theta_i^j)}{(1 - \cos(\theta_i^j))^{1+s}} \le \frac{1}{100}. \tag{5.7}$$

Indeed, suppose this is not the case. Then, for s arbitrary small, (5.6) implies

$$\iint_{\Sigma\times\Sigma} \frac{|\nu_\Sigma(x) - \nu_\Sigma(y)|^2}{|x-y|^{1+s}} \xi(|x|)^2 \, d\sigma(x) d\sigma(y) \ge \frac{cN}{100s^2} \int_0^\infty \int_0^\infty \frac{|\xi(x) - \xi(y)|^2}{|x-y|^{2+s}} \, dxdy.$$

From this inequality, using that E is stable gives

$$\begin{aligned}
&\frac{cN}{100s^2} \int_0^\infty \int_0^\infty \frac{|\xi(x) - \xi(y)|^2}{|x-y|^{2+s}} \, dxdy \\
&\qquad \le \iint_{\Sigma\times\Sigma} \frac{|\nu_\Sigma(x) - \nu_\Sigma(y)|^2}{|x-y|^{1+s}} \xi(|x|)^2 \, d\sigma(x) d\sigma(y) \\
&\qquad \overset{\text{(stability)}}{\le} \iint_{\Sigma\times\Sigma} \frac{|\xi(|x|) - \xi(|y|)|^2}{|x-y|^{2+s}} \, d\sigma(x) d\sigma(y) \\
&\qquad = N \int_0^\infty \int_0^\infty \frac{|\xi(x) - \xi(y)|^2}{|x-y|^{2+s}} \, dxdy + \sum_{i \ne j} \iint_{\Sigma_i \times \Sigma_j} \frac{|\xi(|x|) - \xi(|y|)|^2}{|x-y|^{2+s}} \, d\sigma(x) d\sigma(y).
\end{aligned}$$

Moreover, since $|x - y| \ge ||x| - |y||$, we have

$$\begin{aligned}
\iint_{\Sigma_i \times \Sigma_j} \frac{|\xi(|x|) - \xi(|y|)|^2}{|x-y|^{2+s}} \, d\sigma(x) d\sigma(y) &\le \iint_{\Sigma_i \times \Sigma_j} \frac{|\xi(|x|) - \xi(|y|)|^2}{||x| - |y||^{2+s}} \, d\sigma(x) d\sigma(y) \\
&= \int_0^\infty \int_0^\infty \frac{|\xi(x) - \xi(y)|^2}{|x-y|^{2+s}} \, dxdy,
\end{aligned}$$

for every $i \ne j$. Thus

$$\frac{cN}{100s^2} \int_0^\infty \int_0^\infty \frac{|\xi(x) - \xi(y)|^2}{|x-y|^{2+s}} \, dxdy \le N^2 \int_0^\infty \int_0^\infty \frac{|\xi(x) - \xi(y)|^2}{|x-y|^{2+s}} \, dxdy,$$

which implies, together with (5.5), that

$$s^2 \ge \frac{c}{100N} \ge \frac{c}{50C} s,$$

which gives a contradiction if s is small. Hence, the claim is proved.

Now, we conclude the proof of our theorem by contradiction, with $s_\circ$ the one given by the claim above. Assume by contradiction that $N \ge 2$. Then, for the index j such that (5.7) holds we get that (here, as above, the indices are modulo $2N$)

$$\frac{1 - (-1)^{j+(j+2)} \cos(\theta_j^{j+2})}{(1 - \cos(\theta_j^{j+2}))^{1+s}} = \frac{1}{(1 - \cos(\theta_j^{j+2}))^s} \le \frac{1}{100},$$

holds for every $s \le s_\circ$. Clearly, this is not possible for any value of $\theta_j^{j+2} \in [0, 2\pi)$, and hence we conclude that $N = 1$ and E is made only of one conical sector of angle θ.

At this point, we would formally like to argue that, being E stationary for the fractional perimeter, the first variation formula at $x = 0$ implies that E is a half-space. Nevertheless, the first-variation formula holds only at points $x \in \partial E$ where ∂E is C^2 in a neighborhood of x (see Remark 2.2.6), and this is not the case for the tip of the cone. Hence, we have to argue in this spirit but a bit more indirectly.

With no loss of generality, up to rotation and complementation, assume that $\theta \in (0, \pi]$ and

$$E = \{\rho e^{i\varphi} \,:\, \rho > 0, \varphi \in (0, \theta)\}.$$

Let $(\mathbf{t}_1, \mathbf{t}_2) = \mathbf{t} := e^{i\theta} \in \mathbb{S}^1$, so that $\partial E = \Sigma = \Sigma_0 \cup \Sigma_1$ where $\Sigma_0 = \{\lambda e_1 \,:\, \lambda > 0\}$ and $\Sigma_1 = \{\lambda \mathbf{t} \,:\, \lambda > 0\}$. Observe that, since $\theta \in (0, \pi]$, we have that $\mathbf{t}_2 \geq 0$.

Since $e_1 = (1, 0) \in \Sigma$ and Σ is smooth in a neighborhood of e_1 (see Remark 2.2.6), by the first-variation formula

$$H_s^E(e_1) = P.V.\!\int_{\mathbb{R}^2} \frac{\chi_{E^c}(y) - \chi_E(y)}{|y - e_1|^{2+s}}\, dy = 0.$$

Moreover, by Lemma 5.1.8 this implies

$$P.V.\!\int_{\Sigma} \frac{(y - e_1) \cdot \nu_\Sigma(y)}{|y - e_1|^{2+s}}\, d\sigma(y) = 0.$$

Note that the respective outer-unit normals to Σ_0 and Σ_1 are $-e_2$ and $\mathbf{t}^\perp := i\mathbf{t}$, respectively. Hence $(y - e_1) \cdot \nu_\Sigma(y) = 0$ for all $y \in \Sigma_0$, and $(y - e_1) \cdot \nu_\Sigma(y) = -\mathbf{t}_1^\perp = \mathbf{t}_2 \geq 0$ for all $y \in \Sigma_1$. In particular $(y - e_1) \cdot \nu_\Sigma(y) \geq 0$ for all $y \in \Sigma$. Thus

$$0 = P.V.\!\int_{\Sigma} \frac{(y - e_1) \cdot \nu_\Sigma(y)}{|y - e_1|^{2+s}}\, d\sigma(y) \geq 0.$$

This means that all the inequalities above are equalities, that is

$$\mathbf{t}_2 = 0 \implies \mathbf{t} = \pm e_1 \implies \theta \in \{0, \pi\}.$$

Since $\theta \in (0, \pi]$, necessarily $\theta = \pi$ and thus E is a half-space. $\square$

Chapter 6

Asymptotics of the fractional Laplacian and noncompact manifolds

6.1 Asymptotics of the fractional Laplacian and noncompact manifolds

In this section, we present the results obtained in [CG24]. In this work, we have studied the asymptotics of the fractional Laplacian as $s \to 0^+$ on any complete Riemannian manifold (M, g), both of finite and infinite volume. When M is not stochastically complete, we find that this asymptotics is related to the existence of bounded harmonic functions on M; see, for example, Theorem 1.2.25 in the introduction.

As a corollary, in [CG24] we find the asymptotics of the fractional s-perimeter on (essentially) every complete manifold, generalizing both the existing results [Dip+13] for $\mathbb{R}^n$ and [Car+22] for the Gaussian space.

6.1.1 Preliminaries

Theorem 6.1.1 (Yau)**.** *Let M be a complete Riemannian manifold. Then every $L^2(M)$ harmonic function is constant.*

Proof. Let $u \in L^2(M)$ be harmonic. It is a standard result by Yau (see, for example, [Li12, Lemma 7.1]) that, on every complete Riemannian manifold M, the Caccioppoli-type inequality

$$\int_{B_R(p)} |\nabla u|^2 \, d\mu \leq \frac{4}{R^2} \int_{B_{2R}(p)} |u|^2 \, d\mu$$

holds. Since $u \in L^2(M)$, letting $R \to \infty$ gives that u is constant. □

Definition 6.1.2 ($L^\infty-$Liouville property)**.** *We say that a Riemannian manifold M has the* $L^\infty-$Liouville *property if every bounded harmonic function on M is constant.*

In the next lemma, we give the proof of a result that is known in the case of smooth Riemannian manifolds [CK91; Sim93; Pin92]. Here, we give a proof similar to [Sim93] that extends without modifications to the case of weighted Riemannian manifolds and more general ambient spaces.

Lemma 6.1.3. *Let M be a complete, connected Riemannian manifold. Then* *(i) If* $\mu(M) < +\infty$*, then for all* $x, y \in M$

$$\lim_{t\to+\infty} H_M(x,y,t) = \frac{1}{\mu(M)}\,,$$

and the convergence is uniform in $\Omega \times \Omega$*, for every bounded* $\Omega \subset M$*.*

(ii) If $\mu(M) = +\infty$*, then for all* $x, y \in M$

$$\lim_{t\to+\infty} H_M(x,y,t) = 0\,,$$

and the convergence is uniform in $\Omega \times \Omega$*, for every bounded* $\Omega \subset M$*. Moreover, for every fixed* $p \in M$ *there holds also*

$$\lim_{t\to+\infty} \sup_{x\in M} H_M(x,p,t) = 0\,. \tag{6.1}$$

Proof. To prove the result, we use spectral theory. Let us first do the case $\mu(M) = +\infty$. The spectrum of the Laplacian $\sigma(-\Delta)$ is contained in $[0,\infty)$ and by Theorem 6.1.1, we know that the eigenspace of $\lambda = 0$ contains no constant function except for the function identically 0.

Let $\{E_\lambda\}_{\lambda\geq 0}$ be the spectral resolution of the Laplacian, then for every $f \in L^2(M)$ (here $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the $L^2(M)$ scalar product)

$$\langle e^{t\Delta} f, f\rangle = \int_0^\infty e^{-t\lambda} d\langle E_\lambda f, f\rangle.$$

Since $\lim_{t\to\infty} e^{-\lambda t} = \chi_{\{0\}}(\lambda)$ we can apply dominated convergence to deduce that

$$\lim_{t\to\infty} \langle e^{t\Delta} f, f\rangle = \langle E_0 f, f\rangle\,,$$

and since E_0 projects onto the eigenspace of $\lambda = 0$, made only by the constant function identically zero, we get

$$\lim_{t\to\infty} \langle e^{t\Delta} f, f\rangle = 0\,. \tag{6.2}$$

Now note that for all $f, g \in L^2(M)$ we have $|\langle e^{t\Delta} f, g\rangle| = |\langle e^{t/2\Delta} f, e^{t/2\Delta} g\rangle|$. Thus by Cauchy-Schwartz

$$\begin{aligned}\langle e^{t\Delta} f, g\rangle = \langle e^{t/2\Delta} f, e^{t/2\Delta} g\rangle &\leq \|e^{t/2\Delta} f\|_{L^2}\|e^{t/2\Delta} g\|_{L^2}\\ &= \langle e^{t\Delta} f, f\rangle\langle e^{t\Delta} g, g\rangle\,.\end{aligned}$$

Taking the supremum over $g \in L^2(M)$ with $\|g\|_{L^2} \leq 1$ and sending $t \to \infty$ gives that $e^{t\Delta} f \to 0$ strongly in $L^2(M)$. Since this holds for all $f \in L^2(M)$, this implies $H_M(\cdot, y, t) \to 0$ in $L^2(M)$ as $t \to \infty$.

Now, by a local parabolic Harnack inequality, we can convert this convergence into pointwise convergence that is actually locally uniform. Indeed for $p \in M$, $R \ll 1$ to be chosen depending on p, and $t \geq 10$, taking $f = \chi_{B_R(p)}$ above gives

$$\langle e^{t\Delta}\chi_{B_R(p)}, \chi_{B_R(p)}\rangle = \int_{B_R(p)}\int_{B_R(p)} H_M(x,y,t)\,d\mu(x)d\mu(y) \geq \mu(B_R(p))^2 \inf_{x,y\in B_R(p)} H_M(x,y,t)\,.$$

By the parabolic Harnack inequality (see Remark 6.1.5 after this proof) applied two times

$$\begin{aligned}\inf_{x,y\in B_R(p)} H_M(x,y,t) &\geq C^{-1}\inf_{x\in B_R(p)}\sup_{y\in B_R(p)} H_M(x,y,t-1/2)\\ &\geq C^{-1}\sup_{x\in B_R(p)}\inf_{y\in B_R(p)} H_M(x,y,t-1/2)\\ &\geq C^{-2}\sup_{x,y\in B_R(p)} H_M(x,y,t-1)\,,\end{aligned}$$

for some $C>0$ depending on $B_R(p)\subset M$ but independent of t. Hence

$$\sup_{x,y\in B_R(p)} H_M(x,y,t)\leq C(B_R(p))\langle e^{(t+1)\Delta}\chi_{B_R(p)},\chi_{B_R(p)}\rangle\to 0\,,$$

as $t\to\infty$. Covering any bounded set with small balls allows us to infer the desired local uniform convergence.

We are left to prove (6.1). By the properties of the heat kernel, we have

$$H_M(p,p,t)=\int_M H_M^2(p,z,t/2)d\mu(z)=\|H_M(p,\cdot,t/2)\|^2_{L^2(M)}.$$

Moreover

$$H_M(x,p,t)=\int_M H_M(x,z,t/2)H_M(p,z,t/2)d\mu(z)\leq\sqrt{H_M(p,p,t)}\|H_M(x,\cdot,t/2)\|_{L^2(M)},$$

which concludes the proof if we are able to show that $\sup_{x\in M}\|H_M(x,\cdot,t/2)\|_{L^2(M)}$ is bounded as $t\to\infty$. However, since by the semigroup property $H_M(x,y,t)=e^{(t-1)\Delta}(H_M(x,\cdot,1))(y)$ and we have the contraction estimate $\|e^{s\Delta}(f)\|_{L^2(M)}\leq\|f\|_{L^2(M)}$ for every $s\in(0,\infty)$ and $f\in L^2(M)$ we can write

$$\|H_M(x,\cdot,t)\|_{L^2(M)}=\|e^{(t-1)\Delta}(H_M(x,\cdot,1))\|_{L^2}\leq\|H_M(x,\cdot,1)\|_{L^2}\quad\forall t>1.$$

Therefore, we reach the sought conclusion. This concludes the proof of (ii).

Now assume $\mu(M)<+\infty$. Since the proof in this case is almost identical to the one for infinite volume, we just sketch the argument, highlighting the differences. The only essential difference is that in the case $\mu(M)<+\infty$, the eigenspace relative to $\lambda=0$ is made only by the constant function $\mu(M)^{-1/2}$. Hence

$$E_0 f=\frac{1}{\mu(M)}\int_M f\,d\mu=:\fint_M f,$$

and in place of (6.2) we get

$$\lim_{t\to\infty}\left\langle e^{t\Delta}f-\fint_M f,f\right\rangle\to 0.$$

From here, the proof proceeds exactly the same, showing that $H_M(\cdot,y,t)-1/\mu(M)\to 0$ strongly in $L^2(M)$. Then, one can turn the convergence into pointwise and locally uniform by a similar argument with the parabolic Harnack inequality.

Indeed, denoting by $(f)_+$ the positive part of f, we see that the function

$$v:=(H_M(\cdot,y,t)-1/\mu(M))_+$$

is a nonnegative subsolution to the heat equation. Then, by the parabolic version of the Moser-

Harnack inequality (see, for example, [Sal95, Theorem 5.1]), we have (here C depends on R and the geometry of M in $B_{2R}(p)$)

$$\sup_{[t+R^2/2,t+R^2]} v^2 \leq C \int_t^{t+R^2} \int_{B_R(p)} |v|^2 \, d\mu \to 0 \, . \tag{6.3}$$

Hence $\limsup_{t\to\infty} H_M(\cdot, y, t) \leq 1/\mu(M)$. Arguing similarly with the negative part also gives the $\liminf$ inequality, and hence the pointwise convergence. The fact that the convergence is uniform follows from (6.3).

□

Remark 6.1.4. *Since $\|H_M(x,\cdot,t)\|_{L^1(M)} \leq 1$ and $\|H_M(x,\cdot,t)\|_{L^\infty(M)} \to 0$ as $t \to \infty$, we conclude that also $\|H_M(x,\cdot,t)\|_{L^p(M)} \to 0$ for any $p \in (1,\infty]$. The convergence to zero in $L^1(M)$ is clearly prevented if M is stochastically complete.*

Remark 6.1.5. *We emphasize that we have used only a local (non-uniform) Harnack inequality in $B_R(p) \subset M$, that is where the constant is allowed to depend on the point p and radius R. This is clear since, for fixed $p \in M$ one can take $R \ll 1$ such that, in normal coordinates at p, the metric coefficients satisfy $\|g_{ij} - \delta_{ij}\|_{C^2(B_R(p))} \leq 1/100$. Then, any solution $u : B_R(p) \to \mathbb{R}$ to the heat equation on M satisfies (in coordinates)*

$$u_t - Lu = 0 \, , \quad \textit{in } \mathcal{B}_R(0) \times (0,+\infty) \, ,$$

where $-L$ is a uniformly elliptic operator with uniformly bounded coefficients. Hence, by the standard Harnack inequality on $\mathbb{R}^n$ one can conclude the local estimate.

Remark 6.1.6. *One can turn the previous local uniform convergence in (6.1) into the convergence of solutions of the heat equation. Indeed, in the case $\mu(M) = +\infty$, since $H_M(\cdot, p, t)$ converges uniformly to zero, we get (by dominated convergence)*

$$e^{t\Delta} f(y) = \int_M H_M(x,y,t) f(x) d\mu(x) \to 0 \quad \text{as } t \to \infty \, ,$$

for every $y \in M$ and $f \in L^1(M)$.

6.1.2 The heat density

Now, we shall briefly comment on the following quantity

$$\alpha(E) = \lim_{s\to 0^+} s \int_{E\setminus B_1(0)} \frac{1}{|y|^{n+s}} \, dy,$$

introduced by Dipierro, Figalli, Palatucci, and Valdinoci in [Dip+13] as a measure of the behavior of the set E near infinity, and which is (up to a dimensional constant) the limit in (1.11) in the case $M = \mathbb{R}^n$ with its standard metric. This quantity is invariant by rescaling of E and, at first, can be thought of as a measure of "how conical" E is near infinity. Indeed, if the blow-down E/λ converges in $L^1_{\text{loc}}(\mathbb{R}^n)$ to a regular cone E_∞ as $\lambda \to \infty$, then $\alpha(E) = \mathcal{H}^{n-1}(E_\infty \cap \mathbb{S}^{n-1})$. Nevertheless, the fact that this limit exists is not equivalent to having a conical blow-down. Indeed, one can easily construct examples where the limit in $\alpha(E)$ exists, but the blow-downs of E converge to two different cones along two different subsequences.

