An L^∞-variational problem involving the Fractional Laplacian

Simone Carano and Roger Moser

Abstract

For $s \in (0,1)$ and an open bounded set $\Omega \subset \mathbb{R}^n$, we prove existence and uniqueness of absolute minimisers of the supremal functional

$$E_\infty(u) = \|(-\Delta)^s u\|_{L^\infty(\mathbb{R}^n)},$$

where $(-\Delta)^s$ is the Fractional Laplacian of order s and u has prescribed Dirichlet data in the complement of Ω. We further show that the minimiser u_∞ satisfies the (fractional) PDE

$$(-\Delta)^s u_\infty = E_\infty(u_\infty)\,\mathrm{sgn} f_\infty \qquad \text{in } \Omega,$$

for some analytic function $f_\infty \in L^1(\Omega)$ obtained as the restriction of an s-harmonic measure μ in Ω.

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1 Introduction

For $n \in \mathbb{N}$ and $s \in (0,1)$, consider the supremal functional

$$E_\infty(u) := \|(-\Delta)^s u\|_{L^\infty(\mathbb{R}^n)}, \tag{1.1}$$

where u belongs to the Fréchet potential space

$$\mathcal{W}^{2s,\infty}(\mathbb{R}^n) := \bigcap_{1<p<\infty} \Big\{ u \in W^{s,p}(\mathbb{R}^n) : (-\Delta)^s u \in L^\infty(\mathbb{R}^n) \Big\}.$$

In the above, $(-\Delta)^s u$ is the s-Laplacian of u, which is defined by

$$(-\Delta)^s u(x) := c_{n,s} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy \qquad \forall x \in \mathbb{R}^n,$$

for u smooth in $\mathbb{R}^n$. The previous integral is understood in the Cauchy principal value sense and $c_{n,s} := \frac{s 2^{2s}\Gamma(\frac{n}{2}+s)}{\pi^{n/2}\Gamma(1-s)}$, where Γ denotes the Euler Gamma function. With this choice, the operator $(-\Delta)^s$ approaches the classical Laplacian as $s \to 1^-$.
For $u \in W^{s,2}(\mathbb{R}^n)$, $(-\Delta)^s u$ is understood in the usual weak sense as an element of $W^{-s,2}(\mathbb{R}^n)$. Thus, when $u \in \mathcal{W}^{2s,\infty}(\mathbb{R}^n)$, we are assuming that $(-\Delta)^s u$ is represented by an L^∞-function.
We aim to show existence and uniqueness of global minimisers of (1.1) given Dirichlet conditions on the complement of a fixed bounded open subset $\Omega \subset \mathbb{R}^n$. Explicitely, the problem makes sense for exterior data $u_0 \in \mathcal{W}^{2s,\infty}(\mathbb{R}^n)$ and the competitor class

$$\mathcal{W}^{2s,\infty}_{u_0}(\Omega) := u_0 + \mathcal{W}^{2s,\infty}_0(\Omega)$$

where $\mathcal{W}_0^{2s,\infty}(\Omega) := \{u \in \mathcal{W}^{2s,\infty}(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$, leading to the minimum problem

$$\min_{u \in \mathcal{W}_{u_0}^{2s,\infty}(\Omega)} E_\infty(u) = \min_{u \in \mathcal{W}_{u_0}^{2s,\infty}(\Omega)} \|(-\Delta)^s u\|_{L^\infty(\mathbb{R}^n)}.$$

In addition, we will show that the minimiser satisfies a certain (fractional) PDE. The non-local nature of the operator makes it necessary to consider the behaviour of the minimiser on the whole $\mathbb{R}^n$ instead of Ω. This is in contrast with the case when $s = 1$, corresponding to the classical Laplacian, studied in [22], where the sup norm is taken on the domain Ω. Another important difference with the local case is that no assumptions are required on the regularity of the boundary of Ω.

Before entering into the details of our main result, we briefly describe the mathematical context of our problem. We are dealing with an L^∞-variational problem of fractional differential order $2s$. The state of the art for integer order operators is well established: for the first order, most of the challenges have been succesfully addressed (see [4] for the seminal work of Aronsson and [21] for a survey reference); the higher order case, recently started in [22, 26], is largely in development and still presents many open questions, since most of the approaches used in the first order case do not generalise. Without any pretension of being exhaustive, we refer also to [19, 24, 25] for a glimpse of the literature on higher order problems. On the other hand, the fractional order case seems to be largely unexplored. In [16] (see also [7]), the authors study a notion of infinity Fractional Laplacian, generalising the Aronsson equation to the fractional setting. This PDE is obtained as a limit as $p \to \infty$ of the Euler-Lagrange equation associated to the minimisation problem for the $W^{s,p}$-norm. More broadly, the asymptotic behaviour as $p \to \infty$ of relevant energies of functions in $W^{s,p}$ has been extensively studied in the literature (see e.g. [3, 12, 28, 31]). However, the PDEs arising in these works are not related (at least in an obvious way) to any L^∞-Dirichlet problem of fractional order. For this reason, our approach proceeds in the opposite direction: we begin by considering an L^∞ variational problem of fractional order and aim to derive a fractional PDE that characterises its (unique) minimiser. Supremal functionals involving nonlocal operators have been studied in [30], where the author establishes existence of minimisers for supremal functionals depending on the Riesz fractional gradient, without addressing uniqueness or deriving associated Euler–Lagrange-type equations. In this sense, our work can be viewed as a novel contribution to this framework, as it provides both uniqueness and characterisation of the minimiser via the corresponding governing fractional PDE.

Now we proceed to specify the setting of our problem. We assume that the prescribed data u_0 belong to $C_c^{2s+\gamma}(\mathbb{R}^n)$[1], for some $\gamma > 0$. We could assume milder decay at infinity, however, the aim of this paper is to apply the methods of L^∞-Calculus to the fractional setting, to give further evidence of the robustness of the techniques. For this purpose, in order to keep the technicality as light as possible, we keep the $C_c^{2s+\gamma}$-assumption. Notice that such u_0 belongs to $\mathcal{W}^{2s,\infty}(\mathbb{R}^n)$. Indeed, we have $(-\Delta)^s u_0 \in C_0^\gamma(\mathbb{R}^n)$, with polynomial decay at infinity (see for instance [8, Lemma 3.5] for decay estimates and [33, Prop. 2.6] for Hölder estimates). Also, from the definition of the Gagliardo seminorm (see Section 2), it is easy to see that if $\sigma > s$, then $C_c^\sigma(\mathbb{R}^n) \subset W^{s,p}(\mathbb{R}^n)$ for all $p > 1$.

On the other hand, by [33, Prop. 2.9] and Morrey's inequality [18, Thereom 8.2], we have $\mathcal{W}^{2s,\infty}(\mathbb{R}^n) \subset C^\alpha(\mathbb{R}^n)$ for every $\alpha < 2s$. Thus, the regularity assumption on the data u_0 is not overly restrictive.

Now we can state our main result.

Theorem 1.1. *Fix $s \in (0, 1)$ and $n \in \mathbb{N}$, $n > 2s$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and E_∞ be the functional defined in* (1.1)*. Fix $u_0 \in C_c^{2s+\gamma}(\mathbb{R}^n)$, for some $\gamma > 0$, with $u_0 \not\equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Then*

[1] We refer to Section 2 for the precise definition of the Hölder spaces C^α, $\alpha \in \mathbb{R}^+$.

the problem

$$e_\infty := \inf_{\mathcal{W}^{2s,\infty}_{u_0}(\Omega)} E_\infty$$

is uniquely solvable, namely there exists a unique $u_\infty \in \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$ *such that* $E_\infty(u_\infty) = e_\infty$. *In particular,* $(-\Delta)^s u_\infty \in C^\gamma_{\mathrm{loc}}(\mathbb{R}^n \setminus \overline{\Omega})$ *and* $(-\Delta)^s u_\infty(x) \to 0$ *as* $|x| \to +\infty$.
Moreover, a fractional PDE can be derived as a necessary and sufficient condition for the minimality of u_∞. *Explicitely, there exists a measure* $\mu \in \mathcal{M}(\mathbb{R}^n)$, $\mu \neq 0$, *with compact support and* $|\mu|(\mathbb{R}^n) \leq 1$, *such that* μ *is s-harmonic in* Ω *and*

$$(-\Delta)^s u_\infty = e_\infty \frac{d\mu}{d|\mu|} \qquad \text{in } \mathrm{supp}|\mu| \setminus \partial\Omega. \tag{1.2}$$

The identity above is understood between L^∞*-functions on* $\mathrm{supp}|\mu| \setminus \partial\Omega$.
Moreover, the restriction $\mu \llcorner \Omega$ *is absolutely continuous w.r.t. the Lebesgue measure on* Ω, *i.e.* $\mu \llcorner \Omega = f_\infty \mathscr{L}^n \llcorner \Omega$, *for some function* $f_\infty \in L^1(\Omega) \setminus \{0\}$, *which is real analytic in* Ω. *In particular, there holds*

$$(-\Delta)^s u_\infty = e_\infty \mathrm{sgn} f_\infty \qquad \text{a.e. in } \Omega. \tag{1.3}$$

In the above, a Radon measure μ in $\mathbb{R}^n$ with finite total variation is said to be s-harmonic on Ω if

$$\int_{\mathbb{R}^n} (-\Delta)^s \varphi \, d\mu = 0, \qquad \forall \varphi \in C^\infty_c(\Omega).$$

Notice that the previous integral is well defined since $(-\Delta)^s \varphi \in C_0(\mathbb{R}^n)$, because φ has compact support.

The non trivial choice of the "boundary data" u_0 prevents that the variational problem trivialises, as we will show in the proof of Theorem 1.1.

We prove existence of minimisers via L^p-approximation (i.e. Gamma Convergence), which is a very common technique in this context. The main idea is to consider and solve a variational problem for the (suitably normalised) p-norm of the Fractional Laplacian instead of the functional (1.1). Once the existence of a minimiser u_p of this "p-version" of the problem is established, we recover the original problem for E_∞ in the limit as $p \to \infty$ and, thanks to compactness properties, we show that $u_p \to u_\infty$ in the appropriate convergence, where u_∞ is a minimiser of E_∞. This proof works under the natural assumption that $u_0 \in \mathcal{W}^{2s,\infty}(\mathbb{R}^n)$. On the other hand, the additional $C^{2s+\gamma}_c$-regularity assumption on u_0 is needed in the derivation of the PDE in (1.2). Also in this case, we argue by L^p-approximation, precisely by passing to the limit as $p \to \infty$ in the Euler-Lagrange equation satisfied by u_p. The relevant function f_p that appears in this PDE naturally inherits s-harmonicity properties and a uniform bound on the mass, as consequences of the minimality of u_p. This gives rise to the s-harmonic measure μ, as limit of f_p for $p \to \infty$. At this point, a technical argument based on the $C^{2s+\gamma}_c$-regularity of u_0 ensures that μ is represented by a non-zero analytic function on Ω.

