

THE ISOPERIMETRIC INEQUALITY FOR THE CAPILLARY ENERGY OUTSIDE CONVEX SETS

NICOLA FUSCO, VESA JULIN, MASSIMILIANO MORINI, AND ALDO PRATELLI

ABSTRACT. We study the isoperimetric problem for capillary hypersurfaces with a general contact angle $\theta \in (0, \pi)$, outside arbitrary convex sets. We prove that the capillary energy of any surface supported on any such convex set is larger than that of a spherical cap with the same volume and the same contact angle on a flat support, and we characterize the equality cases. This provides a complete solution to the isoperimetric problem for capillary surfaces outside convex sets at arbitrary contact angles, generalizing the well-known Choe-Ghomi-Ritoré inequality, which corresponds to the case $\theta = \frac{\pi}{2}$.

1. INTRODUCTION

Let $\mathbf{C} \subset \mathbb{R}^N$ be a closed convex set with nonempty interior. Given a set of finite perimeter $E \subset \mathbb{R}^N \setminus \mathbf{C}$ and $\lambda \in (-1, 1)$ we define the capillary energy as

$$J_{\lambda, \mathbf{C}}(E) := P(E; \mathbb{R}^N \setminus \mathbf{C}) - \lambda \mathcal{H}^{N-1}(\partial^* E \cap \mathbf{C}).$$

Here, for any Borel set G , $P(E; G) = \mathcal{H}^{N-1}(\partial E^* \cap G)$ and $\partial^* E$ is the reduced boundary of E (for the definitions and the relevant properties see [2, 21]). The capillary energy has a natural physical motivation as it models a liquid drop supported on a given substrate and we refer to [12] for a comprehensive introduction to the topic.

For every $m > 0$ we consider the isoperimetric problem

$$I_{\mathbf{C}}(m) := \inf \{ J_{\lambda, \mathbf{C}}(E) : E \subset \mathbb{R}^N \setminus \mathbf{C}, |E| = m \}. \quad (1.1)$$

When the convex set $\mathbf{C}$ is bounded the problem (1.1) has a minimizer, and if $\mathbf{C}$ is in addition smooth, then the minimizer is smooth up to a small singular set and the free boundary $\partial E \setminus \mathbf{C}$ meets the surface $\partial \mathbf{C}$ with a contact angle $\theta = \arccos \lambda$ given by the classical Young's law [25, 10]. We also mention the recent work related to Allard type regularity for critical sets [22]. When the convex set is unbounded the problem (1.1) might not admit a minimizer. This happens for instance when $\mathbf{C} = \mathcal{C} \times \mathbb{R} \subset \mathbb{R}^3$, with $\mathcal{C}$ the epigraph of a parabola. In this case, as a consequence of our main Theorem 1.1, minimizing sequences slide upwards to infinity along the boundary of $\mathbf{C}$ and the isoperimetric profile (1.1) agrees with the profile given by the half-space. In the case $\lambda = 0$ the problem for unbounded general convex sets $\mathbf{C}$ is studied in [14].

The issue we want to address here is to find the convex sets $\mathbf{C}$ for which the value of (1.1) is the smallest. In the case $\lambda = 0$ the problem reduces to the relative isoperimetric inequality

outside convex sets proven by Choe-Ghomi-Ritoré in [8]: using the tools developed in [7] they show that the half-space gives the lowest value for (1.1). On the other hand, rather surprisingly the capillary case with general $\lambda \neq 0$ has remained open until now as all the methods devised for the relative isoperimetric problem do not seem to be applicable to (1.1). In this paper, we solve the problem completely for general $\lambda \in (-1, 1)$ by adopting a different approach.

In order to state our main result we denote the half-space by $\mathbf{H} = \{x \in \mathbb{R}^N : x \cdot e_N < \lambda\}$ and by B^λ the (solid) spherical cap

$$B^\lambda = \{x \in B_1 : x \cdot e_N > \lambda\}.$$

Notice that at the points of $B^\lambda \cap \partial\mathbf{H}$ the angle between the outer normal to B^λ and e_N equals $\theta_\lambda = \arccos \lambda$. More in general, given any $m > 0$, we let $B^\lambda[m]$ denote the spherical cap contained in $\mathbb{R}^N \setminus \mathbf{H}$ having volume m and such that the angle between the outer normal and e_N at points of $B^\lambda \cap \partial\mathbf{H}$ is θ_λ . Our main result is the following.



FIGURE 1. Droplet supported on a convex polyhedron

Theorem 1.1. *Let $\lambda \in (-1, 1)$ and let $\mathbf{C}$ be a closed convex set. For every set of finite perimeter $E \subset \mathbb{R}^N \setminus \mathbf{C}$ such that $|E| = m$ we have*

$$J_{\lambda, \mathbf{C}}(E) \geq J_{\lambda, \mathbf{H}}(B^\lambda[m]). \quad (1.2)$$

Moreover the equality holds if and only if E is isometric to $B^\lambda[m]$ and E sits on a flat part of the boundary of $\mathbf{C}$.

In the case where $\mathbf{C}$ has empty interior the capillary energy must be defined as in (4.10).

We highlight also that we do not assume any regularity on $\mathbf{C}$. In particular, the theorem above applies to the case where $\mathbf{C}$ is an infinite wedge and shows that the capillary energy of a droplet sitting outside a wedge and wetting its ridge has energy strictly larger than a spherical cap lying on a flat surface, a fact that, to the best of our knowledge, was not proven before.

Instead, the capillary isoperimetric problem *inside* a convex wedge for $\lambda = 0$ was studied in [20] where it is proved that the minimizer of the capillary energy is a portion of a ball centered at the ridge of the wedge. The same result holds also for critical points, as a consequence of the generalized Heinze-Karcher inequality proven in [16]. Instead, Theorem 1.1 implies that the capillary isoperimetric problem *outside* a convex wedge has the opposite behavior, as explained before.

As we already mentioned, the case $\lambda = 0$ of Theorem 1.1 is the relative isoperimetric inequality due to Choe-Ghomi-Ritoré [8], see also [15] for the rigidity, i.e., the characterization of the equality case for general, possibly nonsmooth convex sets. We also refer to [19] for an alternative proof of the same inequality and to [18] for the problem in higher codimension.

In order to prove Theorem 1.1 we need to introduce some novel ideas, that will be explained in more detail in Section 1.1 below. Indeed, the approach based on normal cones developed in [7, 8] for the case $\lambda = 0$ (and further refined in [15]) gives only information on the free boundary $\partial E \setminus \mathbf{C}$, while the contact region $\partial E \cap \mathbf{C}$ remains invisible. In order to overcome this, we develop the ABP-method for the capillary isoperimetric problem outside convex sets. The ABP approach was originally introduced by Cabré for the standard isoperimetric inequality and extended to the relative isoperimetric problem outside convex sets (the case $\lambda = 0$) in [19]. Here we tackle the general case $\lambda \neq 0$. As we will explain in the next subsection, the core of ABP-method relies on a subtle estimate on the measure of the set of subdifferentials of the function u solving problem (1.3) below. Such an estimate turns out to be significantly difficult to obtain and requires new insights.

We next give an overview of the ABP-argument which we use in the proof of Theorem 1.1.

1.1. Overview of the proof. The proof of Theorem 1.1 is based on the ABP-method applied to the Neumann problem (1.3). We note that in the context of isoperimetric problems this method was used first time by Cabré in [3, 4] and further generalized in [5, 9]. A variant of this method has been used recently in [24] to prove the stability of the isoperimetric inequality for the capillary energy in a half space (see also [17] and the recent preprint [6] for a stronger quantitative version of the same isoperimetric inequality).

In this paper we develop the ABP-method for the capillary isoperimetric problem outside convex sets. Let us sketch the proof and outline the main challenges of the argument.

By scaling we may reduce to the case $|E| = |B^\lambda|$. Assume for simplicity that the set E is regular in which case we denote it by $E = \Omega$. To be more precise we assume that $\Omega \subset \mathbb{R}^N \setminus \mathbf{C}$ is a bounded Lipschitz domain with $|\Omega| = |B^\lambda|$, such that $\Sigma = \partial\Omega \setminus \mathbf{C}$ and $\Gamma = \partial\Omega \cap \mathbf{C}$ are smooth embedded manifolds with a common boundary denoted by γ . Let $u : \overline{\Omega} \rightarrow \mathbb{R}$ be the solution of the Neumann boundary problem

$$\begin{cases} \Delta u = c & \text{in } \Omega \\ \partial_\nu u = 1 & \text{on } \Sigma \\ \partial_\nu u = -\lambda & \text{on } \Gamma, \end{cases} \quad (1.3)$$

where $\lambda \in (-1, 1)$ and the constant

$$c = \frac{\mathcal{H}^{N-1}(\Sigma) - \lambda \mathcal{H}^{N-1}(\Gamma)}{|\Omega|} = \frac{J_{\lambda, \mathbf{C}}(\Omega)}{|\Omega|} \quad (1.4)$$

is the one prescribed by the compatibility condition. We point out that in the case $\Omega = B^\lambda$ up to an additive constant $u(x) = \frac{1}{2}|x|^2$ and $c = N$.

Let us now consider the points $x \in \Omega$ where $|\nabla u(x)| < 1$ and such that the tangent hyperplane to the graph of u at x is also a supporting hyperplane. More precisely, we set

$$\widehat{\Omega} := \{x \in \Omega : |\nabla u(x)| < 1 \text{ and } u(y) - u(x) \geq \nabla u(x) \cdot (y - x) \text{ for all } y \in \Omega\}. \quad (1.5)$$

Clearly it holds $\nabla^2 u(x) \geq 0$ for all $x \in \widehat{\Omega}$. In order to carry on the ABP-argument one needs to show the following crucial estimate

$$|\nabla u(\widehat{\Omega})| \geq |B^\lambda| \quad (1.6)$$

by somehow exploiting the Neumann boundary conditions in (1.3) and the geometry of $\mathbf{C}$.

Once (1.6) is proven, the claim then follows by using the Area Formula, the arithmetic-geometric mean inequality and recalling the value of c in (1.4)

$$|\Omega| = |B^\lambda| \leq |\nabla u(\widehat{\Omega})| \leq \int_{\widehat{\Omega}} \det \nabla^2 u \, dx \leq \int_{\widehat{\Omega}} \frac{(\Delta u)^N}{N^N} \, dx \leq \left(\frac{J_{\lambda, \mathbf{C}}(\Omega)}{|\Omega|N} \right)^N |\Omega|.$$

The above chain of inequalities gives the conclusion as

$$J_{\lambda, \mathbf{H}}(B^\lambda) = N|B^\lambda| = N|\Omega|.$$

The case of a general set of finite perimeter E follows by an approximation argument. In fact, a more refined approximation argument allows us also to characterize the equality case, see Lemma 4.3.

It is then clear that the most relevant estimate is (1.6) which turns out to be challenging to prove. Indeed, the inclusion $B^\lambda \subset \nabla u(\widehat{\Omega})$ does not hold in general and we need to develop a more subtle argument to overcome the problem. The same estimate was already proven in [19] in the case $\lambda = 0$ by reformulating the problem in terms of suitable restricted normal cones to the graph of u in the spirit of [7]. However their argument does not generalize to the case $\lambda \neq 0$.

In order to deal with the case of general λ 's, we proceed as follows. We first show that the (variational) solution to (1.3) is a viscosity supersolution to the same problem, in the sense of Definition 2.1. The latter definition is stronger than the one given in [11] and therefore the supersolution property in the above sense does not follow from known results. Using this property we are able to relate the subdifferentials of u in Ω with the subdifferentials of the restriction of u to Γ proving the following inclusion, see Lemma 2.4,

$$\mathcal{B}_{u_\Gamma}^\lambda \subset \nabla u(\widehat{\Omega}), \quad (1.7)$$

where $\mathcal{B}_{u_\Gamma}^\lambda = \bigcup_{x \in \Gamma} \{\xi \in J_\Gamma u(x) : |\xi| < 1 \text{ and } \xi \cdot \nu_{\mathbf{C}}(x) > \lambda\}$, with $\nu_{\mathbf{C}}(x)$ the outer normal to $\mathbf{C}$ at x and $J_\Gamma u(x)$ the subdifferential at x of the restriction of u to Γ ; that is,

$$J_\Gamma u(x) := \{\xi \in \mathbb{R}^N : u(y) - u(x) \geq \xi \cdot (y - x) \text{ for all } y \in \Gamma\}.$$

The inclusion (1.7) leads to the proof of (1.6) provided we are able to show that

$$|\mathcal{B}_v^\lambda| \geq |B^\lambda| \quad \text{for any } v : K \rightarrow \mathbb{R}, \text{ with } K \subset \partial \mathbf{C} \text{ compact.} \quad (1.8)$$

In fact, it is enough to prove (1.8) only for discrete sets $K \subset \partial \mathbf{C}$, see Lemma 3.3. Note that property (1.8) has nothing to do with Neumann problem (1.3) and it only depends on the geometry of $\mathbf{C}$. By a simple argument based on symmetry one can easily see that closed convex sets satisfy (1.8) for $\lambda = 0$, see Section 3. However, the case $\lambda \neq 0$ is considerably more difficult.

