

# ON BLOW-UPS OF SETS WITH FINITE FRACTIONAL VARIATION

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*To the memory of my beloved friend Diego.*

ABSTRACT. Given  $\alpha \in (0, 1)$  and a set  $E \subset \mathbb{R}^N$  with locally finite fractional  $\alpha$ -variation, we show that for  $|D^\alpha \mathbf{1}_E|$ -a.e.  $x$ , every non-trivial tangent set of  $E$  at  $x$  with locally finite integer perimeter is a half-space oriented by the fractional inner unit normal of  $E$  at  $x$ .

## 1. INTRODUCTION

1.1. **Setting.** Given  $\alpha \in (0, 1)$ , the *fractional  $\alpha$ -gradient* of  $u \in \text{Lip}_c(\mathbb{R}^N)$  is defined as

$$\nabla^\alpha u(x) = c_{N,\alpha} \int_{\mathbb{R}^N} \frac{(u(y) - u(x))(y - x)}{|y - x|^{N+\alpha+1}} dy, \quad x \in \mathbb{R}^N, \quad (1.1)$$

and the *fractional  $\alpha$ -divergence* of  $\varphi \in \text{Lip}_c(\mathbb{R}^N; \mathbb{R}^N)$  is analogously defined as

$$\text{div}^\alpha \varphi(x) = c_{N,\alpha} \int_{\mathbb{R}^N} \frac{(\varphi(y) - \varphi(x)) \cdot (y - x)}{|y - x|^{N+\alpha+1}} dy, \quad x \in \mathbb{R}^N, \quad (1.2)$$

where  $c_{N,\alpha} > 0$  is a suitable normalization constant. The operators (1.1) and (1.2) satisfy the integration-by-parts formula

$$\int_{\mathbb{R}^N} u \text{div}^\alpha \varphi dx = - \int_{\mathbb{R}^N} \varphi \cdot \nabla^\alpha u dx. \quad (1.3)$$

For further details on the operators (1.1) and (1.2) and on the formula (1.3), we refer the reader to [34]. Building on the integration-by-parts formula (1.3), in our previous works [10, 13–19], together with Giovanni E. Comi and several collaborators, we developed a new theory of distributional fractional Sobolev and  $BV$  spaces. The literature around the fractional gradient (1.1) is growing very rapidly in multiple directions, both theoretical and applicative. For a non-exhaustive list of contributions, we refer to [1, 2, 5–9, 11, 12, 21, 23–25, 30–33, 35, 36] and to the references therein.

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**1.2. Fractional variation.** We continue the study of fractional  $BV$  functions. Given  $p \in [1, \infty]$  and an open set  $\Omega \subset \mathbb{R}^N$ , we say that  $u \in BV_{\text{loc}}^{\alpha,p}(\Omega)$  if  $u \in L^p(\mathbb{R}^N)$  and

$$|D^\alpha u|(A) = \sup \left\{ \int_{\mathbb{R}^N} u \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N), \|\varphi\|_{L^\infty} \leq 1, \operatorname{supp} \varphi \subset A \right\} < \infty$$

for every open set  $A \Subset \Omega$ . By Riesz's Representation Theorem,  $u \in BV_{\text{loc}}^{\alpha,p}(\Omega)$  if and only if  $u \in L^p(\mathbb{R}^N)$  and there exists a locally finite (vector-valued) Radon measure  $D^\alpha u \in \mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^N)$  such that (1.3) holds with  $D^\alpha u$  in place of the fractional  $\alpha$ -gradient for every  $\varphi \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$  such that  $\operatorname{supp} \varphi \subset \Omega$ . If  $|D^\alpha u|(\Omega) < \infty$ , then we write  $u \in BV^{\alpha,p}(\Omega)$ . Note that the subscript 'loc' in  $BV_{\text{loc}}^{\alpha,p}$  refers only to the local finiteness of the *fractional variation measure*, since  $BV_{\text{loc}}^{\alpha,p}$  functions are, by definition, in  $L^p(\mathbb{R}^N)$ .