The uniqueness of u_∞ is a consequence of the fact that it satisfies (1.3). To be more precise, the crucial point in proving uniqueness is to guarantee that *any* minimiser u satisfies $|(-\Delta)^s u| = e_\infty$ in Ω. This is suprisingly in complete analogy with the local case (see [14, 22]), despite the non-local nature of the operator $(-\Delta)^s$. In fact, the argument of the proof is similar to the one in [14], inspired in turn to a technique in [29], and consists in the insertion of a penalisation term in the L^p-approximation problem, forcing the approximating minimiser to annul the penalisation and recover the preselected minimiser of the L^∞-problem in the limit. Moreover, in general, the uniqueness property of a global minimiser ensures that it is also an *absolute minimiser* (see for instance [14, Corollary 1.4]).

The behaviour of the measure μ on $\partial\Omega$ remains an open question. However, we believe that, as for the local case [22], it should be possible to rule out concentration phenomena at the boundary by constructing suitable test functions for the Euler-Lagrange equation satisfied by u_p. Such a result may require regularity assumptions on the boundary.

Finally, we observe that if we restrict equation (1.2) to $\mathbb{R}^n \setminus \overline{\Omega}$, we deduce that, thanks to the continuity of $(-\Delta)^s u_\infty$, the measure μ must have constant sign on each connected component of $\text{supp}|\mu|$. However, at the present stage, we are not able to ensure the non-emptyness of $\text{supp}|\mu|$ on $\mathbb{R}^n \setminus \overline{\Omega}$. More in general, to our best knowlegde, behaviour and regularity properties of s-harmonic measures (outside their s-harmonicity domain) are largely unexplored, motivating further investigation.

2 Notation

In the whole paper, $s \in (0,1)$ and $n \in \mathbb{N}$, $n > 2s$, are fixed numbers.
For $k \in \mathbb{N} \cup \{\infty\}$, we denote by $C^k_c(\mathbb{R}^n)$ and $C^k_0(\mathbb{R}^n)$ respectively the space of C^k-functions with compact support and the space of C^k-functions vanishing at infinity with all their derivatives up to order k. If $k = 0$, we write simply $C_c(\mathbb{R}^n)$ and $C_0(\mathbb{R}^n)$.
Let $U \subseteq \mathbb{R}^n$ be an open set and $\alpha \in (0,1)$. We denote by $C^\alpha(U)$ the space of Hölder continuous functions of order α in U. By $C^\alpha_{\text{loc}}(U)$ we refer to the space of locally Hölder continuous functions of order α in U, i.e. we say that $u \in C^\alpha_{\text{loc}}(U)$ if for every $x \in U$ there exists a neighborhood B of x such that $u \in C^\alpha(B)$. More in general, if $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, we denote by $C^\alpha(U)$ (resp. $C^\alpha_{\text{loc}}(U)$) the space of $\lfloor\alpha\rfloor$ times (resp. locally) differentiable functions with derivatives of order $\lfloor\alpha\rfloor$ being (resp. locally) Hölder continuous of order $\alpha - \lfloor\alpha\rfloor$.
Suppose that $\gamma > 0$ is such that $2s + \gamma$ is not an integer and $u \in C^{2s+\gamma}_{\text{loc}}(\mathbb{R}^n) \cap L^1_s(\mathbb{R}^n)$. The space $L^1_s(\mathbb{R}^n)$ is the set of Lebesgue measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ with

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} dx < \infty.$$

Then, the s-Laplacian of u is pointwise defined at every $x \in \mathbb{R}^n$ as

$$(-\Delta)^s u(x) := c_{n,s}\, \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy := c_{n,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy,$$

where $c_{n,s} := \frac{s 2^{2s} \Gamma(\frac{n}{2} + s)}{\pi^{n/2} \Gamma(1-s)}$ and Γ denotes the Euler Gamma function.
For $1 \leq p < \infty$ and an open set $U \subseteq \mathbb{R}^n$, we define the fractional Sobolev space $W^{s,p}(U)$ as

$$W^{s,p}(U) = \left\{ u \in L^p(U) : [u]_{W^{s,p}(U)} < \infty \right\},$$

where $[\cdot]_{W^{s,p}(U)}$ stands for the Gagliardo seminorm, namely

$$[u]_{W^{s,p}(U)} := \left(\int_U dx \int_U \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dy \right)^{\frac{1}{p}}.$$

Also, $W^{s,p}(U)$ is a Banach space, with the norm

$$\|u\|_{W^{s,p}(U)} := \|u\|_{L^p(U)} + [u]_{W^{s,p}(U)}.$$

When $p = 2$ we use the notation $H^s(U)$ to refer to the Hilbert space $W^{s,2}(U)$.
For $u \in H^s(\mathbb{R}^n)$, we can define $(-\Delta)^s u$ in the weak sense as an element of the dual space $H^{-s}(\mathbb{R}^n)$:

$$\langle (-\Delta)^s u, v\rangle := \frac{c_{n,s}}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(x))(v(x)-v(y))}{|x-y|^{n+2s}} dxdy \qquad \forall v \in H^s(\mathbb{R}^n).$$

For $\infty \geq p \geq (2_s)_* := \frac{2n}{n+2s}$, we write $(-\Delta)^s u \in L^p(\mathbb{R}^n)$ if there exists $g \in L^p(\mathbb{R}^n)$ such that

$$\langle (-\Delta)^s u, v\rangle = \int_{\mathbb{R}^n} gv \qquad \forall v \in H^s(\mathbb{R}^n),$$

where we used that $H^s(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, for all $2 \leq p \leq 2_s^* = \frac{2n}{n-2s}$.
If $2s > 1$, write $2s = 1 + \sigma$. Then, for $p \in [1, \infty)$, we define

$$W^{2s,p}(U) := \{u \in W^{1,p}(U) : Du \in W^{\sigma,p}(U)\},$$

equipped with the corresponding suitable norm $\|\cdot\|_{W^{2s,p}(U)}$.
Finally, for a bounded open set $\Omega \subset \mathbb{R}^n$ and $p \in (1, \infty)$, we denote

$$W_0^{s,p}(\overline{\Omega}) := \{u \in W^{s,p}(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}.$$

In particular, we write $H_0^s(\overline{\Omega}) := W_0^{s,2}(\overline{\Omega})$. If Ω has C^2-boundary, then $H_0^s(\overline{\Omega})$ coincide with the closure of $C_c^\infty(\Omega)$ under the norm $\|\cdot\|_{H^s(\mathbb{R}^n)}$.

3 Properties of weak solutions to the fractional Dirichlet problem

In this section, we recall some useful results about weak and very weak solutions to the Dirichlet problem for the Fractional Laplacian. We start by recalling the notion of weak solution.

Definition 3.1. *For a bounded open set $\Omega \subset \mathbb{R}^n$ and $s \in (0, 1)$, let $f \in H^{-s}(\overline{\Omega}) := (H_0^s(\overline{\Omega}))^*$. A function $u \in H_0^s(\overline{\Omega})$ is a weak solution to*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.1}$$

if for every $v \in H_0^s(\overline{\Omega})$ it holds

$$\frac{c_{n,s}}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(x))(v(x)-v(y))}{|x-y|^{n+2s}} dxdy = \langle f, v\rangle_{H^{-s}(\overline{\Omega}),H_0^s(\overline{\Omega})},$$

where $\langle\cdot,\cdot\rangle_{H^{-s}(\overline{\Omega}),H_0^s(\overline{\Omega})}$ stands for the inner product in the duality between $H_0^s(\overline{\Omega})$ and $H^{-s}(\overline{\Omega})$.

Existence and uniqueness of weak solutions to (3.1) are simple consequence of Lax-Milgram's Theorem (see [5, Prop. 2.1]). The unique weak solution to (3.1) enjoys local L^p-regularity: if $f \in L^p(\Omega)$, $p \geq 2$, then $u \in W_{\mathrm{loc}}^{2s,p}(\Omega)$ (see e.g. [5, Theorem 1.3]). Moreover, global L^p-estimates of Calderón-Zygmund type are available, as recently proved in [2, Corollary 1.7], which we state in the following, more particular, version[2].

Theorem 3.2 (Fractional Calderón-Zygmund estimates)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^2 and $s \in (0, 1)$. Let $f \in L^m(\Omega)$ and u be the unique (weak) solution to* (3.1)*. Then, we have:*

[2]The notion of weak solution of Definition 3.1 is more stronger (and then compatible) with the notion of weak solution considered in [2] (at least for C^2-domain, see e.g. [32, Def. 1.2]).

- *if* $1 \le m \le \frac{n}{s}$, *then for all* $1 < p < m_s^* := \frac{nm}{n-ms}$ *there exists* $C > 0$ *such that*

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} \le C\|f\|_{L^m(\Omega)};$$

- *if* $m > \frac{n}{s}$, *then for all* $1 < p < \infty$ *there exists* $C > 0$ *such that*

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} \le C\|f\|_{L^m(\Omega)}.$$

Here, $C > 0$ *are positive constants depending only on* n, s, p, m *and* Ω.

Remark 3.3. We observe that, since $m_s^* > m$, we can choose $p = m$ in Theorem 3.2, obtaining an estimate of the type $W^{s,p}(\mathbb{R}^n) - L^p(\Omega)$ for all $p \in (1, \infty)$:

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} \le C\|f\|_{L^p(\Omega)},$$

for a constant $C = C(n, s, p, \Omega) > 0$. Although always refering to Theorem 3.2, we will use only this estimate throughout the whole paper.

Now, we introduce the notion of *very weak solution* to the Fractional Laplace problem. First, we define the space

$$L_s^\infty(\mathbb{R}^n) := \{u \in L^\infty(\mathbb{R}^n) : \|(1 + |\cdot|^{n+2s})|u|\|_{L^\infty(\mathbb{R}^n)} < \infty\}.$$

Definition 3.4. *Let* $\Omega \subset \mathbb{R}^n$ *be open and* $g \in L_s^1(\mathbb{R}^n)$. *A function* $u \in L_s^1(\mathbb{R}^n)$ *is a very weak solution to*

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.2}$$

if for every φ *with compact support in* Ω, *such that* $(-\Delta)^s\varphi \in L_s^\infty(\mathbb{R}^n)$, *it holds*

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n} u(-\Delta)^s\varphi = 0, \\ u = g \qquad \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.3}$$

Remark 3.5. In the definition above, the regularity of the tests φ is tacitly assumed to be $H^s(\mathbb{R}^n)$. In the literature, it is sometimes required that φ has $L^\infty(\mathbb{R}^n)$ or $C^s(\mathbb{R}^n)$ regularity. However, since $(-\Delta)^s\varphi \in L_s^\infty(\mathbb{R}^n)$, all these assumptions amount to the same class, which are the functions in $W_0^{2s,p}(\overline{\Omega})$, for all $p \in [2, \infty)$, with $(-\Delta)^s\varphi \in L_s^\infty(\mathbb{R}^n)$.
Indeed, assume $\varphi \in L^2(\mathbb{R}^n)$, with $\varphi = 0$ in $\mathbb{R}^n \setminus \Omega$ and (distributional) s-Laplacian $(-\Delta)^s\varphi \in L_s^\infty(\mathbb{R}^n)$. Then, since $\varphi = 0$ outside of Ω, one has $(-\Delta)^s\varphi$ is smooth outside of Ω, decaying as $|x|^{-n-2s}$ at infinity. Therefore, $(-\Delta)^s\varphi \in L^1(\mathbb{R}^n)$ as well, so that $(-\Delta)^s\varphi \in L^p(\mathbb{R}^n)$ for all $p \in [1, \infty]$. Now recall that (see e.g. [1, Corollary 4.56])

$$\{\varphi \in L^2(\mathbb{R}^n) : (-\Delta)^s\varphi \in L^2(\mathbb{R}^n)\} \simeq W^{2s,2}(\mathbb{R}^n).$$

So, in particular, $\varphi \in H^s(\mathbb{R}^n)$ and $(-\Delta)^s\varphi$ is well defined also in the weak sense. By [5, Theorem 2.4]), we conclude that $\varphi \in W^{2s,p}(\mathbb{R}^n)$ for all $p \in [2, \infty)$.

