Existence of Positive Solutions of Some Semilinear Elliptic Equations with Singular Coefficients

Nirmalendu Chaudhuri† Mythily Ramaswamy‡

† *Department of Mathematics, Indian Institute of Science, Bangalore. 560012, India.*

‡ *School of Mathematics, TIFR Centre, IISc, Bangalore. 560012, India.*

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In this article we consider the semilinear elliptic problem in a bounded domain $\Omega \subseteq \mathbb{R}^n$,

$$\left\{\begin{array}{rcll} -\Delta u &=& \frac{\mu}{|x|^\alpha} u^{2^*_\alpha - 1} + f(x)g(u) & \text{in } \Omega \\ u &>& 0 & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega \end{array}\right.$$

where $\mu \geq 0,\ 0 \leq \alpha \leq 2,\ 2^*_\alpha := \frac{2(n-\alpha)}{n-2},\ f : \Omega \to \mathbb{R}^+$ *is measurable,* $f > 0$ *a.e, having a lower order singularity than* $|x|^{-2}$ *at the origin and* $g : \mathbb{R} \to \mathbb{R}$ *is either linear or superlinear. For* $1 < p < n$*, we characterize a class of singular functions* $\Im_p$ *for which the embedding* $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f)$ *is compact. When* $p = 2,\ \alpha = 2$ *,* $f \in \Im_2$ *and* $0 \leq \mu < \left(\frac{n-2}{2}\right)^2$*, we prove that the linear problem has* H_0^1*-discrete spectrum. By improving the Hardy inequality we show that for* f *belonging to a certain subclass of* $\Im_2$*, the first eigenvalue goes to a positive number as* μ *approaches* $\left(\frac{n-2}{2}\right)^2$*. Furthermore, when* g *is superlinear, we show that for the same subclass of* $\Im_2$*, the functional corresponding to the differential equation satisfies Palais-Smale condition if* $\alpha = 2$ *and Brezis-Nirenberg type of phenomenon occurs for the case* $0 \leq \alpha < 2$*.*

Key words: Laplacian, Variational method, Hardy inequality, Sobolev constant, Palais-Smale condition.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 3$ with $0 \in \Omega$. We are concerned with the existence of weak solutions of the following semilinear elliptic problem

$$\left\{\begin{array}{rcll} -\Delta u &=& \frac{\mu}{|x|^\alpha} u^{2^*_\alpha - 1} + f(x)g(u) & \text{in } \Omega \\ u &>& 0 & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega \end{array}\right. \tag{1.1}$$

where $\mu \geq 0,\ 0 \leq \alpha \leq 2,\ 2^*_\alpha = \dfrac{2(n-\alpha)}{n-2},\ f : \Omega \to \mathbb{R}^+$ is measurable, $f > 0$ a.e and $g : \mathbb{R} \to \mathbb{R}$ is either linear or superlinear. For $0 \leq \alpha \leq 2,\ 2^*_\alpha$ is the limiting exponent of the

Hardy-Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*_\alpha}(\Omega, |x|^{-\alpha})$. After the pioneering work of Brezis and Nirenberg on critical exponent problem [4] (the case $\alpha = 0$), it is now well understood that certain lower order terms can reverse the nonexistence and cause positive solutions to exist. Our aim here, is to understand how certain singular coefficients $f(x)$ of the lower order terms can cause the existence of positive solutions.

When $\alpha = 2$, it is well known that for any starshaped domain Ω and any $\mu \geq 0$, the problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & \frac{\mu}{|x|^2} u & \text{in } \Omega \\ u & > & 0 & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right.$$

has no weak solution in $H_0^1(\Omega)$, (see Theorem 3.3 in [8] for bounded domains and section 3 in [11] for $\mathbb{R}^n$). The operator $L_\mu := -(\Delta + \frac{\mu}{|x|^2})$ on $H_0^1(\Omega)$ has discrete spectrum iff $\mu < \beta_{n,2} := \left(\frac{n-2}{2}\right)^2$. This can be seen by using the Hardy-Sobolev inequality

$$\int_\Omega \frac{|u|^2}{|x|^2} dx \leq \left(\frac{2}{n-2}\right)^2 \int_\Omega |\nabla u|^2 dx \qquad \forall\ u \in H_0^1(\Omega)\,. \tag{1.2}$$

For such L_μ, we study the eigenvalue problem with a singular weight $f(x)$ and the behaviour of the first eigenvalue $\lambda_\mu^1(f)$ as $\mu \to \beta_{n,2}$. Notice that the coercivity of L_μ is lost at $\beta_{n,2}$. Since the inequality (1.2) is still valid even after adding the integral $\int_\Omega \frac{|u|^2}{|x|^\beta} dx$ to the left hand side, (see Section 3, the lemma on improved Hardy inequality), we deduce that when $f(x)$ is like $|x|^{-\beta}$ near 0 for some $0 \leq \beta < 2$, the first eigenvalue $\lambda_\mu^1(f)$ goes to a positive number as $\mu \to \beta_{n,2}$. For the case $\beta = 0$, this inequality has been proved by Brezis and Vazquez in [5]. To establish the nonexistence of a positive eigenfunction in $H_0^1(\Omega)$, we show more generally the nonexistence of any positive $H_0^1(\Omega)$ supersolution of $-\Delta u \geq \frac{\mu}{|x|^2} u$ for $\mu > \beta_{n,2}$.

We also study a semilinear problem for L_μ, with singular coefficient. The coefficient $f(x)$ should have lower order singularity than $|x|^{-2}$ at $x = 0$. If $f(x) = |x|^{-2}$, then the problem (1.1) for $\alpha = 2$ does not admit any weak solution, at least for continuous functions $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$2G(u) \leq g(u)u \quad \forall\ u \geq 0$$

in any bounded starshaped domain (see theorem 3.3 in [8]), where $G(u) = \int_0^u g(s)ds$.
For the case $0 < \alpha < 2$, positive solutions in $H_0^1(\Omega)$ exist for the problem

$$-\Delta u = \frac{u}{|x|^\alpha} |u|^{2^*_\alpha - 2} \ \text{ in } \ \Omega$$

only when Ω is invariant under scalings centered at $x = 0$, for example $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$; (see [11], Theorem 2). Using this, an existence result for the Brezis-Nirenberg type problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & \frac{u^{2^*_\alpha - 1}}{|x|^\alpha} + \lambda u & \text{in} \quad \Omega \\ u & > & 0 & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \right.$$

for any bounded domain Ω is also proved in [11]. We study a similar problem with a superlinear lower order term $g(u)$ and a singular coefficient, where an interesting balance between the singularity and nonlinearity is essential for the existence of a solution.

Our main results are:

Theorem 1.1 *Let $0 \leq \mu < \beta_{n,2}$, $\lambda \in \mathbb{R}^+$, $f \in \Im_2$, where*

$$\Im_2 = \left\{ f : \Omega \to \mathbb{R}^+ \;\Big|\; \lim_{|x|\to 0} |x|^2 f(x) = 0 \text{ with } f \in L^\infty_{\text{loc}}(\Omega \setminus \{0\}) \right\} .$$

The eigenvalue problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & \frac{\mu}{|x|^2} u + \lambda u f(x) & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{1.3}$$

admits nontrivial weak solutions in $H^1_0(\Omega)$ corresponding to $\lambda \in (\lambda^k_\mu(f))_{k=1}^\infty$, where $0 < \lambda^1_\mu(f) < \lambda^2_\mu(f) \leq \lambda^3_\mu(f) \leq \ldots \to +\infty$. If Ω is $C^{1,1}$, then any weak solution of (1.3) is in $H^1_0(\Omega) \cap W^{2,r}(\Omega)$ for all r, $1 < r < 2n/(n+2)$.
Furthermore, if $f \in \Re_{2,\beta} := \left\{ f \in \Im_2 \;\Big|\; 0 < \lim_{|x|\to 0} |x|^\beta f(x) < \infty \right\}$ for some $0 \leq \beta < 2$, then $\lambda^1_\mu(f) \to \lambda(f) > 0$ as $\mu \to \beta_{n,2}$.

Remark 1.1 In a recent paper [1], it is shown that $\lambda^1_\mu(f) \to \lambda(f) > 0$ for a larger class of weight functions than $\Re_{2,\beta}$ and determined the border line for $\lambda(f)$ to be zero or positive.