**1.3. Blow-up Theorem.** If  $\mathbf{1}_E \in BV_{\text{loc}}^{\alpha,\infty}(\mathbb{R}^N)$ , then  $|D^\alpha \mathbf{1}_E| \in \mathcal{M}_{\text{loc}}(\mathbb{R}^N)$  is the *distributional fractional  $\alpha$ -perimeter measure* of  $E \subset \mathbb{R}^N$ . In analogy with the classical theory [3, 26], the *fractional reduced  $\alpha$ -boundary*  $\mathcal{F}^\alpha E$  of  $E$  is the set of points  $x \in \operatorname{supp} |D^\alpha \mathbf{1}_E|$  at which the (measure-theoretic) *inner unit fractional normal* exists, namely

$$\nu_E^\alpha(x) = \lim_{r \rightarrow 0^+} \frac{D^\alpha \mathbf{1}_E(B_r(x))}{|D^\alpha \mathbf{1}_E|(B_r(x))} \in \mathbb{S}^{N-1}.$$

The main feature of the fractional reduced boundary we are interested in here is its connection with the set of *tangent* (or *blow-up*) *sets* of  $E$  at  $x$ , that is, the set  $\operatorname{Tan}(E, x)$  of all limit points of  $\left\{ \frac{E-x}{r} : r > 0 \right\}$  in  $L_{\text{loc}}^1(\mathbb{R}^N)$  as  $r \rightarrow 0^+$ .

**Theorem 1.1.** *Let  $\mathbf{1}_E \in BV_{\text{loc}}^{\alpha,\infty}(\mathbb{R}^N)$  and  $x \in \mathcal{F}^\alpha E$ . Then,  $\operatorname{Tan}(E, x) \neq \emptyset$  and any  $F \in \operatorname{Tan}(E, x)$  is such that  $\mathbf{1}_F \in BV_{\text{loc}}^{\alpha,\infty}(\mathbb{R}^N)$  with  $\nu_F^\alpha = \nu_E^\alpha(x)$  for  $|D^\alpha \mathbf{1}_F|$ -a.e.  $y \in \mathcal{F}^\alpha F$ . Moreover, assuming  $x = 0$  and  $\nu_E^\alpha(0) = e_N$  without loss of generality,  $F = \mathbb{R}^{N-1} \times M$  for some measurable set  $M \subset \mathbb{R}$  such that:*

- (i)  $\mathbf{1}_M \in BV_{\text{loc}}^{\alpha,\infty}(\mathbb{R})$  with  $\partial^\alpha \mathbf{1}_M \geq 0$ ;
- (ii)  $|M|, |M^c| \in \{0, \infty\}$ ;
- (iii) if  $|M| = \infty$ , then  $\operatorname{ess\,sup} M = \infty$ ;
- (iv) if  $M \notin \{\emptyset, \mathbb{R}\}$  and  $P(M) < \infty$ , then  $M = (m, \infty)$  for some  $m \in \mathbb{R}$ .

The first part of Theorem 1.1 follows from [13, Th. 5.8 and Prop. 5.9], while the second part is taken from [19, Th. 1.7]. Theorem 1.1 can be regarded as the fractional counterpart of De Giorgi's Blow-up Theorem [26, Th. 15.5], which states that if  $\mathbf{1}_E \in BV_{\text{loc}}(\mathbb{R}^N)$  and  $x \in \mathcal{F}E$ , the *reduced boundary* of  $E$ , then  $\operatorname{Tan}(E, x) = \{H_{\nu_E^+(x)}^+(x)\}$ , where

$$H_{\nu_E^+(x)}^+(x) = \left\{ y \in \mathbb{R}^N : (y - x) \cdot \nu_E^+(x) \geq 0 \right\},$$

and  $\nu_E: \mathcal{F}E \rightarrow \mathbb{S}^{N-1}$  is the (measure-theoretic) *inner unit normal* of  $E$ . In fact, by [19, Th. 1.12(iii)], if  $\mathbf{1}_E \in BV_{\text{loc}}(\mathbb{R}^N)$ , then  $\mathcal{F}E \subset \mathcal{F}^\alpha E$  and  $\nu_E^\alpha = \nu_E$  on  $\mathcal{F}E$ , showing that Theorem 1.1 naturally extends the classical setting. However, the fractional reduced boundary  $\mathcal{F}^\alpha E$  can be substantially larger than  $\mathcal{F}E$ , even when  $E$  is very regular; see [19, Props. 1.13 and 1.15] for the cases of the half-space and the ball.