In fact, due to the aforementioned discretization procedure, proving (1.8) is equivalent to showing that, given any convex partition $A_1, \dots, A_k$ of $\mathbb{R}^N$, with A_i the subdifferential at x_i of a function $v : \{x_1, \dots, x_k\} \subset \partial \mathbf{C} \rightarrow \mathbb{R}$, then

$$\sum_{i=1}^k |A_i \cap \{\xi \in B_1 : \xi \cdot \nu_i > \lambda\}| \geq |B^\lambda|, \quad (1.9)$$

where ν_i is the exterior normal to $\mathbf{C}$ at x_i .

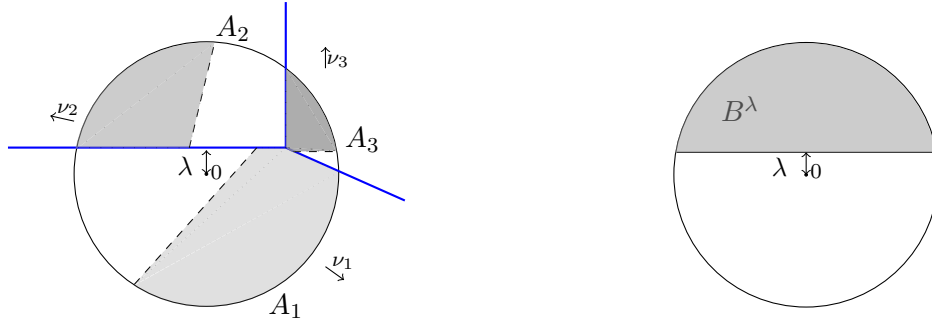


FIGURE 2. An example of partition with the half-line property. The grey regions represent $A_i \cap \{\xi \in B_1 : \xi \cdot \nu_i > \lambda\}$

The proof of the above inequality is a difficult geometric problem that we solve completely in Section 3, see Theorem 3.1. Let us just mention here that the convexity of $\mathbf{C}$ is used only to prove that the subdifferentials satisfy the following *half-line property*: if $\xi \in A_i$ then $\xi + t\nu_i \in A_i$ for all $t > 0$. The arguments of Section 3 show that in fact (1.9) hold for any finite partition of $\mathbb{R}^N$ into essentially disjoint sets satisfying the above half-line property.

2. PRELIMINARY RESULTS

In this section we set up the tools and show some preparatory results that we will need later for the ABP argument. To this aim, in Section 2.1 we establish the crucial viscosity

supersolution property for the variational solutions of the Neumann problem (1.3). In Section 2.2 we introduce a useful notion of restricted subdifferential which enables us to reduce the inequality (1.6) to a property of the convex set $\mathbf{C}$.

As we mentioned in the introduction, we will first prove the relative isoperimetric inequality for regular sets. In order to emphasize this we always denote $E = \Omega$ when the set is assumed to be regular, i.e., it satisfies

$$\mathbf{C} \subset \mathbb{R}^N \text{ is a closed convex set of class } C^2 \quad (2.1)$$

and

$$\begin{aligned} \Omega \subset \mathbb{R}^N \setminus \mathbf{C} \text{ is a bounded Lipschitz open set such that } \Sigma := \partial\Omega \setminus \mathbf{C} \\ \text{is a } (N-1)\text{-manifold with boundary of class } C^2. \end{aligned} \quad (2.2)$$

We call the boundary Σ the *free interface* and denote $\Gamma := \partial\Omega \cap \mathbf{C}$, which we call the *wetted region* which is also an embedded C^2 -regular $(N-1)$ -manifold with boundary. Note that Γ and Σ share the same boundary, which we denote by $\gamma := \bar{\Sigma} \cap \mathbf{C}$ and which by the assumption is a $(N-2)$ -manifold of class C^2 . We call γ the *contact set* of Σ with $\mathbf{C}$. Moreover, we will throughout the section assume $|\Omega| = |B^\lambda|$ if not otherwise mentioned.

We will denote by ν_Ω and $\nu_{\mathbf{C}}$ the outer unit normal to $\partial\Omega$ and to $\partial\mathbf{C}$, respectively. We also denote by $\nu_\Sigma = \nu_\Omega$ the outer unit normal field on Σ , which by our assumptions admits a continuous extension at γ , still denoted by ν_Σ . We also set $\nu_\Gamma = -\nu_{\mathbf{C}}$ on Γ . Note that the Lipschitz regularity of Ω yields

$$\nu_\Gamma \cdot \nu_\Sigma > -1 \quad \text{on } \gamma.$$

Finally, we define the ε -neighborhood of a generic set $F \subset \mathbb{R}^N$ as $(F)_\varepsilon := B_\varepsilon + F = \{x \in \mathbb{R}^N : \text{dist}(x, F) < \varepsilon\}$.

2.1. The Neumann problem. In this section we consider the Neumann problem (1.3) under the assumptions (2.1) and (2.2). In addition, we assume that Ω is connected. Note that even in this case Ω is still merely a Lipschitz domain and therefore the high regularity of u up to the boundary is not granted. However, it turns out that we only need the solution to attain the boundary values in the viscosity sense, for which Hölder continuity up to the boundary is enough. To this aim we consider the variational solution of the problem (1.3) which by definition is a function $u \in H^1(\Omega)$ such that it holds

$$\int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = - \int_{\Omega} c \varphi \, dx + \int_{\partial\Omega} g \varphi \, d\mathcal{H}^{N-1}$$

for all $\varphi \in H^1(\Omega)$, where c is given by (1.4) and

$$g \equiv 1 \text{ on } \Sigma \quad \text{and} \quad g \equiv -\lambda \quad \text{on } \Gamma \setminus \gamma. \quad (2.3)$$

Since Ω is bounded and Lipschitz regular, g is bounded and we have the compatibility condition (1.4) which can be rewritten as

$$c|\Omega| = \int_{\partial\Omega} g d\mathcal{H}^{N-1},$$

there exists a unique (up to an additive constant) variational solution of (1.3). Moreover, by standard elliptic regularity theory the variational solution is Hölder continuous up to the boundary, see for instance [23].

Let us proceed to the notion of viscosity solution. In fact, since we need only the concept of viscosity supersolution for the ABP-argument, we reduce to that. Here is the definition we need.

Definition 2.1. *A lower semicontinuous function $u : \overline{\Omega} \rightarrow \mathbb{R}$ is a viscosity supersolution of (1.3) if whenever $u - \varphi$ has a local minimum at $x_0 \in \overline{\Omega}$ for $\varphi \in C^2(\mathbb{R}^N)$, then*

$$\begin{cases} -\Delta\varphi(x_0) \geq -c & \text{if } x_0 \in \Omega, \\ \partial_{\nu_\Sigma}\varphi(x_0) - 1 \geq 0 & \text{if } x_0 \in \Sigma \setminus \gamma, \\ \partial_{\nu_\Gamma}\varphi(x_0) + \lambda \geq 0 & \text{if } x_0 \in \Gamma \setminus \gamma, \\ \max\{\partial_{\nu_\Sigma}\varphi(x_0) - 1, \partial_{\nu_\Gamma}\varphi(x_0) + \lambda\} \geq 0 & \text{if } x_0 \in \gamma, \end{cases}$$

where $\partial_{\nu_\Sigma}\varphi(x_0) = \nabla\varphi(x_0) \cdot \nu_\Sigma(x_0)$ and $\partial_{\nu_\Gamma}\varphi(x_0) = \nabla\varphi(x_0) \cdot \nu_\Gamma(x_0)$.

We can now prove the main result of the section.

Proposition 2.2. *Let Ω , $\mathbf{C}$ be as in (2.1) and (2.2). Then the variational solution of (1.3) is a viscosity supersolution in the sense of Definition 2.1.*

Proof. Since the equation is satisfied classically in Ω and the Neumann boundary conditions are achieved in a classical sense at $\partial\Omega \setminus \gamma$, it will be enough to check the property on γ . To this aim, assume that $\varphi \in C^2(\mathbb{R}^N)$ and $x_0 \in \gamma$ are such that $(u - \varphi)(x) \geq 0$, with equality achieved only at x_0 . We start by showing that

$$\max\{-\Delta\varphi(x_0) + c, \partial_{\nu_\Sigma}\varphi(x_0) - 1, \partial_{\nu_\Gamma}\varphi(x_0) + \lambda\} \geq 0. \quad (2.4)$$

We argue by contradiction, assuming that

$$\max\{-\Delta\varphi(x_0) + c, \partial_{\nu_\Sigma}\varphi(x_0) - 1, \partial_{\nu_\Gamma}\varphi(x_0) + \lambda\} < 0.$$

By continuity we may find a small ball $B_r(x_0)$ such that

$$-\Delta\varphi + c < 0 \quad \text{in } B_r(x_0) \quad \text{and} \quad \partial_\nu\varphi - g < 0 \quad \text{on } (\partial\Omega \cap B_r(x_0)) \setminus \gamma,$$

with g defined in (2.3). Then, setting $w := u - \varphi$, $h := -c + \Delta\varphi$, $f = \partial_\nu(u - \varphi) = g - \partial_\nu\varphi$, we have that w is a variational solution of

$$\begin{cases} -\Delta w = h & \text{in } \Omega \cap B_r(x_0), \\ \partial_\nu w = f & \text{in } \partial\Omega \cap B_r(x_0), \end{cases}$$

that is,

$$\int_{\Omega} \nabla w \cdot \nabla \psi \, dx = \int_{\Omega} h \psi \, dx + \int_{\partial\Omega} f \psi \, d\mathcal{H}^{N-1} \quad (2.5)$$

for all $\psi \in H^1(\Omega)$ with $\psi = 0$ in $\Omega \setminus B_r(x_0)$. Let us now choose $\psi := \min\{w - \varepsilon, 0\}$ and note that for $\varepsilon > 0$ small enough

$$\psi = \min\{w - \varepsilon, 0\} = 0 \quad \text{in } \Omega \setminus B_r(x_0).$$

Then, (2.5) combined with the fact that $h > 0$ in $\Omega \cap B_r(x_0)$ and $f > 0$ in $\partial\Omega \cap B_r(x_0)$, yield

$$\int_{\Omega} |\nabla(\min\{w - \varepsilon, 0\})|^2 \, dx = \int_{\Omega \cap B_r(x_0)} h \psi \, dx + \int_{\partial\Omega \cap B_r(x_0)} f \psi \, d\mathcal{H}^{N-1} \leq 0$$

and in turn $\min\{w - \varepsilon, 0\} = 0$ in Ω . This is impossible since $w - \varepsilon < 0$ in a neighborhood of x_0 . Thus (2.4) is established.

The inequality (2.4) is not good enough, since we only want information on the boundary. We thus claim that in fact

$$\max\{\partial_{\nu_{\Sigma}}\varphi(x_0) - 1, \partial_{\nu_{\Gamma}}\varphi(x_0) + \lambda\} \geq 0. \quad (2.6)$$

To this aim we observe that for any $x_0 \in \gamma$ there exists a ball $B_r(\bar{x}) \subset \mathbb{R}^N \setminus \overline{\Omega}$ with $x_0 \in \partial B_r(\bar{x})$ (it is enough to take a ball contained in $\mathbf{C}$ and tangent to x_0 , which is possible by the C^2 assumption on $\mathbf{C}$). By translating and dilating we may assume for simplicity that $\bar{x} = 0$ and that $r = 1$. We perturb the test function φ by a functions $\psi_q \in C^2(\mathbb{R}^n \setminus \{0\})$ that we define as

$$\psi_q(x) = \frac{1}{q^{3/2}}(|x|^{-q} - 1)$$

where $q > 0$ is a large number to be chosen. Then by the exterior ball condition we have that $\psi_q(x) \leq 0$ for $x \in \Omega$ while since $x_0 \in \partial B_1$ it holds $\psi_q(x_0) = 0$. Moreover by a direct computation we see that

$$\Delta\psi_q(x_0) \geq \frac{\sqrt{q}}{2} \quad \text{and} \quad |\nabla\psi_q(x_0)| = \frac{1}{\sqrt{q}},$$

for q sufficiently large. We define a new test function

$$\varphi_q(x) = \varphi(x) + \psi_q(x).$$

By construction it holds $\varphi_q \leq \varphi$ in Ω and $\varphi_q(x_0) = \varphi(x_0)$, hence x_0 is still a local minimum for $u - \varphi_q$. Thus

$$-\Delta\varphi_q(x_0) + c \leq -\frac{\sqrt{q}}{2} - \Delta\varphi(x_0) + c < 0$$

when q is large. Therefore, for q large, from (2.4) we obtain (2.6) with φ replaced by φ_q . Finally, letting $q \rightarrow \infty$, since $\nabla\varphi_q(x_0) \rightarrow \nabla\varphi(x_0)$ we obtain (2.6). This concludes the proof. $\square$

2.2. Subdifferentials and restricted subdifferentials. We need some notation in order to proceed. Given $X \subset \mathbb{R}^N$, a function $u : X \rightarrow \mathbb{R}$, a subset $Y \subset X$ and a point $x \in Y$ we define the subdifferential $J_Y u(x)$ as

$$J_Y u(x) := \{ \xi \in \mathbb{R}^N : u(y) - u(x) \geq \xi \cdot (y - x) \text{ for all } y \in Y \}.$$

We note that Y may even be a discrete set.