Theorem 1.2 *For $\mu > \beta_{n,2}$, there exists no $u \in H^1_0(\Omega)$ with $u \geq 0$, $u \not\equiv 0$ such that*

$$-\Delta u \geq \frac{\mu}{|x|^2} u \quad \text{in} \quad \Omega$$

i.e., there does not exist any nonnegative $u \in H^1_0(\Omega)$ with , $u \not\equiv 0$ such that $\int_\Omega \nabla u . \nabla \phi dx \geq \mu \int_\Omega \frac{u}{|x|^2} \phi \, dx$ for all $\phi \in H^1_0(\Omega)$, $\phi \geq 0$.

Remark 1.2 The Theorem 1.2 is still valid even at the critical lavel $\mu = \beta_{n,2}$ and a proof can be found in [7].

Theorem 1.3 *The problem*

$$\left\{ \begin{array}{rcll} -\Delta u & = & \frac{\mu}{|x|^2} u + u^{q-1} f(x) & \text{in} \quad \Omega \\ u & > & 0 & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{1.4}$$

where $0 \leq \mu < \beta_{n,2}$, $f \in \Re_{2,\beta}$ *for some* $0 \leq \beta < 2$ *and* $2 < q < 2^*_\beta$, *admits a weak solution in* $H^1_0(\Omega)$.

Remark 1.3 If Ω is a bounded starshaped domain with respect to 0 and $q \geq 2^*_\beta$, $f(x) = |x|^{-\beta}$ in Ω, $0 \leq \beta < 2$, then by using a Pohozaev type identity, one can easily show that the problem (1.4) does not admit any solution. Thus 2^*_β is critical for this problem.

Remark 1.4 It is therefore natural to ask that, if one relaxes the condition on the local behaviour of $f(x)$, can there exist a solution to the problem (1.4). The counterexample below shows that the relaxation of the condition may not be possible. Consider the domain Ω to be the ball $B(0,r)$, $0 < r << 1$, $q > 2$ and let $f(x) = |x|^{-2}/|\log|x||$. Then, by using a Pohozaev type identity, one can easily show that the problem (1.4) does not admit any solution.

Theorem 1.4 *The problem*

$$\left\{ \begin{array}{rcll} -\Delta u & = & \frac{u^{2^*_\alpha - 1}}{|x|^\alpha} + \lambda u^{q-1} f(x) & \text{in } \Omega \\ u & > & 0 & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right. \tag{1.5}$$

where $0 \leq \alpha < 2$, $f \in \Re_{2,\beta}$, *for some* $0 \leq \beta < 2$, $2 < q < 2^*_\beta$ *and* $\lambda \in \mathbb{R}^+$ *admits a weak solution,*

(a) for all $n \geq 4$ *and* $\lambda > 0$

(b) when $n = 3$, *in the following different cases:*

(i) For all $1 \leq \beta < 2$ *and* $\lambda > 0$.

(ii) For all $0 \leq \beta < 1$ *with* $2(2-\beta) < q < 2^*_\beta$ *and for all* $\lambda > 0$

(iii) For all $0 \leq \beta < 1$ *with* $2 < q \leq 2(2-\beta)$ *and for sufficiently large* $\lambda > 0$.

Remark 1.5 The main difference between the cases $\alpha = 2$ and $\alpha < 2$ for the problem (1.1), is that the functional corresponding to the PDE satisfies Palis-Smale (P.S) condition at any energy level when $\alpha = 2$, but for the case $\alpha < 2$ it satisfies P.S condition only below a certain energy level.

2 Preliminary Results

Let us first recall the well known Hardy-Sobolev inequality. For $1 < p < n$, $D^{1,p}(\mathbb{R}^n)$ is embedded continuously in $L^p(\mathbb{R}^n, |x|^{-p})$ and

$$\int\limits_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx \leq \left(\frac{p}{n-p}\right)^p \int\limits_{\mathbb{R}^n} |\nabla u|^p dx \qquad \forall\ u \in D^{1,p}(\mathbb{R}^n) \tag{2.1}$$

where $D^{1,p}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n;\mathbb{R})$ in the norm $\|u\|_{D^{1,p}} = \|\nabla u\|_{L^p}$ (See [9], for example). If $\Omega = B_R$, a ball, $R > 0$ then also (2.1) is true for any $u \in W_0^{1,p}(B_R)$. If Ω is any general bounded domain, using Schwartz's symmetrization, one can verify the validity of (2.1) for $W_0^{1,p}(\Omega)$. The constant $\beta_{n,p} := \left(\frac{n-p}{p}\right)^p$ is the best for the Hardy-Sobolev embedding for any Ω.

To prove our main results we will make use of the following two propositions which are immediate consequence of Hardy and Sobolev embeddings. Although we use only the case $p = 2$, we prove here for $p,\ 1 < p < n$, just to indicate the possible generalization of our results to the p-laplacian case.

Proposition 2.1 *Let Ω be a bounded domain in $\mathbb{R}^n$ and $1 < p < n$. Let $f \in \Im_p$, where*

$$\Im_p := \left\{ f : \Omega \to \mathbb{R}^+ \;\middle|\; \lim_{|x|\to 0} |x|^p f(x) = 0 \text{ with } f \in L^\infty_{\text{loc}}\left(\Omega \setminus \{0\}\right) \right\}.$$

Then the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f dx)$ is compact.

Proof: Since $\lim_{|x|\to 0} |x|^p f(x) = 0$, for any $\epsilon > 0$, $\quad \exists\ \delta > 0$ such that

$$\sup_{B_\delta \subseteq \Omega} |x|^p f(x) \le \epsilon \ \text{ and } \ f|_{(\Omega \setminus B_\delta)} \ \text{ is bounded}$$

where $B_\delta = B(0,\delta)$. Now the continuity of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f)$ will follow from the Hardy, Sobolev and Poincaré's inequalities. To show the compactness of the embedding let $(g_m) \subseteq W_0^{1,p}(\Omega)$ be a bounded sequence. By reflexivity of the space and the Sobolev embedding we have, for a subsequence (g_{m_k}) of (g_m) as $k \to \infty$

$$\begin{cases} g_{m_k} \rightharpoonup g \ \text{ weakly in } W_0^{1,p}(\Omega)\,, \\ g_{m_k} \to g \ \text{ strongly in } L^p(\Omega)\,. \end{cases} \tag{2.2}$$

Let $C_\delta = \|f\|_{L^\infty(\Omega\setminus B_\delta)}$, so we have

$$\begin{aligned} \int_\Omega |g_{m_k} - g|^p f \, dx &\le \int_{B_\delta} |g_{m_k} - g|^p f(x)\, dx + C_\delta \|g_{m_k} - g\|^p_{L^p(\Omega)} \\ &\le \epsilon \int_{B_\delta} |g_{m_k} - g|^p |x|^{-p} dx + C_\delta \|g_{m_k} - g\|^p_{L^p(\Omega)} \end{aligned}$$

.

Hence by Hardy inequality we have

$$\int_\Omega |g_{m_k} - g|^p f \, dx \le \epsilon \left(\frac{p}{n-p}\right)^p \|\nabla(g_{m_k} - g)\|^p_{L^p(\Omega)} + C_\delta \|g_{m_k} - g\|^p_{L^p(\Omega)}\,.$$

Since $(g_{m_k}) \subseteq W_0^{1,p}(\Omega)$ is bounded,

$$\int_\Omega |g_{m_k} - g|^p f(x)\, dx \le \epsilon M + C_\delta \|g_{m_k} - g\|^p_{L^p(\Omega)}\,,$$

where $M > 0$ be a constant depending on n and p. By (2.2), we have

$$\lim_{k\to\infty} \int_\Omega |g_{m_k} - g|^p f \, dx \leq \epsilon M$$

As $\epsilon > 0$ is arbitrary, $\lim_{k\to\infty} \int_\Omega |g_{m_k} - g|^p f \, dx = 0$. Hence $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f)$ is compact.∎

Remark 2.1 The class of functions $f \in \Im_p$ has lower order singularity than $|x|^{-p}$ at the origin. Here we give some examples of such functions:
(a) any bounded function.
(b) in a small neighbourhood of 0, f is $|x|^{-\beta}, 0 < \beta < p$.
(c) $f(x) = |x|^{-p}/|\log|x||$ in a small neighbourhood of 0.