**1.4. Main result.** By combining [Theorem 1.1\(iv\)](#) with well-known stability properties of the family of tangent sets (see [[22](#), Props. 2.1 and 2.2] for instance), we get that

$$\mathbb{R}^{N-1} \times M \in \text{Tan}(E, 0) \setminus \{\emptyset, \mathbb{R}^N\} \text{ and } P(M) < \infty \implies H_{e_N}^+(0) \in \text{Tan}(E, 0). \quad (1.4)$$

This observation motivates the following question: does [\(1.4\)](#) still hold under the weaker assumption that  $\mathbf{1}_M \in BV_{\text{loc}}(\mathbb{R})$ ? The purpose of the present note is to provide an affirmative answer. Here and below, we write  $BV_{\text{loc}}^*(\mathbb{R}^N) = BV_{\text{loc}}(\mathbb{R}^N) \setminus \{0, 1\}$  for brevity.

**Theorem 1.2.** *Given  $\mathbf{1}_E \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^N)$ , for  $|D^\alpha \mathbf{1}_E|$ -a.e.  $x \in \mathcal{F}^\alpha E$  the following holds. If  $F \in \text{Tan}(E, x) \cap BV_{\text{loc}}^*(\mathbb{R}^N)$ , then  $F = H_{\nu_E^\alpha(x)}^+(y)$  for some  $y \in \mathbb{R}^N$ . In particular,*

$$\text{Tan}(E, x) \cap BV_{\text{loc}}^*(\mathbb{R}^N) \neq \emptyset \implies H_{\nu_E^\alpha(x)}^+(x) \in \text{Tan}(E, x). \quad (1.5)$$

Some comments are in order. First, as shown by the examples in [[19](#), Props. 1.13 and 1.15], the implication [\(1.5\)](#) does not hold when all tangent sets are trivial, so the assumption that a non-trivial blow-up exists is necessary. Second, [Theorem 1.2](#) yields a rigidity property: for  $|D^\alpha \mathbf{1}_E|$ -almost every  $x \in \mathcal{F}^\alpha E$ , every non-trivial tangent set with locally finite integer perimeter is a half-space oriented by  $\nu_E^\alpha(x)$ . This provides the closest available analogue of De Giorgi's Blow-up Theorem in the fractional setting. Third, we do not know whether this rigidity holds at every point of  $\mathcal{F}^\alpha E$ . In particular, extending property [\(iv\)](#) in [Theorem 1.1](#) to the broader case  $\mathbf{1}_M \in BV_{\text{loc}}(\mathbb{R})$  remains open.

[Theorem 1.2](#) was inspired by [[4](#), Th. 1.2], where an analogous statement is established in *Carnot groups*. The key ingredient is the principle, due to Preiss [[27](#), [29](#)], that *tangent measures to tangent measures are tangent measures* (see [Theorem 2.1](#) below). This fact makes it possible to reduce the proof to the situation in which a tangent of  $E$  has locally finite integer perimeter, a case for which the conclusion follows from the classical theory.

A broader motivation for [Theorem 1.2](#) comes from the study of sets with locally finite distributional fractional perimeter. While De Giorgi's theorem describes the infinitesimal structure of sets of finite perimeter, the fractional case lacks an analogous rectifiability theory and even the appropriate infinitesimal models are not fully understood. In this direction, [Theorems 1.2](#) and [3.3](#) suggest that certain rigidity features of the classical theory may persist under suitable assumptions on the existence of non-trivial tangent sets.

## 2. PRELIMINARIES

**2.1. Radon measure.** Given an open set  $\Omega \subset \mathbb{R}^N$ , we let  $\mathcal{M}(\Omega)$  and  $\mathcal{M}_{\text{loc}}(\Omega)$  be the spaces of finite and locally finite signed Radon measures on  $\Omega$ , respectively.