Remark 2.3. *Note that if Y is compact and u is continuous in Y , then*

$$\bigcup_{x \in Y} J_Y u(x) = \mathbb{R}^N.$$

Indeed, for any $\xi \in \mathbb{R}^N$, we may find $c > 0$ so large that $\sup_{x \in Y} (-c + \xi \cdot x - u(x)) < 0$. Setting

$$t_0 := \sup \{ t > 0 : -c + \xi \cdot x + t < u(x) \text{ for all } x \in Y \},$$

we clearly have $-c + \xi \cdot \bar{x} + t_0 = u(\bar{x})$ for some $\bar{x} \in Y$ and $-c + \xi \cdot y + t_0 \leq u(y)$ for all $y \in Y$; that is, $\xi \in J_Y u(\bar{x})$.

Moreover, note that for any $x, x' \in Y$ the intersection $J_Y(x) \cap J_Y(x')$ is contained in a hyperplane orthogonal to $x' - x$.

Let Ω , Γ and Σ be as in (2.1) and (2.2). As we will see, in order to prove the crucial inequality (1.6), it turns out that all relevant information of u (the solution of (1.3)) is contained in its restriction to Γ . In fact, it has to do with the boundary conditions rather than with the PDE. To this aim, we define the following union of *restricted subdifferentials* for functions defined on compact subsets of the boundary of $\mathbf{C}$, $v : K \subset \partial \mathbf{C} \rightarrow \mathbb{R}$

$$\mathcal{B}_v^\lambda := \bigcup_{x \in K} \{ \xi \in J_K v(x) : |\xi| < 1 \text{ and } \xi \cdot \nu_{\mathbf{C}}(x) > \lambda \}, \quad (2.7)$$

where $\lambda \in (-1, 1)$. The following crucial lemma allows us to rewrite the information on the Neumann boundary conditions in (1.3) in terms of a condition on the restricted subdifferentials just introduced.

Lemma 2.4. *Let Ω , Σ and $\mathbf{C}$ be as in (2.1) and (2.2). Let u be the variational solution of (1.3). Then, denoting by u_Γ the restriction of u to Γ , it holds*

$$\mathcal{B}_{u_\Gamma}^\lambda \subset \nabla u(\widehat{\Omega})$$

where $\mathcal{B}_{u_\Gamma}^\lambda$ is as in (2.7), with $v = u_\Gamma$, and $\widehat{\Omega}$ is defined in (1.5).

Proof. Recall that u is continuous up to the boundary and by Proposition 2.2 it is a viscosity supersolution of (1.3) in the sense of Definition 2.1. Fix $\xi \in \mathcal{B}_{u_\Gamma}^\lambda$. Then $\xi \in J_\Gamma u(x_0)$, for some $x_0 \in \Gamma$, $|\xi| < 1$ and $\xi \cdot \nu_{\mathbf{C}}(x_0) > \lambda$. We need to show that $\xi \in \nabla u(\widehat{\Omega})$, which by (1.5) means that $\xi \in J_\Omega u(\bar{x})$ for some $\bar{x} \in \Omega$.

First we claim that $\xi \notin J_{\overline{\Omega}} u(x_0)$. Indeed, assume the opposite, that is

$$u(y) \geq u(x_0) + \xi \cdot (y - x_0) =: \varphi(y) \quad \text{for all } y \in \overline{\Omega}.$$

In particular, x_0 is the minimum point of $u - \varphi$ and since u is a viscosity supersolution and $\nabla\varphi(x_0) = \xi$, see Definition 2.1, we have

$$\begin{cases} \xi \cdot \nu_\Gamma(x_0) + \lambda \geq 0 & \text{if } x_0 \in \Gamma \setminus \gamma, \\ \max\{\xi \cdot \nu_\Sigma(x_0) - 1, \xi \cdot \nu_\Gamma(x_0) + \lambda\} \geq 0 & \text{if } x_0 \in \gamma. \end{cases}$$

From the above condition, since $|\xi| < 1$ we have that $\xi \cdot \nu_\Gamma(x_0) + \lambda \geq 0$, which is impossible since $\xi \in \mathcal{B}_{u_\Gamma}^\lambda$ and so $\xi \cdot \nu_\Gamma(x_0) = -\xi \cdot \nu_\mathbf{C}(x_0) < -\lambda$. Therefore $\xi \notin J_{\overline{\Omega}}u(x_0)$.

By the above it holds that the inequality $u(y) \geq \varphi(y)$ is not true for all $y \in \overline{\Omega}$. This means that $\bar{c} = \min_{y \in \overline{\Omega}} (u(y) - \varphi(y)) < 0$. In turn we have that the graph of $\varphi + \bar{c}$ touches from below the graph of u at some point $\bar{x} \in \overline{\Omega}$. Clearly $\bar{x} \notin \Gamma$, since $\xi \in J_\Gamma u(x_0)$ and $\bar{c} < 0$ imply $\varphi(y) + \bar{c} < u(y)$ for all $y \in \Gamma$. On the other hand, if $\bar{x} \in \Sigma$, again by the supersolution property, we would have that $\xi \cdot \nu_\Sigma(\bar{x}) \geq 1$, which is impossible as $|\xi| < 1$. Therefore $\bar{x} \in \Omega$, which implies $\xi = \nabla u(\bar{x})$, as desired. $\square$

3. THE MAIN GEOMETRIC ESTIMATE

By Lemma 2.4 we know that if $|\mathcal{B}_{u_\Gamma}^\lambda| \geq |B^\lambda|$, where u_Γ is the restriction of u on Γ and $\mathcal{B}_{u_\Gamma}^\lambda$ is as in (2.7), with $v = u_\Gamma$, then we have inequality (1.6). As we will show in this section, the above estimate on restricted subdifferentials holds in fact for generic continuous functions $v : K \rightarrow \mathbb{R}$, where K is any compact subset of $\mathbf{C}$ and not just for u_Γ . This in turn means that such a property pertains solely to the convexity of the set $\mathbf{C}$. Precisely, we will show the following.

Theorem 3.1. *Let $\mathbf{C} \subset \mathbb{R}^N$ be any closed convex set with nonempty interior and of class C^1 and let $\lambda \in (-1, 1)$. If $K \subset \partial\mathbf{C}$ is compact and $v : K \rightarrow \mathbb{R}$ is any continuous function, then*

$$|\mathcal{B}_v^\lambda| \geq |B^\lambda|, \quad (3.1)$$

where $B^\lambda = \{x \in B_1 : x \cdot e_N > \lambda\}$ and $\mathcal{B}_v^\lambda$ is defined in (2.7).

It turns out that the crucial property in establishing the above estimate is the following *half-line property* satisfied by the subdifferentials of functions defined on subsets of $\partial\mathbf{C}$, which is a consequence of the mere convexity of $\mathbf{C}$.

Lemma 3.2. *Let $\mathbf{C} \subset \mathbb{R}^N$ be a closed convex set with nonempty interior and of class C^1 , let $K \subset \partial\mathbf{C}$ and let $v : K \rightarrow \mathbb{R}$. If $\xi \in J_K v(x)$ for some $x \in K$, then*

$$\xi + t\nu_\mathbf{C}(x) \in J_K v(x) \quad \text{for all } t > 0.$$

Proof. If $\xi \in J_K v(x)$, then

$$v(y) - v(x) \geq \xi \cdot (y - x) \quad \text{for all } y \in K.$$

By convexity it holds $\nu_\mathbf{C}(x) \cdot (y - x) \leq 0$ for all $y \in K$. Therefore for any $t > 0$ it holds

$$v(y) - v(x) \geq (\xi + t\nu_\mathbf{C}(x)) \cdot (y - x) \quad \text{for all } y \in K,$$

that is $\xi + t\nu_{\mathbf{C}}(x) \in J_K v(x)$. $\square$

The next useful observation is that by discretization we can reduce the proof of (3.1) to the case of functions v defined on finite subsets of $\partial\mathbf{C}$, as shown in the following lemma. To this aim we recall that a sequence $\{C_n\}$ of closed sets of $\mathbb{R}^N$ converges in the *Kuratowski sense* to a closed set C if the following conditions are satisfied:

- (i) if $x_n \in C_n$ for every n , then any limit point of $\{x_n\}$ belongs to C ;
- (ii) any $x \in C$ is the limit of a sequence $\{x_n\}$ with $x_n \in C_n$.

One can easily see that $C_n \rightarrow C$ in the sense of Kuratowski if and only if $\text{dist}(\cdot, C_n) \rightarrow \text{dist}(\cdot, C)$ locally uniformly in $\mathbb{R}^N$.

Lemma 3.3. *Let $\mathbf{C} \subset \mathbb{R}^N$ be a closed convex set with nonempty interior and with C^1 boundary, and fix $\lambda \in (-1, 1)$. Assume that (3.1) holds for all $v : K \rightarrow \mathbb{R}$, whenever $K \subset \partial\mathbf{C}$ is finite. Then, (3.1) holds for all continuous $v : K \rightarrow \mathbb{R}$, whenever $K \subset \partial\mathbf{C}$ is compact.*

Proof. Fix a compact subset K of $\partial\mathbf{C}$ and a continuous function $v : K \rightarrow \mathbb{R}$. Let $K_n \subset K$ be a sequence of discrete sets such that $K_n \rightarrow K$ in the Kuratowski sense. We define the function $v_n : K_n \rightarrow \mathbb{R}$ as the restriction of v on K_n , i.e., $v_n(x) = v(x)$ for $x \in K_n$.

We claim that

$$\chi_{\mathcal{B}_v^\lambda} \geq \limsup_n \chi_{\mathcal{B}_{v_n}^{\lambda'}} \quad \text{pointwise in } B_1 \quad \forall \lambda' \in (\lambda, 1). \quad (3.2)$$

We argue by contradiction, by assuming that for some $\lambda' \in (\lambda, 1)$ and for some $\xi \in B_1 \setminus \mathcal{B}_v^\lambda$, along a (non relabelled) subsequence $\xi \in \mathcal{B}_{v_n}^{\lambda'}$; i.e., there exist $x_n \in K_n$ such that $\xi \in J_{K_n} v_n(x_n)$ and $\xi \cdot \nu_{\mathbf{C}}(x_n) > \lambda'$. Up to extracting a further (non-relabelled) subsequence we may assume that $x_n \rightarrow x$ for some $x \in K$. Clearly, we also have $\xi \cdot \nu_{\mathbf{C}}(x_n) \rightarrow \xi \cdot \nu_{\mathbf{C}}(x)$, so that

$$\xi \cdot \nu_{\mathbf{C}}(x) \geq \lambda'. \quad (3.3)$$

Fix now $y \in K$. Then, by Kuratowski convergence there exists a sequence $y_n \rightarrow y$ such that $y_n \in K_n$ for all n . Since $\xi \in J_{K_n} v_n(x_n)$, we have

$$v_n(y_n) \geq v_n(x_n) + \xi \cdot (y_n - x_n).$$

Passing to the limit and noticing that $v_n(x_n) \rightarrow v(x)$ and $v_n(y_n) \rightarrow v(y)$, we infer $\xi \in J_K v(x)$. Recalling (3.3), we get in particular $\xi \in \mathcal{B}_v^\lambda$, which is a contradiction. This establishes (3.2), which in turn implies

$$|\mathcal{B}_v^\lambda| \geq \limsup_n |\mathcal{B}_{v_n}^{\lambda'}| \quad \forall \lambda' \in (\lambda, 1), \quad (3.4)$$

by Fatou's lemma.

By assumption for any n we have the estimate

$$|\mathcal{B}_{v_n}^\lambda| \geq |B^\lambda|. \quad (3.5)$$

Let us then prove that there exists a constant C independent of n such that

$$|\mathcal{B}_{v_n}^{\lambda'}| \geq |\mathcal{B}_{v_n}^{\lambda}| - C(\lambda' - \lambda) \quad \text{for all } \lambda' \in (\lambda, \lambda + \delta), \quad (3.6)$$

where $\delta = (1 - \lambda)/4$. For any n and $t \in (0, 1 - \lambda)$ we define A_t (dropping the dependence on n) as

$$A_t := \bigcup_{x \in K_n} \{\xi \in J_{K_n} v_n(x) : \xi \cdot \nu_{\mathbf{C}}(x) = \lambda + t\}$$

and we set for $r > 0$

$$\mathcal{B}_{v_n, r}^t := \bigcup_{x \in K_n} \mathcal{B}_{v_n, r}^t(x) = \bigcup_{x \in K_n} \{\xi \in J_{K_n} v_n(x) : |\xi| < r \text{ and } \xi \cdot \nu_{\mathbf{C}}(x) > t\}.$$

For any $x \in K_n$ the set $\mathcal{B}_{v_n, r}^t(x)$ is the intersection of a fixed bounded set with the half space $\{\xi \cdot \nu_{\mathbf{C}}(x) > t\}$. As a consequence the function $t \mapsto |\mathcal{B}_{v_n, r}^{\lambda+t}|$ is decreasing and Lipschitz continuous (with constant possibly depending on the cardinality of K_n). Moreover, at points of differentiability it holds

$$\frac{d}{dt} |\mathcal{B}_{v_n, r}^{\lambda+t}| = -\mathcal{H}^{N-1}(A_t \cap B_r) \quad (3.7)$$

for all $r > 0$.