Remark 2.2 If either $\lim_{|x|\to 0} |x|^p f(x)$ does not exist or it is different from zero, then the embedding need not be compact. For example, take $f(x)$ to be either $c|x|^{-p}|\sin\frac{1}{|x|}|$, or $c|x|^{-p}/(1+|x|^p)$ where $c > 0$, then the required limit does not exist in the first case and is $c > 0$ in the second case. Since $|\sin\frac{1}{|x|}|$ and $1/(1+|x|^p) \in L^\infty(\Omega)$ and $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, |x|^{-p})$ is not compact, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f)$, is also noncompact.

Proposition 2.2 *Let Ω be a bounded domain and $1 < p < n$. Let $f \in \Re_{p,\beta}$ where*

$$\Re_{p,\beta} := \left\{ f \in \Im_p \;\middle|\; 0 < \lim_{|x|\to 0} |x|^\beta f(x) < \infty \right\}$$

for some $0 \leq \beta < p$. Let $p_\beta^* := \dfrac{p(n-\beta)}{(n-p)}$. Then the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega, f)$ is

$$\begin{array}{lll} \text{(i)} & \text{continuous} & \forall \quad p \leq q \leq p_\beta^* \\ \text{(ii)} & \text{compact} & \forall \quad p \leq q < p_\beta^* \end{array}$$

Proof: (i) For the case $\beta = 0$ and $q = p$ follows from the Sobolev and Hardy's inequality respectively. Let $0 < \beta < p$ and $p < q \leq p_\beta^*$. Since $f \in \Re_{p,\beta}$ there exists $\delta(f) > 0$ and $C_\delta > 0$ such that $f(x) \leq \frac{C_\delta}{|x|^\beta}$ on $B(0,\delta)$. Then for $u \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned} \int_\Omega |u|^q f \, dx &\leq C_\delta \int_{B_\delta} \frac{|u|^q}{|x|^\beta} \, dx + C(\delta) \int_{\Omega\setminus B_\delta} |u|^q \, dx \\ &\leq C_\delta \int_{B_\delta} \left(\frac{|u|^\beta}{|x|^\beta}\right) |u|^{q-\beta} \, dx + C(\delta) \int_\Omega |u|^q dx. \end{aligned}$$

By Hölder, Hardy and Poincaré inequalities we have

$$\begin{aligned} \textstyle\int_\Omega |u|^q f \, dx &\leq C_\delta \left(\int_{B_\delta} \frac{|u|^p}{|x|^p} dx \right)^{\beta/p} \left(\int_{B_\delta} |u|^{p(q-\beta)/(p-\beta)} dx \right)^{(p-\beta)/p} + C_{n,p,\delta} \|\nabla u\|_{L^p(\Omega)}^q \\ &\leq C_{n,p,q,\delta} \|\nabla u\|_{L^p(\Omega)}^\beta \|\nabla u\|_{L^p(\Omega)}^{q-\beta} + C_{n,p,\delta} \|\nabla u\|_{L^p(\Omega)}^q \end{aligned}$$

Here we have used the fact that, $p(q-\beta)/(p-\beta) \le p^* = np/(n-p)$ iff $q \le p^*_\beta$. Hence

$$\left(\int_\Omega |u|^q f \, dx \right)^{1/q} \le C \|\nabla u\|_p$$

where $C > 0$ depending on n, p, q, δ and Ω. Hence $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega, f)$ is continuous.

(ii) For $q = p$ the embedding is compact by Proposition 2.1. So assume $0 < \beta < p$ and $p < q < p^*_\beta$. Let $(g_m) \subseteq W_0^{1,p}(\Omega)$ be bounded. Then there exists a subsequence (g_{m_k}) of (g_m) such that as $k \to \infty$

$$\begin{cases} g_{m_k} \rightharpoonup g & \text{weakly in} \quad W_0^{1,p}(\Omega) \\ g_{m_k} \to g & \text{strongly in} \quad L^r(\Omega) \ \ \forall p \le r < p^*. \end{cases} \tag{2.3}$$

We have

$$\int_\Omega |g_{m_k} - g|^q \, f(x) \, dx \le C_\delta \int_{B_\delta} \frac{|g_{m_k} - g|^q}{|x|^\beta} + C_\delta \|g_{m_k} - g\|^q_{L^p(\Omega)}. \tag{2.4}$$

Choose $r > 0$ such that $q < r < p^*$ and let $p_1 = r/q$. Then $p_1' = r/(r-q)$. By Sobolev embedding, $(g_{m_k} - g) \in L^{p_1}(\Omega)$. If $|x|^{-\beta} \in L^{p_1'}(\Omega)$, then we can use Hölder inequality in (2.4). Now $|x|^{-\beta} \in L^{p_1'}(\Omega)$ iff $\beta p_1' < n$ i.e iff $\beta r/(r-q) < n$. i.e. iff $r > nq/(n-\beta)$. Since $q < p(n-\beta)/(n-p)$ we have $nq/(n-\beta) < np/(n-p) = p^*$. Choose $r > q$ such that $nq/(n-\beta) < r < p^*$. Hence from (2.3) and (2.4), we have

$$\begin{aligned} \int_\Omega |g_{m_k} - g|^q \, f(x) \, dx &\le C_\delta \left(\int_{B_\delta} |g_{m_k} - g|^r \right)^{q/r} \left(\int_{B_\delta} \frac{1}{|x|^{\beta r/(r-q)}} \right)^{(r-q)/r} + C_\delta \|g_{m_k} - g\|^q_{L^q(\Omega)} \\ &\le C \|g_{m_k} - g\|^q_{L^r(\Omega)} \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

Hence the proof follows. ∎

Remark 2.3 If $\Omega = \mathbb{R}^n$ and $f(x) = |x|^{-\alpha}$, for $0 \le \alpha < p$, then the continuity of the embedding

$$D^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*_\alpha}(\mathbb{R}^n, |x|^{-\alpha}) \tag{2.5}$$

follows exactly as part (a) of Proposition 2.2.

Remark 2.4 In the above proposition p^*_β is critical in the sense that if $q > p^*_\beta$, then we will not have any such embedding and for $q = p^*_\beta$ we can show easily the noncompactness of this embedding in the same way as in the case of $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

3 The case $\alpha = 2$

Here we prove the Theorems 1.1 and 1.2.

Proof of Theorem 1.1:
Existence : By Hardy inequality it follows that the operator $L_\mu = -(\Delta + \frac{\mu}{|x|^2})$ on $H_0^1(\Omega)$ is positive definite and self adjoint for all $0 \leq \mu < \beta_{n,2}$. Hence by Lax-Milgram lemma, for any $g \in H^{-1}(\Omega)$

$$\left\{ \begin{array}{rcll} L_\mu u & = & g & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right.$$

has unique solution, and $L_\mu^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ exists and continuous. By proposition 2.1, it follows that

$$\begin{array}{rccc} M_f : & H_0^1(\Omega) & \to H^{-1}(\Omega) \\ & u & \longmapsto uf \end{array}$$

is a compact operator. Since L_μ^{-1} is positive definite and self-adjoint,

$$S_\mu = L_\mu^{-1} \circ M_f : H_0^1(\Omega) \to H_0^1(\Omega)$$

is a compact, self-adjoint, positive definite operator. Hence the problem can be written as

$$\left\{ \begin{array}{rcll} S_\mu u & = & \lambda^{-1} u & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right.$$

and has nontrivial solutions corresponding to $\lambda \in (\lambda_\mu^k(f))$ where $0 < \lambda_\mu^1(f) \leq \lambda_\mu^2(f) \leq \lambda_\mu^3(f) \leq \ldots \to +\infty$, and they are characterized by the usual Rayleigh quotients.

Let $u \in H_0^1(\Omega)$ be a first eigenfunction i.e,

$$L_\mu u = \lambda_\mu^1 uf\,.$$

Since $u^+, u^- \in H_0^1(\Omega)$, we have ,

$$\left\{ \begin{array}{rcll} -\Delta u^+ & \geq & 0 & \text{in } \Omega\,, \\ -\Delta u^- & \geq & 0 & \text{on } \Omega\,. \end{array} \right.$$

Here f need not be smooth enough. So we use the strong maximum principle for weak solutions (Theorem 8.19 in [10]) to conclude that either $u^+ \equiv 0$ a.e on Ω or $u^+ > 0$ a.e on Ω. Thus u cannot change sign.