By the Riesz Representation Theorem,  $\mathcal{M}_{\text{loc}}(\Omega)$  is the dual of  $C_c(\Omega)$ , endowed with the topology of local uniform convergence. Accordingly, we say that a sequence  $(\mu_k)_{k \in \mathbb{N}} \subset \mathcal{M}_{\text{loc}}(\Omega)$  converges to  $\mu \in \mathcal{M}_{\text{loc}}(\Omega)$  in the (local) weak\* sense if

$$\lim_{k \rightarrow \infty} \int_{\Omega} \varphi \, d\mu_k = \int_{\Omega} \varphi \, d\mu \quad \text{for every } \varphi \in C_c(\Omega). \quad (2.1)$$

In this case, we write  $\mu_k \xrightarrow{*} \mu$  in  $\mathcal{M}_{\text{loc}}(\Omega)$  as  $k \rightarrow \infty$ .

For vector-valued measures, we write  $\mathcal{M}(\Omega; \mathbb{R}^m)$  and  $\mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^m)$ , with  $m \in \mathbb{N}$ . The notion of (local) weak\* convergence applies to  $\mathbb{R}^m$ -valued Radon measures by requiring [\(2.1\)](#) to hold componentwise. In this case, we write  $\mu_k \xrightarrow{*} \mu$  in  $\mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^m)$  as  $k \rightarrow \infty$ .

Accordingly, the *total variation* of  $\mu \in \mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^m)$  on an open set  $A \subset \Omega$  is defined as

$$|\mu|(A) = \sup \left\{ \int_{\Omega} \varphi \cdot d\mu : \varphi \in C_c(\Omega; \mathbb{R}^m), \text{supp } \varphi \subset A, \|\varphi\|_{L^\infty} \leq 1 \right\}.$$

We recall that  $|\mu| \in \mathcal{M}_{\text{loc}}(\Omega)$  for every  $\mu \in \mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^m)$ ; see [26, Lem. 4.17]. For a detailed presentation of the theory of Radon measures, we refer to [3, 20, 26, 27].

**2.2. Tangent measures.** Given  $m \in \mathbb{N}$  and  $\mu \in \mathcal{M}_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^m)$ , for every  $x \in \mathbb{R}^N$  and  $r > 0$  we define  $\mu_{x,r} \in \mathcal{M}_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^m)$  by setting

$$\mu_{x,r}(A) = \mu(x + rA) \quad \text{for every Borel set } A \subset \mathbb{R}^N.$$

As customary (see [29, 2.3(1)], [27, Def. 14.1], and [3, Sec. 2.7]), we say that  $\nu \in \text{Tan}(\mu, x)$  if and only if there exist sequences  $(r_k)_{k \in \mathbb{N}}, (c_k)_{k \in \mathbb{N}} \subset (0, \infty)$  such that  $r_k \rightarrow 0^+$  and

$$c_k \mu_{x,r_k} \xrightarrow{*} \nu \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^m) \text{ as } k \rightarrow \infty.$$

Moreover (see [20, Sec. 9.1]), given  $s \geq 0$ , we say that  $\nu \in \text{Tan}_s(\mu, x)$  if there exists an infinitesimal subsequence  $(r_k)_{k \in \mathbb{N}} \subset (0, \infty)$  such that

$$r_k^{-s} \mu_{x,r_k} \rightarrow \nu \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^m) \text{ as } k \rightarrow \infty.$$

By definition, it is clear that  $\text{Tan}_s(\mu, x) \subset \text{Tan}(\mu, x)$  for every  $s \geq 0$ ,  $\mu \in \mathcal{M}_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^m)$ , and  $x \in \mathbb{R}^N$ . For the proof of [Theorem 1.2](#), we will rely on the following crucial result.

**Theorem 2.1.** *Let  $s \geq 0$ ,  $m \in \mathbb{N}$  and  $\mu \in \mathcal{M}_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^m)$ . Then, for  $|\mu|$ -a.e.  $x \in \mathbb{R}^N$ , every  $\nu \in \text{Tan}_s(\mu, x)$  satisfies the following properties:*

- (i)  $\nu_{y,r} \in \text{Tan}_s(\mu, x)$  for every  $y \in \text{supp } |\nu|$  and  $r > 0$ ;
- (ii)  $\text{Tan}_s(\nu, y) \subset \text{Tan}_s(\mu, x)$  for every  $y \in \text{supp } |\nu|$ .