The authors in [Dip+13] refer to $\alpha(E)$ as the weighted volume towards infinity of the set E; however, in light of our results and description, it would be more appropriate to call this quantity *heat density over E.* Indeed, $\alpha(E)$ represents the fraction of heat kernel that flows through the set towards infinity (this explains why $\theta_M \equiv 1$ on stochastically complete manifolds).

Because of this intuitive reason, the limit in the definition of $\alpha(E)$ needs not to exist in general if E, for example, oscillates between two cones near infinity. See [Dip+13, Example 2.8] for the construction of such an example.

On a Riemannian manifold, a similar quantity is needed, but, since no canonical origin (as in $\mathbb{R}^n$) is present, the singular kernel $1/|y|^{n+s}$ has to be replaced with $\mathcal{K}_s(y,p)$ and it has to be proved if and when the limit (1.11) becomes independent of $p \in M$. On Riemannian manifolds, this property of the limit being independent of the base point p turns out to be quite delicate and, as a consequence of Theorem 1.2.23, we will see that is implied by the $L^\infty -$ Liouville property of Definition 6.1.2.

Definition 6.1.7 (Heat density of a set)**.** *Let $E \subset M$ be a measurable set. We define, for every $p \in M$ and $R > 0$, the* heat density of E *as the following limit*

$$\theta_E(p,R) := \lim_{s\to 0^+} \int_{E\setminus B_R(p)} \mathcal{K}_s(x,p)d\mu(x)\,,$$

when it exists. At this level, this may depend on p and R.

Note that, at this point, it is not even clear whether the limit (1.12) of the heat density θ_M of the whole M exists or is different from zero. For example, as a consequence of the proof of Theorem 1.2.28, if there were complete Riemannian manifolds with $\mu(M) = +\infty$ and $\theta_M \neq 1$, then we would see the asymptotic

$$\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E,\Omega) = (\theta_M - \theta_E)\mu(E\cap\Omega) + \theta_E\mu(E^c\cap\Omega)$$

holding (even when $\theta_M \neq 1$), and if $\theta_M = 0$ this would mean that there are Riemannian manifolds where the asymptotic of the fractional s-perimeter of any set E is zero. These type of Riemannian manifolds exist; we describe such a manifold in Example 6.2.4.

First, we show that this does not happen if M is stochastically complete: the limit (1.12) always exists, and it is equal to one. Actually, more is true: if there is a point $p \in M$ for which the limit is 1, then the manifold is stochastically complete.

Before doing so, we state a corollary that follows directly from Lemma 3.4.8.

Corollary 6.1.8. *Let (M^n, g) be a complete Riemannian manifold, and let $R > 0$, $p \in M$. Then*

$$e^{t\Delta}(\chi_{M\setminus B_R(p)})(p) = \int_{M\setminus B_R(p)} H_M(x,p,t)\,d\mu(x) \le Ce^{-c/t}\,,$$

for some $C, c > 0$ depending on the geometry of M in $B_R(p)$.

Proof. Let r be small so that $\mathrm{FA}_0(M,g,r,p,\varphi)$ holds (see Definition 3.4.1), and consider the Riemannian manifold $\widehat{M} := (M, g/(4r)^2)$. With no loss of generality (up to taking a smaller r), we can assume $r \le R$. Clearly, we have that $\mathrm{FA}_0(\widehat{M}, g/(4r)^2, 1/4, p, \varphi(\cdot/(4r)))$ is satisfied (see (b) in Remark 3.4.5). By Lemma 3.4.8 applied to $\widehat{M}$ we have that

$$\int_{\widehat{M}\setminus\widehat{B}_{1/4}(p)} \widehat{H}(x,p,t)\,d\widehat{\mu}(x) \le Ce^{-c/t}.$$

Scaling back this inequality to M (see (b) in Remark 3.4.5) gives

$$\int_{M\setminus B_r(p)} H_M(x,p,t)\,d\mu(x) \le Ce^{-cr^2/t}\,,$$

for some $C, c > 0$ dimensional. Since $M \setminus B_R(p) \subseteq M \setminus B_r(p)$, this concludes the proof. □

Proposition 6.1.9. *Let (M, g) be a complete (possibly weighted) Riemannian manifold with $\mu(M) = +\infty$, and let $\theta_M(p)$ be as in Theorem 1.2.23. If M is stochastically complete, then*

$$\theta_M = \lim_{s\to 0^+} \int_{M\setminus B_1(p)} \mathcal{K}_s(x,p)d\mu(x) = 1 \quad \forall p \in M\,. \tag{6.4}$$

Conversely, if there exists $p \in M$ such that

$$\theta_M(p) = \lim_{s\to 0^+} \int_{M\setminus B_1(p)} \mathcal{K}_s(x,p)d\mu(x) = 1\,,$$

then M is stochastically complete.

Proof. Note that since $\mu(M) = +\infty$ we have $\mu(M \setminus B_1(p)) > 0$. We want to compute the following

$$\lim_{s\to 0^+} \frac{s}{2} \int_{M\setminus B_1(p)} \int_0^\infty H_M(x,p,t)\frac{dt}{t^{1+s/2}}d\mu(x).$$

Claim 1. There holds

$$\lim_{s\to 0^+} \frac{s}{2} \int_{M\setminus B_1(p)} \int_0^1 H_M(x,p,t)\frac{dt}{t^{1+s/2}}d\mu(x) = 0.$$

Indeed, this directly follows by writing

$$\lim_{s\to 0^+} \frac{s}{2} \int_{M\setminus B_1(p)} \int_0^1 H_M(x,p,t)\frac{dt}{t^{1+s/2}}d\mu(x) = \lim_{s\to 0^+} \frac{s}{2} \int_0^1 \frac{dt}{t^{1+s/2}} \left(\int_{M\setminus B_1(p)} H_M(x,p,t)\,d\mu(x) \right)$$

and exploiting the estimate of Corollary 6.1.8.

Claim 2. There holds

$$\lim_{s\to 0^+} \frac{s}{2} \int_{B_1(p)} \int_1^\infty H_M(x,p,t)\frac{dt}{t^{1+s/2}}d\mu(x) = 0. \tag{6.5}$$

By the uniform convergence of the heat kernel to zero (in particular, by the result contained in Remark 6.1.6), we get that $e^{t\Delta}(\chi_{B_1(p)})(p) \to 0$ as $t \to \infty$. Therefore, for all $\varepsilon > 0$ there exists $T = T(\varepsilon)$ such that $e^{t\Delta}(\chi_{B_1(p)})(p) \le \varepsilon$ for all $t \ge T$, whence

$$\limsup_{s\to 0^+} \frac{s}{2} \int_1^\infty e^{t\Delta}(\chi_{B_1(p)})(p)\frac{dt}{t^{1+s/2}}d\mu(x) \le \lim_{s\to 0^+} \frac{s}{2} \int_1^T \frac{dt}{t^{1+s/2}} + \varepsilon \limsup_{s\to 0^+} \frac{s}{2} \int_T^\infty \frac{dt}{t^{1+s/2}} \le \varepsilon\,,$$

for all $\varepsilon > 0$, proving the second claim.

Now, thanks to the first claim, we can reduce ourselves to computing

$$\lim_{s\to 0^+} \frac{s}{2}\int_{M\setminus B_1(p)}\int_1^\infty H_M(x,p,t)\frac{dt}{t^{1+s/2}}d\mu(x).$$

Then we can then add (6.5) to the previous limit, which gives zero contribution, and we end up with

$$\lim_{s\to 0^+} \frac{s}{2}\int_{M}\int_1^\infty H_M(x,p,t)\frac{dt}{t^{1+s/2}}d\mu(x).$$

Using Fubini and the stochastical completeness of M we get

$$\lim_{s\to 0^+} \frac{s}{2}\int_{M}\int_1^\infty H_M(x,p,t)\frac{dt}{t^{1+s/2}}d\mu(x) = \lim_{s\to 0^+} \frac{s}{2}\int_1^\infty \frac{dt}{t^{1+s/2}}d\mu(x) = 1\,,$$

and this concludes the proof.

Conversely, assume that (6.4) holds, then since both the previous claims hold on any connected and geodesically complete Riemannian manifold, we have

$$\lim_{s\to 0^+} \frac{s}{2}\int_{M}\int_1^\infty H_M(x,p,t)\frac{dt}{t^{1+s/2}}d\mu(x) = 1.$$

Setting $\mathcal{M}(t,p) = \int_M H_M(x,p,t)d\mu(x) \leq 1$ we can infer that, for every $T > 0$

$$1 = \lim_{s\to 0^+} \frac{s}{2}\int_T^\infty \frac{\mathcal{M}(t,p)}{t^{1+s/2}}dt \leq \lim_{s\to 0^+} \frac{s}{2}\int_T^\infty \frac{1}{t^{1+s/2}}dt = 1\,.$$

Now, assume by contradiction that M is not stochastically complete. Then since $\mathcal{M}(t,p)$ is nonincreasing in time and nonnegative, there holds $\lim_{t\to\infty}\mathcal{M}(t,p) \leq 1-\delta$ for some $\delta > 0$, and we would have $\mathcal{M}(t,p) \leq 1-\delta/2$ for every $t \geq T = T(\delta)$. This gives

$$1 = \lim_{s\to 0^+} \frac{s}{2}\int_T^\infty \frac{\mathcal{M}(t,p)}{t^{1+s/2}}dt \leq \lim_{s\to 0^+} \frac{s}{2}\int_T^\infty \frac{1-\delta/2}{t^{1+s/2}}dt = 1-\delta/2\,,$$

reaching a contradiction, hence $\lim_{t\to\infty}\mathcal{M}(t,p) = 1$ and since $\mathcal{M}(\cdot,p)$ is nonincreasing, we conclude. ☐

It is an easy consequence of the proof of Proposition 6.1.9 above the following simple result, which we will need in Section 6.1.4.

Lemma 6.1.10. *Let $E \subseteq M$ be a set for which the limit*

$$\lim_{t\to+\infty}\int_E H_M(p,x,t)d\mu(x) \tag{6.6}$$

exists for all $p \in M$. Then $\theta_E(p)$ exists for all $p \in M$ and coincides with the above limit.

Proof. Arguing as in (the first part of) the proof of Proposition 6.1.9 one immediatley gets

$$\lim_{s\to 0^+} \frac{s}{2}\int_{E}\int_0^{1/s} H_M(p,x,t)\frac{dt}{t^{1+s/2}}d\mu(x) = 0.$$

Therefore, to prove that $\theta_E(p)$ exists, we just need to prove the existence of

$$\lim_{s\to 0^+} \frac{s}{2}\int_E \int_{1/s}^{\infty} H_M(p,x,t)\frac{dt}{t^{1+s/2}}d\mu(x),$$

which is a trivial consequence of the existence of (6.6). □

Remark 6.1.11. *Following the proof of Proposition 6.1.9, one can see a clear picture of what happens to the limit in $\theta_M(p)$ even when M is not stochastically complete. Indeed, for every Riemannian manifold (not necessarily stochastically complete) and $p \in M$, the limit $\lim_{t\to\infty} \mathcal{M}(t,p)$ exists. This follows from the fact that $\mathcal{M}(\cdot,p)$ is nonincreasing and nonnegative. Since*

$$\mathcal{M}(t,p) = \int_M H_M(p,x,t)\, d\mu(x) = e^{t\Delta}1$$

is a solution to the heat equation starting from the function equal to one; it follows from the proof above and from standard parabolic estimates that $\mathcal{M}(t,\cdot) \to \theta_M$ in $C^2_{\mathrm{loc}}(M)$ as $t \to \infty$, where $\theta_M : M \to \mathbb{R}$ is a bounded, nonnegative harmonic function on M. Therefore:

(i) If M is stochastically complete, we have $\theta_M \equiv 1$ (in particular, the value of θ_M does not depend on the point), and the proof above shows $\theta_M = 1$.

(ii) If M is not stochastically complete but satisfies the $L^\infty - \mathrm{Liouville}$ property (see Definition 6.1.2) we know that $\theta_M \equiv \theta_\circ \in [0,1)$ and, following the proof of the proposition, one finds that the limit in the definition of θ_M exists, does not depend on the point p and there holds $\theta_M = \theta_\circ$. Note that such Riemannian manifolds exist and were first constructed in [Pin95]. In Example 6.2.4, we describe one with $\theta_\circ = 0$.

(iii) If M is not stochastically complete and does not satisfy the $L^\infty - \mathrm{Liouville}$ property, then in general θ_M is a nonconstant harmonic function on M, and the value of $\theta_M(p)$ can depend on the point p.

6.1.3 Proof of Theorem 1.2.23

Proof of Theorem 1.2.23. With no loss of generality, assume $r < R$.

Step 1. First, we show that the limit does not depend on the radius, that is

$$\theta_E(p,R) = \theta_E(p,r)\,.$$

We have

$$\begin{aligned}\left|\int_{E\setminus B_R(p)} \mathcal{K}_s(x,p)d\mu(x) - \int_{E\setminus B_r(p)} \mathcal{K}_s(x,p)d\mu(x)\right| &\le \int_{B_R(p)\setminus B_r(p)} \mathcal{K}_s(x,p)d\mu(x)\\ &\le Cs\int_{B_R(p)\setminus B_r(p)}\int_0^1 H_M(x,p,t)\frac{dt}{t^{1+s/2}}d\mu(x)\\ &+ Cs\int_{B_R(p)\setminus B_r(p)}\int_1^\infty H_M(x,p,t)\frac{dt}{t^{1+s/2}}d\mu(x) =: I_1 + I_2\,.\end{aligned}$$

For the first integral, by Corollary 6.1.8 as $s \to 0^+$

$$I_1 \le Cs\int_0^1 e^{t\Delta}(\chi_{M\setminus B_r(p)})(p)\frac{dt}{t^{1+s/2}} \le Cs\int_0^1 \frac{e^{-c/t}}{t^{1+s}}\,dt \to 0\,.$$

Regarding the second integral, for all $\varepsilon > 0$ by Lemma 6.1.3 there is $T = T(\varepsilon) > 0$ such that $|H_M(x,p,t)| \le \varepsilon$ for all $x \in B_R(p)$ and $t \ge T$, hence

$$
\begin{aligned}
I_2 &\le Cs \int_1^T \int_{B_R(p)} H_M(x,p,t)d\mu(x)\frac{dt}{t^{1+s/2}} + Cs \int_T^\infty \int_{B_R(p)} H_M(x,p,t)d\mu(x)\frac{dt}{t^{1+s/2}} \\
&\le Cs \int_1^T \frac{dt}{t^{1+s/2}} + Cs\varepsilon\mu(B_R(p)) \int_T^\infty \frac{dt}{t^{1+s/2}} \\
&= C(1-T^{-s/2}) + C\varepsilon\mu(B_R(p))T^{-s/2}\,,
\end{aligned}
$$

letting $s \to 0^+$ (and then $\varepsilon \to 0$) gives $I_2 \to 0$. Hence, taking $s \to 0^+$ shows $\theta_E(p,R) = \theta_E(p,r)$, showing that the limit never depends on the radius. Note that what we have just proved already implies that if E is bounded, then the limit exists and $\theta_E = 0$ since one can just take $R \gg 1$ so that $E \setminus B_R(p) = \varnothing$.

Now fix $q \in M$. For every $p \in B_{1/2}(q)$ we can write

$$
\theta_E(p) = \lim_{s\to 0^+} \int_{E\setminus B_1(q)} \mathcal{K}_s(x,p)d\mu(x).
$$

This is possible because we always have independence on the radius. Indeed

$$
\left| \int_{E\setminus B_{1/2}(p)} \mathcal{K}_s(x,p)d\mu(x) - \int_{E\setminus B_1(q)} \mathcal{K}_s(x,p)d\mu(x) \right| \le \int_{B_1(q)\setminus B_{1/2}(p)} \mathcal{K}_s(x,p)\,d\mu(x)\,,
$$

hence

$$
\limsup_{s\to 0^+} \left| \int_{E\setminus B_{1/2}(p)} \mathcal{K}_s(x,p)d\mu(x) - \int_{E\setminus B_1(q)} \mathcal{K}_s(x,p)d\mu(x) \right| \le \theta_{B_1(q)} = 0\,.
$$

Set

$$
\Theta_{E,s}(p) := \frac{s}{2}\int_0^\infty e^{t\Delta}(\chi_{E\setminus B_1(q)})(p)\frac{dt}{t^{1+s/2}}\,, \tag{6.7}
$$

so that $\theta_E(p) = \lim_{s\to 0^+}\Theta_{E,s}(p)$.

Step 2. θ_E is harmonic.

By Corollary 6.1.8 we have that that $0 \le \Theta_{E,s}(p) \le C$, for some constant $C > 0$ depending only on M. Now fix $\varphi \in C_c^\infty(B_{1/2}(q))$, by dominated convergence

$$
\int_M \theta_E(\Delta\varphi)\,d\mu = \lim_{s\to 0^+}\int_M \Theta_{E,s}(\Delta\varphi)\,d\mu = \lim_{s\to 0^+}\int_M (\Delta\Theta_{E,s})\varphi\,d\mu\,. \tag{6.8}
$$

Note that, for fixed s and $p \in B_{1/2}(q)$, we can write

$$
\Delta\Theta_{E,s}(p) = \frac{s}{2}\int_0^\infty \Delta e^{t\Delta}(\chi_{E\setminus B_1(q)})(p)\frac{dt}{t^{1+s/2}} = \frac{s}{2}\int_0^\infty \partial_t e^{t\Delta}(\chi_{E\setminus B_1(q)})(p)\frac{dt}{t^{1+s/2}},
$$

which, after integration by parts, becomes (note that the boundary term at $t = 0^+$ is zero due to Corollary 6.1.8) equal to

$$
\frac{s}{2}\int_0^\infty e^{t\Delta}(\chi_{E\setminus B_1(q)})(p)\frac{(1+s/2)}{t^{2+s/2}}\,dt\,.
$$

The latter quantity goes to 0 as $s \to 0^+$, and is uniformly bounded for $s \in (0,1)$, for every

$p \in B_{1/2}(q)$. Hence, going back to (6.8) we get

$$\int_M \theta_E(\Delta\varphi)\, d\mu = 0\,.$$

That is, $\theta_E \in L^1_{\mathrm{loc}}(M)$ is a very weak solution of $\Delta\theta_E = 0$. We're left to prove that θ_E is smooth and is a classical solution of $\Delta\theta_E = 0$. In a small chart, in coordinates, one can see that θ_E is a very weak solution of $\partial_i\big(\sqrt{|\det(g)|}g^{ij}\partial_j\theta_E\big) = 0$. Choosing the chart sufficiently small, we get that the coefficients $\sqrt{|\det(g)|}g^{ij}$ are smooth and uniformly elliptic. Then, for example by [ZB12, Theorem 1.3], we get that $\theta_E \in W^{2,2}_{\mathrm{loc}}(M)$ and bootstrapping classical elliptic regularity gives that θ_E is smooth and harmonic.