If possible, let $u, v \in H_0^1(\Omega)$ be two orthogonal eigenfunctions corresponding to $\lambda = \lambda_\mu^1$. Using the equations for u and v we have

$$\int_\Omega u\, v\, (\frac{\mu}{|x|^2} + \lambda_\mu^1 f)\, dx = 0$$

which is a contradiction, because u, v do not change sign in Ω. Hence λ_μ^1 is simple.

Regularity: Without loss of generality let $v \in H_0^1(\Omega)$ be a solution of (1.3) corresponding to $\lambda = \lambda_\mu^1$. Let

$$h(x) = \frac{\mu}{|x|^2} v(x) + \lambda_\mu^1 v(x) f(x) \ \text{ on } \Omega.$$

Since $|x|^{-2} \in L^r, \ \forall\, r < n/2$, and $v(x) \in L^{2^*}$ we have $v(x)|x|^{-2} \in L^r \,\forall\, 1 < r < 2n/(n+2)$. There exists a constant $c > 0$ such that $|f(x)| < c|x|^{-2}$. Hence it follows that $h(x) \in L^r$ for all $r \in (1, 2n/(n+2))$. By Theorem (9.15) in [10] it follows that $v \in W^{2,r}(\Omega)$ for all r $1 < r < 2n/(n+2)$. ∎

Remark 3.1 This H_0^1 eigenfunction need not be in L^∞. In fact it need not be even in L^p for $p > 2^*$. The eigenfunctions in a ball, calculated in [6] for the case $f \equiv 1$ and μ in $(0, (n-2/2)^2)$ behave like $r^{1-(n/2)+\nu}$ near 0, with $\nu = \sqrt{((n-2/2)^2 - \mu)}$. Such eigenfunctions lie in L^p for $p < 2n/(n-2-2\nu)$ only. However if $f \in C^\infty(\Omega \setminus \{0\})$ then by the standard elliptic regularity theory (see corollary 8.11 in [10]) any eigenfunction will be in $C^\infty(\Omega \setminus \{0\})$.

Limit of λ_μ^1 as $\mu \to \beta_{n,2}$: For $f \in \Im_2$ we have for all $0 \le \mu < \beta_{n,2}$,

$$\lambda_\mu^1(f) = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega (|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2})\, dx}{\int_\Omega |u|^2 f\, dx}. \tag{3.1}$$

Let μ_k be increasing to $\beta_{n,2}$ as $k \to \infty$. Then $(\lambda_{\mu_k}^1(f))$ is decreasing a sequence bounded above by λ_0^1, the (first eigenvalue of $-\Delta$ with zero Dirichlet boundary value) and below by 0.Let

$$\lambda_{\mu_k}^1(f) \to \lambda(f) \ \text{ as } \ \mu_k \to \beta_{n,2}. \tag{3.2}$$

Claim: $\lambda(f) > 0$.
We prove this claim through the following improved Hardy inequality.

Lemma (Improved Hardy Inequality) *If $\Omega \subseteq \mathbb{R}^n, n \ge 3$ be a bounded domain then*

$$\beta_{n,2} \int_\Omega \frac{|u|^2}{|x|^2} dx + C_{p,\beta,\Omega} \left(\int_\Omega \frac{|u|^p}{|x|^\beta} dx \right)^{2/p} \le \int_\Omega |\nabla u|^2 dx \tag{3.3}$$

for all $u \in H_0^1(\Omega)$ *and for all* $0 \le \beta < 2$, $1 < p < 2(n-\beta)/(n-2) = 2_\beta^*$ *with* $C_{p,\beta,\Omega} > 0$.

Proof: We replace Ω by a ball B_R with $R = (n|\Omega|/\omega_n)^{1/n}$, where $\omega_n = |S^{n-1}|$ and the function u by its symmetric rearrangement. It is well known that the rearrangement of u, does not change the L^2- norm, decreases H_0^1 norm and increases the integral $\int\limits_\Omega \frac{|u|^2}{|x|^\beta}$ for all $0 < \beta \le 2$, see [3]. Hence it is enough to prove the inequality (3.3) only for radial functions. And moreover by rescaling we can as well take $R = 1$. For $u \in C_0^1(\Omega)$ and u radial we define

$$v(r) = u(r) r^{(n-2)/2}\ , \quad r = |x|$$

to reduce the dimension from n to 2. Now we observe that,

$$\int\limits_B |\nabla u|^2 dx - \beta_{n,2} \int\limits_B \frac{u^2}{|x|^2} dx = \omega_n \left[\int_0^1 (v')^2 r dr - (n-2) \int_0^1 v(r)\, v'(r)\, dr \right]. \tag{3.4}$$

Since $v \in C_0^1(0,1)$ the last integral in (3.4) is zero and hence we have

$$\int\limits_B |\nabla u|^2 dx - \beta_{n,2} \int\limits_B u^2 |x|^{-2} dx = \omega_n \int_0^1 (v')^2 r dr. \tag{3.5}$$

Now for $0 \le \beta < 2$, we have

$$\int\limits_B \frac{|u|^p}{|x|^\beta} dx = \omega_n \int_0^1 |v(r)|^p r^{n-\beta-1-p(n-2)/2} dr. \tag{3.6}$$

Notice that for C^1-radial function $v(r)$,

$$\begin{aligned} |v(r)| &= \left| \int_r^1 v'(t)\, dt \right| = \left| \int_r^1 \left(v'(t)\, t^{1/2} \right) t^{-1/2} dt \right| \\ &\le \left(\int_r^1 |v'(t)|^2\, t\, dt \right)^{1/2} \left(\log \frac{1}{r} \right)^{1/2} \end{aligned}$$

and hence

$$\begin{aligned} \int_0^1 |v|^p r^{n-\beta-1-p(n-2)/2} dr &\le \left(\int_0^1 |v'(r)|^2\, r\, dr \right)^{p/2} \int_0^1 r^{n-\beta-1-p(n-2)/2} \left(\log \frac{1}{r} \right)^{p/2} dr \\ &\le \left(\int_0^1 |v'(r)|^2\, r\, dr \right)^{p/2} \int_0^\infty e^{-\{n-\beta-p(n-2)/2\}r}\, r^{p/2}\, dr. \end{aligned} \tag{3.7}$$

Now the second integral on the right hand side is convergent iff $p < \dfrac{2(n-\beta)}{n-2}$ Call

$$C_{p,\beta} = (\omega_n)^{1-2/p} \left(\int_0^\infty e^{-\{n-\beta-p(n-2)/2\}r} r^{p/2} dr \right)^{-2/p}.$$

Then from (3.6) and (3.7), we get

$$\omega_n \int_0^1 |v'(r)|^2 \, r \, dr \geq C_{p,\beta} \left(\int_B \frac{|u(x)|^p}{|x|^\beta} dx \right)^{2/p}.$$

Combining this with (3.5) the inequality (3.3) holds for for all $u \in C_0^1(\Omega)$. Hence by density argument the inequality (3.3) is true for all $u \in H_0^1(\Omega)$.

Proof of the Claim: Since $2_\beta^* > 2$, as $\beta < 2$, we can take $p = 2$ in the inequality (3.3) reduces to

$$\beta_{n,2} \int_\Omega \frac{|u|^2}{|x|^2} dx + C_{2,\beta} \int_\Omega \frac{|u|^2}{|x|^\beta} dx \leq \int_\Omega |\nabla u|^2 dx.$$

If $f \in \Re_{2,\beta}$, then $0 < l := \lim_{|x|\to 0} |x|^\beta f(x) < \infty$. As $f \in L^\infty_{\text{loc}}(\Omega \setminus \{0\})$ there exists $\alpha(f) > 0$ such that,

$$\alpha(f) \int_\Omega u^2 \, f(x) \, dx \leq \int_\Omega \frac{u^2}{|x|^\beta} dx$$

. Thus

$$\beta_{n,2} \int_\Omega \frac{|u|^2}{|x|^2} dx + \alpha(f) \, C_{2,\beta} \int_\Omega |u|^2 f(x) \, dx \leq \int_\Omega |\nabla u|^2 dx.$$

Hence

$$C(f) \;=\; \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega (|\nabla u|^2 - \beta_{n,2} \frac{|u|^2}{|x|^2}) \, dx}{\int_\Omega |u|^2 f(x) \, dx} \;\geq\; \alpha(f) \, C_{2,\beta} > 0. \tag{3.8}$$