For  $m = 1$  and  $\mu \geq 0$ , [Theorem 2.1](#) can be found in [20, Prop. 9.3] and, with  $\text{Tan}$  in place of  $\text{Tan}_s$ , in [29, Th. 2.12] and [27, Th. 14.16]. Moreover, [Theorem 2.1](#) corresponds to [4, Th. 6.4], stated on a Carnot group  $\mathbb{G}$  in place of  $\mathbb{R}^N$  and under the additional assumption that  $\mu$  is *asymptotically  $s$ -regular*; that is,

$$0 < \liminf_{r \rightarrow 0^+} \frac{|\mu|(B_r(x))}{r^s} \leq \limsup_{r \rightarrow 0^+} \frac{|\mu|(B_r(x))}{r^s} < \infty \quad \text{for } |\mu| \text{-a.e. } x \in \mathbb{G}. \quad (2.2)$$

Finally, [Theorem 2.1](#) also corresponds to [28, Prop. 2.15], with  $m = 1$ ,  $\mathbb{R}^N$  replaced by a homogeneous locally compact metric group  $\mathbf{G}$ , and  $\mu \geq 0$  such that

$$\limsup_{r \rightarrow 0^+} \frac{\mu(B_{2r}(x))}{\mu(B_r(x))} < \infty \quad \text{for } \mu \text{-a.e. } x \in \mathbf{G}. \quad (2.3)$$

The additional assumptions (2.2) and (2.3) are required to ensure the validity of the Differentiation Theorem, which does not generally hold in an arbitrary metric space.

The proof of [Theorem 2.1](#) follows almost *verbatim* the argument of [20, Prop. 9.3] (which, in turn, is nearly identical to that of [27, Th. 14.16]), up to the minor modifications needed to treat the vector-valued case, as in [4, Th. 6.4]. We therefore omit the details.

## 3. PROOF OF THE MAIN RESULT

**3.1. Properties of tangent sets.** From [Theorem 2.1](#), we get the following result.

**Corollary 3.1.** *Let  $\mathbf{1}_E \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^N)$ . Then, for  $|D^\alpha \mathbf{1}_E|$ -a.e.  $x \in \mathcal{F}^\alpha E$ , every  $F \in \text{Tan}(E, x) \setminus \{\emptyset, \mathbb{R}^N\}$  satisfies the following properties:*

- (i)  $\frac{F-y}{r} \in \text{Tan}(E, x)$  for every  $y \in \text{supp } |D^\alpha \mathbf{1}_F|$  and  $r > 0$ ;
- (ii)  $\text{Tan}(F, y) \subset \text{Tan}(E, x)$  for every  $y \in \text{supp } |D^\alpha \mathbf{1}_F|$ .

In order to prove [Corollary 3.1](#), we need the following preliminary result.

**Lemma 3.2.** *If  $\mathbf{1}_E \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^N)$  and  $x \in \mathcal{F}^\alpha E$ , then*

$$F \in \text{Tan}(E, x) \setminus \{\emptyset, \mathbb{R}^N\} \iff D^\alpha \mathbf{1}_F \in \text{Tan}_{N-\alpha}(D^\alpha \mathbf{1}_E, x) \setminus \{0\}.$$

*Proof.* If  $F \in \text{Tan}(E, x) \setminus \{\emptyset, \mathbb{R}^N\}$ , then  $\mathbf{1}_F \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^N)$  by [Theorem 1.1](#). Moreover, we can find an infinitesimal sequence  $(r_k)_{k \in \mathbb{N}} \subset (0, \infty)$  such that  $\frac{E-x}{r_k} \rightarrow F$  in  $L_{\text{loc}}^1(\mathbb{R}^N)$  as  $k \rightarrow \infty$ . Therefore, by the scaling properties of the fractional  $\alpha$ -gradient [\(1.1\)](#) (see also [\[13, Eq. \(4.8\)\]](#)) and [\[19, Th. 1.6\(i\)\]](#), we obtain

$$r_k^{\alpha-N}(D^\alpha \mathbf{1}_E)_{x, r_k} = D^\alpha \mathbf{1}_{\frac{E-x}{r_k}} \xrightarrow{*} D^\alpha \mathbf{1}_F \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N) \text{ as } k \rightarrow \infty,$$

showing that  $D^\alpha \mathbf{1}_F \in \text{Tan}_{N-\alpha}(D^\alpha \mathbf{1}_E, x)$ . In addition, we must have  $D^\alpha \mathbf{1}_F \neq 0$ , since otherwise  $\mathbf{1}_F$  would be constant by [\[19, Prop. 1.8\]](#), and thus  $F \in \{\emptyset, \mathbb{R}^N\}$ , a contradiction.