The regularity of very weak solutions to (3.2) can be improved locally to Sobolev and even to classical regularity. This result is contained in [15], where the authors prove it on the unit ball of $\mathbb{R}^n$. Due to the local nature of the estimates, we may state the result for any open set $\Omega \subset \mathbb{R}^n$.

Theorem 3.6 (Regularity of very weak solutions)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $s \in (0,1)$. Let $g \in L^1_s(\mathbb{R}^n)$ and u a very weak solution to (3.2). Then, we have:*

- *$u \in H^s_{\mathrm{loc}}(\Omega)$ and*

$$\|u\|_{H^s(B)} \le C(B)\|u\|_{L^1_s(\mathbb{R}^n)} \qquad \forall B \Subset \Omega;$$

- *u is real analytic in Ω and*

$$\|D^\alpha u\|_{L^\infty(B')} \le c^{|\alpha|}\alpha! C(B,B',n,s)\left(\|u\|_{L^\infty(B')} + \|u\|_{L^1_s(\mathbb{R}^n)}\right) \qquad \forall \alpha \in \mathbb{N}^n_0 \quad \forall B' \Subset B \Subset \Omega,$$

  for a universal constant $c > 0$.

4 Proof of the main results

We divide the proof of Theorem 1.1 into four parts: We start with ensuring the positivity of e_∞, then we show existence, PDE derivation, and uniqueness of minimisers respectively.

4.1 Strict positivity of the minimum

Let us prove that under the assumption that $u_0 \not\equiv 0$ in $\mathbb{R}^n \setminus \Omega$, we have $e_\infty > 0$.
By seek of contradiction, assume that $e_\infty = 0$. Then, there exists a sequence $(u_k)_k \subset \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$ with $E_\infty(u_k) \to 0$ as $k \to \infty$. Thus,

$$(-\Delta)^s u_k \to 0 \qquad \text{uniformly in } \mathbb{R}^n. \tag{4.1}$$

Set $v_k := u_k - u_0$, then $v_k \in W^{s,p}_0(\overline{\Omega})$ and $f_k := (-\Delta)^s v_k = (-\Delta)^s u_k - (-\Delta)^s u_0 \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, for all $p > 1$. In particular, if $\Omega' \subset \mathbb{R}^n$ is an open bounded set with C^2-boundary such that $\Omega \Subset \Omega'$, we have $f_k \in L^p(\Omega')$. Now, observe that v_k is a weak solution to

$$\begin{cases} (-\Delta)^s v_k = f_k & \text{in } \Omega', \\ \qquad\quad v_k = 0 & \text{in } \mathbb{R}^n \setminus \Omega'. \end{cases}$$

By the Calderón-Zygmund estimates in Theorem 3.2, we have

$$\|v_k\|_{W^{s,p}(\mathbb{R}^n)} \le C\|f_k\|_{L^p(\Omega')} \le C\left(\|(-\Delta)^s u_k\|_{L^p(\Omega')} + \|(-\Delta)^s u_0\|_{L^p(\Omega')}\right)$$

for a constant $C > 0$ independent of k. From these estimates and (4.1), we obtain

$$\limsup_{k\to\infty} \|u_k\|_{W^{s,p}(\mathbb{R}^n)} \le \|u_0\|_{W^{s,p}(\mathbb{R}^n)} + C\|(-\Delta)^s u_0\|_{L^p(\Omega')}.$$

Thus, $(u_k)_k$ is bounded in $W^{s,p}(\mathbb{R}^n)$, for all $p > 1$. By the weak compactness of $W^{s,p}(\mathbb{R}^n)$, there exists $u \in W^{s,p}(\mathbb{R}^n)$ and a (non relabelled) subsequence of u_k such that $u_k \rightharpoonup u$ weakly in $W^{s,p}(\mathbb{R}^n)$, as $k \to \infty$. Clearly, $u = u_0$ in $\mathbb{R}^n \setminus \Omega$ and $(-\Delta)^s u = 0$ in $\mathbb{R}^n$, therefore $u \in \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$. But, since an entire s-harmonic function is necessarily affine (see [20]) and $u \in L^p(\mathbb{R}^n)$ for every $p > 1$, we must have $u = 0$ in $\mathbb{R}^n$, that is a contradiction.

4.2 Existence

Let $p > 1$ and recall that $u_0 \in \mathcal{W}^{2s,\infty}(\Omega)$. Let $w \in C^2(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ be such that $w > 0$ and $\int_{\mathbb{R}^n} w = 1$. Define the weighted potential space

$$\mathscr{L}_w^{2s,p}(\mathbb{R}^n) := \left\{ u \in W^{s,p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |(-\Delta)^s u|^p \, w < \infty \right\},$$

equipped with the norm

$$\|u\|_{\mathscr{L}_w^{2s,p}(\mathbb{R}^n)} := \|u\|_{W^{s,p}(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} |(-\Delta)^s u|^p \, w \right)^{\frac{1}{p}}.$$

This is a reflexive Banach space, since it is isometrically isomorphic to a closed subspace of $W^{s,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$. We want to consider its subspace

$$\mathscr{L}_{w,u_0}^{2s,p}(\Omega) := \{ u \in \mathscr{L}_w^{2s,p}(\mathbb{R}^n) : u = u_0 \quad \text{in } \mathbb{R}^n \setminus \Omega \}$$

and minimise the functional

$$E_p(u) := \left(\int_{\mathbb{R}^n} |(-\Delta)^s u(x)|^p \, w(x) dx \right)^{\frac{1}{p}} \tag{4.2}$$

on $\mathscr{L}_{w,u_0}^{2s,p}(\Omega)$. Notice that $\mathcal{W}^{2s,\infty}(\mathbb{R}^n) \subset \mathscr{L}_w^{2s,p}(\mathbb{R}^n)$, so, in particular, we have $E_p(u_0) < \infty$.
Let us prove that E_p is (weakly) coercive in $\mathscr{L}_{w,u_0}^{2s,p}(\Omega)$. Assume that $E_p(u) \le M$ for some $u \in \mathscr{L}_{w,u_0}^{2s,p}(\Omega)$. Set $v := u - u_0 \in W_0^{s,p}(\overline{\Omega})$ and $f := (-\Delta)^s v = (-\Delta)^s u - (-\Delta)^s u_0 \in L^p_{\text{loc}}(\mathbb{R}^n)$, for all $p > 1$. In particular, if $\Omega' \subset \mathbb{R}^n$ is an open bounded set with C^2-boundary such that $\Omega \Subset \Omega'$, we have $f \in L^p(\Omega')$. Thus, the function v (weakly) solves the Dirichlet problem

$$\begin{cases} (-\Delta)^s v = f & \text{in } \Omega', \\ v = 0 & \text{in } \mathbb{R}^n \setminus \Omega'. \end{cases}$$

Applying the Calderón-Zygmund estimates in Theorem 3.2, we have

$$\|v\|_{W^{s,p}(\mathbb{R}^n)} \le C \|f\|_{L^p(\Omega')},$$

for some constant $C = C(\Omega', s, n, p)$. From this estimate and the trivial inequality

$$\|g\|_{L^p(\Omega')} \le \left(\inf_{\Omega'} w \right)^{-\frac{1}{p}} \left(\int_{\mathbb{R}^n} |g|^p \, w \right)^{\frac{1}{p}} \qquad \forall g \in L^\infty(\mathbb{R}^n),$$

we deduce

$$\begin{aligned} \|u\|_{W^{s,p}(\mathbb{R}^n)} &\le C \left(\|(-\Delta)^s u\|_{L^p(\Omega')} + \|(-\Delta)^s u_0\|_{L^p(\Omega')} + \|u_0\|_{W^{s,p}(\mathbb{R}^n)} \right) \\ &\le C \left(E_p(u) + \|u_0\|_{\mathscr{L}_w^{2s,p}(\mathbb{R}^n)} \right) \\ &\le C \left(M + \|u_0\|_{\mathscr{L}_w^{2s,p}(\mathbb{R}^n)} \right), \end{aligned}$$

for a constant $C = C(\Omega', s, n, p, w)$. The right hand side provides an upper bound also for $\|u\|_{\mathscr{L}_w^{2s,p}(\mathbb{R}^n)}$, ensuring the coercivity of E_p.
Clearly, E_p is weakly lower semicontinuous in $\mathscr{L}_w^{2s,p}(\mathbb{R}^n)$. Therefore, we can apply Direct Methods

on $\mathscr{L}^{2s,p}_{w,u_0}(\Omega)$ to get a minimiser u_p of E_p in this space.
Now fix $q > 1$ and consider the sequence of (extended) functionals $(E_p)_p$ in $\mathscr{L}^{2s,q}_{w,u_0}(\Omega)$ defined as

$$E_p(u) = \begin{cases} \left(\int_{\mathbb{R}^n} |(-\Delta)^s u(x)|^p w(x) \right)^{\frac{1}{p}}, & u \in \mathscr{L}^{2s,q}_{w,u_0}(\Omega) \cap \mathscr{L}^{2s,p}_{w,u_0}(\Omega), \\ +\infty & , \quad u \in \mathscr{L}^{2s,q}_{w,u_0}(\Omega) \setminus \mathscr{L}^{2s,p}_{w,u_0}(\Omega), \end{cases} \qquad \forall p > 1.$$

In a similar way, we can define E_∞ in $\mathscr{L}^{2s,q}_{w,u_0}(\Omega)$ as

$$E_\infty(u) = \begin{cases} \|(-\Delta)^s u\|_{L^\infty(\mathbb{R}^n)}, & u \in \mathcal{W}^{2s,\infty}_{u_0}(\Omega), \\ +\infty & , \quad u \in \mathscr{L}^{2s,q}_{w,u_0}(\Omega) \setminus \mathcal{W}^{2s,\infty}_{u_0}(\Omega). \end{cases}$$