Since for all $u \in H_0^1(\Omega)$ and for every $0 \leq \mu < \beta_{n,2}$,

$$\lambda_\mu^1(f) \;\leq\; \frac{\int_\Omega (|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2}) \, dx}{\int_\Omega |u|^2 f(x) \, dx}, \tag{3.9}$$

we have for every $u \in H_0^1(\Omega)$,

$$0 \;\leq\; \lambda(f) \leq\; \frac{\int_\Omega (|\nabla u|^2 - \beta_{n,2} \frac{|u|^2}{|x|^2}) dx}{\int_\Omega |u|^2 f(x) \, dx}, \tag{3.10}$$

and hence

$$\lambda(f) \leq C(f). \tag{3.11}$$

To prove the other way, observe that for all $u \in H_0^1(\Omega)$ the right hand side of (3.9) is greater than the right hand side of the inequality (3.10) and therefore for each $\mu < \beta_{n,2}$

$$\lambda_\mu^1(f) \geq C(f).$$

Hence

$$\lambda(f) = C(f) > 0. \tag{3.12}$$

Remark 3.2 From the above proof we have seen that in the improved Hardy inequality

$$\beta_{n,2} \int\limits_\Omega \frac{|u|^2}{|x|^2} dx + C_{2,\beta} \int\limits_\Omega \frac{|u|^2}{|x|^\beta} dx \leq \int\limits_\Omega |\nabla u|^2 dx \tag{3.13}$$

the constant $C_{2,\beta}$, is the best and is the limit of the first eigenvalues $\lambda_\mu^1(|x|^{-\beta})$ as $\mu \to \beta_{n,2}$. For the case $f = 1$ on Ω and for $\mu < \beta_{n,2}$ it is proved in [6] that the first eigenvalue of L_μ is the square of the first zero of the Bessel function J_ν where $\nu = (\beta_{n,2} - \mu)^{1/2}$ and the limiting eigenvalue goes to the square of the first zero of the Bessel function J_0, which is nothing but $C(1)$ here.

Remark 3.3 Recenty with Adimurthi [1], we have obtained a best improvement of the inequality (3.13) not only for $p = 2$ but for all $1 < p \leq n$ and precise statement statement is:

Theorem [1]: *Let $R \geq \sup_\Omega (|x|\ e^{2/p})$ and $1 < p \leq n$, then there exists $C > 0$, depending on n, p and R such that*

$$\left(\frac{n-p}{p}\right)^p \int\limits_\Omega \frac{|u(x)|^p}{|x|^p}\ dx + C \int\limits_\Omega \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx \leq \int\limits_\Omega |\nabla u|^p dx$$

for any $u \in W_0^{1,p}(\Omega)$ if and only if

(i) $\gamma \geq 2$ *when* $1 < p < n$,

(ii) $\gamma \geq n$ *when* $p = n$.

Proof of Theorem 1.2 : Let $\mu > \beta_{n,2}$ and if possible let there exist nonnegative $u \in H_0^1(\Omega)$, $u \not\equiv 0$ such that for all $\phi \in H_0^1(\Omega)$, $\phi \geq 0$,

$$\int\limits_\Omega \nabla u.\nabla \phi\, dx \geq \mu \int\limits_\Omega \frac{u}{|x|^2} \phi dx. \tag{3.14}$$

Then by strong maximum principle $u > 0$ a.e., in Ω. Since $0 \in \Omega$ there exists $0 < r < 1$ such that $B_r = B(0, r) \subseteq \Omega$. From Theorem 1.1, for any $0 \leq \beta < 2$, there exists $\phi_\beta > 0$ in

$H_0^1(B_r)$ and $\lambda(\beta) > 0$, such that

$$-\Delta\phi_\beta = \frac{\lambda(\beta)}{|x|^\beta}\phi_\beta \quad \text{in} \quad B_r. \tag{3.15}$$

It is easy to check that, $\lambda(\beta)$ is a decreasing function on $[0, 2)$. By Hardy inequality and (3.15), we have $\lambda(\beta) \geq \beta_{n,2}$, for any $\beta < 2$. Hence

$$\lim_{\beta\to 2} \lambda(\beta) \geq \beta_{n,2}. \tag{3.16}$$

Since the inequality (3.14) is true for any $\phi \in H_0^1(B_r),\ \phi \geq 0$ we get

$$-\int_{B_r} u\Delta\phi_\beta \, dx \, + \int_{\partial B(0,r)} \frac{\partial\phi_\beta}{\partial\nu} d\sigma \, \geq \, \mu \int_{B_r} \frac{u}{|x|^2}\phi_\beta \, dx.$$

From above inequality and (3.15),

$$\lambda(\beta) \int_{B_r} \frac{u}{|x|^\beta}\phi_\beta \, dx + \int_{\partial B(0,r)} \frac{\partial\phi_\beta}{\partial\nu} d\sigma \, \geq \, \mu \int_{B_r} \frac{u}{|x|^2}\phi_\beta \, dx$$

By Hopf Lemma, $\frac{\partial\phi_\beta}{\partial\nu} < 0$ on $\partial B(0, r)$ and hence we have

$$\mu \, < \, \lambda(\beta) \quad \text{for all} \quad \beta \in [0, 2). \tag{3.17}$$

Since for any $\beta < 2$

$$\lambda(\beta) \leq \frac{\int_{B_r} |\nabla v|^2 dx}{\int_{B_r} \frac{v^2}{|x|^\beta} dx}$$

for all $v \in H_0^1(B_r)$, we conclude that

$$\lim_{\beta\to 2} \lambda(\beta) \leq \beta_{n,2}. \tag{3.18}$$

Hence, from (3.16) and (3.18) we obtain

$$\lim_{\beta\to 2} \lambda(\beta) = \beta_{n,2}. \tag{3.19}$$

Since $\mu \geq \beta_{n,2}$, (3.17) and (3.19) lead to a contradiction. Hence the theorem. ∎

Proof of Theorem 1.3 : For $0 \leq \mu < \beta_{n,2}$, we define, $J_\mu : H_0^1(\Omega) \to \mathbb{R}$ by

$$J_\mu(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \mu\frac{|u|^2}{|x|^2}) \, dx - \frac{1}{q}\int_\Omega |u|^q f(x) \, dx \qquad \forall \ \ u \in H_0^1(\Omega) \, .$$

By Proposition (2.2) and Hardy's inequality it follows that $J_\mu \in C^1(H_0^1(\Omega), \mathbb{R})$ and for any $u, v \in H_0^1(\Omega)$ and we have

$$\langle DJ_\mu(u), v\rangle = \int_\Omega [\nabla u.\nabla v - \mu \frac{u}{|x|^2} v - |u|^{q-2} u\, v\, f(x)]\, dx\,.$$

To get a nontrivial solution of the problem (1.4), we look for critical points of J_μ. We first show that J_μ satisfies the geometric conditions of the mountain pass lemma and P.S condition. We will take $\|\nabla u\|_{L^2(\Omega)}$ as the norm in $H_0^1(\Omega)$.
(i) Clearly $J_\mu(0) = 0$
(ii) To prove there exists $\delta > 0$ and $\rho > 0$ such that $J_\mu(u) > \rho$ wherever $\|u\|_{H_0^1(\Omega)} = \delta$. observe that,

$$J_\mu(u) \geq \frac{1}{2}(1 - \beta_{n,2}^{-1}\, \mu)\, \|u\|^2_{H_0^1(\Omega)} - \frac{1}{q} \int_\Omega |u|^q f(x)\, dx. \tag{3.20}$$

By Proposition 2.2 and inequality (3.20) we have for all $u \in H_0^1(\Omega)$,

$$J_\mu(u) \geq C_1 \|u\|^2_{H_0^1(\Omega)} - C_2 \|u\|^q_{H_0^1(\Omega)}, \tag{3.21}$$

where $C_1 = \frac{1}{2}(1 - \mu\beta_{n,2}^{-1}) > 0$ and $C_2 > 0$ are constants. So for sufficiently small $\delta > 0$ there exists $\rho > 0$ such that $J_\mu(u) > \rho$ whenever $\|u\|_{H_0^1(\Omega)} = \delta$.
(iii) Since $q > 2$, from (3.21), it follows that $\exists\ v \in H_0^1(\Omega)$ with $\|v\|_{H_0^1(\Omega)} > \delta$ such that $J_\mu(v) < 0$. Hence J_μ satisfies the geometric conditions of the mountain pass theorem.