Conversely, if  $D^\alpha \mathbf{1}_F \in \text{Tan}_{N-\alpha}(D^\alpha \mathbf{1}_E, x) \setminus \{0\}$ , then clearly  $\mathbf{1}_F \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^N)$ , and we can find an infinitesimal sequence  $(r_k)_{k \in \mathbb{N}} \subset (0, \infty)$  such that

$$r_k^{\alpha-N}(D^\alpha \mathbf{1}_E)_{x, r_k} \xrightarrow{*} D^\alpha \mathbf{1}_F \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N) \text{ as } k \rightarrow \infty.$$

Thus, letting  $E_k = (E-x)/r_k$  for every  $k \in \mathbb{N}$ , the sequence  $(E_k)_{k \in \mathbb{N}}$  satisfies  $\mathbf{1}_{E_k} \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^N)$  for every  $k \in \mathbb{N}$ , and

$$D^\alpha \mathbf{1}_{E_k} = r_k^{\alpha-N}(D^\alpha \mathbf{1}_E)_{x, r_k} \xrightarrow{*} D^\alpha \mathbf{1}_F \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N) \text{ as } k \rightarrow \infty.$$

Since  $x \in \mathcal{F}^\alpha E$ , possibly passing to a subsequence (which we do not relabel), by [Theorem 1.1](#) there exists  $G \in \text{Tan}(E, x)$  such that  $\mathbf{1}_G \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^N)$  and  $E_k \rightarrow G$  in  $L_{\text{loc}}^1(\mathbb{R}^N)$  as  $k \rightarrow \infty$ . Again by [\[19, Th. 1.6\(i\)\]](#), possibly passing to a further subsequence (which we do not relabel), we also have that

$$D^\alpha \mathbf{1}_{E_k} \xrightarrow{*} D^\alpha \mathbf{1}_G \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N) \text{ as } k \rightarrow \infty,$$

from which it follows that  $D^\alpha \mathbf{1}_G = D^\alpha \mathbf{1}_F$ . By [\[19, Prop. 1.8\]](#), this implies that  $\mathbf{1}_G - \mathbf{1}_F$  is constant, so that either  $F = G$  or  $F = \mathbb{R}^N \setminus G$ . The latter possibility is ruled out, since otherwise  $D^\alpha \mathbf{1}_F = -D^\alpha \mathbf{1}_G$  and thus  $D^\alpha \mathbf{1}_F = 0$ , a contradiction. Hence  $F = G$ , and therefore  $F \in \text{Tan}(E, x)$ . Moreover,  $F \notin \{\emptyset, \mathbb{R}^N\}$ , since  $D^\alpha \mathbf{1}_F \neq 0$ .  $\square$

*Proof of Corollary 3.1.* If  $F \in \text{Tan}(E, x) \setminus \{\emptyset, \mathbb{R}^N\}$ , then by [Lemma 3.2](#) we deduce that  $D^\alpha \mathbf{1}_F \in \text{Tan}_{N-\alpha}(D^\alpha \mathbf{1}_E, x) \setminus \{0\}$ . Therefore, by [Theorem 2.1\(i\)](#), we get that  $(D^\alpha \mathbf{1}_F)_{y, r} \in \text{Tan}_{N-\alpha}(D^\alpha \mathbf{1}_E, x)$  for every  $y \in \text{supp } |D^\alpha \mathbf{1}_F|$  and every  $r > 0$ . Since  $(D^\alpha \mathbf{1}_F)_{y, r} = D^\alpha \mathbf{1}_{(F-y)/r}$  by the scaling properties of [\(1.1\)](#), again by [Lemma 3.2](#) this implies that  $(F-y)/r \in \text{Tan}(E, x)$  for every  $y \in \text{supp } |D^\alpha \mathbf{1}_F|$  and  $r > 0$ . Finally, since  $\text{Tan}(E, x)$  is closed with respect to convergence in  $L_{\text{loc}}^1(\mathbb{R}^N)$  (see [\[22, Prop. 2.2\]](#)), we also obtain that  $\text{Tan}(F, y) \subset \text{Tan}(E, x)$  for every  $y \in \text{supp } |D^\alpha \mathbf{1}_F|$ , concluding the proof.  $\square$