The idea is to show that in fact the Lipschitz constant is independent of n . To this aim, we fix $x \in K_n$ and study the set

$$A_t(x) = \{\xi \in J_{K_n} v_n(x) : \xi \cdot \nu_{\mathbf{C}}(x) = \lambda + t\}.$$

Assume that $\xi \in A_t(x)$, so in particular $\xi \in J_{K_n} v_n(x)$. By Lemma 3.2 for every $s > 0$ it holds $\xi + s\nu_{\mathbf{C}}(x) \in J_{K_n} v_n(x)$, so $\xi + s\nu_{\mathbf{C}}(x) \in A_{t+s}(x)$. Therefore for every $s > 0$ it holds

$$A_t(x) + s\nu_{\mathbf{C}}(x) \subset A_{t+s}(x).$$

Then, it holds $\mathcal{H}^{N-1}(A_t(x) \cap B_{1+t}) \leq \mathcal{H}^{N-1}(A_{t+s}(x) \cap B_{1+t+s})$. Therefore, the function

$$t \mapsto \mathcal{H}^{N-1}(A_t \cap B_{1+t}) = \sum_{x \in K_n} \mathcal{H}^{N-1}(A_t(x) \cap B_{1+t})$$

is non-decreasing. Notice that the equality holds true since for any $x, x' \in K_n$ one has $\mathcal{H}^{N-1}(A_t(x) \cap A_t(x')) = 0$. Indeed by Remark 2.3 $A_t(x) \cap A_t(x') \subset J_{K_n} v_n(x) \cap J_{K_n} v_n(x')$ is contained in a hyperplane orthogonal to $x' - x$ and neither $\nu_{\mathbf{C}}(x)$ nor $\nu_{\mathbf{C}}(x')$ can be parallel to $x' - x$. Integrating (3.7) we have

$$\int_{\delta}^{2\delta} \mathcal{H}^{N-1}(A_t \cap B_{1+t}) dt \leq \int_{\delta}^{2\delta} \mathcal{H}^{N-1}(A_t \cap B_{1+2\delta}) dt = - \int_{\delta}^{2\delta} \frac{d}{dt} |\mathcal{B}_{v_n, 1+2\delta}^{\lambda+t}| dt \leq |B_2|.$$

By the mean value theorem there is $\hat{t} \in [\delta, 2\delta]$ such that $\mathcal{H}^{N-1}(A_{\hat{t}} \cap B_{1+\hat{t}}) \leq C = 2^{N+2} \Omega_N / (1 - \lambda)$. In turn, using the monotonicity obtained above, we have

$$\mathcal{H}^{N-1}(A_t \cap B_1) \leq C \quad \text{for all } t \in [0, \delta].$$

The claim (3.6) then follows by integrating (3.7) from $t = 0$ to $t = \lambda' - \lambda$. Finally the statement of the lemma follows from (3.4), (3.5) and (3.6) and letting $\lambda' \rightarrow \lambda$. $\square$

Thanks to the previous result we are reduced to consider discrete functions defined on finite subsets $K \subset \partial \mathbf{C}$. In this situation, there are only finitely many subdifferentials $J_K v(x)$ and it is also immediate that they are essentially disjoint, i.e., they have disjoint interiors. Hence, by Remark 2.3 and Lemma 3.2 the subdifferentials $J_K v(x)$ for $x \in K$ form a finite partition of the space $\mathbb{R}^N$ made of sets having the half-line property. As we will see, Theorem 3.1 follows from these properties only, but the argument is rather involved in the general case. However, the situation becomes particularly simple in the Choe-Ghomi-Ritoré case $\lambda = 0$.

Proof of Theorem 3.1 for $\lambda = 0$. By Lemma 3.3 it is enough to consider functions defined on finite sets. Let $K = \{x_1, \dots, x_n\}$ be any finite subset of $\partial \mathbf{C}$ and let $v : K \rightarrow \mathbb{R}$. Recall that $\bigcup_{i=1}^n J_K v(x_i) = \mathbb{R}^N$ and that the sets $J_K v(x_i)$ have disjoint interiors. Moreover Lemma 3.2 implies that they have the half-line property

$$\xi \in J_K v(x_i) \implies \xi + t\nu_{\mathbf{C}}(x_i) \in J_K v(x_i) \text{ for all } t > 0. \quad (3.8)$$

Now, up to a set of Lebesgue measure zero, we may split $J_K v(x_i) = J_K v(x_i)^+ \cup J_K v(x_i)^-$, where

$$J_K v(x_i)^\pm := \{\xi \in J_K v(x_i) : \pm \xi \cdot \nu_{\mathbf{C}}(x_i) > 0\}.$$

Let us then fix $\xi \in J_K v(x_i)^-$ and denote $\tau = -\xi \cdot \nu_{\mathbf{C}}(x_i) > 0$. Then by (3.8) the symmetric point $\hat{\xi} = \xi + 2\tau\nu_{\mathbf{C}}(x_i)$ belongs to $J_K v(x_i)^+$ and has the same norm as ξ . Hence $|J_K v(x_i)^+ \cap B_1| \geq \frac{1}{2}|J_K v(x_i) \cap B_1|$. In turn,

$$|\mathcal{B}_v^0| = \sum_{i=1}^n |J_K v(x_i)^+ \cap B_1| \geq \frac{1}{2} \sum_{i=1}^n |J_K v(x_i) \cap B_1| = \frac{1}{2}|B_1|, \quad (3.9)$$

which is the desired estimate. $\square$

Remark 3.4. *Combining the previous proof with the ABP argument sketched in the introduction and rigorously developed in the next section, we recover an ABP-proof of the relative isoperimetric inequality outside convex sets obtained by Choe, Ghomi and Ritoré in [8]. Note that an ABP-argument for the same inequality has been already provided in [19]. However, in their argument our crucial estimate (3.9) is replaced by a geometric estimate based on normal cones and inspired by the techniques in [7], see [19, Proposition 2.4].*

We turn to the general case. To this aim let $K = \{x_1, \dots, x_n\}$ be any finite subset of $\mathbf{C}$ and let $v : K \rightarrow \mathbb{R}$ be any function. It is obvious that Theorem 3.1 follows if we show that

$$\mathcal{H}^{N-1}\left(\bigcup_{x \in K} \{\xi \in \partial B_\varrho \cap J_K v(x) : \xi \cdot \nu_{\mathbf{C}}(x) > \lambda\}\right) \geq \mathcal{H}^{N-1}(\{\xi \in \partial B_\varrho : \xi \cdot e_N > \lambda\}),$$

for all $\varrho \in (0, 1)$. By rescaling, this inequality is equivalent to

$$\mathcal{H}^{N-1}\left(\bigcup_{x \in K} \{\xi \in \partial B_1 \cap J_K v_\varrho(x) : \xi \cdot \nu_{\mathbf{C}}(x) > \lambda'\}\right) \geq \mathcal{H}^{N-1}(\{\xi \in \partial B_1 : \xi \cdot e_N > \lambda'\}), \quad (3.10)$$

where we have set $\lambda' = \frac{\lambda}{\varrho} \in \mathbb{R}$, $v_\varrho = \frac{v}{\varrho}$. Note that (3.10) is trivially satisfied if $\lambda' \geq 1$ or $\lambda' \leq -1$.

Therefore, we are ultimately bound to show that for any function $v : K \rightarrow \mathbb{R}$ and any finite set $K \subset \partial\mathbf{C}$ we have

$$\mathcal{H}^{N-1}\left(\bigcup_{x \in K} \{\xi \in \partial B_1 \cap J_K v(x) : \xi \cdot \nu_{\mathbf{C}}(x) > \lambda\}\right) \geq \mathcal{H}^{N-1}(\{\xi \in \partial B_1 : \xi \cdot e_N > \lambda\}). \quad (3.11)$$

In the case $\lambda < 0$, it will be in fact be convenient to rewrite the previous inequality as

$$\mathcal{H}^{N-1}\left(\bigcup_{x \in K} \{\xi \in \partial B_1 \cap J_K v(x) : \xi \cdot \nu_{\mathbf{C}}(x) < \lambda\}\right) \leq \mathcal{H}^{N-1}(\{\xi \in \partial B_1 : \xi \cdot e_N < \lambda\}) \quad (3.12)$$

as $\mathcal{H}^{N-1}\left(\bigcup_{x \in K} \{\xi \in \partial B_1 \cap J_K v(x) : \xi \cdot \nu_{\mathbf{C}}(x) = \lambda\}\right) = 0$.

Proof of Theorem 3.1. From the previous discussion it is enough to consider functions defined on finite subsets of $\partial\mathbf{C}$. Let us fix $K = \{x_1, x_2, \dots, x_n\} \subset \partial\mathbf{C}$ and $v : K \rightarrow \mathbb{R}$. Then, keeping in mind Remark 2.3, we have that the sets A_i defined as

$$A_i := \text{int}\{\xi \in \mathbb{R}^N : \xi \in J_K v(x_i)\}$$

are disjoint and

$$\mathcal{N} := \mathbb{R}^N \setminus \bigcup_i^n A_i \text{ satisfies } |\mathcal{N}| = 0. \quad (3.13)$$

Moreover, as they are obtained as a finite intersection of open half-spaces, we have

$$\mathcal{H}^{N-1}(\partial A_i \cap \partial B_\varrho) = 0 \quad \text{for all } i = 1, \dots, n \text{ and all } \varrho > 0. \quad (3.14)$$

Thus, for any $\xi \in \bigcup_i A_i$ we can define the associated normal

$$\nu(\xi) := \nu_{\mathbf{C}}(x_i),$$

where i is such that $\xi \in A_i$. Recall that by the half-line property stated in Lemma 3.2, we have

$$\xi \in A_i \Rightarrow \xi + t\nu(\xi) \in A_i \quad \text{for all } t > 0. \quad (3.15)$$

For $\lambda \in (-1, 1)$ denote $r = r(\lambda) := \sqrt{1 - \lambda^2}$ and define

$$\partial B_r^- = \{\xi \in \partial B_r \setminus \mathcal{N} : \xi \cdot \nu(\xi) < 0\} \quad \text{and} \quad \partial B_r^+ = \{\xi \in \partial B_r \setminus \mathcal{N} : \xi \cdot \nu(\xi) > 0\}$$

Now observe that, by the half-line property, for every $\xi \in \mathbb{R}^N \setminus \mathcal{N}$ we have

$$\nu(\xi + t\nu(\xi)) = \nu(\xi) \quad \forall t > 0. \quad (3.16)$$

In particular, if $\xi \in \partial B_r^-$, then the symmetric point $\hat{\xi} := \xi - 2(\xi \cdot \nu(\xi))\nu(\xi) \in \partial B_r^+$. Thus, $\mathcal{H}^{N-1}(\partial B_r^-) \leq \mathcal{H}^{N-1}(\partial B_r^+)$ and in turn, since $\mathcal{H}^{N-1}(\{\xi \in \partial B_r \setminus \mathcal{N} : \xi \cdot \nu(\xi) = 0\}) = 0$, we conclude that

$$\mathcal{H}^{N-1}(\partial B_r^-) \leq \frac{1}{2} \mathcal{H}^{N-1}(\partial B_r) \leq \mathcal{H}^{N-1}(\partial B_r^+). \quad (3.17)$$

We now distinguish the two cases $\lambda < 0$ and $\lambda > 0$.

The case $\lambda < 0$. Let us set

$$S_\lambda := \{\xi \in \partial B_1 \setminus \mathcal{N} : \xi \cdot \nu(\xi) < \lambda\}.$$

and

$$\varphi(\lambda) := \mathcal{H}^{N-1}(\{\xi \in \partial B_1 : \xi \cdot e_N < \lambda\}).$$

By the discussion above it is enough to prove (3.12), which may be rewritten as

$$\mathcal{H}^{N-1}(S_\lambda) \leq \varphi(\lambda) \quad \text{for all } \lambda \in (-1, 0).$$

In fact, as mentioned in the introduction, we will show that the above inequality holds for any finite collection of open sets (A_i) satisfying (3.13), (3.14), and (3.15).

Let us fix $\varepsilon > 0$ and prove that for all $\lambda \in (-1, 0)$ it holds

$$\mathcal{H}^{N-1}(S_\lambda) < \varphi(\lambda) + \varepsilon. \quad (3.18)$$

By continuity (3.18) holds when λ is close to -1 . Let us assume by contradiction that there exists $\lambda < 0$ such that

$$\mathcal{H}^{N-1}(S_\lambda) = \varphi(\lambda) + \varepsilon \quad \text{and} \quad \mathcal{H}^{N-1}(S_{\lambda'}) < \varphi(\lambda') + \varepsilon \quad \text{for all } \lambda' \in [-1, \lambda). \quad (3.19)$$

We define the map $\Psi : S_\lambda \rightarrow \partial B_r^-$,

$$\Psi(\xi) = \xi + \left(-\sqrt{r^2 - 1 + (\xi \cdot \nu(\xi))^2} - (\xi \cdot \nu(\xi)) \right) \nu(\xi).$$

The geometric meaning of the map is the following: let σ_ξ be the half-line $\{\xi + t\nu(\xi) : t > 0\}$. Then $\xi \in S_\lambda$ if and only if σ_ξ intersects ∂B_r at two points (see Figure 3), and in this case $\Psi(\xi)$ is the intersection point closer to ξ .