To prove J_μ satisfies P.S condition let $(u_m) \subseteq H_0^1(\Omega)$ be a P.S sequence at some level $C \in \mathbb{R}^+$ i.e., as $m \to \infty$

$$J_\mu(u_m) \to C,\ DJ_\mu(u_m) \to 0 \text{ strongly in } H^{-1}(\Omega)\,. \tag{3.22}$$

Observe that

$$J_\mu(v) - \frac{1}{2}\langle DJ_\mu(v), v\rangle = \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega |v|^q f\, dx \qquad \forall\ \ v \in H_0^1(\Omega)\,. \tag{3.23}$$

This, together with (3.22) implies that (u_m) is bounded in $H_0^1(\Omega)$. By Proposition 2.2 and reflexivity, there exists a subsequence (u_{m_k}) of (u_m) such that as $k \to \infty$

$$\begin{cases} u_{m_k} \rightharpoonup u \ \text{ weakly in } H_0^1(\Omega) \\ u_{m_k} \rightharpoonup u \ \text{ weakly in } L^2(\Omega, |x|^{-2}) \\ u_{m_k} \to u \ \text{ strongly in } L^q(\Omega, f)\,. \end{cases} \tag{3.24}$$

Since $J_\mu(u_{m_k}) \to C$ as $k \to \infty$, we have

$$\begin{aligned} C &= J_\mu(u_{m_k}) + o(1)\,, \text{ where } o(1) \to 0 \text{ as } k \to \infty \\ &= \frac{1}{2}(\|\nabla u_{m_k}\|_2^2 - \mu\|u_{m_k}\|^2_{L^2(|x|^{-2})}) - \frac{1}{q}\|u_{m_k}\|^q_{L^q(f)} + o(1) \\ &= \frac{1}{2}\left(\|\nabla u\|_2^2 - \mu\|u\|^2_{L^2(|x|^{-2})}\right) - \frac{1}{q}\|u\|^q_{L^q(f)} \\ &\quad + \frac{1}{2}\left(\|\nabla(u_{m_k} - u)\|_2^2 - \mu\|u_{m_k} - u\|^2_{L^2(|x|^{-2})}\right) + o(1) \\ &\geq J_\mu(u) + \frac{1}{2}(1 - \mu\beta_{n,2}^{-1})\|\nabla(u_{m_k} - u)\|_2^2 + o(1)\,. \end{aligned} \tag{3.25}$$

We will now prove that $J_\mu(u) = C$, so that from (3.25), we have $\|\nabla(u_{m_k} - u)\|^2_{L^2(\Omega)} \to 0$ as $k \to \infty$ and hence J_μ satisfies P.S condition. From (3.22) and (3.24), we have

$$\langle DJ_\mu(u), v\rangle = 0 \qquad \forall \;\; v \in H_0^1(\Omega)\,. \tag{3.26}$$

Therefore

$$\begin{aligned} C &= \lim_{k\to\infty} J_\mu(u_{m_k}) \\ &= \lim_{k\to\infty}\left[J_\mu(u_{m_k}) - \frac{1}{2}\langle DJ_\mu(u_{m_k}),\, u_{m_k}\rangle\right] \qquad \text{by} \quad (3.22) \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\int_\Omega |u|^q f\, dx \qquad \text{by} \quad (3.23) \quad \text{and} \quad (3.24) \\ &= J_\mu(u) - \frac{1}{2}\langle DJ_\mu(u), u\rangle \qquad \text{by (3.23)} \\ &= J_\mu(u) \qquad \text{by (3.26).} \end{aligned}$$

Now define,

$$\Gamma = \left\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = v\right\}$$

. Then $d = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} J_\mu(\gamma(t)) \geq \rho > 0$. By the mountain pass theorem of Ambrosetti-Rabinowitz [2], d is a critical value i.e. there exists $u \in H_0^1(\Omega)$ such that $J'_\mu(u) = 0$ and $J_\mu(u) = d > 0$. Since $J_\mu(u) = d = J_\mu(|u|)$, the problem (1.4) admits a solution. ∎

4 Proof of Theorem 1.4 (The Case $0 \leq \alpha < 2$)

For $\lambda \geq 0$, define, $J_\lambda : H_0^1(\Omega) \to \mathbb{R}$ by

$$J_\lambda(u) := \frac{1}{2}\int\limits_\Omega |\nabla u|^2 dx - \frac{1}{2^*_\alpha}\int\limits_\Omega |u|^{2^*_\alpha}|x|^{-\alpha}dx - \frac{\lambda}{q}\int\limits_\Omega |u|^q f\, dx \;\; \forall\, u \in H_0^1(\Omega).$$

By Proposition 2.2, $J_\lambda \in C^1(H_0^1, \mathbb{R})$, and for any $u, v \in H_0^1(\Omega)$ we have

$$\langle DJ_\lambda(u), v\rangle = \int\limits_\Omega \Big(\nabla u.\nabla v - |u|^{2^*_\alpha - 2}uv|x|^{-\alpha} - \lambda|u|^{q-2}uvf\Big)\, dx.$$

Any nontrivial critical point of J_λ will give us a solution to the problem (1.5). For $\alpha = 0$ and $\beta = 0$, the existence of solution for the problem (1.5) was shown by Brezis-Nirenberg; (See [4]). They showed that J_λ satisfies P.S condition below the energy level $\frac{1}{n}S^{n/2}$, where S is the best Sobolev constant. In our case also we will get a certain energy level to conclude J_λ satisfies P.S condition below that level.

For $0 \leq \alpha < 2$ the embedding $D^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^n, |x|^{-\alpha})$ is continuous, (See Remark 2.3). Define

$$S_\alpha := \inf_{u \in H^1(\mathbb{R}^n)} \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^n)}}{\|u\|^2_{L^{2^*_\alpha}(\mathbb{R}^n, |x|^{-\alpha})}} \tag{4.1}$$

Since $f \in \Re_{2,\beta}$, there exist $r > 0$ and $K_1(f),\ K_2(f) > 0$ such that $K_1|x|^{-\beta} \leq f(x) \leq K_2|x|^{-\beta}$ on $B(0, r)$. For any $\epsilon > 0$, the family u_ϵ on $\mathbb{R}^n$ defined by,

$$u_\epsilon(x) := \frac{C_\epsilon}{(\epsilon^{2-\alpha} + |x|^{2-\alpha})^{(n-2)/(2-\alpha)}},\quad C_\epsilon = [\epsilon^{2-\alpha}(n-2)(n-\alpha)]^{(n-2)/2(2-\alpha)} \tag{4.2}$$

satisfies

$$-\Delta u_\epsilon = u_\epsilon^{2^*_\alpha - 1}|x|^{-\alpha} \text{ in } \mathbb{R}^n$$

and is minimizer for (4.1). Hence we have

$$\|\nabla u_\epsilon\|^2_{L^2(\mathbb{R}^n)} = \|u_\epsilon\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^n, |x|^{-\alpha})} = S_\alpha^{(n-\alpha)/(2-\alpha)}. \tag{4.3}$$

Choose $\phi \in C_0^\infty(\Omega)$ such that $\phi \equiv 1$ on B_r. For $\epsilon > 0$, define the family $\tilde{u}_\epsilon$ on Ω by

$$\tilde{u}_\epsilon(x) := \phi(x)u_\epsilon(x).$$

Then by (4.2) and (4.3),

$$\|\nabla \tilde{u}_\epsilon\|^2_{L^2(\Omega)} = S_\alpha^{(n-\alpha)/(2-\alpha)} + O(\epsilon^{n-2}) \tag{4.4}$$

$$\|\tilde{u}_\epsilon\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega, |x|^{-\alpha})} = S_\alpha^{(n-\alpha)/(2-\alpha)} + O(\epsilon^{n-\alpha}). \tag{4.5}$$

Define, $v_\epsilon : \Omega \to \mathbb{R}$ by

$$v_\epsilon(x) = \frac{\tilde{u}_\epsilon(x)}{\|\tilde{u}_\epsilon\|_{L^{2^*_\alpha}(\Omega,|x|^{-\alpha})}}, \tag{4.6}$$

so from (4.4) and (4.5) we have,

$$\|v_\epsilon\|_{L^{2^*_\alpha}(\Omega,|x|^{-\alpha})} = 1 \tag{4.7}$$

and

$$\|\nabla v_\epsilon\|^2_{L^2(\Omega)} = S_\alpha + O(\epsilon^{n-2}) \tag{4.8}$$

We can show that for $\lambda \geq 0$, J_λ satisfies the geometric conditions of mountain pass lemma as in the proof of Theorem 1.2. We will use the Theorem of Brezis-Nirenberg, (See Theorem 2.2 in [4]) to get a nontrivial critical point for J_λ. We need the following two lemmas which are similar to Theorem 2.1 and Lemma 2.1 in [4]. We indicate the proof here as the calculations are quite different and are important to understand the new phenomenon here, namely the balance between the singularity and the nonlinearity.