**3.2. Iterated tangent sets.** For the proof of [Theorem 1.2](#), we rely on the notion of *iterated* tangent sets. Precisely, given  $\mathbf{1}_E \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^N)$  and  $x \in \mathcal{F}^\alpha E$ , we define  $\text{Tan}^1(E, x) = \text{Tan}(E, x)$  and

$$\text{Tan}^{k+1}(E, x) = \bigcup \left\{ \text{Tan}(F) : F \in \text{Tan}^k(E, x) \right\} \quad (3.1)$$

for all  $k \in \mathbb{N}$ , where

$$\text{Tan}(E) = \bigcup \left\{ \text{Tan}(E, x) : x \in \mathcal{F}^\alpha E \right\}.$$

The proof of [Theorem 1.2](#) is based on the following result, which may be of independent interest (and rephrases [\[4, Th. 6.1\]](#) in the present setting).

**Theorem 3.3.** *If  $\mathbf{1}_E \in BV_{\text{loc}}^{\alpha, \infty}(\mathbb{R}^N)$ , then for  $|D^\alpha \mathbf{1}_E|$ -a.e.  $x \in \mathcal{F}^\alpha E$  it holds that*

$$\bigcup_{k=2}^{\infty} \text{Tan}^k(E, x) \subset \text{Tan}(E, x).$$

*Proof.* Let  $x \in \mathcal{F}^\alpha E$  be such that [Corollary 3.1\(ii\)](#) holds and let  $F \in \text{Tan}(E, x)$ . If  $F \in \{\emptyset, \mathbb{R}^N\}$ , then  $\mathcal{F}^\alpha F = \emptyset$  and there is nothing to prove. If instead  $F \notin \{\emptyset, \mathbb{R}^N\}$ , then for every  $y \in \mathcal{F}^\alpha F$  we have that  $\text{Tan}(F, y) \subset \text{Tan}(E, x)$ . By the definition in [\(3.1\)](#), we infer that  $\text{Tan}^2(E, x) \subset \text{Tan}^1(E, x)$ , and the conclusion follows by iteration.  $\square$

**3.3. Proof of [Theorem 1.2](#).** We can now prove our main result.

*Proof of [Theorem 1.2](#).* Let  $x \in \mathcal{F}^\alpha E$  be such that [Theorem 3.3](#) holds. By assumption, there exists  $F \in \text{Tan}(E, x) \setminus \{\emptyset, \mathbb{R}^N\}$  such that  $\mathbf{1}_F \in BV_{\text{loc}}(\mathbb{R}^N)$ . Hence, we have that  $\text{Tan}(F, y) = \{H_{\nu_F(y)}^+(y)\}$  for every  $y \in \mathcal{F}F$ . Since  $\mathcal{F}F \subset \mathcal{F}^\alpha F$  by [\[19, Th. 1.12\(iii\)\]](#), from the definition in [\(3.1\)](#) we deduce that  $H_{\nu_F(y)}^+(y) \in \text{Tan}^2(E, x)$  for every  $y \in \mathcal{F}F$ . This, in turn, by [Theorem 3.3](#), implies that  $H_{\nu_F(y)}^+(y) \in \text{Tan}(E, x)$  for every  $y \in \mathcal{F}F$ . Combining [\[19, Prop. 1.13\]](#) with [Theorem 1.1](#), we then obtain that, for every  $y \in \mathcal{F}F$ ,

$$\nu_F(y) = \nu_{H_{\nu_F(y)}^+(y)}^\alpha(z) = \nu_E^\alpha(x) \quad \text{for a.e. } z \in \mathbb{R}^N.$$

As a consequence,  $\nu_F(y) = \nu_E^\alpha(x)$  for every  $y \in \mathcal{F}F$ , which implies that  $F = H_{\nu_E^\alpha(x)}^+(x_0)$  for some  $x_0 \in \mathbb{R}^N$  by [\[26, Prop. 15.15\]](#), and [\(1.5\)](#) follows by [\[22, Props. 2.1 and 2.2\]](#).  $\square$

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