Using the Hölder inequality, it is not difficult to prove that $(E_p)_p$ is monotone increasing and

$$\lim_{p\to\infty} E_p(u) = E_\infty(u) \qquad \forall u \in \mathscr{L}^{2s,q}_{w,u_0}(\Omega).$$

Let us show that $(E_p)_p$ is equi-coercive in $\mathscr{L}^{2s,q}_{w,u_0}(\Omega)$ with respect to p. Assume $E_p(u) \leq M$ for all $p \geq q$. By fixing a domain $\Omega' \Supset \Omega$ with C^2-boundary, we can apply the fractional Calderon-Zygmund estimates as before. Together with the Hölder inequality, this yields

$$\begin{aligned} \|u\|_{\mathscr{L}^{2s,q}_w(\mathbb{R}^n)} &\leq C\left(\|u_0\|_{\mathscr{L}^{2s,q}_w(\mathbb{R}^n)} + E_q(u)\right) \\ &\leq C\left(\|u_0\|_{\mathscr{L}^{2s,q}_w(\mathbb{R}^n)} + E_p(u)\right) \\ &\leq C\left(\|u_0\|_{\mathscr{L}^{2s,q}_w(\mathbb{R}^n)} + M\right), \end{aligned}$$

for a constant $C = C(\Omega', s, n, q, w)$, giving the equi-coercivity of $(E_p)_p$.
Since $(E_p)_p$ is a monotone increasing sequence of weakly lower semicontinuous functionals on $\mathscr{L}^{2s,q}_{w,u_0}(\Omega)$, its Γ-limit is given by (see [10, Remark 2.12])

$$\Gamma - \lim_p E_p = \lim_p E_p = E_\infty.$$

By the Fundamental Theorem of Gamma Convergence [10, Thm. 2.10], if $(u_p)_p$ is a sequence of minimisers of E_p, up to passing to a subsequence, we have $u_p \rightharpoonup u_\infty$ weakly $\mathscr{L}^{2s,q}_{w,u_0}(\Omega)$ where u_∞ is a minimiser of E_∞ on $\mathcal{W}^{2s,\infty}_{u_0}(\Omega)$. By definition of E_∞, we necessarily have $u_\infty \in \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$. Furthermore, Theorem 2.10 in [10] also implies

$$E_p(u_p) \to E_\infty(u_\infty) \quad \text{as } p \to \infty. \tag{4.3}$$

4.3 PDE derivation

Now we work towards the proof of (1.3). Let us begin by writing down the Euler-Lagrange equation satisfied by u_p:

$$\int_{\mathbb{R}^n} f_p(x)(-\Delta)^s v(x)\,dx = 0, \quad \forall v \in \mathscr{L}^{2s,p}_{w,0}(\Omega), \tag{4.4}$$

where, for $e_p := E_p(u_p)$, $f_p : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$f_p := e_p^{1-p} w|(-\Delta)^s u_p|^{p-2}(-\Delta)^s u_p. \tag{4.5}$$

By the non-trivial choice of the boundary data u_0, we have $e_\infty > 0$. Thus, we may assume without loss of generality that $e_p \geq e_1 > 0$ for all $p > 1$. So, f_p is a well defined measurable function on $\mathbb{R}^n$. Let us prove that that $(f_p)_p$ is bounded in $L^1(\mathbb{R}^n)$: using the Hölder inequality w.r.t. the measure $w\mathscr{L}^n \llcorner \mathbb{R}^n$ and recalling that $\int_{\mathbb{R}^n} w = 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |f_p| &= e_p^{1-p} \int_{\mathbb{R}^n} |(-\Delta)^s u_p|^{p-1} w \\ &\leq e_p^{1-p} \left(\int_{\mathbb{R}^n} |(-\Delta)^s u_p|^p w \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} w \right)^{\frac{1}{p}} \\ &= e_p^{1-p} e_p^{p-1} \cdot 1 \\ &= 1.\end{aligned} \tag{4.6}$$

Observe that by Remark 3.5, a test function φ in Definition 3.4 is such that $\varphi \in W_0^{2s,p}(\overline{\Omega})$. In particular, $\varphi \in \mathscr{L}^{2s,p}_{w,0}(\Omega)$, so it is a good test function also for (4.4). Hence, f_p is a *very weak s-harmonic* function on Ω. Equivalently, it is a very weak solution to the Fractional Laplace problem on Ω w.r.t its own Dirichlet data.
Now we have all the tools to construct the function f_∞ appearing in (1.3). Thanks to Theorem 3.6, we have $f_p \in H^s_{\text{loc}}(\Omega)$ and the following estimate holds: for any open set $B \Subset \Omega$ there exists a constant $C = C(B) > 0$ such that

$$\|f_p\|_{H^s(B)} \leq C \|f_p\|_{L^1(\mathbb{R}^n)}. \tag{4.7}$$

By (4.6) and compact Sobolev embeddings (see [18, Corollary 7.2]), if $n > 2s$, there exist a subsequence of (f_p), which we denote by (f_p) as well, and a function f_∞ such that

$$f_p \to f_\infty \quad \text{in } L^q(B) \quad \forall q \in [1, 2_s^*),$$

where $2_s^* := \frac{2n}{n-2s}$. Observe that, thanks to (4.6), $f_\infty \in L^1(\Omega) \cap L^q(B)$ for all $q \in [1, 2_s^*)$ and up to passing to a further subsequence

$$f_p \to f_\infty \quad \text{a.e. in } \Omega. \tag{4.8}$$

Actually, the convergence is locally uniform in Ω. Indeed, by classical regularity (Theorem 3.6), we have $f_p \in C^\infty(\Omega)$. Thus, f_p is s-harmonic in Ω in the classical sense. In particular, it is a local weak solution to the Fractional Laplace problem in Ω (w.r.t. its own Dirichlet boundary data). So, from Theorems 1.4 and 3.2 in [11], for any open set $B \Subset \Omega$, there exist $\sigma \in (0,1)$ and a constant $C = C(\sigma, n, s, B)$ such that

$$\|f_p\|_{C^{0,\sigma}(B)} \leq C \left(\|f_p\|_{L^2(B)} + \|f_p\|_{L^1(\mathbb{R}^n)} \right).$$

Hence, by (4.7) we obtain

$$\|f_p\|_{C^{0,\sigma}(B)} \leq C \|f_p\|_{L^1(\mathbb{R}^n)}. \tag{4.9}$$

Thus, the L^1-bound (4.6) of f_p in $\mathbb{R}^n$ ensures

$$f_p \to f_\infty \quad \text{locally uniformly in } \Omega, \tag{4.10}$$

up to passing to a further subsequence.
Let us prove that f_∞ is real analytic in Ω. By Theorem 3.6, for any pair of open sets $B' \Subset B \Subset \Omega$ and multindex $\alpha \in \mathbb{N}^n$, we have

$$\|D^\alpha f_p\|_{L^\infty(B')} \leq c^{|\alpha|} \alpha! C(B, B', n, s) \left(\|f_p\|_{L^\infty(B)} + \|f_p\|_{L^1(\mathbb{R}^n)} \right).$$

for a universal constant $c > 0$. Thus, using (4.9) and the L^1-bound of f_p, we can pass to the limit in the above estimate, obtaining

$$\|D^\alpha f_\infty\|_{L^\infty(B')} \le c^{|\alpha|}\alpha! C(B, B', n, s, \sigma),$$

which provides the analyticity of f_∞ in Ω.
Moreover, again from (4.6), we infer the existence of a Radon measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ with $|\mu|(\mathbb{R}^n) \le 1$ and such that, up to passing to a subsequence,

$$f_p \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\mathbb{R}^n).$$

In particular, by (4.4), f_p is s-harmonic in the sense of distributions, namely

$$\int_{\mathbb{R}^n} f_p(-\Delta)^s\varphi\, dx = 0 \qquad \forall \varphi \in C_c^\infty(\Omega). \tag{4.11}$$

Notice that if $\varphi \in C_c^\infty(\Omega)$, then $(-\Delta)^s\varphi \in C^\infty(\mathbb{R}^n)$ and

$$|(-\Delta)^s\varphi(x)| \le c_{n,s}\|\varphi\|_\infty |\Omega| d(x, \partial\Omega)^{-n-2s} \qquad \forall x \in \mathbb{R}^n \setminus \Omega.$$

So, $(-\Delta)^s\varphi \in C_0(\mathbb{R}^n)$ and we can pass to the limit in (4.11), obtaining

$$\int_{\mathbb{R}^n} (-\Delta)^s\varphi\, d\mu = 0 \qquad \forall \varphi \in C_c^\infty(\Omega),$$

i.e. μ is s-harmonic in Ω. In addition, thanks to (4.8), we can write

$$\mu \llcorner \Omega = f_\infty \mathscr{L}^n \llcorner \Omega. \tag{4.12}$$

The next result is central in our argument.

Lemma 4.1. (Non-triviality of f_∞) *The map f_∞ constructed above is such that $f_\infty \not\equiv 0$ in Ω.*

Before proving Lemma 4.1, we need a couple of technical results. They can be seen essentially as a consequence of Theorem 1.1 in [17]. The latter can be stated as "all functions are locally s-harmonic up to a small error". We recall (and briefly prove) it here in a more general version, for Hölder continuous functions, which can be easily obtained from Theorem 1.1 in [27], which is another version of [17, Theorem 1.1].

Theorem 4.2. (All functions are locally s-harmonic up to a small error) *Let $\alpha \in \mathbb{R}^+$ and suppose that $u \in C^\alpha(\overline{B}_1)$. Then, for every $\varepsilon > 0$, there exists $u_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ such that*

$$\begin{cases} (-\Delta)^s u_\varepsilon = 0 & \text{in } \overline{B}_1, \\ \|u - u_\varepsilon\|_{C^\alpha(B_1)} \le \varepsilon. \end{cases}$$

Proof. For integer α, the result reduces to Theorem 1.1 in [27]. So, we can assume $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Consider a mollification $u_\eta \in C^\infty(\mathbb{R}^n)$ of u, with $\eta > 0$, such that

$$\|u - u_\eta\|_{C^\alpha(B_1)} < \eta.$$

Notice that $u_{\eta_{|\bar{B}_1}} \in C^k(\bar{B}_1)$, for all $k \in \mathbb{N}$. So, by applying [27, Theorem 1.1] to $u_{\eta_{|\bar{B}_1}}$, for any $\varepsilon > 0$ and $k \in \mathbb{N}$, we can find $u_\eta^\varepsilon \in C_c^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} (-\Delta)^s u_\eta^\varepsilon = 0 & \text{in } \overline{B}_1, \\ \|u_\eta - u_\eta^\varepsilon\|_{C^k(B_1)} \le \dfrac{\varepsilon}{2}. \end{cases}$$

For $k > \alpha$, we can estimate

$$\|u - u_\eta^\varepsilon\|_{C^\alpha(B_1)} \le \|u_\eta - u_\eta\|_{C^\alpha(B_1)} + \|u_\eta - u_\eta^\varepsilon\|_{C^k(B_1)} \le \eta + \frac{\varepsilon}{2}.$$

We conclude by choosing $\eta < \frac{\varepsilon}{2}$. □

The following lemma provides instead that "all functions are locally almost s-harmonic".