Lastly, (1.12) follows from the last part of the proof of Proposition 6.1.9, and the fact that $p \mapsto \theta_M(p)$ is harmonic is verbatim the proof we did for $E \subset M$ above. □

6.1.4 On the dimension of the space of bounded harmonic functions

On a complete Riemannian manifold M, denote by

$$\mathcal{H}_b(M) := \big\{\text{bounded, harmonic function on } M\big\}.$$

In this subsection, we explain how Theorem 1.2.23 can be used to bound from below the dimension of $\mathcal{H}_b(M)$.

Definition 6.1.12 (Bulky set)**.** *Let M be a stochastically complete Riemannian manifold (thus, in particular, $\theta_M = 1$) with $\mu(M) = +\infty$, and let $E \subset M$. We say that E is a bulky set if θ_E exists and*

$$\sup_M \theta_E = 1.$$

Lemma 6.1.13. *Let $E, F \subset M$ be disjoint sets such that θ_E and θ_F exists. Then $\theta_{E\cup F}$ exists and $\theta_{E\cup F} = \theta_E + \theta_F$.*

Proof. The proof is trivial by the additivity of the integral in the definition of $\theta_{E\cup F}$. □

Theorem 6.1.14. *Let $m \geq 1$ and M be a stochastically complete Riemannian manifold with $\mu(M) = +\infty$. Assume that there exist m disjoint $E_1, \dots, E_m$ bulky sets (see Definition 6.1.12). Then $\dim_{\mathbb{R}} \mathcal{H}_b(M) \geq m$.*

Proof. By Theorem 1.2.23 we see that $\theta_{E_1}, \dots, \theta_{E_m}$ are m harmonic functions with values in $(0, 1]$. By Lemma 6.1.13, for every $x \in M$ there holds

$$\theta_{E_1}(x) + \ldots + \theta_{E_m}(x) = \theta_{E_1\cup\ldots\cup E_m}(x) \leq \theta_M(x) = 1. \tag{6.9}$$

We show that these m harmonic functions are linearly independent. Assume by contradiction that there are $\alpha_1 \dots, \alpha_m \in \mathbb{R}$ such that

$$\alpha_1\theta_{E_1} + \ldots + \alpha_m\theta_{E_m} = 0. \tag{6.10}$$

With no loss of generality (up to relabeling the sets E_i) assume also $\alpha_1 \neq 0$.

By the very definition of bulky set, there is a sequence $x_k \in E_1$ with $\theta_{E_1}(x_k) \to 1$ as $k \to \infty$. Hence, taking $x = x_k$ in (6.9) and letting $k \to \infty$ gives

$$\lim_{k\to\infty} \theta_{E_i}(x_k) = 0, \quad \forall\, i = 2, \dots, m.$$

Thus, evaluating (6.10) at $x = x_k$ and letting $k \to \infty$ we get

$$\alpha_1 \lim_{k\to\infty} \theta_{E_1}(x_k) = \alpha_1 = 0,$$

and this contradicts $\alpha_1 \neq 0$. $\square$

The relationship between bulky sets and the notion of massive set

Definition 6.1.15 (Massive set). *A subset $E \subset M$ is said to be massive if there exists a bounded superharmonic function $v \geq 0$ such that $v \equiv 1$ on $M \setminus E$ and $\inf_E v = 0$.*

We call such a function v an admissible function for E, and we call

$$s_E(x) := \sup\left\{v(x) \,|\, v \leq 1 \text{ is admissible for } E\right\}$$

the *superharmonic potential* of E. Let also

$$e_E(x) := \mathbb{P}(\exists\, t \geq 0 \,|\, \beta_t^x \in E)\,,$$

and

$$h_E(x) := \mathbb{P}(\exists\, t_k \uparrow +\infty \,|\, \beta_{t_k}^x \in E)\,,$$

where β_t^x is a standard Brownian motion on M. We recall the following result, whose proof can be found in [Gri99a, Section 4.5].

Theorem 6.1.16. *Let $E \subset M$ be open with ∂E smooth. Then for every $x \in M$*

(i) $e_E(x) = s_{M\setminus E}(x)$.

(ii) $h_E(x) = \lim_{t\to\infty} e^{t\Delta}(s_{M\setminus E})$.

It follows from the definition of massiveness that E is massive if and only if $s_E \not\equiv 1$, hence from (i) we get that E is massive if and only if $e_{M\setminus E} \not\equiv 1$. Note that this is not equivalent to the fact that $\mathbb{P}(\,\beta_t^x \in E,\ \forall\, t \geq 0) > 0$ for some $x \in M$, since M could be stochastically incomplete.

The next result shows that bulky sets (see Definition 6.1.12) are essentially massive sets for which the limit in θ_E exists.

Lemma 6.1.17. *Let M be a stochastically complete Riemannian manifold with $\mu(M) = +\infty$. Let $E \subset M$ be open with ∂E smooth. Then*

(i) If E is massive and θ_E exists, then E is bulky.

(ii) If E is bulky, then E is massive.

Proof. (i). Let E be massive and θ_E exists. By Theorem 6.1.16 we have $\theta_{M\setminus E} \leq e_{M\setminus E} = s_E$. Moreover, by [Gri99a, Proposition 4.3] there holds $\inf_M s_E = 0$, thus also $\inf_M \theta_{M\setminus E} = 0$. Since $\theta_{M\setminus E} = 1 - \theta_E$ we get $\sup_M \theta_E = 1$, that is E is bulky.

(ii). Suppose by contradiction that E is not massive. Then by Theorem 6.1.16 we get $s_E = e_{M\setminus E} \equiv 1$, and thus $s_{M\setminus E} \equiv 0$. Hence, by Theorem 6.1.16 again, $h_E = \lim_{t\to\infty} e^{t\Delta}(s_{M\setminus E}) \equiv 0$, which by definition of h_E means that

$$\mathbb{P}(\exists\, t_k \uparrow +\infty \,|\, \beta_{t_k}^x \in E) = 0, \quad \forall\, x \in M.$$

This implies that almost surely there exists $\tau > 0$ such that $\beta_t^x \in M \setminus E$ for all $t \geq \tau$, and hence

$$0 = \lim_{t\to+\infty} \mathbb{P}(\beta_t^x \in E) = \lim_{t\to\infty} \int_E H_M(x, y, t)\, d\mu(y)\,.$$

By Lemma 6.1.10 we get $\theta_E \equiv 0$ and hence E is not bulky, contradiction. □

Remark 6.1.18. *It is clearly not true that E massive implies that E is bulky. That is, the assumption that θ_E exists is necessary. This is because massiveness is a notion that "bounds from below" the size of E (to say that adding an arbitrary set to a massive set results in a massive set), while the notion of a bulky set asks for a suitable weighted limit at infinity to exist.*

Recall the following celebrated result by A. Grigor'yan [Gri90].

Theorem 6.1.19 ([Gri90])**.** *$\dim_{\mathbb{R}} \mathcal{H}_b(M)$ is equal to the supremum of the number of disjoint massive sets (see Definition 6.1.15) which can be put on M.*

Lastly, let us point out that we believe our Theorem 6.1.14 holds in the sharp version similar to Theorem 6.1.19. That is, we conjecture the following to hold:

Conjecture 6.1.20. *Let M be a stochastically complete manifold of infinite volume. Then $\dim_{\mathbb{R}} \mathcal{H}_b(M)$ is equal to the supremum of the number of disjoint bulky sets (see Definition 6.1.12) which can be put on M.*

6.2 Infinite volume asymptotics and proof of Theorem 1.2.28

Now we turn to the proof of Theorem 1.2.25. To prove this result, we will need Lemma 6.2.1 below, which essentially says that for manifolds with $\mu(M) = +\infty$, the singular kernel $\mathcal{K}_s$ locally behaves like that of $\mathbb{R}^n$ as $s \to 0^+$. This is not the case for finite volume manifolds[1].

Recall the notation of Remark 2.2.3, where we denote by $\frac{\alpha_{n,s}}{|x-y|^{n+s}}$ the singular kernel of $\mathbb{R}^n$ with its standard metric. Note also that

$$c_n s(2-s) \leq \alpha_{n,s} \leq C_n s(2-s).$$

The following lemma is a sharpening of Lemma 3.4.13 for manifolds with infinite volume. Indeed, in Lemma 3.4.13, we were not interested in characterizing the sharp dependence from s of $\mathcal{K}_s$ as $s \to 0^+$. Moreover, in Lemma 3.4.13, we estimate $\mathcal{K}_s$ locally on every complete Riemannian manifold M (both with finite and infinite volume), but the result stated in Lemma 6.2.1 below is not true on manifolds with finite volume.

Lemma 6.2.1. *Let (M, g) be a complete n-dimensional Riemannian manifold with $\mu(M) = +\infty$, and let $p \in M$. Assume that in normal coordinates at p there holds $\frac{99}{100}|v|^2 \leq g_{ij}(q)v^i v^j \leq \frac{101}{100}|v|^2$ and $|\nabla g_{ij}(q)| \leq 1/100$ for all $v \in \mathbb{R}^n$ and $q \in B_1(p)$. Then there exists $\mathcal{K}'_s : B_1(p) \times B_1(p) \to [0, \infty)$ such that*

$$\lim_{s\to 0^+} \sup_{x,y \in B_{1/8}(p)} \left|\mathcal{K}_s(x, y) - \mathcal{K}'_s(x, y)\right| = 0\,,$$

and for all $x, y \in B_{1/8}(p)$

$$c\frac{\alpha_{n,s}}{d(x, y)^{n+s}} \leq \mathcal{K}'_s(x, y) \leq C\frac{\alpha_{n,s}}{d(x, y)^{n+s}}\,, \tag{6.11}$$

[1] Indeed, for finite volume manifolds, the same conclusion (6.11) holds with constants depending on s, but as $s \to 0^+$ the constants do not behave like the ones of $\mathbb{R}^n$.

for some dimensional constants $c, C > 0$.

Proof of Lemma 6.2.1. Let $\varphi^{-1} : B_1(p) \to \mathbb{R}^n$ be the inverse of the exponential map at p. Take $\eta \in C_c^\infty(\mathcal{B}_{4/5}(0))$ with $\chi_{\mathcal{B}_{2/5}(0)} \le \eta \le \chi_{\mathcal{B}_{4/5}(0)}$ and let $g'_{ij} := g_{ij}\eta + (1-\eta)\delta_{ij}$. This is a metric on $\mathbb{R}^n$ with $g'_{ij} = g_{ij}$ in $\mathcal{B}_{2/5}(0)$. Denote by $\mathcal{K}_s, \mathcal{K}'_s$ the singular kernels of (M,g) and $M' := (\mathbb{R}^n, g')$ respectively. Let $\Lambda := \sup_{q\in B_{1/5}(p)} H_M(q,q,1)$ and $\Lambda' := \sup_{x\in \mathcal{B}_{1/5}(0)} H_{M'}(x,x,1)$, and note that

$$\sup_{t\ge 1}\sup_{q\in B_{1/5}(p)} H_M(q,q,t) \le \Lambda, \qquad \sup_{t\ge 1}\sup_{x\in \mathcal{B}_{1/5}(0)} H_{M'}(x,x,t) \le \Lambda',$$

since the maps $t \mapsto H_M(q,q,t)$ and $t \mapsto H_{M'}(x,x,t)$ are decreasing. For $x, y \in \mathcal{B}_{1/5}(0)$ we have

$$\begin{aligned}
\left|\mathcal{K}_s(\varphi(x),\varphi(y)) - \mathcal{K}'_s(x,y)\right| &\le \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty \left|H_M(\varphi(x),\varphi(y),t) - H_{M'}(x,y,t)\right| \frac{dt}{t^{1+s/2}} \\
&\le Cs(2-s)\int_0^1 \left|H_M(\varphi(x),\varphi(y),t) - H_{M'}(x,y,t)\right| \frac{dt}{t^{1+s/2}} \\
&+ Cs(2-s)\int_1^{1/s} \left|H_M(\varphi(x),\varphi(y),t) - H_{M'}(x,y,t)\right| \frac{dt}{t^{1+s/2}} \\
&+ Cs(2-s)\int_{1/s}^\infty \left|H_M(\varphi(x),\varphi(y),t) - H_{M'}(x,y,t)\right| \frac{dt}{t^{1+s/2}} \\
&:= Cs(2-s)\big(I_1 + I_2 + I_3\big)\,.
\end{aligned}$$

By Lemma 3.4.11 there holds

$$I_1 = \int_0^1 \left|H_M(\varphi(x),\varphi(y),t) - H_{M'}(x,y,t)\right| \frac{dt}{t^{1+s/2}} \le C\int_0^1 e^{-c/t}\frac{dt}{t^{1+s/2}} \le C\,,$$

for some dimensional $C > 0$. Regarding the second integral

$$I_2 \le \int_1^{1/s} (\Lambda + \Lambda')\frac{dt}{t^{1+s/2}} = (\Lambda+\Lambda')\frac{1-s^{s/2}}{s/2}\,,$$

and lastly

$$\begin{aligned}
I_3 &= \int_{1/s}^\infty \left|H_M(\varphi(x),\varphi(y),t) - H_{M'}(x,y,t)\right| \frac{dt}{t^{1+s/2}} \\
&\le s^{s/2}\int_1^\infty \Big[H_M(\varphi(x),\varphi(y),\xi/s) + H_{M'}(x,y,\xi/s)\Big]\frac{d\xi}{\xi^{1+s/2}} \to 0\,,
\end{aligned}$$

as $s \to 0^+$, since both M and M' have infinite volume, and thus, their heat kernel tends to zero as $t \to +\infty$ (see Lemma 6.1.3). Hence, as $s \to 0^+$

$$\left|\mathcal{K}_s(\varphi(x),\varphi(y)) - \mathcal{K}'_s(x,y)\right| \le Cs + C(\Lambda+\Lambda')(1-s^{s/2}) + o(1) \to 0,$$

and note that this estimate is uniform in $x, y \in \mathcal{B}_{1/5}(0)$. This follows, for example, from the parabolic Harnack inequality since one can locally estimate the supremum of H_M and $H_{M'}$ with the L^1 norm at later times; see the end of the proof of Lemma 6.1.3. Then

$$\lim_{s\to 0^+}\sup_{x,y\in B_{1/8}(p)} \left|\mathcal{K}_s(x,y) - \mathcal{K}'_s(x,y)\right| = 0\,.$$

Lastly, by Lemma 3.4.7 there exists dimensional constants $c, C > 0$ such that

$$c\frac{\alpha_{n,s}}{d(x,y)^{n+s}} \le \mathcal{K}'_s(x,y) \le C\frac{\alpha_{n,s}}{d(x,y)^{n+s}}\,,$$

and this concludes the proof. □

6.2.1 Proof of Theorem 1.2.25

Proof of Theorem 1.2.25. As we can assume $s < s_\circ/2$, it follows from the proof of Proposition A.1.4 that the integral in $(-\Delta)^{s/2}_{\mathrm{Si}} u$ is absolutely convergent[2] for a.e. $x \in M$, and the principal value is not needed. Moreover, since $u \in H^{s_\circ/2}(M)$ we have

$$\int_M (u(x) - u(y))^2 \mathcal{K}_{s_\circ}(x,y)\, d\mu(y) < +\infty,$$

for a.e. $x \in M$. Fix $x \in M$ in the intersection of these two sets of full measure, and take R such that $\mathrm{supp}(u) \subset B_R(x)$. Then

$$\begin{aligned}(-\Delta)^{s/2}_{\mathrm{Si}} u(x) &= \int_M (u(x) - u(y))\mathcal{K}_s(x,y)\, d\mu(y) \\ &= \int_{B_R(x)} (u(x) - u(y))\mathcal{K}_s(x,y)\, d\mu(y) + u(x)\int_{M\setminus B_R(x)} \mathcal{K}_s(x,y)\, d\mu(y)\,. \end{aligned} \tag{6.12}$$

Note that being $\mu(M) = +\infty$ we have

$$\int_{M\setminus B_R(x)} \mathcal{K}_s(x,y)\, d\mu(y) \neq 0\,.$$

Claim. As $s \to 0^+$ there holds

$$\lim_{s\to 0^+} \int_{B_R(x)} (u(x) - u(y))\mathcal{K}_s(x,y)\, d\mu(y) = 0\,.$$

Indeed, let $\rho \ll 1$ small that will be chosen later. We denote here by C a constant which does not depend on s. Then

$$\begin{aligned}&\left|\int_{B_R(x)} (u(x) - u(y))\mathcal{K}_s(x,y)\, d\mu(y)\right| \\ &\qquad = \left|\int_{B_\rho(x)} (u(x) - u(y))\mathcal{K}_s(x,y)\, d\mu(y) + \int_{B_R(x)\setminus B_\rho(x)} (u(x) - u(y))\mathcal{K}_s(x,y)\, d\mu(y)\right| \\ &\qquad \le \int_{B_\rho(x)} |u(x) - u(y)|\mathcal{K}_s(x,y)\, d\mu(y) + 2\|u\|_{L^\infty}\int_{B_R(x)\setminus B_\rho(x)} \mathcal{K}_s(x,y)\, d\mu(y)\,.\end{aligned}$$

We estimate these two integrals separately. Let $\mathcal{K}'_s$ be the singular kernel given by Lemma 6.2.1, applied with ρ sufficiently small and suitably rescaled. For the first integral, Lemma 6.2.1 gives

$$\limsup_{s\to 0^+} \int_{B_\rho(x)} |u(x) - u(y)|\big(\mathcal{K}_s(x,y) - \mathcal{K}'_s(x,y)\big)\, d\mu(y) = 0\,. \tag{6.13}$$

[2]Here we are not assuming M being stochastically complete, but in Proposition A.1.4 stochastical completeness is only used to have that $(-\Delta)^{s/2}_{\mathrm{B}} u = (-\Delta)^{s/2}_{\mathrm{Si}} u$ a.e., not to show the absolute convergence of the integrals.