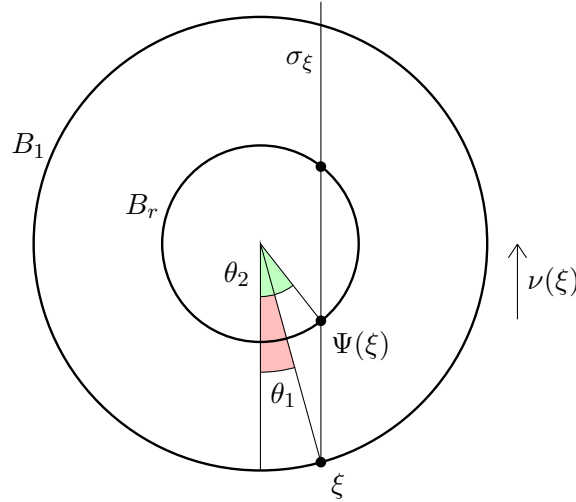


FIGURE 3. The meaning of the radius r and the definition of Ψ . Note that $\xi \cdot \nu(\xi) < \lambda < 0$ if and only if the half-line σ_ξ intersects B_r . Note also that $J_\Psi^{N-1}(\xi) = \cos(\theta_1)/\cos(\theta_2)$.

An elementary calculation shows that indeed $|\Psi(\xi)| = r$ and that, also by (3.16),

$$\Psi(\xi) \cdot \nu(\Psi(\xi)) = \Psi(\xi) \cdot \nu(\xi) < 0$$

for all $\xi \in S_\lambda$.

Note also that Ψ is smooth and thus we may calculate its tangential $(N-1)$ -Jacobian J_Ψ^{N-1} . By a tedious but straightforward computation, recalling also that $\xi \mapsto \nu(\xi)$ is locally constant, we get for all $\xi \in S_\lambda$

$$J_\Psi^{N-1}(\xi) = \frac{r|\xi \cdot \nu(\xi)|}{\sqrt{r^2 - 1 + (\xi \cdot \nu(\xi))^2}}.$$

For a “geometric” interpretation of this fomula, see Figure 3.

We now compute $\mathcal{H}^{N-1}(\Psi(S_\lambda))$ using the Area formula (see for instance [2, Theorem 2.91]):

$$\begin{aligned} \mathcal{H}^{N-1}(\Psi(S_\lambda)) &= \int_{S_\lambda} J_\Psi^{N-1} d\mathcal{H}^{N-1} = \int_{S_\lambda} \frac{r|\xi \cdot \nu(\xi)|}{\sqrt{r^2 - 1 + (\xi \cdot \nu(\xi))^2}} d\mathcal{H}^{N-1} \\ &= \int_0^\infty \mathcal{H}^{N-1}\left(\left\{\xi \in S_\lambda : \frac{r|\xi \cdot \nu(\xi)|}{\sqrt{r^2 - 1 + (\xi \cdot \nu(\xi))^2}} \geq t\right\}\right) dt, \end{aligned}$$

where the last equality follows by the layer-cake formula.

Notice that $\frac{r|\xi \cdot \nu(\xi)|}{\sqrt{r^2 - 1 + (\xi \cdot \nu(\xi))^2}} \geq 1$ for all $\xi \in S_\lambda$, while for $t > 1$

$$\frac{r|\xi \cdot \nu(\xi)|}{\sqrt{r^2 - 1 + (\xi \cdot \nu(\xi))^2}} \geq t \quad \text{is equivalent to} \quad \xi \cdot \nu(\xi) \geq -\frac{t\sqrt{1-r^2}}{\sqrt{t^2-r^2}} =: \ell(t),$$

where we used also that $\xi \cdot \nu(\xi) < 0$ for $\xi \in S_\lambda$. Recalling that $r = \sqrt{1-\lambda^2}$ we may write the latter condition as

$$\xi \cdot \nu(\xi) \geq \ell(t) = \frac{t\lambda}{\sqrt{t^2 + \lambda^2 - 1}}.$$

Note that $\ell(1) = -1$, $\lim_{t \rightarrow \infty} \ell(t) = \lambda$, and $\ell(\cdot)$ is increasing, so that $-1 < \ell(t) < \lambda$ for all $t \in (1, +\infty)$. Therefore, we have

$$\begin{aligned} \mathcal{H}^{N-1}(\Psi(S_\lambda)) &= \int_1^\infty \mathcal{H}^{N-1}\left(\left\{\xi \in S_\lambda : \xi \cdot \nu(\xi) \geq \ell(t)\right\}\right) dt + \mathcal{H}^{N-1}(S_\lambda) \\ &= \int_1^\infty \mathcal{H}^{N-1}\left(\left\{\xi \in \partial B_1 : \lambda > \xi \cdot \nu(\xi) \geq \ell(t)\right\}\right) dt + \mathcal{H}^{N-1}(S_\lambda) \quad (3.20) \\ &= \int_1^\infty \left(\mathcal{H}^{N-1}(S_\lambda) - \mathcal{H}^{N-1}(S_{\ell(t)})\right) dt + \mathcal{H}^{N-1}(S_\lambda). \end{aligned}$$

By the contradiction assumption (3.19) it holds

$$\mathcal{H}^{N-1}(S_\lambda) = \varphi(\lambda) + \varepsilon \quad \text{and} \quad \mathcal{H}^{N-1}(S_{\ell(t)}) < \varphi(\ell(t)) + \varepsilon$$

for all $t > 1$ and therefore by (3.20)

$$\mathcal{H}^{N-1}(\Psi(S_\lambda)) > \int_1^\infty (\varphi(\lambda) - \varphi(\ell(t))) dt + \varphi(\lambda) + \varepsilon. \quad (3.21)$$

A direct computation of the last integral seems to be tricky. However, we can overcome the difficulty by the following geometric argument: we pick any point $\bar{x} \in \partial \mathbf{C}$ and consider the trivial situation $v : \{\bar{x}\} \rightarrow \mathbb{R}$. Without loss of generality we may assume that $\nu_{\mathbf{C}}(\bar{x}) = e_N$. In this case, we have only one subdifferential $A_1 = \mathbb{R}^N$, the map $\nu(\xi)$ introduced above trivialises to the constant map e_N , the set S_λ and the map Ψ are replaced respectively by $\tilde{S}_\lambda = \{\xi \in \partial B_1 : \xi \cdot e_N < \lambda\}$ and

$$\tilde{\Psi}(\xi) = \xi + \left(-\sqrt{r^2 - 1 + (\xi \cdot e_N)^2} - (\xi \cdot e_N) \right) e_N.$$

It is immediate to verify that in this case $\tilde{\Psi}(\tilde{S}_\lambda) = \{\xi \in \partial B_r : \xi \cdot e_N < 0\}$ and thus, by applying (3.20) to $\tilde{\Psi}$ and $\tilde{S}_\lambda$, we get

$$\int_1^\infty (\varphi(\lambda) - \varphi(\ell(t))) dt + \varphi(\lambda) = \frac{1}{2} \mathcal{H}^{N-1}(\partial B_r).$$

Combining with (3.21) yields

$$\mathcal{H}^{N-1}(\Psi(S_\lambda)) > \frac{1}{2} \mathcal{H}^{N-1}(\partial B_r) + \varepsilon$$

which contradicts the first inequality in (3.17) as $\Psi(S_\lambda) \subset \partial B_r^-$. Hence, we have (3.18) and the conclusion follows from the arbitrariness of ε .

The case $\lambda > 0$. The argument for this case is “symmetric” to the previous one. To this aim, let us set

$$S_\lambda^+ := \{\xi \in \partial B_1 \setminus n : \xi \cdot \nu(\xi) > \lambda\}$$

and

$$\varphi^+(\lambda) := \mathcal{H}^{N-1}(\{\xi \in \partial B_1 : \xi \cdot e_N > \lambda\}).$$

Recalling (3.11), it is enough to prove that for any small $\varepsilon > 0$

$$\mathcal{H}^{N-1}(S_\lambda^+) > \varphi^+(\lambda) - \varepsilon \quad \text{for all } \lambda \in (0, 1)$$

and as before, we argue by contradiction by assuming that there exists $\lambda > 0$ such that

$$\mathcal{H}^{N-1}(S_\lambda^+) = \varphi^+(\lambda) - \varepsilon \quad \text{and} \quad \mathcal{H}^{N-1}(S_{\lambda'}^+) > \varphi^+(\lambda') - \varepsilon \quad \text{for all } \lambda' \in (\lambda, 1). \quad (3.22)$$

We define the map $\Psi^+ : S_\lambda^+ \rightarrow \partial B_r$,

$$\Psi^+(\xi) = \xi + \left(\sqrt{r^2 - 1 + (\xi \cdot \nu(\xi))^2} - (\xi \cdot \nu(\xi)) \right) \nu(\xi).$$

The geometric meaning of the map is the following: let σ_ξ^- be the half-line $\{\xi - t\nu(\xi) : t > 0\}$. Then $\xi \in S_\lambda^+$ implies that σ_ξ^- intersects ∂B_r at two points and $\Psi^+(\xi)$ is the intersection point closer to ξ . This time, we claim that

$$\partial B_r^+ \subset \Psi^+(S_\lambda^+). \quad (3.23)$$

Indeed, let $\xi' \in \partial B_r^+$. Then, the half-line $\{\xi' + t\nu(\xi') : t > 0\}$ intersects ∂B_1 at one point $\hat{\xi}$ such that $\hat{\xi} \cdot \nu(\xi') > \lambda$. By the half-line property (3.15), ξ' and $\hat{\xi}$ belong to the same subdifferential

and thus $\nu(\xi') = \nu(\widehat{\xi})$. Therefore, by the very definition of Ψ^+ we have $\Psi^+(\widehat{\xi}) = \xi'$ and thus (3.23) follows. In turn, recalling (3.17), it follows that

$$\mathcal{H}^{N-1}(\Psi^+(S_\lambda^+)) \geq \frac{1}{2} \mathcal{H}^{N-1}(\partial B_r). \quad (3.24)$$

Arguing as before, by the Area and layer-cake formulas, observing that Ψ^+ is injective, we get

$$\mathcal{H}^{N-1}(\Psi^+(S_\lambda^+)) = \int_0^\infty \mathcal{H}^{N-1}\left(\left\{\xi \in S_\lambda^+ : \frac{r \xi \cdot \nu(\xi)}{\sqrt{r^2 - 1 + (\xi \cdot \nu(\xi))^2}} \geq t\right\}\right) dt.$$

Now for $t > 1$, $\frac{r \xi \cdot \nu(\xi)}{\sqrt{r^2 - 1 + (\xi \cdot \nu(\xi))^2}} \geq t$ is equivalent to

$$\xi \cdot \nu(\xi) \leq \ell(t) := \frac{t\lambda}{\sqrt{t^2 + \lambda^2 - 1}},$$

where $\lambda < \ell(t) < 1$ for all $t \in (1, +\infty)$. Arguing as in the previous case, we may write

$$\mathcal{H}^{N-1}(\Psi^+(S_\lambda^+)) = \int_1^\infty \left(\mathcal{H}^{N-1}(S_\lambda^+) - \mathcal{H}^{N-1}(S_{\ell(t)}^+) \right) dt + \mathcal{H}^{N-1}(S_\lambda^+)$$

and, using the contradiction assumption (3.22), we arrive at

$$\mathcal{H}^{N-1}(\Psi(S_\lambda^+)) < \int_1^\infty (\varphi^+(\lambda) - \varphi^+(\ell(t))) dt + \varphi^+(\lambda) - \varepsilon = \frac{1}{2} \mathcal{H}^{N-1}(\partial B_r) - \varepsilon,$$

contradicting (3.24). $\square$

4. THE CAPILLARY ISOPERIMETRIC INEQUALITY OUTSIDE CONVEX SETS

In this section we finally prove Theorem 1.1. We will need the following approximation lemmas whose delicate and technical proofs are postponed until the final Appendix.

Lemma 4.1. *Let $\mathbf{C} \subset \mathbb{R}^N$ be a closed convex set with nonempty interior. Let $E \subset \mathbb{R}^N \setminus \mathbf{C}$ be a bounded set of finite perimeter. Then there exist a sequence of smooth closed convex sets $\mathbf{C}_n$ and a sequence of open sets $\Omega_n \subset \mathbb{R}^N \setminus \mathbf{C}_n$, satisfying the following properties:*

- (i) $\mathbf{C}_n \rightarrow \mathbf{C}$ in the Kuratowski sense;
- (ii) Ω_n is a bounded Lipschitz domain s.t. $\Sigma_n := \partial\Omega_n \setminus \mathbf{C}_n$ is a $(N-1)$ -manifold with boundary of class C^∞ ;
- (iii) $|\Omega_n \triangle E| \rightarrow 0$ as $n \rightarrow \infty$, $\partial\Omega_n \subset \left\{x : \text{dist}(x, \partial E) < \frac{1}{n}\right\}$;
- (iv) $P(\Omega_n; \mathbb{R}^N \setminus \mathbf{C}_n) \rightarrow P(E; \mathbb{R}^N \setminus \mathbf{C})$;
- (v) $\mathcal{H}^{N-1}(\partial\Omega_n \cap \mathbf{C}_n) \rightarrow \mathcal{H}^{N-1}(\partial^* E \cap \mathbf{C})$ and $\mathcal{H}^{N-1} \llcorner (\partial\Omega_n \cap \mathbf{C}_n) \xrightarrow{*} \mathcal{H}^{N-1} \llcorner (\partial^* E \cap \mathbf{C})$ in the sense of measures.