Lemma 4.3. (All functions are locally almost s-harmonic) *Suppose that $u \in C^\alpha(\overline{B}_1)$ for some $\alpha > 2s$. Then, for every $\varepsilon > 0$, there exists $u_\varepsilon \in C_c^\alpha(\mathbb{R}^n)$ such that*

$$\begin{cases} |(-\Delta)^s u_\varepsilon| \le \varepsilon & \text{in } \overline{B}_1, \\ u_\varepsilon = u & \text{in } B_1. \end{cases}$$

Proof. By suitably extending u to the whole $\mathbb{R}^n$, we can assume that $u \in C_c^\alpha(\mathbb{R}^n)$. Consider a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ in B_2, $\varphi = 0$ in $\mathbb{R}^n \setminus B_3$, and $0 < \varphi < 1$ elsewhere. Let us fix $\varepsilon > 0$ and apply Theorem 4.2 to $u_{|\bar{B}_3} \in C^\alpha(\overline{B}_3)$. Then, we can find a function $v_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ which is s-harmonic in B_3 and

$$\|u - v_\varepsilon\|_{C^\alpha(B_3)} \le \varepsilon. \tag{4.13}$$

Now set

$$u_\varepsilon := \varphi(u - v_\varepsilon) + v_\varepsilon \in C_c^\alpha(\mathbb{R}^n).$$

Note that $u_\varepsilon = u$ in B_2. Moreover, we can compute its s-Laplacian using the Leibniz formula (see for instance [9]):

$$(-\Delta)^s u_\varepsilon = (u - v_\varepsilon)(-\Delta)^s\varphi + \varphi\,(-\Delta)^s(u - v_\varepsilon) - B(\varphi, u - v_\varepsilon) + (-\Delta)^s v_\varepsilon \qquad \text{in } \mathbb{R}^n.$$

From the s-harmonicity of v_ε in B_3, we have

$$(-\Delta)^s u_\varepsilon = (u - v_\varepsilon)(-\Delta)^s\varphi + \varphi(-\Delta)^s u - B(\varphi, u - v_\varepsilon) \qquad \text{in } B_3. \tag{4.14}$$

Now, for $x \in \overline{B}_1$, we have

$$\begin{aligned} B(\varphi, u - v_\varepsilon)(x) =& c_{n,s} \int_{\mathbb{R}^n} \frac{(\varphi(x) - \varphi(y))((u - v_\varepsilon)(x) - (u - v_\varepsilon)(y))}{|x - y|^{n+2s}}\, dy \\ =& \varphi(x)\, c_{n,s} \left(\int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\, dy + \int_{\mathbb{R}^n} \frac{v_\varepsilon(x) - v_\varepsilon(y)}{|x - y|^{n+2s}}\, dy \right) - \\ &- c_{n,s} \int_{\mathbb{R}^n} \frac{\varphi(y)((u - v_\varepsilon)(x) - (u - v_\varepsilon)(y))}{|x - y|^{n+2s}}\, dy \\ =& \varphi(x)(-\Delta)^s u(x) - c_{n,s} \int_{\mathbb{R}^n} \frac{\varphi(y)((u - v_\varepsilon)(x) - (u - v_\varepsilon)(y))}{|x - y|^{n+2s}}\, dy, \end{aligned}$$

where we used that v_ε is s-harmonic in $\overline{B}_1$ and each integral is understood in the principal value sense.

Thus, by using (4.14) and setting $w_\varepsilon := u - v_\varepsilon$, we have

$$(-\Delta)^s u_\varepsilon(x) = w_\varepsilon(x)(-\Delta)^s\varphi(x) + c_{n,s} \int_{\mathbb{R}^n} \frac{\varphi(y)(w_\varepsilon(x) - w_\varepsilon(y))}{|x - y|^{n+2s}}\, dy \qquad \forall x \in \overline{B}_1.$$

Since $\varphi = 1$ in B_2 and vanishes out of B_3, that the last integral can be written as

$$\int_{\mathbb{R}^n} \frac{\varphi(y)(w_\varepsilon(x) - w_\varepsilon(y))}{|x-y|^{n+2s}}\, dy = \int_{B_3} \frac{\varphi(y)(w_\varepsilon(x) - w_\varepsilon(y))}{|x-y|^{n+2s}}\, dy$$
$$= \int_{B_{\frac{1}{2}}(x)} \frac{w_\varepsilon(x) - w_\varepsilon(y)}{|x-y|^{n+2s}}\, dy + \int_{B_3 \setminus B_{\frac{1}{2}}(x)} \frac{\varphi(y)(w_\varepsilon(x) - w_\varepsilon(y))}{|x-y|^{n+2s}}\, dy.$$

In order to estimate the first integral in the right hand side, we distinguish between the following cases. It is not restrictive to assume that $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $\alpha < 2$.

- $2s < \alpha < 1$.
  By using the Hölder continuity of w_ε and (4.13), we have

$$\left| \int_{B_{\frac{1}{2}}(x)} \frac{w_\varepsilon(x) - w_\varepsilon(y)}{|x-y|^{n+2s}}\, dy \right| \le [w_\varepsilon]_{C^\alpha(B_3)} \int_{B_{\frac{1}{2}}(x)} \frac{dy}{|x-y|^{n+2s-\alpha}} \le \varepsilon\, C,$$

  where $C = C(n,s,\alpha) := \int_{B_{\frac{1}{2}}(x)} \frac{dy}{|x-y|^{n+2s-\alpha}}$ is a positive constant.

- $2s \le 1 < \alpha$.
  This case is similar to the previous one.

- $1 < 2s < \alpha$.
  By using the symmetry of $B_{\frac{1}{2}}(x)$, the Lagrange mean value Theorem, and the fact that $\nabla w_\varepsilon \in C^{\alpha-1}(B_3, \mathbb{R}^n)$, we have

$$\left| \int_{B_{\frac{1}{2}}(x)} \frac{w_\varepsilon(x) - w_\varepsilon(y)}{|x-y|^{n+2s}}\, dy \right| = \left| \int_{B_{\frac{1}{2}}(x)} \frac{w_\varepsilon(x) - w_\varepsilon(y) - \nabla w_\varepsilon(\xi)\cdot(x-y)}{|x-y|^{n+2s}}\, dy \right|$$
$$\le [\nabla w_\varepsilon]_{C^{\alpha-1}(B_3)} \int_{B_{\frac{1}{2}}(x)} \frac{dy}{|x-y|^{n+2s-\alpha}}$$
$$\le \varepsilon\, C.$$

On the other hand,

$$\left| \int_{B_3 \setminus B_{\frac{1}{2}}(x)} \frac{\varphi(y)(w_\varepsilon(x) - w_\varepsilon(y))}{|x-y|^{n+2s}}\, dy \right| \le \int_{B_3 \setminus B_{\frac{1}{2}}(x)} \frac{|w_\varepsilon(x) - w_\varepsilon(y)|}{|x-y|^{n+2s}}\, dy$$
$$\le 2\|w_\varepsilon\|_{L^\infty(B_3)} 2^{n+2s} |B_3|$$
$$\le C'\varepsilon,$$

for a constant $C' = C'(n,s) > 0$.
Therefore, we can estimate the s-Laplacian of u_ε as

$$|(-\Delta)^s u_\varepsilon| \le C''\varepsilon \qquad \text{in } \overline{B}_1,$$

for a constant $C'' = C''\left(\|(-\Delta)^s\varphi\|_{L^\infty(B_1)}, C, C'\right) > 0$ independent of ε, providing the statement. □

As a consequence of Lemma 4.3, we are able to construct a competitor $w_\varepsilon \in \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$ with small Fractional Laplacian in $\mathbb{R}^n \setminus \Omega$.

Corollary 4.4. (Improving boundary conditions) *Suppose that $u_0 \in C^{2s+\gamma}_c(\mathbb{R}^n)$, for some $\gamma > 0$. Then, for every $\varepsilon > 0$, there exists $w_\varepsilon \in C^{2s+\gamma}_c(\mathbb{R}^n)$ such that*

$$\begin{cases} |(-\Delta)^s w_\varepsilon| \leq \varepsilon & \text{in } \mathbb{R}^n \setminus \Omega, \\ w_\varepsilon = u_0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Proof. Step 1: case $\Omega = B_1$.
Consider the Kelvin Transform $\mathcal{K} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ defined as

$$\mathcal{K}(x) := \frac{x}{|x|^2}.$$

Observe that $\mathcal{K}(B_1) = \mathbb{R}^n \setminus \overline{B}_1$ and $\mathcal{K}_{|\partial B_1} = \mathrm{id}_{|\partial B_1}$. Now, consider the function $v := (u_0)_{\mathcal{K}} : \mathbb{R}^n \to \mathbb{R}$ defined as

$$v(x) := (u_0)_{\mathcal{K}}(x) := |x|^{2s-n} u_0(\mathcal{K}(x)), \qquad \forall x \in \mathbb{R}^n.$$

Since u_0 has compact support, the origin is not a singularity for v, which is then well defined and of class $C^{2s+\gamma}$ in $\mathbb{R}^n$.
Let us fix $\varepsilon > 0$. We can apply Lemma 4.3 to the restriction $v_{|B_1} \in C^{2s+\gamma}(\overline{B}_1)$, obtaining a function $v_\varepsilon \in C^{2s+\gamma}_c(\mathbb{R}^n)$ such that

$$\begin{cases} |(-\Delta)^s v_\varepsilon| \leq \varepsilon & \text{in } \overline{B}_1, \\ v_\varepsilon = v & \text{in } B_1. \end{cases} \tag{4.15}$$

Now define $w_\varepsilon := (v_\varepsilon)_{\mathcal{K}}$. Again, since v_ε has compact support, w_ε is well defined and $C^{2s+\gamma}$ in $\mathbb{R}^n$. Moreover, it is immediate to see that

$$w_\varepsilon = u_0 \qquad \text{in } \mathbb{R}^n \setminus B_1.$$

Finally, using in [1, Prop. 1.22(ii)] (which holds also for $C^{2s+\gamma}$-functions) and (4.15), we can compute and estimate the Fractional Laplacian of w_ε for $x \in \mathbb{R}^n \setminus B_1$ as

$$|(-\Delta)^s w_\varepsilon(x)| = |x|^{-2s-n} |(-\Delta)^s v_\varepsilon(\mathcal{K}(x))| \leq \varepsilon.$$

Step 2: general case.
It is enough to prove the thesis in $\mathbb{R}^n \setminus B_r(x)$, for a fixed ball $B_r(x) \Subset \Omega$. To this purpose, we can consider the generalised Kelvin Transform $\mathcal{K}_{r,x} : \mathbb{R}^n \setminus \{x\} \to \mathbb{R}^n \setminus \{x\}$, defined as

$$\mathcal{K}_{r,x}(y) = r^2 \frac{y-x}{|y-x|^2} + x$$

and the transformation

$$u_{\mathcal{K}_{r,x}}(y) = |y-x|^{2s-n} u(\mathcal{K}_{r,x}(y)),$$

for a function $u \in C^{2s+\gamma}_c(\mathbb{R}^n)$. By repeating the argument of the previous case, the conclusion follows. □

Remark 4.5. We recall that our non-trivial choice of u_0, i.e. $u_0 \not\equiv 0$ in $\mathbb{R}^n \setminus \Omega$, makes the result of Corollary 4.4 not obvious. Moreover, one cannot expect to pass to the limit as $\varepsilon \to 0$ in the thesis. In other words, the result does not hold for $\varepsilon = 0$, since otherwise one would find an extension w of $u_{0|\mathbb{R}^n \setminus \Omega}$ to the whole $\mathbb{R}^n$ which is s-harmonic in $\mathbb{R}^n \setminus \Omega$. Since s-harmonic functions satisfy the unique continuation property on their s-harmonicity domain, w is forced to vanish on $\mathbb{R}^n \setminus \Omega$, contradicting the non-trivial choice of u_0.