Lemma 4.1 *Let $(u_m) \subseteq H^1_0(\Omega)$ be a sequence such that*

$$J_\lambda(u_m) \to \beta_\lambda < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{(n-\alpha)/(2-\alpha)} \text{ and } DJ_\lambda(u_m) \to 0 \text{ strongly} \quad \text{in } H^{-1}\,. \tag{4.9}$$

Then (u_m) is relatively compact in $H^1_0(\Omega)$ i.e. J_λ satisfies $(P.S)_{\beta_\lambda}$.

Proof: Let us prove first (u_m) is bounded. From (4.9) we have for all $m \geq k$,

$$\begin{cases} J_\lambda(u_m) = \beta_\lambda + o(1) \\ \\ \langle DJ_\lambda(u_m),\ u_m\rangle \leq o(1)\|u_m\|_{H^1_0(\Omega)} \end{cases} \tag{4.10}$$

Now we have

$$2J_\lambda(u_m) - \langle DJ_\lambda(u_m), u_m\rangle = \left(1 - \frac{2}{2^*_\alpha}\right) \int\limits_\Omega |u|^{2^*_\alpha}|x|^{-\alpha}dx + \lambda\left(1 - \frac{2}{q}\right) \int\limits_\Omega |u_m|^q f\ dx \tag{4.11}$$

Since $\lambda \geq 0,\ 2^*_\alpha > 2,\ q > 2$, by standard arguments, for example see(lemma 2.3 in [12]), using (4.10) and (4.11), we conclude (u_m) is bounded in $H^1_0(\Omega)$. By Propositions (2.1) and (2.2) and the reflexivity of the spaces there exists a subsequence (u_{m_k}) of (u_m) such that as $k \to \infty$

$$\begin{cases} u_{m_k} \rightharpoonup u & \text{weakly in } H^1_0(\Omega) \\ u_{m_k} \to u & \text{strongly in } L^p(\Omega, |x|^{-\alpha}),\ \forall\ 2 \leq p < 2^*_\alpha \\ u_{m_k} \rightharpoonup u & \text{weakly in } L^{2^*_\alpha}(\Omega, |x|^{-\alpha}) \\ u_{m_k} \to u & \text{strongly in } L^q(\Omega, f),\ \ \forall\ 2 < q < 2^*_\beta \\ u_{m_k} \to u & \text{a.e. } [|x|^{-\alpha}dx] \text{ in } \Omega. \end{cases} \tag{4.12}$$

We write, $d\mu(x) = \frac{dx}{|x|^\alpha}$. Since $DJ_\lambda(u_m) \to 0$ strongly in H^{-1}, by (4.12) we have

$$\langle DJ_\lambda(u),\, v\rangle = 0 \;\; \forall\, v \in H_0^1.(\Omega)$$

In particular $\langle DJ_\lambda(u),\, u\rangle = 0$, and hence we have from (4.11)

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{2^*_\alpha}\right) \int\limits_\Omega |u|^{2^*_\alpha} d\mu(x) + \lambda \left(\frac{1}{2} - \frac{1}{q}\right) \int\limits_\Omega |u|^q \, f \, dx \geq 0 \tag{4.13}$$

Again by (4.12) we have

$$\int\limits_\Omega |\nabla u_{m_k}|^2 dx = \int\limits_\Omega |\nabla(u_{m_k} - u)|^2 dx + \int\limits_\Omega |\nabla u|^2 dx + o(1) \tag{4.14}$$

$$\int\limits_\Omega |u_{m_k}|^{2^*_\alpha} d\mu(x) = \int\limits_\Omega |u_{m_k} - u|^{2^*_\alpha} d\mu(x) + \int\limits_\Omega |u|^{2^*_\alpha} d\mu(x) + o(1) \tag{4.15}$$

and

$$\int\limits_\Omega \left(u_{m_k}|u_{m_k}|^{2^*_\alpha - 2} - u|u|^{2^*_\alpha - 2}\right)(u_{m_k} - u) d\mu(x) = \int\limits_\Omega |u_{m_k} - u|^{2^*_\alpha} d\mu(x) + o(1). \tag{4.16}$$

By (4.14) - (4.15) we have

$$J_\lambda(u_{m_k}) = J_\lambda(u) + J_0(u_{m_k} - u) + o(1). \tag{4.17}$$

and by (4.16)

$$\begin{aligned} o(1) &= \langle u_{m_k} - u, DJ_\lambda(u_{m_k})\rangle = \langle u_{m_k} - u, DJ_\lambda(u_{m_k}) - DJ_\lambda(u)\rangle \\ &= \int\limits_\Omega |\nabla(u_{m_k} - u)|^2 dx - \int\limits_\Omega |u_{m_k} - u|^{2^*_\alpha} d\mu(x) + o(1) \end{aligned} \tag{4.18}$$

Hence it follows from (4.18) that

$$J_0(u_{m_k} - u) = \frac{(2-\alpha)}{2(n-\alpha)} \int\limits_\Omega |\nabla(u_{m_k} - u)|^2 dx + o(1) \tag{4.19}$$

and thus from (4.9), (4.13) and (4.17)

$$J_0(u_{m_k} - u) \leq J_\lambda(u_{m_k}) + o(1) \leq \beta_\lambda < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{\frac{n-\alpha}{2-\alpha}} \quad \forall\, k \geq k_0.$$

Combining (4.19) with this, we have $\forall\, k \geq k_0$

$$\|\nabla(u_{m_k} - u)\|_2^2 < S_\alpha^{\frac{n-\alpha}{2-\alpha}} \tag{4.20}$$

and hence

$$\left(1 - S_\alpha^{-2_\alpha^*/2} \|u_{m_k} - u\|_{H_0^1}^{2_\alpha^* - 2}\right) > 0\,. \tag{4.21}$$

Now (4.1) implies

$$\|w\|^2_{L^{2_\alpha^*}(\Omega,|x|^{-\alpha})} \le S_\alpha^{-1} \|\nabla w\|_2^2 \qquad \forall \;\; w \in H_0^1(\Omega)\,.$$

Thus

$$\begin{aligned} \|u_{m_k} - u\|_{H_0^1}^2 \left(1 - S_\alpha^{-2_\alpha^*/2} \|u_{m_k} - u\|_{H_0^1}^{2_\alpha^* - 2}\right) &\le \int\limits_\Omega |\nabla(u_{m_k} - u)|^2 dx - \int\limits_\Omega |u_{m_k} - u|^{2_\alpha^*} d\mu(x) \\ &= o(1). \end{aligned} \tag{4.22}$$

From (4.21) and (4.22), it follows that $u_{m_k} \to u$ strongly in $H_0^1(\Omega)$. ∎

Lemma 4.2 *There exists some $v \in H_0^1(\Omega), v \neq 0$ such that*

$$\sup_{t \ge 0} J_\lambda(tv) < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{\frac{n-\alpha}{2-\alpha}}$$

holds:

(a) For all $n \ge 4$ and $\lambda > 0$.

(b) When $n = 3$, in the following three cases:

(i) For all $1 \le \beta < 2$, $2 < q < 2_\beta^*$ and $\lambda > 0$.

(ii) For all $0 \le \beta < 1$ with $2(2-\beta) < q < 2_\beta^*$ and for all $\lambda > 0$.

(iii) For all $0 \le \beta < 1$ with $2 < q \le 2(2-\beta)$ and for sufficiently large $\lambda > 0$.