Remark 4.2. *Note that if $E \subset \mathbb{R}^N \setminus \mathbf{C}$ is a bounded set of finite perimeter and $\Omega_n, \mathbf{C}_n$ are the approximating sets given in the previous lemma, then clearly*

$$J_{\lambda, \mathbf{C}_n}(\Omega_n) \rightarrow J_{\lambda, \mathbf{C}}(E).$$

Next approximation lemma is needed to deal with the characterization of the equality case in Theorem 1.1.

Lemma 4.3. *Let $\mathbf{C} \subset \mathbb{R}^N$ be a closed convex set with nonempty interior. Let $\Omega \subset \mathbb{R}^N \setminus \mathbf{C}$ be a minimizer of problem (1.1) for some $m > 0$. Then there exist a sequence of smooth closed convex sets $\mathbf{C}_n$ and a sequence of open sets $\Omega_n \subset \mathbb{R}^N \setminus \mathbf{C}_n$, satisfying properties (i)–(v) of Lemma 4.1 with E replaced by Ω , and in addition:*

(vi) *if $x \in \partial\Omega \setminus \mathbf{C}$, $\overline{B}_r(x) \subset \mathbb{R}^N \setminus \mathbf{C}$, and $\overline{B}_r(x) \cap \Sigma_{\text{sing}} = \emptyset$, where $\Sigma_{\text{sing}} \subset \partial\Omega \setminus \mathbf{C}$ is the set of singular points of $\partial\Omega \setminus \mathbf{C}$, then $\partial\Omega_n \cap \overline{B}_r(x)$ converge to $\partial\Omega \cap \overline{B}_r(x)$ in C^∞ .*

Proof of Theorem 1.1. For simplicity, we consider here the case when $\mathbf{C}$ has nonempty interior. The extension to the general case is described in Remark 4.4. We start by showing the inequality (1.2) if $\mathbf{C}$ and $E = \Omega \subset \mathbb{R}^N \setminus \mathbf{C}$ satisfy (2.1) and (2.2), respectively, and Ω is connected.

By scaling we may assume $m = |B^\lambda|$. Let $u : \overline{\Omega} \rightarrow \mathbb{R}$ be the variational solution of the Neumann boundary problem (1.3) and denote its restriction to $\Gamma \subset \partial\mathbf{C}$ by u_Γ . Let $\mathcal{B}_{u_\Gamma}^\lambda$ be the set defined in (2.7), with $v = u_\Gamma$, and let $\widehat{\Omega}$ be the set defined in (1.5). Then by Lemma 2.4 and Theorem 3.1 we have

$$|\nabla u(\widehat{\Omega})| \geq |\mathcal{B}_{u_\Gamma}^\lambda| \geq |B^\lambda|.$$

Therefore we have

$$\begin{aligned} |\Omega| = |B^\lambda| &\leq |\nabla u(\widehat{\Omega})| \leq \int_{\widehat{\Omega}} \det \nabla^2 u \, dx \\ &\leq \int_{\widehat{\Omega}} \frac{(\Delta u)^N}{N^N} \, dx = \left(\frac{J_{\lambda, \mathbf{C}}(\Omega)}{|\Omega|N} \right)^N |\widehat{\Omega}| \leq \left(\frac{J_{\lambda, \mathbf{C}}(\Omega)}{|\Omega|N} \right)^N |\Omega|. \end{aligned} \quad (4.1)$$

Since

$$J_{\lambda, \mathbf{H}} = N|B^\lambda| = N|\Omega|,$$

the inequality (1.2) follows.

We now remove the connectedness assumption, that is we consider $\mathbf{C}$ and $E = \Omega \subset \mathbb{R}^N \setminus \mathbf{C}$ satisfying (2.1) and (2.2). Decomposing $\Omega = \cup_{i=1}^n \Omega_i$, where Ω_i are the connected components, and setting $m_i := |\Omega_i|$, we have

$$J_{\lambda, \mathbf{C}}(\Omega) = \sum_{i=1}^n J_{\lambda, \mathbf{C}}(\Omega_i) \geq \sum_{i=1}^n J_{\lambda, \mathbf{H}}(B^\lambda[m_i]) > J_{\lambda, \mathbf{H}}(B^\lambda[m]), \quad (4.2)$$

where the last inequality follows from strict concavity of the map $m \mapsto J_{\lambda, \mathbf{H}}(B^\lambda[m])$. The general case of a bounded set E of finite perimeter now follows by approximation, using Lemma 4.1 and Remark 4.2. And finally, the case of a possible unbounded set follows by approximating E in L^1 with a sequence $E_n \subset \mathbb{R}^N \setminus \mathbf{C}$ such that $P(E_n) \rightarrow P(E)$, since this implies that $P(E_n; \mathbb{R}^N \setminus \mathbf{C})$ and $\mathcal{H}^{N-1}(\partial^* E_n \cap \mathbf{C})$ converge to $P(E; \mathbb{R}^N \setminus \mathbf{C})$ and $\mathcal{H}^{N-1}(\partial^* E \cap \mathbf{C})$, respectively.

We now analyse the case of equality. Assume that E is a set of finite perimeter with $|E| = m = |B^\lambda|$ for which equality in (1.2) holds. In particular, E is a minimizer of the isoperimetric problem (1.1) and therefore it coincides (up to negligible sets) with an open set Ω , see for instance [13]. Moreover, by the same argument as in (4.2), we know that Ω is connected.

Let Ω_n and $\mathbf{C}_n$ be the two approximating sequences provided by Lemma 4.3, and denote the connected components of Ω_n as Ω_n^i , with $i = 1, 2, \dots, K_n$, where $|\Omega_n^1| \geq |\Omega_n^i|$ for every i . Let now $u_{n,i}$ be the variational solution of

$$\begin{cases} \Delta u_{n,i} = \frac{J_{\lambda, \mathbf{C}_n}(\Omega_n^i)}{|\Omega_n^i|} & \text{in } \Omega_n^i \\ \partial_\nu u_{n,i} = 1 & \text{on } \Sigma_n^i \\ \partial_\nu u_{n,i} = -\lambda & \text{on } \Gamma_n^i, \end{cases} \quad (4.3)$$

where $\Sigma_n^i = \partial\Omega_n^i \setminus \mathbf{C}_n$, $\Gamma_n^i := \partial\Omega_n^i \cap \mathbf{C}_n$. Arguing as in the first part of the proof (see (4.1)), for every $1 \leq i \leq K_n$ we have

$$\begin{aligned} |B^\lambda| &\leq |\mathcal{B}_{u_{n,i}}^\lambda| \leq |\nabla u_{n,i}(\widehat{\Omega}_n^i)| \leq \int_{\widehat{\Omega}_n^i} \det \nabla^2 u_{n,i} dx \\ &\leq \int_{\widehat{\Omega}_n^i} \frac{(\Delta u_{n,i})^N}{N^N} dx \leq \frac{(J_{\lambda, \mathbf{C}_n}(\Omega_n^i))^N}{N^N |\Omega_n^i|^N} |\widehat{\Omega}_n^i| \leq \frac{(J_{\lambda, \mathbf{C}_n}(\Omega_n^i))^N}{N^N |\Omega_n^i|^N} |\Omega_n^i|, \end{aligned} \quad (4.4)$$

where

$$\widehat{\Omega}_n^i := \{x \in \Omega_n^i : J_{\widehat{\Omega}_n^i} u_{n,i}(x) \neq \emptyset \text{ and } \nabla u_{n,i}(x) \in B_1\}.$$

Summing up the inequality we deduce

$$J_{\lambda, \mathbf{C}_n}(\Omega_n) = \sum_{i=1}^{K_n} J_{\lambda, \mathbf{C}_n}(\Omega_n^i) \geq N|B^\lambda|^{\frac{1}{N}} \sum_{i=1}^{K_n} |\Omega_n^i|^{\frac{N-1}{N}} \geq N|B^\lambda|^{\frac{1}{N}} |\Omega_n|^{\frac{N-1}{N}}. \quad (4.5)$$

Note that by properties (iii), (iv) and (v) of Lemma 4.1 we have that $|\Omega_n| \rightarrow |\Omega| = |B^\lambda|$ and

$$J_{\lambda, \mathbf{C}_n}(\Omega_n) \rightarrow J_{\lambda, \mathbf{C}}(\Omega) = J_{\lambda, \mathbf{C}}(B^\lambda) = N|B^\lambda|, \quad (4.6)$$

and combining this with the last inequality in (4.5) we deduce that $|\Omega_n^1| \rightarrow |B^\lambda|$, while $\sum_{i=2}^{K_n} |\Omega_n^i| \rightarrow 0$. Therefore, up to replacing Ω_n with Ω_n^1 , we may from now on assume that Ω_n is connected, and we simply write u_n and $\widehat{\Omega}_n$ in place of $u_{n,1}$ and $\widehat{\Omega}_{n,1}$.

Let us observe that (up to additive constants), we may assume that the functions u_n vanish at points x_n of $\widehat{\Omega}_n$ uniformly far from $\partial\Omega_n$. Thus, we have

$$u_n(y) \geq \nabla u_n(x_n) \cdot (y - x_n) \geq -\text{diam}(\Omega_n) \quad \text{for all } y \in \Omega_n,$$

where we used the fact that $|\nabla u_n(x_n)| < 1$. Thus the u_n 's are uniformly bounded from below. Since they solve the equation $\Delta u_n = c_n$, with c_n uniformly bounded, it follows from a standard Harnack inequality that

$$\sup_n \|u_n\|_{L^\infty(\Omega')} < +\infty \quad \text{for all } \Omega' \subset\subset \Omega.$$

In turn, recalling also (4.6) and by standard elliptic regularity, we may assume that there exists $u \in C^\infty(\Omega)$ such that up to extracting a (non relabelled) subsequence

$$\Delta u = N \text{ in } \Omega \quad \text{and} \quad u_n \rightarrow u \in C^\infty(\overline{\Omega}') \text{ for all } \Omega' \subset\subset \Omega.$$

Note now that by (4.6), the inequalities in (4.4) (with $K_n = 1$) become equalities in the limit for u . In particular, $|\widehat{\Omega}_n| \rightarrow |\Omega| = |B^\lambda|$ and since $|\Omega_n \Delta \Omega| \rightarrow 0$, we have (up to a non relabelled subsequence)

$$\chi_{\widehat{\Omega}_n} \rightarrow \chi_\Omega \text{ almost everywhere.} \quad (4.7)$$

Thus, by Fatou Lemma applied to the positive functions $\frac{(\Delta u_n)^N}{N^N} - \det \nabla^2 u_n$, we may pass to the limit in (4.4) to conclude that

$$\int_\Omega \det \nabla^2 u \, dx = \int_\Omega \frac{(\Delta u)^N}{N^N} \, dx$$

and, in turn, since $\det \nabla^2 u \leq \left(\frac{\Delta u}{N}\right)^N = 1$,

$$\det \nabla^2 u = 1 \text{ in } \Omega.$$

The above equality in the arithmetic-geometric mean inequality implies that all the eigenvalues of $\nabla^2 u$ are equal to 1 in Ω . Thus, $\nabla^2 u = I$ in Ω and in turn by the connectedness of Ω , there exist $x_0 \in \mathbb{R}^N$ and $b \in \mathbb{R}$ such that

$$u(x) = \frac{1}{2}|x - x_0|^2 + b \text{ for all } x \in \Omega. \quad (4.8)$$

Note also that since $|\nabla u_n| < 1$ in $\widehat{\Omega}_n$ and recalling (4.7), we have also that $|\nabla u| \leq 1$ in Ω and thus (4.8) implies $\Omega \subset B_1(x_0)$.