Now we are ready to ensure the non-triviality of f_∞ in Ω.

Proof of Lemma 4.1 Let us fix $\varepsilon > 0$ and let $w_\varepsilon \in C_c^{2s+\gamma}(\mathbb{R}^n)$ be the function given by Corollary 4.4, corresponding to our data $u_0 \in C_c^{2s+\gamma}(\mathbb{R}^n)$. By testing (4.4) with $u_p - w_\varepsilon \in \mathscr{L}^{2s,p}_{w,0}(\Omega)$, we get

$$\int_{\mathbb{R}^n} f_p(-\Delta)^s u_p = \int_{\mathbb{R}^n} f_p(-\Delta)^s w_\varepsilon.$$

On the other hand, from the definition of f_p, we can write

$$\int_{\mathbb{R}^n} f_p(-\Delta)^s u_p = e_p^{1-p} \int_{\mathbb{R}^n} w|(-\Delta)^s u_p|^p = e_p.$$

Thus,

$$\int_{\mathbb{R}^n} f_p(-\Delta)^s w_\varepsilon = e_p.$$

Passing to the limit as $p \to \infty$, by recalling (4.3), (4.12), and that $|\mu|(\mathbb{R}^n) \leq 1$, we obtain

$$\begin{aligned} e_\infty &= \int_{\mathbb{R}^n} (-\Delta)^s w_\varepsilon \, d\mu \\ &= \int_\Omega f_\infty(-\Delta)^s w_\varepsilon + \int_{\mathbb{R}^n \setminus \Omega} (-\Delta)^s w_\varepsilon \, d\mu \\ &\leq \int_\Omega f_\infty(-\Delta)^s w_\varepsilon + \|(-\Delta)^s w_\varepsilon\|_{L^\infty(\mathbb{R}^n \setminus \Omega)} |\mu|(\mathbb{R}^n) \\ &\leq \int_\Omega f_\infty(-\Delta)^s w_\varepsilon + \varepsilon. \end{aligned}$$

Therefore, if $\varepsilon < e_\infty$, we have

$$\int_\Omega f_\infty(-\Delta)^s w_\varepsilon \geq e_\infty - \varepsilon > 0,$$

providing that $f_\infty \not\equiv 0$ in Ω. □

We are ready to derive the PDE in (1.3). Thanks to Lemma 4.1, f_∞ is a non-trivial analytic function in Ω, so the set $\Gamma := f_\infty^{-1}(0) \subset \Omega$ has null Lebesgue measure. Moreover, recalling that, for all $q \in (1, \infty)$, $u_p \rightharpoonup u_\infty$ weakly in $\mathscr{L}^{2s,q}_{w,u_0}(\Omega)$, up to passing to a further subsequence, we have

$$(-\Delta)^s u_p \rightharpoonup (-\Delta)^s u_\infty \quad \text{weakly in } L^q_{\text{loc}}(\mathbb{R}^n). \tag{4.16}$$

In particular, the convergence in (4.16) holds (weakly) in $L^q(\Omega)$. Recalling (4.5), we can write

$$|f_p|^{\frac{1}{p-1}} \operatorname{sgn}(f_p) = e_p^{-1} w^{\frac{1}{p-1}} (-\Delta)^s u_p \qquad \text{in } \Omega.$$

If we restrict the above equation to a compact set $K \subset \Omega \setminus \Gamma$ and pass to the limit as $p \to \infty$, thanks to (4.3), (4.10), and Lemma 4.1, we infer

$$\operatorname{sgn}(f_\infty) = e_\infty^{-1}(-\Delta)^s u_\infty \qquad \text{in } K,$$

which provides the validity of (1.3), since $|\Gamma| = 0$.

Now we aim to derive (1.2). Having proved (1.3), it is enough to restrict our attention to $\mathbb{R}^n \setminus \overline{\Omega}$. First, let us show that $(-\Delta)^s u_\infty \in C^\gamma_{\text{loc}}(\mathbb{R}^n \setminus \overline{\Omega})$[3] and $(-\Delta)^s u_\infty(x) \to 0$ as $|x| \to \infty$: Since $u_\infty = u_0$ in $\mathbb{R}^n \setminus \Omega$, we can write

$$(-\Delta)^s u_\infty(x) = (-\Delta)^s u_0(x) + c_{n,s} \int_\Omega \frac{u_0(y) - u_\infty(y)}{|x-y|^{n+2s}} dy \qquad \forall x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

[3]With the notation C^γ_{loc} we understand the space of locally Hölder continuous functions (see Section 2).

Set

$$A_\infty(x) := \int_\Omega \frac{u_0(y) - u_\infty(y)}{|x-y|^{n+2s}} dy \qquad \forall x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

Then, for any $\beta \in \mathbb{N}_0^n$, we can estimate the derivatives of order β of A_∞ as

$$|D^\beta A_\infty(x)| \le d(x, \partial\Omega)^{-n-2s-|\beta|}(\|u_0\|_{L^1(\Omega)} + \|u_\infty\|_{L^1(\Omega)}) \qquad \forall x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

In particular, $A_\infty(x) \to 0$ as $|x| \to \infty$. Hence, since $u_0 \in C_c^{2s+\gamma}(\mathbb{R}^n)$, we have that $(-\Delta)^s u_0(x) \to 0$ as $|x| \to \infty$, thus the same holds for $(-\Delta)^s u_\infty$. Moreover, if $x \in \mathbb{R}^n \setminus \overline{\Omega}$ and $r < \frac{1}{2} d(x, \partial\Omega)$, the triangle inequality implies

$$|D^\beta A_\infty| \le r^{-n-2s-|\beta|}(\|u_0\|_{L^1(\Omega)} + \|u_\infty\|_{L^1(\Omega)}) \qquad \text{in } B_r(x).$$

Thus, $A_\infty \in C^\infty(\mathbb{R}^n \setminus \overline{\Omega})$ and so, being $(-\Delta)^s u_0 \in C^\gamma(\mathbb{R}^n)$ (see [33, Prop. 2.6]), we have that $(-\Delta)^s u_\infty \in C^\gamma_{\text{loc}}(\mathbb{R}^n \setminus \overline{\Omega})$.
As a consequence, we can show that μ has compact support in $\mathbb{R}^n$: Applying the previous argument to $(-\Delta)^s u_p$, we infer $(-\Delta)^s u_p(x) \to 0$ as $|x| \to \infty$. More precisely, since $u_p \rightharpoonup u_\infty$ weakly in $\mathscr{L}^{2s,q}_{w,u_0}(\Omega)$ for all $q > 1$, the norm $\|u_p\|_{L^1(\Omega)}$ is uniformly bounded w.r.t. p, thus the following uniform estimate holds: for every $\varepsilon > 0$ there exists $R > 0$, independent of p, such that

$$\|(-\Delta)^s u_p\|_{L^\infty(\mathbb{R}^n \setminus B_R)} < \varepsilon.$$

Since e_p is bounded from below by $e_1 > 0$, by taking $\varepsilon < e_1$, we have that there exists $\delta \in (0,1)$, independent of p, so that

$$\frac{|(-\Delta)^s u_p|}{e_p} < 1 - \delta \qquad \text{in } \mathbb{R}^n \setminus B_R.$$

Therefore,

$$\left(\frac{|(-\Delta)^s u_p|}{e_p}\right)^{p-1} \to 0 \quad \text{uniformly in } \mathbb{R}^n \setminus B_R, \quad \text{as } p \to \infty.$$

Recalling the definition of f_p in (4.5), we get

$$|f_p| = e_p^{1-p} w |(-\Delta)^s u_p|^{p-1} \le \|w\|_{L^\infty(\mathbb{R}^n)} \left(\frac{|(-\Delta)^s u_p|}{e_p}\right)^{p-1}.$$

Thus,

$$f_p \to 0 \quad \text{uniformly in } \mathbb{R}^n \setminus B_R, \quad \text{as } p \to \infty.$$

In particular, by lower semicontinuity of the mass, μ cannot concentrate out of B_R, providing that $\text{supp}(\mu) \Subset \mathbb{R}^n$.
Now we are ready to derive the PDE (1.2). As we have seen for $(-\Delta)^s u_\infty$, we can prove that $(-\Delta)^s u_p \in C^\gamma_{\text{loc}}(\mathbb{R}^n \setminus \overline{\Omega})$ and for every ball $B \Subset \mathbb{R}^n \setminus \overline{\Omega}$, there exists $C = C(B, n, s)$ such that

$$\|(-\Delta)^s u_p\|_{C^\gamma(B)} \le \|(-\Delta)^s u_0\|_{C^\gamma(B)} + C(\|u_0\|_{L^1(\Omega)} + \|u_p\|_{L^1(\Omega)}).$$

Thus, the uniform bound of $\|u_p\|_{L^1(\Omega)}$ and (4.16) ensure

$$(-\Delta)^s u_p \to (-\Delta)^s u_\infty \quad \text{locally uniformly in } \mathbb{R}^n \setminus \overline{\Omega} \quad \text{as } p \to \infty. \tag{4.17}$$

By writing

$$|f_p|^{\frac{1}{p-1}} = e_p^{-1} w^{\frac{1}{p-1}} |(-\Delta)^s u_p|,$$

and, by recalling that $e_p \to e_\infty$, we have

$$|f_p|^{\frac{1}{p-1}} \to e_\infty^{-1}|(-\Delta)^s u_\infty| \quad \text{locally uniformly in } \mathbb{R}^n \setminus \overline{\Omega} \quad \text{as } p \to \infty.$$