Moreover, by the bounds of Lemma 6.2.1 and since $u \in H^{s_\circ/2}(M)$, for a.e. $x \in M$

$$\int_{B_\rho(x)} \frac{(u(x)-u(y))^2}{d(x,y)^{n+s_\circ}}\,dy \le C(s_\circ)\int_{B_\rho(x)} (u(x)-u(y))^2 \mathcal{K}_{s_\circ}(x,y)\,dy < +\infty.$$

Hence, by Lemma 6.2.1 again and Hölder's inequality

$$\begin{aligned}\int_{B_\rho(x)} |u(x)-u(y)|\mathcal{K}_s'(x,y)\,d\mu(y) &\le Cs\int_{B_\rho(x)} \frac{|u(x)-u(y)|}{d(x,y)^{n+s}}\,d\mu(y)\\ &\le Cs\left(\int_{B_\rho(x)} \frac{(u(x)-u(y))^2}{d(x,y)^{n+s_\circ}}\,dy\right)^{1/2}\left(\int_{B_\rho(x)} \frac{1}{d(x,y)^{n+2s-s_\circ}dy}\right)^{1/2}\\ &\le Cs\left(\frac{\rho^{s_\circ-2s}}{s_\circ-2s}\right)^{1/2} \to 0\,,\end{aligned}$$

as $s \to 0^+$, where in the second-last inequality we have used polar coordinates for ρ sufficiently small (possibly depending on x). Thus, with (6.13) we have that the first integral tends to zero.

Regarding the second integral, one can note that we have proved in part (i) of Theorem 1.2.23 that, for every $x \in M$ and $r, R > 0$

$$\lim_{s\to 0^+}\int_{B_R(x)\setminus B_r(x)} \mathcal{K}_s(x,y)d\mu(y) = 0\,,$$

since $B_R(x)$ is a bounded set, and this concludes the proof of the claim.

By the very definition of θ_M we have

$$\lim_{s\to 0^+}\int_{M\setminus B_R(x)} \mathcal{K}_s(x,y)\,d\mu(y) = \theta_M(x),$$

hence letting $s \to 0^+$ in (6.12) gives

$$\lim_{s\to 0^+}(-\Delta)^{s/2}_{\mathrm{Si}}u(x) = \theta_M(x)u(x)\,,$$

for a.e. $x \in M$, and this concludes the proof. □

6.2.2 Global asymptotics

To prove our result Theorem 1.2.30 on the asymptotics for infinite volume, one needs also to know the asymptotics as $s \to 0^+$ of the fractional s-perimeter on the entire M, that is, when $\Omega \equiv M$. This is addressed by Theorem 1.2.26 on the asymptotics of the fractional Sobolev seminorms, which we now prove, and which is the counterpart of Theorem 6.2.9 in the case of infinite volume.

Proof of Theorem 1.2.26. Formally, one would like to infer that

$$\begin{aligned}\frac{1}{2}[u]^2_{H^{s/2}(M)} &\triangleq \frac{1}{2}\iint_{M\times M}(u(x)-u(y))^2\mathcal{K}_s(x,y)\,d\mu(x)d\mu(y)\\ &= \int_M u(-\Delta)^{s/2}_{\mathrm{Si}}u\,d\mu \xrightarrow{s\to 0^+} \int_M u^2\theta_M\,d\mu\,,\end{aligned}$$

where the first equality is the very definition of the seminorm (see Definition 2.2.1). The second inequality is nontrivial since the integrals one would write in the few lines of a proof are not absolutely convergent in general. Moreover, for the last step of taking the limit as $s \to 0^+$, one needs to show that the a.e. convergence $(-\Delta)^{s/2}_{\mathrm{Si}} u \to \theta_M u$ of Theorem 1.2.25 can be upgraded to weak convergence in $L^2(M)$. Now we shall justify both steps.

Step 1. We have

$$\frac{1}{2} \iint_{M\times M} (u(x)-u(y))^2 \mathcal{K}_s(x,y)\, d\mu(x) d\mu(y) = \int_M u(-\Delta)^{s/2}_{\mathrm{Si}} u \, d\mu \,. \tag{6.14}$$

Fix $\varepsilon > 0$ and let

$$(-\Delta)^{s/2}_{\varepsilon} u(x) := \int_{M\setminus B_\varepsilon(x)} (u(x)-u(y)) \mathcal{K}_s(x,y)\, d\mu(y) \,.$$

Let also $D := \{(z,z) \,:\, z \in M\}$ denote the diagonal of $M \times M$ and D_δ a δ-neighborhood of D. We have

$$\begin{aligned}
&\iint_{M\times M\setminus D_{\varepsilon/\sqrt{2}}} (u(x)-u(y))^2 \mathcal{K}_s(x,y)\, d\mu(x) d\mu(y) \\
&\qquad = \iint_{M\times M\setminus D_{\varepsilon/\sqrt{2}}} u(x)(u(x)-u(y)) \mathcal{K}_s(x,y)\, d\mu(x) d\mu(y) \\
&\qquad\quad - \iint_{M\times M\setminus D_{\varepsilon/\sqrt{2}}} u(y)(u(x)-u(y)) \mathcal{K}_s(x,y)\, d\mu(x) d\mu(y) \\
&\qquad = 2 \iint_{M\times M\setminus D_{\varepsilon/\sqrt{2}}} u(x)(u(x)-u(y)) \mathcal{K}_s(x,y)\, d\mu(x) d\mu(y) \\
&\qquad = 2 \int_M \int_{M\setminus B_\varepsilon(x)} u(x)(u(x)-u(y)) \mathcal{K}_s(x,y)\, d\mu(y) d\mu(x) \\
&\qquad = 2 \int_M u(-\Delta)^{s/2}_{\varepsilon} u \, d\mu \,,
\end{aligned}$$

where splitting the integral and Fubini are justified since the integrals are absolutely convergent. Indeed

$$\begin{aligned}
&\int_M \int_{M\setminus B_\varepsilon(x)} |u(x)(u(x)-u(y))| \mathcal{K}_s(x,y)\, d\mu(y) d\mu(x) \\
&\le \int_M |u(x)|^2 \int_{M\setminus B_\varepsilon(x)} \mathcal{K}_s(x,y)\, d\mu(y) d\mu(x) + \int_M |u(x)| \int_{M\setminus B_\varepsilon(x)} |u(y)| \mathcal{K}_s(x,y)\, d\mu(y) d\mu(x) \,,
\end{aligned}$$

but by Corollary 6.1.8

$$\begin{aligned}
\int_{M\setminus B_\varepsilon(x)} \mathcal{K}_s(x,y)\, d\mu(y) &= C \int_0^\infty \left(\int_{M\setminus B_\varepsilon(x)} H_M(x,y,t)\, d\mu(y) \right) \frac{dt}{t^{1+s/2}} \\
&\le C \int_0^\infty \frac{e^{-c/t}}{t^{1+s/2}} dt \le C \,,
\end{aligned}$$

for some C depending on s and ε. Hence

$$\int_M \int_{M\setminus B_\varepsilon(x)} |u(x)(u(x)-u(y))| \mathcal{K}_s(x,y)\, d\mu(y) d\mu(x) \le C(\|u\|_{L^\infty}, \mu(\mathrm{supp}(u)), \varepsilon, s) < +\infty \,,$$

and this shows the absolute convergence.

Moreover, by Proposition A.1.4 for a.e. $x \in M$ the integral in $(-\Delta)^{s/2}_{\mathrm{Si}} u$ is absolutely convergent, then

$$\int_M \left|(-\Delta)^{s/2}_{\mathrm{Si}} u - (-\Delta)^{s/2}_{\varepsilon} u\right|^2 d\mu \leq \int_M \left| \int_{B_\varepsilon(x)} |u(x) - u(y)| \mathcal{K}_s(x,y) d\mu(y) \right|^2 d\mu(x)\,,$$

and the right hand side ternds to 0 as $\varepsilon \to 0$. Indeed, as $\varepsilon \to 0$, by the very same argument at the end of the proof of Theorem 1.2.25, there holds

$$\int_{B_\varepsilon(x)} |u(x) - u(y)| \mathcal{K}_s(x,y) d\mu(y) \to 0\,,$$

for a.e. $x \in M$, and for x fixed, the convergence is monotone (decreasing) since the integrand is positive. Hence we have proved $(-\Delta)^{s/2}_{\varepsilon} u \to (-\Delta)^{s/2}_{\mathrm{Si}} u$ in $L^2(M)$ as $\varepsilon \to 0$. Now, letting $\varepsilon \to 0$ in

$$\frac{1}{2} \iint_{M \times M \setminus D_{\varepsilon/\sqrt{2}}} (u(x) - u(y))^2 \mathcal{K}_s(x,y)\, d\mu(x) d\mu(y) = \int_M u (-\Delta)^{s/2}_{\varepsilon} u\, d\mu\,,$$

together with the monotone convergence theorem on the left-hand side, we get the equality of the seminorms, and this completes the proof of Step 1.

Step 2. As $s \to 0^+$, there holds

$$(-\Delta)^{s/2}_{\mathrm{Si}} u \rightharpoonup \theta_M u \;\; \text{weakly in } L^2(M)\,.$$

The convergence a.e. to $\theta_M u$ is given by Theorem 1.2.25. To prove that the convergence holds weakly in $L^2(M)$, we show that $(-\Delta)^{s/2}_{\mathrm{Si}} u$ is equibounded in $L^2(M)$. By (A.4) there is C depending only on $s_\circ$ such that

$$\|(-\Delta)^{s/2}_{\mathrm{Si}} u\|^2_{L^2(M)} \leq C \|u\|^2_{L^2(M)} + C s^2 \|u\|^2_{H^{s_\circ}(M)}\,,$$

and hence

$$\limsup_{s \to 0^+} \|(-\Delta)^{s/2}_{\mathrm{Si}} u\|^2_{L^2(M)} \leq C \|u\|^2_{L^2(M)} < +\infty\,.$$

This concludes Step 2 and, sending $s \to 0^+$ in (6.14) concludes the proof. □

Corollary 6.2.2. *Let (M,g) be stochastically complete and with $\mu(M) = +\infty$. Let $E \subset M$ be bounded and such that $\mathrm{Per}_{s_\circ}(E) < +\infty$ for some $s_\circ \in (0,1)$. Then*

$$\lim_{s \to 0^+} \frac{1}{2} \mathrm{Per}_s(E) = \mu(E)\,.$$

Proof. Since M is stochastically complete, by Proposition 6.1.9, we have $\theta_M \equiv 1$. Then the result follows taking $u = \chi_E$ in Theorem 1.2.26. □

Remark 6.2.3. *Note that the equivalence of the seminorms (6.14) always holds for characteristic functions, without any assumption. Indeed for every measurable $E \subset M$*

$$2 \int_M \chi_E \cdot (-\Delta)^{s/2}_{\mathrm{Si}} \chi_E \, dx = 2 \int_E \left(\lim_{\varepsilon \to 0} \int_{M \setminus B_\varepsilon(x)} (1 - \chi_E(y)) \mathcal{K}_s(x,y) dy \right) dx$$

$$= 2\int_E \left(\lim_{\varepsilon\to 0} \int_{(M\setminus B_\varepsilon(x))\cap E^c} \mathcal{K}_s(x,y)dy \right) dx$$
$$= 2\int_E \int_{E^c} \mathcal{K}_s(x,y)dy = [\chi_E]^2_{H^{s/2}(M)}\,,$$

where the second-last equality follows by the monotone convergence theorem.

One can note that stochastic completeness is not really needed in Corollary 6.2.2. Even when M is not stochastically complete, by Theorem 1.2.23, we know that θ_M is a (possibly nonconstant) bounded harmonic function with values in $[0,1]$. Then, by Theorem 1.2.26 with $u = \chi_E$ we deduce

$$\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E) = \int_E \theta_M \, d\mu,$$

which is Corollary 1.2.27. Consequently, if in particular $\theta_M \equiv \theta_\circ \in [0,1]$ we have

$$\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E) = \theta_\circ \mu(E)\,, \tag{6.15}$$

for every E bounded with $\mathrm{Per}_{s_\circ}(E) < +\infty$. This feature led us to note the following example, which shows that, interestingly enough, Riemannian manifolds with $\theta_M \equiv 0$ exist.

Example 6.2.4. *There exists a complete Riemannian manifold N where the asymptotics of the fractional s-perimeter as $s \to 0^+$ is zero for every set, that is: for every bounded E with $\mathrm{Per}_{s_\circ}(E) < +\infty$ for some $s_\circ \in (0,1)$ there holds*

$$\lim_{s\to 0^+} \mathrm{Per}_s(E) = 0\,.$$

*By (6.15) above, we see that it is enough to provide an example of a Riemannian manifold N with $\theta_N(p) \equiv 0$, meaning that the limit does not depend on the point p and is always zero. Moreover, by part (ii) of Remark 6.1.11 this is satisfied if N has the $L^\infty-$*Liouville *property, is not stochastically complete and*

$$\mathcal{N}(t,p) := \int_N H_N(x,p,t)\, d\mu(x) \to 0\,, \quad \text{as } t \to \infty\,.$$

A complete Riemannian manifold N with these properties actually exists, and we now sketch how it is constructed. We want N such that

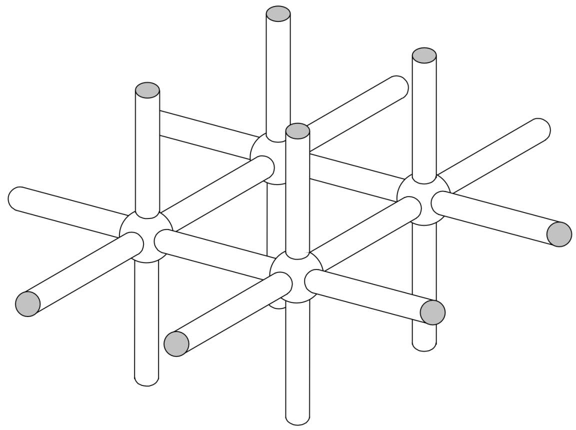
- (i) *N has the $L^\infty-$*Liouville *property.*
- (ii) *N is not stochastically complete.*
- (iii) *For every $p \in N$ we have $\mathcal{N}(p,t) \to 0$ as $t \to \infty$.*

The construction of N that satisfies $(i),(ii)$ is taken from [Gri99a, Section 13.5], which in turn builds on the first such example found by Pinchover in [Pin95]. Here, we note that it satisfies also (iii).

Start from the two-dimensional jungle-gym JG^2 in $\mathbb{R}^3$ as in Figure 6.1. This is done by smoothly connecting the lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ with necks. Let g be the standard metric on JG^2 induced by the embedding in $\mathbb{R}^3$. Fix $o \in JG^2$ and let $r := \mathrm{dist}(o,x)$ be the geodesic distance from x to o. It can be checked that JG^2 satisfies:

- (a) *$\mu(B_R(o)) \le CR^3$ for R large.*

Figure 6.1: The two dimensional jungle-gym in $\mathbb{R}^3$. Picture taken from [Gri99a].



(*b*) $G(o,x) \le C/r$ *for large* $r = \operatorname{dist}(o,x)$.

(*c*) JG^2 *has the* $L^\infty-$Liouville *property.*

Let $\rho : JG^2 \to [0,+\infty)$ *be a smooth positive function with* $\rho = 1$ *in* $[0,1]$ *and* $\rho(r) = \frac{1}{r\log(r)}$ *for* $r \ge 10$, *and consider the conformal metric* $\widehat{g} := \rho^2 g$ *on* JG^2. *We claim that* $N := (JG^2, \widehat{g})$ *has the desired properties.*

First, we note that N *is complete. Indeed, let* $\gamma_\star(s) : [0,\infty) \to JG^2$ *be a* $\widehat{g}$*-minimizing geodesic parametrized by the arclength of* g. *This means that* $\gamma_\star$ *minimizes the* $\widehat{g}$*-length between every couple of points on it. We refer to [CMR24, Section 2.2] for a proof that such an object exists. Then, for every divergent geodesic* $\gamma : [0,\infty) \to N$ *(i.e., a properly embedded half-line in* N *that is a geodesic with respect to* $\widehat{g}$*) we have*

$$\mathrm{Length}_{\widehat{g}}(\gamma) \ge \mathrm{Length}_{\widehat{g}}(\gamma_\star) = \int_{\gamma_\star} \rho \, ds \ge \int_{10}^{\infty} \frac{1}{t\log(t)}\, dt = +\infty.$$

Thus, N *is geodesically complete and hence complete.*

Denote by $\widehat{\Delta}, \widehat{G}, \widehat{\mu}$ *the Laplace operator, the Green function and the Riemannian volume form on* N *respectively. Since* $\dim(N) = 2$, *we have*

$$\widehat{\Delta} = \rho^{-2}\Delta_g \quad \textit{and} \quad \widehat{\mu} = \rho^2 \mu_g. \tag{6.16}$$

In particular, (JG^2, g) *and* N *have the same harmonic functions, and thus* N *also has the* $L^\infty-$Liouville *property. Moreover, the relations (6.16) imply that* JG^2 *and* N *have the same heat kernel, and thus their Green's function coincide* $\widehat{G} = G$. *Then, by the choice of* ρ, *for* R *fixed big*

$$\int_{N\setminus \widehat{B}_R(o)} \widehat{G}(o,x)\, d\widehat{\mu}(x) = \int_{JG^2\setminus \widehat{B}_R(o)} G(o,x)\rho^2(r(x))\, d\mu(x) < +\infty,$$

and by [Gri99a, Corollary 6.7] this implies that N *is not stochastically complete. Moreover, since* $\widehat{G}(o,x) \le C/\operatorname{dist}(o,x)$ *and* $\rho(r) = 1$ *for* $r \in (0,1)$, *it is easily seen that also*

$$\int_{\widehat{B}_R(o)} \widehat{G}(o,x)\, d\widehat{\mu}(x) < +\infty.$$

Consequently, by Tonelli's theorem

$$\int_0^\infty \mathcal{N}(p,t)\,dt = \int_0^\infty \int_N H_N(x,p,t)\,d\widehat{\mu}(x)dt = \int_N \left(\int_0^\infty H_N(x,p,t)\,dt \right) d\widehat{\mu}(x)$$
$$= \int_N \widehat{G}(o,x)\,d\widehat{\mu}(x) = \int_{N\setminus \widehat{B}_R(o)} \widehat{G}(o,x)\,d\widehat{\mu}(x) + \int_{\widehat{B}_R(o)} \widehat{G}(o,x)\,d\widehat{\mu}(x) < +\infty,$$

and since the function $\mathcal{N}(p,\cdot)$ is nonincreasing this implies that $\mathcal{N}(p,t) \to 0$ as $t \to \infty$.

6.2.3 Localized asymptotics and proof of Theorem 1.2.28

We now show (among other things) that (1.11) is well posed as in $\mathbb{R}^n$ for manifolds with the L^∞ – Liouville property, in the sense that it does not depend on the choice of p.

Lemma 6.2.5. *Let (M,g) be a complete Riemannian manifold with $\mu(M) = +\infty$ and $E \subset M$ be a set for which the limit (1.11) exists for some $p \in M$. If M has the L^∞ –* Liouville *property, then $\theta_E(p) \equiv \theta_E$ is constant, meaning that the limit in $\theta_E(q)$ exists for all $q \neq p$ and equals $\theta_E(p)$.*

Proof. We adopt the notation in the proof of Theorem 1.2.23. In particular, let $q \mapsto \Theta_{E,s}(q)$ be defined in (6.7). Arguing exactly as in the proof of Theorem 1.2.23, every subsequential limit (say, in $C^2_{\mathrm{loc}}(M)$) of $\Theta_{E,s}$ as $s \to 0^+$ is a bounded harmonic function on M.