We now study the boundary conditions satisfied by u . To this aim, let $\overline{B}_{2r}(x) \subset \mathbb{R}^N \setminus \mathbf{C}$, with $x \in \partial\Omega$ and $\overline{B}_{2r}(x) \cap \Sigma_{sing} = \emptyset$, where Σ_{sing} is the possibly empty singular set of $\partial\Omega \setminus \mathbf{C}$. By (vi) of Lemma 4.3 we have that $\partial\Omega_n \cap \overline{B}_{2r}(x)$ converge in C^∞ to $\partial\Omega \cap \overline{B}_{2r}(x)$. Since by (4.3)

$$\begin{aligned} \int_{B_{2r}(x) \cap \Omega_n} \nabla u_n \cdot \nabla \varphi &= -c_n \int_{B_{2r}(x) \cap \Omega_n} \varphi \, dx \\ &+ \int_{\partial\Omega_n \cap B_{2r}(x)} \varphi \, d\mathcal{H}^{N-1} \text{ for all } \varphi \in H_0^1(B_{2r}(x)), \end{aligned} \quad (4.9)$$

and since $\partial\Omega_n \cap B_{2r}(x)$ are uniformly Lipschitz boundaries, by a standard Harnack Inequality up to boundary we have that, up to possibly replacing u_n by $\tilde{u}_n := u_n + d_n$, $d_n \in \mathbb{R}$, we have

$$\sup_n \|\tilde{u}_n\|_{L^\infty(B_{3r/2}(x) \cap \Omega_n)} < +\infty.$$

In turn, by a Caccioppoli Inequality argument and exploiting that Trace Theorem holds on $\partial\Omega_n \cap B_{\frac{3}{2}r}(x)$ with uniform constants, we deduce

$$\sup_n \|\tilde{u}_n\|_{H^1(\Omega_n \cap B_r(x))} < +\infty.$$

Thus we may extend each $\tilde{u}_n$ to the whole $B_r(x)$, in such a way that

$$\sup_n \|\tilde{u}_n\|_{H^1(B_r(x))} < +\infty.$$

Hence, up to a not relabelled subsequence, $\tilde{u}_n \rightharpoonup \tilde{u}$ weakly in $H^1(B_r(x))$, with $\nabla \tilde{u} = \nabla u$ in $\Omega \cap B_r(x)$. Therefore, we can pass to the limit in (4.9) to get

$$\int_{B_r(x) \cap \Omega} \nabla u \cdot \nabla \varphi = -N \int_{B_r(x) \cap \Omega} \varphi dx + \int_{\partial\Omega \cap B_r(x)} \varphi d\mathcal{H}^{N-1} \quad \text{for all } \varphi \in C_c^\infty(B_r(x)),$$

which yields $\partial_\nu u = 1$ on $\partial\Omega \cap B_r(x)$ and thus on $\partial\Omega \setminus (\mathbf{C} \cup \Sigma_{\text{sing}})$ by the arbitrariness of $B_r(x)$. In turn, since $1 = \partial_\nu u(x) = (x - x_0) \cdot \nu_\Omega(x)$ for all $x \in \partial\Omega \setminus (\mathbf{C} \cup \Sigma_{\text{sing}})$ and recalling that $\Omega \subset B_1(x_0)$, we have necessarily $\partial\Omega \setminus (\mathbf{C} \cup \Sigma_{\text{sing}}) \subset \partial B_1(x_0)$. Hence, $\Sigma_{\text{sing}} = \emptyset$ and $\partial\Omega \setminus \mathbf{C} \subset \partial B_1(x_0)$.

Note now that, since $\mathbf{C}_n \rightarrow \mathbf{C}$ in the Kuratowski sense, it follows that the boundaries $\partial\mathbf{C}_n$ are locally equi-Lipschitz. Consequently, for every ball $B_r(x)$ such that $\overline{B_r(x)} \cap \partial\Omega = (\Gamma \setminus \gamma) \cap \overline{B_r(x)}$, we eventually obtain $\overline{B_r(x)} \cap \partial\Omega_n = (\Gamma_n \setminus \gamma_n) \cap \overline{B_r(x)}$, allowing us to extend each u_n to functions $\tilde{u}_n \in H^1(B_r)$ having uniformly bounded H^1 -norms. Hence, up to extracting a subsequence (not explicitly relabeled), we may assume $\tilde{u}_n \rightharpoonup \tilde{u}$ weakly in $H^1(B_r(x))$, where $\tilde{u} = u$ within $\Omega \cap B_r(x)$. Proceeding similarly to before, we deduce that $\partial_\nu u = -\lambda$ almost everywhere on $\Gamma \setminus \gamma$.

Take now a point $\hat{x} \in \Gamma \setminus \gamma$ and introduce the half-space

$$H := \{y \in \mathbb{R}^N : (y - x_0) \cdot \nu_{\mathbf{C}}(\hat{x}) > \lambda\}.$$

Considering that $-\lambda = \partial_\nu u(\hat{x}) = (\hat{x} - x_0) \cdot \nu_\Omega(\hat{x}) = -(\hat{x} - x_0) \cdot \nu_{\mathbf{C}}(\hat{x})$, we infer that $\mathbf{C}$ must be contained within $\mathbb{R}^N \setminus H$, and that the boundary ∂H is tangent to $\mathbf{C}$ exactly at the point $\hat{x}$. Additionally, for any non-tangential direction $v \in \mathbb{S}^{N-1}$ satisfying $v \cdot \nu_{\mathbf{C}}(\hat{x}) > 0$, all points along the half-line $\hat{x} + tv$, $t > 0$, within the ball $B_1(x_0)$ lie entirely within Ω . Indeed, otherwise this half-line would intersect $\partial\Omega$ at a point belonging to $B_1 \setminus \mathbf{C}$, leading to a contradiction to the fact that $\partial\Omega \setminus \mathbf{C} \subset \partial B_1(x_0)$. Thus, we conclude $B_1(x_0) \cap H \subseteq \Omega$. Finally, as the set $B_1(x_0) \cap H$ is a spherical cap isometric to B^λ and since $|\Omega| = |B^\lambda|$, we deduce that $\Omega = B_1(x_0) \cap H$, concluding the proof of the theorem. $\square$

Remark 4.4. *Let us here describe how the proof of Theorem 1.1 can be extended to the case of a convex set $\mathbf{C}$ with empty interior. First of all, we point out that in this case the capillary energy must be defined as follows*

$$J_{\lambda, \mathbf{C}}(E) := P(E; \mathbb{R}^N \setminus \mathbf{C}) - \lambda \int_{\mathbf{C}} (\text{Tr}^+(\chi_E) + \text{Tr}^-(\chi_E)) d\mathcal{H}^{N-1}, \quad (4.10)$$

where $\text{Tr}^\pm(\chi_E)$ denote the traces of the characteristic function χ_E on both sides of $\mathbf{C}$, see for instance [2, Theorem 3.77]. Then, inequality (1.2) immediately follows by approximating $\mathbf{C}$ with convex sets with nonempty interior. Concerning the equality case, it can be obtained as in the proof for the nonempty interior case, since both Lemma 4.1 and 4.3 extend to the

empty interior case. For Lemma 4.1, this is readily obtained with a further approximation of $\mathbf{C}$. A direct argument to extend Lemma 4.3 is to argue exactly as in the proof presented in the Appendix, replacing $\mathbf{C}$ by its ε -neighborhood $\mathbf{C}_\varepsilon$, and Ω by $\Omega_\varepsilon = \Omega \setminus \mathbf{C}_\varepsilon$ where Ω is a solution of problem (1.1) in $\mathbb{R}^N \setminus \mathbf{C}$.

5. APPENDIX

In this section we give the proofs of the two approximation lemmas that were used to prove the main theorem.

Proof of Lemma 4.1. Let B_R be a ball such that $E \subset\subset B_R$ and assume without loss of generality that the interior of $\mathbf{C}$ contains the origin. Moreover, we can assume without loss of generality that $\mathcal{H}^{N-1}(\partial^* E \cap \mathbf{C}) > 0$, since otherwise the claim is obvious.

Given $\sigma > 0$ we begin by constructing a sequence of smooth convex sets $\mathbf{C}_\sigma^k \subset \mathbb{R}^N$, with $\mathbf{C} \subset \mathbf{C}_\sigma^k$, converging to $\mathbf{C}_\sigma := (1 + \sigma)\mathbf{C}$ in the Kuratowski sense as $k \rightarrow \infty$ and such that $(1 + \sigma)\mathbf{C} \cap B_R \subset \mathbf{C}_\sigma^k \cap B_R$. Up to slightly dilating $\mathbf{C}_\sigma^k$ if needed, we may always assume that

$$\mathcal{H}^{N-1}(\partial^* E \cap \partial \mathbf{C}_\sigma^k) = 0 \quad \text{for all } k, \sigma. \quad (5.1)$$

We consider the signed distance function $\text{sd}_{\mathbf{C}_\sigma^k}(x)$ from $\partial \mathbf{C}_\sigma^k$, which is a C^∞ function in $O_\sigma^k = \{x : \text{sd}_{\mathbf{C}_\sigma^k}(x) > -\eta_\sigma^k\}$ for some $\eta_\sigma^k > 0$. Consider the smooth convex sets $\mathbf{C}_{\sigma,s}^k := \{x : \text{sd}_{\mathbf{C}_\sigma^k}(x) \leq s\}$ for $s > -\eta_\sigma^k$.

To approximate E we first extend $\chi_E|_{\mathbb{R}^N \setminus \mathbf{C}}$ to a function $u \in BV(\mathbb{R}^N)$, with compact support, such that $|Du|(\partial \mathbf{C}) = 0$, $0 \leq u \leq 1$, see [2, Proposition 3.21] or [21, Theorem 13.8]. Note that for all $t \in (0, 1)$, $\{u > t\} \setminus \mathbf{C} = E$. For any $\varepsilon > 0, t \in (0, 1)$ we set $U_{\varepsilon,t} = \{x : u_\varepsilon(x) > t\}$, where $u_\varepsilon = \varrho_\varepsilon * u$, for a standard mollifier ϱ_ε . Note that for a.e. $t \in (0, 1)$ there exists a sequence ε_n converging to zero such that

$$\begin{aligned} \lim_{n \rightarrow \infty} |U_{\varepsilon_n,t} \Delta \{u > t\}| &= 0, \quad \lim_{n \rightarrow \infty} P(U_{\varepsilon_n,t}) = P(\{u > t\}), \\ \partial U_{\varepsilon_n,t} &\subset \left\{x : \text{dist}(x, \partial \{u > t\}) < \frac{1}{n}\right\}, \end{aligned} \quad (5.2)$$

see [2, Theorem 3.42].

Consider now the C^∞ map $x \mapsto (\text{sd}_{\mathbf{C}_\sigma^k}(x), u_\varepsilon(x))$ defined for all $x \in O_\sigma^k$. By Sard's theorem we have that

$$\text{rank} \begin{pmatrix} \nabla \text{sd}_{\mathbf{C}_\sigma^k}(x) \\ \nabla u_{\varepsilon_n}(x) \end{pmatrix} = 2 \quad \text{on } \{x : \text{sd}_{\mathbf{C}_\sigma^k}(x) = s, u_{\varepsilon_n}(x) = t\} \text{ for a.e. } (s, t) \in (0, \infty) \times (0, 1).$$

Keeping in mind that $\mathcal{H}^{N-1}(\partial^* E \cap \mathbf{C}) > 0$, and provided that σ is small enough, a simple argument shows that there exists $\delta > 0$ such that for any $t \in (0, 1)$ and $s \in (0, \delta)$ the intersection $\{\text{sd}_{\mathbf{C}_\sigma^k} = s\} \cap \{u_{\varepsilon_n} = t\}$ is not empty for all n sufficiently large. Hence, we may fix from now on $t \in (0, 1)$ satisfying (5.2) and such that for a.e. $s \in (0, \delta)$ the above rank condition holds for all n . Therefore for a.e. $s \in (0, \delta)$ the open set $\Omega_{\sigma,\varepsilon_n,s}^k = U_{\varepsilon_n,t} \setminus \mathbf{C}_{\sigma,s}^k$ is a Lipschitz domain such that $\partial \Omega_{\sigma,\varepsilon_n,s}^k \setminus \mathbf{C}_{\sigma,s}^k$ is a C^∞ manifold with boundary. Note that for

any σ and k we have that for a.e. s , $\mathcal{H}^{N-1}(\partial^* E \cap \partial \mathbf{C}_{\sigma,s}^k) = 0$. Therefore for all such s , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} P(\Omega_{\sigma, \varepsilon_n, s}^k; \mathbb{R}^N \setminus \mathbf{C}_{\sigma,s}^k) &= \lim_{n \rightarrow \infty} P(U_{\varepsilon_n, t}; \mathbb{R}^N \setminus \mathbf{C}_{\sigma,s}^k) \\ &= P(E; \mathbb{R}^N \setminus \mathbf{C}_{\sigma,s}^k) = P(E \setminus \mathbf{C}_{\sigma,s}^k; \mathbb{R}^N \setminus \mathbf{C}_{\sigma,s}^k). \end{aligned} \quad (5.3)$$

From the above convergence and the continuity of the trace operator for BV functions, see [2, Theorem 3.88] we have that

$$\lim_{n \rightarrow \infty} \mathcal{H}^{N-1}(\partial \Omega_{\sigma, \varepsilon_n, s}^k \cap \partial \mathbf{C}_{\sigma,s}^k) = \mathcal{H}^{N-1}(\partial^*(E \setminus \mathbf{C}_{\sigma,s}^k) \cap \partial \mathbf{C}_{\sigma,s}^k) = \mathcal{H}^{N-1}(E \cap \partial \mathbf{C}_{\sigma,s}^k), \quad (5.4)$$

where the last equality follows from the fact that $\mathcal{H}^{N-1}(\partial^* E \cap \partial \mathbf{C}_{\sigma,s}^k) = 0$. Observe now that, since $\mathbf{C}_{\sigma,s}^k$ converge to $\mathbf{C}_\sigma^k$ in the Kuratowski sense, as $s \rightarrow 0$, we have in particular that $\mathcal{H}^{N-1} \llcorner \partial \mathbf{C}_{\sigma,s}^k \xrightarrow{*} \mathcal{H}^{N-1} \llcorner \partial \mathbf{C}_\sigma^k$, see for instance [13, Remark 2.2]. Therefore, thanks to (5.1) we conclude that $\mathcal{H}^{N-1}(E \cap \partial \mathbf{C}_{\sigma,s}^k) \rightarrow \mathcal{H}^{N-1}(E \cap \partial \mathbf{C}_\sigma^k) = \mathcal{H}^{N-1}(\partial^*(E \setminus \mathbf{C}_\sigma^k) \cap \partial \mathbf{C}_\sigma^k)$. Thus we have

$$\begin{aligned} \lim_{s \rightarrow 0} P(E \setminus \mathbf{C}_{\sigma,s}^k; \mathbb{R}^N \setminus \mathbf{C}_{\sigma,s}^k) &= P(E \setminus \mathbf{C}_\sigma^k; \mathbb{R}^N \setminus \mathbf{C}_\sigma^k) \\ \lim_{s \rightarrow 0} \mathcal{H}^{N-1}(\partial^*(E \setminus \mathbf{C}_{\sigma,s}^k) \cap \partial \mathbf{C}_{\sigma,s}^k) &= \mathcal{H}^{N-1}(\partial^*(E \setminus \mathbf{C}_\sigma^k) \cap \partial \mathbf{C}_\sigma^k). \end{aligned} \quad (5.5)$$