In particular, we observe that if $x \in \mathbb{R}^n \setminus \overline{\Omega}$ is such that $(-\Delta)^s u_\infty(x) \neq 0$, then, in a neighborhood of x, by continuity, $(-\Delta)^s u_p$ and $(-\Delta)^s u_\infty$ have (equal) constant sign. Therefore, since $\text{sgn}(f_p) = \text{sgn}(-\Delta)^s u_p$, the uniform convergence in (4.17) implies

$$\text{sgn}(f_p) \to \text{sgn}(-\Delta)^s u_\infty \quad \text{locally uniformly in } (\mathbb{R}^n \setminus \overline{\Omega}) \cap \{(-\Delta)^s u_\infty \neq 0\} \quad \text{as } p \to \infty. \tag{4.18}$$

Now, from the L^1 bound of f_p in $\mathbb{R}^n$, we can also assume that there exists a measure $\nu \in \mathcal{M}^+(\mathbb{R}^n)$, with $|\nu|(\mathbb{R}^n) \leq 1$ and $|\mu| << \nu$, such that

$$|f_p| \overset{*}{\rightharpoonup} \nu \quad \text{in } \mathcal{M}(\mathbb{R}^n).$$

So, by using (4.18), we get

$$f_p = |f_p| \, \text{sgn} f_p \overset{*}{\rightharpoonup} \nu \, \text{sgn}(-\Delta)^s u_\infty \quad \text{in } (\mathbb{R}^n \setminus \overline{\Omega}) \cap \{(-\Delta)^s u_\infty \neq 0\}.$$

On the other hand,

$$f_p \overset{*}{\rightharpoonup} \mu = |\mu| \frac{d\mu}{d|\mu|} \quad \text{in } \mathbb{R}^n.$$

Therefore, by uniqueness of the polar decomposition of measures, we have

$$\nu = |\mu| \quad \text{and} \quad \frac{d\mu}{d|\mu|} = \text{sgn}(-\Delta)^s u_\infty \quad |\mu| - \text{a.e. in } (\mathbb{R}^n \setminus \overline{\Omega}) \cap \{(-\Delta)^s u_\infty \neq 0\}. \tag{4.19}$$

Now, we claim that $\text{supp}(|\mu| \llcorner \mathbb{R}^n \setminus \overline{\Omega}) \subset \{x \in \mathbb{R}^n \setminus \overline{\Omega} : |(-\Delta)^s u_\infty(x)| = e_\infty\}$. Indeed, suppose by contradiction that $|(-\Delta)^s u_\infty(x)| < e_\infty$ for some $x \in \text{supp}(|\mu| \llcorner \mathbb{R}^n \setminus \overline{\Omega})$. Then, by (4.17), there exists $\varepsilon \in (0,1)$, independent of p, and a radius $r > 0$, such that

$$\frac{|(-\Delta)^s u_p|}{e_p} \leq 1 - \varepsilon \quad \text{in } B_r(x).$$

But then

$$\left(\frac{|(-\Delta)^s u_p|}{e_p} \right)^{p-1} \to 0 \quad \text{uniformly in } B_r(x) \quad \text{as } p \to \infty,$$

implying that

$$|f_p| \to 0 \quad \text{uniformly in } B_r(x) \quad \text{as } p \to \infty.$$

So, we have $|\mu|(B_r(x)) = 0$, leading to a contradiction. Therefore, taking into account (4.19), for every $x \in \text{supp}(|\mu| \llcorner \mathbb{R}^n \setminus \overline{\Omega})$, we conclude that

$$(-\Delta)^s u_\infty(x) = |(-\Delta)^s u_\infty(x)| \, \text{sgn}(-\Delta)^s u_\infty(x) = e_\infty \frac{d\mu}{d|\mu|}.$$

4.4 Uniqueness

We are left with proving the uniqueness of the minimiser u_∞ in $\mathcal{W}^{2s,\infty}_{u_0}(\Omega)$. The key step is to show that every minimiser of E_∞ satisfies a fractional PDE in Ω of the same structure as (1.3). The main idea is to introduce a penalisation term into the Gamma Convergence argument of Section 4.2, forcing the minimiser of the L^p-problem to convergence to a preselected minimiser of E_∞. More explicitly, we prove the following result.

Lemma 4.6. (Necessity of the PDE) *Let u_0 and Ω be as in Theorem 1.1. Suppose that $u \in \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$ is a minimiser of the functional E_∞ defined in (1.1). Then there exists an analytic function $f \in L^1(\Omega) \setminus \{0\}$ such that*

$$(-\Delta)^s u = e_\infty \mathrm{sgn} f \qquad a.e.\ in\ \Omega. \tag{4.20}$$

Proof. Let $u \in \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$ be a minimiser of E_∞. For any $p \geq 2$, we consider the auxiliary functional

$$A_p(v) := E_p(v) + \frac{1}{2} \fint_\Omega |v-u|^2, \qquad v \in \mathscr{L}^{2s,p}_{w,u_0}(\Omega),$$

where the functional E_p is defined in (4.2).
Using the Direct Method, as we did in Section 4.2, it is not difficult to see that A_p attains a minimum point. Let $v_p \in \mathscr{L}^{2s,p}_{w,u_0}(\Omega)$ be a minimiser of A_p and u_p be a minimiser of E_p in the same space. We have

$$e_p = E_p(u_p) \leq E_p(v_p) \leq A_p(v_p) \leq A_p(u) = E_p(u) \leq E_\infty(u) = e_\infty.$$

Since $e_p \to e_\infty$, we have

$$\lim_{p\to\infty} E_p(v_p) = \lim_{p\to\infty} A_p(v_p) = e_\infty. \tag{4.21}$$

Now, if $(v_p)_p$ is a sequence of minimisers of A_p, by the Calderon-Zygmund estimates, we can prove as in Section 4.2 that $(v_p)_p$ is bounded in $\mathscr{L}^{2s,q}_{w,u_0}(\Omega)$ for any fixed $q > 1$. So, up to considering a subsequence, we have $v_p \rightharpoonup v_\infty$ weakly in $\mathscr{L}^{2s,q}_{w,u_0}(\Omega)$ for some $v_\infty \in \mathscr{L}^{2s,q}_{w,u_0}(\Omega)$.
Let us prove that $v_\infty = u$ in $\mathbb{R}^n$: Passing to the limit as $p \to \infty$ in the following inequality

$$e_p + \frac{1}{2} \fint_\Omega |v_p - u|^2 \leq A_p(v_p),$$

by (4.21), we get

$$e_\infty + \frac{1}{2} \fint_\Omega |v_\infty - u|^2 \leq e_\infty,$$

which necessarily implies $v_\infty = u$ in Ω, and therefore in $\mathbb{R}^n$, since they share the same data in the complement of Ω.
Now, denote $a_p := E_p(v_p)$ and write down the Euler-Lagrange equation satisfied by v_p:

$$(-\Delta)^s g_p + v_p - u = 0 \qquad \text{in } \left(\mathscr{L}^{2s,p}_{w,0}(\Omega)\right)^*, \tag{4.22}$$

where g_p is defined by

$$g_p := a_p^{1-p} w |(-\Delta)^s v_p|^{p-2} (-\Delta)^s v_p \qquad \text{a.e. in } \mathbb{R}^n. \tag{4.23}$$

We can prove that g_p is well defined and uniformly bounded in $L^1(\mathbb{R}^n)$ as showed for f_p in Section 4.2. Moreover, by solving the Dirichlet problem

$$\begin{cases} (-\Delta)^s h_p = v_p - u & \text{in } \Omega, \\ \qquad h_p = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

and applying Thorem 3.2, the unique solution h_p is such that

$$\|h_p\|_{W^{s,q}(\mathbb{R}^n)} \leq C \|v_p - u\|_{L^q(\Omega)}.$$

In particular, the function $g_p + h_p \in L^1(\mathbb{R}^n)$ satisfies

$$(-\Delta)^s (g_p + h_p) = 0 \qquad \text{in } \left(\mathscr{L}^{2s,p}_{w,0}(\Omega)\right)^*.$$

Hence, $g_p + h_p$ is a very weak solution to the Fractional Laplace problem in Ω (w.r.t. its own Dirichlet boundary data). By repeating the same argument of Section 4.3, elliptic estimates imply that g_p+h_p is actually a classical solution. In particular, for every ball $B \Subset \Omega$, there exists $\sigma \in (0,1)$ such that

$$\|g_p + h_p\|_{C^{0,\sigma}(B)} \leq C\,\|g_p + h_p\|_{L^1(\mathbb{R}^n)} \leq C\big(\|g_p\|_{L^1(\mathbb{R}^n)} + \|v_p - u\|_{L^q(\Omega)}\big) \leq C,$$

for some constant $C = C(\sigma, n, s, B) > 0$ independent of p (we are using the L^1-bound of g_p and that $v_p \rightharpoonup u$ weakly in $\mathscr{L}^{2s,q}_{w,u_0}(\Omega)$ for any $q > 1$). Since $h_p \to 0$ in $W^{s,q}(\mathbb{R}^n)$ as $p \to \infty$, we have $g_p \to g_\infty$ locally uniformly as $p \to \infty$, for some $g_\infty \in L^1(\Omega)$, up to passing to a subsequence. One can guarantee the analiticity of g_∞ as we did for f_∞ in Section 4.3, since it is the restriction to Ω of an s-harmonic measure $\nu \in \mathcal{M}(\mathbb{R}^n)$ in Ω (the limit measure of g_p). Furthermore, as in Lemma 4.1, we can prove that $g_\infty \not\equiv 0$, since the same argument holds even if $(-\Delta)^s g_p \rightharpoonup 0$ in the sense of distributions only (which is guaranteed by (4.22)). Finally, since $a_p \to e_\infty$ as $p \to \infty$, we can pass to the limit in (4.23) exactly as we did for f_p, obtaining (4.20) with $f = g_\infty$. □

We conclude the section with showing the uniqueness of minimisers of E_∞. Consider two minimisers $u_1, u_2 \in \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$. Then, thanks to Lemma 4.6 they both satisfy

$$|(-\Delta)^s u_1| = |(-\Delta)^s u_2| = e_\infty \qquad \text{a.e. in } \Omega. \tag{4.24}$$

Now, notice that the average $\frac{u_1+u_2}{2} \in \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$. Hence, we have

$$\left\|(-\Delta)^s\left(\frac{u_1+u_2}{2}\right)\right\|_{L^\infty(\mathbb{R}^n)} \geq e_\infty.$$

On the other hand, the triangle inequality gives

$$\left\|(-\Delta)^s\left(\frac{u_1+u_2}{2}\right)\right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2}\|(-\Delta)^s u_1\|_{L^\infty(\mathbb{R}^n)} + \frac{1}{2}\|(-\Delta)^s u_2\|_{L^\infty(\mathbb{R}^n)} = e_\infty.$$

Therefore, we obtain

$$\left\|(-\Delta)^s\left(\frac{u_1+u_2}{2}\right)\right\|_{L^\infty(\mathbb{R}^n)} = e_\infty,$$

giving that $\frac{u_1+u_2}{2}$ is a minimiser as well. By Lemma (4.6), we have

$$\left|(-\Delta)^s\left(\frac{u_1+u_2}{2}\right)\right| = e_\infty \qquad \text{a.e. in } \Omega. \tag{4.25}$$

Putting together (4.24) and (4.25), the triangle inequality implies

$$(-\Delta)^s u_1 = (-\Delta)^s u_2 \qquad \text{a.e. in } \Omega.$$

In particular, the function $v = u_1 - u_2$ solves

$$\begin{cases} (-\Delta)^s v = 0 & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

By uniqueness of weak solutions to the Dirichlet problem for the Fractional Laplacian, we conclude that $u_1 = u_2$ in $\mathbb{R}^n$.