Since M has the L^∞ – Liouville property, every such subsequential limit is constant. Then, since the limit $\lim_{s\to 0^+} \Theta_{E,s}(p) = \theta_E(p)$ exists by hypothesis, all subsequential limits must coincide with $\theta_E(p)$ everywhere. □

Let us note that the conclusion of Lemma 6.2.5 is not completely trivial in general and is particular to Riemannian manifolds that have the L^∞ – Liouville property. Indeed, we believe that on a general complete Riemannian manifold, it can happen that the limit in $\theta_E(\cdot)$ exists for some $p \in M$ but does not exist for some other $q \neq p$.

Lemma 6.2.6. *In the hypothesis of Lemma 6.2.5, for every bounded $F \subset M$ and $R > 0$ with $F \subset B_{R/2}(p)$ there holds*

$$\mu(F)\theta_E = \lim_{s\to 0^+} \mathcal{J}_s(F, E \setminus B_R(p)) = \lim_{s\to 0^+} \int_F \int_{E\setminus B_R(p)} \mathcal{K}_s(x,y)\,d\mu(x)d\mu(y).$$

Proof. Now since $F \subset B_{R/2}(p)$, we have that $B_{R/10}(y) \subset B_R(p) \subset B_{10R}(y)$ for every $y \in F$. Since the kernel $\mathcal{K}_s$ is nonnegative we get

$$\int_{E\setminus B_{10R}(y)} \mathcal{K}_s(x,y)\,d\mu(x) \le \int_{E\setminus B_R(p)} \mathcal{K}_s(x,y)\,d\mu(x) \le \int_{E\setminus B_{R/10}(y)} \mathcal{K}_s(x,y)\,d\mu(x)\,.$$

By the very definition of θ_E (1.11) and the fact that the limit does not depend on the radius whenever it exists (see part (i) of Theorem 1.2.23) both the left-hand side and right-hand side of the last inequality converge to $\theta_E(y) = \theta_E$ since θ_E is constant by Lemma 6.2.5, as $s \to 0^+$. Hence, integrating in $y \in F$ and letting $s \to 0^+$, by dominated convergence

$$\lim_{s\to 0^+} \int_F \int_{E\setminus B_R(p)} \mathcal{K}_s(x,y)\,d\mu(x)d\mu(y) = \int_F \theta_E\,d\mu(y) = \mu(F)\theta_E\,,$$

which is what we wanted to prove. □

Lemma 6.2.7. *Let (M, g) be complete with $\mu(M) = +\infty$, and let $A, B \subset M$ be two disjoint measurable sets with $\mu(A), \mu(B) < +\infty$ and with $\mathcal{J}_{s_\circ}(A, B) < +\infty$, for some $s_\circ \in (0, 1)$. Then*

$$\lim_{s\to 0^+} \mathcal{J}_s(A, B) = 0\,.$$

Proof. First, by Lemma 6.2.11 we have

$$\limsup_{s\to 0^+} \mathcal{J}_s(A, B) \leq \limsup_{s\to 0^+} \frac{s}{2} \iint_{A\times B} \int_{1/s}^{\infty} H_M(x, y, t) \frac{dt}{t^{1+s/2}} d\mu(x) d\mu(y)\,.$$

Then

$$\begin{aligned}\frac{s}{2} \iint_{A\times B} \int_{1/s}^{\infty} H_M(x, y, t) \frac{dt}{t^{1+s/2}} d\mu(x) d\mu(y) &= C s^{1+s/2} \int_A \int_1^{\infty} e^{(\xi/s)\Delta}(\chi_B)(x) \frac{d\xi}{\xi^{1+s/2}} d\mu(x) \\ &\leq C \int_A \left(s \int_1^{\infty} e^{(\xi/s)\Delta}(\chi_B)(x) \frac{d\xi}{\xi^{1+s/2}} \right) d\mu(x)\,.\end{aligned}$$

Since $\chi_B \in L^1(M)$, for every $x \in A$ (see Remark 6.1.6) there holds by dominated convergence

$$s \int_1^{\infty} e^{(\xi/s)\Delta}(\chi_B)(x) \frac{d\xi}{\xi^{1+s/2}} \to 0\,,$$

as $s \to 0^+$. From here, the result follows by dominated convergence using that $\mu(A) < +\infty$. □

The results above directly imply the following.

Corollary 6.2.8. *Let (M, g) be complete with $\mu(M) = +\infty$ and with the $L^\infty-$*Liouville *property, and let $\Omega \subset M$ be bounded. Then, for every $F \subset \Omega$ with $\mathrm{Per}_{s_\circ}(F, \Omega) < +\infty$, for some $s_\circ \in (0, 1)$, there holds*

$$\lim_{s\to 0^+} \mathcal{J}_s(F, E \cap \Omega^c) = \mu(F)\theta_E\,.$$

Proof. Let $p \in M$ and $R \gg 1$ be such that $\Omega \subset B_R(p)$, then

$$\begin{aligned}\mathcal{J}_s(F, E \cap \Omega^c) &= \mathcal{J}_s(F, E \cap \Omega^c \cap B_R(p)) + \mathcal{J}_s(F, E \cap \Omega^c \cap B_R^c(p)) \\ &= \mathcal{J}_s(F, E \cap \Omega^c \cap B_R(p)) + \mathcal{J}_s(F, E \cap B_R^c(p))\,.\end{aligned}$$

From here, since $\Omega^c \cap B_R(p)$ and F are disjoint and both with finite volume, the first term tends to zero as

$$\mathcal{J}_s(F, E \cap \Omega^c \cap B_R(p)) \leq \mathcal{J}_s(F, \Omega^c \cap B_R(p)) \to 0\,,$$

as $s \to 0^+$. Moreover, the second term tends to $\mu(F)\theta_E$ by Lemma 6.2.6. □

The proof of our main theorem in the infinite volume case is just a simple application of all the results we have derived above.

Proof of Theorem 1.2.28. Write

$$\begin{aligned}\frac{1}{2}\mathrm{Per}_s(E, \Omega) &= \mathcal{J}_s(E \cap \Omega, E^c \cap \Omega) + \mathcal{J}_s(E \cap \Omega, E^c \cap \Omega^c) + \mathcal{J}_s(E \cap \Omega^c, E^c \cap \Omega) \\ &= \frac{1}{2}\mathrm{Per}_s(E \cap \Omega) - \mathcal{J}_s(E \cap \Omega, E \cap \Omega^c) + \mathcal{J}_s(E^c \cap \Omega, E \cap \Omega^c)\,.\end{aligned}$$

By Corollary 6.2.2 applied to the first term, and by Corollary 6.2.8 applied with $F = E \cap \Omega$ and $F = E^c \cap \Omega$ respectively on the second and third term, taking the limit as $s \to 0^+$ we get

$$\begin{aligned}\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E,\Omega) &= \mu(E\cap\Omega) - \theta_E\mu(E\cap\Omega) + \theta_E\mu(E^c\cap\Omega)\\ &= (1-\theta_E)\mu(E\cap\Omega) + \theta_E\mu(E^c\cap\Omega)\,,\end{aligned}$$

and this shows (i).

To prove (ii) we follow closely the proof of in [Dip+13, Theorem 2.7], which deals with the analogous property in the case of the Euclidean space $\mathbb{R}^n$. We just sketch the argument since in the reference [Dip+13], the proof is carried on in full detail, and in our case, it is analogous. Again, let us denote

$$\Theta_{E,s}(p) := \int_{E\setminus B_R(p)} \mathcal{K}_s(x,p)d\mu(x)\,,$$

and fix $R > 0$ such that $\Omega \subset B_{R/2}(p)$. Note that

$$\begin{aligned}&\int_{\Omega\setminus E}\int_{E\setminus B_R(p)} \mathcal{K}_s(x,y)\,d\mu(x)d\mu(y) - \int_{\Omega\cap E}\int_{E\setminus B_R(p)} \mathcal{K}_s(x,y)\,d\mu(x)d\mu(y)\\ &= \frac{1}{2}\mathrm{Per}_s(E,\Omega) - \frac{1}{2}\mathrm{Per}_s(E\cap\Omega,\Omega) - \mathcal{J}_s(\Omega\setminus E,(E\setminus\Omega)\cap B_R(p)) + \mathcal{J}_s(\Omega\cap E,(E\setminus\Omega)\cap B_R(p))\,.\end{aligned}$$

Now, arguing exactly as in the proof of Lemma 6.2.6 we have that for every $F \subset \Omega$ there holds

$$\lim_{s\to 0^+}\left|\mu(F)\Theta_{E,s}(p) - \int_F\int_{E\setminus B_R(p)} \mathcal{K}_s(x,y)\,d\mu(x)d\mu(y)\right| = 0\,. \tag{6.17}$$

Since $\Omega \setminus E$ and $(E \setminus \Omega) \cap B_R(p)$ are disjoint and both with finite volume (since they are bounded), by Lemma 6.2.7 we have

$$\lim_{s\to 0^+} \mathcal{J}_s(\Omega\setminus E,(E\setminus\Omega)\cap B_R(p)) = 0\,,$$

and similarly

$$\lim_{s\to 0^+} \mathcal{J}_s(\Omega\cap E,(E\setminus\Omega)\cap B_R(p)) = 0\,.$$

Hence, taking the limit as $s \to 0^+$ above using (6.17) for the left-hand side with $F = \Omega \setminus E$ and $F = \Omega \cap E$ respectively gives

$$\lim_{s\to 0^+} \Theta_{E,s}\big(\mu(\Omega\setminus E) - \mu(\Omega\cap E)\big) = \lim_{s\to 0^+}\frac{1}{2}\big(\mathrm{Per}_s(E,\Omega) - \mathrm{Per}_s(E\cap\Omega,\Omega)\big)\,.$$

Since $E \cap \Omega \subset \Omega$ is bounded, by Corollary 6.2.2 we have

$$\lim_{s\to 0^+}\frac{1}{2}\mathrm{Per}_s(E\cap\Omega,\Omega) = \lim_{s\to 0^+}\frac{1}{2}\mathrm{Per}_s(E\cap\Omega) = \mu(E\cap\Omega)\,,$$

thus

$$\lim_{s\to 0^+} \Theta_{E,s}(p)\big(\mu(\Omega\setminus E) - \mu(\Omega\cap E)\big) = \left(\lim_{s\to 0^+}\frac{1}{2}\mathrm{Per}_s(E,\Omega)\right) - \mu(E\cap\Omega)\,.$$

From here, the conclusion of the theorem easily follows using that $\lim_{s\to 0^+}\Theta_{E,s}(p) = \theta_E$. Indeed, if $\mu(\Omega \setminus E) = \mu(\Omega \cap E)$ then the limit $\lim_{s\to 0^+}\frac{1}{2}\mathrm{Per}_s(E,\Omega)$ always exists and is equal to

$\mu(E \cap \Omega)$. On the other hand, if the limit $\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E,\Omega)$ exists then from above the limit in θ_E also exists and there holds

$$\theta_E = \frac{\left(\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E,\Omega)\right) - \mu(E\cap\Omega)}{\mu(\Omega\setminus E) - \mu(E\cap\Omega)},$$

and this concludes the proof. □

6.2.4 Finite volume asymptotic: proof of Theorem 1.2.30

We first give a simple proof of Theorem 1.2.30 in the case $\Omega = M$, using our results from Subsection 7.3 in our work [CG24] on the equivalence of the spectral fractional Laplacian and ours defined by the singular integral.

Theorem 6.2.9. *Let (M,g) be a complete Riemannian manifold with $\mu(M) < +\infty$ and let $s_\circ \in (0,1)$. Then, for every $u \in H^{s_\circ/2}(M)$ there holds*

$$\lim_{s\to 0^+} \frac{1}{2}[u]^2_{H^{s/2}(M)} = \|u\|^2_{L^2(M)} - \frac{1}{\mu(M)}\left(\int_M u\, d\mu\right)^2.$$

Proof. Let $\{E_\lambda\}_{\lambda\geq 0}$ be the spectral resolution of the Laplacian $-\Delta$ on $L^2(M)$, and let $\sigma(-\Delta) \subset [0,\infty)$ be the spectrum of $-\Delta$. In particular, for every $u \in L^2(M)$, $d\langle E_\lambda u, u\rangle$ is a regular Borel (real valued) measure on $[0,\infty)$ concentrated on $\sigma(-\Delta)$, and with

$$\|u\|^2_{L^2(M)} = \int_{\sigma(-\Delta)} d\langle E_\lambda u, u\rangle\,.$$

We refer to [Gri09, Appendix A.5] for an introduction and properties of the spectral resolution. Since $\mu(M) < +\infty$, we have that $0 \in \sigma(-\Delta)$ lies in the point spectrum with eigenfunction $\phi_0 = \mu(M)^{-1/2}$. Then

$$-\Delta = \int_{\sigma(-\Delta)} \lambda dE_\lambda\,, \quad \text{and} \quad (-\Delta)^{s/2}_{\mathrm{Spec}} = \int_{\sigma(-\Delta)} \lambda^{s/2} dE_\lambda\,,$$

on $\mathrm{Dom}((-\Delta)^{s/2}_{\mathrm{Spec}}) := \left\{u \in L^2(M) \,:\, \int_{\sigma(-\Delta)} \lambda^s\, d\langle E_\lambda u, u\rangle < +\infty\right\}$.

Hence, for all $s < s_\circ$, by Corollary [CG24, Corollary 7.9] we have

$$\frac{1}{2}[u]^2_{H^{s/2}(M)} = \int_M u(-\Delta)^{s/2}_{\mathrm{Si}} u\, d\mu = \int_{\sigma(-\Delta)\setminus\{0\}} \lambda^{s/2} d\langle E_\lambda u, u\rangle.$$

Taking the limit as $s \to 0^+$ gives

$$\lim_{s\to 0^+} \frac{1}{2}[u]^2_{H^{s/2}(M)} = \int_{\sigma(-\Delta)\setminus\{0\}} d\langle E_\lambda u, u\rangle = \|u\|^2_{L^2(M)} - \langle E_0 u, u\rangle = \|u\|^2_{L^2(M)} - \frac{1}{\mu(M)}\left(\int_M u\, d\mu\right)^2,$$

where in the last line we have used that E_0 is the projector onto the eigenspace of $-\Delta$ relative to the eigenvalue $\lambda = 0$, but by a result of Yau (see Theorem 6.1.1) on a complete manifold every $L^2(M)$ harmonic function is constant and then $\langle E_0 u, u\rangle = \langle \phi_0, u\rangle^2_{L^2(M)} = \frac{1}{\mu(M)}\left(\int_M u\, d\mu\right)^2$. □

Remark 6.2.10. *This result allows us to prove our Theorem 1.2.30 in the case $\Omega = M$. Indeed, if $E \subset M$ is such that $\mathrm{Per}_{s_\circ}(E) < +\infty$ for some $s_\circ \in (0,1)$, then taking $u = \chi_E$ in Theorem 6.2.9* *gives*

$$\lim_{s\to 0^+} \frac{1}{2}\mathrm{Per}_s(E) = \mu(E) - \frac{1}{\mu(M)}\mu(E)^2 = \frac{\mu(E)\mu(E^c)}{\mu(M)}\,.$$

Lemma 6.2.11. *Let (M,g) be a complete Riemannian manifold, and let $A,B\subset M$ two disjoint measurable sets with (say) $\mu(A)<+\infty$. If $\mathcal{J}_{s_\circ}(A,B)<+\infty$ for some $s_\circ\in(0,1)$ then*

$$\lim_{s\to 0^+}\left|\mathcal{J}_s(A,B) - \frac{1}{|\Gamma(-s/2)|}\iint_{A\times B}\int_{1/s}^{\infty} H_M(x,y,t)\frac{dt}{t^{1+s/2}}\,d\mu(x)d\mu(y)\right| = 0\,.$$

Proof. Since $\int_M H_M(x,y,t)\,d\mu(x)\le 1$ for all $y\in M$ and $t\in(0,\infty)$ we have

$$\begin{aligned}
&\left|\mathcal{J}_s(A,B) - \frac{1}{|\Gamma(-s/2)|}\iint_{A\times B}\int_{1/s}^{\infty} H_M(x,y,t)\frac{dt}{t^{1+s/2}}\,d\mu(x)d\mu(y)\right| \\
&\quad= \iint_{A\times B}\left(\mathcal{K}_s(x,y) - \frac{1}{|\Gamma(-s/2)|}\int_{1/s}^{\infty} H_M(x,y,t)\frac{dt}{t^{1+s/2}}\right)\,d\mu(x)d\mu(y)\\
&\quad= \frac{1}{|\Gamma(-s/2)|}\iint_{A\times B}\left(\int_0^1 H_M(x,y,t)\frac{dt}{t^{1+s/2}} + \int_1^{1/s} H_M(x,y,t)\frac{dt}{t^{1+s/2}}\right)d\mu(x)d\mu(y)\\
&\quad= \frac{1}{|\Gamma(-s/2)|}\iint_{A\times B}\int_0^1 H_M(x,y,t)\frac{dt}{t^{1+s/2}}d\mu(x)d\mu(y)\\
&\qquad+\frac{1}{|\Gamma(-s/2)|}\int_A\int_1^{1/s}\left(\int_B H_M(x,y,t)\,d\mu(x)\right)\frac{dt}{t^{1+s/2}}d\mu(y)\\
&\quad\le Cs\iint_{A\times B}\int_0^{\infty} H_M(x,y,t)\frac{dt}{t^{1+s_\circ/2}} + Cs\mu(A)\int_1^{1/s}\frac{dt}{t^{1+s/2}}\\
&\quad= Cs\mathcal{J}_{s_\circ}(A,B) + C\mu(A)(1-s^{s/2})\,,
\end{aligned}$$

and taking $s\to 0^+$ concludes the proof. □

Now, we can prove the main result of this subsection.