By a similar argument, if $\sigma > 0$ is such that $\mathcal{H}^{N-1}(\partial^* E \cap \partial(1+\sigma)\mathbf{C}) = 0$, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} P(E \setminus \mathbf{C}_\sigma^k; \mathbb{R}^N \setminus \mathbf{C}_\sigma^k) &= P(E \setminus \mathbf{C}_\sigma; \mathbb{R}^N \setminus \mathbf{C}_\sigma), \\ \lim_{k \rightarrow \infty} \mathcal{H}^{N-1}(\partial^*(E \setminus \mathbf{C}_\sigma^k) \cap \partial \mathbf{C}_\sigma^k) &= \mathcal{H}^{N-1}(\partial^*(E \setminus \mathbf{C}_\sigma) \cap \partial \mathbf{C}_\sigma). \end{aligned} \quad (5.6)$$

Finally, we note that by monotone convergence

$$\lim_{\sigma \rightarrow 0} P(E \setminus \mathbf{C}_\sigma; \mathbb{R}^N \setminus \mathbf{C}_\sigma) = \lim_{\sigma \rightarrow 0} P(E; \mathbb{R}^N \setminus \mathbf{C}_\sigma) = P(E; \mathbb{R}^N \setminus \mathbf{C}). \quad (5.7)$$

By scaling, this is equivalent to say that

$$\lim_{\sigma \rightarrow 0} P(((1+\sigma)^{-1}E) \setminus \mathbf{C}; \mathbb{R}^N \setminus \mathbf{C}) = P(E; \mathbb{R}^N \setminus \mathbf{C}).$$

Therefore, the trace Theorem again implies that

$$\begin{aligned} &\lim_{\sigma \rightarrow 0} \mathcal{H}^{N-1}(\partial^*(E \setminus \mathbf{C}_\sigma) \cap \partial \mathbf{C}_\sigma) \\ &= \lim_{\sigma \rightarrow 0} (1+\sigma)^{N-1} \mathcal{H}^{N-1}(\partial^*((1+\sigma)^{-1}E) \setminus \mathbf{C}) \cap \partial \mathbf{C} = \mathcal{H}^{N-1}(\partial^* E \cap \mathbf{C}). \end{aligned} \quad (5.8)$$

From this equality, together with (5.3)-(5.7) we conclude, by a diagonal argument, that there exist sequences $s_n \rightarrow 0^+$, $k_n \rightarrow \infty$ and $\sigma_n \rightarrow 0^+$ such that, setting $\Omega_n = \Omega_{\sigma_n, \varepsilon_n, s_n}^{k_n}$, $\mathbf{C}_n = \mathbf{C}_{\sigma_n, s_n}^{k_n}$, (i)-(iv) and the first part of (v) hold. Concerning the second part of (v), namely, the weak* convergence of $\mathcal{H}^{N-1} \llcorner (\partial \Omega_n \cap \mathbf{C}_n)$ to $\mathcal{H}^{N-1} \llcorner (\partial^* E \cap \mathbf{C})$, it easily follows by observing that, from (5.3) and the continuity of the trace operator, for every φ in a countable dense subset of $C_c(\mathbb{R}^N)$ we can generalise (5.4) as

$$\lim_{n \rightarrow \infty} \int_{\partial \Omega_{\sigma_n, \varepsilon_n, s_n}^{k_n} \cap \partial \mathbf{C}_{\sigma_n, s_n}^{k_n}} \varphi(x) d\mathcal{H}^{N-1}(x) = \int_{E \cap \partial \mathbf{C}_{\sigma_n, s_n}^{k_n}} \varphi(x) d\mathcal{H}^{N-1}(x),$$

and in the same way we get the analogous generalisations of (5.5)–(5.8). Thus, we conclude again by a diagonal argument. $\square$

We now prove the second approximation lemma.

Proof of Lemma 4.3. We start by observing that by the classical regularity for volume-constrained perimeter minimizers $\partial\Omega \setminus \mathbf{C}$ is smooth outside a (relatively closed) singular set Σ_{sing} of Hausdorff dimension at most $N - 8$. Moreover, since Ω is a Λ -minimiser of the capillary energy (see for instance [13]), it can be shown that Ω is open and satisfies the following perimeter density estimates: there exist $c_0, r_0 > 0$ such that for all $x \in \partial\Omega$ and for all $r \in (0, r_0)$

$$\mathcal{H}^{N-1}(\partial\Omega \cap B_r(x)) \geq c_0 r^{N-1}. \quad (5.9)$$

Note that this estimate in particular implies that Ω is bounded, so we can take a ball B_R such that $\Omega \subset\subset B_R$; moreover, we assume without loss of generality that the interior of $\mathbf{C}$ contains the origin of $\mathbb{R}^N$.

Let $\mathbf{C}_\sigma^k \subset \mathbb{R}^N$, $\sigma > 0$ and $k \in \mathbb{N}$, be a family of smooth convex sets exactly as in the proof of Lemma 4.1, satisfying in particular (5.1).

We consider the signed distance function $\text{sd}_{\mathbf{C}_\sigma^k}(x)$ from $\partial\mathbf{C}_\sigma^k$, which is a C^∞ function in $O_\sigma^k = \{x : \text{sd}_{\mathbf{C}_\sigma^k}(x) > -\eta_\sigma^k\}$ for some $\eta_\sigma^k > 0$. Consider the smooth convex sets $\mathbf{C}_{\sigma,s}^k := \{x : \text{sd}_{\mathbf{C}_\sigma^k}(x) \leq s\}$ for $s > -\eta_\sigma^k$.

Note that if Ω is an open set of finite perimeter satisfying (5.9), then by [1, Theorem 5] we have that the outer $(N - 1)$ -dimensional Minkowski content of $\partial\Omega$ coincides with $P(\Omega)$; that is,

$$P(\Omega) = \lim_{\varrho \rightarrow 0^+} \frac{\mathcal{L}^N((\Omega)_\varrho \setminus \Omega)}{\varrho}.$$

Thus, by the coarea formula, the above equality may be rewritten as

$$P(\Omega) = \lim_{\varrho \rightarrow 0^+} \frac{1}{\varrho} \int_0^\varrho P(\{\text{sd}_\Omega < t\}) dt, \quad (5.10)$$

where sd_Ω stands for the signed distance function from $\partial\Omega$. For $\varepsilon > 0$ set $\text{sd}_\Omega^\varepsilon := \rho_\varepsilon * \text{sd}_\Omega$, where ρ_ε is a standard mollifier, and note that by the Dominated Convergence Theorem

$$\int_{\{0 < \text{sd}_\Omega^\varepsilon < \rho\}} |\nabla \text{sd}_\Omega^\varepsilon| dx \rightarrow \mathcal{L}^N((\Omega)_\rho \setminus \Omega)$$

as $\varepsilon \rightarrow 0^+$. Therefore, by (5.10) and again the Coarea Formula, we have for any sequence $\varepsilon_m \rightarrow 0^+$

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_0^1 P\left(\left\{\text{sd}_\Omega^{\varepsilon_m} < \frac{t}{n}\right\}\right) dt = P(\Omega).$$

In turn, by Fatou's Lemma, the lower semicontinuity of the perimeter and by Sard's Theorem, we obtain that for almost every $t \in (0, 1)$ we have

$$\left\{\text{sd}_\Omega^{\varepsilon_m} < \frac{t}{n}\right\} \text{ is smooth} \quad \text{and} \quad \liminf_{n \rightarrow \infty} \liminf_{m \rightarrow \infty} P\left(\left\{\text{sd}_\Omega^{\varepsilon_m} < \frac{t}{n}\right\}\right) = P(\Omega). \quad (5.11)$$

For any m, n we consider the C^∞ map $x \mapsto (\text{sd}_{\mathbf{C}_\sigma^k}(x), n \text{sd}_\Omega^{\varepsilon_m}(x))$ defined for all $x \in O_\sigma^k$. By Sard's theorem we have that

$$\text{rank} \begin{pmatrix} \nabla \text{sd}_{\mathbf{C}_\sigma^k}(x) \\ n \nabla \text{sd}_\Omega^{\varepsilon_m}(x) \end{pmatrix} = 2 \text{ on } \{x : \text{sd}_{\mathbf{C}_\sigma^k}(x) = s, n \text{sd}_\Omega^{\varepsilon_m}(x) = t\} \text{ for a.e. } (s, t) \in (0, \infty) \times (0, 1).$$

Since by the minimality of Ω , we have $\mathcal{H}^{N-1}(\partial\Omega \cap \mathbf{C}) > 0$, a simple argument shows that there exists $\delta > 0$ such that for any $t \in (0, 1)$ and $s \in (0, \delta)$ the intersection $\{\text{sd}_{\mathbf{C}_\sigma^k}(x) = s\} \cap \{n \text{sd}_\Omega^{\varepsilon_m}(x) = t\} \neq \emptyset$ for all m, n sufficiently large. Hence we can fix $t \in (0, 1)$ satisfying (5.11) such that the above rank condition holds for all $s \in (0, \delta) \setminus \mathcal{Z}$ and for all m, n sufficiently large, where $\mathcal{Z}$ is a negligible set. Note that possibly replacing $\mathcal{Z}$ with a larger (not renamed) negligible set we may also assume that

$$\mathcal{H}^{N-1}(\partial\Omega \cap \partial\mathbf{C}_{\sigma,s}^k) = 0 \quad \text{for all } s \in (0, \delta) \setminus \mathcal{Z}. \quad (5.12)$$

At this point, we may also extract subsequences m_j, n_j such that

$$\lim_{j \rightarrow \infty} P\left(\left\{\text{sd}_\Omega^{\varepsilon_{m_j}} < \frac{t}{n_j}\right\}\right) = P(\Omega). \quad (5.13)$$

For any $s \in (0, \delta) \setminus \mathcal{Z}$ and j sufficiently large, we can define

$$\Omega_{\sigma,j,s}^k := \left\{\text{sd}_\Omega^{\varepsilon_{m_j}} < \frac{t}{n_j}\right\} \setminus \mathbf{C}_{\sigma,s}^k.$$

Note that (5.12) and (5.13) imply that for all $s \in (0, \delta) \setminus \mathcal{Z}$, we have

$$\lim_{j \rightarrow \infty} P(\Omega_{\sigma,j,s}^k; \mathbb{R}^N \setminus \mathbf{C}_{\sigma,s}^k) = P(\Omega; \mathbb{R}^N \setminus \mathbf{C}_{\sigma,s}^k) = P(\Omega \setminus \mathbf{C}_{\sigma,s}^k; \mathbb{R}^N \setminus \mathbf{C}_{\sigma,s}^k). \quad (5.14)$$

From the above convergence and the continuity of the trace operator we have that

$$\lim_{j \rightarrow \infty} \mathcal{H}^{N-1}(\partial\Omega_{\sigma,j,s}^k \cap \partial\mathbf{C}_{\sigma,s}^k) = \mathcal{H}^{N-1}(\partial(\Omega \setminus \mathbf{C}_{\sigma,s}^k) \cap \partial\mathbf{C}_{\sigma,s}^k) = \mathcal{H}^{N-1}(\Omega \cap \partial\mathbf{C}_{\sigma,s}^k), \quad (5.15)$$

where the last equality follows from (5.12).

We can argue exactly as in the second part of the proof of Lemma 4.1, to infer that (5.5)–(5.8) hold. From these equalities, together with (5.14) and (5.15), by a diagonal argument, that there exist sequences $k_j \rightarrow \infty$, $s_j, \sigma_j \rightarrow 0$, with $s_j \in (0, \delta) \setminus \mathcal{Z}$, such that $\Omega_j := \Omega_{\sigma_j,j,s_j}^{k_j}$ and $\mathbf{C}_j := \mathbf{C}_{\sigma_j,s_j}^{k_j}$ satisfy (i)–(v) with E replaced by Ω .

Finally, property (vi) follows by observing that if $x \in \partial\Omega \setminus \Sigma_{\text{sing}}$ and sd_Ω is smooth in $\overline{B}_{2\rho}(x)$, then $\text{sd}_\Omega^{\varepsilon_{m_j}} \rightarrow \text{sd}_\Omega$ in $C^\infty(\overline{B}_\rho(x))$, which in turn easily implies (vi). $\square$

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DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI NAPOLI FEDERICO II, ITALY

E-mail address: `n.fusco@unina.it`

MATEMATIKAN JA TILASTOTIETEEN LAITOS, JYVÄSKYLÄN YLIOPISTO, FINLAND

E-mail address: `vesa.julin@jyu.fi`

DIPARTIMENTO DI SCIENZE MATEMATICHE FISICHE E INFORMATICHE, UNIVERSITÀ DI PARMA, ITALY

E-mail address: `massimiliano.morini@unipr.it`

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA, ITALY

E-mail address: `aldo.pratelli@unipi.it`