Proof : Take $\{v_\epsilon\}$, for $\epsilon > 0$ as defined in (4.6). Call $X_\epsilon = \|\nabla v_\epsilon\|^2_{L^2(\Omega)}$.

$$J_\lambda(tv_\epsilon) = \frac{t^2}{2} X_\epsilon - \frac{t^{2_\alpha^*}}{2_\alpha^*} - \frac{\lambda t^q}{q} \int\limits_\Omega |v_\epsilon|^q f \, dx\,. \tag{4.23}$$

Since $q,\ 2_\alpha^* > 2$, so from (4.23) it follows that $J_\lambda(tv_\epsilon) \to -\infty$ on $t \to +\infty$. Hence $\sup_{t \ge 0} J_\lambda(tv_\epsilon)$ achieves at some $t_\epsilon > 0$ (If $t_\epsilon = 0$, then we are done).Therefore

$$X_\epsilon - t_\epsilon^{2_\alpha^* - 2} - \lambda t_\epsilon^{q-2} \int\limits_\Omega |v_\epsilon|^q f(x) dx = 0\,. \tag{4.24}$$

Let $Y_\epsilon = \sup_{t \ge 0} J_\lambda(tv_\epsilon) = J_\lambda(t_\epsilon v_\epsilon)$. Therefore from (4.24), we have

$$t_\epsilon \le X_\epsilon^{\frac{1}{2_\alpha^* - 2}}$$

and

$$Y_\epsilon \le \frac{(2-\alpha)}{2(n-\alpha)} X_\epsilon^{(n-\alpha)/(2-\alpha)} - \frac{\lambda t_\epsilon^q}{q} \int\limits_\Omega |v_\epsilon|^q f \, dx \, .$$

Hence by (4.8), we have

$$Y_\epsilon \le \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{(n-\alpha)/(2-\alpha)} + O(\epsilon^{n-2}) - \frac{\lambda t_\epsilon^q}{q} \int\limits_\Omega |v_\epsilon|^q f \, dx \, . \tag{4.25}$$

The aim is to show that the third term in (4.25) is small and dominates the second term. We will first show that

$$t_\epsilon \to S_\alpha^{1/(2_\alpha^*-2)} = S_\alpha^{(n-2)/2(2-\alpha)} \quad as \quad \epsilon \to 0 \, . \tag{4.26}$$

Since $X_\epsilon \to S_\alpha$ as $\epsilon \to 0$, it is enough to prove that

$$\int\limits_\Omega |v_\epsilon|^q f(x) dx \to 0 \ \text{ as } \ \epsilon \to 0 \, . \tag{4.27}$$

From (4.2), (4.6) and (4.7), we have

$$\int\limits_\Omega |v_\epsilon|^q f(x) dx \le C \epsilon^{q(n-2)/2} \int\limits_0^r \frac{s^{n-1-\beta}}{\left[\epsilon^{2-\alpha} + s^{2-\alpha}\right]^{q(n-2)/(2-\alpha)}} ds + O(\epsilon^{q(n-2)/2}) \, , \tag{4.28}$$

where $C > 0$ is a constant independent of ϵ. Now, for $\epsilon < r$,

$$\begin{aligned}
\int\limits_0^r \frac{s^{n-1-\beta}}{\left[\epsilon^{2-\alpha} + s^{2-\alpha}\right]^{q(n-2)/(2-\alpha)}} ds &= \int\limits_0^\epsilon \frac{s^{n-1-\beta}}{\left[\epsilon^{2-\alpha} + s^{2-\alpha}\right]^{q(n-2)/(2-\alpha)}} ds + \int\limits_\epsilon^r \frac{s^{n-1-\beta}}{\left[\epsilon^{2-\alpha} + s^{2-\alpha}\right]^{q(n-2)/(2-\alpha)}} ds \\
&\le \epsilon^{n-1-\beta} \int\limits_0^\epsilon \frac{ds}{\epsilon^{q(n-2)}} + \int\limits_\epsilon^r s^{n-1-\beta-q(n-2)ds} \\
&= \epsilon^{n-\beta-q(n-2)} + I_{\epsilon,n}
\end{aligned} \tag{4.29}$$

where

$$I_{\epsilon,n} = \int\limits_\epsilon^r s^{n-\beta-1-q(n-2)} ds$$

. For all $n \ge 4$, we have $n - \beta - q(n-2) < 0$ and for $n = 3$, $\ n - \beta - q(n-2) = 3 - \beta - q$. Therefore,

$$I_{\epsilon,n} = \begin{cases} |\log \epsilon| - |\log r| & \text{for} \quad n = 3, \ 0 \le \beta < 1 \ \text{ with } \ q = 3 - \beta \\ C_1(\epsilon^{3-\beta-q} - C_1') & \text{for} \quad n = 3, \ q \ne 3 - \beta \\ C_2(\epsilon^{n-\beta-q(n-2)} - C_2') & \text{for all} \quad n \ge 4. \end{cases} \tag{4.30}$$

where $C_1', C_2, C_2' > 0$ and $C_1 \ge 0$ are constants. Since $q < 2_\beta^*$ i.e., $q(n-2)/2 < (n-\beta)$, from (4.28)- (4.30), we have

$$\int\limits_\Omega |v_\epsilon| f(x) dx \to 0 \, \text{as } \, \epsilon \to 0 \, .$$

Now to prove,

$$Y_\epsilon < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{(n-\alpha)/(2-\alpha)} \tag{4.31}$$

for sufficiently small $\epsilon > 0$, it is enough to prove that $\lambda \int\limits_\Omega |v_\epsilon|^q f\, dx$ goes zero slower than ϵ^{n-2} as $\epsilon \to 0$, or equivalently

$$\lambda \lim_{\epsilon \to 0} \epsilon^{2-n} \int\limits_\Omega |v_\epsilon|^q f\, dx = +\infty\,. \tag{4.32}$$

From (4.2), (4.6) , we have

$$\begin{aligned} \int\limits_\Omega |v_\epsilon|^q f(x) &\geq \int\limits_{B_\epsilon} |v_\epsilon|^q f(x) dx = \int\limits_{B_\epsilon} |v_\epsilon|^q |x|^{-\beta} dx \\ &\geq C\epsilon^{q(n-2)/2} \epsilon^{n-\beta-q(n-2)} \\ &= C\epsilon^{n-\beta-q(n-2)/2}. \end{aligned}$$

Therefore,

$$\epsilon^{2-n} \int\limits_\Omega |v_\epsilon|^q f(x) dx \geq C\epsilon^{2-\beta-q(n-2)/2}, \tag{4.33}$$

where $C > 0$ is a constant. From (4.33), we have

Case (a): When $n \geq 4$, we have $(2-\beta) - q(n-2)/2 < 0$, because $q > 2$ and hence (4.32) holds for any $\lambda > 0$.

Case (b): When $n = 3$, $\;2 - \beta - q(n-2)/2 = 2 - \beta - q/2$.

(i) Since for $1 \leq \beta < 2$ we have $2 - \beta - q/2 < 0$, (4.32) holds for all $\lambda > 0$.

(ii) For, $0 \leq \beta < 1$ with $2(2-\beta) < q < 2^*_\beta$, we have $2 - \beta - q/2 < 0$, so (4.32) holds for all $\lambda > 0$.

(iii) For, $0 \leq \beta < 1$ with $2 < q \leq 2(2-\beta)$, we have $2 - \beta - q/2 \geq 0$. Hence we can conclude the validity of (4.32) only for large $\lambda > 0$.

Hence the lemma follows. ∎

Proof of Theorem 1.4: Choose t_0 sufficiently large such that $J_\lambda(t_0\, v_\epsilon) < \rho = \inf\limits_{\|u\|=\delta} J_\lambda(u)$. Call $v_0 = t_0 v_\epsilon$ and define,

$$\Gamma = \Big\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = v_0\Big\}$$

and

$$\beta_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in I} J_\lambda(\gamma(t)) \geq \rho > 0.$$

Therefore, we have

$$\beta_\lambda \leq \max_{t\geq 0} J_\lambda(tv_\epsilon) < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{(n-\alpha)/(2-\alpha)}. \tag{4.34}$$

By Theorem 2.2 of Brezis-Nirenberg [4] there exists a sequence $(u_m) \subseteq H_0^1(\Omega)$ such that as $m \to \infty$

$$J_\lambda(u_m) \to \beta_\lambda \text{ and } DJ_\lambda(u_m) \to 0 \text{ strongly in } H^{-1}(\Omega).$$

Since β_λ satisfies (4.34), using Lemma 4.1 and Lemma 4.2 we complete the proof. ∎

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