Proof of Theorem 1.2.30. First, we claim that

$$\lim_{s\to 0^+}\frac{1}{|\Gamma(-s/2)|}\iint_{A\times B}\int_{1/s}^{\infty} H_M(x,y,t)\frac{dt}{t^{1+s/2}}\,d\mu(x)d\mu(y) = \frac{\mu(A)\mu(B)}{\mu(M)}\,. \tag{6.18}$$

Indeed

$$s\int_{1/s}^{\infty} H_M(x,y,t)\frac{dt}{t^{1+s/2}} = s^{1+s/2}\int_1^{\infty} H(x,y,r/s)\frac{dr}{r^{1+s/2}}\,,$$

and since by Lemma 6.1.3 as $t\to+\infty$ the heat kernel $H_M(x,y,t)$ converges to $1/\mu(M)$ for all $x,y\in M$, we get

$$\begin{aligned}
\lim_{s\to 0^+}\frac{s}{2}\iint_{A\times B}\int_{1/s}^{\infty} H_M(x,y,t)\frac{dt}{t^{1+s/2}}\,d\mu(x)d\mu(y) &= \frac{\mu(A)\mu(B)}{\mu(M)}\lim_{s\to 0^+}(s/2)\,s^{s/2}\int_1^{\infty}\frac{dr}{r^{1+s/2}}\\
&= \frac{\mu(A)\mu(B)}{\mu(M)}\,.
\end{aligned}$$

Then, putting together Lemma 6.2.11 and (6.18) readily implies

$$\lim_{s\to 0^+} \mathcal{J}_s(A,B) = \frac{\mu(A)\mu(B)}{\mu(M)}\,. \tag{6.19}$$

Lastly, since $\mathrm{Per}_{s_\circ}(E,\Omega) < +\infty$ and

$$\frac{1}{2}\mathrm{Per}_s(E,\Omega) = \mathcal{J}_s(E\cap\Omega, E^c\cap\Omega) + \mathcal{J}_s(E\cap\Omega, E^c\cap\Omega^c) + \mathcal{J}_s(E\cap\Omega^c, E^c\cap\Omega)\,,$$

the theorem follows by letting $s \to 0^+$ and using (6.19) on each term. □

Weighted manifolds

Our result for finite volume manifolds, that is, Theorem 1.2.30, extends with proofs mutatis mutandis, to the case of weighted manifolds with finite volume, implying the one in [Car+22].

A weighted manifold is a Riemannian manifold (M,g) endowed with a measure μ that has a smooth positive density with respect to the Riemannian volume form dV_g. The space (M,g,μ) features the so-called weighted Laplace operator $-\Delta_\mu$, generalizing the Laplace-Beltrami operator, which is symmetric with respect to measure μ. It is possible to extend $-\Delta_\mu$ to a self-adjoint operator in $L^2(M,\mu)$, which allows one to define the heat semigroup $e^{t\Delta_\mu}$ as one would on a classical Riemannian manifold. The heat semigroup has the integral kernel $H_\mu(x,y,t)$, which is called the heat kernel of (M,g,μ), and has completely analogous properties to the classical one. For every detail regarding the heat kernel on weighted manifolds, we refer to the survey [Gri06].

In this case, we see that our proof applies since Lemma 6.1.3 also holds (with the same proof) on geodesically complete weighted manifolds, and also Theorem 6.2.9 holds with the same proof. For example, since the total mass of the Gaussian space is one, in the case of the Gaussian space (which is a weighted manifold of finite volume), we see that our Theorem 1.2.30 recovers the main result in [Car+22].

Appendix

Proposition A.1.1. *Let $s \in (0,1)$ and $u \in BV(\mathcal{B}_1)$. Then*

$$\iint_{\mathcal{B}_1 \times \mathcal{B}_1} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\, dx\, dy \le \frac{C(n)}{(1-s)s}[u]^s_{BV(\mathcal{B}_1)} \|u\|^{1-s}_{L^1(\mathcal{B}_1)}.$$

Proof. See, for instance, Proposition 4.2 in [BLP14]. □

Lemma A.1.2. *Let $v : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function with $\|v\|_{L^\infty(\mathbb{R}^n)} \le C_\circ$ satisfying $|Lv(x)| \le C_\circ$ for every $x \in \mathcal{B}_1(0)$, where L is an integro-differential operator of order $s \in (0,1)$ of the integral form*

$$Lu(x) = \int_{\mathbb{R}^n} (u(x)-u(y))K(x,y)\, dxdy\,,$$

and K is a nonnegative kernel comparable to that of the fractional s-Laplacian, that is satisfying

$$\frac{c}{|y|^{n+s}} \le K(x, x-y) \le \frac{C}{|y|^{n+s}} \quad \forall\, x, y \in \mathbb{R}^n, \tag{A.1}$$

for some constants $c, C > 0$. Then

$$[v]_{C^\alpha(\mathcal{B}_{1/2}(0))} \le C(n,s)C_\circ\,, \tag{A.2}$$

for some small positive $\alpha = \alpha(n,s)$.

Proof. The result is a standard consequence of [Sil06, Theorem 5.1]. Let us point out that Theorem 5.1 in [Sil06] would seem to require assumption [Sil06, (2.2)] to hold for all $r > 0$. However, it is clear from its (very short) proof that (A.2) only requires assumption [Sil06, (2.2)] to be verified at "small" scales $r \in (0,1)$ (and in our setting, this can be easily verified using (A.1)). □

Lemma A.1.3. *Let $s \in (0,2)$, $u \in H^{s/2}(\mathbb{R}^n)$ and $U : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ be the Caffarelli-Silvestre extension of u (in the sense of Theorem 3.7). Let $\widetilde{X} \in C^1_c(\mathbb{R}^{n+1}_+; \mathbb{R}^{n+1})$ be a vector field such that $X := \widetilde{X}|_{z=0}$ is tangent to $\mathbb{R}^n \times \{0\}$. Then*

$$\frac{d}{dt}\bigg|_{t=0} 2\beta_s \int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}(U \circ \phi^{\widetilde{X}}_t)|^2 z^{1-s}\, dxdz = \frac{d}{dt}\bigg|_{t=0} [u \circ \phi^X_t]^2_{H^{s/2}(\mathbb{R}^n)}.$$

Proof. Let V_t be the Caffarelli-Silvestre extension of $u \circ \phi^X_t$, for any $t \in \mathbb{R}$. By the minimality of the extension in the energy space, we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} 2\beta_s \int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}(U\circ\phi_t^{\widetilde{X}})|^2 z^{1-s} &= \lim_{t\to 0^+}\frac{2\beta_s}{t}\left(\int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}(U\circ\phi_t^{\widetilde{X}})|^2 z^{1-s} - \int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}U|^2 z^{1-s}\right)\\ &\geq \lim_{t\to 0^+}\frac{2\beta_s}{t}\left(\int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}V_t|^2 z^{1-s} - \int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}U|^2 z^{1-s}\right)\\ &= \lim_{t\to 0^+}\frac{[u\circ\phi_t^X]^2_{H^{s/2}(\mathbb{R}^n)} - [u]^2_{H^{s/2}(\mathbb{R}^n)}}{t}\\ &= \frac{d}{dt}\Big|_{t=0}[u\circ\phi_t^X]^2_{H^{s/2}(\mathbb{R}^n)}.\end{aligned}$$

Since this holds for every $\widetilde{X}$, applying this inequality with $-\widetilde{X}$ in place of $\widetilde{X}$ gives

$$\frac{d}{dt}\Big|_{t=0} 2\beta_s \int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}(U\circ\phi_t^{\widetilde{X}})|^2 z^{1-s} \leq \frac{d}{dt}\Big|_{t=0}[u\circ\phi_t^X]^2_{H^{s/2}(\mathbb{R}^n)}.$$

thus equality holds. □

The results below are not sharp. In particular, we believe that Propositions A.1.4 and A.1.5 hold also for $s=\sigma$ since this is the case for domains in $\mathbb{R}^n$. Here we focus on providing proofs that apply verbatim to the case of weighted manifolds, and we avoid using any local Euclidean-like structure of M.

Proposition A.1.4. *Let (M,g) be a stochastically complete Riemannian manifold, $\sigma\in(0,1)$ and $u\in H^\sigma(M)$ (as defined in Definition 2.2.1). Then, for every $s<\sigma$ the singular integral $(-\Delta)^{s/2}_{\mathrm{Si}}u$ and the Bochner $(-\Delta)^{s/2}_{\mathrm{B}}u$ definition coincide a.e. Moreover $(-\Delta)^{s/2}_{\mathrm{B}}u = (-\Delta)^{s/2}_{\mathrm{Si}}u \in L^2(M)$.*

Proof. Let $u\in H^\sigma(M)$ and $x\in M$. Since M is stochastically complete, if we could exchange the order of integration, we would have

$$\begin{aligned}(-\Delta)^{s/2}_{\mathrm{B}}u(x) &= \frac{1}{\Gamma(-s/2)}\int_0^\infty (e^{t\Delta}u(x)-u(x))\frac{dt}{t^{1+s/2}}\\ &= \frac{1}{\Gamma(-s/2)}\int_0^\infty \left(\int_M H_M(x,y,t)(u(y)-u(x))d\mu(y)\right)\frac{dt}{t^{1+s/2}}\\ &= \int_M (u(y)-u(x))\mathcal{K}_s(x,y)d\mu(y) = (-\Delta)^{s/2}_{\mathrm{Si}}u(x)\,.\end{aligned}$$

Now we shall justify the steps above, showing that the integral is absolutely convergent. Note that this will also justify the last equality since we have defined $(-\Delta)^{s/2}_{\mathrm{Si}}$ with the Cauchy principal value. In particular, we show that

$$\int_M\left(\int_M |u(x)-u(y)|\mathcal{K}_s(x,y)d\mu(y)\right)^2 d\mu(x) < +\infty\,.$$

This will prove at the same time that the integral above is absolutely convergent for a.e. $x\in M$ and that $(-\Delta)^{s/2}_{\mathrm{Si}}u\in L^2(M)$. Let us call

$$\mathfrak{I}(t) := \int_M |u(x)-u(y)|H_M(x,y,t)\,d\mu(y)\,,$$

and denote by C a constant that depends at most on σ.

Note that, by Jensen's inequality

$$\begin{aligned}\int_0^\infty \mathfrak{I}(t)^2 \frac{dt}{t^{1+\sigma}} &= \int_0^\infty \left(\int_M |u(x)-u(y)| H_M(x,y,t)\, d\mu(y) \right)^2 \frac{dt}{t^{1+\sigma}} \\ &\le \int_0^\infty \int_M |u(x)-u(y)|^2 H_M(x,y,t)\, d\mu(y) \frac{dt}{t^{1+\sigma}} \\ &= C \int_M |u(x)-u(y)|^2 \mathcal{K}_{2\sigma}(x,y)\, d\mu(y)\,. \end{aligned} \tag{A.3}$$

Write

$$\begin{aligned}&\int_M \left(\int_M |u(x)-u(y)| \mathcal{K}_s(x,y) d\mu(y) \right)^2 d\mu(x) \\ &\qquad = Cs^2 \int_M \left(\int_0^\infty \mathfrak{I}(t) \frac{dt}{t^{1+s/2}} \right)^2 d\mu \\ &\qquad \le Cs^2 \int_M \left(\int_0^1 \mathfrak{I}(t) \frac{dt}{t^{1+s/2}} \right)^2 d\mu + Cs^2 \int_M \left(\int_1^\infty \mathfrak{I}(t) \frac{dt}{t^{1+s/2}} \right)^2 d\mu\,. \end{aligned}$$

For the first integral, since $s < \sigma$, by Hölder's inequality and (A.3) we have

$$\int_M \left(\int_0^1 \mathfrak{I}(t) \frac{dt}{t^{1+s/2}} \right)^2 d\mu \le \int_M \left(\int_0^1 \mathfrak{I}(t)^2 \frac{dt}{t^{1+\sigma}} \right) \left(\int_0^1 \frac{dt}{t^{1-\sigma+s}} \right) d\mu \le C[u]^2_{H^\sigma(M)} < +\infty.$$

For the second integral, let us first renormalize the measure $\nu := C dt/t^{1+s/2}$ in a way that it becomes a probability measure on $[1,\infty)$. Then, by Jensen again (applied two times: to $d\nu(t)$ and then $H_M(x,y,t)d\mu(y)$)

$$\begin{aligned}\int_M \left(\int_1^\infty \mathfrak{I}(t) \frac{dt}{t^{1+s/2}} \right)^2 d\mu &\le \frac{C}{s^2} \iint_{M\times M} \int_1^\infty |u(x)-u(y)|^2 H_M(x,y,t)\, d\nu(t) d\mu(y) d\mu(x) \\ &\le \frac{4C}{s^2} \iint_{M\times M} \int_1^\infty |u(x)|^2 H_M(x,y,t)\, d\nu(t) d\mu(y) d\mu(x) \\ &\le \frac{4C}{s^2} \|u\|^2_{L^2(M)} < +\infty\,. \end{aligned}$$

Hence, we have proved

$$\begin{aligned}\|(-\Delta)^{s/2}_{\mathrm{Si}} u\|^2_{L^2(M)} &\le \int_M \left(\int_M |u(x)-u(y)| \mathcal{K}_s(x,y) d\mu(y) \right)^2 d\mu(x) \\ &\le C\|u\|^2_{L^2(M)} + Cs^2 \|u\|^2_{H^\sigma(M)}, \end{aligned} \tag{A.4}$$

and this concludes the proof. □

Next, we address the equivalence of the spectral fractional Laplacian $(-\Delta)^{s/2}_{\mathrm{Spec}}$ with the other definitions. We refer to [Gri09] and [Eri+22, Section 2.6] and the references therein for an introduction to the spectral theory of the fractional Laplacian on general spaces.

Let E_λ be the spectral resolvent of (minus) the Laplacian on (M,g). Then, for $s \in (0,2)$ in the classical sense of spectral theory

$$\mathrm{Dom}((-\Delta)^{s/2}_{\mathrm{Spec}}) := \left\{ u \in L^2(M) \,:\, \int_{\sigma(-\Delta)} \lambda^s \, d\langle E_\lambda u, u\rangle < +\infty \right\},$$

and for $u \in \mathrm{Dom}((-\Delta)^{s/2}_{\mathrm{Spec}})$

$$(-\Delta)^{s/2}_{\mathrm{Spec}} u := \int_{\sigma(-\Delta)} \lambda^{s/2} d\langle E_\lambda u, \cdot \rangle \,.$$

Proposition A.1.5. *Let (M, g) be a stochastically complete Riemannian manifold, $\sigma \in (0, 1)$ and $s < \sigma$. Then $H^\sigma(M) \subseteq \mathrm{Dom}((-\Delta)^{s/2}_{\mathrm{Spec}})$.*

Proof. Let $u \in H^\sigma(M)$, and let

$$\varphi(\lambda) := \lambda^{s/2} = \frac{1}{\Gamma(-s/2)} \int_0^\infty (e^{-\lambda t} - 1) \frac{dt}{t^{1+s/2}} \,.$$

Since $u \in L^2(M)$, by standard spectral theory (see [Gri09] for example)

$$\begin{aligned} \int_0^\infty \lambda^s d\langle E_\lambda u, u\rangle &= \int_0^\infty |\varphi(\lambda)|^2 d\langle E_\lambda u, u\rangle = \|\varphi(-\Delta)u\|^2_{L^2(M)} \\ &= \left\| \int_0^\infty (e^{t\Delta}u - u) \frac{dt}{t^{1+s/2}} \right\|^2_{L^2(M)} = \|(-\Delta)^{s/2}_{\mathrm{B}} u\|^2_{L^2(M)} = \|(-\Delta)^{s/2}_{\mathrm{Si}} u\|^2_{L^2(M)} < +\infty \,, \end{aligned}$$

where we have used that by Proposition A.1.4 to infer $(-\Delta)^{s/2}_{\mathrm{B}} u = (-\Delta)^{s/2}_{\mathrm{Si}} u \in L^2(M)$. □

Bibliography

[AAC01] Giovanni Alberti, Luigi Ambrosio, and Xavier Cabré. "On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property". In: vol. 65. 1-3. Special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday. 2001, pp. 9–33. DOI: 10.1023/A:1010602715526. URL: https://doi.org/10.1023/A:1010602715526.

[ABO03] G. Alberti, S. Baldo, and G. Orlandi. "Functions with prescribed singularities". In: *J. Eur. Math. Soc. (JEMS)* 5.3 (2003), pp. 275–311. ISSN: 1435-9855,1435-9863. DOI: 10.1007/s10097-003-0053-5. URL: https://doi.org/10.1007/s10097-003-0053-5.

[ABO05] G. Alberti, S. Baldo, and G. Orlandi. "Variational convergence for functionals of Ginzburg-Landau type". In: *Indiana Univ. Math. J.* 54.5 (2005), pp. 1411–1472. ISSN: 0022-2518,1943-5258. DOI: 10.1512/iumj.2005.54.2601. URL: https://doi.org/10.1512/iumj.2005.54.2601.

[AC00] Luigi Ambrosio and Xavier Cabré. "Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi". In: *J. Amer. Math. Soc.* 13.4 (2000), pp. 725–739. ISSN: 0894-0347,1088-6834. DOI: 10.1090/S0894-0347-00-00345-3. URL: https://doi.org/10.1090/S0894-0347-00-00345-3.

[ADM11] L. Ambrosio, G. De Philippis, and L. Martinazzi. "Gamma-convergence of nonlocal perimeter functionals". In: *Manuscripta Math.* 134.3-4 (2011), pp. 377–403. ISSN: 0025-2611,1432-1785. DOI: 10.1007/s00229-010-0399-4. URL: https://doi.org/10.1007/s00229-010-0399-4.

[Alm62] Frederick Justin Almgren Jr. "The homotopy groups of the integral cycle groups". In: *Topology* 1 (1962), pp. 257–299. ISSN: 0040-9383. DOI: 10.1016/0040-9383(62)90016-2. URL: https://doi.org/10.1016/0040-9383(62)90016-2.

[Alm65] Frederick J. Almgren. "The theory of varifolds : a variational calculus in the large for the k-dimensional area integrand". In: *mimeographed notes, Princeton University library* (1965).

[Alm66] F. J. Almgren Jr. "Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem". In: *Ann. of Math. (2)* 84 (1966), pp. 277–292. ISSN: 0003-486X. DOI: 10.2307/1970520. URL: https://doi.org/10.2307/1970520.

[Aro67] D. G. Aronson. "Bounds for the fundamental solution of a parabolic equation". In: *Bull. Amer. Math. Soc.* 73 (1967), pp. 890–896. ISSN: 0002-9904. DOI: 10.1090/S0002-9904-1967-11830-5. URL: https://doi.org/10.1090/S0002-9904-1967-11830-5.

[BBM01] J. Bourgain, H. Brezis, and P. Mironescu. "Another look at Sobolev spaces". In: *Optimal control and partial differential equations.* IOS, Amsterdam, 2001, pp. 439–455.

[BBM02] Jean Bourgain, Haïm Brezis, and Petru Mironescu. "Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications". In: vol. 87. Dedicated to the memory of Thomas H. Wolff. 2002, pp. 77–101. DOI: 10.1007/BF02868470. URL: https://doi.org/10.1007/BF02868470.

[Ber15] S. Bernstein. "Sur un théorème de géométrie et son application aux équations aux dérivées partielles du type elliptique". French. In: *Communications de la Société Mathématique de Kharkow. 2-éme série* 15.1 (1915), pp. 38–45.