Remark 4.7 (Pointwise representative of $(-\Delta)^s u_\infty$)**.** We conclude this section by observing that $(-\Delta)^s u_\infty \in L^\infty(\mathbb{R}^n)$ holds actually in the pointwise sense, not only in the weak one. Indeed, by arguing as in Remark 3.5, one can easily infer that $u_\infty - u_0 \in W^{2s,p}(\mathbb{R}^n)$ for all $p \in [2, \infty)$. So that $u_\infty \in W^{2s,p}(\mathbb{R}^n)$ as well, since $u_0 \in C_c^{2s+\gamma}(\mathbb{R}^n) \subset W^{2s,p}(\mathbb{R}^n)$. In particular, $(-\Delta)^s u_\infty \in L^2(\mathbb{R}^n)$ in the usual sense, so it possesses a pointwise representative.

5 More general supremands

In this last section, we point out that Theorem 1.1 can be generalised to supremands of the form $F(x, (-\Delta)^s u(x))$, where $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that, for almost every $x \in \mathbb{R}^n$, $F(x, 0) = 0$, $F(x, \cdot)$ is of class C^1, and $|F(x, \cdot)|$ is strictly convex. Assume also:

$$\exists c > 0 : \quad c \le F_\xi(x, \xi) \le \frac{1}{c} \qquad \text{a.e. } x \in \mathbb{R}^n,\ \forall \xi \in \mathbb{R}. \tag{5.1}$$

More explicitely, by suitably modifying the proof in Section 4 according to the assumptions on F (in the same spirit of [22]), one obtains the following result for the functional

$$E_\infty(u) := \| F(\cdot, (-\Delta)^s u) \|_{L^\infty(\mathbb{R}^n)}.$$

Theorem 5.1. *Fix $s \in (0, 1)$ and $n \in \mathbb{N}$, $n > 2s$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and assume that $u_0 \in C_c^{2s+\gamma}(\mathbb{R}^n)$, for some $\gamma > 0$, with $u_0 \not\equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Then the problem*

$$e_\infty := \inf_{\mathcal{W}^{2s,\infty}_{u_0}(\Omega)} E_\infty$$

admits a unique solution $u_\infty \in \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$.
In particular, $(-\Delta)^s u_\infty \in C^\gamma_{\mathrm{loc}}(\mathbb{R}^n \setminus \overline{\Omega})$ and $(-\Delta)^s u_\infty(x) \to 0$ as $|x| \to +\infty$.
Moreover, a system of PDEs can be derived as a necessary and sufficient condition for the minimality of u_∞. Explicitely, there exists a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$, $\mu \neq 0$, with compact support and $|\mu|(\mathbb{R}^n) \le \frac{1}{c}$, such that μ is s-harmonic in Ω and

$$F(\cdot, (-\Delta)^s u_\infty) = e_\infty \frac{d\mu}{d|\mu|} \qquad \text{in } supp|\mu| \setminus \partial\Omega. \tag{5.2}$$

The identity above is understood between L^∞-functions on $supp|\mu| \setminus \partial\Omega$.
Moreover, the restriction $\mu \llcorner \Omega$ is absolutely continuous w.r.t. the Lebesgue measure on Ω, i.e. $\mu \llcorner \Omega = f_\infty \mathscr{L}^n \llcorner \Omega$, for some function $f_\infty \in L^1(\Omega) \setminus \{0\}$, which is real analytic in Ω. In particular, there holds

$$F(\cdot, (-\Delta)^s u_\infty) = e_\infty \mathrm{sgn} f_\infty \qquad \text{a.e. in } \Omega. \tag{5.3}$$

Remark 5.2. The assumptions on F may seem restrictive at first, however, due to the L^∞-structure of the problem, this shares the same solutions with any other problem associated to a reparametrised supremand of the form $g \circ F$, for any lower semicontinuous, strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ (compare [14,22]). Thus, the result of Theorem 5.1 actually applies for a wider class of supremands.

Proof of Theorem 5.1:
i) Strict positivity of e_∞. Assume by contradiction that $e_\infty = 0$. Then, there exists a sequence $(u_k)_k \subset \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$ such that $F(\cdot, (-\Delta)^s u_k) \to 0$ uniformly in $\mathbb{R}^n$. From the assumptions on F, we must have

$$(-\Delta)^s u_k \to 0 \qquad \text{uniformly in } \mathbb{R}^n.$$

Thus, we conclude as in 4.1.

ii) Existence. For $p > 1$ and w as in Section 4.2, minimise the functional

$$E_p(u) := \left(\int_{\mathbb{R}^n} |F(x, (-\Delta)^s u(x))|^p \, w(x) dx \right)^{\frac{1}{p}} \tag{5.4}$$

on $\mathscr{L}^{2s,p}_{w,u_0}(\Omega)$.
By integration of (5.1), we have

$$c|\xi| \le F(x,\xi) \le \frac{|\xi|}{c} \qquad \text{a.e. } x \in \mathbb{R}^n, \, \forall \xi \in \mathbb{R}. \tag{5.5}$$

This ensures that E_p is well defined on $\mathscr{L}^{2s,p}_{w,u_0}(\Omega)$ and (weakly) coercive. In addition, by convexity of $|F(x,\cdot)|$, E_p is also weakly lower semicontinuous in $\mathscr{L}^{2s,p}_{w}(\mathbb{R}^n)$. Thus, Direct Methods provide the existence of a minimiser u_p. The Gamma Convergence argument applies as in Section 4.2, giving the existence of a minimiser $u_\infty \in \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$ of E_∞. Up to passing to a subsequence, as $p \to \infty$, we have

$$\begin{cases} u_p \rightharpoonup u_\infty \text{ weakly in } \mathscr{L}^{2s,q}_{w,u_0}(\Omega), \\ e_p := E_p(u_p) \to E_\infty(u_\infty) = e_\infty. \end{cases}$$

iii) PDE derivation. The Euler-Lagrange equation satisfied by u_p is

$$\int_{\mathbb{R}^n} f_p(x)(-\Delta)^s v(x) \, dx = 0, \quad \forall v \in \mathscr{L}^{2s,p}_{w,0}(\Omega), \tag{5.6}$$

where $f_p : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$f_p := e_p^{1-p} w |F(\cdot, (-\Delta)^s u_p)|^{p-2} F(\cdot, (-\Delta)^s u_p) F_\xi(\cdot, (-\Delta)^s u_p).$$

By using (5.5), we obtain

$$\int_{\mathbb{R}^n} |f_p| \le \frac{1}{c},$$

thus $(f_p)_p$ is bounded in $L^1(\mathbb{R}^n)$. Moreover, f_p is a very weak s-harmonic function on Ω. So, up to extracting a subsequence, it converges in the sense of measure to an s-harmonic measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ with $|\mu|(\mathbb{R}^n) \le \frac{1}{c}$.
Moreover, by elliptic estimates, as in Section 4.3, we have that f_p converges uniformly in Ω to a real analytic function $f_\infty \in L^1(\Omega)$. Of course, $\mu \llcorner \Omega = f_\infty \mathscr{L}^n \llcorner \Omega$.
Let us show the non-triviality of f_∞. Let w_ε be as in Corollary 4.4. By testing (5.6) with $u_p - w_\varepsilon$, using that $\text{sgn} F(\cdot, (-\Delta)^s u_p) = \text{sgn}(-\Delta)^s u_p$ and that $F_\xi(\cdot,\xi)|\xi| \ge c^2 |F(x,\xi)|$, we have

$$\int_{\mathbb{R}^n} f_p (-\Delta)^s w_\varepsilon = \int_{\mathbb{R}^n} f_p (-\Delta)^s u_p \ge c^2 e_p^{1-p} \int_{\mathbb{R}^n} w |F(\cdot, (-\Delta)^s u_p)|^p = c^2 e_p.$$

So that, passing to the limit as $p \to \infty$ and using Corollary 4.4, we obtain

$$c^2 e_\infty = \int_{\mathbb{R}^n} (-\Delta)^s w_\varepsilon \, d\mu \le \int_\Omega f_\infty (-\Delta)^s w_\varepsilon + \frac{\varepsilon}{c},$$

providing $f_\infty \not\equiv 0$ in Ω, for ε small enough.
To derive (5.3), we write

$$|f_p|^{\frac{1}{p-1}} \text{sgn}(f_p) = e_p^{-1} w^{\frac{1}{p-1}} F(\cdot, (-\Delta)^s u_p) F_\xi(\cdot, (-\Delta)^s u_p)^{\frac{1}{p-1}} \qquad \text{in } \Omega.$$

Thanks to (5.5), $F_\xi(\cdot, (-\Delta)^s u_p)$ is bounded from above and below, so we can pass to the limit in the previous equation obtaining (5.3).
We can argue similarly to Section 4.2 to derive (5.2).

iv) Uniqueness. As in Section 4.4, we can prove that any minimiser $u \in \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$ must satisfy

$$F(\cdot, (-\Delta)^s u) = e_\infty \text{sgn} f \qquad \text{a.e. in } \Omega,$$

for some $f \in L^1(\Omega)$ real analytic.
Now, let $u_1, u_2 \in \mathcal{W}^{2s,\infty}_{u_0}(\Omega)$ be two minimisers. Then

$$|F(\cdot, (-\Delta)^s u_1)| = |F(\cdot, (-\Delta)^s u_2)| = e_\infty \qquad \text{a.e. in } \Omega.$$

By reasoning as in Section 4.4, the strict convexity of $|F(x, \cdot)|$ allows to conclude

$$(-\Delta)^s u_1 = (-\Delta)^s u_2 \qquad \text{a.e. in } \Omega.$$

So that $u_1 = u_2$ in $\mathbb{R}^n$ by the uniqueness of the fractional Dirichlet problem. □

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Simone Carano
Department of Mathematical Sciences, University of Bath
Bath BA2 7AY, UK
E-mail: sc3705@bath.ac.uk

Roger Moser
Department of Mathematical Sciences, University of Bath
Bath BA2 7AY, UK
E-mail: r.moser@bath.ac.uk