[BFV14] Begoña Barrios, Alessio Figalli, and Enrico Valdinoci. "Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces". In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 13.3 (2014), pp. 609–639. ISSN: 0391-173X,2036-2145.

[BGL14] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and geometry of Markov diffusion operators*. Vol. 348. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Cham, 2014, pp. xx+552. ISBN: 978-3-319-00226-2. DOI: 10.1007/978-3-319-00227-9. URL: https://doi.org/10.1007/978-3-319-00227-9.

[BGS15] Valeria Banica, María del Mar González, and Mariel Sáez. "Some constructions for the fractional Laplacian on noncompact manifolds". In: *Rev. Mat. Iberoam.* 31.2 (2015), pp. 681–712. ISSN: 0213-2230,2235-0616. DOI: 10.4171/RMI/850. URL: https://doi.org/10.4171/RMI/850.

[BLP14] L. Brasco, E. Lindgren, and E. Parini. "The fractional Cheeger problem". In: *Interfaces Free Bound.* 16.3 (2014), pp. 419–458. ISSN: 1463-9963,1463-9971. DOI: 10.4171/IFB/325. URL: https://doi.org/10.4171/IFB/325.

[BM21] Haïm Brezis and Petru Mironescu. *Sobolev maps to the circle—from the perspective of analysis, geometry, and topology*. Vol. 96. Progress in Nonlinear Differential Equations and their Applications. Birkhäuser/Springer, New York, 2021, pp. xxxi+530. ISBN: 978-1-0716-1510-2. DOI: 10.1007/978-1-0716-1512-6. URL: https://doi.org/10.1007/978-1-0716-1512-6.

[BR14] Jaeyoung Byeon and Paul H. Rabinowitz. "A Note on Mountain Pass Solutions for a Class of Allen-Cahn Models (Nonlinear Partial Differential Equations, Dynamical Systems and Their Applications)". In: *Publications of the Research Institute for Mathematical Sciences* 1881 (Apr. 2014), pp. 1–17. URL: http://hdl.handle.net/2433/195655.

[Bra02] Andrea Braides. *Gamma-Convergence for Beginners*. Oxford University Press, 2002. ISBN: 9780198507840. DOI: 10.1093/acprof:oso/9780198507840.001.0001.

[Cab+18] X. Cabré et al. "Curves and surfaces with constant nonlocal mean curvature: meeting Alexandrov and Delaunay". In: *J. Reine Angew. Math.* 745 (2018), pp. 253–280. ISSN: 0075-4102,1435-5345. DOI: 10.1515/crelle-2015-0117. URL: https://doi.org/10.1515/crelle-2015-0117.

[Car+22] Alessandro Carbotti et al. "Asymptotics of the s-fractional Gaussian perimeter as $s \to 0^+$". In: *Fract. Calc. Appl. Anal.* 25.4 (2022), pp. 1388–1403. ISSN: 1311-0454,1314-2224. DOI: 10.1007/s13540-022-00066-8. URL: https://doi.org/10.1007/s13540-022-00066-8.

[Cas25] Michele Caselli. "Stable s-minimal cones in $\mathbb{R}^2$ are flat for $s \sim 0$". In: *Nonlinear Analysis* 259 (2025). ISSN: 0362-546X. DOI: https://doi.org/10.1016/j.na.2025.113828.

[CC14] Xavier Cabré and Eleonora Cinti. "Sharp energy estimates for nonlinear fractional diffusion equations". In: *Calc. Var. Partial Differential Equations* 49.1-2 (2014), pp. 233–269. ISSN: 0944-2669,1432-0835. DOI: 10.1007/s00526-012-0580-6. URL: https://doi.org/10.1007/s00526-012-0580-6.

[CCS20] X. Cabré, E. Cinti, and J. Serra. "Stable s-minimal cones in $\mathbb{R}^3$ are flat for $s \sim 1$". In: *J. Reine Angew. Math.* 764 (2020), pp. 157–180. ISSN: 0075-4102,1435-5345. DOI: 10.1515/crelle-2019-0005. URL: https://doi.org/10.1515/crelle-2019-0005.

[CCS21] X. Cabré, E. Cinti, and J. Serra. "Stable solutions to the fractional Allen-Cahn equation in the nonlocal perimeter regime". In: 2021. URL: https://api.semanticscholar.org/CorpusID:243985987.

[CFP24] Michele Caselli, Mattia Freguglia, and Nicola Picenni. "A nonlocal approximation of the area in codimension two". In: *arXiv:2406.13696* (2024).

[CFP25] Michele Caselli, Mattia Freguglia, and Nicola Picenni. "Coercivity and Gamma-convergence of the p-energy of sphere-valued Sobolev maps". In: *arXiv:2503.14142* (2025).

[CFS24a] M. Caselli, E. Florit-Simon, and J. Serra. "Fractional Sobolev spaces on Riemannian manifolds". In: *Math. Ann.* 390.4 (2024), pp. 6249–6314. ISSN: 0025-5831,1432-1807. DOI: 10.1007/s00208-024-02894-w. URL: https://doi.org/10.1007/s00208-024-02894-w.

[CFS24b] M. Caselli, E. Florit-Simon, and J. Serra. "Yau's conjecture for nonlocal minimal surfaces". In: *arXiv:2306.07100* (2024).

[CG18] X. Cabré and Poggesi G. "Stable Solutions to Some Elliptic Problems: Minimal Cones, the Allen-Cahn Equation, and Blow-Up Solutions". In: *Lecture Notes in Mathematics: Geometry of PDEs and Related Problems* (2018), pp. 1–45.

[CG24] M. Caselli and L. Gennaioli. "Asymptotics as $s \to 0^+$ of the fractional perimeter on Riemannian manifolds." In: *Annali della Scuola normale superiore di Pisa - Classe di scienze* (2024).

[Cha+23] H. Chan et al. "Nonlocal approximation of minimal surfaces: optimal estimates from stability". In: *arXiv:2308.06328* (2023).

[Cha84] Isaac Chavel. *Eigenvalues in Riemannian geometry.* Vol. 115. Pure and Applied Mathematics. Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk. Academic Press, Inc., Orlando, FL, 1984, pp. xiv+362. ISBN: 0-12-170640-0.

[Che69] S. S. Chern. "Minimal submanifolds of a Riemannian manifold". In: *mimeographed lecture notes, Univ. Kansas, Lawrence, Kan.* (1969).

[Cic+24] M. Cicalese et al. "A notion of s-fractional mass for 1-currents in higher codimension". In: *arXiv:2403.03901* (2024).

[Cir+18] G. Ciraolo et al. "Rigidity and sharp stability estimates for hypersurfaces with constant and almost-constant nonlocal mean curvature". In: *J. Reine Angew. Math.* 741 (2018), pp. 275–294. ISSN: 0075-4102,1435-5345. DOI: 10.1515/crelle-2015-0088. URL: https://doi.org/10.1515/crelle-2015-0088.

[CK91] Isaac Chavel and Leon Karp. "Large time behavior of the heat kernel: the parabolic λ-potential alternative". In: *Comment. Math. Helv.* 66.4 (1991), pp. 541–556. ISSN: 0010-2571,1420-8946. DOI: 10.1007/BF02566664. URL: https://doi.org/10.1007/BF02566664.

[CL24] Otis Chodosh and Chao Li. "Stable minimal hypersurfaces in $\mathbf{R}^4$". In: *Acta Math.* 233.1 (2024), pp. 1–31. ISSN: 0001-5962,1871-2509. DOI: 10.4310/acta.2024.v233.n1.a1. URL: https://doi.org/10.4310/acta.2024.v233.n1.a1.

[CLY81] Siu Yuen Cheng, Peter Li, and Shing Tung Yau. "On the upper estimate of the heat kernel of a complete Riemannian manifold". In: *Amer. J. Math.* 103.5 (1981), pp. 1021–1063. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2374257. URL: https://doi.org/10.2307/2374257.

[CM11] Tobias Holck Colding and William P. Minicozzi II. *A course in minimal surfaces*. Vol. 121. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011, pp. xii+313. ISBN: 978-0-8218-5323-8. DOI: 10.1090/gsm/121. URL: https://doi.org/10.1090/gsm/121.

[CM20] Otis Chodosh and Christos Mantoulidis. "Minimal surfaces and the Allen-Cahn equation on 3-manifolds: index, multiplicity, and curvature estimates". In: *Ann. of Math. (2)* 191.1 (2020), pp. 213–328. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2020.191.1.4. URL: https://doi.org/10.4007/annals.2020.191.1.4.

[CM23] O. Chodosh and C. Mantoulidis. "The p-widths of a surface". In: *Publ. Math. Inst. Hautes Études Sci.* 137 (2023), pp. 245–342. ISSN: 0073-8301,1618-1913. DOI: 10.1007/s10240-023-00141-7. URL: https://doi.org/10.1007/s10240-023-00141-7.

[CMR24] Giovanni Catino, Paolo Mastrolia, and Alberto Roncoroni. "Two rigidity results for stable minimal hypersurfaces". In: *Geom. Funct. Anal.* 34.1 (2024), pp. 1–18. ISSN: 1016-443X,1420-8970. DOI: 10.1007/s00039-024-00662-1. URL: https://doi.org/10.1007/s00039-024-00662-1.

[CP79] M. do Carmo and C. K. Peng. "Stable complete minimal surfaces in $\mathbf{R}^3$ are planes". In: *Bull. Amer. Math. Soc. (N.S.)* 1.6 (1979), pp. 903–906. ISSN: 0273-0979,1088-9485. DOI: 10.1090/S0273-0979-1979-14689-5. URL: https://doi.org/10.1090/S0273-0979-1979-14689-5.

[CRS10] L. Caffarelli, J.-M. Roquejoffre, and O. Savin. "Nonlocal minimal surfaces". In: *Comm. Pure Appl. Math.* 63.9 (2010), pp. 1111–1144. ISSN: 0010-3640,1097-0312. DOI: 10.1002/cpa.20331. URL: https://doi.org/10.1002/cpa.20331.

[CS07] Luis Caffarelli and Luis Silvestre. "An extension problem related to the fractional Laplacian". In: *Comm. Partial Differential Equations* 32.7-9 (2007), pp. 1245–1260. ISSN: 0360-5302,1532-4133. DOI: 10.1080/03605300600987306. URL: https://doi.org/10.1080/03605300600987306.

[CS11] Luis Caffarelli and Luis Silvestre. "Regularity results for nonlocal equations by approximation". In: *Arch. Ration. Mech. Anal.* 200.1 (2011), pp. 59–88. ISSN: 0003-9527,1432-0673. DOI: 10.1007/s00205-010-0336-4. URL: https://doi.org/10.1007/s00205-010-0336-4.

[CS22] Xavier Cabré and Tomás Sanz-Perela. "A universal Hölder estimate up to dimension 4 for stable solutions to half-Laplacian semilinear equations". In: *J. Differential Equations* 317 (2022), pp. 153–195. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2022.02.001. URL: https://doi.org/10.1016/j.jde.2022.02.001.

[CSV19] E. Cinti, J. Serra, and E. Valdinoci. "Quantitative flatness results and BV-estimates for stable nonlocal minimal surfaces". In: *J. Differential Geom.* 112.3 (2019), pp. 447–504. ISSN: 0022-040X,1945-743X. DOI: 10.4310/jdg/1563242471. URL: https://doi.org/10.4310/jdg/1563242471.

[CV13] L. Caffarelli and E. Valdinoci. “Regularity properties of nonlocal minimal surfaces via limiting arguments”. In: *Adv. Math.* 248 (2013), pp. 843–871. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2013.08.007. URL: https://doi.org/10.1016/j.aim.2013.08.007.

[Dáv02] J. Dávila. “On an open question about functions of bounded variation”. In: *Calc. Var. Partial Differential Equations* 15.4 (2002), pp. 519–527.

[Dav89] E. B. Davies. *Heat kernels and spectral theory.* Vol. 92. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1989, pp. x+197. ISBN: 0-521-36136-2. DOI: 10.1017/CBO9780511566158. URL: https://doi.org/10.1017/CBO9780511566158.

[Dip+13] Serena Dipierro et al. “Asymptotics of the s-perimeter as $s \searrow 0$”. In: *Discrete Contin. Dyn. Syst.* 33.7 (2013), pp. 2777–2790. ISSN: 1078-0947. DOI: 10.3934/dcds.2013.33.2777. URL: https://doi.org/10.3934/dcds.2013.33.2777.

[Dip20] Serena Dipierro. “A comparison between the nonlocal and the classical worlds: minimal surfaces, phase transitions, and geometric flows”. In: *Notices Amer. Math. Soc.* 67.9 (2020), pp. 1324–1335. ISSN: 0002-9920,1088-9477. DOI: 10.1090/noti. URL: https://doi.org/10.1090/noti.

[DPV12] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. “Hitchhiker’s guide to the fractional Sobolev spaces”. In: *Bull. Sci. Math.* 136.5 (2012), pp. 521–573. ISSN: 0007-4497,1952-4773. DOI: 10.1016/j.bulsci.2011.12.004. URL: https://doi.org/10.1016/j.bulsci.2011.12.004.

[DPW18] J. Dávila, M. del Pino, and J. Wei. “Nonlocal s-minimal surfaces and Lawson cones”. In: *J. Differential Geom.* 109.1 (2018), pp. 111–175. ISSN: 0022-040X,1945-743X. DOI: 10.4310/jdg/1525399218. URL: https://doi.org/10.4310/jdg/1525399218.

[DSV20] Serena Dipierro, Joaquim Serra, and Enrico Valdinoci. “Improvement of flatness for nonlocal phase transitions”. In: *Amer. J. Math.* 142.4 (2020), pp. 1083–1160. ISSN: 0002-9327,1080-6377. DOI: 10.1353/ajm.2020.0032. URL: https://doi.org/10.1353/ajm.2020.0032.

[Eri+22] Sylvester Eriksson-Bique et al. “Regularity of solutions to the fractional Cheeger-Laplacian on domains in metric spaces of bounded geometry”. In: *J. Differential Equations* 306 (2022), pp. 590–632. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2021.10.029. URL: https://doi.org/10.1016/j.jde.2021.10.029.

[Eva10] Lawrence C. Evans. *Partial differential equations.* Second. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, pp. xxii+749. ISBN: 978-0-8218-4974-3. DOI: 10.1090/gsm/019. URL: https://doi.org/10.1090/gsm/019.

[FF60] H. Federer and W. H. Fleming. “Normal and integral currents”. In: *Ann. of Math. (2)* 72 (1960), pp. 458–520. ISSN: 0003-486X. DOI: 10.2307/1970227. URL: https://doi.org/10.2307/1970227.

[Fig+15] A. Figalli et al. “Isoperimetry and stability properties of balls with respect to nonlocal energies”. In: *Comm. Math. Phys.* 336.1 (2015), pp. 441–507. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s00220-014-2244-1. URL: https://doi.org/10.1007/s00220-014-2244-1.

[Flo24] E. Florit-Simon. “Weyl Law and convergence in the classical limit for min-max nonlocal minimal surfaces”. In: *arXiv:2406.12162* (2024).

[FS08a] R. L. Frank and R. Seiringer. "Non-linear ground state representations and sharp Hardy inequalities". In: *J. Funct. Anal.* 255.12 (2008), pp. 3407–3430. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2008.05.015. URL: https://doi.org/10.1016/j.jfa.2008.05.015.

[FS08b] Rupert L. Frank and Robert Seiringer. "Non-linear ground state representations and sharp Hardy inequalities". In: *J. Funct. Anal.* 255.12 (2008), pp. 3407–3430. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2008.05.015. URL: https://doi.org/10.1016/j.jfa.2008.05.015.

[FS20a] Juan Carlos Felipe-Navarro and Tomás Sanz-Perela. "Uniqueness and stability of the saddle-shaped solution to the fractional Allen-Cahn equation". In: *Rev. Mat. Iberoam.* 36.6 (2020), pp. 1887–1916. ISSN: 0213-2230,2235-0616. DOI: 10.4171/rmi/1185. URL: https://doi.org/10.4171/rmi/1185.

[FS20b] Alessio Figalli and Joaquim Serra. "On stable solutions for boundary reactions: a De Giorgi-type result in dimension $4+1$". In: *Invent. Math.* 219.1 (2020), pp. 153–177. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-019-00904-2. URL: https://doi.org/10.1007/s00222-019-00904-2.

[FS25] F. Franceschini and J. Serra. "Flat nonlocal minimal surfaces are smooth". In: *to appear* (2025).

[FS80] Doris Fischer-Colbrie and Richard Schoen. "The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature". In: *Comm. Pure Appl. Math.* 33.2 (1980), pp. 199–211. ISSN: 0010-3640,1097-0312. DOI: 10.1002/cpa.3160330206. URL: https://doi.org/10.1002/cpa.3160330206.

[FZ24] Alessio Figalli and Yi Ru-Ya Zhang. "Uniform boundedness for finite Morse index solutions to supercritical semilinear elliptic equations". In: *Comm. Pure Appl. Math.* 77.1 (2024), pp. 3–36. ISSN: 0010-3640,1097-0312. DOI: 10.1002/cpa.22132. URL: https://doi.org/10.1002/cpa.22132.

[Gas20] Pedro Gaspar. "The second inner variation of energy and the Morse index of limit interfaces". In: *J. Geom. Anal.* 30.1 (2020), pp. 69–85. ISSN: 1050-6926,1559-002X. DOI: 10.1007/s12220-018-00134-7. URL: https://doi.org/10.1007/s12220-018-00134-7.

[GG18] Pedro Gaspar and Marco A. M. Guaraco. "The Allen-Cahn equation on closed manifolds". In: *Calc. Var. Partial Differential Equations* 57.4 (2018), Paper No. 101, 42. ISSN: 0944-2669,1432-0835. DOI: 10.1007/s00526-018-1379-x. URL: https://doi.org/10.1007/s00526-018-1379-x.

[GG19] Pedro Gaspar and Marco A. M. Guaraco. "The Weyl law for the phase transition spectrum and density of limit interfaces". In: *Geom. Funct. Anal.* 29.2 (2019), pp. 382–410. ISSN: 1016-443X,1420-8970. DOI: 10.1007/s00039-019-00489-1. URL: https://doi.org/10.1007/s00039-019-00489-1.

[GG98] N. Ghoussoub and C. Gui. "On a conjecture of De Giorgi and some related problems". In: *Math. Ann.* 311.3 (1998), pp. 481–491. ISSN: 0025-5831,1432-1807. DOI: 10.1007/s002080050196. URL: https://doi.org/10.1007/s002080050196.

[Gho91] N. Ghoussoub. "Location, multiplicity and Morse indices of min-max critical points". In: *J. Reine Angew. Math.* 417 (1991), pp. 27–76. ISSN: 0075-4102,1435-5345. DOI: 10.1515/crll.1991.417.27. URL: https://doi.org/10.1515/crll.1991.417.27.

[Gho93] Nassif Ghoussoub. *Duality and perturbation methods in critical point theory.* Vol. 107. Cambridge Tracts in Mathematics. With appendices by David Robinson. Cambridge University Press, Cambridge, 1993, pp. xviii+258. ISBN: 0-521-44025-4. DOI: 10.1017/CBO9780511551703. URL: https://doi.org/10.1017/CBO9780511551703.

[GIM20] Gabriele Grillo, Kazuhiro Ishige, and Matteo Muratori. "Nonlinear characterizations of stochastic completeness". In: *J. Math. Pures Appl. (9)* 139 (2020), pp. 63–82. ISSN: 0021-7824. DOI: 10.1016/j.matpur.2020.05.008. URL: https://doi.org/10.1016/j.matpur.2020.05.008.

[Gri+23] Gabriele Grillo et al. *A general nonlinear characterization of stochastic incompleteness.* 2023. arXiv: 2301.07942 [math.AP].

[Gri06] Alexander Grigor'yan. "Heat kernels on weighted manifolds and applications". In: *The ubiquitous heat kernel.* Vol. 398. Contemp. Math. Amer. Math. Soc., Providence, RI, 2006, pp. 93–191. DOI: 10.1090/conm/398/07486. URL: https://doi.org/10.1090/conm/398/07486.

[Gri09] Alexander Grigor'yan. *Heat kernel and analysis on manifolds.* Vol. 47. AMS/IP Studies in Advanced Mathematics. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009, pp. xviii+482. ISBN: 978-0-8218-4935-4. DOI: 10.1090/amsip/047. URL: https://doi.org/10.1090/amsip/047.

[Gri90] Alexander Grigor'yan. "Dimensions of spaces of harmonic functions". In: *Mat. Zametki* 48.5 (1990), pp. 55–61, 159. ISSN: 0025-567X. DOI: 10.1007/BF01236296. URL: https://doi.org/10.1007/BF01236296.

[Gri99a] Alexander Grigor'yan. "Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds". In: *Bull. Amer. Math. Soc. (N.S.)* 36.2 (1999), pp. 135–249. ISSN: 0273-0979. DOI: 10.1090/S0273-0979-99-00776-4. URL: https://doi.org/10.1090/S0273-0979-99-00776-4.

[Gri99b] Alexander Grigor'yan. "Estimates of heat kernels on Riemannian manifolds". In: *Spectral theory and geometry (Edinburgh, 1998).* Vol. 273. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1999, pp. 140–225. ISBN: 0-521-77749-6. DOI: 10.1017/CBO9780511566165.008. URL: https://doi.org/10.1017/CBO9780511566165.008.

[Gro88] M. Gromov. "Dimension, nonlinear spectra and width". In: *Geometric aspects of functional analysis (1986/87).* Vol. 1317. Lecture Notes in Math. Springer, Berlin, 1988, pp. 132–184. ISBN: 3-540-19353-7. DOI: 10.1007/BFb0081739. URL: https://doi.org/10.1007/BFb0081739.

[GT77] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order.* Vol. Vol. 224. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin-New York, 1977, pp. x+401. ISBN: 3-540-08007-4.

[Gua18] Marco A. M. Guaraco. "Min-max for phase transitions and the existence of embedded minimal hypersurfaces". In: *J. Differential Geom.* 108.1 (2018), pp. 91–133. ISSN: 0022-040X,1945-743X. DOI: 10.4310/jdg/1513998031. URL: https://doi.org/10.4310/jdg/1513998031.

[Gut09] Larry Guth. "Minimax problems related to cup powers and Steenrod squares". In: *Geom. Funct. Anal.* 18.6 (2009), pp. 1917–1987. ISSN: 1016-443X,1420-8970. DOI: 10.1007/s00039-009-0710-2. URL: https://doi.org/10.1007/s00039-009-0710-2.

[IMN18] Kei Irie, Fernando C. Marques, and André Neves. "Density of minimal hypersurfaces for generic metrics". In: *Ann. of Math. (2)* 187.3 (2018), pp. 963–972. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2018.187.3.8. URL: https://doi.org/10.4007/annals.2018.187.3.8.

[JS02] Robert L. Jerrard and Halil Mete Soner. "The Jacobian and the Ginzburg-Landau energy". In: *Calc. Var. Partial Differential Equations* 14.2 (2002), pp. 151–191. ISSN: 0944-2669,1432-0835. DOI: 10.1007/s005260100093. URL: https://doi.org/10.1007/s005260100093.

[Li12] Peter Li. *Geometric analysis*. Vol. 134. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2012, pp. x+406. ISBN: 978-1-107-02064-1. DOI: 10.1017/CBO9781139105798. URL: https://doi.org/10.1017/CBO9781139105798.

[Lin11] Fang Hua Lin. "Topology of Sobolev mappings and associated variational problems". In: *Surveys in geometric analysis and relativity*. Vol. 20. Adv. Lect. Math. (ALM). Int. Press, Somerville, MA, 2011, pp. 363–393. ISBN: 978-1-57146-230-5.

[LMN18] Yevgeny Liokumovich, Fernando C. Marques, and André Neves. "Weyl law for the volume spectrum". In: *Ann. of Math. (2)* 187.3 (2018), pp. 933–961. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2018.187.3.7. URL: https://doi.org/10.4007/annals.2018.187.3.7.

[LS88] A. C. Lazer and S. Solimini. "Nontrivial solutions of operator equations and Morse indices of critical points of min-max type". In: *Nonlinear Anal.* 12.8 (1988), pp. 761–775. ISSN: 0362-546X,1873-5215. DOI: 10.1016/0362-546X(88)90037-5. URL: https://doi.org/10.1016/0362-546X(88)90037-5.

[LY86] Peter Li and Shing-Tung Yau. "On the parabolic kernel of the Schrödinger operator". In: *Acta Math.* 156.3-4 (1986), pp. 153–201. ISSN: 0001-5962,1871-2509. DOI: 10.1007/BF02399203. URL: https://doi.org/10.1007/BF02399203.

[Mag12] F. Maggi. *Sets of finite perimeter and geometric variational problems*. Vol. 135. Cambridge Studies in Advanced Mathematics. An introduction to geometric measure theory. Cambridge University Press, Cambridge, 2012, pp. xx+454. ISBN: 978-1-107-02103-7. DOI: 10.1017/CBO9781139108133. URL: https://doi.org/10.1017/CBO9781139108133.

[MM77a] L. Modica and S. Mortola. "Un esempio di Γ^--convergenza". In: *Boll. Un. Mat. Ital. B (5)* 14.1 (1977), pp. 285–299.

[MM77b] Luciano Modica and Stefano Mortola. "Il limite nella Γ-convergenza di una famiglia di funzionali ellittici". In: *Boll. Un. Mat. Ital. A (5)* 14.3 (1977), pp. 526–529. ISSN: 0392-4033.

[MN17a] Fernando C. Marques and André Neves. "Existence of infinitely many minimal hypersurfaces in positive Ricci curvature". In: *Invent. Math.* 209.2 (2017), pp. 577–616. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-017-0716-6. URL: https://doi.org/10.1007/s00222-017-0716-6.

[MN17b] Fernando C. Marques and André Neves. "Existence of infinitely many minimal hypersurfaces in positive Ricci curvature". In: *Invent. Math.* 209.2 (2017), pp. 577–616. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-017-0716-6. URL: https://doi.org/10.1007/s00222-017-0716-6.

[MNS19] Fernando C. Marques, André Neves, and Antoine Song. "Equidistribution of minimal hypersurfaces for generic metrics". In: *Invent. Math.* 216.2 (2019), pp. 421–443. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-018-00850-5. URL: https://doi.org/10.1007/s00222-018-00850-5.

[Mod87] L. Modica. "The gradient theory of phase transitions and the minimal interface criterion". In: *Arch. Ration. Mech. Anal.* 98.2 (1987), pp. 123–142. ISSN: 0003-9527. DOI: 10.1007/BF00251230. URL: https://doi.org/10.1007/BF00251230.

[Moy25] Julien Moy. "$C^{1,\alpha}$ regularity of hypersurfaces of bounded nonlocal mean curvature in Riemannian manifolds". In: *Calc. Var. Partial Differential Equations* 64.1 (2025), Paper No. 9, 36. ISSN: 0944-2669,1432-0835. DOI: 10.1007/s00526-024-02846-x. URL: https://doi.org/10.1007/s00526-024-02846-x.

[MS02] V. Mazýa and T. Shaposhnikova. "On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces". In: *J. Funct. Anal.* 195.2 (2002), pp. 230–238. ISSN: 0022-1236,1096-0783. DOI: 10.1006/jfan.2002.3955. URL: https://doi.org/10.1006/jfan.2002.3955.

[MS23] C. Mihaila and B. Seguin. "A definition of fractional k-dimensional measure: bridging the gap between fractional length and fractional area". In: *arXiv:2303.11542* (2023).

[MSS24] J. Marx-Kuo, L. Sarnataro, and D. Stryker. "Index, Intersections, and Multiplicity of Min-Max Geodesics". In: *arXiv:2410.02580* (2024).

[MSW19] V. Millot, Y. Sire, and K. Wang. "Asymptotics for the fractional Allen-Cahn equation and stationary nonlocal minimal surfaces". In: *Arch. Ration. Mech. Anal.* 231.2 (2019), pp. 1129–1216. ISSN: 0003-9527,1432-0673. DOI: 10.1007/s00205-018-1296-3. URL: https://doi.org/10.1007/s00205-018-1296-3.

[Pin92] Yehuda Pinchover. "Large time behavior of the heat kernel and the behavior of the Green function near criticality for nonsymmetric elliptic operators". In: *J. Funct. Anal.* 104.1 (1992), pp. 54–70. ISSN: 0022-1236,1096-0783. DOI: 10.1016/0022-1236(92)90090-6. URL: https://doi.org/10.1016/0022-1236(92)90090-6.

[Pin95] Yehuda Pinchover. "On nonexistence of any λ_0-invariant positive harmonic function, a counterexample to Stroock's conjecture". In: *Comm. Partial Differential Equations* 20.9-10 (1995), pp. 1831–1846. ISSN: 0360-5302. DOI: 10.1080/03605309508821153. URL: https://doi.org/10.1080/03605309508821153.

[Pit74] Jon T. Pitts. "Regularity and singularity of one dimensional stationary integral varifolds on manifolds arising from variational methods in the large". In: *Symposia Mathematica, Vol. XIV (Convegno di Geometria Simplettica e Fisica Matematica & Convegno di Teoria Geometrica dell'Integrazione e Varietà Minimali, INDAM, Rome, 1973)*. Istituto Nazionale di Alta Matematica, Rome, 1974, pp. 465–472.

[Pit81] Jon T. Pitts. *Existence and regularity of minimal surfaces on Riemannian manifolds.* Vol. 27. Mathematical Notes. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1981, pp. iv+330. ISBN: 0-691-08290-1.

[PKW13] M. del Pino, M. Kowalczyk, and J. Wei. "Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$". In: *J. Differential Geom.* 93.1 (2013), pp. 67–131. ISSN: 0022-040X,1945-743X. URL: http://projecteuclid.org/euclid.jdg/1357141507.

[Pog81] A. V. Pogorelov. "On the stability of minimal surfaces". In: *Dokl. Akad. Nauk SSSR* 260.2 (1981), pp. 293–295. ISSN: 0002-3264.

[Pon04] A. C. Ponce. "A new approach to Sobolev spaces and connections to Γ-convergence". In: *Calc. Var. Partial Differential Equations* 19.3 (2004), pp. 229–255. ISSN: 0944-2669,1432-0835. DOI: 10.1007/s00526-003-0195-z. URL: https://doi.org/10.1007/s00526-003-0195-z.

[PPS24] Davide Parise, Alessandro Pigati, and Daniel Stern. "Convergence of the self-dual $U(1)$-Yang-Mills-Higgs energies to the $(n-2)$-area functional". In: *Comm. Pure Appl. Math.* 77.1 (2024), pp. 670–730. ISSN: 0010-3640,1097-0312. DOI: 10.1002/cpa.22150. URL: https://doi.org/10.1002/cpa.22150.

[PS21] A. Pigati and D. Stern. "Minimal submanifolds from the abelian Higgs model". In: *Invent. Math.* 223.3 (2021), pp. 1027–1095. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-020-01000-6. URL: https://doi.org/10.1007/s00222-020-01000-6.

[PS23] A. Pigati and D. Stern. "Quantization and non-quantization of energy for higher-dimensional Ginzburg-Landau vortices". In: *Ars Inven. Anal.* (2023), Paper No. 3, 55. ISSN: 2769-8505.

[PSV13] Giampiero Palatucci, Ovidiu Savin, and Enrico Valdinoci. "Local and global minimizers for a variational energy involving a fractional norm". In: *Ann. Mat. Pura Appl. (4)* 192.4 (2013), pp. 673–718. ISSN: 0373-3114,1618-1891. DOI: 10.1007/s10231-011-0243-9. URL: https://doi.org/10.1007/s10231-011-0243-9.

[Ros97] Steven Rosenberg. *The Laplacian on a Riemannian manifold.* Vol. 31. London Mathematical Society Student Texts. An introduction to analysis on manifolds. Cambridge University Press, Cambridge, 1997, pp. x+172. DOI: 10.1017/CBO9780511623783. URL: https://doi.org/10.1017/CBO9780511623783.

[Sal92] L. Saloff-Coste. "A note on Poincaré, Sobolev, and Harnack inequalities". In: *Internat. Math. Res. Notices* 2 (1992), pp. 27–38. ISSN: 1073-7928,1687-0247. DOI: 10.1155/S1073792892000047. URL: https://doi.org/10.1155/S1073792892000047.

[Sal95] L. Saloff-Coste. "Parabolic Harnack inequality for divergence-form second-order differential operators". In: *Potential Anal.* 4.4 (1995), pp. 429–467. ISSN: 0926-2601,1572-929X. DOI: 10.1007/BF01053457. URL: https://doi.org/10.1007/BF01053457.

[Sav09] Ovidiu Savin. "Regularity of flat level sets in phase transitions". In: *Ann. of Math. (2)* 169.1 (2009), pp. 41–78. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2009.169.41. URL: https://doi.org/10.4007/annals.2009.169.41.

[Ser23] J. Serra. "Nonlocal minimal surfaces: recent developments, applications, and future directions". In: *SeMA Journal* (2023). DOI: 10.1007/s40324-023-00345-1. URL: https://doi.org/10.1007/s40324-023-00345-1.

[Ser83] L. Serge. *Real analysis.* Second. Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1983, pp. xv+533. ISBN: 0-201-14179-5.

[Sil06] Luis Silvestre. "Hölder estimates for solutions of integro-differential equations like the fractional Laplace". In: *Indiana Univ. Math. J.* 55.3 (2006), pp. 1155–1174. ISSN: 0022-2518,1943-5258. DOI: 10.1512/iumj.2006.55.2706. URL: https://doi.org/10.1512/iumj.2006.55.2706.

[Sim68] J. Simons. "Minimal varieties in riemannian manifolds". In: *Ann. of Math.* 88 (1968), pp. 62–105. ISSN: 0003-486X. DOI: 10.2307/1970556. URL: https://doi.org/10.2307/1970556.

[Sim93] Barry Simon. "Large time behavior of the heat kernel: on a theorem of Chavel and Karp". In: *Proc. Amer. Math. Soc.* 118.2 (1993), pp. 513–514. ISSN: 0002-9939,1088-6826. DOI: 10.2307/2160331. URL: https://doi.org/10.2307/2160331.

[Son23] Antoine Song. "Existence of infinitely many minimal hypersurfaces in closed manifolds". In: *Ann. of Math. (2)* 197.3 (2023), pp. 859–895. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2023.197.3.1. URL: https://doi.org/10.4007/annals.2023.197.3.1.

[SS81] Richard Schoen and Leon Simon. "Regularity of stable minimal hypersurfaces". In: *Comm. Pure Appl. Math.* 34.6 (1981), pp. 741–797. ISSN: 0010-3640,1097-0312. DOI: 10.1002/cpa.3160340603. URL: https://doi.org/10.1002/cpa.3160340603.

[Sti10] Pablo Raúl Stinga. "Fractional powers of second order partial differential operators: extension problem and regularity theory". In: 2010. URL: https://api.semanticscholar.org/CorpusID:120574484.

[SV13a] O. Savin and E. Valdinoci. "Regularity of nonlocal minimal cones in dimension 2". In: *Calc. Var. Partial Differential Equations* 48.1-2 (2013), pp. 33–39. ISSN: 0944-2669,1432-0835. DOI: 10.1007/s00526-012-0539-7. URL: https://doi.org/10.1007/s00526-012-0539-7.

[SV13b] O. Savin and E. Valdinoci. "Some monotonicity results for minimizers in the calculus of variations". In: *J. Funct. Anal.* 264.10 (2013), pp. 2469–2496. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2013.02.005. URL: https://doi.org/10.1016/j.jfa.2013.02.005.

[Tho24] J. Thompson. "Density estimates and the fractional Sobolev inequality for sets of zero s-mean curvature". In: *arXiv:2406.04618* (2024).

[TW12] Yoshihiro Tonegawa and Neshan Wickramasekera. "Stable phase interfaces in the van der Waals–Cahn–Hilliard theory". In: *J. Reine Angew. Math.* 668 (2012), pp. 191–210. ISSN: 0075-4102,1435-5345. DOI: 10.1515/crelle.2011.134. URL: https://doi.org/10.1515/crelle.2011.134.

[Var67] S. R. S. Varadhan. "On the behavior of the fundamental solution of the heat equation with variable coefficients". In: *Comm. Pure Appl. Math.* 20 (1967), pp. 431–455. ISSN: 0010-3640,1097-0312. DOI: 10.1002/cpa.3160200210. URL: https://doi.org/10.1002/cpa.3160200210.

[Wic14] N. Wickramasekera. "A general regularity theory for stable codimension 1 integral varifolds". In: *Ann. of Math. (2)* 179.3 (2014), pp. 843–1007. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2014.179.3.2. URL: https://doi.org/10.4007/annals.2014.179.3.2.

[Yau82] Shing Tung Yau, ed. *Seminar on Differential Geometry.* Vol. No. 102. Annals of Mathematics Studies. Papers presented at seminars held during the academic year 1979–1980. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982, pp. ix+706. ISBN: 0-691-08268-5.

[ZB12] Wei Zhang and Jiguang Bao. "Regularity of very weak solutions for elliptic equation of divergence form". In: *J. Funct. Anal.* 262.4 (2012), pp. 1867–1878. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2011.11.027. URL: https://doi.org/10.1016/j.jfa.2011.11.027.